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Nijenhuis Operators on Trusses and Lie Affgebras

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Abstract

Hochschild cohomology theory of rings is extended from the ring case to abelian heaps with distributing multiplication or trusses.

This new definition of cohomology is then utilised to give the required conditions for a Nijenhuis product on a truss to be associative, which was previously defined by the extension of the Nijenhuis product on an associative ring introduced by Cariñena, Grabowski and Marmo in [Quantum Bi-Hamiltonian Systems, Int. J. Mod. Phys. A **15**, 4797–4810, 2000, arXiv:math-ph/0610011].

The definition of Nijenhuis product and operators on trusses is then linearised to the case of affine spaces with compatible associative multiplications or associative affgebras. It is shown that this construction leads to compatible Lie brackets on an affine space.

This is then taken further, with Lie algebras extended to the affine case using the heap operation, giving them a definition that is not dependent on the unique element 0, but so that they still adhere to antisymmetry and Jacobi properties. It is then looked at how Nijenhuis brackets function on these Lie affgebras and demonstrated how they fulfil the compatibility condition in the affine case.

Declarations

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

Signed [redacted]

Date 29/04/2026

This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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I hereby give consent for my thesis, if accepted, to be available for electronic sharing.

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The University's ethical procedures have been followed and, where appropriate, that ethical approval has been granted.

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Contents

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Chapter 1

Introduction

1.1 Key Results

This thesis looks to both encompass and expand on the ideas presented in the articles *Affine Nijenhuis operators and Hochschild cohomology of trusses* [12] and *Lie and Nijenhuis Brackets on Affine Spaces* [13], both of which were written by Tomasz Brzeziński and myself with the primary motivation of expanding and generalising concepts presented in the papers *Quantum Bi-Hamiltonian Systems* [16] and *Poisson-Nijenhuis structures* [32] respectively.

Some key results of this thesis are:

1). The formulation of an extension of Hochschild cohomology from the usual ring case to the trusses case (Chapter 3).

2). Using the Hochschild cohomology of trusses (Chapter 3) to then give the necessary and sufficient conditions for a Nijenhuis product on a truss to be associative (defined by the extension of the Nijenhuis product on an associative ring in [16]) (Chapter 4).

3). Formulation of the affine Nijenhuis operator (Chapter 5). We observe in Corollary 5.3 that If P is a multiplicative idempotent in $\mathrm{Aff}(A)$, then for all $\alpha \in \mathbb{F}$, $(1-\alpha)P + \alpha\,\mathrm{id}$ is an affine Nijenhuis operator.

4). Expanding on our findings in Corollary 5.3, we propose an intrinsic definition of the left and right Lie bracket in affine space rather than vector space. Such algebraic structures are referred to as *Lie affgebras* (Chapter 6).

5). Defining of the *Nijenhuis operator over a Lie affgebra*, and defining the *Nijenhuis bracket* using definitions adapted from *Poisson-Nijenhuis structures* [32]. It is demonstrated how Nijenhuis operators give rise to a family of compatible Lie brackets on an affine space. (Chapter 7).

Now that these key points have been covered in brief, the rest of this introduction chapter will look to give a more verbose summary and explanation of this thesis and its contents.

1.2 Notation & Convention

A brief note on the notation that is used through this thesis. We are dealing with many different algebraic structures and operations and the best effort has been made to make these differences as clear to the reader as possible. One specific area that could become confusing is the use of different brackets to denote different types of operation. The exact meaning of each bracket will of course be contextual, but as a general rule we are dealing with:

1). *Angled brackets*, or *chevron brackets* denoting ternary operations, such as that of a *truss* (see Definition 2.11), denoted generically as:

$$\langle -, -, - \rangle.$$

Elements of a truss $a, b, c \in T$ can be expressed to determine a fourth element:

$$\langle a, b, c \rangle = d \in T.$$

2). *Square brackets* are used to denote binary operations such as Lie brackets or commutator brackets, again these are highly context dependent, an example would be the classic Lie bracket (see Definition 2.28):

$$[-, -] : A \to A,$$

$$[x, x] = 0,$$

$$[x, y] = -[y, x],$$

where $x, y \in A$ and A is a vector space.

3). *Regular brackets*, *curved brackets* or *parentheses* are also frequently employed as one would expect. These are by far the most varied by context, but should also be the most self-explanatory to the reader. For completeness, here is an example of them being used, as can be seen in the definition of a Nijenhuis product μ (Definition 2.45):

$$\mu(a, b) = ab.$$

In other texts square brackets are often used to denote the ternary operation of a heap or truss, but due to the more prevalent nature of the Lie bracket and commutator brackets notation we employ the angled brackets for the less commonly seen ternary operation for readability.

Furthermore, throughout this thesis, when making reference to a *ring*, it is important to note that we are referring specifically to a *non-unital ring*.

1.3 Motivation

This thesis began with a curiosity around possible applications of *trusses*, a truss $(T, \langle -, -, - \rangle, \cdot)$ is a type of algebraic structure that is composed of a set T, together with both a binary operation $\cdot$ and the less commonly seen ternary operation, which we denote generically by $\langle -, -, - \rangle$. The truss was proposed in *Trusses: between braces and rings* [10] by *Tomasz Brzeziński* as a way to bridge the distributive laws of rings with the distributive laws of braces, hence the denomination *truss*.

Being a relatively new algebraic structure, the opportunities for applications of trusses form a vast frontier, which anyone with a love of mathematics, both pure and applied, should revel in the opportunity to explore.

Upon reading *Quantum Bi-Hamiltonian Systems* [16], a paper by *Cariñena*, *Grabowski* and *Marmo*, it seemed that the theorems of this paper could possibly be extended to the world of trusses, with some modifications. [16, Theorem 1] states that the Nijenhuis product μ_N is associative if and only if:

$$\delta_\mu T_N(a, b, c) := aT_N(b, c) - T_N(ab, c) + T_N(a, bc) - T_N(a, b)c = 0. \tag{1.1}$$

Now when working with trusses we are unable to use the unique element 0, as no such element exists in the world of trusses that is equivalent to the way the element 0 functions within the ring $(\mathbb{R}, +, \times)$, however, we can of course observe that (1.1) can be rearranged to give us the expression:

$$aT_N(b, c) = T_N(ab, c) - T_N(a, bc) + T_N(a, b)c,$$

which can then be expressed as the ternary operation:

$$aT_N(b, c) = \langle T_N(ab, c), T_N(a, bc), T_N(a, b)c \rangle.$$

Suffice it to say that although this is a seemingly a straightforward starting point, to actually prove the equivalence of the associativity of μ_N and the *Hochschild 2-cocycle* condition $\delta_\mu T_N(a, b, c) = 0$ requires the notion of a Hochschild cohomology of trusses. Formulation of a Hochschild cohomology of trusses became a key result of this thesis, allowing us to formulate the condition (see Theorem 4.6) of the ν-relative Hochschild 2-cocycle for all (equivalently for any) neutral element $\nu \in T$, which we denote:

$$\delta^2_\mu T^\nu_N(a, b, c) := aT^\nu_N(b, c) - T^\nu_N(ab, c) + T^\nu_N(a, bc) - T^\nu_N(a, b)c = \nu.$$

Further work thereafter involved adapting other theorems from [16] in a similar manner and adapting results from [16] from the case of linear to affine maps. Again a core motivation here, as above, is generalisation and rethinking of different concepts. We look at affine Nijenhuis operators in Chapter 5 as a continuation of this work.

Later in the thesis we shift our focus towards Lie affgebras, a term coined in [28]. This required formulations of properties equivalent to those of the standard Lie bracket, this is detailed in Definition 6.1 and Remark 6.2.

Finally these workings come full circle in Chapter 7, *Nijenhuis Operators on Lie Affgebras*, not just motivated by previous findings, but also from definitions found in *Poisson-Nijenhuis structures* [32] by *Y. Kosmann-Schwarzbach* and *F. Magri*. Here we define Nijenhuis operators over Lie affgebras, and the Nijenhuis bracket.

1.4 Expository Introduction

Here we provide an overview of the entire contents of this thesis.

We begin with Chapter 2, *Preliminaries.* This chapter contains a substantial amount of prerequisite information that is essential for understanding the later chapters of this thesis. This chapter looks to not only provide the reader with knowledge of the many different algebraic structures and concepts that are used in this thesis, but to make clear through notation how these give rise to the more general structures and concepts that we define later on in the thesis. A lot of the core notation required to understand the later chapters is explained in Chapter 2 also, which is critical when there is such a vast array of algebraic structures and different functions, mapping and operators at work.

Chapter 2 opens with information on *heaps* and *trusses* (Section 2.1). Understanding of these structures is essential and almost all the results of this thesis hinge on the generalisation of concepts from rings to either heaps or trusses.

Heaps (Definition 2.1) are algebraic structures composed of a non-empty set H, together with a ternary operation:

$$\langle -,-,- \rangle : H \times H \times H \to H, \qquad (a,b,c) \mapsto \langle a,b,c \rangle,$$

and conforms to properties of *associativity*, for all $a,b,c,d,e \in H$,

$$\langle \langle a,b,c \rangle, d, e \rangle = \langle a,b,\langle c,d,e \rangle \rangle,$$

and conform to what are called *Mal'cev identities*, namely:

$$\langle a,b,b \rangle = a = \langle b,b,a \rangle.$$

Furthermore, heaps with the *commutative* property:

$$\langle a,b,c \rangle = \langle c,b,a \rangle,$$

are referred to as *abelian heaps.* The definition of a *heap homomorphism* is also given in Definition 2.3.

We then explain how heaps can be constructed from a group structure $(G, \cdot, 1)$ in Remark 2.4 by defining the ternary operation for $a, b, c \in G$:

$$\langle -, -, - \rangle : G \times G \times G \longrightarrow G, \qquad \langle a, b, c \rangle = a \cdot b^{-1} \cdot c.$$

Following this we discuss the letter ν which we use throughout this thesis to denote a neutral element, using the Greek letter "nu" (Remark 2.5) for its phonetic similarity to the English word "neutral". This neutral element is used immediately to define an important structure, the *retract* or *retracted group* (Definition 2.6) of a heap H. The retract starts with a heap structure $(H, \langle -, -, - \rangle)$ and the neutral element $\nu \in H$ and then defines the binary operation:

$$\cdot_\nu : H \times H \longrightarrow H, \qquad a \cdot_\nu b = \langle a, \nu, b \rangle,$$

giving rise to a group structure denoted by $G(H, \nu)$, the neutral is denoted due to it being dependent on a given neutral element.

From this we may derive a convenient way of expressing the ternary operation when we are dealing with an abelian heap, that looks a lot more familiar to what we are used to as mathematicians (Remark 2.7), we may express ternary operations $\langle a, b, c \rangle$ by:

$$\langle a, b, c \rangle = a - b + c.$$

This notation is used often in this thesis, as the "+" and "−" symbols make calculations with many terms a lot easier to keep track of, allowing one to see if a term is in the even position if it is preceded by a "−" or the odd position if it is preceded by a "+".

We then define the *translation isomorphism* (Definition 2.8), this isomorphism allows one to compare structures with different neutral elements, for example it acts as an isomorphism of the groups $G(H; \nu') \to G(H, \nu)$. Such an isomorphism is defined, where H is a heap by:

$$\tau_{\nu'}^{\nu} : H \to H, \qquad \tau_{\nu'}^{\nu}(a) := \langle a, \nu', \nu \rangle.$$

Definition 2.9 outlines the concept of *sub-heaps*. Following this is the definition of a *truss* (Definition 2.11), the central algebraic structure to this thesis. A truss $(T, \langle -, -, - \rangle, \cdot)$, similarly to a heap, is a set T together with a ternary operation $\langle -, -, - \rangle$, this forms an abelian heap $(T, \langle -, -, - \rangle)$ however trusses also have a second operation, an associative binary operation $\cdot$ that distributes over $\langle -, -, - \rangle$, where for all $a, b, c, d \in T$, we have:

$$a\langle b, c, d \rangle = \langle ab, ac, ad \rangle, \qquad \langle a, b, c \rangle d = \langle ad, bd, cd \rangle.$$

If multiplication $\cdot$ admits identity, then we refer to a *unital truss*. Furthermore, if multiplication $\cdot$ is commutative, then we refer to a *commutative truss*.

We then explain in Remark 2.13 the richness of the world of trusses when compared to that of rings, with examples provided. The relationship between trusses and ring extensions is touched upon in Remark 2.15, which can be seen in more detail in [4].

Then we come to Section 2.2, *Affine Modules*. Here we lay the groundwork for many of the extensions from traditional vector space to affine space that are discussed in the latter chapters of this thesis.

We begin by defining both *left* and *right modules* in Definitions 2.16-2.17. Then we give a definition of *affine space* (Definition 2.18) which we then expand on in Remarks 2.19-2.20, where we look at how affine spaces can be represented using the ternary operation in the form:

$$\langle a, b, c\rangle = a + \overrightarrow{bc}.$$

where $\overrightarrow{bc} \in \vec{A}$ is the unique vector from b to c. Noting also that $c = b + \overrightarrow{bc}$. Remark 2.20 expands on Definition 2.18 and Remark 2.19, demonstrating how abelian heaps can be interpreted to be affine versions of abelian groups.

We then define an *affine module* and discuss its properties, homomorphisms between such structures, the isomorphic nature of two such structures with differing "neutral elements" (see Definition 2.21 - 2.25).

We then define an *(associative) affgebra* as an affine module A, together with a bi-affine (associative) multiplication $A \times A \to A$ in Definition 2.26.

Following this are some brief notes on *Lie algebras* (Section 2.3). These are present due to the importance of *Lie affgebras* to the results of this thesis (see Definition 2.28).

Following this is Section 2.4, *Homology & Cohomology*. This section covers numerous definitions that are required to understand the Hochschild cohomology of trusses in Chapter 3 onwards. Here we cover definitions of *chain complexes*, *differentials*, *cycles boundaries*, *homology modules*, *cochain complexes*, *cocycles coboundaries*, *cohomology modules* in Definitions 2.30-2.31. We also define *bimodules* in Definition 2.32 and then we define the *coboundary operator* δ (Definition 2.33). This is of particular importance, as is the foundation to the later definition of *ν-relative Hochschild n-coboundary operator on a truss T* (Definition 3.1). It is then explained how the coboundary operator δ relates to cocycles, coboundaries and cochains from Theorem 2.34 to Example 2.36.

Then the necessary definitions required to properly define Hochschild cohomology are given. This includes definitions of the *opposite algebra*, *enveloping algebra*, as well as some other terms. These are all covered between Definition 2.37 and Remark 2.41. Then we come to the definition of *Hochschild homology*, *Hochschild cycles*, *Hochschild boundaries* (Definition 2.42) and, most importantly for the results of this thesis, we come to definitions of *Hochschild cohomology*, as well as *Hochschild cocycles* and *Hochschild coboundaries* (Definition 2.43). Finally, for this section, we note that we are interested in the case where for our Hochschild cohomology:

$$HH^n(A:B) = H^n(Hom_{\mathbb{K}}(A^{\otimes *}:B)),$$

we are interested in the case where $B = A$, which we denote by (Remark 2.44):

$$HH^n(A:B) = HH^n(A).$$

Next we have Section 2.5 covering the preliminaries on *Nijenhuis Operators*. We begin with definitions adapted from [16], which are then built upon in the latter chapters of this thesis. Definition 2.45 states that given an associative algebra (A, μ) over a field $\mathbb{K}$, with product $\mu : A \times A \longrightarrow A,\ \mu(a, b) = ab$, for all $a, b \in A$. If we then have the linear map $N : A \to A$, we define the *Nijenhuis product* as:

$$\mu_N : (a, b) \longmapsto a \circ_N b = N(a)b - N(ab) + aN(b).$$

This gives rise to a new algebraic structure (A, μ_N). Furthermore, we may then note that:

$$a \circ_N b = \delta_\mu N(a, b), \tag{1.2}$$

where δ_μ is the *Hochschild coboundary operator associated with* μ, that is defined by the series:

$$\begin{aligned}\delta_\mu f(a_1, ..., a_{n+1}) = a_1 f(a_2, ..., a_{n+1})& \\ + \sum_{i=1}^{n} (-1)^i f(a_1, ..., \mu(a_i, a_{i+1}), ..., a_{n+1})& \\ + (-1)^{n+1} f(a_1, ..., a_n)a_{n+1},&\end{aligned}$$

with:

$$\delta_\mu : C^n(A : V) \longrightarrow C^{n+1}(A : V),$$

where $a_1, ..., a_n \in A$ with $f \in V$ where V is an A-bimodule and $C^n(A : V)$ denotes the space of *n-cochains*. These definitions are the first things that we are interested in moving to the truss case in Chapters 3 and 4.

Furthermore, in this section we define *derivations* in Definition 2.46, which are core to results in Chapter 3 onwards.

We then remark on cohomology groups (Remark 2.47) and then move to define the *μ-Nijenhuis torsion* of a linear map $N : A \to A$ is defined as:

$$T_N(a, b) = N(a \circ_N b) - N(a)N(b),$$

or equivalently:

$$T_N(a, b) = N(N(a)b) - N^2(ab) + N(aN(b)) - N(a)N(b).$$

In cases where we have $T_N(a, b) = 0$, for all $a, b \in A$ we refer to N as a *μ-Nijenhuis tensor* (Definition 2.48). These definitions of the Nijenhuis torsion and the Nijenhuis tensor are once more core to the results in this thesis.

We then go through some remarks and theorems adapted from [16], with perhaps the most important being Theorem 2.51, which is later expanded on in Theorem 4.6.

This section is then closed with some remarks on what exactly Nijenhuis operators are, why they were used in [16] and how they relate to *weak quantum bi-Hamiltonian systems* and *quantum bi-Hamiltonian systems* (Definition 2.53).

Chapter 3 begins with a definition for the *ν-relative Hochschild n-coboundary operator* on a truss T. This is an interpretation of the standard coboundary operator that would be used in ring like structures, but tailored for trusses where there is no unique element with the properties like that of the element $0 \in (\mathbb{R}, +, \times)$.

Following this we have Theorem 3.2, which states that both $\delta_\nu^{n+1} \circ \delta_\nu^n = \nu$ and $\delta_\nu^n(\nu) = \nu$, meaning that each δ_ν^n is a homomorphism of abelian groups $\Delta_\nu^n : G(C^n(T); \nu) \to G(C^{n+1}(T); \nu)$.

Following directly on from this, we have Definition 3.3, which defines the heap of *ν-relative n-cocycles*:

$$Z_\nu^n(T) := \{f \in C^n(T) \mid \delta_\nu^n(f) = \nu\},$$

and the heap of *ν-relative n-coboundaries*:

$$B_\nu^n(T) := \operatorname{Im} \delta_\nu^{n-1}.$$

These concepts are then combined into the quotient heap:

$$H_\nu^n(T) = Z_\nu^n(T)/B_\nu^n(T),$$

which we call the *n-th ν-relative Hochschild cohomology heap* of T (see Definition 3.4).

Following this are some workings that largely employ our definition of δ_ν^n, the full results of which, can be seen between Remark 3.5 and Remark 3.9.

The Section 3.2, *Derivations on Trusses* largely focuses on further workings that can be performed with the aid of Definition 3.10, which outlines a definition for the derivation D on a truss T, where for all $a, b \in T$, we have the heap homomorphism $D : T \to T$ such that:

$$D(ab) = \langle D(a)b, ab, aD(b)\rangle.$$

We also note that derivations on T form a heap which is denoted by $\mathrm{Der}(T)$. The rest of this section largely deals with results based on this definition.

Chapter 4 contains several results that build upon findings in [16]. This chapter opens with definitions of the *Nijenhuis product* (given as a ternary operation), for a truss T, for all heap homomorphisms $N : T \to T$, the binary operation $\circ_N$ on T, defined by:

$$a \circ_N b = \langle N(a)b, N(ab), aN(b)\rangle.$$

In the case where, for all $a, b \in T$:

$$N(a \circ_N b) = N(a)N(b),$$

we refer to the *Nijenhuis operator* N *(over a truss)*.

Finally, for all $\nu \in T$, the ν*-Nijenhuis torsion* of N is defined as:

$$T_N^\nu(a, b) = \langle N(a \circ_N b), N(a)N(b), \nu\rangle.$$

These definitions (see Definition 4.1), which are now in the truss case, provide the foundation for much of this thesis.

Section 4.2 largely details results arising from our new Nijenhuis definitions (Definition 4.1), with key results being Theorem 4.6, which is an expansion of [16, Theorem 1], Theorem 4.9, which is an expansion of [16, Theorem 2] and Theorem 4.12, which is an expansion of [16, Theorem 3, Theorem 4]. Expanding [16, Theorem 1] (Theorem 4.6) specifically acted as one of the leading motivations to this thesis.

It should also be noted that Definition 4.10 introduces a notion of *compatibility* of Nijenhuis operators on a truss, another necessary definition, the creation of which was required to prove the theorems we have adapted from [16].

Chapter 5 follows on from the previous results, it begins by defining the *affine Nijenhuis operator* (Definition 5.1). The majority of the chapter consists of results from Theorem 5.2 to Theorem 5.7. A key result from this chapter is Corollary 5.3, as it provides the foundation for the work of Chapter 6.

Chapter 6 follows from the findings in Chapter 5, specifically Corollary 5.3. Here we propose an intrinsic definition of the left and right Lie bracket in affine space rather than vector space. We shall refer not to a *Lie algebra*, but to a *Lie affgebra*. First we define the *Lie affgebra*, and the *Lie truss* (Definition 6.1, Remark 6.2), which are expansions of the usual Lie algebra, but rather than being defined over a vector space, we instead define the Lie affgebra over an affine space.

Following our definition for Lie affgebras are examples of these structures and comparisons to the structure introduced in [28] and showing the reduction of Lie affgebras to Lie algebras (Example 6.3-Proposition 6.10).

Section 6.2 introduces once more the notion of *derivations* (Definition 6.11), but this time as affine homomorphisms, with the rest of this section largely focusing on results arising from Definition 6.11.

Then in Section 6.3, we look at the relationship between Lie affgebras and Lie algebras, with a key result being how any Lie affgebra can be retracted to a Lie algebra.

In Chapter 7, all our workings start to come full circle, as we look to how Nijenhuis operators behave on Lie affgebras, with definitions largely adapted from *Poisson-Nijenhuis structures* [32] by *Kosmann-Schwarzbach* and *Magri*. In particular we show that Nijenhuis operators give rise to a family of compatible Lie brackets on an affine space.

The chapter begins by defining (Definition 7.1) *Nijenhuis operator (over a Lie affgebra)* N, where $(A, [-,-])$ is a Lie affgebra by an affine homomorphism $N : A \to A$ such that, for all $a, b \in A$:

$$[N(a), N(b)] = N\big(\langle [N(a), b], N([a, b]), [a, N(b)]\rangle\big).$$

Where the binary operation $[-,-]_N$ on A, given by:

$$[a, b]_N = \langle [N(a), b], N([a, b]), [a, N(b)]\rangle,$$

is called the *Nijenhuis bracket*.

With this new concept defined, we continue the chapter with various results in the form of examples and theorems.

This leads us to our final Chapter 8 which contains some closing remarks, as well as some recommended reading of the papers: *Special normalised affine matrices. An example of a Lie affgebra* [9]; *On matrix Lie affgebras* [14] and *Lie affgebras vis-à-vis Lie algebras* [3]. These papers were all written concurrently with this thesis and contain further research into Lie affgebras. Brief summaries of these papers can be found in Section 8.2, *Recommended Further Reading*.

Finally, at the very end in Section 8.3, we have suggestions for potential further work within this area of algebra. It is worth reiterating that the potential for further work is vast and offers a tremendous opportunity to further develop our understanding of mathematics.

Chapter 2

Preliminaries

Understanding the results in the latter chapters of this thesis is contingent on the reader having knowledge of a large number of algebraic structures and the relations we can draw between those algebraic structures, including heaps, trusses, affine modules, lie algebras, cohomology, Hochschild cohomology and Nijenhuis operators.

This preliminary section endeavours to not only explain these concepts in literal form, but to contextualize and give analogues to these different algebras and algebraic relations.

2.1 Heaps & Trusses

We begin with preliminary information on two types of algebraic structures *heaps* [5], [11], [17], [39] and *trusses* [4], [10], [11]. These structures are fundamental to the entirety of this thesis.

Assuming that the reader is familiar with group theory, it is quite simple to define a heap, though the structure is not immediately intuitive to those unfamiliar with it, especially when first presented with the *Mal'cev identities* and the bracket notation. Here we will use angled brackets $\langle -, -, - \rangle$ to denote this operation.

It is perhaps most simple to think of the ternary operations within heaps as a series of binary operations, then we may better understand the two main properties of a heap. Using conventions mentioned below in Definition 2.1, we may think of the expression $\langle a, b, c \rangle$ as the linear combination $a - b + c$, this should make the associative property, as well as the Mal'cev identities, much more comprehendible. It is, however, important for the reader to note that although we borrow the standard addition and subtraction symbols, this is not an exact equivalency to how these symbols are most frequently used, but instead they play the part of a visual aid. An example of where this difference can be noted is that not all heaps are abelian (see Remark 2.1-2.2). Expressing heaps using addition and subtraction symbols is outlined in more detail in Remark 2.7.

Definition 2.1. A **heap** $(H, \langle -,-,- \rangle)$ is an algebraic structure composed of a non-empty set H, together with a ternary operation:

$$\langle -,-,- \rangle : H \times H \times H \longrightarrow H, \qquad (a,b,c) \longmapsto \langle a,b,c \rangle,$$

that conforms to the following properties, for all $a,b,c,d,e \in H$,

Associativity: $\langle \langle a,b,c \rangle, d, e \rangle = \langle a,b,\langle c,d,e \rangle \rangle$,

Mal'cev identities: $\langle a,b,b \rangle = a = \langle b,b,a \rangle$.

Furthermore, a heap is said to be an **abelian heap** if it also has the following commutative property, for all $a,b,c \in H$,

$$\langle a,b,c \rangle = \langle c,b,a \rangle.$$

Following from this definition, note that abelian heaps allow you to *rearrange certain elements.* This is very convenient when making large calculations. However, there are, of course, restrictions around which elements can be exchanged with others.

When thinking of our ternary operation as the linear combination:

$$a - b + c - d + e,$$

with $a,b,c,d,e \in H$, it is clear that we cannot just swap the position of the elements a and b. However, a permutation of the position of a, c and e would cause no issue at all, assuming again that we are working within an abelian heap. Remark 2.2 below will explain this in more rigorous terms using the bracket notation, demonstrating the workings of the Mal'cev identities in abelian heaps.

Remark 2.2. When dealing with an abelian heap $(H, \langle -,-,- \rangle)$, one can see from the associative property of heaps that the internal placement of brackets in the ternary operation have no effect. Thus, we may write:

$$\langle \langle a,b,c \rangle, d, e \rangle = \langle a,b,\langle c,d,e \rangle \rangle = \langle a,b,c,d,e \rangle,$$

for such cases. However, the positioning of elements within the ternary operation does matter, elements in odd and even positions within the operation may be exchanged with each other, but only if the exchange is between, either two odd or two even elements, for example, we may write for any $a,b,c,x \in H$:

$$\langle a,x,c,b,x \rangle = \langle a,b,c,x,x \rangle = \langle a,b,c \rangle.$$

Definition 2.3. A **heap homomorphism** is a mapping between two heaps $(H, \langle -,-,- \rangle)$ and $(H', \langle -,-,- \rangle)$ that preserves the ternary operation for all $a,b,c \in H$:

$$f : H \longrightarrow H', \qquad f(\langle a,b,c \rangle) \longmapsto \langle f(a), f(b), f(c) \rangle.$$

The set of all heap homomorphisms from $(H, \langle -,-,- \rangle)$ to $(H', \langle -,-,- \rangle)$ is denoted $\mathrm{Heap}(H, H')$.

The set $\mathrm{Heap}(H, H')$ notably includes all constant functions. In the case where both H and H' are abelian, $\mathrm{Heap}(H, H')$ is a heap by a pointwise operation $\langle f, g, h \rangle(a) = \langle f(a), g(a), h(a) \rangle$. Any singleton set is a (abelian) heap with a trivial (only possible) operation. The unique function that goes from any heap to the singleton heap is a heap homomorphism. This means that the singleton set is a terminal object in the category of heaps, which we denote by $*$. Similarly, the empty set is an abelian heap, the initial object in the category of heaps. Note that to maintain the correspondence between heaps and groups, described in the following remark as well as Definition 2.6, we will assume that the heaps being discussed are non-empty.

Now we shall look at how groups and heaps relate to each other, most mathematicians will be far more familiar with the concept of the group than that of the heap, making this a useful analogy. Furthermore, the concept of the retract or a retracted group will be used in some of the results of this thesis.

Remark 2.4. Given a group $(G, \cdot, 1)$, for any $a, b, c \in G$ we define a ternary operation:

$$\langle -,-,- \rangle : G \times G \times G \longrightarrow G, \qquad \langle a, b, c \rangle = a \cdot b^{-1} \cdot c.$$

This gives rise to a heap structure. We will denote this heap by $H(G)$. Note also that if our underlying group structure $(G, \cdot, 1)$ is abelian, then so is $H(G)$.

Remark 2.5. Before the introduction to *retracts* in Definition 2.6, the reader should note that throughout this thesis the Greek letter ν is frequently employed as a denotation of a *neutral element*, this is due to both heaps and trusses based on number systems not interacting in the usual manner with the number *zero*.

To explain this further, consider the group $(\mathbb{Z}, +)$, the neutral element $0 \in (\mathbb{Z}, +)$ has the unique property that it can simply be excluded from calculations, for example, if we have elements $z_1, z_2, z_3 \in (\mathbb{Z}, +)$, we may write:

$$z_1 + z_2 + z_3 = z_1 + z_2 + z_3 + 0,$$

the inclusion or exclusion of this element in an equation from this type of algebraic structure has absolutely no bearing on the meaning of the equation. However, given a heap $(\mathbb{Z}, \langle -,-,- \rangle)$ we may note that the element 0 behaves differently. To illustrate this, given $0, a, b, c, d, e \in (\mathbb{Z}, \langle -,-,- \rangle)$, these equations of varying sizes cannot be reduced further due to the nature of the ternary operation:

$$\langle 0, b, c \rangle, \qquad \langle a, 0, c \rangle, \qquad \langle a, b, 0 \rangle,$$

and similarly for longer expressions:

$$\langle a, b, c, 0, e \rangle, \qquad \langle a, b, c, d, 0 \rangle.$$

This is because all of these elements must be expressed with an *odd* number of terms, they simply cannot be reduced to an even number within a heap system. A good analogue for this would be matrices. If you have a matrix with three entries, let's say for example, a set of coordinates in three-dimensional space (x, y, z), these can never be reduced to just two elements without fundamentally changing the system that you are working within (from three-dimensional to two-dimensional). Also note that in this analogy, the number 0, in such a system, represents a tangible position in space and cannot be ignored.

The only way that an element can be *cancelled out* in the world of heaps is with the *Mal'cev identities* (Definition 2.1), where any two adjacent elements can be cancelled out. Thus all the two elements need to do is match each other, there is no special element that acts as the usual 0.

