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ABSTRACT

Meixner [J. London Math. Soc. **s1-9**(1), 6–13 (1934)] proved that there exist exactly five classes of orthogonal Sheffer sequences: Hermite polynomials which are orthogonal with respect to Gaussian distribution, Charlier polynomials orthogonal with respect to Poisson distribution, Laguerre polynomials orthogonal with respect to gamma distribution, Meixner polynomials of the first kind, orthogonal with respect to negative binomial distribution, and Meixner polynomials of the second kind, orthogonal with respect to Meixner distribution. The Segal–Bargmann transform provides a unitary isomorphism between the L^2-space of the Gaussian distribution and the Fock or Segal–Bargmann space of entire functions. This construction was also extended to the case of the Poisson distribution. The present paper deals with the latter three classes of orthogonal Sheffer sequences. By using a set of nonlinear coherent states, we construct and study a generalized Segal–Bargmann transform which is a unitary isomorphism between the L^2-space of the orthogonality measure and a certain Fock space of entire functions. To derive our results, we use normal ordering in generalized Weyl algebras that are naturally associated with the orthogonal Sheffer sequences.

I. INTRODUCTION

Fock spaces play a fundamental role in quantum mechanics as well as in infinite-dimensional analysis and probability, both classical and noncommutative (quantum), see e.g., Refs. 13, 31, and 33. Roughly speaking, a symmetric Fock space is an infinite orthogonal sum of symmetric n-particle Hilbert spaces. There exists an alternative description of a symmetric Fock space as a space of holomorphic functions. Such a space is usually called the Segal–Bargmann space.

Let us briefly discuss the Segal–Bargmann construction in the one-dimensional case. Bargmann[9] defined a Hilbert space $\mathbb{F}(\mathbb{C})$ as the closure of polynomials over $\mathbb{C}$ in the L^2-space $L^2(\mathbb{C}, \nu)$. Here ν is the Gaussian measure on $\mathbb{C}$ given by $\nu(dz) = \pi^{-1} \exp(-|z|^2) dA(z)$, where $dA(z)$ is the Lebesgue measure on $\mathbb{C}$. The monomials $(z^n)_{n=0}^{\infty}$ form an orthogonal basis for $\mathbb{F}(\mathbb{C})$ with $(z^m, z^n)_{\mathbb{F}(\mathbb{C})} = n!\, \delta_{m,n}$. Here and below, $\delta_{m,n}$ denotes the Kronecker delta. The $\mathbb{F}(\mathbb{C})$ consists of entire functions $\varphi(z) = \sum_{n=0}^{\infty} f_n z^n$ that satisfy $\sum_{n=0}^{\infty} |f_n|^2\, n! < \infty$. The $\mathbb{F}(\mathbb{C})$ is a reproducing kernel Hilbert space with reproducing kernel $\mathbb{K}(z, w) = \sum_{n=0}^{\infty} (n!)^{-1} (\bar{z} w)^n$.

Let μ be the standard Gaussian distribution on $\mathbb{R}$ and let $(h_n)_{n=0}^{\infty}$ be the sequence of monic Hermite polynomials that form an orthogonal basis for $L^2(\mathbb{R}, \mu)$. The Segal–Bargmann transform is the unitary operator $\mathbb{S} : L^2(\mathbb{R}, \mu) \to \mathbb{F}(\mathbb{C})$ that satisfies $(\mathbb{S}\, h_n)(z) = z^n$. This operator has a representation through the coherent states:

$$\mathbb{E}(x, z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}\, h_n(x) = \exp\left(-\frac{1}{2}(z^2 - 2xz)\right), \quad x \in \mathbb{R},\ z \in \mathbb{C}.$$

More precisely, for $f \in L^2(\mathbb{R},\mu)$ and $z \in \mathbb{C}$, one has $(\mathbb{S}f)(z) = \int_{\mathbb{R}} f(x)\mathbb{E}(x,z)\,\mu(dx)$. For a fixed $z \in \mathbb{C}$, $\mathbb{E}(\cdot,z)$ is an eigenfunction of the lowering operator in $L^2(\mathbb{R},\mu)$ with eigenvalue z. More exactly, if we define the (unbounded) lowering operator ∂^- in $L^2(\mathbb{R},\mu)$ by $\partial^- h_n = nh_{n-1}$, then $\partial^-\mathbb{E}(\cdot,z) = z\,\mathbb{E}(\cdot,z)$. For z real, the operator $\mathbb{S}$ can also be written as

$$(\mathbb{S}f)(z) = \int_{\mathbb{R}} f(x+z)\,\mu(dx), \quad f \in L^2(\mathbb{R},\mu),\ z \in \mathbb{R}. \tag{1.1}$$

Let also ∂^+ denote the raising operator for the Hermite polynomials: $\partial^+ h_n = h_{n+1}$. Then, the operator of multiplication by the variable x in $L^2(\mathbb{R},\mu)$ has the form $\partial^+ + \partial^-$. Hence, under the Segal–Bargmann transform $\mathbb{S}$, this operator goes over to the operator $Z + D$, where Z is the multiplication by the variable z in $\mathbb{F}(\mathbb{C})$, and D is the differentiation in $\mathbb{F}(\mathbb{C})$. In this setting, the operators Z and D are adjoint of each other. Note that these operators satisfy the commutation relation $[D,Z] = 1$, hence they are generators of a Weyl algebra, see e.g., Ref. 29, Chap. 5.

The Segal–Bargmann transform for the Gaussian measure admits an extension to both the multivariate case[9] and an infinite-dimensional case, see e.g., Refs. 20 and 32, Sec. 3.3.

Asai *et al.*[8] constructed a counterpart of the Segal–Bargmann transform in the case of the Poisson distribution with parameter $\sigma > 0$: $\pi_\sigma(d\xi) = e^{-\sigma}\sum_{n=0}^{\infty}\frac{1}{n!}\,\sigma^n\,\delta_n(d\xi)$ (δ_n denoting the Dirac measure at n). Define the Gaussian measure ν_σ on $\mathbb{C}$ by

$$\nu_\sigma(dz) = \frac{1}{\pi\sigma}\,\exp\left(-\frac{|z|^2}{\sigma}\right)dA(z). \tag{1.2}$$

Let the Hilbert space $\mathbb{F}_\sigma(\mathbb{C})$ be the closure of polynomials over $\mathbb{C}$ in $L^2(\mathbb{C},\nu_\sigma)$. The monomials $(z^n)_{n=0}^{\infty}$ form an orthogonal basis for $\mathbb{F}_\sigma(\mathbb{C})$ with $(z^m,z^n)_{\mathbb{F}_\sigma(\mathbb{C})} = \sigma^n\,n!\,\delta_{n,m}$. The $\mathbb{F}_\sigma(\mathbb{C})$ consists of entire functions $\varphi(z) = \sum_{n=0}^{\infty} f_n\,z^n$ that satisfy $\sum_{n=0}^{\infty}|f_n|^2\,\sigma^n\,n! < \infty$. Let $(c_n)_{n=0}^{\infty}$ be the sequence of monic Charlier polynomials that form an orthogonal basis for $L^2(\mathbb{N}_0,\pi_\sigma)$ (here and below we denote $\mathbb{N}_0 = \{0,1,2,\dots\}$). The generalized Segal–Bargmann transform is a unitary operator $\mathbb{S} : L^2(\mathbb{N}_0,\pi_\sigma) \to \mathbb{F}_\sigma(\mathbb{C})$ satisfying $(\mathbb{S}c_n)(z) = z^n$. The corresponding coherent states are

$$\mathbb{E}(\xi,z) = \sum_{n=0}^{\infty}\frac{z^n}{n!\,\sigma^n}c_n(\xi) = e^{-z}\left(1+\frac{z}{\sigma}\right)^{\xi}, \quad \xi \in \mathbb{N}_0,\ z \in \mathbb{C}.$$

[In this paper, we always denote a (generalized) Segal–Bargmann transform by $\mathbb{S}$ and the corresponding coherent states by $\mathbb{E}(\cdot,\cdot)$. This should not lead to confusion, since it will always be clear from the context which particular choice of the distribution on $\mathbb{R}$ we are dealing with.] It holds that $\sigma\partial^-\mathbb{E}(\cdot,z) = z\,\mathbb{E}(\cdot,z)$, where ∂^- is the lowering operator for the Charlier polynomials $(c_n)_{n=0}^{\infty}$. Note that $\sigma\partial^-$ is the adjoint of the raising operator ∂^+ for the polynomials $(c_n)_{n=0}^{\infty}$.