However, there is some nuance to that statement. The underlying group structure could absolutely have a neutral element. Using this neutral element we discuss the idea of a *retract* in Definition 2.6.

Definition 2.6. Given a heap $(H, \langle -, -, - \rangle)$ and an element $\nu \in H$, we define the following binary operation on H:

$$\cdot_\nu : H \times H \longrightarrow H, \qquad a \cdot_\nu b = \langle a, \nu, b \rangle.$$

This yields a group structure which we denote $G(H; \nu)$. Such a structure is known as the **retract** of H. Further, note that if $(H, \langle -, -, - \rangle)$ is abelian, then the retract $G(H; \nu)$ is also abelian. Finally, note that we always have $H(G(H; \nu)) = H$, regardless of the chosen element ν, meaning that calculations in a heap H can always be performed in any retract.

Remark 2.7. We may also derive some further properties from the axioms of an abelian heap H. The following examples can be clearly understood when using the notion of a retract and interpreting the ternary operation as:

$$\langle a, b, c \rangle = a - b + c.$$

In particular we have that any three elements within the ternary operation determine the fourth element:

$$\langle a, b, c \rangle = d.$$

Furthermore,

$$\langle a, b, c \rangle = d \iff \langle a, d, c \rangle = b.$$

Additionally we have:

$$a = b \iff \langle a, b, c \rangle = c \iff \langle c, a, b \rangle = c,$$

for all or equivalently any $c \in H$. Transitioning from the any case to the all case can be demonstrated by the following equivalencies using the associative property and Mal'cev identities:

$$\langle a, b, c \rangle = c \iff \langle \langle a, b, c \rangle, c, d \rangle = \langle c, c, d \rangle \iff \langle a, b, d \rangle = d.$$

Furthermore, we have the property:

$$\langle a, \langle a, b, c\rangle, c\rangle = b,$$

which leads us to see that:

$$\langle a, \langle a, \langle a, b, c\rangle, c\rangle, c\rangle = b,$$

et cetera. As well as other basic observations such as:

$$\langle c, \langle a, b, c\rangle, a\rangle = \langle c, d, a\rangle,$$

and further:

$$\langle \langle c, c, b\rangle, a, a\rangle = b = \langle c, d, a\rangle = \langle a, d, c\rangle.$$

Furthermore, we may also note that an expression of the form $\langle a, b, c\rangle = d$ holds under the cyclic permutation of its elements $a \to b \to c \to d \to a$, to put this explicitly, if we have:

$$\langle a, b, c\rangle = d,$$

then we also have:

$$\langle d, a, b\rangle = c,$$
$$\langle c, d, a\rangle = b,$$
$$\langle b, c, d\rangle = a.$$

One may see that using these principles, we may extend this notion to any size of expression:

$$\langle a_1, ..., a_{2n+1}\rangle = \langle a_{\alpha(1)}, a_{\beta(2)}, a_{\alpha(3)}, a_{\beta(4)}, ..., a_{\alpha(2n+1)}\rangle,$$

for any permutations α of $\{1, 3, 5, ..., 2n+1\}$ and β of $\{2, 4, 6, ..., 2n\}$.

Now we will introduce the notion of the *translation isomorphism.* This isomorphism is a useful tool when translating two systems with differing neutral elements.

Where relevant, this thesis looks to see which results are completely independent of the choice of neutral element(s) in given algebraic systems, including when differing systems with differing neutral elements are interacting, such as can be found in the calculations of Chapter 3.

Definition 2.8. Let H be a heap. For all $\nu, \nu' \in H$, the **translation isomorphism** $\tau^{\nu}_{\nu'} : H \to H$ is defined as:

$$\tau^{\nu}_{\nu'}(a) := \langle a, \nu', \nu\rangle,$$

where the inverse of $\tau^{\nu}_{\nu'}$ is given by $\tau^{\nu'}_{\nu}$. Furthermore, $\tau^{\nu'}_{\nu}$ is an isomorphism of groups $G(H;\nu) \to G(H;\nu')$. The set of all translation isomorphisms of H is a group with respect to the composition. This group is isomorphic to any of the retracts of H.

Definition 2.9. A (non-empty) sub-set K of a heap $(H, \langle -, -, - \rangle)$ is called a **sub-heap** if it is closed under the heap operation, that is, for all $a, b, c \in K$, we have $\langle a, b, c \rangle \in K$.

Definition 2.10. A (non-empty) sub-heap K of a heap $(H, \langle -, -, - \rangle)$ is said to be a **normal sub-heap**, if for all $a, b \in K$ and $h \in H$ there exists an element $k \in K$ such that:

$$\langle h, a, b \rangle = \langle k, a, h \rangle.$$

A sub-heap (as in Definition 2.9) defines an equivalence relation on H, $a \sim b$ if, and only if for all (or equivalently any) $x \in K$, we have $\langle a, b, x \rangle \in K$. The set of equivalence classes is denoted H/K. One easily shows that in the case of a *normal sub-heap*, this is the same as the heap associated to the quotient of retracts, that is, for any $\nu \in K$, we have $H/K = H(G(H; \nu)/G(K; \nu))$. If H is abelian, then H/K is an abelian heap with the inherited structure $[\bar{a}, \bar{b}, \bar{c}] = \overline{[a, b, c]}$, where $\bar{a}$ denotes the class of $\bar{a}$ (or other element respectively) in H/K.

Now we look to define the structure that is central to this thesis, the *truss*. Similar to how we build on the idea of a *group*, forming a more complex structure that we call a *ring*, we look to build on the idea of the *heap*, specifically an *abelian heap*, together with binary operation often denoted by $\cdot$ or juxtaposition.

Trusses possess an interesting property which help the reader to contextualise further its place among other algebraic structures. In [10] it is explained how the truss connects the distributive laws of *rings* with the distributive laws of *braces*. However, in this thesis we will largely be thinking of trusses, via their relation to heaps, as both of these structures are core to our results.

Definition 2.11. A **truss** is an algebraic structure $(T, \langle -, -, - \rangle, \cdot)$, comprised of a set T, a ternary operation $\langle -, -, - \rangle$ such that $(T, \langle -, -, - \rangle)$ forms an abelian heap, and an associative binary operation $\cdot$ that distributes over $\langle -, -, - \rangle$, that is, for all $a, b, c, d \in T$, we have:

$$a\langle b, c, d \rangle = \langle ab, ac, ad \rangle, \qquad \langle a, b, c \rangle d = \langle ad, bd, cd \rangle.$$

If multiplication $\cdot$ admits identity, then the truss is said to be **unital**. Furthermore, if multiplication $\cdot$ is commutative, then we refer to such a truss as a **commutative truss**.

Now that we have our definition for a truss, we can make an observation that follows from Remark 2.5. In this remark we discussed the nature of heaps and how they interact with the element 0. A natural question would be to ask how trusses interact with neutral elements due to their differing structure. It is clear that we still lack a traditional neutral element such as 0, instead only able to perform cancellations by way of Mal'cev identities. However, note that throughout this thesis we employ the use of the neutral element, relative to a given retract and usually denoted by ν. Many calculations will also be performed using two different neutral elements from two different retracts, often denoting these as ν and ν'.

Remark 2.12. Using the definition of a retract (Definition 2.6), we can find an alternative way of viewing the heap operation and the truss distributive laws in the retract. In $G(T;\nu)$ we have:

$$a(b+c) = a\langle b,\nu,c\rangle = \langle ab, a\nu, ac\rangle = ab - a\nu + ac,$$

$$(a+b)c = ac - \nu c + bc,$$

$$a(b-c) = a\langle b,c,\nu\rangle = \langle ab, ac, a\nu\rangle = ab - ac + a\nu,$$
$$(a-b)c = ac - bc + \nu c,$$

$$a(-b) = a(\nu - b) = a\nu - ab + a\nu,$$
$$(-a)b = \nu b - ab + \nu b.$$

There are often times where thinking of the heap operation as a combination of the binary operations $+$ and $-$ will prove useful; these particular distributive laws will be used throughout this thesis. It can also be noted that these equations can be equivalently stated using bracket notation.

Remark 2.13. Comparing trusses and rings reveals how much wider the world of trusses really is. Given an abelian group A, understood as a heap $H(A)$, there are at least four non-isomorphic truss multiplications, only one of which, $ab = 0$, gives rise to a ring in all circumstances. They are:

$$ab = 0, \qquad ab = a, \qquad ab = b, \qquad ab = a + b = \langle a, 0, b\rangle, \tag{2.1}$$

for all $a, b \in A$. There are additional truss structures on specific groups. For example, commutative truss multiplications $\cdot$ on $\mathbb{Z}$ are given in terms of the standard arithmetic operations on $\mathbb{Z}$, for all $m, n \in \mathbb{Z}$, by:

$$m \cdot n = amn + b(m+n) + c, \tag{2.2}$$

where $a, b, c \in \mathbb{Z}$ are such that:

$$ac = b(b-1); \tag{2.3}$$

see [11, Theorem 3.51]. These trusses are denoted by $T(\mathbb{Z}; a, b, c)$. These are split into isomorphism classes:

1). For all $a \in \mathbb{N}$,

$$m \cdot n = amn,$$

$$m \cdot n = amn + m + n,$$

2). For all $a \in \mathbb{Z}^+, b \in \{2, 3, \ldots, a-1\}$ and $c \in \mathbb{Z}^+$ such that $ac = b(b-1)$,

$$m \cdot n = amn + b(m+n) + c;$$

see [11, Corollary 3.53]. As explained in [4, Example 7.4] there exists 23 non-isomorphic truss structures on the group $\mathbb{Z}_p \oplus \mathbb{Z}_p$ (where p is a prime number), as opposed to the 8 possible ring structures.

Theorem 2.14. *See [4, Theorem 4.3] Given a truss $(T, \langle -,-,-\rangle, \cdot)$ and an element $\nu \in T$, we may define the following operations:*

$$a +_\nu b = \langle a, \nu, b\rangle,$$

$$a \cdot_\nu b = \langle ab, a\nu, \nu^2, \nu b, \nu\rangle.$$

Such operations allow us to create retractions from trusses to rings. Here we have the ring structure $(T, +_\nu, \cdot_\nu)$. Furthermore, the rings $(T, +_\nu, \cdot_\nu)$ and $(T, +_{\nu'}, \cdot_{\nu'})$ are isomorphic for any $\nu, \nu' \in T$.

Proof. We know that the operation $+_\nu$ gives rise to a group structure, known as the retract, from Definition 2.6. Furthermore, we are dealing with an abelian heap structure, and so the retract must also be abelian.

Let us check that $\cdot_\nu$ adheres to associative and distributive properties we would expect to see within a ring structure:

1. Associativity:

$$\begin{aligned}(a \cdot_\nu b) \cdot_\nu c &= \langle ab, a\nu, \nu^2, \nu b, \nu\rangle \cdot_\nu c,\\ &= \langle \langle ab, a\nu, \nu^2, \nu b, \nu\rangle c, \langle ab, a\nu, \nu^2, \nu b, \nu\rangle \nu, \nu^2, \nu c, \nu\rangle,\\ &= \langle abc, a\nu c, \nu^2 c, \nu bc, \nu c, ab\nu, a\nu^2, \nu^3, \nu b\nu, \nu^2, \nu^2, \nu c, \nu\rangle,\end{aligned}$$

$$\begin{aligned}a \cdot_\nu (b \cdot_\nu c) &= a \cdot_\nu \langle bc, b\nu, \nu^2, \nu c, \nu\rangle,\\ &= \langle a\langle bc, b\nu, \nu^2, \nu c, \nu\rangle, a\nu, \nu^2, \nu\langle bc, b\nu, \nu^2, \nu c, \nu\rangle, \nu\rangle,\\ &= \langle abc, ab\nu, a\nu^2, a\nu c, a\nu, a\nu, \nu^2, \nu bc, \nu b\nu, \nu^3, \nu^2 c, \nu^2, \nu\rangle,\end{aligned}$$

comparing terms and using Remark 2.7 yields:

$$(a \cdot_\nu b) \cdot_\nu c = a \cdot_\nu (b \cdot_\nu c).$$

2. Distributivity:

$$\begin{aligned}(a +_\nu b) \cdot_\nu c &= \langle a, \nu, b\rangle \cdot_\nu c,\\ &= \langle\langle a, \nu, b\rangle c, \langle a, \nu, b\rangle\nu, \nu^2, \nu c, \nu\rangle,\\ &= \langle ac, \nu c, bc, a\nu, \nu^2, b\nu, \nu^2, \nu c, \nu\rangle,\end{aligned}$$

$$\begin{aligned}a \cdot_\nu c +_\nu b \cdot_\nu c &= \langle ac, a\nu, \nu^2, \nu c, \nu\rangle +_\nu \langle bc, b\nu, \nu^2, \nu c, \nu\rangle,\\ &= \langle ac, a\nu, \nu^2, \nu c, \nu, \nu, bc, b\nu, \nu^2, \nu c, \nu\rangle,\\ &= \langle ac, a\nu, \nu^2, \nu c, bc, b\nu, \nu^2, \nu c, \nu\rangle. && \text{(Mal'cev)}\end{aligned}$$

$$\begin{aligned}a \cdot_\nu (b +_\nu c) &= a \cdot_\nu \langle b, \nu, c\rangle,\\ &= \langle a\langle b, \nu, c\rangle, a\nu, \nu^2, \nu\langle b, \nu, c\rangle, \nu\rangle,\\ &= \langle ab, a\nu, ac, a\nu, \nu^2, \nu b, \nu^2, \nu c, \nu\rangle,\end{aligned}$$

$$\begin{aligned}a \cdot_\nu b +_\nu a \cdot_\nu c &= \langle ab, a\nu, \nu^2, \nu b, \nu\rangle +_\nu \langle ac, a\nu, \nu^2, \nu c, \nu\rangle,\\ &= \langle ab, a\nu, \nu^2, \nu b, \nu, \nu, ac, a\nu, \nu^2, \nu c, \nu\rangle,\\ &= \langle ab, a\nu, \nu^2, \nu b, ac, a\nu, \nu^2, \nu c, \nu\rangle. && \text{(Mal'cev)}\end{aligned}$$

we again compare terms and use Remark 2.7 to see that:

$$(a +_\nu b) \cdot_\nu c = a \cdot_\nu c +_\nu b \cdot_\nu c,$$

$$a \cdot_\nu (b +_\nu c) = a \cdot_\nu b +_\nu a \cdot_\nu c.$$

Thus one can see that we may retract a ring-like structure $(T, +_\nu, \cdot_\nu)$ that adheres to the above laws of associativity and distributivity, all with respect to the chosen element ν. Now let us check that $(T, +_\nu, \cdot_\nu)$ is isomorphic to $(T, +_{\nu'}, \cdot_{\nu'})$, recalling the translation isomorphism $\tau_\nu^{\nu'} : T_\nu \to T_{\nu'}$ (Definition 2.8), we first look at our chosen element ν:

$$\tau_\nu^{\nu'}(\nu) = \langle \nu, \nu, \nu'\rangle = \nu',$$

clearly. Now let us see if $\tau_\nu^{\nu'}$ preserves our operations $+_\nu$ and $\cdot_\nu$, we must confirm that both are such that:

$$\tau_\nu^{\nu'}(a +_\nu b) = \tau_\nu^{\nu'}(a) +_{\nu'} \tau_\nu^{\nu'}(b),$$

$$\tau_\nu^{\nu'}(a \cdot_\nu b) = \tau_\nu^{\nu'}(a) \cdot_{\nu'} \tau_\nu^{\nu'}(b),$$

or equivalently:

$$\tau_\nu^{\nu'}\langle a, \nu, b\rangle = \langle \tau_\nu^{\nu'}(a), \nu', \tau_\nu^{\nu'}(b)\rangle,$$

$$\tau_\nu^{\nu'}\langle ab, a\nu, \nu^2, \nu b, \nu\rangle = \langle \tau_\nu^{\nu'}(a)\tau_\nu^{\nu'}(b), \tau_\nu^{\nu'}(a)\nu', (\nu')^2, \nu'\tau_\nu^{\nu'}(b), \nu'\rangle.$$

we begin by computing:

$$\tau_\nu^{\nu'}\langle a,\nu,b\rangle = \langle a,\nu,b,\nu,\nu'\rangle,$$

$$\begin{aligned}\langle \tau_\nu^{\nu'}(a),\nu',\tau_\nu^{\nu'}(b)\rangle &= \langle\langle a,\nu,\nu'\rangle,\nu',\langle b,\nu,\nu'\rangle\rangle,\\ &= \langle a,\nu,\nu',\nu',b,\nu,\nu'\rangle,\\ &= \langle a,\nu,b,\nu,\nu'\rangle.\end{aligned}$$

as required. Now we compute:

$$\begin{aligned}\tau_\nu^{\nu'}\langle ab,a\nu,\nu^2,\nu b,\nu\rangle &= \langle ab,a\nu,\nu^2,\nu b,\nu,\nu,\nu'\rangle,\\ &= \langle ab,a\nu,\nu^2,\nu b,\nu'\rangle,\end{aligned}$$

$$\begin{aligned}\langle \tau_\nu^{\nu'}(a)\tau_\nu^{\nu'}(b),\tau_\nu^{\nu'}(a)\nu',(\nu')^2,\nu'\tau_\nu^{\nu'}(b),\nu'\rangle &= \langle\langle a,\nu,\nu'\rangle\langle b,\nu,\nu'\rangle,\langle a,\nu,\nu'\rangle\nu',\\ &\quad (\nu')^2,\nu'\langle b,\nu,\nu'\rangle,\nu'\rangle,\\ &= \langle ab,a\nu,a\nu',\nu b,\nu^2,\nu\nu',\nu' b,\\ &\quad \nu'\nu,(\nu')^2,a\nu',\nu\nu',(\nu')^2,(\nu')^2,\\ &\quad \nu' b,\nu'\nu,(\nu')^2,\nu'\rangle,\\ &= \langle ab,a\nu,\nu^2,\nu b,\nu'\rangle,\end{aligned}$$

as required. Clearly the inverse can be proven by similar calculation. We can see that our translation isomorphism preserves our chosen element ν as well as the corresponding operations $+_\nu$ and $\cdot_\nu$, thus we have $(T,+_\nu,\cdot_\nu) \cong (T,+_{\nu'},\cdot_{\nu'})$.

Thus, both the associativity and the distributivity properties hold as required, this concludes the proof. □

Remark 2.15. In the paper [4], the close relation between trusses and ring extensions is explained. Specifically, let R be an associative ring, let I be an ideal in R, and let $q \in R$ be an idempotent element. Then:

$$T(I;q) := q + I,$$

is a truss with the heap operation $\langle a,b,c\rangle = a-b+c$ and the same multiplication as in R. Any truss may be embedded in an associative ring in this way. Furthermore, there is an isomorphism between $R(T(I,q),+_q,\cdot_q)$ and I as rings.

Proof. To demonstrate isomorphism $R(T(I,q),+_q,\cdot_q) \cong I$, we must verify that we have preservation of both addition and multiplication, we begin with $+$ and $\cdot$ taken from I, and first demonstrate:

$$\begin{aligned}(q+a)\cdot_q(q+b) &= (q+a)(q+b) - q(q+b) + q^2 - (q+a)q + q, &(2.4)\\ &= q^2 + aq + qb + ab - q^2 - qb + q^2 - q^2 - aq + q, &(2.5)\\ &= ab + q. &(2.6)\end{aligned}$$

We then define:

$$\varphi : q + I \longrightarrow I, \qquad q + a \longmapsto a,$$

with the inverse:

$$\varphi^{-1} : I \longrightarrow q + I, \qquad a \longmapsto q + a.$$

Now to check the preservation of operations, starting with addition we have:

$$\begin{aligned} \varphi((q+a) +_q (q+b)) &= \varphi(q + a - q + q + b), \\ &= a + b, \\ &= \varphi(q+a) + \varphi(q+b), \end{aligned}$$

thus addition is preserved. Finally, we check the preservation of multiplication, using equations (2.5-2.6) we have:

$$\begin{aligned} \varphi((q+a) \cdot_q (q+b)) &= \varphi(q + ab), \\ &= ab, \\ &= \varphi(q+a)\varphi(q+b), \end{aligned}$$

thus we have demonstrated the preservation of both addition and multiplication, and hence we have isomorphism $R(T(I,q), +_q, \cdot_q) \cong I$. □

2.2 Affine Modules

Knowledge of affine space is also fundamental to understanding this thesis and its results. In the previous section some examples and analogies are given in the context of coordinates, vectors and Cartesian systems, this was intentional, both to give a clear and simple mental image of how these systems could be understood in a real world context, and so that reader can begin thinking about fundamentals of affine geometry.

An affine space can be seen as a generalization of the more familiar *vector space*. Affine spaces ask the question: "Why are we assuming the point of origin in a vector space?"

A clear example of these differences would be to assume we have a standard vector space V and a generic vector $\overrightarrow{b} \in V$, then note that this is of course the vector that goes from the origin point to the coordinates labelled b. In a two-dimensional vector space our origin would be $(0,0)$ and our point would be of the form $b = (x, y)$ where x, y are elements from our underlying field. The difference when working in affine space is that to write the vector $\overrightarrow{b}$ we must instead denote it in such a way that information about the point of origin is encoded or alternatively, any other point, as is clear in Definition 2.18. Essentially there is always a point of reference rather than assuming a point of origin. We would write an element from such an affine space as $a + \overrightarrow{ab} = b$ where in this case we would have $a = 0$. This is all explained rigorously in Definition 2.18 below, which has been adapted from [22].

For completeness we define first the traditional module over a ring [1], [2].

Definition 2.16. Let R be a ring. A **right R-module** is an abelian group M, together with a ring homomorphism $\rho : M \times R \to M$ with the properties that for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$:

1).
$$\rho(m_1 + m_2, r_1) = \rho(m_1, r_1) + \rho(m_2, r_1),$$
$$\rho(m_1, r_1 + r_2) = \rho(m_1, r_1) + \rho(m_1, r_2),$$

2).
$$\rho(m_1, r_1 r_2) = \rho(\rho(m_1, r_1), r_2),$$

3). where 1_R is the identity in R:
$$\rho(m_1, 1_R) = m_1.$$

Definition 2.17. Let R be a ring. A **left R-module** is an abelian group N, together with a ring homomorphism $\lambda : R \times N \to N$ with the properties that for all $r_1, r_2 \in R$ and $n_1, n_2 \in N$:

1).
$$\lambda(r_1 + r_2, n_1) = \lambda(r_1, n_1) + \lambda(r_2, n_1),$$
$$\lambda(r_1, n_1 + n_2) = \lambda(r_1, n_1) + \lambda(r_1, n_2),$$

2).
$$\lambda(r_1 r_2, n_1) = \lambda(r_1, \lambda(r_2, n_1)),$$

3). where 1_R is the identity in R:
$$\lambda(1_R, n_1) = n_1.$$

Definition 2.18. An **affine space** is an algebraic structure $(A, \vec{A}, +)$, consisting of a non-empty set of points A, a $\mathbb{K}$-vector space $\vec{A}$, and an action $+ : A \times \vec{A} \to A$, such that the following properties hold:

1). $a + 0 = a,$ for all $a \in A$.

2). $(a + \vec{u}) + \vec{v} = a + (\vec{u} + \vec{v}),$ for all $a \in A,$ and all $\vec{u}, \vec{v} \in \vec{A}$.

3). For any two points $a, b \in A$, there exists a unique vector, which we will denote by $\overrightarrow{ab} \in \vec{A}$ such that $a + \overrightarrow{ab} = b$.

Remark 2.19. As seen in, for example [7, Section 4] or [8] when looking at affine spaces the action + of $\vec{A}$ on A turns the latter into an abelian heap, where for all $a, b, c \in A$, we have the operation:

$$\langle a, b, c\rangle = a + \overrightarrow{bc}, \tag{2.7}$$

where $\overrightarrow{bc} \in \vec{A}$ is the unique vector from b to c. Note that $c = b + \overrightarrow{bc}$.

The following notion arises from [39] as an affine approach to abelian groups.

Remark 2.20. We may now see from the above remarks and definitions how an abelian heap can be viewed as an affine version of an abelian group. First: any abelian group is an abelian heap with the operation $\langle a, b, c\rangle = a - b + c$, second; any coset of an abelian group is an abelian heap and third; any non-empty abelian heap H gives rise to the family of isomorphism groups by retracting at any element, that is by reducing the operation from a ternary one to a binary one by setting $a + b = \langle a, \nu, b\rangle$ for any fixed element $\nu \in H$.

Furthermore, as seen in Remark 2.12, the heap operation can be understood using this definition of addition as:

$$\langle a, b, c\rangle = a - b + c. \tag{2.8}$$

This interpretation allows one to view the heap operation as a combination of addition and subtraction. Such an interpretation can make many of the calculations in this thesis much more readable and so this representation will be applied throughout this thesis. However, one must be conscious of the corresponding neutral elements chosen for such binary operations, which in this case would be $\nu \in H$ for our representation (2.8).

Abelian heaps may also be interpreted as affine $\mathbb{Z}$-modules in the sense of [38]. This will now be explained by first defining affine modules through the lens of heaps of modules [7], allowing one to define an affine space intrinsically without any reference to vector spaces.

Definition 2.21. Let $\mathbb{K}$ be a commutative ring with identity. Then an **affine $\mathbb{K}$-module** is a non-empty abelian heap A, together with the ternary action:

$$\triangleright : \mathbb{K} \times A \times A \longrightarrow A, \qquad (\alpha, a, b) \longmapsto \alpha \triangleright_a b,$$

such that the following properties hold for all $\alpha, \beta \in \mathbb{K}$ and $a, b, c \in A$:

1). $\alpha \triangleright_a - : A \to A$ and $- \triangleright_a b : \mathbb{K} \to A$ are both heap homomorphisms, where $\mathbb{K}$ is understood as a heap with the operation $\langle \alpha, \beta, \gamma\rangle = \alpha - \beta + \gamma$.

2). $(\alpha\beta) \triangleright_a b = \alpha \triangleright_a (\beta \triangleright_a b)$.

3). $\alpha \triangleright_a b = \langle \alpha \triangleright_c b, \alpha \triangleright_c a, a\rangle$.

4). $0 \triangleright_a b = a$ and $1 \triangleright_a b = b$.

Here the element a in the expression $\alpha \triangleright_a b$ is called the **base** of the action. A homomorphism of affine $\mathbb{K}$-modules is a heap homomorphism f such that:

$$f(\alpha \triangleright_a b) = \alpha \triangleright_{f(a)} f(b).$$

When $\mathbb{K}$ is a field, we refer to an **affine space** rather than an affine module.

Furthermore, we note that properties (1) and (3) imply that for all $\alpha \in \mathbb{K}$ and $a, b, c, d \in A$:

$$\alpha \triangleright_{\langle a,b,c\rangle} d = \langle \alpha \triangleright_a d, \alpha \triangleright_b d, \alpha \triangleright_c d\rangle, \tag{2.9}$$

which is a heap homomorphism, see [7, Lemma 3.5]. Additionally, one may note that property (3) alone implies that:

$$\alpha \triangleright_a a = a. \tag{2.10}$$

One can also see that an abelian heap A is an affine $\mathbb{Z}$-module with the unique action determined by the conditions (3) and (4):

$$n \triangleright_a b = \begin{cases} \langle b, a, b, \ldots, a, b\rangle_{2n-1} & n > 0, \\ \langle a, b, a, \ldots, b, a\rangle_{|2n+1|} & n \leq 0. \end{cases}$$

Any morphism of heaps is automatically a morphism of corresponding affine $\mathbb{Z}$-modules. This establishes a categorical isomorphism of affine $\mathbb{Z}$-modules and abelian heaps in the same way as an isomorphism of the categories of abelian groups and $\mathbb{Z}$-modules.

Lemma 2.22. *Let $f, g, h : A \to B$ be homomorphisms of affine $\mathbb{K}$-modules. Then:*

$$\langle f, g, h\rangle : A \longrightarrow B, \qquad a \longmapsto \langle f(a), g(a), h(a)\rangle,$$

is a homomorphism of affine $\mathbb{K}$-modules.

Proof. This lemma follows directly from property (3) from Definition 2.21. Specifically, for all $\alpha \in \mathbb{K}$ and $a, b \in A$, the actions:

$$\langle f, g, h\rangle(\alpha \triangleright_a b) = \langle \alpha \triangleright_{f(a)} f(b), \alpha \triangleright_{g(a)} g(b), \alpha \triangleright_{h(a)} h(b)\rangle,$$

can be brought to a common base, let's say $f(a)$ yielding:

$$\langle \alpha \triangleright_{f(a)} f(b), \alpha \triangleright_{f(a)} g(b), \alpha \triangleright_{f(a)} g(a), g(a), \alpha \triangleright_{f(a)} h(b), \alpha \triangleright_{f(a)} h(a), h(a)\rangle.$$

The rearrangements and heap homomorphism properties (1) in Definition 2.21, together with (2.10) and property (3) yield:

$$\langle \alpha \triangleright_{f(a)} \langle f, g, h\rangle(b), \alpha \triangleright_{f(a)} \langle f, g, h\rangle(a), \langle f, g, h\rangle(a)\rangle = \alpha \triangleright_{\langle f,g,h\rangle(a)} \langle f, g, h\rangle(b),$$

as required. □

Remark 2.23. Let A be an affine $\mathbb{K}$-module. Take any element $\nu \in A$ and consider the abelian group A_ν. We may then use property (3) from Definition 2.21, the bases of all actions can be related to the base ν. If we write $\alpha a := \alpha \triangleright_\nu a$, we may then immediately find that in terms of operations in A_ν:

$$\alpha \triangleright_a b = \alpha b - \alpha a + a, \qquad \text{for all } a, b \in A, \alpha \in \mathbb{K}. \tag{2.11}$$

As the left-hand side of (2.11) is independent of the choice of the element ν, then clearly right-hand side is also independent of the choice of ν. The meanings of the binary operations do however depend on the element ν.

Using these ideas, we may further understand the interplay between affine spaces and the ternary operation. The definition of an affine space presented here is equivalent to that which can be seen in Definition 2.18 using a set A and a free transitive action $+$ of a $\mathbb{K}$-vector space $\vec{A}$. In particular, for all $b, c \in A$, there exists a unique vector going from b to c, and every vector in $\vec{A}$ can be found in this way. Furthermore, any point $a \in A$ can be uniquely shifted or translated to another point $a + \overrightarrow{bc} \in A$. This combination of three points equips A with the abelian heap structure:

$$\langle a, b, c\rangle = a + \overrightarrow{bc}.$$

Now A becomes an affine $\mathbb{K}$-module in the sense of Definition 2.21 with the action:

$$\alpha \triangleright_a b = a + \alpha \overrightarrow{ab}.$$

Conversely, if we start with an affine $\mathbb{K}$-space A as in Definition 2.21, we can reconstruct the underlying vector space by picking any $\nu \in A$ and defining the addition of vectors and multiplication by scalars by:

$$a + b = \langle a, \nu, b\rangle, \qquad \alpha a = \alpha \triangleright_\nu a. \tag{2.12}$$

The resulting vector space (for fields) or $\mathbb{K}$-module (for commutative rings) will be denoted A_ν. Here we would have, $\overrightarrow{bc} = \langle \nu, b, c\rangle \in A_\nu$.

Remark 2.24. When looking to the affine case, the translation isomorphism from Definition 2.8 shows how different choices of the zero element in an affine $\mathbb{K}$-module A lead to isomorphic $\mathbb{K}$-modules. Specifically, $\tau_\nu^{\nu'} : A \to A$ is an automorphism of heaps and an isomorphism of abelian groups $A_\nu \cong A_{\nu'}$. Furthermore, for all $a \in A$ and $\alpha \in \mathbb{K}$:

$$\begin{aligned}\alpha \triangleright_{\nu'} \tau_\nu^{\nu'}(a) &= \langle \alpha \triangleright_{\nu'} a, \alpha \triangleright_{\nu'} \nu, \alpha \triangleright_{\nu'} \nu'\rangle,\\ &= \langle \alpha \triangleright_{\nu'} a, \alpha \triangleright_{\nu'} \nu, \nu, \nu, \nu'\rangle,\\ &= \langle \alpha \triangleright_{\nu} a, \nu, \nu'\rangle = \tau_\nu^{\nu'}(\alpha \triangleright_\nu a).\end{aligned}$$

With the first equality coming from the fact that $\alpha \triangleright_{\nu'} -$ is a heap homomorphism. The second equality is a consequence of the Mal'cev identity and (2.10). The third equality is a result of the base change property (c) in Definition 2.21. Thus, we have shown that $\tau_\nu^{\nu'}$ is an isomorphism of $\mathbb{K}$-modules $A_\nu \cong A_{\nu'}$.

Remark 2.25. We can look at how the translation isomorphism interacts with notions of dimension and linearisation.

Given a field $\mathbb{K}$, the **dimension** of an affine space A, written $\dim(A)$, is defined as the dimension of the vector space A_ν. Using what we understand from Definition 2.8 and Remark 2.24, we see that the dimension does not depend on the choice of the element $\nu \in A$.

Furthermore, the traditional definition of affine maps involve a pair, a function $f : A \to B$, together with a (uniquely defined) linear transformation $\vec{f} : \vec{A} \to \vec{B}$, called the **linearisation** of f such that $\vec{f}(\overrightarrow{ab}) = \overrightarrow{f(a)f(b)}$. This is equivalent to saying that f is a homomorphism of affine spaces in the sense of Definition 2.21. If we fix $\nu \in A$ and $\nu' \in B$, then define the vector space structures A_ν and $B_{\nu'}$ as in (2.12), then the linearisation of $f : A \to B$ is given by:

$$\vec{f} : A_\nu \longrightarrow B_{\nu'}, \qquad a \longmapsto \langle f(a), f(\nu), \nu' \rangle. \tag{2.13}$$

Definition 2.26. Let $\mathbb{K}$ be a commutative ring. An **(associative) $\mathbb{K}$-affgebra**, or simply an **affgebra**, is an affine $\mathbb{K}$-module A, together with a bi-affine (associative) multiplication $A \times A \to A$.

The bi-affine property of the multiplication in an affgebra implies that it is a bi-heap homomorphism, so we may conclude that an affgebra is also a truss. Using the above interpretation of heaps as affine $\mathbb{Z}$-modules, trusses can be seen as associative $\mathbb{Z}$-affgebras in the same way that rings are associative $\mathbb{Z}$-algebras.

Example 2.27. Let A be an affine $\mathbb{K}$-module. The set $\mathrm{Aff}(A)$ of all affine endomorphisms of A is an affgebra with the pointwise heap operation, the action:

$$(\alpha \triangleright_f g)(a) = \alpha \triangleright_{f(a)} g(a), \tag{2.14}$$

for all $\alpha \in \mathbb{K}$, $f, g \in \mathrm{Aff}(A)$, $a \in A$, with multiplication given by composition.

Proof. Using Lemma 2.22, we see that $\mathrm{Aff}(A)$ is a heap and the general properties of endomorphism trusses affirm that $\mathrm{Aff}(A)$ is a truss. Since the action is defined pointwise, it satisfies the requirements of Definition 2.21. All the remains is to show that $\alpha \triangleright_f g$ is an affine map. That it is a homomorphism follows from (2.9). By [7, Lemma 3.15], for all $\alpha, \beta \in \mathbb{K}$ and $w, x, y, z \in A$:

$$\alpha \triangleright_{\beta \triangleright_x y} (\beta \triangleright_z w) = \beta \triangleright_{\alpha \triangleright_x z} (\alpha \triangleright_y w).$$

Setting $x = f(a)$, $y = f(b)$, $z = g(a)$ and $w = g(b)$ for any $a, b \in A$ and $f, g \in \mathrm{Aff}(A)$ we obtain:

$$(\alpha \triangleright_f g)(\beta \triangleright_a b) = \beta \triangleright_{(\alpha \triangleright_f g)(a)} (\alpha \triangleright_f g)(b).$$

Thus we may conclude that that $\alpha \triangleright_f g$ is an affine map, completing our proof. $\square$

2.3 Lie Algebras

Throughout this thesis there is frequent reference to Lie algebras, as well as their extensions to affine space, what we will call Lie affgebras. These algebras and their properties are essential in the latter chapters of this thesis. It is important to be familiar with their properties listed as (1-4) in Definition 2.28 as later on in the thesis we tackle ways to adapt these rules to affine space in Chapter 6.

To ensure that everyone is on the same page, especially in regards to notation due to the vast number of different operations using various differing bracket notations, we begin with a definition of Lie algebras adapted from [21].

Definition 2.28. Let $\mathbb{K}$ be a field. A **Lie algebra** over $\mathbb{K}$ is a $\mathbb{K}$-vector space A together with a bilinear mapping $[-,-] : A \to A$, which we call a **Lie bracket**. In such a structure, the following properties must also hold for all $x, y, z \in A,\ \alpha, \beta \in \mathbb{K}$:

1).

$$[x, x] = 0, \qquad \text{for all } x \in A,$$

2). Bilinearity:

$$[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z],$$

$$[z, \alpha x + \beta y] = \alpha[z, x] + \beta[z, y],$$

3). Jacobi Identity:

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0,$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0,$$

4). Antisymmetry:

$$[x, y] = -[y, x].$$

Note that property (4) may be derived from the other properties.

Definition 2.29. Let $\mathbb{K}$ be a field and let $[-,-] : A \to A$ be a Lie bracket on the $\mathbb{K}$-vector space A. If the Lie bracket has the property:

$$[a, a] = a, \qquad \text{for all } a \in A,$$

then we refer to an **idempotent Lie bracket**.

In this thesis we look to expand the idea of a Lie algebra to what we will call a Lie affgebra, looking at the affine case, as opposed to the common definition rooted in vector spaces.

2.4 Homology & Cohomology

The concept of *homology* is core to this thesis, specifically *Hochschild cohomology.* Homology looks at measuring the amount of obstructions present for the creation of specific algebraic constructions by yielding a numerical output. An output of 0 would indicate that there are no such obstructions for a specified construction [43]. For emphasis, the key reason that cohomology theory is important is its utility when looking at the deformation of algebras. Cohomology can be used as a tool to measure the degree to which two algebras differ. Observing the similarities and differences of different algebras is an obviously compelling task, and having a sophisticated method to record just how similar or different two algebras are is incredibly powerful.

In the paper *Affine Nijenhuis operators and Hochschild cohomology of trusses* [12] that preceded this thesis, a large portion of work went into creating a Hochschild cohomology of trusses. The homology of algebras generally deals with structures that are more "ring-like" than trusses, so retooling these concepts for the world of trusses became a core result of that paper, as it was necessary work to then expand the work of Cariñena, J. Grabowski, G. Marmo in [16] to the world of trusses and to affine space. As such, many of the results in Chapter 3 onward are critically dependent on the reimagining of the concepts presented in this section.

There are several terms and definitions that the reader must familiarize themselves with in this section. Some fundamentals have been adapted from *An Introduction to Homological Algebra* [43], *Basic Algebra II* [30], *Cohomology Theory in Abstract Groups. I* [20], with others adapted from *On the cohomology groups of an associative algebra* [29] by *Gerhard Hochschild*, with more modern definitions of these ideas adapted from *Hochschild Cohomology for Algebras* [45].

Definition 2.30. Let $\mathbb{K}$ be a commutative ring. A **chain complex** denoted commonly by $C_\bullet$ or simply C of $\mathbb{K}$-modules is a family of $\mathbb{K}$-modules C_n where $n \in \mathbb{Z}$, together with the $\mathbb{K}$-module maps $d_n : C_n \to C_{n-1}$ such that for all compositions we have:

$$d_n \circ d_{n+1} : C_{n+1} \longrightarrow C_{n-1}, \qquad x \longmapsto 0,$$

that is:

$$d_n \circ d_{n+1} = 0.$$

The mappings d_n are known as the **differentials** of C. Both the kernel of d_n and the image of d_{n+1} are also of interest. The kernel of d_n is the module of **n-cycles** of C, which we will denote $Z_n = Z_n(C)$. The image of $d_{n+1} : C_{n+1} \to C_n$ is the module of **n-boundaries** of C denoted by $B_n = B_n(C)$. We may also note that because $d_n \circ d_{n+1} = 0$, it follows that for all $n \in \mathbb{Z}$:

$$0 \subseteq B_n \subseteq Z_n \subseteq C_n.$$

The n**-th homology module** of C is the sub-quotient:

$$H_n(C) = Z_n(C)/B_n(C).$$

Now we take our definitions of the *chain complex*, *n-cycles* and *n-boundaries* and expand on them below. Note that the definition of the below is obtained somewhat trivially by reindexing such that $C^n = C_{-n}$.

Definition 2.31. Let $\mathbb{K}$ be a commutative ring. A **cochain complex** $C^\bullet$ of $\mathbb{K}$-modules is a family C^n, where $n \in \mathbb{Z}$ of $\mathbb{K}$-modules, together with the maps $d^n : C^n \to C^{n+1}$, such that for all compositions, similarly to Definition 2.30, we have:

$$d^{n+1} \circ d^n : C^n \longrightarrow C^{n+2} = 0,$$

that is:

$$d^{n+1} \circ d^n = 0.$$

Similarly to Definition 2.30, we may then define $Z^n(C^\bullet) = \mathrm{Ker}(d^n)$ as the module of n-**cocycles**, and $B^n(C^\bullet) = \mathrm{Im}(d^{n-1}) \subseteq C^n$ as the module of n-**coboundaries**. Finally we may write the sub-quotient:

$$H^n(C^\bullet) = Z^n(C^\bullet)/B^n(C),$$

which is the n**-th cohomology module** of $C^\bullet$.

The reader should note that the notation for both a chain complex and cochain complex can be written more formally as:

$$C_n = \{C_n\}_{n\in\mathbb{Z}}, \qquad C^n = \{C^n\}_{n\in\mathbb{Z}}.$$

This would be perhaps the more traditional notation. However, for ease we will be using C_n and C^n in this thesis.

To understand Definition 2.33 the reader must first have an understanding of *bimodules*. This is briefly covered below [15], [30].

Definition 2.32. Let R and S be rings. An S-R-**bimodule** is an abelian group M such that:

1). M is also a left S-module with operation $\cdot$,

2). M is also a right R-module with operation $*$,

3). for all $r \in R$, $s \in S$, $x \in M$ we have the compatibility relation,

$$(s \cdot x) * r = s \cdot (x * r).$$

In the definitions below we will frequently use the notation that A is an associative algebra over an arbitrary commutative ring $\mathbb{K}$, and the notation $\mathrm{Hom}_{\mathbb{K}}(A^{\otimes(n)} : B)$ will represent the linear homomorphisms from $A^{(n)}$ to B, where B is an A-bimodule and $n \geq 0$, we refer to these morphisms as *cochains*.

Definition 2.33. A **coboundary operator** δ, operating on the set of all cochains of A is a mapping $\delta : \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(n)} : B) \to \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(n+1)} : B)$ where:

$$\begin{aligned} \delta f(a_0, ..., a_n) = a_0 f(a_1, ..., a_n) \\ + \sum_{i=1}^{n} (-1)^i f(a_0, ..., a_{i-1}a_i, ..., a_n) \\ + (-1)^{n+1} f(a_0, ..., a_{n-1})a_n. \end{aligned} \tag{2.15}$$

for all $a_0, a_1, a_2, ..., a_n \in A$, $f \in \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(n)} : B)$.

To be explicit, the sequence formed on the right-hand side of equation (2.15) is one with alternating signs, which will be helpful when we look to expand these definitions to the truss case at the beginning of Chapter 3. The reader should also note that in the case where $n = 0$ our expression takes a form that looks similar to the output of the commutator bracket:

$$\delta f(a_0) = a_0 f - f a_0,$$

for $f \in B$. Then for $n = 1$ we have:

$$\delta f(a_0, a_1) = a_0 f(a_1) - f(a_0 a_1) + f(a_0)a_1,$$

for $f \in \mathrm{Hom}_{\mathbb{K}}(A : B)$ it should then be clear how expressions take form where $n > 1$, with an ever increasing number of middle terms generated by the series in equation (2.15) for increasing values of n in Definition 2.33.

Theorem 2.34. *Given a coboundary operator δ, for every cochain f, we have:*

$$\delta\delta f = 0, \tag{2.16}$$

where δ and f are as in Definition 2.33.

Proof. Adapting a proof from Hochschild's paper [29], we can see that this theorem can be proven by induction on the dimension n of $f \in B$.

If we start with the dimension $n = 0$, then our equation (2.15) takes the form:

$$\delta f(a_0) = a_0 f - f a_0,$$

and the equation (2.16) takes the form:

$$\begin{aligned} \delta\delta f(a_0, a_1) &= a_0 \delta f(a_1) - \delta f(a_0 a_1) + \delta f(a_0)a_1, \\ &= a_0(a_1 f - f a_1) - (a_0 a_1 f - f a_0 a_1) + (a_0 f - f a_0)a_1, \\ &= a_0 a_1 f - a_0 f a_1 - a_0 a_1 f + f a_0 a_1 + a_0 f a_1 - f a_0 a_1, \\ &= 0. \end{aligned}$$

Now we define an element $\overline{f} \in \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(n-1)} : \mathrm{Hom}_{\mathbb{K}}(A : B))$ as the relation:

$$\overline{f}(a_0, ..., a_{n-1})(a_n) = f(a_0, ..., a_n),$$

where $f \in \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(n)} : B)$ and $n \geq 1$, with the convention that for $n = 0$ we assign $\mathrm{Hom}_{\mathbb{K}}(A^{\otimes(0)} : \mathrm{Hom}_{\mathbb{K}}(A : B)) = \mathrm{Hom}_{\mathbb{K}}(A : B)$.

Then we may derive from Definition 2.33 that for $f \in \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(n)} : B)$ and $n \geq 1$ we have:

$$\begin{aligned} \delta f(a_0, ..., a_n) = & (a_0\overline{f}(a_1, ..., a_{n-1}))(a_n) \\ & + \sum_{i=1}^{n-1} (-1)^i \overline{f}(a_0, ..., a_{i-1}a_i, ..., a_{n-1})(a_n) \\ & + (-1)^n (\overline{f}(a_0, ..., a_{n-2})a_{n-1})(a_n), \\ \\ & = \delta\overline{f}(a_0, ..., a_{n-1})(a_n), \end{aligned}$$

if we then consider:

$$\overline{\delta f}(a_0, ..., a_{n-1})(a_n) = \delta f(a_0, ..., a_n),$$

it follows that we may equate:

$$\overline{\delta f} = \delta\overline{f},$$

it then follows that we may equate:

$$\overline{\delta\delta f} = \delta\overline{\delta f} = \delta\delta\overline{f}.$$

Now we note that in all cases the dimension $\dim(\overline{f}) = \dim(f) - 1$, and $\overline{f} = 0$ only when $f = 0$, thus we now apply mathematical induction, proving that we must have $\delta\delta f = 0$ for every cochain f. □

Definition 2.35. Let $f \in \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(n)} : B)$ be cochain similarly to that in Definition 2.33. A cochain f is said to be a **cocycle** if:

$$\delta f = 0.$$

Furthermore, f is said to be a **coboundary** if there exists a cochain $g \in \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(n)} : B)$ such that $f = \delta g$.

Example 2.36. Furthermore, we can be more detailed with our Definition 2.35, instead we can refer to our cochains f as n**-cochains** of the n**-cocycles** δf:

$$\delta f(a_0, ..., a_n) = 0,$$

where $a_0, ..., a_n \in A$. Below are examples of 2-, 1- and 0-*cocycles*.

1). First let $B = A$, in the case of a 2-*cocycle* then we are dealing with:

$$\delta f(a_0, a_1, a_2) = a_0 f(a_1, a_2) - f(a_0a_1, a_2) + f(a_0, a_1a_2) - f(a_0, a_1)a_2 = 0,$$

if we are then to take the case where f is the usual ring multiplication, which we shall denote in this example by $\times$ for clarity, then we are dealing with the expression:

$$a_0(a_1 \times a_2) - (a_0a_1) \times a_2 + a_0 \times (a_1a_2) - (a_0 \times a_1)a_2 = 0,$$

we can then also write our juxtaposed elements with the explicit $\times$ symbol between them:

$$a_0 \times (a_1 \times a_2) - (a_0 \times a_1) \times a_2 + a_0 \times (a_1 \times a_2) - (a_0 \times a_1) \times a_2 = 0,$$

and it is clear that our positive and negative terms cancel each other out, meaning that the standard ring multiplication $\times$ is a 2-cocycle.

2). If we then consider the case of a 1-*cocycle*, then we are dealing with f and δf such that:

$$\delta f(a_0, a_1) = a_0 f(a_1) - f(a_0a_1) + f(a_0)a_1 = 0,$$

or equivalently, we can say that f here is a derivation of A.

3). We may also consider the case of the 0-cocycle, which would imply that for our f and δf, we have:

$$\delta f(a_0) = a_0 f - f a_0 = 0.$$

or equivalently, f is the central element of B.

Recalling the notation used in Definitions 2.30-2.31, we adapt definitions of *Hochschild homology* and *Hochschild cohomology* from the book *Hochschild Cohomology for Algebras* [45], but first we require the definition of the module $B \otimes A^{\otimes *}$ adapted from [43], [45].

Definition 2.37. Let $\mathbb{K}$ be a commutative associative ring with the multiplicative identity 1, and let A be an associative ring also with the multiplicative identity 1 such that A is also a $\mathbb{K}$-module for which multiplication is a $\mathbb{K}$-bilinear mapping. That is to say, we have $\mathrm{id}_{\mathbb{K}} : \mathbb{K} \to A$, where $\mathrm{id}_{\mathbb{K}}(k) \mapsto k \cdot 1$ for all $k \in \mathbb{K}$. We shall denote by A^{op} the **opposite algebra** of A, this is defined as A as a module over the ring $\mathbb{K}$ with the multiplication $a \cdot_{\mathrm{op}} b = ba$ for all $a, b \in A$.

Definition 2.38. Following from Definition 2.37, we define the tensor product $A^e = A \otimes A^{\mathrm{op}}$ with tensor product multiplication:

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1a_2 \otimes b_2b_1,$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in A^{\mathrm{op}}$ where $\otimes$ is a tensor product taken over the ring $\mathbb{K}$. Note also that we could equivalently write $a_1, a_2, b_1, b_2 \in A$ as both A and A^{op} have the same underlying K-modules. We call A^e the **enveloping algebra** of A.

Definition 2.39. Given our above definition of bimodules (Definition 2.32), we may note that an A-bimodule is equivalent to our definition of the enveloping algebra A^e module if we have for all $a, b \in A$ and $m \in M$:

$$(a \otimes b) \cdot m = amb.$$

A similar observation can also be made for a right A^e-module with the action instead defined by:

$$m \cdot (a \otimes b) = bma.$$

As is convention, when referring to a module it will be assumed to be a *left* module unless otherwise stated.

Definition 2.40. Following from Definitions 2.37-2.39, we must introduce two isomorphisms, the full origin of which can be seen in [45]. First we define $\mathbb{K}$ to be a commutative ring with multiplicative identity 1 and let A be an associative ring again with the multiplicative identity 1, and let B be an A-bimodule.

Our first isomorphism to make note of is the $\mathbb{K}$-module isomorphism, where for all $n \geq 0$ we define:

$$C_n(A, B) \cong B \otimes A^{\otimes(n)}. \tag{2.17}$$

Additionally we must introduce the isomorphisms of left A^e-modules:

$$A^{\otimes(n+2)} \cong A^e \otimes A^{\otimes(n)}, \tag{2.18}$$

such that, for all $a_0, ..., a_{n+1} \in A$ we have:

$$a_0 \otimes a_1 \otimes ... \otimes a_n \otimes a_{n+1} \longmapsto (a_0 \otimes a_{n+1}) \otimes (a_1 \otimes ... \otimes a_n),$$

where the action of A^e on $A^e \otimes A^{\otimes(n)}$ is the multiplication on the leftmost factor A^e.

Remark 2.41. Now using our above isomorphisms (2.17-2.18), the induced differential on the complex $B \otimes A^{\otimes *}$ corresponding to the map $1_B \otimes d_n$ on $B \otimes_{A^e} A^{\otimes(n+2)}$ where $1_B \in B$ is the identity map in B, for every $n > 0$ is the differential mapping $d_n : B \otimes A^{\otimes(n)} \to B \otimes A^{\otimes(n-1)}$ such that:

$$\begin{aligned} d_n(b \otimes a_1 \otimes a_2 \otimes ... \otimes a_n) &= ba_1 \otimes a_2 \otimes ... \otimes a_n \\ &+ \sum_{i=1}^{n-1} (-1)^i b \otimes a_1 \otimes a_2 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n \\ &+ (-1)^n a_n b \otimes a_1 \otimes a_2 \otimes ... \otimes a_{n-1}, \end{aligned}$$

where $a_1, ..., a_n \in A$ and $b \in M$. We will then define *Hochschild homology* as the homology of this complex. The full explanation of how we reach this definition can be seen in the opening pages of [45], but this is quite lengthy. For the purposes of this thesis, we just need to understand what Hochschild homology itself is, then we look to extend this definition later in this thesis to the truss case in Chapter 3.

Definition 2.42. Given A and B as in Remark 2.41. The **Hochschild homology** $HH_*(A:B)$ of A with coefficients in an A-bimodule B is defined:

$$HH_n(A:B) = H_n(B \otimes A^{\otimes *}),$$

equivalently, that is to say that Hochschild homology is the homology of the complex in Remark 2.41. Furthermore, that is also to say that we may define Hochschild homology by:

$$HH_n(A:B) = H_n(B \otimes A^{\otimes *}) = Z_n(B \otimes A^{\otimes *})/B_n(B \otimes A^{\otimes *}),$$

where $Z_n(B \otimes A^{\otimes *}) = \mathrm{Ker}(d_n)$ and $B_n(B \otimes A^{\otimes *}) = \mathrm{Im}(d_{n+1})$ for all $n \geq 0$, and the convention that d_0 is the zero map, and where for all other $n > 0$ we have d_n as in Remark 2.41. Following convention from Definition 2.30, we also define the elements in $Z_n(B \otimes A^{\otimes *})$ as the **Hochschild n-cycles**, and the elements of $B_n(B \otimes A^{\otimes *})$ as the **Hochschild n-boundaries**.

Definition 2.43. Given A and B as in Remark 2.41. The **Hochschild cohomology** $HH^*(A:B)$ of A with coefficients in an A-bimodule B is defined:

$$HH^n(A:B) = H^n(\mathrm{Hom}_{\mathbb{K}}(A^{\otimes *}:B)),$$

or equivalently:

$$HH^n(A:B) = Z^n(\mathrm{Hom}_{\mathbb{K}}(A^{\otimes *}:B))/B^n(\mathrm{Hom}_{\mathbb{K}}(A^{\otimes *}:B)),$$

where $Z^n(\mathrm{Hom}_{\mathbb{K}}(A^{\otimes *}:B)) = \mathrm{Ker}(d^{(n+1)})$ and $B^n(\mathrm{Hom}_{\mathbb{K}}(A^{\otimes *}:B)) = Im(d^{(n)})$ for all $n \geq 0$, using the convention that $d^{(0)}$ is the zero map and for all other $n > 0$ we have differentials such that $d^{(n)} : \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(n-1)}:B) \to \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(n)}:B)$ where:

$$\begin{aligned} d^{(n)}(f)(a_1 \otimes ... \otimes a_n) &= a_1 f(a_2 \otimes ... \otimes a_n) \\ &+ \sum_{i=1}^{n-1} (-1)^i f(a_1 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n) \\ &+ (-1)^n f(a_1 \otimes ... \otimes a_{n-1}) a_n, \end{aligned}$$

for all $f \in Hom_{\mathbb{K}}(A^{\otimes(n-1)}:B)$ and $a_1, ..., a_n \in A$. We refer to the elements of $Z^n(\mathrm{Hom}_{\mathbb{K}}(A^{\otimes *}:B))$ as **Hochschild n-cocycles** and the elements of $B^n(\mathrm{Hom}_{\mathbb{K}}(A^{\otimes *}:B))$ as **Hochschild n-coboundaries**.

Remark 2.44. In this thesis, we are interested in the case where for our Hochschild cohomology we have $B = A$. For such a case we use the notation:

$$HH^n(A:B) = HH^n(A).$$

2.5 Nijenhuis Operators

In this section we cover key information on Nijenhuis operators. This section is largely derived from information found in *Quantum Bi-Hamiltonian Systems*

[16], latter chapters of this thesis will then try to build upon findings from [16], expanding theorems built upon ring-like structures to the world of trusses.

Our first definition follows directly from the definitions outlined in Section 2.4.

Definition 2.45. Given an associative algebra (A, μ) over a field $\mathbb{K}$, with product:

$$\mu : A \times A \longrightarrow A, \qquad (a, b) \longmapsto ab,$$

or equivalently:

$$\mu(a, b) = ab,$$

for all $a, b \in A$. Given the linear map $N : A \to A$, we define the **Nijenhuis product** as:

$$\mu_N : (a, b) \longmapsto a \circ_N b = N(a)b - N(ab) + aN(b),$$

this gives rise to a new algebraic structure (A, μ_N). Furthermore, we may then note that:

$$a \circ_N b = \delta_\mu N(a, b), \tag{2.19}$$

where δ_μ is the **Hochschild coboundary operator associated with** μ, that is defined by the series seen in equation (2.15) in Definition 2.33:

$$\begin{aligned}\delta_\mu f(a_1, ..., a_{n+1}) = {} & a_1 f(a_2, ..., a_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(a_1, ..., \mu(a_i, a_{i+1}), ..., a_{n+1}) \\ & + (-1)^{n+1} f(a_1, ..., a_n) a_{n+1},\end{aligned}$$

with:

$$\delta_\mu : C^n(A : V) \longrightarrow C^{n+1}(A : V),$$

where $a_1, ..., a_n \in A$ with $f \in V$, where V is an A-bimodule, and $C^n(A : V)$ denotes the space of n-**cochains**, that is to say a space, equivalent to an additive group, consisting of the n-linear mappings from $A \times ... \times A$, n times into V.

We later look to expand the idea of the Hochschild coboundary operator to the truss case in Definition 3.1.

We may also note that, as seen in Theorem 2.34, we have:

$$\delta_\mu \circ \delta_\mu = 0. \tag{2.20}$$

Looking at equality (2.19), we may observe that N is a derivation of our original algebra if, and only if, N is a 1-cocycle with respect to the Hochschild coboundary operator δ_μ; the standard definition for derivations will be formally defined in Definition 2.46 below adapted from [30]. Later we look to derivations of trusses in Definition 3.10.

Definition 2.46. Let B be an algebra over the commutative ring $\mathbb{K}$, and let A be a sub-algebra of B. Then we define a **derivation** D of A into B as a $\mathbb{K}$-homomorphism of A into B, such that for all $x, y \in A$, we have:

$$D(xy) = xD(y) + D(x)y.$$

Here $\mathrm{Der}_{\mathbb{K}}(A : B)$ denotes the set of derivations of A into B. If we have $A = B$, then we refer to this as a derivation in A over $\mathbb{K}$, this set of derivations is denoted by $\mathrm{Der}_{\mathbb{K}}(A)$.

In the paper [16], the authors bring together, in just a few lines, many of the concepts we have tackled in Sections 2.4 and 2.5, and suddenly it all fits neatly together, note Remark 2.47 below.

Remark 2.47. The cohomology groups may be defined as follows. Given an associative algebra (A, μ), with product μ as in Definition 2.45, and an A-bimodule V, and $\delta_\mu : C^n(A : V) \longrightarrow C^{n+1}(A : V)$ also as in Definition 2.45.

1). An n-cochain $f \in C^n(A : V)$ is an *n-cocycle* if we have:

$$\delta_\mu f = 0.$$

2). An element of the form $\delta_\mu \beta$ where $\beta \in C^{n-1}(A : V)$ is an *n-coboundary.*

3). Together, these form the sub-group $B^n(A : V)$ of the additive group $Z^n(A : V)$ of n-cocycles.

4). The cohomology group $H^n(A : V)$ is defined by the quotient group:

$$H^n(A : V) = Z^n(A : V)/B^n(A : V).$$

The reader should now see how this comes full circle with what was discussed in Definition 2.31.

Following this, we may then define the μ-Nijenhuis torsion of a map N. This torsion allows us to measure the obstruction for the linear map N to be a homomorphism of these products.

Definition 2.48. The μ-**Nijenhuis torsion** of a linear map $N : A \to A$ is defined as:

$$T_N(a, b) = N(a \circ_N b) - N(a)N(b), \tag{2.21}$$

or equivalently:

$$T_N(a, b) = N(N(a)b) - N^2(ab) + N(aN(b)) - N(a)N(b). \tag{2.22}$$

In cases where we have $T_N(a, b) = 0$, for all $a, b \in A$, we refer to N as a μ-**Nijenhuis tensor**.

The following lemma and theorem are both taken from [16]. Theorem 2.50 is later expanded upon greatly, extending it to the truss case in Theorem 4.9. These workings are included here for completeness.

Lemma 2.49. *Given a μ-Nijenhuis tensor $N : A \to A$ and the products $\mu_{N^{k+r}}$ and μ_{N^k} are what we will call N^r-related, that is to say that we have:*

$$N^r(a \circ_{N^{k+r}} b) = N^r(a) \circ_{N^k} N^r(b), \tag{2.23}$$

for all $k, r = 0, 1, 2, 3, ...$ and $a, b \in A$.