A key difference with the Gaussian case is that, under the transformation $\mathbb{S}$, the operator of multiplication by the variable ξ goes over to the operator $\rho = \mathcal{U}\mathcal{V}$ in $\mathbb{F}_\sigma(\mathbb{C})$, where

$$\mathcal{U} = Z + \sigma, \quad \mathcal{V} = D + 1. \tag{1.3}$$

Note that the operators $\mathcal{U}$ and $\mathcal{V}$ still satisfy the commutation relation $[\mathcal{V},\mathcal{U}] = 1$, hence $\mathcal{U}$ and $\mathcal{V}$ generate a Weyl algebra.

Both Hermite polynomials $(h_n)_{n=0}^{\infty}$ and Charlier polynomials $(c_n)_{n=0}^{\infty}$ belong to the class of orthogonal Sheffer sequences. Recall that a monic polynomial sequence $(s_n)_{n=0}^{\infty}$ over $\mathbb{R}$ is called a Sheffer sequence if its (exponential) generating function is of the form

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\,s_n(x) = \exp\left[A(t) + xB(t)\right], \tag{1.4}$$

where $A(t)$ and $B(t)$ are formal power series over $\mathbb{R}$ satisfying $A(0) = B(0) = 0$ and $B'(0) = 1$.

Meixner[30] proved that there exist exactly five classes of orthogonal Sheffer sequences. In fact, a monic polynomial sequence $(s_n)_{n=0}^{\infty}$ is an orthogonal Sheffer sequence if and only if it satisfies the recurrence relation

$$xs_n(x) = s_{n+1}(x) + (\lambda n + l)s_n(x) + (\sigma n + \eta n(n-1))s_{n-1}(x), \tag{1.5}$$

where $\lambda \in \mathbb{R}$, $l \in \mathbb{R}$, $\sigma > 0$ and $\eta \geq 0$. The transformation of the constants $(\lambda,l) \mapsto (-\lambda,-l)$ corresponds to the push-forward of the orthogonality measure under the map $\mathbb{R} \ni x \mapsto -x \in \mathbb{R}$. Hence, we may assume that $\lambda \geq 0$. The constant l corresponds to the shift of the orthogonality measure by l, so it can be chosen appropriately, depending on the other three constants. It is also convenient to introduce parameters $\alpha,\beta \in \mathbb{C}$ that satisfy $\alpha + \beta = \lambda$, $\alpha\beta = \eta$. In the case of both Hermite and Charlier polynomials, we have $\eta = 0$.

In this paper, we will deal with the case $\eta > 0$, which corresponds to the other three classes of orthogonal Sheffer sequences. More exactly, for $\alpha = \beta > 0$ and $l = \sigma/\alpha$, we obtain the sequence of Laguerre polynomials which are orthogonal with respect to the following gamma distribution on $\mathbb{R}_+ = (0,\infty)$:

$$\mu_{\alpha,\alpha,\sigma}(dx) = \frac{1}{\Gamma\left(\frac{\sigma}{\eta}\right)}\,\alpha^{-\frac{\sigma}{\eta}}x^{-1+\frac{\sigma}{\eta}}\,e^{-\frac{x}{\alpha}}\,dx. \tag{1.6}$$

For $\alpha > \beta > 0$ and $l = \sigma/\alpha$, we obtain the sequence of Meixner polynomials of the first kind which are orthogonal with respect to the following negative binomial (Pascal) distribution on $(\alpha - \beta)\mathbb{N}_0$:

$$\mu_{\alpha,\beta,\sigma}(dx) = \left(1 - \frac{\beta}{\alpha}\right)^{\frac{\sigma}{\eta}} \sum_{n=0}^{\infty} \left(\frac{\beta}{\alpha}\right)^n \frac{1}{n!} \left(\frac{\sigma}{\eta}\right)^{(n)} \delta_{(\alpha-\beta)n}\ (dx). \tag{1.7}$$

Finally, for $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$ and $l = 0$, we obtain the sequence of Meixner polynomials of the second kind (or Meixner–Pollaczak polynomials) which are orthogonal with respect to the following Meixner distribution on $\mathbb{R}$:

$$\mu_{\alpha,\beta,\sigma}(dx) = C_{\alpha,\beta,\sigma}\ \exp\left(\frac{\left(\frac{\pi}{2} - \mathrm{Arg}\ (\alpha)\right)x}{\Im(\alpha)}\right) \left|\Gamma\left(\frac{ix}{2\Im(\alpha)} + \frac{i\sigma\beta}{2\eta\Im(\alpha)}\right)\right|^2 dx, \tag{1.8}$$

where $\mathrm{Arg}(\alpha) \in (0, \pi/2]$ and the constant $C_{\alpha,\beta,\sigma}$ is given by

$$C_{\alpha,\beta,\sigma} = \frac{\left(2\cos\left(\frac{\pi}{2} - \mathrm{Arg}\ (\alpha)\right)\right)^{\frac{\sigma}{\eta}}}{4\Im(\alpha)\pi\ \Gamma\left(\frac{\sigma}{\eta}\right)}\ \exp\left(\frac{\left(\frac{\pi}{2} - \mathrm{Arg}\ (\alpha)\right)\sigma\Re(\alpha)}{\Im(\alpha)\eta}\right). \tag{1.9}$$

The aim of the paper is to study a generalized Segal–Bargmann transform which is a unitary operator $\mathbb{S} : L^2(\mu_{\alpha,\beta,\sigma}) \to \mathbb{F}_{\eta,\sigma}(\mathbb{C})$ satisfying $(\mathbb{S}s_n)(z) = z^n$, where $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ is a Fock space of entire functions to be defined below. This Segal–Bargmann transform has been previously discussed by Feinsilver[15] and Asai[6,7] from the viewpoints of orthogonal polynomials, quantum probability, and representation theory. See also Refs. 4, 5, 23, and 24.

For $h \in \mathbb{C}$, let $((\cdot \mid h)_n)_{n=0}^{\infty}$ denote the sequence of generalized factorials with increment h,[21] i.e., for $z \in \mathbb{C}$, $(z \mid h)_0 = 1$ and

$$(z \mid h)_n = z(z-h)(z-2h)\cdots(z-(n-1)h), \quad n \in \mathbb{N}. \tag{1.10}$$

In particular, $(z \mid 1)_n = (z)_n$ is a falling factorial and $(z \mid -1)_n = (z)^{(n)}$ is a rising factorial. Note that the so-called h-derivative, $(D_h f)(z) = h^{-1}(f(z+h) - f(z))$, is the lowering operator for this polynomial sequence: $(D_h(\cdot \mid h)_n)(z) = n(z \mid h)_{n-1}$.

For $\sigma > 0$ and $\eta \geq 0$, we define $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ as the Hilbert space of entire functions $\varphi(z) = \sum_{n=0}^{\infty} f_n\, z^n$ that satisfy

$$\sum_{n=0}^{\infty} |f_n|^2\ n!\ (\sigma \mid -\eta)_n < \infty, \tag{1.11}$$

and $(z^m, z^n)_{\mathbb{F}_{\eta,\sigma}(\mathbb{C})} = \delta_{m,n}(\sigma \mid -\eta)_n\ n!$. Note that, for $\eta = 0$, we have $(\sigma \mid 0)_n = \sigma^n$ and so $\mathbb{F}_{0,\sigma}(\mathbb{C}) = \mathbb{F}_{\sigma}(\mathbb{C})$.