Proof. We begin by proving the case where we have $r = 1$, that is:

$$N^r(a \circ_{N^{k+1}} b) = N^r(a) \circ_{N^k} N(b), \tag{2.24}$$

we start by applying N^k to the Nijenhuis torsion which we express in its expanded form, as in equation (2.22):

$$N(N(a)b) - N^2(ab) + N(aN(b)) - N(a)N(b) = 0, \tag{2.25}$$

giving us:

$$N^{k+1}(N(a)b) - N^{k+2}(ab) + N^{k+1}(aN(b)) - N^k(N(a)N(b)) = 0, \tag{2.26}$$

because we are dealing specifically with a Nijenhuis torsion, where our $T_N(a, b) = 0$, we can perform rearrangements:

$$N^{k+2}(ab) - N^{k+1}(aN(b)) = N^{k+1}(N(a)b) - N^k(N(a)N(b)) = 0, \tag{2.27}$$

we then use induction on line (2.27) with $k := k - 1$, we have:

$$N^{k+2}(ab) - N^{k+1}(aN(b)) = N(N^{k+1}(a)b) - N^{k+1}(a)N(b) = 0, \tag{2.28}$$

similarly we see that:

$$N^{k+2}(ab) - N^{k+1}(N(a)b) = N(aN^{k+1}(b)) - N(a)N^{k+1}(b), \tag{2.29}$$

which when combined with equation (2.27) gives us:

$$N^{k+1}(aN(b)) - N^k(N(a)N(b)) = N(aN^{k+1}(b)) - N(a)N^{k+1}(b), \tag{2.30}$$

Now we combine the equations (2.27) and (2.30), yielding:

$$N^{k+2}(ab) - N^k(N(a)N(b)) = \tag{2.31}$$

$$N(N^{k+1}(a)b + aN^{k+1}(b)) - N^{k+1}(a)N(b) - N(a)N^{k+1}(b), \tag{2.32}$$

which may be rearranged to:

$$N^k(N(a))N(b) + N(a)N^k(N(b)) - N^k(N(a)N(b)) \tag{2.33}$$

$$= N(N^{k+1}(a)b + aN^{k+1}(b) - N^{k+1}(ab)). \tag{2.34}$$

The reader can now see that (2.34) is exactly the same as (2.24). We then apply 2.24 inductively, yielding:

$$N^r(a \circ_{N^{k+r}} b) = N^{r-1}N(a \circ_{N^{k+r}} b) = N^{r-1}(N(a) \circ_{N^{k+r-1}} N(b)), \tag{2.35}$$

giving us equation (2.23). □

Using the above results, we can now demonstrate another theorem from [16]. As mentioned, the below theorem is expanded upon significantly in Theorem 4.9 of this thesis.

Theorem 2.50. *If we have a μ-Nijenhuis tensor $N : A \to A$, then it follows that:*

$$(\mu_{N^i})_{N^k} = \mu_{N^{i+k}}, \tag{2.36}$$

furthermore, N^r is a μ_{N^i}-Nijenhuis tensor, that is to say:

$$N^r(a(\circ_{N^i})_{N^r}b) = N^r(a) \circ_{N^i} N^r(b), \tag{2.37}$$

for all $i, k = 0, 1, 2, 3, ...$ and $a, b \in A$. Note also that all of the products μ_{N^k} are associative and compatible.

Proof. The proof of this theorem illustrates more clearly exactly what is being stated. We begin by looking at the case where $k = 1$. We shall prove first that:

$$(\mu_{N^i})_N = \mu_{N^{i+1}}, \tag{2.38}$$

We compute:

$$\begin{aligned} a(\circ_{N^i})_N b &= N(a) \circ_{N^i} b + a \circ_{N^i} N(b) - N(a \circ_{N^i} b), \\ &= N^{i+1}(a)b + N(a)N^i(b) - N^i(N(a)b) + N^i(a)N(b) \\ &+ aN^{i+1}(b) - N^i(aN(b)) - N(a) \circ_{N^{i-1}} N(b), \\ &= N^{i+1}(a)b + aN^{i+1}(b) - N^{i+1}(ab) - N^i(N(a)b + aN(b) - N(ab)) \\ &+ N^{i-1}(N(a)N(b)), \\ &= N^{i+1}(a)b + aN^{i+1}(b) - N^{i+1}(ab) - N^{i-1}(N(a \circ_N b) - N(a)N(b)), \\ &= A \circ_{N^{i+1}} B, \end{aligned}$$

using our previous workings in Lemma 2.49, we know that:

$$N(a \circ_{N^i} b) = N(a) \circ_{N^{i-1}} N(b).$$

holds. If we then use equations (2.38) and (2.24), we can see that N is indeed a μ_{N^i}-Nijenhuis tensor, this then gives us a compatible associative product $(\mu_{N^i})_N = \mu_{N^{i+1}}$, and thus we can apply Lemma 2.49 and equation (2.38) to μ_{N^i}, instead of μ, proving that our theorem holds. □

Now we look to another theorem from the *Quantum Bi-Hamiltonian Systems* paper, specifically [16, Theorem 1]. This theorem serves as somewhat of an impetus to this thesis, adapting this theorem to the truss case upon first glance does seem like it would be a fairly straightforward piece of work, one could naïvely say that it just needs some rearranging. However, the need to then develop expansions of cohomology theory and Nijenhuis operators to the affine case before even beginning to prove the theorem, does lead to a larger task than one first anticipated. So, in a sense, this theorem can be thanked for leading to the extension of several mathematical concepts to a more general case, which is surely one of the core motivations in the study of algebra. Such expansion now means that, not only can this theorem be expanded to the truss case, but so can many other theorems be expanded as well. The expanded truss version of this theorem appears later in this thesis in Chapter 4 as Theorem 4.6, and was a key result of [12] as well as this thesis. It is included here with an expanded proof for completeness.

Theorem 2.51. *Let μ_N be a Nijenhuis product as in Definition 2.45. The Nijenhuis product $\circ_N$ is associative if, and only if, the μ-Nijenhuis torsion T_N of N is a 2-Hochschild cocycle of the algebra A, i.e. $\delta_\mu T_N(a,b,c) = 0$. Equivalently, we may write:*

$$(a \circ_N b) \circ_N c = a \circ_N (b \circ_N c) \tag{2.39}$$

$$\iff$$

$$\delta_\mu T_N(a,b,c) = aT_N(b,c) - T_N(ab,c) + T_N(a,bc) - T_N(a,b)c = 0. \tag{2.40}$$

Here μ_N is an associative product compatible with μ, meaning that $\mu + \lambda\mu_N$ are associative for all $\lambda \in \mathbb{K}$, where $\mathbb{K}$ is a field. If μ is unital with the unit 1, then μ_N also has the unit 1, providing that $N(1) = 1$.

Furthermore, if N is a μ-Nijenhuis tensor, then the Nijenhuis product μ_N is an associative product on A that is compatible with μ.

Proof. We shall prove that statement (2.39-2.40) from both directions.

1. Let us first make the assumption that $\circ_N$ is associative, this implies that:

$$(a \circ_N b) \circ_N c - a \circ_N (b \circ_N c) = 0, \tag{2.41}$$

holds. Let us now expand our terms. We have:

$$\begin{aligned}
(a \circ_N b) \circ_N c &= (N(a)b - N(ab) + aN(b)) \circ_N c,\\
\\
&= N(N(a)b - N(ab) + aN(b))c\\
&\quad - N((N(a)b - N(ab) + aN(b))c)\\
&\quad + (N(a)b - N(ab) + aN(b))N(c),\\
\\
&= N(N(a)b)c - N(N(ab))c + N(aN(b))c\\
&\quad - N(N(a)bc) + N(N(ab)c) - N(aN(b)c)\\
&\quad + N(a)bN(c) - N(ab)N(c) + aN(b)N(c),
\end{aligned}$$

for our first term. Then expanding our other term yields:

$$\begin{aligned}
a \circ_N (b \circ_N c) &= a \circ_N (N(b)c - N(bc) + bN(c)),\\
&= N(a)(N(b)c - N(bc) + bN(c))\\
&\quad - N(a(N(b)c - N(bc) + bN(c)))\\
&\quad + aN(N(b)c - N(bc) + bN(c)),\\
&= N(a)N(b)c - N(a)N(bc) + N(a)bN(c)\\
&\quad - N(aN(b)c) + N(aN(bc)) - N(abN(c))\\
&\quad + aN(N(b)c) - aN(N(bc)) + aN(bN(c)),
\end{aligned}$$

we then combine these calculations, cancelling terms and expressing equation (2.41) as:

$$(a \circ_N b) \circ_N c - a \circ_N (b \circ_N c) =$$

$$\begin{aligned}
&= N(N(a)b)c - N(N(ab))c + N(aN(b))c - N(N(a)bc)\\
&+ N(N(ab)c) + N(a)bN(c) - N(ab)N(c) + aN(b)N(c)\\
&- N(a)N(b)c + N(a)N(bc) - N(a)bN(c) - N(aN(bc))\\
&+ N(abN(c)) - aN(N(b)c) + aN(N(bc)) - aN(bN(c)).
\end{aligned}$$

Then if we calculate the terms of $\delta_\mu T_N(a, b, c)$, we find:

$$aT_N(b, c) = aN(N(b)c) - aN(N(bc)) + aN(bN(c)) - aN(b)N(c),$$

$$T_N(ab, c) = N(N(ab)c) - N(N(abc)) + N(abN(c)) - N(ab)N(c),$$

$$T_N(a, bc) = N(N(a)bc) - N(N(abc)) + N(aN(bc)) - N(a)N(bc),$$

$$T_N(a, b)c = N(N(a)b)c - N(N(ab))c + N(aN(b))c - N(a)N(b)c.$$

One can now compare these terms and see that if, (2.41) holds, then we must have $\delta_\mu T_N(a, b, c) = 0$, as the two equations share identical terms.

2. Now, if we begin instead with the assumption that $\delta_\mu T_N(a, b, c) = 0$, it is clear that we must also have the condition (2.41), as we may simply rearrange the terms of $\delta_\mu T_N(a, b, c)$ so that they are exactly (2.41). □

Now that we have covered the key preliminary information for this thesis, the reader should now be in a position where they are quite familiar with Nijenhuis operators. It would be poignant to discuss why Nijenhuis operators were included in the paper *Quantum Bi-Hamiltonian Systems* [16] by Cariñena, Grabowski and Marmo, and the relation of Nijenhuis operators to bi-Hamiltonian systems.

The below definitions and remarks are taken from [16, Section 3] and are included for completeness, with some updated notation as to be consistent with the rest of this thesis.

Definition 2.52. In the classical case, a bi-Hamiltonian system consists of two compatible Poisson brackets and a system that is Hamiltonian with respect to both brackets. When defining a *weak quantum bi-Hamiltonian system*, we mean that we have two Lie algebra structures on the space $\mathrm{Op}(\mathcal{H})$ of operators on a Hilbert space $\mathcal{H}$ which are compatible in the sense that the corresponding commutators are compatible Lie brackets, which is to say that the sum of these Lie brackets also form a Lie bracket, and a derivation $D \in \mathrm{Der}(\mathrm{Op}(\mathcal{H}))$ which is an inner derivation with respect to both associative structures [16], [19].

Since we want the Leibniz rule:

$$[a, b \circ c] = [a, b] \circ c + b \circ [a, c],$$

in view of Dirac's proof [18], that derivations of a sufficiently non-degenerate associative algebra are just adjoint operators, we want to have a new bracket in the form of the commutator of a new associative structure. We shall call these pairs of associative structures **weak quantum bi-Hamiltonian**.

Definition 2.53. Following from Definition 2.52, we may then create the additional requirement that both associative structures have the same unit 1. Further to this condition, one can also consider a stronger version of compatibility of associative products $\circ_1$ and $\circ_2$, requiring that $\circ_1 + \lambda \circ_2$ is associative for all $\lambda \in \mathbb{K}$, where $\mathbb{K}$ is the ground field; we shall call such a pair a **quantum bi-Hamiltonian system**.

Following this, [16] makes some algebraic observations that connect the concepts of *quantum bi-Hamiltonian systems* and *Nijenhuis operators*.

Remark 2.54. Let $(A, \cdot)$ be a unital associative algebra. One can define a new associative product on A by taking an element $k \in A$ and defining a new product:

$$a \circ_k b = akb.$$

Where the product for the original associative structure is shown simply by juxtaposition. It is clear that the unit is only preserved when $k = 1$, and that we have a homomorphism of the products:

$$T_k(a \circ_k b) = T_k(a)T_k(b),$$

where T_k is the linear map:

$$T_k : A \longrightarrow A, \qquad T_k(a) = ka,$$

which is a non-unital isomorphism in the case that k is invertible. We can then generalise this if we deform the associative structure by an associative analogue of the Nijenhuis tensor.

Let (A, μ) be an associative algebra over a field $\mathbb{K}$, where μ is the product:

$$\mu : A \times A \longrightarrow A, \qquad \mu(a, b) \longmapsto ab,$$

where $a, b \in A$, and let $N : A \to A$ be a linear map, where $N \in A^* \otimes A$. If N is a derivation of the algebra (A, μ), then we have $N(a)b + aN(b) - N(ab) = 0$. In any case, the map:

$$\mu_N : (a, b) \longmapsto a \circ_N b = N(a)b + aN(b) - N(ab),$$

is a bilinear map and therefore it defines a new algebraic structure (A, μ_N). If we then consider the A-bimodule structure in A as given by left and right multiplication, we may say that we have:

$$a \circ_N b = \delta_\mu N(a, b),$$

where δ_μ is the Hochschild coboundary operator associated with μ. Therefore, N is a derivation of the original algebra if and only if N is a 1-cocycle with respect to the Hochschild coboundary operator δ_μ.

Finally, we note that the obstruction for the linear map N to be a homomorphism of these products can be measured by the μ-Nijenhuis torsion of N:

$$T_N(a, b) = N(a \circ_N b) - N(a)N(b).$$

Thus, we can now see what Nijenhuis operators are, how they relate to bi-Hamiltonian systems, and why they were first used in the *Quantum Bi-Hamiltonian Systems* paper [16].

In the coming chapters, we look to not just use the concepts defined in this preliminary chapter, but to extend many of the definitions themselves to more general cases. We look to extend Hochschild cohomology from the ring case to the truss case (Chapter 3); we look to extend Nijenhuis products and Nijenhuis operators from the ring case to the truss case (Chapter 4); we extend Lie algebras and the Lie bracket to the affine case (Chapter 6); we look at affine Nijenhuis operators and also define Nijenhuis operators over Lie affgebras, as well as defining the Nijenhuis bracket (Chapter 7).

Chapter 3

Hochschild Cohomology of Trusses

This chapter will look to make a proposal for the Hochschild cohomology of trusses. This section not only provides a necessary foundation for the theorems present in latter chapters, but also stands on its own as a novel look at performing Hochschild Cohomology on trusses as opposed to ring-like structures.

3.1 Hochschild Coboundaries & Cohomology on Trusses

Definition 3.1. Let T be a truss and let $C^0(T) = T$, and for all positive integers n, let $C^n(T)$ be the set of all multi-heap functions $T^n \to T$, i.e. heap morphisms in each argument. For all $n \in \mathbb{N}, C^n(T)$ are viewed as heaps with the operation defined pointwise that is inherited from T.

For all $n \in N$ and $\nu \in T$, the heap homomorphism $\delta^n_\nu : C^n(T) \to C^{n+1}(T)$ is defined by:

$$\delta^n_\nu f(a_0, ..., a_n) = \begin{cases} \langle \nu, a_0\nu, a_0 f(a_1, ..., a_n), f(a_0a_1, a_2, \dots, a_n), \dots, \\ \qquad f(a_0, a_1, a_2, \dots, a_{n-1}a_n), f(a_0, ..., a_{n-1})a_n, \nu a_n \rangle, & n \text{ even}, \\ \\ \langle \nu, a_0\nu, a_0 f(a_1, ..., a_n), f(a_0a_1, a_2, \dots, a_n), \dots, \\ \qquad f(a_0, a_1, a_2, \dots, a_{n-1}a_n), f(a_0, ..., a_{n-1})a_n, \nu a_n, \nu \rangle, & n \text{ odd}, \end{cases}$$

for all $f \in C^n(T)$ and $a_0, ..., a_n \in T$. We call this the **ν-relative Hochschild n-coboundary operator on T**.

Also note that the maps δ^n_ν are heap homomorphisms by the truss distributive law and the fact that T is an abelian heap. The use of the term 'coboundary' is justified by the following theorem.

Theorem 3.2. *For all $n \in \mathbb{N}$ and $\nu \in T$:*

$$\delta_\nu^{n+1} \circ \delta_\nu^n = \nu,$$

where in each case the constant function with the value ν is denoted by ν as well.

Furthermore, for all n, $\delta_\nu^n(\nu) = \nu$, and hence each δ_ν^n is a homomorphism of abelian groups $\delta_\nu^n : G(C^n(T);\nu) \to G(C^{n+1}(T);\nu)$.

Proof. Here we shall represent the heap operation $\langle -,-,- \rangle$ as a linear combination of $+$ and $-$ and use the distributive laws from the retract $G(T;\nu)$ where $(T, \langle -,-,- \rangle, \cdot)$ is a truss. Starting with the case where n is even and $n+1$ is odd, we have:

$$\begin{aligned}\delta_\nu^n f(a_0,...,a_n) &= -a_0\nu + a_0 f(a_1,...,a_n) \\ &+ \sum_{j=1}^{n} (-1)^j f(a_0,...,a_{j-1}a_j,...,a_n) \\ &- f(a_0,...,a_{n-1})a_n + \nu a_n,\end{aligned}$$

$$\begin{aligned}\delta_\nu^{n+1} f(a_0,...,a_{n+1}) &= -a_0\nu + a_0 f(a_1,...,a_{n+1}) \\ &+ \sum_{i=1}^{n+1} (-1)^i f(a_0,...,a_{i-1}a_i,...,a_{n+1}) \\ &+ f(a_0,...,a_n)a_{n+1} - \nu a_{n+1}.\end{aligned}$$

We compose these functions:

$$\begin{aligned}\delta_\nu^{n+1}\delta_\nu^n f(a_0,...,a_{n+1}) &= -a_0\nu + a_0\delta_\nu^n f(a_1,...,a_{n+1}) \\ &+ \sum_{i=1}^{n+1} (-1)^i \delta_\nu^n f(a_0,...,a_{i-1}a_i,...,a_{n+1}) \\ &+ \delta_\nu^n f(a_0,...,a_n)a_{n+1} - \nu a_{n+1}.\end{aligned}$$

Cancelling terms with alternating signs we find:

$$\begin{aligned}\sum_{i=1}^{n+1} (-1)^i \delta_\nu^n f(a_0,...,a_{i-1}a_i,...,a_{n+1}) &= a_0a_1\nu - a_0a_1 f(a_2,...,a_{n+1}) \\ &- \sum_{i=2}^{n+1} (-1)^{i+1} a_0 f(a_1,...,a_{i-1}a_i,...,a_{n+1}) \\ &- \sum_{j=1}^{n} (-1)^j f(a_0,...,a_{j-1}a_j,...,a_n)a_{n+1} \\ &+ f(a_0,...,a_{n-1})a_na_{n+1} - \nu a_na_{n+1}.\end{aligned}$$

This can then be simplified further by applying the definition of δ_ν^n and the truss distributive laws:

$$\sum_{i=1}^{n+1}(-1)^i\delta_\nu^n f(a_0,...,a_{i-1}a_i,...,a_{n+1}) = a_0\nu - a_0\delta_\nu^n f(a_1,...,a_{n+1})$$
$$- \delta_\nu^n f(a_0,...,a_n)a_{n+1} + \nu a_{n+1}.$$

Making the substitution in the formula for the composition we thus find:

$$\begin{aligned}\delta_\nu^{n+1}\delta_\nu^n f(a_0,...,a_{n+1}) = {} & -a_0\nu + a_0\delta_\nu^n f(a_1,...,a_{n+1}) \\ & + a_0\nu - a_0\delta_\nu^n f(a_1,...,a_{n+1}) - \delta_\nu^n f(a_0,...,a_n)a_{n+1} + \nu a_{n+1} \\ & + \delta_\nu^n f(a_0,...,a_n)a_{n+1} - \nu a_{n+1} = \nu.\end{aligned}$$

We may then compare terms to find:

$$\delta_\nu^{n+1}\delta_\nu^n f(a_0,...,a_{n+1}) = \nu.$$

Now we may look at the case where n is odd and $n+1$ is even.

$$\begin{aligned}\delta_\nu^n f(a_0,...,a_n) = {} & \nu - a_0\nu + a_0 f(a_1,...,a_n) \\ & + \sum_{j=1}^{n}(-1)^j f(a_0,...,a_{j-1}a_j,...,a_n) \\ & + f(a_0,...,a_{n-1})a_n - \nu a_n + \nu,\end{aligned}$$

$$\begin{aligned}\delta_\nu^{n+1} f(a_0,...,a_{n+1}) = {} & \nu - a_0\nu + a_0 f(a_1,...,a_{n+1}) \\ & + \sum_{i=1}^{n+1}(-1)^i f(a_0,...,a_{i-1}a_i,...,a_{n+1}) \\ & - f(a_0,...,a_n)a_{n+1} + \nu a_{n+1}.\end{aligned}$$

In this case the composition comes out as:

$$\begin{aligned}\delta_\nu^{n+1}\delta_\nu^n f(a_0,...,a_{n+1}) = {} & -a_0\nu + a_0\delta_\nu^n f(a_1,...,a_{n+1}) \\ & + \sum_{i=1}^{n+1}(-1)^i\delta_\nu^n f(a_0,...,a_{i-1}a_i,...,a_{n+1}) \\ & - \delta_\nu^n f(a_0,...,a_n)a_{n+1} + \nu a_{n+1}.\end{aligned}$$

Once more cancelling the alternating terms, using the definition of functions δ_ν^n, as well as the distributive laws for trusses we find:

$$
\begin{aligned}
\sum_{i=1}^{n+1}(-1)^i\delta_\nu^n f(a_0,...,a_{i-1}a_i,...,a_{n+1}) &= a_0a_1\nu - a_0a_1f(a_2,...,a_{n+1})\\
&- \sum_{i=2}^{n+1}(-1)^{i+1}a_0f(a_1,...,a_{i-1}a_i,...,a_{n+1})\\
&+ \sum_{j=1}^{n}(-1)^j f(a_0,...,a_{j-1}a_j,...,a_n)a_{n+1}\\
&+ \nu a_{n+1} - a_0\nu + f(a_0,...,a_{n-1})a_na_{n+1} - \nu a_na_{n+1},\\
\\
&= a_0\nu - a_0\delta_\nu^n f(a_1,...,a_{n+1}) + \delta_\nu^n f(a_0,...,a_n)a_{n+1} - \nu a_{n+1}.
\end{aligned}
$$

Using the above calculation, we then expand:

$$
\begin{aligned}
\delta_\nu^{n+1}\delta_\nu^n f(a_0,...,a_{n+1}) &= -a_0\nu + a_0\delta_\nu^n f(a_1,...,a_{n+1})\\
&+ a_0\nu - a_0\delta_\nu^n f(a_1,...,a_{n+1}) + \delta_\nu^n f(a_0,...,a_n)a_{n+1} - \nu a_{n+1}\\
&- \delta_\nu^n f(a_0,...,a_n)a_{n+1} + \nu a_{n+1} = \nu,
\end{aligned}
$$

We may compare terms to find:

$$\delta_\nu^{n+1}\delta_\nu^n f(a_0,...,a_{n+1}) = \nu.$$

as required.

Since ν is the neutral element in $G(T;\nu)$, we immediately find that, for all $a_0,\dots,a_n$ and the constant function $\nu : T^n \to T$, $(a_0\dots a_{n-1}) \mapsto \nu$:

$$\delta_\nu^n(\nu)(a_0,\dots,a_n) = -a_0\nu + a_0\nu + (-1)^{n+1}\nu a_n + (-1)^n\nu a_n = \nu.$$

Now, the observation that any homomorphism of heaps $f : H \to K$ is a homomorphism of groups $f : G(H;\nu) \to G(K;f(\nu))$, for all $\nu \in H$ confirms the final assertion. □

Definition 3.3. In the setup of Definition 3.1, we define the heap of **ν-relative n-cocycles**:

$$Z_\nu^n(T) := \{f \in C^n(T) \mid \delta_\nu^n(f) = \nu\},$$

and the heap of **ν-relative n-coboundaries**:

$$B_\nu^n(T) := \operatorname{Im}\delta_\nu^{n-1}.$$

Since Theorem 3.2 implies that $B_\nu^n(T)$ is a sub-heap of $Z_\nu^n(T)$, we can formulate the core definition of this chapter.

Definition 3.4. For all $n \in \mathbb{N}$ and $\nu \in T$, the quotient heap:

$$H_\nu^n(T) = Z_\nu^n(T)/B_\nu^n(T),$$

is called the **n-th ν-relative Hochschild cohomology heap** of T.

Remark 3.5. The last assertion of Theorem 3.2 ensures the existence of the cochain complex of abelian groups $(\delta^n_\nu : G(C^n(T);\nu) \to G(C^{n+1}(T);\nu))$. This allows one for an abelian group interpretation of ν-retracts of ν-relative Hochschild cohomology heaps, namely, $G(H^n_\nu(T);\bar{\nu})$, where $\bar{\nu}$ is the class of the constant heap map ν, is the n-th cohomology group of the above complex of abelian groups $G(C^n(T);\nu)$.

We next show that the relative Hochschild cohomology heaps for different ν can be identified up to isomorphism. We start with the following:

Lemma 3.6. *For all heap homomorphisms $f \in C^n(T)$ and $\nu, \nu' \in T$:*

$$\delta^n_{\nu'}(f) = \tau^{\nu'}_\nu \circ \delta^n_\nu(\tau^\nu_{\nu'} \circ f).$$

Proof. First we look at the case where n is even:

$$\delta^n_{\nu'} f(a_0, ..., a_n) = \tau^{\nu'}_\nu(\delta^n_\nu(\tau^\nu_{\nu'} \circ f)(a_0, ..., a_n)).$$

Let us first calculate:

$$\begin{aligned}
\delta^n_\nu(\tau^\nu_{\nu'} \circ f)(a_0, ..., a_n) &= \langle \nu, a_0\nu, a_0 f(a_1, ..., a_n), a_0\nu', a_0\nu, f(a_0a_1, ..., a_n), \nu', \nu, ... \\
&\quad ..., f(a_0, ..., a_{n-1}a_n), \nu', \nu, f(a_0, ..., a_{n-1})a_n, \nu' a_n, \nu a_n, \nu a_n\rangle, \\
&= \langle \nu, a_0\nu', a_0 f(a_1, ..., a_n), f(a_0a_1, ..., a_n), ... \\
&\quad ..., f(a_0, ..., a_{n-1}a_n), f(a_0, ..., a_{n-1})a_n, \nu' a_n\rangle, \\
&= \langle \nu, \nu', \delta^n_{\nu'} f(a_0, ..., a_n)\rangle.
\end{aligned}$$

Now, for the case where we have n odd, by similar calculation:

$$\begin{aligned}
\delta^n_\nu(\tau^\nu_{\nu'} \circ f)(a_0, ..., a_n) &= \langle \nu, a_0\nu, a_0 f(a_1, ..., a_n), a_0\nu', a_0\nu, f(a_0a_1, ..., a_n), \nu', \nu, \\
&\quad f(a_0, a_1a_2, ..., a_n), \nu', \nu ..., f(a_0, ..., a_{n-1}a_n), \nu', \nu, \\
&\quad f(a_0, ..., a_{n-1})a_n, \nu' a_n, \nu a_n, \nu a_n, \nu\rangle, \\
&= \langle \nu, a_0\nu, a_0 f(a_1, ..., a_n), f(a_0a_1, ..., a_n), \nu', \nu, \\
&\quad f(a_0, a_1a_2, ..., a_n), ..., f(a_0, ..., a_{n-1}a_n), \\
&\quad f(a_0, ..., a_{n-1})a_n, \nu' a_n, \nu\rangle, \\
&= \langle \nu, \nu', \delta^n_{\nu'} f(a_0, ..., a_n)\rangle.
\end{aligned}$$

Thus, for both cases we have:

$$\begin{aligned}
\tau^{\nu'}_\nu(\delta^n_\nu(\tau^\nu_{\nu'} \circ f)(a_0, ..., a_n)) &= \langle \nu, \nu', \delta^n_{\nu'} f(a_0, ..., a_n), \nu, \nu'\rangle, \\
&= \delta^n_{\nu'} f(a_0, ..., a_n).
\end{aligned}$$

□

As a consequence of the commutativity of the heap operations and the Mal'cev identities, for all $\nu, \nu' \in T$ and $n \in \mathbb{N}$ we can consider heap isomorphisms:

$$\overrightarrow{\tau}_{\nu}^{\nu'} : C^n(T) \longrightarrow C^n(T), \qquad f \longmapsto \tau_{\nu}^{\nu'} \circ f.$$

In terms of these isomorphisms, the statement of Lemma 3.6 can be rephrased as:

$$\delta_{\nu'}^n \left(\overrightarrow{\tau}_{\nu}^{\nu'}(f) \right) = \overrightarrow{\tau}_{\nu}^{\nu'} \left(\delta_{\nu}^n(f) \right), \tag{3.1}$$

for all $f \in C^n(T)$.

We could then refer to property (3.1) as being "translation invariant".

Lemma 3.7. *For all $\nu, \nu' \in A$, the maps $\overrightarrow{\tau}_{\nu}^{\nu'}$ restrict to isomorphisms $B_{\nu}^n(T) \to B_{\nu'}^n(T)$ and $Z_{\nu}^n(T) \to Z_{\nu'}^n(T)$.*

Proof. This follows (almost) immediately from Lemma 3.6 or its equivalent formulation in (3.1). Specifically, if $g = \delta_{\nu}^n(f) \in B_{\nu}^n(T)$, then $\overrightarrow{\tau}_{\nu}^{\nu'}(g) = \delta_{\nu'}^n \left(\overrightarrow{\tau}_{\nu}^{\nu'}(f) \right) \in B_{\nu'}^n(T)$. If $\delta_{\nu}^n(f) = \nu$, then:

$$\nu' = \langle \nu, \nu, \nu' \rangle = \overrightarrow{\tau}_{\nu}^{\nu'} \left(\delta_{\nu}^n(f) \right) = \delta_{\nu'}^n \left(\overrightarrow{\tau}_{\nu}^{\nu'}(f) \right),$$

i.e. $\overrightarrow{\tau}_{\nu}^{\nu'}(f) \in Z_{\nu'}^n(T)$. □

When combined, Lemma 3.6 and Lemma 3.7 yield the identification of cohomology heaps sought for.

Theorem 3.8. *Let T be a truss. Then, for all $n \in \mathbb{N}$ and $\nu, \nu' \in T$:*

$$H_{\nu}^n(T) \cong H_{\nu'}^n(T).$$

Proof. By Lemma 3.6 and Lemma 3.7 the isomorphisms $\overrightarrow{\tau}_{\nu}^{\nu'}$ descent to the isomorphisms $H_{\nu}^n(T) \to H_{\nu'}^n(T)$; for $f, g \in Z_{\nu}^n(T)$ represent the same class in $H_{\nu}^n(T)$ if and only if:

$$\langle f, g, \delta_{\nu}^{n-1}(h) \rangle \in B_{\nu}^n(T),$$

for some (equivalently all) $h \in C^{n-1}(T)$. Hence:

$$\left\langle \overrightarrow{\tau}_{\nu}^{\nu'}(f), \overrightarrow{\tau}_{\nu}^{\nu'}(g), \delta_{\nu'}^{n-1} \left(\overrightarrow{\tau}_{\nu}^{\nu'}(h) \right) \right\rangle = \left\langle \overrightarrow{\tau}_{\nu}^{\nu'}(f), \overrightarrow{\tau}_{\nu}^{\nu'}(g), \overrightarrow{\tau}_{\nu}^{\nu'} \left(\delta_{\nu}^{n-1}(h) \right) \right\rangle,$$

$$= \overrightarrow{\tau}_{\nu}^{\nu'} \left(\langle f, g, \delta_{\nu}^{n-1}(h) \rangle \right) \in B_{\nu'}^n(T).$$

□

In view of Theorem 3.8, rather than talking about ν-relative Hochschild cohomology heaps, we might talk just as well about simply Hochschild cohomology heaps and drop the subscript ν from the notation.

Remark 3.9. One easily checks that $\overset{\rightarrow\nu'}{\tau}_{\nu}(\nu) = \nu'$, and hence the isomorphism described in Theorem 3.8 is an isomorphism of abelian groups $G(H^n_\nu(T); \bar{\nu}) \simeq G(H^n_{\nu'}(T); \overline{\nu'})$.

In case of the Hochschild cohomology of algebras, one-cocycles correspond to derivations. A similar statement can be made in the case of the cohomology of trusses, although this correspondence is not quite as direct as in the ring case.

3.2 Derivations on Trusses

Definition 3.10 ([8])**.** Let T be a truss. A heap homomorphism $D : T \to T$ is called a **derivation** if, for all $a, b \in T$:

$$D(ab) = \langle D(a)b, ab, aD(b)\rangle.$$

Derivations on T form a heap which is denoted by $\mathrm{Der}(T)$.

Note that we later revisit derivations, but within the context of affgebras in Definition 6.11.

Proposition 3.11. *For all $\nu \in T$, $\mathrm{Der}(T) \cong Z^1_\nu(T)$ as heaps.*

Proof. The isomorphism and its inverse are given by:

$$\Theta : \mathrm{Der}(T) \to Z^1_\nu(T), \qquad \Theta(D) : a \mapsto \langle D(a), a, \nu\rangle, \tag{3.2}$$
$$\Theta^{-1} : Z^1_\nu(T) \to \mathrm{Der}(T), \qquad \Theta^{-1}(f) : a \mapsto \langle f(a), \nu, a\rangle. \tag{3.3}$$

It is clear that the defined maps are inverses of each other. Thus, it remains to be checked if their domains and codomains are as stated:

If D is a derivation, then, for all $a, b \in T$:

$$\begin{aligned}
\delta^1_\nu(\Theta(D))(a,b) &= \langle \nu, a\nu, a\Theta(D)(b), \Theta(D)(ab), \Theta(D)(a)b, \nu b, \nu\rangle, \\
&= \langle \nu, a\nu, aD(b), ab, a\nu, D(ab), ab, \nu, D(a)b, ab, \nu b, \nu b, \nu\rangle, \\
&= \langle \nu, D(ab), aD(b), ab, D(a)b\rangle = \langle \nu, D(ab), D(ab)\rangle = \nu,
\end{aligned}$$

where the second equality uses the truss distributive law, the third one arises from the cancellation and reshuffling rules described in Remark 2.2, and the penultimate equality is the definition of the derivation. Thus, $\Theta(D) \in Z^1_\nu(T)$ as required.

In the converse direction, if $f \in Z^1_\nu(T)$, then, for all $a, b \in T$:

$$\langle \nu, a\nu, af(b), f(ab), f(a)b, \nu b, \nu\rangle = \nu,$$

and so by Remark 2.7, equivalently:

$$f(ab) = \langle f(a)b, \nu b, af(b), a\nu, \nu \rangle.$$

Therefore:

$$\begin{aligned} \langle \Theta^{-1}(f)(a)b, ab, a\Theta^{-1}(f)b \rangle &= \langle f(a)b, \nu b, ab, ab, af(b), a\nu, ab \rangle, \\ &= \langle f(ab), \nu, ab \rangle = \Theta^{-1}(f)(ab), \end{aligned}$$

so that $\Theta^{-1}(f)$ is a derivation on T, as required. □

As an illustration of Hochschild cohomology of trusses we compute the cohomology heaps of the second of the trusses (2.1) in Remark 2.13.

Now let us look at a derivation as an endomorphism of heaps.

Proposition 3.12. *Let T be an abelian heap, viewed as a truss, by means of the product $a \cdot b = a$ for all $a, b \in T$, it follows that $Der(T) = End(T)$.*

Proof. We compute:

$$D(a \cdot b) = \langle D(a) \cdot b, a \cdot b, a \cdot D(b) \rangle = \langle D(a), a, a \rangle = D(a),$$

then we can see that:

$$D(a) \cdot D(b) = D(a) = D(a \cdot b).$$

□

We may also look at derivations as endomorphisms of abelian groups.

Proposition 3.13. *Let T be an abelian heap, viewed as a truss by means of the product:*

$$a \cdot b = a + b,$$

for all $a, b \in T$, it follows that, $Der(T) = End(A)$, where $End(A)$ is the set of endomorphisms over an abelian group.

Proof. If D is a derivation, than we may compute:

$$\begin{aligned} D(a \cdot b) = \langle D(a) \cdot b, a \cdot b, a \cdot D(b) \rangle &= \langle D(a) + b, a + b, a + D(b) \rangle, \\ &= D(a) + b - a - b + a + D(b), \\ &= D(a) + D(b), \end{aligned}$$

thus:

$$D(a \cdot b) = D(a) + D(b).$$

implying that we have an equality if D is a endomorphism of abelian groups.

Now let us check the proof in the other direction. If D is a morphism, such that:

$$D(a \cdot b) = D(a) + D(b),$$

then it follows that:

$$\begin{aligned} D(a \cdot b) = D(a) + D(b) &= D(a) + b - a - b + a + D(b), \\ &= \langle D(a) + b, a + b, a + D(b) \rangle, \\ &= \langle D(a) \cdot b, a \cdot b, a \cdot D(b) \rangle. \end{aligned}$$

It clearly follows that:

$$D(a \cdot b) = \langle D(a) \cdot b, a \cdot b, a \cdot D(b) \rangle,$$

and thus D is a derivative if, and only if, it is an endomorphism of abelian groups. □

Theorem 3.14. *Given a derivation $D : T \to T$, that is to say:*

$$D(ab) = \langle D(a)b, ab, aD(b) \rangle,$$

and a retract $R(T; \nu)$, where $\nu \in T$ and the operation $\cdot_\nu$, such that:

$$a \cdot_\nu b = ab -_\nu a\nu +_\nu \nu^2 -_\nu \nu b.$$

If we then consider a group homomorphism $D_\nu : R(T; \nu) \to R(T; \nu)$, where:

$$D_\nu(a) = \langle D(\nu), D(a), a \rangle.$$

It follows that:

$$D_\nu(a \cdot_\nu b) = D_\nu(a) \cdot_\nu b +_\nu a \cdot_\nu D_\nu(b). \tag{3.4}$$

Proof. We start by noting that D_ν is an additive map, this can clearly be seen as we have:

$$D_\nu(\nu) = \langle D(\nu), D(\nu), \nu \rangle = \nu.$$

Following on from this, we can see that:

$$D_\nu(a +_\nu b) = \langle D_\nu(a), D_\nu(\nu), D_\nu(b) \rangle = \langle D_\nu(a), \nu, D_\nu(b) \rangle = D_\nu(a) +_\nu D_\nu(b).$$

Now we want to show that D_ν is a derivation. We begin by computing the left-hand side of equation (3.4):

$$\begin{aligned} D_\nu(a \cdot_\nu b) &= D_\nu(ab -_\nu a\nu +_\nu \nu^2 -_\nu \nu b), \\ &= D_\nu(ab) -_\nu D_\nu(a\nu) +_\nu D_\nu(\nu^2) -_\nu D_\nu(\nu b), \\ &= D(\nu) -_\nu D(ab) +_\nu ab -_\nu D(\nu) +_\nu D(a\nu) -_\nu a\nu \\ &\quad +_\nu D(\nu) -_\nu D(\nu^2) +_\nu \nu^2 -_\nu D(\nu) +_\nu D(\nu b) -_\nu \nu b, \\ &= ab -_\nu a\nu +_\nu \nu^2 -_\nu \nu b -_\nu D(ab) +_\nu D(a\nu) -_\nu D(\nu^2) +_\nu D(\nu b), \\ &= ab -_\nu a\nu +_\nu \nu^2 -_\nu \nu b -_\nu D(a)b +_\nu ab -_\nu aD(b) +_\nu D(a)\nu -_\nu a\nu \\ &\quad +_\nu aD(\nu) -_\nu D(\nu)\nu +_\nu \nu^2 -_\nu \nu D(\nu) +_\nu D(\nu)b -_\nu \nu b +_\nu \nu D(b). \end{aligned}$$

Similarly, we compute the right-hand side of equation (3.4):

$$\begin{aligned}
D_\nu(a)\cdot_\nu b +_\nu a\cdot_\nu D_\nu(b) &= D_\nu(a)b -_\nu D_\nu(a)\nu +_\nu \nu^2 -_\nu \nu b \\
&\quad +_\nu aD_\nu(b) -_\nu a\nu +_\nu \nu^2 -_\nu \nu D_\nu(b), \\
&= D(\nu)b -_\nu D(a)b +_\nu ab -_\nu D(\nu)\nu +_\nu D(a)\nu -_\nu a\nu +_\nu \nu^2 \\
&\quad -_\nu \nu b +_\nu aD(\nu) -_\nu aD(b) +_\nu ab -_\nu a\nu +_\nu \nu^2 -_\nu \nu D(\nu) \\
&\quad +_\nu \nu D(b) -_\nu \nu b, \\
&= 2ab -_\nu 2a\nu +_\nu 2\nu^2 -_\nu 2\nu b -_\nu aD(b) +_\nu aD(\nu) -_\nu \nu D(\nu) \\
&\quad +_\nu \nu D(b) -_\nu D(a)b +_\nu D(a)\nu -_\nu D(\nu)\nu +_\nu D(\nu)b, \\
&= ab -_\nu a\nu +_\nu \nu^2 -_\nu \nu b -_\nu D(ab) \\
&\quad +_\nu D(a\nu) -_\nu D(\nu^2) +_\nu D(\nu b), \\
&= ab -_\nu a\nu +_\nu \nu^2 -_\nu \nu b -_\nu D(a)b +_\nu ab -_\nu aD(b) \\
&\quad +_\nu D(a)\nu -_\nu a\nu +_\nu aD(\nu) -_\nu D(\nu)\nu +_\nu \nu^2 -_\nu \nu D(\nu) \\
&\quad +_\nu D(\nu)b -_\nu \nu b +_\nu \nu D(b).
\end{aligned}$$

Thus, it is clear that equation (3.4) holds. □

We now look to expand on Proposition 3.12.

Example 3.15. For an abelian group A, let us denote by $\mathcal{L}(A)$ the truss with the product given by the left projection, that is $ab = a$, then:

$$H^n(\mathcal{L}(A)) \cong \begin{cases} \mathrm{End}(A) & \text{for } n = 1, \\ * & \text{otherwise.} \end{cases} \tag{3.5}$$

Proof. We perform all computations relative to the neutral element 0 in A. In view of the product in $\mathcal{L}(A)$, the formula for the coboundary operator (relative to 0 and written in the abelian group form) comes out as:

$$\delta^n f(a_0, \dots, a_n) = \sum_{i=1}^{n-1} (-1)^i f(a_0, \dots, \widehat{a_i}, \dots, a_n), \tag{3.6}$$

where $\widehat{a_i}$ indicates the absence of a_i.

First note that, for all $a, b \in A$:

$$\delta^0(a)(b) = -a, \tag{3.7}$$

and hence: $H^0(\mathcal{L}(A)) = Z^0(\mathcal{L}(A)) = \{0\}$.

For $n \geq 2$ and $f \in C^n(\mathcal{L}(A))$, $\delta^n(f) = 0$ if and only if, for all $a_0, \dots, a_n \in A$:

$$\begin{aligned} f(a_0, a_2, \dots, a_n) &= f(a_0, a_1, a_3, \dots, a_n) - f(a_0, a_1, a_2, a_4, \dots, a_n) \\ &\quad + \dots + (-1)^{n+1} f(a_0, a_1, a_2, \dots, a_{n-2}, a_n). \end{aligned} \tag{3.8}$$

The left-hand side of equation (3.8) is independent of a_1. Thus, setting $a_1 = 0$ and relabelling the indices, we find that f is an n-cocycle if, and only if, for all $a_0, \dots, a_{n-1}$:

$$\begin{aligned} f(a_0, a_1, \dots, a_{n-1}) &= f(a_0, 0, a_2, \dots, a_{n-1}) - f(a_0, 0, a_1, a_3, \dots, a_{n-1}) \\ &\quad + \dots + (-1)^{n+1} f(a_0, 0, a_1, \dots, a_{n-3}, a_{n-1}). \end{aligned} \tag{3.9}$$

Set:

$$g : A^{n-1} \to A, \qquad (a_0, a_1, \dots, a_{n-2}) \mapsto -f(a_0, 0, a_1, \dots, a_{n-2}).$$

Since $f \in C^n(\mathcal{L}(A))$, the map g is a heap homomorphism in all arguments, and hence $g \in C^{n-1}(\mathcal{L}(A))$. The formula (3.9) immediately yields $\delta^{n-1}(g) = f$, and hence every n-cocycle is also an n-coboundary. Therefore, all Hochschild heaps are trivial whenever $n \geq 2$.

Finally, elements of $C^1(\mathcal{L}(A))$ are heap endomorphisms of f, i.e. any functions $f : A \to A$ such that $f(a - b + c) = f(a) - f(b) + f(c)$. Given such an f, the map $g(a) = f(a) - f(0)$ is additive. Conversely, given a group endomorphism g of A and $c \in A$, the map $f(a) = g(a) + c$, is a heap endomorphism. The formula (3.6) implies that every heap endomorphism $f : A \to A$ is a one-cocycle, while (3.7) yields that two one-cocycles belong to the same cohomology class if, and only if, they differ by a constant (i.e., they correspond to the same abelian group endomorphism). This establishes the isomorphism of $H^1(\mathcal{L}(A))$ with the heap of additive endomorphisms of A. □

Chapter 4

Nijenhuis Products & Operators on Trusses

In this chapter we build on notions of Nijenhuis products and operators introduced in [16] and discussed in Chapter 2, expanding them from the framework of rings into that of trusses. We also determine the sufficient and necessary conditions for the associativity of the Nijenhuis product, study the compatibility of Nijenhuis operators and give examples, as well as classify all Nijenhuis operators on commutative trusses built on the group of all integers (see Remark 2.13).

4.1 Nijenhuis Products, Operators & Torsions over Trusses

Our first definition defines the *Nijenhuis product*, *operator*, and the *torsion* relative to a given element ν. This definition is an expansion from the ring based definitions found in [16] to the truss case. Similar expansions, made instead to Lie affgebras, can be found later in this thesis in Definition 7.1 and are used throughout Chapter 7.

Definition 4.1. Let T be a truss. For all heap homomorphisms $N : T \to T$, the binary operation $\circ_N$ on T, defined by:

$$a \circ_N b = \langle N(a)b, N(ab), aN(b)\rangle,$$

is called the **Nijenhuis product**.

Furthermore, N is called a **Nijenhuis operator (over a truss)** if for all $a, b \in T$:

$$N(a \circ_N b) = N(a)N(b).$$

For all $\nu \in T$, the **ν-Nijenhuis torsion** of N is defined as:

$$T_N^{\nu}(a, b) = \langle N(a \circ_N b), N(a)N(b), \nu\rangle.$$

Note that in view of Remark 2.7, N is a Nijenhuis operator if, and only if, for any $\nu \in T$, its ν-torsion is a constant function equal to ν, that is, for all $\nu, a, b \in T$, $T_N^\nu(a,b) = \nu$. Such a ν-torsion is said to be **trivial**.

Remark 4.2. The authors of [16] use the term Nijenhuis tensor rather than Nijenhuis operator. However, as the former is commonly used to describe the obstruction of an almost complex structure to origin from a complex structure and is closer to the Nijenhuis torsion (see [31, footnote 1 p. 627]), we prefer the latter. In addition, the term Nijenhuis operator is now widely used to describe a way of deforming a given algebraic structure (typically a Lie bracket, but associative products too) into a structure of the same kind, which also extends to trusses, as argued in the present text.

4.2 Resulting Examples & Theorems

Example 4.3. Let T be a truss.

1). The identity map $\mathrm{id} : T \to T$ is a Nijenhuis operator.

2). Let $P : T \to T$ be a multiplicative idempotent homomorphism of heaps. Then P is a Nijenhuis operator on T. In particular, for any idempotent elements $q \in T$, the constant map $a \mapsto q$ is a Nijenhuis operator.

Proof. In the first case the Nijenhuis product is the same as the original multiplication in T and, hence, clearly the identity map is a Nijenhuis operator. In the second example, since, for all $a, b, c \in T$:

$$P(\langle a,b,c\rangle) = \langle P(a), P(b), P(c)\rangle, \quad P(ab) = P(a)P(b), \quad P(P(a)) = P(a),$$

we can easily compute:

$$\begin{aligned} P(a \circ_P b) &= P(\langle P(a)b, P(ab), aP(b)\rangle) = \langle P(P(a)b), P(a)P(b), P(aP(b)\rangle, \\ &= \langle P(a)P(b), P(a)P(b), P(a)P(b)\rangle = P(a)P(b), \end{aligned}$$

as required.

Any constant map is a heap homomorphism, and if the image is an idempotent element of T, then such a map is a multiplicative idempotent homomorphism of heaps. □

A direct connection between Nijenhuis operators on rings and trusses is given in the following proposition.

Proposition 4.4. *Let $T(q; I)$ be a truss associated to an ideal I and idempotent q in a ring R; see Remark 2.15. Let $\bar{N}$ be a Nijenhuis operator on I, such that, for all $x \in I$:*

$$\bar{N}(xq) = \bar{N}(x)q, \qquad \bar{N}(qx) = q\bar{N}(x). \tag{4.1}$$

Then:

$$N : T(q; I) \to T(q; I), \qquad q + x \mapsto q + \bar{N}(x),$$

is a Nijenhuis operator on $T(q; I)$.

Proof. It is immediate from the definition of N that it is a heap homomorphism. The Nijenhuis product comes out as, for all $x, y \in I$:

$$\begin{aligned}(q+x) \circ_N (q+y) &= q + \bar{N}(x)q + qy + \bar{N}(x)y - q - \bar{N}(xq) - \bar{N}(qy) - \bar{N}(xy) \\ &\quad + q + x\bar{N}(y) + xq + q\bar{N}(y), \\ \\ &= q + x \circ_{\bar{N}} y + xq + qy,\end{aligned}$$

by the fact that q is an idempotent and by (4.1). Since $\bar{N}$ is the Nijenhuis operator on the ring I, we conclude:

$$N\left((q+x) \circ_N (q+y)\right) = q + \bar{N}(x)\bar{N}(y) + \bar{N}(x)q + q\bar{N}(y) = N(q+x)N(q+y),$$

that is N is a Nijenhuis operator on the truss $T(I;q)$. □

Example 4.5. An explicit example of a Nijenhuis operator of the type described in Proposition 4.4 can be constructed as follows. Let R be the sub-ring of the ring $M_{(n+1)\times(n+1)}(\mathbb{F})$ of $(n+1)\times(n+1)$-matrices over a field $\mathbb{F}$ consisting of matrices of the following block form:

$$A(\mathbf{a}, \mathbf{b}, \alpha) = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ 0 & \alpha \end{pmatrix},$$

where $\mathbf{a} \in M_{n\times n}(\mathbb{F})$, $\mathbf{b} \in M_{n\times 1}(\mathbb{F})$, and $\alpha \in \mathbb{F}$. Set:

$$I = \{A(\mathbf{a}, \mathbf{b}, 0) \mid \mathbf{a} \in M_{n\times n}(\mathbb{F}), \mathbf{b} \in M_{n\times 1}(\mathbb{F})\}, \qquad q = A(0,0,1),$$

so that:

$$T(I;q) = \{A(\mathbf{a}, \mathbf{b}, 1) \mid \mathbf{a} \in M_{n\times n}(\mathbb{F}), \mathbf{b} \in M_{n\times 1}(\mathbb{F})\}.$$

Let P_+ denote the projection in $M_{n\times n}(\mathbb{F})$ onto the sub-ring of upper triangular matrices. As argued in [16, Example 1], the map:

$$\bar{N} : I \to I, \qquad A(\mathbf{a}, \mathbf{b}, 0) \mapsto A(P_+(\mathbf{a}), \mathbf{b}, 0),$$

is a Nijenhuis operator on I. Since $\bar{N}$ affects only the upper $n \times n$ block of $A(\mathbf{a}, \mathbf{b}, 0)$, and in view of the general multiplication rules in R:

$$A(\mathbf{a}, \mathbf{b}, \alpha)A(\mathbf{a}', \mathbf{b}', \alpha') = A(\mathbf{a}\mathbf{a}', \mathbf{a}\mathbf{b}' + \alpha'\mathbf{b}, \alpha\alpha'),$$

the condition (4.1) is satisfied. Therefore:

$$N : T(I;q) \to T(I;q), \qquad A(\mathbf{a}, \mathbf{b}, 1) \mapsto A(P_+(\mathbf{a}), \mathbf{b}, 1),$$

is a Nijenhuis operator on $T(I;q)$.

We now extend Theorem 2.51, as originally stated in Quantum Bi-Hamiltonian Systems [16, Theorem 1], to the truss case.

Theorem 4.6. *Let T be a truss and $N : T \to T$ a heap homomorphism. The Nijenhuis product $\circ_N$ is associative if, and only if, the ν-Nijenhuis torsion of N is a ν-relative Hochschild 2-cocycle for all (equivalently for any) $\nu \in T$. If this is the case then T is a truss with the Nijenhuis product $\circ_N$.*

Proof. First note that whether associative or not, the Nijenhuis product distributes over the heap operation. Indeed, for all $\nu, a, b, c, d \in T$:

$$\begin{aligned} a \circ_N \langle b, c, d\rangle &= \langle N(a)\langle b, c, d\rangle, N(a\langle b, c, d\rangle), aN(\langle b, c, d\rangle)\rangle, \\ &= \langle N(a)b, N(a)c, N(a)d, N(ab), N(ac), N(ad), aN(b), aN(c), aN(d)\rangle, \\ &= \langle N(a)b, N(ab), aN(b), N(a)c, N(ac), aN(c), N(a)d, N(ad), aN(d)\rangle, \\ &= \langle a \circ_N b, a \circ_N c, a \circ_N d\rangle, \end{aligned}$$

since T is a truss and N is a heap homomorphism. The rearrangement of terms leading to the third equality follows by the fact that T is an abelian heap and hence the terms in odd (resp. even) positions in the angle brackets can be reshuffled freely; see Remark 2.2. This proves the left distributive law. The right distributive law is proven by similar calculations.

Using the heap homomorphism property of N, the associativity of the product in T as well as its distributivity over the heap operation, we find, for all $a, b, c \in T$:

$$\begin{aligned} (a \circ_N b) \circ_N c = \langle & N(N(a)b)c, N(N(ab))c, N(aN(b))c, N(N(a)bc), \\ & N(N(ab)c), N(aN(b)c), N(a)bN(c), N(ab)N(c), aN(b)N(c)\rangle, \end{aligned}$$

and:

$$\begin{aligned} a \circ_N (b \circ_N c) = \langle & N(a)N(b)c, N(a)N(bc), N(a)bN(c), N(aN(b)c), \\ & N(aN(bc)), N(abN(c)), aN(N(b)c), aN(N(bc)), aN(bN(c))\rangle. \end{aligned}$$

Therefore, cancelling repeated terms we obtain:

$$\begin{aligned} &\langle a \circ_N (b \circ_N c), (a \circ_N b) \circ_N c, \nu\rangle = \\ &\qquad \langle N(a)N(b)c, N(a)N(bc), N(aN(bc)), N(abN(c)), \\ &\qquad aN(N(b)c), aN(N(bc)), aN(bN(c)), N(N(a)b)c, N(N(ab))c, \\ &\qquad N(aN(b))c, N(N(a)bc), N(N(ab)c), N(ab)N(c), aN(b)N(c), \nu\rangle. \end{aligned}$$

Next, using the truss distributive laws, we compute:

$$aT_N^\nu(b, c) = \langle aN(N(b)c), aN(N(bc)), aN(bN(c)), aN(b)N(c), a\nu\rangle,$$

$$T_N^\nu(ab, c) = \langle N(N(ab)c), N(N(abc)), N(abN(c)), N(ab)N(c), \nu\rangle,$$

$$T_N^\nu(a, bc) = \langle N(N(a)bc), N(N(abc)), N(aN(bc)), N(a)N(bc), \nu\rangle,$$

and:

$$T_N^\nu(a,b)c = \langle N(N(a)b)c, N(N(ab))c, N(aN(b))c, N(a)N(b)c, \nu c\rangle.$$

Thus we have:

$$\begin{aligned}\delta_\nu^2 T_N^\nu(a,b,c) = \langle & aN(N(b)c), aN(N(bc)), aN(bN(c)), aN(b)N(c), a\nu,\\ & N(N(ab)c), N(N(abc)), N(abN(c)), N(ab)N(c), \nu,\\ & N(N(a)bc), N(N(abc)), N(aN(bc)), N(a)N(bc), \nu,\\ & N(N(a)b)c, N(N(ab))c, N(aN(b))c, N(a)N(b)c, \nu c, \nu c, a\nu, \nu\rangle.\end{aligned}$$

We can then perform cancellations and rearrangements allowed by the definition of an abelian heap (see Remark 2.2), yielding:

$$\langle a \circ_N (b \circ_N c), (a \circ_N b) \circ_N c, \nu\rangle = \delta_\nu^2 T_N^\nu(a,b,c).$$

Therefore, by Remark 2.7:

$$a \circ_N (b \circ_N c) = (a \circ_N b) \circ_N c,$$

if, and only if, $\delta_\nu^2 T_N^\nu(a,b,c) = \nu$, as required. □

Corollary 4.7. *If N is a Nijenhuis operator on a truss $(T, \langle -,-,-\rangle, \cdot)$, it follows that $(T, \langle -,-,-\rangle, \circ_N)$ is also a truss. We denote this truss by $T[N]$.*

Proof. By the definition of a Nijenhuis operator, its ν-torsion is trivial for all ν and hence, it is a 2-cocycle, as needed. □

As an illustration of Theorem 4.6, we classify all Nijenhuis operators on $T(\mathbb{Z}; a, b, c)$ on $\mathbb{Z}$, described in Remark 2.13.

Proposition 4.8. *The following table lists all Nijenhuis operators on commutative trusses $T(\mathbb{Z}; a, b, c)$ on $\mathbb{Z}$, described in Remark 2.13.*

$T(\mathbb{Z}; a,b,c)$	Nijenhuis operators	
$c \neq 0$	$N(m) = \left(\frac{b}{\gcd(b,c)}\, q + 1\right) m + \frac{c}{\gcd(b,c)}\, q,$ $N(m) = \left(\frac{b-1}{\gcd(b-1,c)}\, q + 1\right) m + \frac{c}{\gcd(b-1,c)}\, q,$	$q \in \mathbb{Z}$
$c = 0$	$N(m) = qm,$ $N(m) = \left(\frac{a}{2b-1}\, q + 1\right) m + q,$	$q \in \mathbb{Z}$

Proof. A heap homomorphism $N : \mathbb{Z} \to \mathbb{Z}$ is necessarily of the form, for all $m \in \mathbb{Z}$:

$$N(m) = pm + q, \tag{4.2}$$

for some $p, q \in \mathbb{Z}$. We need to determine what conditions p and q have to satisfy in order for N to be a Nijenhuis operator. We take the most general commutative truss on $\mathbb{Z}$, $T(\mathbb{Z}; a, b, c)$, with multiplication (2.2), where a, b, c satisfy the constraint (2.3), and compute, for all $m, n \in \mathbb{Z}$:

$$N(m) \cdot n = apmn + (aq + b)n + bpm + bq + c,$$
$$N(m \cdot n) = apmn + bpm + bpn + cp + q.$$

These yield the Nijenhuis product:

$$m \circ_N n = apmn + (aq + b)(m + n) + 2bq + 2c - q - cp. \tag{4.3}$$

In view of Theorem 4.6, the product (4.3) is associative if, and only if, the ν-Nijenhuis torsion is a cocycle for any fixed $\nu \in \mathbb{Z}$, in particular for 0. The 0-Nijenhuis torsion comes out as:

$$T_N^0(m, n) = -cp^2 + ((2b - 1)q + 2c)p - aq^2 - (2b - 1)q - c. \tag{4.4}$$

On the other hand, as recalled in Remark 2.13, the product (4.3) is associative if, and only if:

$$2abpq + 2acp - apq - acp^2 = (aq + b)(aq + b - 1). \tag{4.5}$$

Since $ac = b(b - 1)$, and in view of (4.4), this can be rewritten as:

$$aT_N^0(m, n) = 0. \tag{4.6}$$

Thus, if $a \neq 0$, the associativity of the Nijenhuis product is equivalent to the vanishing of the Nijenhuis torsion, i.e. to N being a Nijenhuis operator.

We consider the equation $T_N^0(m, n) = 0$ as an equation with the unknown p. If $c \neq 0$ this is a quadratic equation with the discriminant q^2, and hence the solutions are:

$$p = \left(\frac{b}{c}\, q + 1\right) \quad \text{or} \quad p = \left(\frac{b-1}{c}\, q + 1\right).$$

To ensure that the solutions are integer, the numbers q must be multiples of the denominators of the fractions divided by the greatest common multiple of the numerator and the denominator. By rescaling accordingly, we obtain the following Nijenhuis operators:

$$N(m) = \left(\frac{b}{\gcd(b, c)}\, q + 1\right) m + \frac{c}{\gcd(b, c)}\, q, \qquad q \in \mathbb{Z},$$

and:

$$N(m) = \left(\frac{b-1}{\gcd(b-1,c)}\, q + 1 \right) m + \frac{c}{\gcd(b-1,c)}\, q, \qquad q \in \mathbb{Z}.$$

If $c = 0$, then $T_N^0(m,n) = 0$ is equivalent to:

$$(2b-1)pq - aq^2 - (2b-1)q = 0.$$

Thus, $q = 0$ or $p = \frac{a}{2b-1}\, q + 1$, which is an integer for all q, since the constraint (4.5) implies that $b = 0$ or $b = 1$ in this case.

Putting all these cases together we obtain the table as stated. □

The following theorem is the truss version of [16, Theorem 2]. This theorem was included in this thesis in the preliminaries section as Theorem 2.50 for completeness.