For general $\sigma > 0$ and $\eta > 0$, we prove that $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ is the closure of the polynomials over $\mathbb{C}$ in the L^2-space $L^2(\mathbb{C}, \lambda_{\eta,\sigma})$. Here $\lambda_{\eta,\sigma}$ is the random Gaussian measure ν_r [see formula (1.2)] where the random variable r (the variance of ν_r) is distributed according to the gamma distribution $\mu_{\eta,\eta,\eta\sigma}$. The $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ is a reproducing kernel Hilbert space with reproducing kernel $\mathbb{K}(z,w) = \sum_{n=0}^{\infty} \frac{(z\,w)^n}{n!\ (\sigma|-\eta)_n}$.

We note that Asai[6] derived a representation of the density of the measure $\lambda_{\eta,\sigma}$ which involves the modified Bessel function. Furthermore, it was shown in Ref. 6 that $\lambda_{\eta,\sigma}$ is the unique probability measure on $\mathbb{C}$ whose L^2-space contains the Hilbert space $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ as its subspace. In the case $\sigma = \eta = 1$, the space $\mathbb{F}_{1,1}(\mathbb{C})$ was also studied by Alpay *et al.*,[4] Sec. 9 and Alpay and Porat,[5] see also Refs. 23 and 24.

The generalized Segal–Bargmann transform $\mathbb{S} : L^2(\mu_{\alpha,\beta,\sigma}) \to \mathbb{F}_{\eta,\sigma}(\mathbb{C})$ admits a representation

$$(\mathbb{S}f)(z) = \int_{\mathbb{R}} f(x)\mathbb{E}(x,z)\ \mu_{\alpha,\beta,\sigma}(dx),$$

where

$$\mathbb{E}(x,z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\ (\sigma \mid -\eta)_n}\ s_n(x), \tag{1.12}$$

and $\mathbb{E}(\cdot, z) \in L^2(\mu_{\alpha,\beta,\sigma})$ for each $z \in \mathbb{C}$. Hence, $(\mathbb{E}(\cdot, z))_{z\in\mathbb{C}}$ are nonlinear coherent states corresponding to the sequence of numbers $\rho_n = n!(\sigma \mid -\eta)_n$ $(n \in \mathbb{N}_0)$. See e.g., Refs. 3, 18, and 37 for studies of nonlinear coherent states. For applications of (generalized) coherent states in physics, see e.g., Refs. 17 and 34.

In the special case where $\eta = 1$ and $\sigma = 2j$ with $j \in \{1, \frac{1}{2}, 2, \frac{2}{3}, \ldots\}$, we get $\rho_n = n!(2j)^{(n)}$. Nonlinear coherent states with such a choice of ρ_n are called the Barut–Girardello states,[10] see also Ref. 3, Sec. 1.1.3. Such states appeared in Ref. 10 in a study of coherent states associated with the Lie algebra of the group $SU(1,1)$. For the general choice of the parameters λ, η and σ, Feinsilver,[15] Secs. 1 and 3.8 obtained a representation of the function $\mathbb{E}(x,z)$ through a hypergeometric function.

We note that, for each $z \in \mathbb{C}$, $\mathbb{E}(\cdot, z)$ is an eigenfunction (belonging to the eigenvalue z) of the annihilation operator $\sigma\partial^- + \eta\partial^+(\partial^-)^2$, which is the adjoint of the operator ∂^+. Here ∂^+ and ∂^- are the raising and lowering operators for the Sheffer sequence $(s_n)_{n=0}^{\infty}$:

$$\partial^+ s_n = s_{n+1}, \quad \partial^- s_n = n s_{n-1} \quad n \in \mathbb{N}_0. \tag{1.13}$$

For each $\zeta \in \mathbb{C}$, we define a complex-valued Poisson measure on $\mathbb{N}_0$ with parameter ζ by

$$\pi_\zeta(d\xi) = e^{-\zeta} \sum_{n=0}^{\infty} \frac{1}{n!}\, \zeta^n\, \delta_n(d\xi). \tag{1.14}$$

We prove that the nonlinear coherent states can be written in the form $\mathbb{E}(x,z) = \int_{\mathbb{N}_0} \mathcal{E}(x,\beta\xi)\, \pi_{\frac{z}{\beta}}(d\xi)$, where

$$\mathcal{E}(x,\beta\xi) = \sum_{n=0}^{\infty} \frac{\beta^n(\xi)_n}{n!\,(\sigma \mid -\eta)_n}\, s_n(x),$$

and we derive explicit formulas for $\mathcal{E}(x,\beta\xi)$.

Furthermore, in the cases of the gamma distribution and the negative binomial distribution, we prove that, for each $f \in L^2(\mu_{\alpha,\beta,\sigma})$,

$$(\mathbb{S}f)(z) = \int_{\mathbb{N}_0} \int f(x)\, \mu_{\alpha,\beta,\eta\xi+\sigma}(dx)\, \pi_{\frac{z}{\beta}}(d\xi), \quad z \in \mathbb{C}.$$

In particular, for $z > 0$, $(\mathbb{S}f)(z)$ is the expectation of f with respect to the random measure $\mu_{\alpha,\beta,\eta\xi+\sigma}$, where the random variable ξ has Poisson distribution $\pi_{\frac{z}{\beta}}$. Similarly, in the case of the Meixner distribution, we show that

$$(\mathbb{S}f)(z) = \int_{\mathbb{N}_0} \int f(x+\beta\xi)\, \mu_{\alpha,\beta,\eta\xi+\sigma}(dx)\, \pi_{\frac{z}{\beta}}(d\xi), \quad z \in \mathbb{C}.$$

However, this formula holds only for functions f from $\mathcal{E}^1_{\min}(\mathbb{C})$, the space of entire functions of order at most 1 and minimal type.[19] [The set $\mathcal{E}^1_{\min}(\mathbb{C})$ is dense in $L^2(\mu_{\alpha,\beta,\sigma})$.] Note that, for $r > 0$, $(\mathbb{S}f)(\beta r)$ is the expectation of the function $f(x+\beta\xi)$ with respect to the probability measure $\mu_{\alpha,\beta,\eta\xi+\sigma}(dx)\pi_r(d\xi)$.

Similarly to the Gaussian and Poisson cases, under the generalized Segal–Bargmann transform $\mathbb{S}$, the operator of multiplication by the variable x in $L^2(\mu_{\alpha,\beta,\sigma})$ goes over to an operator in $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ that admits a representation through the operators Z and D.

Let us now briefly describe our strategy to prove these results. Let $\mathcal{P}(\mathbb{C})$ denote the vector space of polynomials over $\mathbb{C}$. Consider the polynomials s_n as elements of $\mathcal{P}(\mathbb{C})$ (with real coefficients), and consider ∂^+ and ∂^- as linear operators in $\mathcal{P}(\mathbb{C})$ defined by (1.13). Define linear operators U and V in $\mathcal{P}(\mathbb{C})$ by

$$U = \partial^+ + \beta\partial^+\partial^- + \frac{\sigma}{\alpha}, \quad V = \alpha\partial^- + 1. \tag{1.15}$$

Let also Z denote the operator of multiplication by variable z in $\mathcal{P}(\mathbb{C})$. In view of (1.5), we get, in the case $\alpha \geq \beta > 0$ (hence $l = \frac{\sigma}{\alpha}$): $Z = UV$. Similarly, in the case $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$ (hence $l = 0$), we have $Z + \frac{\sigma}{\alpha} = UV$. Since $[\partial^-, \partial^+] = 1$, the operators U and V satisfy the commutation relation

$$[V,U] = \beta V + (\alpha - \beta). \tag{1.16}$$

Hence, they generate a generalized Weyl algebra, see e.g., Ref. 29, Chap. 8 and the references therein.