Theorem 4.9. *If N is a Nijenhuis operator on a truss T, then, for all $j,k \in \mathbb{N}$:*

1). N^k is a Nijenhuis operator on T and hence $T[N^k]$ is a truss,

2). $T[N^k][N^l] = T[N^{k+l}]$,

3). N^l is a Nijenhuis operator on $T[N^k]$,

where N^k means the k-fold composition of N, and $N^0 = \mathrm{id}$.

Proof. To simplify the notation we will write $\underset{k}{\circ}$ for the product $\circ_{N^k}$ in $T[N^k]$. Obviously, $\underset{0}{\circ} = \cdot$, the original multiplication in T.

First we prove that for all $a,b \in T$, $k \in \mathbb{N}$:

$$N^k(a)N(b) = \langle N(N^k(a)b), N^{k+1}(ab), N^k(aN(b))\rangle, \tag{4.7}$$

$$N(a)N^k(b) = \langle N(aN^k(b)), N^{k+1}(ab), N^k(N(a)b)\rangle, \tag{4.8}$$

$$\begin{aligned} N^{k+1}(ab) &= \langle N(N^k(a)b), N^k(a)N(b), N^k(aN(b))\rangle, \\ &= \langle N^k(N(a)b), N(a)N^k(b), N(aN^k(b))\rangle. \end{aligned} \tag{4.9}$$

We will prove equality (4.7) by induction. The equality (4.8) can be proven symmetrically, while (4.9) is an equivalent restatement of (4.7) and (4.8); see Remark 2.7.

For $k = 0$, (4.7) is automatically satisfied, while for $k = 1$ this is the definition of a Nijenhuis operator. Assume that (4.7) is true for k, then first, using the Nijenhuis condition and then the inductive assumption we obtain:

$$\begin{aligned} N^{k+1}(a)N(b) &= N\left(\left\langle N^{k+1}(a)b, N(N^k(a)b), N^k(a)N(b)\right\rangle\right), \\ &= N\left(\left\langle N^{k+1}(a)b, N(N^k(a)b), N(N^k(a)b), N^{k+1}(ab), N^k(aN(b))\right\rangle\right), \\ &= \left\langle N^{k+2}(a)b, N^{k+2}(ab), N^{k+1}(aN(b))\right\rangle. \end{aligned}$$

The final equality follows by the Mal'cev identity (the cancellation rule in the heap operation) and the fact that N is a heap homomorphism. Therefore, (4.7) is true for all natural k by the principle of mathematical induction.

Using equations (4.7-4.9) we can compute:

$$
\begin{aligned}
N(a)\underset{k}{\circ}N(b) &= \langle N^{k+1}(a)N(b), N^k(N(a)N(b)), N(a)N^{k+1}(b)\rangle, \\
&= \langle N(N^{k+1}(a)b), N^{k+2}(ab), N^{k+1}(aN(b)), N^{k+1}(aN(b)), N^{k+2}(ab) \\
&\quad N^{k+1}(N(a)b), N(aN^{k+1}(b)), N^{k+2}(ab), N^{k+1}(N(a)b)\rangle, \\
&= \langle N(N^{k+1}(a)b), N^{k+2}(ab), N(aN^{k+1}(b))\rangle = N(a\underset{k+1}{\circ}b).
\end{aligned}
$$

The penultimate equality follows by the cancellation rules for an abelian heap operation. The last equality is a consequence of the fact that N is a heap homomorphism and the definition of the Nijenhuis product. Starting with this we can employ the inductive argument to prove that, for all $a, b \in T$, $k, l \in \mathbb{N}$:

$$
N^l(a\underset{k+l}{\circ}b) = N^l(a)\underset{k}{\circ}N^l(b). \tag{4.10}
$$

In particular, the case $k = 0$ yields assertion (1).

The second assertion is also proved inductively on l. The inductive step is straightforward, so only the case $l = 1$ needs to be proven. For all $a, b \in T$:

$$
\begin{aligned}
\langle N(a)\underset{k}{\circ}b, N(a\underset{k}{\circ}b), a\underset{k}{\circ}N(b)\rangle &= \langle N^{k+1}(a)b, N^k(N(a)b), N(a)N^k(b), \\
&\quad N(N^k(a)b), N^{k+1}(ab), N(aN^k(b)), \\
&\quad N^k(a)N(b), N^k(aN(b)), aN^{k+1}(b)\rangle, \\
&= \langle N^{k+1}(a)b, N^{k+1}(ab), N^{k+1}(ab), N^{k+1}(ab), aN^{k+1}(b)\rangle, \\
&= \langle N^{k+1}(a)b, N^{k+1}(ab), aN^{k+1}(b)\rangle = a\underset{k+1}{\circ}b.
\end{aligned}
$$

The second equality follows by (4.9). This completes the proof of statement (2). The last assertion follows immediately from (2) and (4.10). □

Following [16] we propose:

Definition 4.10. Nijenhuis operators N_1, N_2 on a truss T are said to be **compatible** if, for all $a, b \in T$:

$$
N_1(a)N_2(b) = \langle N_1(a \circ_{N_2} b), N_2(a)N_1(b), N_2(a \circ_{N_1} b)\rangle. \tag{4.11}
$$

Example 4.11. The identity operator id on T is compatible with any Nijenhuis operator on T.

The next statement is the truss version of [16, Theorem 3 & Theorem 4].

Theorem 4.12. *Let T be a truss.*

1). If $N_1, N_2, \ldots, N_{2n+1}$ are pairwise compatible Nijenhuis operators on T, then:

$$\langle N_1, N_2, \ldots, N_{2n+1}\rangle,$$

is a Nijenhuis operator on T.

2). For all Nijenhuis operators N on T:

a). the operators N^k and N^l are compatible, for all $k, l \in \mathbb{N}$,

b). $\langle N^{k_1}, N^{k_2}, \ldots, N^{k_{2n+1}}\rangle$ is a Nijenhuis operator for all $k_i, n \in \mathbb{N}$.

Proof. (1) We first note that, since the multiplication in the truss distributes over the heap operation and abelian heaps satisfy the rearrangement rules described in Remark 2.2, for any heap homomorphisms $N_1, N_2, \ldots, N_{2n+1} : T \to T$, and all $a, b \in T$:

$$a \circ_{\langle N_1, N_2, \ldots, N_{2n+1}\rangle} b = \langle a \circ_{N_1} b, a \circ_{N_2} b, \ldots, a \circ_{N_{2n+1}} b\rangle. \tag{4.12}$$

With (4.12) at hand we can prove the statement by induction on n. For $n = 1$:

$$\begin{aligned}
&\langle N_1, N_2, N_3\rangle(a)\ \langle N_1, N_2, N_3\rangle(b) = \\
&\quad = \langle N_1(a)N_1(b), N_1(a)N_2(b), N_1(a)N_3(b), N_2(a)N_1(b), N_2(a)N_2(b), \\
&\qquad N_2(a)N_3(b), N_3(a)N_1(b), N_3(a)N_2(b), N_3(a)N_3(b)\rangle, \\
&\quad = \langle N_1(a \circ_{N_1} b), N_1(a \circ_{N_2} b), N_2(a)N_1(b), N_2(a \circ_{N_1} b), N_1(a \circ_{N_3} b), \\
&\qquad N_3(a)N_1(b), N_3(a \circ_{N_1} b), N_2(a)N_1(b), N_2(a \circ_{N_2} b), N_2(a \circ_{N_3} b), \\
&\qquad N_3(a)N_2(b), N_3(a \circ_{N_2} b), N_3(a)N_1(b), N_3(a)N_2(b), N_3(a \circ_{N_3} b)\rangle, \\
&\quad = \langle N_1(a \circ_{N_1} b), N_1(a \circ_{N_2} b), N_1(a \circ_{N_3} b), N_2(a \circ_{N_1} b), N_2(a \circ_{N_2} b), \\
&\qquad N_2(a \circ_{N_3} b), N_3(a \circ_{N_1} b), N_3(a \circ_{N_2} b), N_3(a \circ_{N_3} b)\rangle, \\
&\quad = \langle N_1(a \circ_{\langle N_1, N_2, N_3\rangle} b), N_2(a \circ_{\langle N_1, N_2, N_3\rangle} b), N_3(a \circ_{\langle N_1, N_2, N_3\rangle} b)\rangle, \\
&\quad = \langle N_1, N_2, N_3\rangle(a \circ_{\langle N_1, N_2, N_3\rangle} b),
\end{aligned}$$

where the definition of the heap bracket on operators and the truss distributive laws were used to derive the first equality, next the pairwise compatibility was employed. The third equality arises from the rearrangement and cancellation rules outlined in Remark 2.2, while the next equality is a consequence of (4.12). Therefore, $\langle N_1, N_2, N_3\rangle$ is a Nijenhuis operator as required.

Next assume that the statement is true for $n = k-1$ and note that (4.12), together with the rearrangement rules in Remark 2.2, imply that if $N_1, N_2, \ldots, N_{2k+1}$ are pairwise compatible Nijenhuis operators then N_{2k} and N_{2k+1} are compatible with $N = \langle N_1, N_2, \ldots, N_{2k-1}\rangle$. Hence $\langle N, N_{2k}, N_{2k+1}\rangle$ is a Nijenhuis operator by the same arguments as those used above to establish the $n = 1$ case.

(2) Without any loss of generality, we may assume that $k \geq l$. As in the proof of Theorem 4.9 we will write $\underset{k}{\circ}$ for $\circ_{N^k}$, etc. In view of Theorem 4.9 and the definition of the Nijenhuis product $\underset{k-l}{\circ}$ we can compute, for all $a, b \in T$:

$$\begin{aligned}
\langle N^k(a\underset{l}{\circ}b), N^l(a)N^k(b), N^l(a\underset{k}{\circ}b)\rangle &= \langle N^{k-l}(N^l(a)N^l(b)), N^l(a)N^k(b), N^l(a)\underset{k-l}{\circ}N^l(b)\rangle, \\
&= \langle N^{k-l}(N^l(a)N^l(b)), N^l(a)N^k(b), N^k(a)N^l(b), N^l(a)N^k(b), N^l(a)N^k(b)\rangle, \\
&= N^k(a)N^l(b).
\end{aligned}$$

The last equality follows by the cancellation and rearrangement rules recalled in Remark 2.2. This completes the proof of statement (a). Statement (b) then follows by assertion (1). □

Chapter 5

Affine Nijenhuis Operators

In this section we apply the above discussion to trusses and operators, arising from associative algebras, and in this way extend the results of [16] from the case of linear to affine maps. Next we construct an affine version of (weak) quantum bi-Hamiltonian systems. That is, we construct an affine Lie bracket (in the sense of [28, Definition 1]) which can be represented as the commutator of a deformed associative bi-affine product on an affine space.

A key result in this chapter is Corollary 5.3, as it provides the foundation for Chapter 6.

5.1 Defining Affine Nijenhuis Operators

Definition 5.1. An associative algebra A over a field $\mathbb{F}$ can be viewed as a truss with the original multiplication of the heap structure arising from the additive group, that is, $\langle a, b, c\rangle = a - b + c$. To indicate this ternary point of view we write $\mathrm{T}(A)$. From this perspective an affine map $N : A \to A$ is a homomorphism of heaps that preserves affine or barycentric combinations. That is, for all $a, b, c \in A$ and $\lambda \in \mathbb{F}$:

$$N(a - b + c) = N(a) - N(b) + N(c), \tag{5.1}$$

$$N((1 - \lambda)a + \lambda b) = (1 - \lambda)N(a) + \lambda N(b). \tag{5.2}$$

The set of all affine maps $A \to A$ is denoted by $\mathrm{Aff}(A)$. This is a truss with the product given by composition and the heap operation defined pointwise. One easily checks that $\mathrm{Aff}(A)$ is an affine space over the vector space of all linear endomorphisms of A with the operations defined pointwise. If $N \in \mathrm{Aff}(A)$ is a Nijenhuis operator on $\mathrm{T}(A)$ we refer to it as an **affine Nijenhuis operator** on A.

Although, given $N \in \mathrm{Aff}(A)$ and $\lambda \in \mathbb{F}$, the function $\lambda N : A \to A$, $a \mapsto \lambda N(a)$, is not an affine map, it is still a homomorphism of heaps, i.e. the first of conditions (5.1-5.2) is satisfied. Hence the following theorem can be stated.

5.2 Resulting Examples & Theorems

Theorem 5.2. *Let A be an associative $\mathbb{F}$-algebra.*

1). If N is an affine Nijenhuis operator on A, then for all $\lambda \in \mathbb{F}$, λN is a Nijenhuis operator on $\mathrm{T}(A)$ compatible with N.

2). If $N_1, \ldots, N_n \in \mathrm{Aff}(A)$ are pairwise compatible affine Nijenhuis operators on $\mathrm{T}(A)$, then for all $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ such that $\sum_{i=1}^n \lambda_i = 1$, $N = \sum_{i=1}^n \lambda_i N_i$, is an affine Nijenhuis operator on A.

Proof. (1) First note that for all $a, b \in A$:

$$a \circ_{\lambda N} b = \lambda N(a)b - \lambda N(ab) + a\lambda N(b) = \lambda a \circ_N b. \tag{5.3}$$

Hence, if N is a Nijenhuis operator on $\mathrm{T}(A)$, then:

$$\lambda N(a \circ_{\lambda N} b) = \lambda^2 N(a)N(b) = \lambda N(a)\lambda N(b),$$

as required. The compatibility property likewise follows by (5.3).

(2) The correspondence between Nijenhuis products in (5.3) implies that the Nijenhuis operators $\lambda_i N_i$, $i = 1, \ldots n$ are pairwise compatible. Since $\sum_{i=1}^n \lambda_i = 1$,

$$\begin{aligned} N &= \sum_{i=1}^{n} \lambda_i N_i = N_1 - \lambda_2 N_1 + \lambda_2 N_2 - \lambda_3 N_1 + \ldots - \lambda_n N_1 + \lambda_n N_n, \\ &= \langle N_1, \lambda_2 N_1, \lambda_2 N_2, \lambda_3 N_1, \ldots, \lambda_n N_1, \lambda_n N_n \rangle, \end{aligned}$$

and hence the affine map $\sum_{i=1}^n \lambda_i N_i$ is a Nijenhuis operator on $\mathrm{T}(A)$ and hence an affine Nijenhuis operator on A by assertion (1) in Theorem 4.12. ☐

Corollary 5.3. *If P is a multiplicative idempotent in $\mathrm{Aff}(A)$, then for all $\alpha \in \mathbb{F}$, $(1-\alpha)P + \alpha\,\mathrm{id}$ is an affine Nijenhuis operator.*

Proof. This follows immediately from Theorem 5.2 and Example 4.11. ☐

Following on from Remark 2.19, with this interpretation, an affine map from B over $\vec{B}$ to A corresponds to a heap homomorphism $f : A \to B$ such that, for all $a, b \in A$ and $\lambda \in \mathbb{F}$:

$$f(a + \lambda\overrightarrow{ab}) = f(a) + \lambda\overrightarrow{f(a)f(b)}. \tag{5.4}$$

Any such map defines uniquely linear transformation $\vec{f} : \vec{A} \to \vec{B}$ by $\overrightarrow{ab} \mapsto \overrightarrow{f(a)f(b)}$. We refer to it as a **linearisation** of f.

If A is a vector space, then it is an affine space over itself, with the vector from a to b being simply the difference $b - a$. The heap operation (2.7) coincides then with $a - b + c$, while to be affine map from A to A in the sense of (5.4) is equivalent to satisfying conditions (5.1-5.2).

Recall from [28, Definition 1] that a Lie bracket on an affine space A is an anti-symmetric bi-affine map $[-,-] : A \times A \to \vec{A}$ satisfying the Jacobi identity:

$$\overrightarrow{[[a,b],c]} + \overrightarrow{[[b,c],a]} + \overrightarrow{[[c,a],b]} = 0.$$

The arrows over the brackets indicate the linearisations of affine maps $[-,b] : A \to \vec{A}$.

Let A be an affine space with a bi-affine associative multiplication $\cdot : A \times A \to A$ (we will keep writing the dot between the elements of A, in order to avoid the confusion with the end points of the vector in $\vec{A}$). The fact that, for all $a \in A$, the function $A \to A$, $b \mapsto a \cdot b$ is an affine map implies, in particular, that it is a heap homomorphism, which is equivalent to say that the multiplication left-distributes over the heap operation (2.7). Similarly, the heap homomorphism property of maps $b \mapsto b \cdot a$ yield the right truss distributive law. In short, A is a truss, which might be called an **affine truss** or an associative **affgebra** – the term coined in [28].

Remark 5.4. In the same way as a truss can be embedded in a ring (see Remark 2.15), any associative affgebra can be obtained as a coset in an associative algebra. Explicitly, given an algebra A, an ideal I of A and an idempotent element $q \in A$, $T(I;q) = q + I$ is an affine space over I with $\overrightarrow{(q+x)(q+y)} = y - x$, to which the multiplication on A restricts as a bi-affine map.

With no additional effort, the notion of an affine Nijenhuis operation and the statement (2) of Theorem 5.2 can be extended to affgebras.

Proposition 5.5. *If $N_1, \ldots, N_n \in \mathrm{Aff}(A)$ are pairwise compatible affine Nijenhuis operators on an associative $\mathbb{F}$-affgebra A, then, for all $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ such that $\sum_{i=1}^{n} \lambda_i = 1$, $N = \sum_{i=1}^{n} \lambda_i N_i$, is an affine Nijenhuis operator on A.*

Example 5.6. Let A be an associative algebra, I be an ideal in A and $q \in A$ and idempotent element. Assume that I decomposes into a sum of two ideals in A, $I = I_1 \oplus I_2$. Let $P_i : I \to I_i$, $i = 1, 2$ be corresponding projections, such that $P_i(xq) = P_i(x)q$ and $P_i(qx) = qP_i(x)$, for all $x \in I$, $i = 1, 2$. By [16, Theorem 5], for all $\lambda_1, \lambda_2 \in \mathbb{F}$, $\lambda_1 P_1 + \lambda_2 P_2$ is a Nijenhuis operator on I, in particular each of the P_i is a Nijenhuis operator. In view of Proposition 4.4:

$$N : T(q;I) \to T(q;I), \qquad q + x \mapsto q + \lambda_1 P_1(x) + \lambda_2 P_2(x),$$

is an Nijenhuis operator on $T(q;I)$. On the other hand, N can be understood as an affine combination of operators $N_i(q+x) = q + P_i(x)$ and $Q(q+x) = q$, as:

$$N = (1 - \lambda_1 - \lambda_2)\, Q + \lambda_1 N_1 + \lambda_2 N_2.$$

For an explicit example we can take the algebra R, its ideal I and an idempotent q described in Example 4.5. Every element of I can be uniquely decomposed into the sum of an upper triangular and strictly lower triangular matrix. If P_- denotes the projection on the latter, for all scalars λ_1, λ_2 we obtain the following affine Nijenhuis operator on $T(I;q)$:

$$N(A(\mathbf{a},\mathbf{b},1)) = A(\lambda_1 P_+(\mathbf{a}) + \lambda_2 P_-(\mathbf{a}), \lambda_1 \mathbf{b}, 1).$$

Any associative affgebra A admits a Lie bracket given by the linearised commutator, for all $a, b \in A$:

$$[a,b] = \overrightarrow{(b\cdot a)(a\cdot b)}. \tag{5.5}$$

Indeed, $[a,b]$ is clearly anti-symmetric, and, for all $a,b,c \in A$:

$$\begin{aligned}\overrightarrow{[[a,b],c]} + \text{cycl.} &= \overrightarrow{\left[\overrightarrow{(b\cdot a)(a\cdot b)}, c\right]} + \text{cycl.} = [a\cdot b, c] - [b\cdot a, c] + \text{cycl.},\\ &= \overrightarrow{(c\cdot a\cdot b)(a\cdot b\cdot c)} - \overrightarrow{(c\cdot b\cdot a)(b\cdot a\cdot c)} + \text{cycl.} = 0.\end{aligned}$$

In the case of the affgebra $T(I;q)$ of Remark 5.4, the Lie bracket comes out as the translation of the standard commutator, i.e. $[q+x, q+y] = q + [x,y]$.

With all these preliminaries at hand we can state the following affine version of [16, Theorem 8].

Theorem 5.7. *Let N be an affine Nijenhuis operator on an associative affgebra A. Let $[-,-]$ be the Lie bracket* (5.5)*.*

1). The multiplication $\circ_N$ is a bi-affine operation, thus making $A[N]$ into an associative affgebra.

2). The operation $[-,-]_N : A\times A \to \vec{A}$, given by:

$$[a,b]_N := [N(a), b] - \vec{N}([a,b]) + [a, N(b)],$$

for all $a,b\in A$, is a Lie bracket on A such that:

$$[a,b]_N = \overrightarrow{(b\circ_N a)(a\circ_N b)}.$$

3). For all $a,b \in A$:

$$\vec{N}([a,b]_N) = [N(a), N(b)].$$

Proof. First, note that in view of the definition of the heap operation (2.7), the Nijenhuis product comes out as:

$$a\circ_N c = N(a)\cdot c + \overrightarrow{N(a\cdot c)(a\cdot N(c))}.$$

We will use repeatedly the following elementary facts from the theory of affine spaces. For all points $a, b, c, d \in A$ and all vectors $v, w \in \vec{A}$:

$$\overrightarrow{(a+v)(b+w)} = \overrightarrow{ab} - v + w, \qquad \overrightarrow{cd} - \overrightarrow{ab} = \overrightarrow{bd} - \overrightarrow{ac}. \tag{5.6}$$

To check if $\circ_N$ is a bi-affine multiplication, take any $a, b, c \in A$ and $\lambda \in \mathbb{F}$, and using the facts that N is an affine map, the multiplication $\cdot$ is bi-affine, and (5.6) compute:

$$(a + \lambda\overrightarrow{ab}) \circ_N c =$$

$$= \left(N(a) + \lambda\overrightarrow{N(a)N(b)}\right) \cdot c + \overrightarrow{N\left(a \cdot c + \lambda\overrightarrow{(a\cdot c)(b\cdot c)}\right)\left((a+\lambda\overrightarrow{ab}) \cdot N(c)\right)},$$

$$= N(a) \cdot c + \lambda\overrightarrow{(N(a)\cdot c)(N(b)\cdot c)}$$

$$+ \overrightarrow{\left(N(a\cdot c) + \lambda\overrightarrow{N(a\cdot c)N(b\cdot c)}\right)\left(a \cdot N(c) + \lambda\overrightarrow{(a\cdot N(c))(b\cdot N(c))}\right)},$$

$$= N(a) \cdot c + \lambda\overrightarrow{(N(a)\cdot c)(N(b)\cdot c)} + \overrightarrow{N(a\cdot c)(a\cdot N(c))}$$

$$- \lambda\overrightarrow{N(a\cdot c)N(b\cdot c)} + \lambda\overrightarrow{(a\cdot N(c))(b\cdot N(c))},$$

$$= a \circ_N c + \lambda\left(\overrightarrow{(N(a)\cdot c)(N(b)\cdot c)} - \overrightarrow{N(a\cdot c)(a\cdot N(c))} + \overrightarrow{N(b\cdot c)(b\cdot N(c))}\right),$$

$$= a \circ_N c + \lambda\overrightarrow{\left(N(a)\cdot c + \overrightarrow{N(a\cdot c)(a\cdot N(c))}\right)\left(N(b)\cdot c + \overrightarrow{N(b\cdot c)(b\cdot N(c))}\right)},$$

$$= a \circ_N c + \lambda\overrightarrow{(a \circ_N c)(b \circ_N c)},$$

as required. The second compatibility condition is proven in a symmetric way. Therefore, $A[N]$ is an associative affgebra.

Using properties (5.6), we find for all $a, b \in A$:

$$
\begin{aligned}
\overrightarrow{(b \circ_N a)(a \circ_N b)} &= \overrightarrow{\left(N(b) \cdot a + \overrightarrow{N(b \cdot a)(b \cdot N(a))}\right)\left(N(a) \cdot b + \overrightarrow{N(a \cdot b)(a \cdot N(b))}\right)}, \\
&= \overrightarrow{(N(b) \cdot a)(N(a) \cdot b)} - \overrightarrow{N(b \cdot a)(b \cdot N(a))} + \overrightarrow{N(a \cdot b)(a \cdot N(b))}, \\
&= \overrightarrow{(N(b) \cdot a)(a \cdot N(b))} - \overrightarrow{N(b \cdot a)N(a \cdot b)} + \overrightarrow{(b \cdot N(a))(N(a) \cdot b)}, \\
&= [N(a), b] - \vec{N}([a, b]) + [a, N(b)], \\
&= [a, b]_N.
\end{aligned}
$$

In view of the fact that the Nijenhuis product $\circ_N$ makes A an associative affgebra, this proves both assertions in statement (2).

Finally, since N is an affine map and a Nijenhuis operator on A:

$$
\begin{aligned}
\vec{N}([a, b]_N) = \vec{N}\left(\overrightarrow{(b \circ_N a)(a \circ_N b)}\right) &= \overrightarrow{N(b \circ_N a)N(a \circ_N b)}, \\
&= \overrightarrow{(N(b) \cdot N(a))\,(N(a) \cdot N(b))} = [N(a), N(b)].
\end{aligned}
$$

This completes the proof of the theorem. □

Chapter 6

Lie Affgebras

Following on from our findings in Chapter 5, specifically Corollary 5.3, here we will propose an intrinsic definition of the left and right Lie bracket in affine space, rather than vector space. We shall refer, not to a *Lie algebra*, but to a *Lie affgebra*. Following this definition are examples of these structures and comparisons to the structure introduced in [28], and showing the reduction of Lie affgebras to Lie algebras.

6.1 Lie Affgebra Definitions & Results

Definition 6.1. Let A be an affine $\mathbb{K}$-module. A **left Lie bracket** on A is a bi-affine map $[-,-] : A \times A \to A$, such that for all $a, b, c \in A$:

$$\langle [a,b], [a,a], [b,a] \rangle = [b,b], \tag{6.1}$$

$$\langle [a,[b,c]], [a,a], [b,[c,a]], [b,b], [c,[a,b]] \rangle = [c,c]. \tag{6.2}$$

Similarly, we call the bi-affine map $[-,-]$ a **right Lie bracket** if it satisfies condition (6.2) and

$$\langle [[a,b],c], [a,a], [[b,c],a], [b,b], [[c,a],b] \rangle = [c,c].$$

An affine $\mathbb{K}$-module with the left (resp. right) Lie bracket is called a **left** (resp. **right**) **Lie affgebra**.

In the case that $\mathbb{K} = \mathbb{Z}$, the affine structure of A is uniquely determined by the heap structure and we refer to a **Lie truss**, rather than a Lie $\mathbb{Z}$-affgebra.

When making remarks on properties of left Lie brackets and left Lie affgebras, symmetric properties may then also be stated when discussing right Lie brackets and affgebras. Thus, throughout the rest of this thesis, we shall focus on the left cases and drop the use of the adjective 'left'.

Remark 6.2. Note that the antisymmetry property (6.1) can be stated equivalently as:

$$\langle [a,b], [b,b], [b,a] \rangle = [a,a].$$

Similarly, the Jacobi identity (6.2) holds under any permutation of the three terms $[a,a]$, $[b,b]$, $[c,c]$.

Example 6.3. Let A be an affine $\mathbb{K}$-module A and let $\sigma : A \to A$ be an endomorphism of affine modules. Define the binary operation $[-,-] : A \times A \to A$ by $[a,b] = \sigma(a)$. As both the constant map and σ are affine transformations, this operation is bi-affine. One can also easily check to see that $[-,-]$ is a Lie bracket. Now if we consider two endomorphisms σ_1, σ_2, then the corresponding Lie affgebras are isomorphic if, and only if, there exists an automorphism of affine spaces $f : A \to A$ such that, for all $a, b \in A$:

$$\sigma_2(f(a)) = [f(a), f(b)]_2 = f([a,b]_1) = f(\sigma_1(a)),$$

that is, if and only if:

$$\sigma_2 = f \circ \sigma_1 \circ f^{-1}.$$

In particular, the Lie affgebras given by $[a,b] = a$ and $[a,b] = \text{const.}$ are not isomorphic if A has more than one element.

Proposition 6.4. *Given an affine $\mathbb{K}$-module A and $\zeta \in \mathbb{K}$, define:*

$$[-,-] : A \times A \longrightarrow A, \qquad [a,b] = \zeta \triangleright_a b. \tag{6.3}$$

Then $[-,-]$ is a Lie bracket on A. If $\mathbb{K}$ is a field and $\dim A \geq 1$, then the Lie affgebras corresponding to ζ_1 and ζ_2 are isomorphic if, and only if, $\zeta_1 = \zeta_2$.

Proof. In view of the condition (1), in Definition 2.21 and equation (2.9), $[-,-]$ is a bi-heap homomorphism. We will use the base change property (3) in Definition 2.21 repeatedly to prove that $[-,-]$ is a bi-affine transformation. First:

$$[\alpha \triangleright_a b, c] = \zeta \triangleright_{\alpha \triangleright_a b} c = \big\langle \zeta \triangleright_a c, \zeta \triangleright_a (\alpha \triangleright_a b), \alpha \triangleright_a b \big\rangle = \big\langle \zeta \triangleright_a c, (\zeta\alpha) \triangleright_a b), \alpha \triangleright_a b \big\rangle,$$

by the associative law (2) in Definition 2.21. On the other hand:

$$\begin{aligned} \alpha \triangleright_{[a,c]} ([b,c]) &= \big\langle \alpha \triangleright_a (\zeta \triangleright_b c), \alpha \triangleright_a (\zeta \triangleright_a c), \zeta \triangleright_a c \big\rangle, \\ &= \big\langle (\alpha\zeta) \triangleright_a c, (\alpha\zeta) \triangleright_a b, \alpha \triangleright_a b, (\alpha\zeta) \triangleright_a c, \zeta \triangleright_a c \big\rangle = [\alpha \triangleright_a b, c]. \end{aligned}$$

Therefore, $[-,c]$ is an affine transformation. The other affine homomorphism property is shown similarly:

$$\begin{aligned} [a, \alpha \triangleright_b c] &= \zeta \triangleright_a (\alpha \triangleright_b c)\big\rangle = \big\langle (\alpha\zeta) \triangleright_a c, (\alpha\zeta) \triangleright_a b, \zeta \triangleright_a b \big\rangle, \\ &= \big\langle \alpha \triangleright_a [a,c], \alpha \triangleright_a [a,b], [a,b] \big\rangle = \alpha \triangleright_{[a,b]} [a,c], \end{aligned}$$

as required. Next we need to check the condition (6.1). Note that, for all $a, b \in A$, $[a,a] = a$ by (2.10), and hence:

$$\big\langle [a,b], [a,a], [b,a] \big\rangle = \big\langle \zeta \triangleright_a b, a, \zeta \triangleright_a a, \zeta \triangleright_a b, b \big\rangle = b = [b,b].$$

Finally, by fixing $\nu \in A$, and using (3) in Definition 2.21, we can express any action $\triangleright_a$ etc. in terms of $\triangleright_\nu$ to obtain:

$$[a,[b,c]] = \big\langle \zeta^2 \triangleright_\nu c, \zeta^2 \triangleright_\nu b, \zeta \triangleright_\nu b, \zeta \triangleright_\nu a, a \big\rangle.$$

The Jacobi identity (6.2) can now be verified by varying a, b, c cyclically and using the reshuffling and cancellation rules for the abelian heap operation.