Consider the linear bijective operator $\mathcal{S}$ in $\mathcal{P}(\mathbb{C})$ that satisfies $(\mathcal{S}s_n)(z) = (z \mid \beta)_n$ $(n \in \mathbb{N}_0)$, see (1.10). Define operators $\mathcal{U} = \mathcal{S}U\mathcal{S}^{-1}$ and $\mathcal{V} = \mathcal{S}V\mathcal{S}^{-1}$. An easy calculation shows that

$$\mathcal{U} = Z + \frac{\sigma}{\alpha}, \quad \mathcal{V} = \alpha D_\beta + 1, \tag{1.17}$$

compare with (1.3). Obviously, $\mathcal{U}$ and $\mathcal{V}$ also satisfy the commutation relation $[\mathcal{V},\mathcal{U}] = \beta\mathcal{V} + (\alpha - \beta)$. Hence, they also generate a generalized Weyl algebra. Compare it with Feinsilver's finite difference algebra.[14]

Let us remark that orthogonal Sheffer sequences with $\eta > 0$ already appeared in studies related to the square of white noise algebra, see e.g., Ref. 1 and the references therein. It was shown in Ref. 2 that the square of white noise algebra contains a subalgebra generated by elements fulfilling the relations of Feinsilver's finite difference algebra, see also Refs. 11 and 12. For further studies of Lie algebras related to orthogonal Sheffer sequences, see Refs. 6, 7, Appendix A, and 15.

Similarly to Katriel's theorem about the normal ordering in the Weyl algebra,[25] we discuss the normal (Wick) ordering for the operator $(UV)^n$ in terms of U^k and V^k, compare with Ref. 29, Sec. 8.2 and the references therein. This allows us to derive explicit formulas for $s_n(z)$ and a representation of monomials z^n through the polynomials $s_k(z)$. In these formulas, we use Stirling numbers and Lah numbers. As a corollary, we find useful formulas for the moments of the orthogonality measure $\mu_{\alpha,\beta,\sigma}$. These results are presented in the Appendix, and the reader may find them of independent interest.

We explicitly construct an open unbounded domain $\mathcal{D}_{\alpha,\beta,\sigma}$ in $\mathbb{C}$ that contains 0. We define a reproducing kernel Hilbert space $\mathcal{F}_{\alpha,\beta,\sigma}$ of analytic functions on $\mathcal{D}_{\alpha,\beta,\sigma}$ that have representation $\varphi(z) = \sum_{n=0}^{\infty} f_n(z \mid \beta)_n$ with coefficients $f_n \in \mathbb{C}$ satisfying (1.11). We extend $\mathcal{S}$ to a unitary operator $\mathcal{S} : L^2(\mu_{\alpha,\beta,\sigma}) \to \mathcal{F}_{\alpha,\beta,\sigma}$ that satisfies $(\mathcal{S}s_n)(z) = (z \mid \beta)_n$. Thus, under the unitary operator $\mathcal{S}$, the operator of multiplication

by the variable x in $L^2(\mu_{\alpha,\beta,\sigma})$ goes over to the operator $\mathcal{U}\mathcal{V}$ in $\mathcal{F}_{\alpha,\beta,\sigma}$ for $\alpha \geq \beta > 0$ and to the operator $\mathcal{U}\mathcal{V} - \frac{\sigma}{\alpha}$ for $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$. We study the unitary operator $\mathcal{S}$ by using the results obtained through the normal ordering in the generalized Weyl algebras.

Next, we construct a unitary operator $\mathbb{T} : \mathcal{F}_{\alpha,\beta,\sigma} \to \mathbb{F}_{\sigma,\eta}(\mathbb{C})$ that satisfies

$$(\mathbb{T}(\cdot \mid \beta)_n)(z) = z^n, \quad n \in \mathbb{N}_0.$$

We prove that this operator has a representation

$$(\mathbb{T}f)(z) = \int_{\mathbb{N}_0} f(\beta\xi)\, \pi_{\frac{z}{\beta}}(d\xi), \quad f \in \mathcal{F}_{\alpha,\beta,\sigma}.\ z \in \mathbb{C}. \tag{1.18}$$

Finally, we use that $\mathbb{S} = \mathbb{T}\mathcal{S}$.

As a consequence of our considerations, we also derive explicit formulas for the action of the operators U and V, defined by (1.15). Compare with Ref. 28, Sec. 4.

The paper is organized as follows. In Sec. II, we define and discuss the Fock space $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ and the topological space of entire functions $\mathcal{E}^1_{\min}(\mathbb{C})$. In Sec. III, we present our main results. In Sec. IV, we present the proofs of the main results. Finally, in the Appendix, we discuss the normal ordering in the generalized Weyl algebra generated by operators U, V satisfying the commutation relation $[V, U] = aV + b$ with $a, b \in \mathbb{C}$. We apply the obtained result to an orthogonal Sheffer sequences $(s_n)_{n=0}^{\infty}$ and find useful formulas for the moments of its orthogonality measure.

We expect that the key ideas of this paper can be extended to an infinite-dimensional setting, compare with Ref. 28. This will be a topic of our future research.

II. THE SPACES $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ AND $\mathcal{E}^1_{\min}(\mathbb{C})$

For $\eta > 0$ and $\sigma \geq 0$, we denote by $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ the vector space of all entire functions $\varphi : \mathbb{C} \to \mathbb{C}$, $\varphi(z) = \sum_{n=0}^{\infty} f_n z^n$ with coefficients $f_n \in \mathbb{C}$ $(n \in \mathbb{N}_0)$ satisfying (1.11). Consider $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ as a Hilbert space equipped with the inner product $(\varphi, \psi)_{\mathbb{F}_{\eta,\sigma}(\mathbb{C})} = \sum_{n=0}^{\infty} f_n\, \overline{g_n}\, n!\, (\sigma \mid -\eta)_n$ for $\varphi(z) = \sum_{n=0}^{\infty} f_n z^n$, $\psi(z) = \sum_{n=0}^{\infty} g_n z^n \in \mathbb{F}_{\eta,\sigma}(\mathbb{C})$. This is a reproducing kernel Hilbert space with reproducing kernel $\mathbb{K}(z, w) = \sum_{n=0}^{\infty} \frac{(\bar{z}\, w)^n}{n!\, (\sigma \mid -\eta)_n}$, i.e., for each $\varphi \in \mathbb{F}_{\eta,\sigma}(\mathbb{C})$, we have $(\varphi, \mathbb{K}(z, \cdot))_{\mathbb{F}_{\eta,\sigma}(\mathbb{C})} = \varphi(z)$.

Consider the following gamma distribution on $\mathbb{R}_+$:

$$\mu_{\eta,\eta,\,\eta\sigma}(dr) = \frac{1}{\Gamma\left(\frac{\sigma}{\eta}\right)} \left(\frac{1}{\eta}\right)^{\frac{\sigma}{\eta}} r^{-1+\frac{\sigma}{\eta}} e^{-\frac{r}{\eta}}\, dr.$$

Let $\lambda_{\eta,\sigma}$ be the random Gaussian measure ν_r [see formula (1.2)] where the random variable r is distributed according to $\mu_{\eta,\eta,\eta\sigma}$, i.e.,

$$\lambda_{\eta,\sigma}(dz) = \int_{\mathbb{R}_+} \nu_r(dz)\, \mu_{\eta,\eta,\,\eta\sigma}(dr) = \Lambda_{\eta,\sigma}(z)\, A(dz), \tag{2.1}$$

where

$$\Lambda_{\eta,\sigma}(z) = \frac{1}{\pi\, \Gamma\left(\frac{\sigma}{\eta}\right)} \left(\frac{1}{\eta}\right)^{\frac{\sigma}{\eta}} \int_{\mathbb{R}_+} \exp\left(-\frac{|z|^2}{r} - \frac{r}{\eta}\right) r^{-2+\frac{\sigma}{\eta}}\, dr. \tag{2.2}$$

In the following proposition, we will use the modified Bessel function

$$K_\theta(x) = \frac{\pi}{2\, \sin(\theta\pi)} (I_{-\theta}(x) - I_\theta(x)),$$

where

$$I_\theta(x) = \left(\frac{x}{2}\right)^{\theta} \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{n!\, \Gamma(\theta + n + 1)}.$$

In these formulas, the parameter θ is assumed to be not an integer. When θ is an integer, the limit is used to define $K_\theta(x)$.