Now, assume that $\mathbb{K}$ is a field, $\dim A \geq 1$ and consider two Lie brackets on A:

$$[a, b]_1 = \zeta_1 \triangleright_a b \quad \text{and} \quad [a, b]_2 = \zeta_2 \triangleright_a b, \qquad \text{for all } a, b \in A.$$

An affine isomorphism $f : A \to A$ is an isomorphism of Lie affgebras $(A, [-,-]_1) \to (A, [-,-]_2)$ if, and only if:

$$\zeta_2 \triangleright_{f(a)} f(b) = [f(a), f(b)]_2 = f([a, b]_1) = \zeta_1 \triangleright_{f(a)} f(b).$$

Since f is bijective this is equivalent to the statement that $\zeta_1 \triangleright_a b = \zeta_2 \triangleright_a b$, for all $a, b \in A$. In the vector space A_a, the above equality implies that $\zeta_1 b = \zeta_2 b$, for all b, and since A_a is at least one-dimensional this is equivalent to $\zeta_1 = \zeta_2$, as required. □

Remark 6.5. As previously referenced in Corollary 5.3, the term *Lie affgebra* was introduced by Grabowska, Grabowski and Urbański in [28] to describe a Lie-type structure on an affine space and its relation to Definition 6.1 should be clarified. In [28] an affine space is understood in the traditional manner as a set A over the vector space $\vec{A}$. A Lie bracket on $(A, \vec{A})$ is then defined as an anti-symmetric bi-affine map:

$$[-,-]_u : A \times A \longrightarrow \vec{A},$$

that satisfies the Jacobi identity in $\vec{A}$:

$$\overrightarrow{[a}, [b, c]_u]_u + \overrightarrow{[b}, [c, a]_u]_u + \overrightarrow{[c}, [a, b]_u]_u = 0, \tag{6.4}$$

where the arrow indicates the linearisation of the maps $[a, -]_u$ etc. We will refer to the map $[-,-]_u$ as to the **vector-valued Lie bracket** on $(A, \vec{A})$.

Proposition 6.6. *Let A be an affine space over the field $\mathbb{K}$ of characteristic different from 2. For any $\nu \in A$, there is a bijective correspondence between idempotent Lie brackets on A and vector-valued Lie brackets on (A, A_ν).*

Proof. Assume that $[-,-]$ is an idempotent Lie bracket on A (see Definition 2.29), that is $[a, a] = a$, for all $a \in A$, and, for all $a, b \in A$, define $[a, b]_u \in A_\nu$ by:

$$[a, b]_u = \langle [a, b], b, \nu \rangle = [a, b] - b.$$

Then, using the idempotent property of $[-,-]$ and (6.1) we obtain:

$$[a, b]_u + [b, a]_u = [ab] - b + [b, a] - a = \langle [a, b], [b, b], [b, a] \rangle - a = [a, a] - a = \nu,$$

so the bracket is anti-symmetric. The linearisation of $[a, -]_u$ comes out as:

$$\begin{aligned}
\overrightarrow{[a}, [b, c]_u]_u &= \overrightarrow{[a}, \langle [b, c], c, \nu \rangle]_u = \langle [a, [b, c]]_u, [a, c]_u, \nu \rangle, \\
&= \langle [a, [b, c]], [b, c], \nu, [a, c]_u, \nu \rangle, \\
&= \langle [a, [b, c]], c, \nu, \nu, c, [b, c], \nu, [a, c]_u, \nu \rangle, \\
&= \langle [a, [b, c]], c, \nu, [b, c]_u, \nu, [a, c]_u, \nu \rangle, \\
&= \langle [a, [b, c]], [c, c], \nu, [b, c]_u, \nu, [a, c]_u, \nu \rangle, \\
&= [a, [b, c]] - [c, c] - [b, c]_u - [a, c]_u,
\end{aligned}$$

where the last expression is written in A_ν. The Jacobi identity (6.2) written in A_ν reads:

$$[a,[b,c]] - [a,a] + [b,[c,a]] - [b,b] + [c,[a,b]] - [c,c] = \nu, \tag{6.5}$$

and its combination with the anti-symmetry of $[-,-]_v$ implies the Jacobi identity (6.4).

In the opposite direction one sets:

$$[a,b] = [a,b]_u + b.$$

This is an idempotent by the antisymmetry of $[-,-]_u$ (and the fact that the characteristic of the field is different from 2). Antisymmetry also implies that $[-,-]$ satisfies condition (6.1). Noting that for an affine map $f : A \to A$ in (A, A_ν) and $a \in A_\nu$ and $b \in A$:

$$f(a+b) = \vec{f}(a) + f(b),$$

we can compute:

$$\begin{aligned}[a,[b,c]] &= [a,[b,c]]_u + [b,c] = [a,[b,c]_u + c]_u + [b,c]_u + c,\\ &= \vec{[a},[b,c]_u]_u + [a,c]_u + [b,c]_u + [c,c].\end{aligned}$$

Now the Jacobi identity (6.4) together with the antisymmetry of $[-,-]_u$ imply the Jacobi identity (6.5). ☐

Example 6.7. In view of Proposition 6.6, Proposition 6.4 gives a family of vector-valued Lie brackets on an affine space $(A, \vec{A})$, labelled by scalars $\alpha \in \mathbb{K}$:

$$[a,b]_u = \alpha\,\overrightarrow{ab}, \qquad a, b \in A.$$

Every associative algebra is a Lie algebra with the commutator bracket. An analogous statement is true for associative affgebras.

Proposition 6.8. *An associative $\mathbb{K}$-affgebra A is a Lie affgebra with the bracket:*

$$[a,b] = \langle ab, ba, b\rangle, \tag{6.6}$$

for all $a, b \in A$.

Proof. The maps $[a,-]$ and $[-,b]$, induced by the bracket (6.6), are affine homomorphisms by Lemma 2.22.

Checking the anti-symmetry property (6.1) is straightforward. For the Jacobi identity, note that the bracket (6.6) is an idempotent operation and that:

$$[a,[b,c]] = \langle abc, acb, ac, bca, cba, ca, bc, cb, c\rangle.$$

Permuting this cyclically one concludes that the Jacobi identity (6.2) holds. ☐

Again, as is the case for Lie algebras, a commutator Lie bracket needs not be defined on an associative algebra, it is sufficient to consider *pre-Lie algebras* introduced in [24] and [42] (see [35] for a review).

Definition 6.9. A **left pre-Lie affgebra** is an affine space A, together with the bi-affine map $\cdot : A \times A \to A$, such that, for all $a, b, c \in A$:

$$(a \cdot b) \cdot c = \big\langle a \cdot (b \cdot c), b \cdot (a \cdot c), (b \cdot a) \cdot c \big\rangle. \tag{6.7}$$

Similarly, a **right pre-Lie affgebra** is an affine space with bi-affine binary operation $\cdot$, satisfying the condition:

$$a \cdot (b \cdot c) = \big\langle (a \cdot b) \cdot c, (a \cdot c) \cdot b, a \cdot (c \cdot b) \big\rangle, \tag{6.8}$$

for all $a, b, c \in A$.

We note in passing that, when written in terms of addition $a + b = \langle a, \nu, b\rangle$ the conditions (6.7) and (6.8) coincide exactly with the corresponding pre-Lie algebras conditions.

Proposition 6.10. *Let $(A, \cdot)$ be a right (or left) pre-Lie affgebra. Then A is a Lie affgebra with the bracket:*

$$[a, b] = \langle a \cdot b, b \cdot a, b\rangle,$$

for all $a, b \in A$.

Proof. The only point worth mentioning here, is that in the case of a right pre-Lie affgebra:

$$\begin{aligned}[a, [b, c]] &= \big\langle a \cdot (b \cdot c), a \cdot (c \cdot b), a \cdot c, (b \cdot c) \cdot a, (c \cdot b) \cdot a, c \cdot a, b \cdot c, c \cdot b, c \big\rangle, \\ &= \big\langle (a \cdot b) \cdot c, (a \cdot c) \cdot b, a \cdot c, (b \cdot c) \cdot a, (c \cdot b) \cdot a, c \cdot a, b \cdot c, c \cdot b, c \big\rangle,\end{aligned}$$

by (6.8). Once the brackets are redistributed in a uniform way, as above, the cancellation of the cyclic combinations follows by the same arguments as in the associative case. The left pre-Lie affgebra condition (6.7) allows one to distribute the brackets in double operation to the form $- \cdot (- \cdot -)$. □

Now we will look at derivations within the context of affgebras. The following definition closely mirrors Definition 3.10 from earlier in this thesis, however, it is a more general definition.

6.2 Affine Derivations

Definition 6.11. Let A be a non-necessarily associative $\mathbb{K}$-affgebra and let $\sigma : A \to A$ be an affine map such that, for all $a, b \in A$:

$$\sigma(ab) = \big\langle \sigma(a)b, \sigma(ab), a\sigma(b) \big\rangle. \tag{6.9}$$

A **derivation along** σ is an affine homomorphism $X : A \to A$, such that:

$$X\sigma = \sigma X, \tag{6.10}$$

$$X(ab) = \big\langle X(a)b, \sigma(ab), aX(b)\big\rangle, \qquad \text{for all } a, b \in A. \tag{6.11}$$

The set of all derivations along σ on A is denoted by $\mathrm{Der}_\sigma(A)$. Note that $\sigma \in \mathrm{Der}_\sigma(A)$.

Following this generalised definition, we can see how a derivation of this form X can be reduced to resemble derivations as seen in Definition 3.10.

Example 6.12. When σ is the identity map $\sigma(a) = a$ we have the following:

$$\langle \sigma(a)b, \sigma(ab), a\sigma(b)\rangle = \langle ab, ab, ab\rangle = ab = \sigma(ab).$$

Furthermore, we may see that a derivation along σ would yield the following:

$$\begin{aligned} X(ab) &= \langle X(a)b, \sigma(ab), aX(b)\rangle, \\ &= \langle X(a)b, ab, aX(b)\rangle. \end{aligned}$$

Hence when σ is the identity map, our generalised definition for derivations reduces to the kind seen earlier in this thesis in Definition 3.10.

Example 6.13. Let $[-,-]$ be the commutator Lie bracket (6.6) on an associative $\mathbb{K}$-affgebra A. Then, for all $a \in A$, the map $X_a : A \to A$, $b \mapsto [a, b]$ is a derivation along the identity on A.

Proof. Obviously X_a is an affine transformation. Checking the derivation property (with $\sigma = \mathrm{id}$) is straightforward. □

Example 6.14. Let A be an affine $\mathbb{K}$-module, take $\nu \in A$ and view it as an associative affgebra with multiplication:

$$ab = \langle a, \nu, b\rangle = a + b,$$

that is the multiplication of A is addition in the $\mathbb{K}$-module A_ν (with the $\mathbb{K}$-action $\alpha a = \alpha \triangleright_\nu a$). Then $\mathrm{Der}_{\mathrm{id}}(A) = \mathrm{End}_{\mathbb{K}}(A_\nu)$.

Proof. An affine endomorphism $X : A \to A$ is a derivation along the identity if, and only if:

$$X(a{+}b) = X(ab) = \big\langle X(a), \nu, b, a, \nu, b, a, \nu, X(b)\big\rangle = \big\langle X(a), \nu, X(b)\big\rangle = X(a){+}X(b),$$

i.e., X is an additive map. In particular $X(\nu) = \nu$, and since X is an affine transformation, for all $\alpha \in \mathbb{K}$ and $a \in A_\nu$:

$$X(\alpha a) = \alpha \triangleright_{X(\nu)} X(a) = \alpha X(\nu).$$

Therefore, X is a derivation along the identity if, and only if, it is an endomorphism of $\mathbb{K}$-modules. □

Example 6.15. Given a truss T and a mapping $D : T \to T$ where:

$$D(ab) = \langle D(a)b, ab, aD(b)\rangle,$$

and $e \in T$ such that we have a retracted ring $R(T; e)$ where:

$$a + b = \langle a, e, b\rangle, \qquad a \cdot b = ab - ae + e^2 - eb,$$

where $\cdot$ denotes our multiplication in the ring and juxtaposition denotes multiplications in T. Then if we have the mapping:

$$d_e : T \to T, \qquad d_e(a) = \langle D(a), D(e), e, a, e\rangle,$$

which may be denoted in $R(T; e)$ as:

$$d_e(a) = D(a) - D(e) - a,$$

then we may also note that we can express $d_e(e)$ as:

$$d_e(e) = D(e) - D(e) - e = e,$$

thus d_e is a homomorphism of abelian groups $R(T; e) \to R(T; e)$. Then we may show that d_e is a derivation on $R(T; e)$, that is to say that the following holds:

$$d_e(a \cdot b) = d_e(a) \cdot b + a \cdot d_e(b). \tag{6.12}$$

Proof. We begin by computing the left-hand side of equation 6.12:

$$\begin{aligned} d_e(a \cdot b) &= D(a \cdot b) - D(e) - a \cdot b, \\ &= D(ab - ae + e^2 - eb + e) - D(e) - ab + ae - e^2 + eb, \\ &= D(ab) - D(ae) + D(e^2) - D(eb) + D(e) - D(e) - ab + ae - e^2 + eb, \\ &= D(a)b - ab + aD(b) - D(a)e + ae + aD(e) + D(e)e - e^2 \\ &+ eD(e) - D(e)b + eb - eD(b) - ab + ae - e^2 + eb. \end{aligned}$$

We then compute the right-hand side of equation 6.12,

$$\begin{aligned} d_e(a) \cdot b + a \cdot d_e(b) &= D(a) \cdot b - D(e) \cdot b - a \cdot b + a \cdot D(b) - a \cdot D(e) - a \cdot b, \\ \\ &= D(a)b - eb + e^2 - D(a)e - D(e)b + eb - e^2 + D(e)e \\ &+ aD(b) - eD(b) - aD(e) + eD(e) - 2(a \cdot b), \\ \\ &= D(a)b - eb + e^2 - D(a)e - D(e)b + eb - e^2 + D(e)e \\ &+ aD(b) - eD(b) - aD(e) + eD(e) \\ &- 2ab + 2ae - 2e^2 + 2eb - 2e. \end{aligned}$$

Here we can see that equation 6.12 holds. Thus we may conclude that a derivation D on a truss T induces a derivation d_e on any retract $R(T; e)$ respectively. □

Example 6.16. Consider the derivation D, and the truss build on $(\mathbb{Z}, \langle - - - \rangle, *)$ with:

$$m * n = amn + b(m+n) + c,$$

where $a, b \in \mathbb{Z}$. Then there exists a bijective correspondence between the derivation d on the retract $R(\mathbb{Z}, 0)$, and a derivation D on $(\mathbb{Z}, \langle - - - \rangle, *)$ such that:

$$D \longmapsto d + id,$$

$$d \longmapsto D - id.$$

Proof. Given the above conditions, we may write:

$$D(m * n) = \langle D(m) * n, D(m * n), m * D(n) \rangle,$$

$$d(m) = \langle D(m), D(0), 0, m, 0 \rangle$$

□

Example 6.17. Following on from these calculations, given a truss T and its retracted ring $R(T;0)$, if we have the property that $R(T;0)$ has the identity $D(1) = 1$ and we define:

$$d(a) := D(a) - D(0) - a,$$

then we may compute the following:

$$\begin{aligned} d(a+b) &= D(a - 0 + b) - D(0) - a - b, \\ &= D(a) - D(0) + D(b) - D(0) - a - b, \\ &= d(a) + d(b). \end{aligned}$$

Then using the following properties:

$$d(ab) = d(a)b + ad(b),$$

$$D(ab) = D(a)b - ab + aD(b),$$

we compute:

$$\begin{aligned} d(ab) &= D(ab) - D(0) - ab, && (6.13) \\ &= D(a)b - ab + aD(b) - D(0) - ab. && (6.14) \end{aligned}$$

Then we compute:

$$d(a)b + ad(b) = D(a)b - D(0)b - ab + aD(b) - aD(0) - ab. \quad (6.15)$$

Finally we compute:

$$\begin{aligned} d(0) = D(0a) &= D(0)a - 0a + 0D(a), \\ = D(a0) &= D(a)0 - a0 + aD(0), \end{aligned}$$

thus:

$$d(0) = D(0)a = aD(0). \quad (6.16)$$

we can now see that if we have $D(0) = 0$, it follows that in equation (6.16) we have $d(0) = 0$, and that the lines (6.13-6.15) equate. Hence, D is a derivation of R, if $D(0) = 0$. Now let us check if the inverse holds. If we set:

$$D(0) = 0,$$

it follows that:

$$\begin{aligned} d(a) &= D(a) - D(0) - a, \\ &= D(a) - 0 - a, \\ &= D(a) - a. \end{aligned}$$

If we then compute:

$$\begin{aligned} d(a+b) &= D(a-0+b) - D(0) - a - b, \\ &= D(a) - D(0) + D(b) - D(0) - a - b, \\ &= D(a) + D(b) - a - b, \\ &= d(a) + d(b). \end{aligned}$$

Then again we check:

$$\begin{aligned} d(ab) &= D(ab) - ab, && (6.17) \\ &= D(a)b - ab + aD(b) - ab. && (6.18) \end{aligned}$$

We again show that, as above, equations (6.17) and (6.19) are equal:

$$d(a)b + ad(b) = D(a)b - ab + aD(b) - ab. \tag{6.19}$$

Finally it is trivial to see that:

$$\begin{aligned} d(0) &= D(0a) = D(0)a - 0a + 0D(a), \\ &= D(a0) = D(a)0 - a0 + aD(0) =, \\ &= 0. \end{aligned}$$

Thus, we may state that D is a derivation of R if, and only if, $D(0) = 0$.

Proposition 6.18. *Let L be a Lie affgebra with an idempotent bracket. Then for all $a \in L$, the map:*

$$X_a : L \longrightarrow L, \qquad b \mapsto [a, b],$$

is a derivation on L along the identity.

Proof. By the definition of a Lie bracket, the map X_a is an affine transformation. Take any $b, c \in L$, and using the Jacobi identity (6.2), the antisymmetry (6.1), and the idempotent property of the Lie bracket, compute:

$$\begin{aligned} X_a([b,c]) = [a,[b,c]] &= \Big\langle [a,a],[b,[c,a]],[b,b],[c,[a,b]],[c,c] \Big\rangle, \\ &= \Big\langle [a,a],[b,[c,c]],[b,[a,c]],[b,[a,a]],[b,b],[[a,b],[a,b]],[[a,b],c],[c,c],[c,c] \Big\rangle, \\ &= \Big\langle a,[b,c],[b,X_a(c)],[b,a],[b,b],[a,b],[X_a(b),c] \Big\rangle, \\ &= \Big\langle [X_a(b),c],[b,c],[b,X_a(c)],[a,a],a \Big\rangle = \Big\langle [X_a(b),c],[b,c],[b,X_a(c)] \Big\rangle, \end{aligned}$$

as needed. ∎

Theorem 6.19. *Let A be a non-necessarily associative $\mathbb{K}$-affgebra and let $\sigma : A \to A$ be an affine map. Then:*

1). Provided σ satisfies (6.9), $\mathrm{Der}_\sigma(A)$ is a Lie affgebra with the $\mathbb{K}$-affine structure arising from $\mathrm{Aff}(A)$ and the Lie bracket:

$$[X, Y] = \big\langle XY, YX, \sigma \big\rangle, \tag{6.20}$$

for all $X, Y \in \mathrm{Der}_\sigma(A)$.

2). If the affine $\mathbb{K}$-sub-module:

$$\mathrm{Aff}(A)_\sigma := \{X \in \mathrm{Aff}(A) \mid X \text{ satisfies } (6.11)\} \subseteq \mathrm{Aff}(A),$$

is a Lie affgebra with bracket (6.20), then $\sigma \in \mathrm{Aff}(A)_\sigma$ and $\mathrm{Aff}(A)_\sigma = \mathrm{Der}_\sigma(A)$.

Proof. 1). First we need to check whether $\mathrm{Der}_\sigma(A)$ is closed under the $\mathbb{K}$-action (2.14) (it is clear that $\mathrm{Der}_\sigma(A)$ is a heap with the pointwise operation). In view of the base change property (3) in Definition 2.21, actions with different bases (elements under the symbol $\triangleright$) can be converted to the action with a fixed base. Hence, take $\nu \in A$, then for all $X, Y \in \mathrm{Der}_\sigma(A)$, $a \in A$ and $\alpha \in \mathbb{K}$:

$$(\alpha \triangleright_X Y)(a) = \big\langle \alpha \triangleright_\nu Y(a), \alpha \triangleright_\nu X(a), X(a) \big\rangle.$$

To simplify notation further we will denote $\triangleright_\nu$ by $\cdot$. We compute, for all $a, b \in A$:

$$\begin{aligned}
(\alpha \triangleright_X Y)(ab) &= \big\langle \alpha \cdot Y(ab), \alpha \cdot X(ab), X(ab) \big\rangle, \\
&= \big\langle \alpha \cdot (Y(a)b), \alpha \cdot \sigma(ab), \alpha \cdot (aY(b)), \\
&\qquad \alpha \cdot (X(a)b), \alpha \cdot \sigma(ab), \alpha \cdot (aX(b)), \\
&\qquad X(a)b, \sigma(ab), aX(b) \big\rangle, \\
&= \big\langle (\alpha \cdot Y(a))b, (\alpha \cdot X(a))b, X(a)b, \sigma(ab), a\alpha \cdot Y(b), a\alpha \cdot X(b), aX(b) \big\rangle, \\
&= \Big\langle \big\langle \alpha \cdot Y(a), \alpha \cdot X(a), X(a) \big\rangle b, \sigma(ab), a\big\langle \alpha \cdot Y(b), \alpha \cdot X(b), X(b) \big\rangle \Big\rangle, \\
&= \big\langle (\alpha \triangleright_X Y)(a)b, \sigma(ab), a(\alpha \triangleright_X Y)(b) \big\rangle,
\end{aligned}$$

which proves that $\alpha \triangleright_X Y$ is a derivation along σ, as required.

Next we need to check that $[X, Y]$ is a derivation along σ. Since all the maps involved in the definition of $[X, Y]$ are homomorphisms of affine modules, $[X, Y]$ is an affine module homomorphism as well by Lemma 2.22. The derivation property is checked as follows. For all $a, b \in A$:

$$[X,Y](ab) = \big\langle XY(ab), YX(ab), \sigma(ab)\big\rangle =$$

$$= \big\langle X(aY(b)), X\sigma(ab), X(Y(a)b), Y(aX(b)), Y\sigma(ab), Y(X(a)b), \sigma(ab)\big\rangle,$$

$$= \big\langle X(a)Y(b), \sigma(aY(b)), aXY(b), X\sigma(ab), XY(a)b, \sigma(Y(a)b), Y(a)X(b), \\ Y(a)X(b), \sigma(aX(b)), aYX(b), Y\sigma(ab), YX(a)b, \sigma(X(a)b), X(a)Y(b), \sigma(ab)\big\rangle,$$

$$= \big\langle \sigma(ab), \sigma(aY(b)), aXY(b), X\sigma(ab), XY(a)b, \sigma(Y(a)b), \\ \sigma(aX(b)), aYX(b), Y\sigma(ab), YX(a)b, \sigma(X(a)b)\big\rangle,$$

$$= \big\langle \sigma(ab), \sigma Y(ab), a[X,Y](b), X\sigma(ab), [X,Y](a)b, \sigma^2(ab) \\ \sigma X(ab), a\sigma(b), Y\sigma(ab), \sigma(a)b, \sigma^2(ab)\big\rangle,$$

$$= \big\langle a[X,Y](b), a\sigma(b), \sigma(ab), \sigma(a)b, [X,Y](a)b\big\rangle,$$

$$= \big\langle a[X,Y](b), \sigma(ab), [X,Y](a)b\big\rangle.$$

Here we used the derivation property in obtaining the second, third, fourth and the fifth equalities; equation (6.10) to obtain the sixth one; and (6.9) to derive the last equality. We also used freely the cancellation and rearranging rules stemming from the definition of an abelian heap. This proves that $[X,Y]$ is a derivation along σ.

The functions $[-,Y]$ and $[X,-]$ are affine maps by Lemma 2.22. It remains to check that $[-,-]$ is a Lie bracket. First, for all $X, Y \in \mathrm{Der}_\sigma(A)$, note that $[X,X] = \sigma = [Y,Y]$, and hence:

$$\big\langle [X,Y], [X,X], [Y,X]\big\rangle = \big\langle XY, YX, \sigma, \sigma, YX, XY\big\rangle = \sigma = [Y,Y].$$

Finally, taking in addition $Z \in \mathrm{Der}_\sigma(A)$:

$$\big\langle [X,[Y,Z]], [X,X], [Y,[Z,X]], [Y,Y], [Z,[X,Y]]\big\rangle \\ = \big\langle X[Y,Z], [Y,Z]X, \sigma, \sigma, Y[Z,X], [Z,X]Y, \sigma, \sigma, Z[X,Y], [X,Y]Z, \sigma\big\rangle, \\ = \big\langle XYZ, XZY, X\sigma, \sigma X, ZYX, YZX, YZX, YXZ, Y\sigma, \sigma Y, \\ XZY, ZXY, ZXY, ZYX, Z\sigma, \sigma Z, YXZ, XYZ, \sigma\big\rangle = \sigma = [Z,Z],$$

where the condition (6.10) was used. This completes the proof that the set of all derivations on A along σ is a Lie affgebra.

2). Since $[X,X] = \sigma$ and $\mathrm{Aff}(A)_\sigma$ is closed under the Lie bracket, $\sigma \in \mathrm{Der}_\sigma(A)$. Furthermore, exploring the Jacobi identity for $X, Y = Z \in \mathrm{Aff}(A)_\sigma$ we find:

$$\langle \sigma, YX^2, \sigma X, X\sigma, YX^2\rangle = \sigma,$$

which implies that $X\sigma = \sigma X$ and hence, every element of $\mathrm{Aff}(A)_\sigma$ is a derivation along σ. □

6.3 From Lie Affgebras to Lie Algebras

The following theorem reveals that, just as any associative truss can be retracted to an associative ring [4, Theorem 4.3], any Lie affgebra can be retracted to a Lie algebra.

Theorem 6.20. *Let $(A, [-,-])$ be a Lie $\mathbb{K}$-affgebra. Then, for all $\nu \in A$ the $\mathbb{K}$-module A_ν together with the bracket:*

$$[a,b]_\nu = \langle [a,b],[a,\nu],[\nu,\nu],[\nu,b],\nu\rangle = [a,b]-[a,\nu]+[\nu,\nu]-[\nu,b], \tag{6.21}$$

is a Lie $\mathbb{K}$-algebra. Furthermore, any two Lie $\mathbb{K}$-algebras $(A_\nu, [-,-]_\nu)$ and $(A_{\nu'}, [-,-]_{\nu'})$ are mutually isomorphic.

Proof. The bracket $[-,-]_\nu$ is a bi-linearisation of the bi-affine bracket $[-,-]$ by the repetitive use of the linearisation formula (2.13) (step-by-step for each argument), and hence is a bilinear operation on A_ν.

Written in terms of addition and subtraction in A_ν, the antisymmetry property (6.1) and Jacobi identity (6.2) of the Lie bracket $[-,-]$, for all $a,b,c \in A_\nu$ come out as:

$$[a,b]-[a,a]+[b,a] = [b,b], \quad [a,[b,c]]-[a,a]+[b,[c,a]]-[b,b]+[c,[a,b]] = [c,c].$$

The first equality immediately implies that $[a,a]_\nu = \nu$ and $[a,b]_\nu = -[b,a]_\nu$.

The second equality is used to check the Jacobi identity for $[-,-]_\nu$:

$$\begin{aligned} [a,[b,c]_\nu]_\nu + \text{cycl.} &= [a,[b,c]-[b,\nu]+[\nu,\nu]-[\nu,c]]_\nu + \text{cycl.}, \\ &= [a,[b,c]]_\nu - [a,[b,\nu]]_\nu + [a,[\nu,\nu]]_\nu - [a,[\nu,c]]_\nu + \text{cycl.}, \\ &= [a,[b,c]] - [\nu,[b,c]] - [a,[b,\nu]] + [\nu,[b,\nu]] \\ &\quad + [a,[\nu,\nu]] - [\nu,[\nu,\nu]] - [a,[\nu,c]] + [\nu,[\nu,c]] + \text{cycl.}, \\ &= \nu. \end{aligned}$$

Finally, using the translation isomorphism $\tau_\nu^{\nu'}$ from Remark 2.8, we obtain the following equality of terms (binary operations in $A_{\nu'}$):

$$\begin{aligned} [\tau_\nu^{\nu'}(a), \tau_\nu^{\nu'}(b)]_{\nu'} &= [a-\nu, b-\nu]_{\nu'}, \\ &= [a,b]_{\nu'} - [a,\nu]_{\nu'} - [\nu,b]_{\nu'} + [\nu,\nu]_{\nu'}, \\ &= \langle [a,b],[a,\nu],[\nu,\nu],[\nu,b],\nu'\rangle = \tau_\nu^{\nu'}([a,b]_\nu). \end{aligned}$$

This proves the isomorphism of Lie algebras. □

Example 6.21. Let A be an associative $\mathbb{K}$-affgebra with the commutator bracket (6.6) in Proposition 6.8. Take any $\nu \in A$. By Theorem 2.14 A_ν is an associative algebra with the multiplication:

$$a \cdot_\nu b = ab - a\nu + \nu^2 - \nu b.$$

One easily checks that:

$$[a, b]_\nu = a \cdot_\nu b - b \cdot_\nu a,$$

the usual commutator Lie bracket on an associative algebra.

Example 6.22. Let A be a Lie affgebra with the bracket $[a, b] = \zeta \triangleright_a b$, $\zeta \in \mathbb{K}$ (see Proposition 6.4). Take any $\nu \in A$. Then, irrespective of the choice of ζ:

$$[a, b]_\nu = \nu,$$

the trivial Lie bracket on A_ν. As shown in Proposition 6.4, in the case where $\mathbb{K}$ is a field, the Lie brackets on A depend heavily on the choice of ζ. Thus non-isomorphic Lie affgebras can reduce to isomorphic (or even identical) Lie algebras.

In a similar way all Lie affgebra structures $[a, b] = \sigma(a)$, discussed in Example 6.3, collapse to the zero Lie bracket on A_ν.

Chapter 7

Nijenhuis Operators on Lie Affgebras

We now look to how Nijenhuis operators behave on Lie affgebras, with definitions largely adapted from *Poisson-Nijenhuis structures* [32]. In particular, we show that Nijenhuis operators give rise to a family of compatible Lie brackets on an affine space.

7.1 Nijenhuis Operators over Lie Affgebras & Nijenhuis Brackets

Our first definition is an alternative interpretation of the Nijenhuis operator to the ring based interpretation used in [16] and the truss based interpretation Definition 4.1 applied throughout Chapter 4 and Chapter 5.

Definition 7.1. Let $(A, [-,-])$ be a Lie affgebra. An affine homomorphism $N : A \to A$ is called a **Nijenhuis operator (over a Lie affgebra)** if, for all $a, b \in A$:

$$[N(a), N(b)] = N\big(\langle [N(a), b], N([a, b]), [a, N(b)]\rangle\big). \tag{7.1}$$

The binary operation $[-,-]_N$ on A, given by:

$$[a, b]_N = \langle [N(a), b], N([a, b]), [a, N(b)]\rangle, \tag{7.2}$$

is called the **Nijenhuis bracket**.

Note that, since N is an affine homomorphism, the Nijenhuis bracket $[-,-]_N$ is affine on each argument by Lemma 2.22 and hence a bi-affine map.

Remark 7.2. To make the following calculations more legible, from this point onward, we will be using the operator notation convention $N(a) = Na$. We will also use the additive notation for ternary heap operation (2.8). In this notation the Nijenhuis bracket (7.2) and the Nijenhuis operator condition (7.1) come out as:

$$[a, b]_N = [Na, b] - N[a, b] + [a, Nb], \qquad [Na, Nb] = N[a, b]_N.$$

7.2 Resulting Examples & Theorems

Theorem 7.3. *A Nijenhuis bracket on a Lie affgebra $(A, [-,-])$ is a Lie bracket.*

Proof. First we see if the Nijenhuis bracket satisfies the antisymmetry property. We start with:

$$\begin{aligned}\langle [b,b]_N, [b,a]_N, [a,a]_N\rangle &= [Nb,b] - N[b,b] + [b,Nb]\\ &\quad - [Nb,a] + N[b,a] - [b,Na]\\ &\quad + [Na,a] - N[a,a] + [a,Na],\end{aligned}$$

and then use the antisymmetry property (6.1) of $[-,-]$:

$$\langle [b,b]_N, [b,a]_N, [a,a]_N\rangle =$$

$$\begin{aligned}&= [Nb,b] - N[b,b] + [b,Nb] - [a,a] + [a,Nb] - [Nb,Nb] + N[a,a] - N[a,b]\\ &+ N[b,b] - [Na,Na] + [Na,b] - [b,b] + [Na,a] - N[a,a] + [a,Na].\end{aligned}$$

Next we perform cancellations and apply again the antisymmetry of $[-,-]$:

$$\langle [b,b]_N, [b,a]_N, [a,a]_N\rangle = [Na,b] - N[a,b] + [a,Nb] = [a,b]_N,$$

thus the antisymmetry property holds for $[-,-]_N$.

It remains to check that the Nijenhuis bracket satisfies the Jacobi identity. To this end we start with developing the left-hand side of the Jacobi identity:

$$\text{LHS} = \langle [a,[b,c]_N]_N, [a,a]_N, [b,[c,a]_N]_N, [b,b]_N, [c,[a,b]_N]_N\rangle,$$

$$\begin{aligned}&= [Na,[Nb,c]] - [Na,N[b,c]] + [Na,[b,Nc]] - N[a,[Nb,c]] + N[a,N[b,c]]\\ &- N[a,[b,Nc]] + [a,[Nb,Nc]] - [a,a] + N[a,a] - [Na,Na] + [Nb,[Nc,a]]\\ &- [Nb,N[c,a]] + [Nb,[c,Na]] - N[b,[Nc,a]] + N[b,N[c,a]] - N[b,[c,Na]]\\ &+ [b,[Nc,Na]] - [b,b] + N[b,b] - [Nb,Nb] + [Nb,[Nc,a]] - [Nb,N[c,a]]\\ &+ [Nb,[c,Na]] - N[b,[Nc,a]] + N[b,N[c,a]] - N[b,[c,Na]] + [b,[Nc,Na]],\end{aligned}$$

and then apply the Jacobi identity for $[-,-]$ several times to obtain:

$$\begin{aligned}\text{LHS} &= [c,c] + [Na,Na] + [Nc,Nc] + [Nb,Nb] + [Nc,Nc] + N[Na,[b,c]]\\ &- N[Na,Na] - N[b,b] - N[c,c] + N[Nb,[c,a]] - N[a,a] - N[Nb,Nb]\\ &- N[c,c] + N[Nc,[a,b]] - N[a,a] - N[b,b] - N[Nc,Nc] - [Na,N[b,c]]\\ &+ N[a,N[b,c]] + N[a,a] - [Nb,N[c,a]] + N[b,N[c,a]] + N[b,b]\\ &- [Nc,N[a,b]] + N[c,N[a,b]].\end{aligned}$$

In the next step we perform cancellations and substitutions of the form:

$$N^2[x,y] = N[Nx,y] - [Nx,Ny] + N[x,Ny].$$

This results in:

$$\begin{aligned}\text{LHS} = {} & [c,c] + [Na,Na] + [Nc,Nc] + [Nb,Nb] + [Nc,Nc] - N[Na,Na] \\ & - N[a,a] - N[Nb,Nb] - N[b,b] - N[Nc,Nc] - N[c,c] - N[c,c] \\ & + N^2[a,[b,c]] + N^2[b,[c,a]] + N^2[c,[a,b]].\end{aligned}$$

The application of the Jacobi identity for $[-,-]$ yields:

$$\begin{aligned}\text{LHS} = {} & [c,c] + [Na,Na] + [Nc,Nc] + [Nb,Nb] + [Nc,Nc] - N[Na,Na] - N[a,a] \\ & - N[Nb,Nb] - N[b,b] - N[Nc,Nc] - N[c,c] - N[c,c] \\ & + N^2[a,a] + N^2[b,b] + N^2[c,c].\end{aligned}$$

It remains to use the definition of the Nijenhuis bracket (7.2), antisymmetry and the Nijenhuis operator (7.1) condition to conclude that:

$$\text{LHS} = [c,c] - N[c,c] + [Nc,Nc] = [Nc,c] - N[c,c] + [c,Nc] = [c,c]_N.$$

i.e.

$$\langle [a,[b,c]_N]_N, [a,a]_N, [b,[c,a]_N]_N, [b,b]_N, [c,[a,b]_N]_N \rangle = [c,c]_N,$$

as required. □

Example 7.4. Let A be an associative $\mathbb{K}$-affgebra A, and $N : A \to A$ an affine map such that, for all $a, b \in A$:

$$N(a)N(b) = N\langle N(a)b, N(ab), aN(b)\rangle.$$

A straightforward computation:

$$\begin{aligned}N\langle [N(a),b], N[a,b], [a,N(b)]\rangle &= N(N(a)b - bN(a) + b - N(ab) + N(ba) - N(b) \\ &\qquad + aN(b) - N(b)a + N(b)), \\ &= N(a)N(b) - N(b)N(a) + N(b) = [N(a),N(b)],\end{aligned}$$

affirms that N is a Nijenhuis operator on the Lie affgebra A with the bracket given in Proposition 6.8.

Example 7.5. Let A be the $\mathbb{K}$-affine module with the Lie bracket $[a,b] = \zeta \triangleright_a b$ as in Proposition 6.4. For any affine endomorphism $N : A \to A$, using the homomorphism property of N and the base change property (c) in Definition 2.21 we find:

$$\begin{aligned}[a,b]_N &= \langle \zeta \triangleright_{Na} b, N(\zeta \triangleright_a b), \zeta \triangleright_a Nb\rangle, \\ &= \langle \zeta \triangleright_a b, \zeta \triangleright_a Na, Na, \zeta \triangleright_{Na} Nb, \zeta \triangleright_a Nb\rangle, \\ &= \langle \zeta \triangleright_a b, \zeta \triangleright_a Na, Na, Na, \zeta \triangleright_a Na, \zeta \triangleright_a Nb, \zeta \triangleright_a Nb\rangle, \\ &= \zeta \triangleright_a b = [a,b].\end{aligned}$$

Again, by using the homomorphism property of N, we immediately find that:

$$[Na, Nb] = N[a,b] = N[a,b]_N.$$

Therefore, any affine endomorphism N of A is a Nijenhuis operator for $(A, [-,-])$ and the Nijenhuis Lie bracket coincides with the original Lie bracket on A.

Theorem 7.6. *Let N be a Nijenhuis operator on a Lie $\mathbb{K}$-affgebra $(A, [-,-])$. For any $k \in \mathbb{N}$ we denote by N^k the k-th composition power of N, with the convention that $N^0 = \mathrm{id}$. For all $k, l \in \mathbb{N}$:*

1). N^k is a Nijenhuis operator on $(A, [-,-])$.

2). N^l is a Nijenhuis operator on $(A, [-,-]_{N^k})$.

3). For all $\alpha \in \mathbb{K}$, the map:

$$N_\alpha^{k,l} := \alpha \triangleright_{N^k} N^l : A \longrightarrow A, \qquad a \longmapsto \alpha \triangleright_{N^k a} N^l a,$$

is a Nijenhuis operator on $(A, [-,-])$.

Proof. Using similar methodology to that of Theorem 4.9 we first show that, for all $a, b \in A, k, l \in \mathbb{N}$:

$$\begin{aligned}
[N^k a, Nb] &= \langle N[N^k a, b], N^{k+1}[a, b], N^k[a, Nb]\rangle, &(7.3)\\
[Na, N^k b] &= \langle N[a, N^k b], N^{k+1}[a, b], N^k[Na, b]\rangle, &(7.4)\\
N^{k+1}[a, b] &= \langle N[N^k a, b], [N^k a, Nb], N^k[a, Nb]\rangle, &(7.5)\\
&= \langle N^k[Na, b], [Na, N^k b], N[a, N^k b]\rangle. &(7.6)
\end{aligned}$$

We will show how (7.3) can be proven by induction and only note that (7.4) can be proven symmetrically, and that proofs of (7.5) and (7.6) follow trivially. Starting with (7.3), for $k = 0$ we have $N^0 = \mathrm{id}$, and for $k = 1$ we have the Nijenhuis operator. If we assume that equation (7.3) holds for k, then the Nijenhuis condition yields:

$$\begin{aligned}
[N^{k+1}a, Nb] = [N(N^k a), Nb] &= N\langle [N^{k+1}a, b], N[N^k a, b], [N^k a, Nb]\rangle, &(7.7)\\
&= N\langle [N^{k+1}a, b], N[N^k a, b], N[N^k a, b], N^{k+1}[a, b], N^k[a, Nb]\rangle, &(7.8)\\
&= \langle N[N^{k+1}a, b], N^{k+2}[a, b], N^{k+1}[a, Nb]\rangle. &(7.9)
\end{aligned}$$

Thus, (7.3) holds for all $k \in \mathbb{N}$, by induction. Then using the above, we compute:

$$\begin{aligned}
[Na, Nb]_{N^k} &= \langle [N^{k+1}a, Nb], N^k[Na, Nb], [Na, N^{k+1}b]\rangle,\\
\\
&= \langle N[N^{k+1}a, b], N^{k+2}[a, b], N^{k+1}[a, Nb],\\
&\qquad N^{k+1}[Na, b], N^{k+2}[a, b], N^{k+1}[a, Nb],\\
&\qquad N^{k+1}[Na, b], N^{k+2}[a, b], N[a, N^{k+1}b]\rangle,\\
\\
&= \langle N[N^{k+1}a, b], N^{k+2}[a, b], N[a, N^{k+1}b]\rangle = N[a, b]_{N^{k+1}}.
\end{aligned}$$

Now we will show that:

$$N^l[a, b]_{N^{k+l}} = [N^l a, N^l b]_{N^k}. \tag{7.10}$$

This is proven through a straightforward induction, and we need only demonstrate the case where $l = 1$. Using (7.5) and (7.6), we find:

$$\langle [Na, b]_{N^k}, N[a, b]_{N^k}, [a, Nb]_{N^k} \rangle =$$

$$\begin{aligned}
&= \langle [N^{k+1}a, b], N^k[Na, b], [Na, N^k b], N[N^k a, b], N^{k+1}[a, b], \\
&\qquad N[a, N^k b], [N^k a, Nb], N^k[a, Nb], [a, N^{k+1}b] \rangle, \\
&= \langle [N^{k+1}a, b], N^{k+1}[a, b], N^{k+1}[a, b], N^{k+1}[a, b], [a, N^{k+1}b] \rangle, \\
&= \langle [N^{k+1}a, b], N^{k+1}[a, b], [a, N^{k+1}b] \rangle = [a, b]_{N^{k+1}}.
\end{aligned}$$

Lastly, we must demonstrate:

$$[N_\alpha^{k,l}a, N_\alpha^{k,l}b] = N_\alpha^{k,l} \langle [N_\alpha^{k,l}a, b], N_\alpha^{k,l}[a, b], [a, N_\alpha^{k,l}b] \rangle. \tag{7.11}$$

To make the following of the argument easier, we use the additive notation explained in Remark 7.2, write the affine action relative to the element determining the binary operations as in Remark 2.23, and we compute:

$$\begin{aligned}
[N_\alpha^{k,l}a, N_\alpha^{k,l}b] &= \alpha^2[N^l a, N^l b] - \alpha^2[N^l a, N^k b] + \alpha[N^l a, N^k b] \\
&\quad - \alpha^2[N^k a, N^l b] + \alpha^2[N^k a, N^k b] - \alpha[N^k a, N^k b] \\
&\quad + \alpha[N^k a, N^l b] - \alpha[N^k a, N^k b] + [N^k a, N^k b].
\end{aligned}$$

Now compute, applying the Nijenhuis condition:

$$N_\alpha^{k,l} \langle [N_\alpha^{k,l}a, b], N_\alpha^{k,l}[a, b], [a, N_\alpha^{k,l}b] \rangle =$$

$$\begin{aligned}
&= \alpha^2 N^l[N^l a, b] - \alpha^2 N^l[N^k a, b] + \alpha N^l[N^k a, b] - \alpha^2 N^k[N^l a, b] + \alpha^2 N^k[N^k a, b] \\
&- \alpha N^k[N^k a, b] + \alpha N^k[N^l a, b] - \alpha N^k[N^k a, b] + N^k[N^k a, b] - \alpha^2 N^l N^l[a, b] \\
&+ \alpha^2 N^l N^k[a, b] - \alpha N^l N^k[a, b] + \alpha^2 N^k N^l[a, b] - \alpha^2 N^k N^k[a, b] + \alpha N^k N^k[a, b] \\
&- \alpha N^k N^l[a, b] + \alpha N^k N^k[a, b] - N^k N^k[a, b] + \alpha^2 N^l[a, N^l b] - \alpha^2 N^l[a, N^k b] \\
&+ \alpha N^l[a, N^k b] - \alpha^2 N^k[a, N^l b] + \alpha^2 N^k[a, N^k b] - \alpha N^k[a, N^k b] + \alpha N^k[a, N^l b] \\
&- \alpha N^k[a, N^k b] + N^k[a, N^k b], \\
&= -\alpha^2 N^l[N^k a, b] + \alpha^2 N^l N^k[a, b] - \alpha^2 N^l[a, N^k b] + \alpha N^l[N^k a, b] - \alpha N^l N^k[a, b] \\
&+ \alpha N^l[a, N^k b] - \alpha^2 N^k[N^l a, b] + \alpha^2 N^k N^l[a, b] - \alpha^2 N^k[a, N^l b] + \alpha N^k[N^l a, b] \\
&- \alpha N^k N^l[a, b] + \alpha N^k[a, N^l b] + \alpha^2[N^l a, N^l b] + \alpha^2[N^k a, N^k b] + [N^k a, N^k b] \\
&- 2\alpha[N^k a, N^k b].
\end{aligned}$$

To see how the terms of our two equations compare, we require some further calculation. For the case where $k \leq l$ (or similarly where $k \geq l$) we use equation (7.10), the definition of the Nijenhuis bracket and part (1) of this theorem to see:

$$\begin{aligned} N^k[a,b]_{N^l} &= [N^k a, N^k b]_{N^{l-k}} = [N^k a, N^l b] - N^{l-k}[N^k a, N^k b] + [N^l a, N^k b], \\ &= [N^k a, N^l b] - N^l[N^k a, b] + N^{l+k}[a,b] - N^l[a, N^k b] + [N^l a, N^k b], \\ &= [N^k a, N^l b] - N^l[a,b]_k + [N^l a, N^k b], \end{aligned}$$

thus we can use the equality:

$$\begin{aligned} [N^k a, N^l b] + [N^l a, N^k b] = N^k([N^l a, b] - N^l[a,b] + [a, N^l b]) \\ + N^l([N^k a, b] - N^k[a,b] + [a, N^k b]), \end{aligned}$$

to see that we have satisfied the equality (7.11). □

Corollary 7.7. *Let N be a Nijenhuis operator on a Lie $\mathbb{K}$-affgebra $(A, [-,-])$. Then, for all $k, l \in \mathbb{N}$ the Lie brackets $[-,-]_{N^k}$ and $[-,-]_{N^l}$ are compatible in the sense that for all $\alpha \in \mathbb{K}$, the map:*

$$A \times A \to A, \qquad (a,b) \mapsto \alpha \triangleright_{[a,b]_{N^k}} [a,b]_{N^l}, \tag{7.12}$$

is a Lie bracket on A.

Proof. One easily checks that the operation (7.12) is the Nijenhuis bracket corresponding to $N_\alpha^{k,l}$ in Theorem (7.6), and hence it is a Lie bracket by the statement (3) of Theorem (7.6) and by Theorem 7.3. □

Example 7.8. Let A be an affine space and $\sigma \in \mathrm{Aff}(A)$. Take the Lie bracket $[a,b] = \sigma(a)$ of Example 6.3. An affine map $N : A \to A$ is a Nijenhuis operator for this bracket, provided $N \circ \sigma = \sigma \circ N$. The corresponding Nijenhuis bracket is $[a,b]_N = \sigma(Na)$. By Corollary 7.7, we have a family of pairwise compatible Lie structures that includes the original bracket, $[a,b]_{N^k} = \sigma(N^k a)$. Of course, this is not an exhaustive list of compatible structures. Let us take any $\xi \in \mathrm{Aff}(A)$ and consider $[a,b]_\xi = \xi(a)$. Then for all $a, b \in A$ and $\alpha \in \mathbb{K}$:

$$\alpha \triangleright_{[a,b]} [a,b]_\xi = \alpha \triangleright_{\sigma(a)} \xi(a) = (\alpha \triangleright_\sigma \xi)(a),$$

which is the Lie bracket of the type Example 6.3 with σ replaced by the affine homomorphism $\alpha \triangleright_\sigma \xi$.

Finally, we prove that Nijenhuis operators on Lie affgebras retract to Nijenhuis operators on Lie algebras. That is to say, the linearisation of a Nijenhuis operator on a Lie affgebra is a Nijenhuis operator on the corresponding Lie algebra.

Theorem 7.9. *Let $(A, [-,-])$ be a Lie affgebra and N a Nijenhuis operator. If we have an element $\nu \in A$, which acts as the neutral element of addition, then we have the Lie algebra $(A_\nu, [-,-]_\nu)$, as seen in Theorem 6.20. Let $N_\nu : A_\nu \to A_\nu$ be defined by:*

$$N_\nu a = \langle Na, N\nu, \nu \rangle = Na - N\nu,$$

then N_ν is a Nijenhuis operator on the Lie algebra $(A_\nu, [-,-]_\nu)$.

Proof. The function N_ν is a linearisation of the affine map N (see (2.13)) and hence a linear operator on the vector space A_ν. We must demonstrate that the following equality holds:

$$[N_\nu a, N_\nu b]_\nu = N_\nu \left([N_\nu a, b]_\nu - N_\nu[a, b]_\nu + [a, N_\nu b]_\nu\right). \tag{7.13}$$

Compute the left-hand side:

$$\begin{aligned}
[N_\nu a, N_\nu b]_\nu &= [N_\nu a, N_\nu b] - [N_\nu a, \nu] + [\nu, \nu] - [\nu, N_\nu b], \\
\\
&= [Na, Nb] - [Na, N\nu] + [Na, \nu] - [N\nu, Nb] + [N\nu, N\nu] - [N\nu, \nu] \\
&+ [\nu, Nb] - [\nu, N\nu] + [\nu, \nu] - [Na, \nu] + [N\nu, \nu] - [\nu, \nu] + [\nu, \nu] \\
&- [\nu, Nb] + [\nu, N\nu] - [\nu, \nu], \\
\\
&= [Na, Nb] - [Na, N\nu] + [N\nu, N\nu] - [N\nu, Nb].
\end{aligned}$$

Then compute the right-hand side:

$$\begin{aligned}
&N_\nu \left([N_\nu a, b]_\nu - N_\nu[a, b]_\nu + [a, N_\nu b]_\nu\right) = \\
\\
&= N_\nu([Na, b]_\nu - [N\nu, b]_\nu + [\nu, b]_\nu - N[a, b]_\nu + N\nu + [a, Nb]_\nu - [a, N\nu]_\nu + [a, \nu]_\nu), \\
\\
&= N([Na, b] - [Na, \nu] + [\nu, \nu] - [\nu, b] - [N\nu, b] + [N\nu, \nu] - [\nu, \nu] + [\nu, b] \\
&+ [\nu, b] - [\nu, \nu] + [\nu, \nu] - [\nu, b] - N[a, b] + N[a, \nu] - N[\nu, \nu] + N[\nu, b] \\
&+ [a, Nb] - [a, \nu] + [\nu, \nu] - [\nu, Nb] - [a, N\nu] + [a, \nu] - [\nu, \nu] + [\nu, N\nu] \\
&+ [a, \nu] - [a, \nu] + [\nu, \nu] - [\nu, \nu]) - N\nu, \\
\\
&= N([Na, b] - N[a, b] + N[a, Nb]) - N([Na, \nu] - N[a, \nu] + N[a, N\nu]) \\
&+ N([N\nu, \nu] - N[\nu, \nu] + N[\nu, N\nu]) - N([N\nu, b] - N[\nu, b] + N[\nu, Nb]).
\end{aligned}$$

We may then use the Nijenhuis condition, when comparing terms, to see that both sides of equation (7.13) are equal. □

Chapter 8

Conclusion & Further Works

8.1 Closing Remarks

If we want to solve problems that are unintuitive to us, then we should strive to create tools that also seem unintuitive at first glance - a strangely shaped key for a strangely shaped lock. In mathematics we should, in fact, be used to strange concepts unlocking further understanding of a subject area, be it the accidental utility of the once purely academic *imaginary numbers*, or the rich understanding we now have of the quantum world that behaves in such drastically different ways to things on the macro scale. It is for these reasons that I see a strong argument for the further study of trusses and their applications. The truss is a relatively new algebraic structure that demands further investigation and curiosity.

8.2 Recommended Further Reading

As mentioned in the opening chapters, this thesis is very much an expansion of two articles, *Affine Nijenhuis operators and Hochschild cohomology of trusses* [12], and *Lie and Nijenhuis Brackets on Affine Spaces* [13], with the latter being a continuation from the findings in the former.

However, in the interim between the publishing of those original two papers and the completion of this thesis, there have been other endeavours into these areas of algebra that further contribute to some of the ideas presented here.

Key papers to look into for further work would be: *Special normalised affine matrices. An example of a Lie affgebra* [9], *On matrix Lie affgebras* [14], and *Lie affgebras vis-à-vis Lie algebras* [3]. The following is a very brief summary of each.

Special normalised affine matrices. An example of a Lie affgebra [9] looks to how certain matrices can form Lie affgebras.

On matrix Lie affgebras [14] studies Lie affgebras on several classes of affine spaces of matrices.

Lie affgebras vis-à-vis Lie algebras [3] is a slightly longer paper. This shows how any Lie affgebra is isomorphic to a Lie algebra together with an element and a specific generalised derivation. That is to say, that this paper looks into the very relationship between Lie algebras and Lie affgebras in both directions.

8.3 Potential for Further Work

The potential for future work is vast, there are countless theorems and concepts that could be expanded to the general Lie affgebra case. An example of future work would be to look at work such as: *When Leibniz algebras are Nijenhuis?* [33] and expand Nijenhuis brackets (Definition 7.1) to Leibniz affgebras.

There is also potential in looking to some of the papers by K. Grabowska, J. Grabowski and P. Urbański, taking some of the results and extending them to the general Lie affgebra case.

Bibliography

[1] W. A. Adkins, S. H. Weintraub, *Algebra: An Approach via Module Theory*, Springer-Verlag New York, Inc., 1992.

[2] F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag New York, Inc., 1992.

[3] R. Andruszkiewicz, T. Brzeziński, K. Radziszewski, *Lie affgebras vis-à-vis Lie algebras*, Results Math. **80**, 61, 2025.

[4] R. Andruszkiewicz, T. Brzeziński, B. Rybołowicz, *Ideal Ring Extensions and Trusses*, J. Algebra, **600**, 237–278, 2022.

[5] R. Baer, *Zur Einführung des Scharbegriffs*. J. Reine Angew. Math. **160**, 199–207, 1929.

[6] S. Benenti, *Fibrés affines canoniques et mécanique newtonienne*. In Action hamiltoniennes de groupes. Troisième théorème de Lie (Lyon, 1986), **27** Travaux en Cours, Hermann, Paris, 13–37, 1988.

[7] S. Breaz, T. Brzeziński, B. Rybołowicz, P. Saracco, *Heaps of modules and affine spaces*, Ann. Mat. Pura Appl. **203**, 403–445, 2024.

[8] T. Brzeziński, *Lie trusses and heaps of Lie affebras*, Proc. Sci., **307**, PoS(CORFU2021), 2022.

[9] T. Brzeziński, *Special normalised affine matrices. An example of a Lie affgebra*, In Geometric Methods in Physics XL, Trends Math., Birkhäuser/Springer, Cham, 115–125, 2024.

[10] T. Brzeziński, *Trusses: between braces and rings*, Trans. Amer. Math. Soc. **372**, 4149–4176, 2019.

[11] T. Brzeziński, *Trusses: Paragons, ideals and modules*, J. Pure Appl. Algebra **224**, 106258, 2020.

[12] T. Brzeziński, J. Papworth, *Affine Nijenhuis operators and Hochschild cohomology of trusses*, SIGMA **19**, 056, 2023.

[13] T. Brzeziński, J. Papworth, *Lie and Nijenhuis brackets on affine spaces*, Bull. Belg. Math. Soc. Simon Stevin **30** 683–704, 2023.

[14] T. Brzeziński, K. Radziszewski, *On matrix Lie affgebras*, Contemp. Math. **826**, 2025.

[15] H. Bourlès, *Fundamentals of Advanced Mathematics 1*, ISTE-Elsevier, 2017.

[16] J.F. Cariñena, J. Grabowski, G. Marmo, *Quantum Bi-Hamiltonian Systems*, Int. J. Mod. Phys. A **15**, 4797–4810, 2000.

[17] J. Certaine, *The ternary operation* $(abc) = ab^{-1}c$ *of a group.* Bull. Amer. Math. Soc. **49**, 869–877, 1943.

[18] P. A. M. Dirac, *The Principles of Quantum Mechanics, 4th ed.*, Clarendon Press, Oxford, 1958.

[19] B. A. Dubrovin, G. Marmo, A. Simoni, *Alternative Hamiltonian description for quantum systems*, Mod. Phys. Lett. A, **5**, 1229–1234, 1990.

[20] S. Eilenberg, S. MacLane, *Cohomology Theory in Abstract Groups. I*, Ann. Math., **48**, 51–78, 1947.

[21] K. Erdmann, M.J. Wildon, *Introduction to Lie Algebras*, Springer Undergrad. Math. Ser., 2006.

[22] J. Gallier, *Curves and Surfaces in Geometric Modeling*, Morgan Kaufmann Publishing, 1999.

[23] M. Gerstenhaber, *On the Deformation of Rings and Algebras.* Ann. Math. **79**, 59–103, 1964.

[24] M. Gerstenhaber, *The cohomology structure of an associative ring.* Ann. Math. **78**, 267–288, 1963.

[25] K. Grabowska, J. Grabowski, P. Urbański, *AV-differential geometry: Euler-Lagrangian equations*, J. Geom. Phys. **57**, 1984–1998, 2007.

[26] K. Grabowska, J. Grabowski, P. Urbański, *AV-differential geometry: Poisson and Jacobi structures*, J. Geom. Phys. **52**, 398–446, 2004.

[27] K. Grabowska, J. Grabowski, P. Urbański, *Frame-independent mechanics: geometry on affine bundles*, Travaux mathematiques. Fasc. XVI, 107–120, *Trav. Math.*, **16**, Univ. Luxemb., Luxembourg, 2005.

[28] K. Grabowska, J. Grabowski, P. Urbański, *Lie brackets on affine bundles*, Ann. Global Anal. Geom. **24**, 101–130, 2003.

[29] G. Hochschild, *On the cohomology groups of an associative algebra.* Ann. Math. **46**, 58–67, 1945.

[30] N. Jacobson, *Basic Algebra II*, W. H. Freeman and Company, 1989.

[31] Y. Kosmann-Schwarzbach, *Nijenhuis structures on Courant algebroids.* Bull. Braz. Math. Soc. (N. S.), **42**, 625–649, 2011.

[32] Y. Kosmann-Schwarzbach, F. Magri, *Poisson-Nijenhuis structures*, Ann. Inst. Henri Poincaré **53**, 35–81, 1990.

[33] H. Li, T. Ma, S. Wang, *When Leibniz algebras are Nijenhuis?*, arXiv:2310.14267, 2024

[34] F. Magri, *A simple model of the integrable Hamiltonian equation.* J. Math. Phys. **19**, 1156–1162, 1978.

[35] D. Manchon, *A short survey on pre-Lie algebras*, ESI Lect. Math. Phys., Zürich, 89–102, 2011.

[36] E. Massa, S. Vignolo & D. Bruno, *Non-holonomic Lagrangian and Hamiltonian mechanics: an intrinsic approach.* J. Phys. A **35**, 6713–6742, 2002.

[37] A. Nijenhuis, R.W. Richardson Jr., *Deformations of Lie algebra structures*, J. Math. Mech. **17**, 89–105, 1967.

[38] F. Ostermann, J. Schmidt, *Der baryzentrische Kalkül als axiomatische Grundlage der affinen Geometrie*, J. Reine Angew. Math. **224**, 44–57, 1966.

[39] H. Prüfer, *Theorie der Abelschen Gruppen. I. Grundeigenschaften.* Math. Z. **20**, 165–187, 1924.

[40] W. Tulczyjew, *Frame independence of analytical mechanics*, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **119**, 273–279, 1985.

[41] P. Urbański, *Affine Poisson structures in analytical mechanics*, In Quantization and infinite-dimensional systems (Białowieża, 1993), Plenum, New York, 123–129, 1994.

[42] E.B. Vinberg, *The theory of homogeneous convex cones*, Trans. Moscow Math. Soc. **12**, 340–403, 1963.

[43] C.A. Weibel, *An Introduction to Homological Algebra*, Cambridge Stud. Adv. Math., 1994.

[44] A. Weinstein, *A universal phase space for particles in Yang-Mills fields*, Lett. Math. Phys. **2**, 417–420, 1978.

[45] S. J. Witherspoon, *Hochschild Cohomology for Algebras*, Amer. Math. Soc., 2020.