Proposition 2.1. *Let $\eta > 0$ and $\sigma > 0$. Then $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ is the closed subspace of $L^2(\mathbb{C}, \lambda_{\eta,\sigma})$ constructed as the closure of $\mathcal{P}(\mathbb{C})$. Furthermore,*

$$\Lambda_{\eta,\sigma}(z) = \frac{2\eta^{-\frac{1}{2}\left(1+\frac{\sigma}{\eta}\right)}}{\pi\Gamma\left(\frac{\sigma}{\eta}\right)} |z|^{\frac{\sigma}{\eta}-1} K_{1-\frac{\sigma}{\eta}}\left(2\eta^{-\frac{1}{2}}|z|\right). \tag{2.3}$$

Proof. Recall that, for $m, n \in \mathbb{N}_0$, we have $\int_{\mathbb{C}} z^m\, \overline{z^n}\, \nu_r(dz) = \delta_{n,m}\, r^n\, n!$. By formula (A13) in the Appendix, we get $\int_{\mathbb{R}_+} r^n\, \mu_{\alpha,\alpha,\sigma}(dr) = \left(\frac{\sigma}{\alpha} \mid -\alpha\right)_n$. Hence, by (2.1),

$$\begin{aligned}\int_{\mathbb{C}} z^m\, \overline{z^n}\, \lambda_{\eta,\sigma}(dz) &= \int_{\mathbb{R}_+} \int_{\mathbb{C}} z^m\, \overline{z^n}\, \nu_r(dz)\, \mu_{\eta,\eta,\,\eta\sigma}(dr) \\ &= \delta_{m,n}\, n! \int_{\mathbb{R}_+} r^n\, \mu_{\eta,\eta,\,\eta\sigma}(dr) = \delta_{m,n}\, n! (\sigma \mid -\eta)_n.\end{aligned}$$

Formula (2.3) for the density $\Lambda_{\eta,\sigma}(z)$ of the measure $\lambda_{\eta,\sigma}$ was proved by Asai,[6] Theorem 3.1. □

Remark 2.2. In fact, $\lambda_{\eta,\sigma}$ is the unique probability measure on $\mathbb{C}$ which satisfies

$$\int_{\mathbb{C}} z^m\, \overline{z^n}\, \lambda_{\eta,\sigma}(dz) = \delta_{m,n}\, n! (\sigma \mid -\eta)_n,$$

see Ref. 6, Theorem 3.1.

Following Ref. 5, let us recall some basic facts about the Mellin transform and the Mellin convolution. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be such that, for some interval $(a, b) \subset \mathbb{R}$, the function $f(r) r^{c-1}$ is integrable on $\mathbb{R}_+$ for all $c \in (a, b)$. Then the Mellin transform of f is defined by $\mathcal{M}(f)(c) = \int_{\mathbb{R}_+} r^{c-1} f(r)\, dr$ for $c \in (a, b)$. Obviously, for $\eta > 0$ and $f(r) = e^{-r/\eta}$, we have $\mathcal{M}(f)(c) = \eta^c\, \Gamma(c)$ for $c > 0$. The Mellin convolution of functions f and g is the function $f * g$ that satisfies $\mathcal{M}(f * g)(c) = \mathcal{M}(f)(c)\, \mathcal{M}(g)(c)$. Explicitly, the function $f * g$ is given by

$$(f * g)(r) = \int_{\mathbb{R}_+} f\left(\frac{r}{t}\right) g(t)\, \frac{1}{t}\, dt = \int_{\mathbb{R}_+} f(t) g\left(\frac{r}{t}\right) \frac{1}{t}\, dt, \quad r > 0. \tag{2.4}$$

Lemma 2.3. Assume that $\eta = \sigma$. Then the function $\Lambda_{\sigma,\sigma}$ in Proposition 2.1 has the form $\Lambda_{\sigma,\sigma}(z) = (\pi\sigma)^{-1} \psi(|z|^2)$, where $\psi(r) = (f_1 * f_2)(r)$ with $f_1(r) = e^{-r}$ and $f_2(r) = e^{-r/\sigma}$.

Proof. Immediate by formulas (2.2) and (2.4). □

Remark 2.4. In the special case $\eta = \sigma = 1$, the statement of Lemma 2.3 was proved in Refs. 4 and 5. By Ref. 5, p. 5, $\psi(r) = \int_{\mathbb{R}} \exp\left(-\sqrt{r}\, 2 \cosh(x)\right) dx$ is a modified Bessel function of the second kind.

Let $\varphi : \mathbb{C} \to \mathbb{C}$ be an entire function. One says that φ is of order at most 1 and minimal type (when the order is equal to 1) if φ satisfies

$$\sup_{z \in \mathbb{C}} |\varphi(z)| \exp\left(-t\, |z|\right) < \infty \quad \forall t > 0.$$

One denotes by $\mathcal{E}^1_{\min}(\mathbb{C})$ the vector space of all such functions.

For each $t > 0$, $\|\varphi\|_t = \sup_{z \in \mathbb{C}} |\varphi(z)| \exp\left(-t\, |z|\right)$ is a norm on $\mathcal{E}^1_{\min}(\mathbb{C})$, and denote by B_t the completion of $\mathcal{E}^1_{\min}(\mathbb{C})$ in this norm. For any $0 < t_1 < t_2$, the Banach space B_{t_1} is continuously embedded into B_{t_2}. Note that, as a set, $\mathcal{E}^1_{\min}(\mathbb{C}) = \bigcap_{t>0} B_t$. One defines the projective topology on $\mathcal{E}^1_{\min}(\mathbb{C})$ induced by the B_t spaces, i.e., one chooses the coarsest locally convex topology on $\mathcal{E}^1_{\min}(\mathbb{C})$ for which the embedding of $\mathcal{E}^1_{\min}(\mathbb{C})$ into B_t is continuous for each $t > 0$. Equipped with this topology, $\mathcal{E}^1_{\min}(\mathbb{C})$ is a Fréchet space. The following theorem is proved by Grabiner,[19] see also Ref. 16.

Theorem 2.5 (Ref. 19). *Let $(s_n)_{n=0}^{\infty}$ be a Sheffer sequence with generating function (1.4). Assume that the formal power series $A(t)$ and $B(t)$ in (1.4) determine analytic functions in a neighborhood of zero. Then the following statements hold.*

(i) *An entire function $\varphi : \mathbb{C} \to \mathbb{C}$ belongs to $\mathcal{E}^1_{\min}(\mathbb{C})$ if and only if it can be represented in the form*

$$\varphi(z) = \sum_{n=0}^{\infty} f_n\, s_n(z), \tag{2.5}$$

where $\sum_{n=0}^{\infty} |f_n|^2\, (n!)^2\, 2^{nk} < \infty$ for all $k \in \mathbb{N}$. The representation of the function φ as in (2.5) is unique, and the series on the right-hand side of formula (2.5) converges in $\mathcal{E}^1_{\min}(\mathbb{C})$.

(ii) *For each $k \in \mathbb{N}$, denote by $\mathcal{H}_k$ the completion of $\mathcal{E}^1_{\min}(\mathbb{C})$ in the Hilbertian norm $|||\varphi|||_k = \left(\sum_{n=0}^{\infty} |f_n|^2\, (n!)^2\, 2^{nk}\right)^{1/2}$, where f_n $(n \in \mathbb{N}_0)$ are the coefficient from (2.5). Then, $\mathcal{E}^1_{\min}(\mathbb{C})$ is the projective limit of the $\mathcal{H}_k$ spaces.*

Corollary 2.6. *(i) For each $\eta \geq 0$ and $\sigma > 0$, the Fréchet space $\mathcal{E}^1_{\min}(\mathbb{C})$ is continuously embedded into $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$.*

(ii) Let $(s_n)_{n=0}^{\infty}$ be an orthogonal Sheffer sequence and let $\mu_{\alpha,\beta,\sigma}$ be its orthogonality measure. Then the Fréchet space $\mathcal{E}^1_{\min}(\mathbb{C})$ is continuously embedded into $L^2(\mu_{\alpha,\beta,\sigma})$. Furthermore, $\mathcal{E}^1_{\min}(\mathbb{C})$ is a dense subset of $L^2(\mu_{\alpha,\beta,\sigma})$.

Proof. (i) The sequence of monomials $(z^n)_{n=0}^{\infty}$ is a Sheffer sequence for which $A(t) = 0$ and $B(t) = t$, hence it satisfies the conditions of Theorem 2.5. Therefore, the statement follows from the definition of $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ and Theorem 2.5.

(ii) It follows from Ref. 30 that each orthogonal Sheffer sequence satisfies the conditions of Theorem 2.5. Next, it follows from the recurrence formula (1.5) that $\|s_n\|^2_{L^2(\mu_{\alpha,\beta,\sigma})} = n!\,(\sigma \mid -\eta)_n$. Hence, $\varphi \in L^2(\mu_{\alpha,\beta,\sigma})$ if and only if $\varphi(x) = \sum_{n=0}^{\infty} f_n s_n(x)$ with f_n satisfying (1.11), and the series $\sum_{n=0}^{\infty} f_n s_n(x)$ converges in $L^2(\mu_{\alpha,\beta,\sigma})$. Since $(\sigma \mid -\eta)_n \leq n!(\min\{\eta,\sigma\})^n$, the statement follows from Theorem 2.5.

□

III. MAIN RESULTS

Let $\sigma > 0$. We assume that either $\alpha \geq \beta > 0$ and $l = \sigma/\alpha$ or $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$, and $l = 0$. Let $(s_n)_{n=0}^{\infty}$ be the Sheffer sequence satisfying the recurrence formula (1.5), and let $\mu_{\alpha,\beta,\sigma}$ be its orthogonality measure. We denote by $X_{\alpha,\beta}$ the support of $\mu_{\alpha,\beta,\sigma}$, i.e., $X_{\alpha,\beta} = \mathbb{R}_+$ if $\alpha = \beta > 0$, $X_{\alpha,\beta} = (\alpha - \beta)\mathbb{N}_0$ if $\alpha > \beta > 0$, and $X_{\alpha,\beta} = \mathbb{R}$ if $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$.

We define a generalized Segal–Bargmann transform $\mathbb{S} : L^2(X_{\alpha,\beta}, \mu_{\alpha,\beta,\sigma}) \to \mathbb{F}_{\eta,\sigma}(\mathbb{C})$ as a unitary operator satisfying $(\mathbb{S}s_n)(z) = z^n$ for $n \in \mathbb{N}_0$.

Theorem 3.1. *The generalized Segal–Bargmann transform $\mathbb{S}$ has a representation through the nonlinear coherent states*

$$\mathbb{E}(x,z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,(\sigma \mid -\eta)_n}\, s_n(x), \quad x \in X_{\alpha,\beta},\ z \in \mathbb{C}, \tag{3.1}$$

i.e., $\mathbb{E}(\cdot, z) \in L^2(X_{\alpha,\beta}, \mu_{\alpha,\beta,\sigma})$ for each $z \in \mathbb{C}$ and

$$(\mathbb{S}f)(z) = \int_{X_{\alpha,\beta}} f(x)\, \mathbb{E}(x,z)\, \mu_{\alpha,\beta,\sigma}(dx), \quad z \in \mathbb{C}. \tag{3.2}$$

Furthermore, if $\alpha = \beta > 0$,

$$\mathbb{E}(x,z) = \int_{\mathbb{N}_0} \left[(\sigma/\eta)^{(\xi)}\right]^{-1} \left(\frac{x}{\alpha}\right)^{\xi} \pi_{\frac{z}{\alpha}}(d\xi), \quad x \in \mathbb{R}_+,\ z \in \mathbb{C}, \tag{3.3}$$

if $\alpha > \beta > 0$,

$$\mathbb{E}((\alpha-\beta)n, z) = \int_{\mathbb{N}_0} \left(1 - \frac{\beta}{\alpha}\right)^{\xi} \frac{(\xi\eta + \sigma \mid -\eta)_n}{(\sigma \mid -\eta)_n}\, \pi_{\frac{z}{\beta}}(d\xi), \quad n \in \mathbb{N}_0,\ z \in \mathbb{C}, \tag{3.4}$$

and if $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$,

$$\mathbb{E}(x,z) = \int_{\mathbb{N}_0} \left(2\cos\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\right)^{\xi} \left((\sigma/\eta)^{(\xi)}\right)^{-1} \exp\left(i\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\xi\right) \left(-\frac{ix}{2\Im(\alpha)} - \frac{i\sigma\alpha}{2\eta\Im(\alpha)}\right)^{(\xi)} \pi_{\frac{z}{\beta}}(d\xi), \quad x \in \mathbb{R},\ z \in \mathbb{C}, \tag{3.5}$$

where $\operatorname{Arg}(\alpha) \in [0, \pi/2)$.

Let ∂^+ and ∂^- denote the raising and lowering operators for the Sheffer sequence $(s_n)_{n=0}^{\infty}$, see (1.13). Denote $A^- = \sigma\partial^- + \eta\partial^+(\partial^-)^2$.

Corollary 3.2. *For any $p, q \in \mathcal{P}(\mathbb{C})$, $(\partial^+ p, q)_{L^2(\mu_{\alpha,\beta,\sigma})} = (p, A^- q)_{L^2(\mu_{\alpha,\beta,\sigma})}$ and the operator A^- with domain $\mathcal{P}(\mathbb{C})$ is closable in $L^2(\mu_{\alpha,\beta,\sigma})$. Keep the notation A^- for the closure of A^-. Then, for each $z \in \mathbb{C}$, $\mathbb{E}(\cdot, z)$ is an eigenvector of A^- belonging to the eigenvalue z.*

For $\alpha \geq \beta > 0$ and $z \in \mathbb{C}$, we define a complex-valued measure $\rho_{\alpha,\beta,\sigma,z}$ on $X_{\alpha,\beta}$ by

$$\rho_{\alpha,\beta,\sigma,z}(dx) = \int_{\mathbb{N}_0} \mu_{\alpha,\beta,\eta\xi+\sigma}(dx)\, \pi_{\frac{z}{\beta}}(d\xi). \tag{3.6}$$

In particular, if $z > 0$, $\rho_{\alpha,\beta,\sigma,z}$ is the random measure $\mu_{\alpha,\beta,\eta\xi+\sigma}$, where the random variable ξ has Poisson distribution $\pi_{\frac{z}{\beta}}$.

Theorem 3.3. *Let $\alpha \geq \beta > 0$. For each $f \in L^2(X_{\alpha,\beta}, \mu_{\alpha,\beta,\sigma})$,*

$$(\mathbb{S}f)(z) = \int_{X_{\alpha,\beta}} f(x)\, \rho_{\alpha,\beta,\sigma,z}(dx), \quad z \in \mathbb{C}. \tag{3.7}$$

In the case where α and β have non-zero imaginary part, a counterpart of Theorem 3.3 has the following form.

Theorem 3.4. *Let $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$. The operator $\mathbb{S}$, considered as a linear operator in $\mathcal{P}(\mathbb{C})$, admits an extension to a continuous linear operator $\mathbb{S}$ in $\mathcal{E}^1_{\min}(\mathbb{C})$, and for each $f \in \mathcal{E}^1_{\min}(\mathbb{C})$ and $z \in \mathbb{C}$,*

$$(\mathbb{S}f)(z) = \int_{\mathbb{N}_0} \int_{\mathbb{R}} f(x + \beta\xi)\, \mu_{\alpha,\beta,\eta\xi+\sigma}(dx)\, \pi_{\frac{z}{\beta}}(d\xi).$$

In particular, for each $r > 0$,

$$(\mathbb{S}f)(\beta r) = \int_{\mathbb{N}_0} \int_{\mathbb{R}} f(x + \beta\xi)\, \mu_{\alpha,\beta,\eta\xi+\sigma}(dx)\, \pi_r(d\xi).$$

Remark 3.5. Using the approach to the generalized Segal–Bargmann transform developed in this paper, one can easily show that, in the case of the monic Charlier polynomials $(c_n)_{n=0}^{\infty}$ that are orthogonal with respect to the Poisson distribution π_σ ($\sigma > 0$, $\alpha = 1$, $\beta = 0$, $l = \sigma$), the corresponding Segal–Bargmann transform $\mathbb{S} : L^2(\mathbb{N}_0, \pi_\sigma) \to \mathbb{F}_\sigma(\mathbb{C})$, satisfying $\mathbb{S}c_n = z^n$ ($n \in \mathbb{N}_0$), admits the following representation:

$$(\mathbb{S}f)(z) = \int_{\mathbb{N}_0} f(x)\, \pi_{\sigma+z}(dx), \quad f \in L^2(\mathbb{N}_0, \pi_\sigma),\ z \in \mathbb{C},$$

compare with formula (1.1), which holds in the Gaussian case. Note that, for $z \in (-\sigma, +\infty)$, $\pi_{\sigma+z}$ is the (usual) Poisson distribution with parameter $\sigma + z$.

We define linear operators $\mathbb{U} = Z + \beta ZD + \frac{\sigma}{\alpha}$ and $\mathbb{V} = \alpha D + 1$, acting in $\mathcal{P}(\mathbb{C})$ and satisfying $[\mathbb{V}, \mathbb{U}] = \beta\mathbb{V} + (\alpha - \beta)$. We also define the operator

$$\rho = \mathbb{U}\mathbb{V} = Z + \lambda ZD + \frac{\sigma}{\alpha} + \sigma D + \eta ZD^2.$$

Proposition 3.6. The operator ρ is essentially self-adjoint in $\mathbb{F}_{\eta,\sigma}(\mathbb{C})$ and we keep the notation ρ for its closure. If $\alpha \geq \beta > 0$, then $\mathbb{S}\rho\mathbb{S}^{-1}$ is the operator of multiplication by the variable x in $L^2(X_{\alpha,\beta}, \mu_{\alpha,\beta,\sigma})$. If $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$ and $\beta = \overline{\alpha}$, then $\mathbb{S}(\rho - \frac{\sigma}{\alpha})\mathbb{S}^{-1}$ is the operator of multiplication by the variable x in $L^2(\mathbb{R}, \mu_{\alpha,\beta,\sigma})$.

The proof of the above statements will be based on Lemmas 3.7–3.11 below.

Similarly to (1.14), we will now define a complex-valued measure $\mu_{\alpha,\beta,\zeta}$ for a complex parameter ζ. First, we define a domain $\mathfrak{D}_{\alpha,\beta}$ in $\mathbb{C}$ as follows. If $\alpha > \beta > 0$, we define $\mathfrak{D}_{\alpha,\beta} = \mathbb{C}$, if either $\alpha = \beta > 0$ or $\Re(\alpha) = 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$, we define $\mathfrak{D}_{\alpha,\beta} = \{z \in \mathbb{C} \mid \Re(z) > 0\}$, and if $\Re(\alpha) > 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$, we define

$$\mathfrak{D}_{\alpha,\beta} = \{z \in \mathbb{C} \mid \Re(z) > 0,\ |\Im(z)| < \Re(z)\Im(\alpha)/\Re(\alpha)\}. \tag{3.8}$$

Now, if $\alpha \geq \beta > 0$ and $\zeta \in \mathfrak{D}_{\alpha,\beta}$, we define the complex-valued measure $\mu_{\alpha,\beta,\zeta}$ on $X_{\alpha,\beta}$ by replacing the positive parameter σ in formulas (1.6) and (1.7) with ζ. Next, if $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$, we use the formula $\overline{\Gamma(z)} = \Gamma(\overline{z})$ for $z \in \mathbb{C}$, $\Re(z) > 0$, to write formula (1.8) in the form

$$\mu_{\alpha,\beta,\sigma}(dx) = C_{\alpha,\beta,\sigma}\ \exp\left((\pi/2 - \operatorname{Arg}(\alpha))x/\Im(\alpha)\right)\Gamma\left(\frac{ix}{2\Im(\alpha)} + \frac{i\sigma\beta}{2\eta\Im(\alpha)}\right)\Gamma\left(-\frac{ix}{2\Im(\alpha)} - \frac{i\sigma\alpha}{2\eta\Im(\alpha)}\right)dx. \tag{3.9}$$

Now, for $\zeta \in \mathfrak{D}_{\alpha,\beta}$, we define the complex-valued measure $\mu_{\alpha,\beta,\zeta}$ on $\mathbb{R}$ by replacing the positive parameter σ in formulas (1.9) and (3.9) with ζ.

Furthermore, we define an open domain $\mathcal{D}_{\alpha,\beta,\sigma}$ in $\mathbb{C}$ as follows. If $|\alpha| = |\beta|$ [i.e., either $\alpha = \beta > 0$ or $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$],

$$\mathcal{D}_{\alpha,\beta,\sigma} = \{z \in \mathbb{C} \mid \Re(\alpha z) > -\sigma/2\}, \tag{3.10}$$

and if $\alpha > \beta > 0$, $\mathcal{D}_{\alpha,\beta,\sigma} = \mathbb{C}$. We will use below the following obvious observation: for each $z \in \mathcal{D}_{\alpha,\beta,\sigma}$ and $n \in \mathbb{N}$, $z + \beta n \in \mathcal{D}_{\alpha,\beta,\sigma}$. In particular, $\beta\mathbb{N}_0 \subset \mathcal{D}_{\alpha,\beta,\sigma}$.

Lemma 3.7. Let $(f_n)_{n=0}^{\infty}$ be a sequence of complex numbers such that (1.11) holds. Then the series $\sum_{n=0}^{\infty} f_n (z \mid \beta)_n$ converges uniformly on compact sets in $\mathcal{D}_{\alpha,\beta,\sigma}$, hence it is a holomorphic function on $\mathcal{D}_{\alpha,\beta,\sigma}$. Denote by $\mathcal{F}_{\alpha,\beta,\sigma}$ the vector space of all holomorphic functions on $\mathcal{D}_{\alpha,\beta,\sigma}$ that have representation

$$\varphi(z) = \sum_{n=0}^{\infty} f_n \, (z \mid \beta)_n, \tag{3.11}$$

with $(f_n)_{n=0}^{\infty}$ satisfying (1.11). Then

$$f_n = \frac{1}{n!}(D_\beta^n \varphi)(0) = \frac{(-1)^n}{n!\,\beta^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \varphi(\beta k). \tag{3.12}$$

In particular, a function $\varphi \in \mathcal{F}_{\alpha,\beta,\sigma}$ has a unique representation (3.11), and φ is completely determined by its values on the set $\beta\mathbb{N}_0$.

Let us consider $\mathcal{F}_{\alpha,\beta,\sigma}$ as a Hilbert space equipped with the inner product $(\varphi, \psi)_{\mathcal{F}_{\alpha,\beta,\sigma}} = \sum_{n=0}^{\infty} f_n \, \overline{g_n} \, n! \, (\sigma \mid -\eta)_n$ for $\varphi(z) = \sum_{n=0}^{\infty} f_n (z \mid \beta)_n$, $\psi(z) = \sum_{n=0}^{\infty} g_n (z \mid \beta)_n \in \mathcal{F}_{\alpha,\beta,\sigma}$. Let $\mathcal{S} : L^2(X_{\alpha,\beta}, \mu_{\alpha,\beta,\sigma}) \to \mathcal{F}_{\alpha,\beta,\sigma}$ be the unitary operator satisfying

$$(\mathcal{S}s_n)(z) = (z \mid \beta)_n, \quad n \in \mathbb{N}_0.$$

Define

$$\mathcal{E}(x,z) = \sum_{n=0}^{\infty} \frac{(z \mid \beta)_n}{n! \, (\sigma \mid -\eta)_n} \, s_n(x), \qquad x \in X_{\alpha,\beta}, \; z \in \mathcal{D}_{\alpha,\beta,\sigma}. \tag{3.13}$$

Lemma 3.8. For each $z \in \mathcal{D}_{\alpha,\beta,\sigma}$, we have $\mathcal{E}(\cdot, z) \in L^2(X_{\alpha,\beta}, \mu_{\alpha,\beta,\sigma})$ and

$$(\mathcal{S}f)(z) = \int_{X_{\alpha,\beta}} f(x) \, \mathcal{E}(x,z) \, \mu_{\alpha,\beta,\sigma}(dx), \quad f \in L^2(X_{\alpha,\beta}, \mu_{\alpha,\beta,\sigma}). \tag{3.14}$$

Furthermore, if $\alpha = \beta > 0$,

$$\mathcal{E}(x,z) = \frac{\Gamma\left(\frac{\sigma}{\alpha^2}\right)}{\Gamma\left(\frac{\alpha z + \sigma}{\alpha^2}\right)} \left(\frac{x}{\alpha}\right)^{\frac{z}{\alpha}}, \quad x \in \mathbb{R}_+, \; z \in \mathcal{D}_{\alpha,\alpha,\sigma}, \tag{3.15}$$

if $\alpha > \beta > 0$,

$$\mathcal{E}((\alpha - \beta)n, z) = \left(1 - \frac{\beta}{\alpha}\right)^{\frac{z}{\beta}} \frac{(\alpha z + \sigma \mid -\eta)_n}{(\sigma \mid -\eta)_n} \quad n \in \mathbb{N}_0, \; z \in \mathbb{C}, \tag{3.16}$$

and if $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$,

$$\mathcal{E}(x,z) = \left(2 \cos\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\right)^{\frac{\alpha z}{\eta}} \frac{\Gamma\left(\frac{\sigma}{\eta}\right)}{\Gamma\left(\frac{\sigma + \alpha z}{\eta}\right)} \exp\left(\frac{i\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\alpha z}{\eta}\right) \frac{\Gamma\left(-\frac{ix}{2\Im(\alpha)} - \frac{i\sigma\alpha}{2\eta\Im(\alpha)} + \frac{\alpha z}{\eta}\right)}{\Gamma\left(-\frac{ix}{2\Im(\alpha)} - \frac{i\sigma\alpha}{2\eta\Im(\alpha)}\right)}, \quad x \in \mathbb{R}, \; z \in \mathcal{D}_{\alpha,\beta,\sigma}. \tag{3.17}$$

The following lemma provides alternative formulas for the action of the operator $\mathcal{S}$.

Lemma 3.9. (i) Let $\alpha \geq \beta > 0$. Then we have, for each $f \in L^2(X_{\alpha,\beta}, \mu_{\alpha,\beta,\sigma})$ and $z \in \mathcal{D}_{\alpha,\beta,\sigma}$,

$$(\mathcal{S}f)(z) = \int_{X_{\alpha,\beta}} f(x) \, \mu_{\alpha,\beta,\alpha z+\sigma}(dx). \tag{3.18}$$

(ii) *Let $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$ and $\beta = \overline{\alpha}$. The operator $\mathcal{S}$, considered as a linear operator in $\mathcal{P}(\mathbb{C})$, admits an extension to a continuous linear operator $\mathcal{S}$ in $\mathcal{E}^1_{\min}(\mathbb{C})$, and for each $f \in \mathcal{E}^1_{\min}(\mathbb{C})$,*

$$(\mathcal{S}f)(z) = \int_{\mathbb{R}} f(x+z) \, \mu_{\alpha,\beta,\alpha z+\sigma}(dx), \quad z \in \Psi_{\alpha,\beta,\sigma}, \tag{3.19}$$

where

$$\Psi_{\alpha,\beta,\sigma} = \{z \in \mathbb{C} \mid \alpha z + \sigma \in \mathfrak{D}_{\alpha,\beta}\}. \tag{3.20}$$

Recall the operators $\mathcal{U}$ and $\mathcal{V}$, given by (1.17) and satisfying $[\mathcal{V},\mathcal{U}] = \beta\mathcal{V} + (\alpha-\beta)$. We define the operator $\mathcal{R} = \mathcal{U}\mathcal{V}$ acting in $\mathcal{P}(\mathbb{C})$.

Lemma 3.10. The operator $\mathcal{R}$ is essentially self-adjoint in $\mathcal{F}_{\alpha,\beta,\sigma}$ and we keep the notation $\mathcal{R}$ for its closure. Then, if $\alpha \geq \beta > 0$, $\mathcal{S}\mathcal{R}\mathcal{S}^{-1}$ is the operator of multiplication by the variable x in $L^2(X_{\alpha,\beta}, \mu_{\alpha,\beta,\sigma})$, and if $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$, $\mathcal{S}(\mathcal{R} - \frac{\sigma}{\alpha})\mathcal{S}^{-1}$ is the operator of multiplication by the variable x in $L^2(X_{\alpha,\beta}, \mu_{\alpha,\beta,\sigma})$.

Next, we define a unitary operator $\mathbb{T} : \mathcal{F}_{\alpha,\beta,\sigma} \to \mathbb{F}_{\eta,\sigma}(\mathbb{C})$ satisfying

$$(\mathbb{T}(\cdot \mid \beta)_n)(z) = z^n, \quad n \in \mathbb{N}_0.$$

Lemma 3.11. For each $f \in \mathcal{F}_{\alpha,\beta,\sigma}$ and $z \in \mathbb{C}$, formula (1.18) holds.

Recall the operators U and V, defined by (1.15) and satisfying the commutation relation (1.16).

Proposition 3.12. The operators U and V acting in $\mathcal{P}(\mathbb{C})$ can be (uniquely) extended to continuous linear operators acting in $\mathcal{E}^1_{\min}(\mathbb{C})$. We preserve the notations U and V for these extensions. Let also Z denote the continuous linear operator in $\mathcal{E}^1_{\min}(\mathbb{C})$ of multiplication by variable z, If $\alpha \geq \beta > 0$, then $Z = UV$ and $U = Z(1 - \alpha D_{\beta-\alpha})$ (where D_0 denotes the differentiation D). If $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$ and $\beta = \overline{\alpha}$, then $Z + \frac{\sigma}{\alpha} = UV$ and $U = (Z + \frac{\sigma}{\alpha})(1 - \alpha D_{\beta-\alpha})$. In either case, the operator $1 - \alpha D_{\beta-\alpha}$ is a self-homeomorhism of $\mathcal{E}^1_{\min}(\mathbb{C})$ and $V = (1 - \alpha D_{\beta-\alpha})^{-1}$.

The following proposition provides explicit formulas for the action of the operator $V = (1 - \alpha D_{\beta-\alpha})^{-1}$ in $\mathcal{E}^1_{\min}(\mathbb{C})$.

Proposition 3.13. Let $f \in \mathcal{E}^1_{\min}(\mathbb{C})$. If $\alpha = \beta > 0$, then

$$(Vf)(z) = \int_{\mathbb{R}_+} f(z+x)\, \mu_{\alpha,\alpha,\eta}(dx), \quad z \in \mathbb{C}, \tag{3.21}$$

if $\alpha > \beta > 0$, then

$$(Vf)(z) = \int_{(\alpha-\beta)\mathbb{N}_0} \left(f(z+x)\alpha/\beta - f(z)(\alpha-\beta)/\beta\right) \mu_{\alpha,\beta,\eta}(dx), \quad z \in \mathbb{C}, \tag{3.22}$$

and if $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$ and $\beta = \overline{\alpha}$, then

$$(Vf)(z) = \int_{\mathbb{R}} \left(f(z+x+\beta)\alpha/\beta - f(z)(\alpha-\beta)/\beta\right) \mu_{\alpha,\beta,\eta}(dx), \quad z \in \mathbb{C}. \tag{3.23}$$

IV. PROOFS

A. Proof of Lemmas 3.7–3.10

1. The case of the gamma distribution and the negative binomial distribution

First, we will prove Lemmas 3.7–3.10 in the case $\alpha \geq \beta > 0$. We divide the proof into several steps.
Step 1. By Corollary A.5, Proposition A.6, and formula (A13) in the Appendix, we get, for $z \in (-\sigma/\alpha, +\infty)$,

$$\begin{aligned}(\mathcal{S}x^n)(z) &= (\mathcal{S}\mathcal{S}^{-1}\mathcal{R}^n\mathcal{S}1)(z) = (\mathcal{R}^n 1)(z) = \sum_{k=1}^{n} (\alpha-\beta)^{n-k} S(n,k)\, (z+\sigma/\alpha \mid -\beta)_k \\ &= \int_{X_{\alpha,\beta}} x^n\, \mu_{\alpha,\beta,\sigma+\alpha z}(dx).\end{aligned} \tag{4.1}$$

Hence, for each polynomial $p \in \mathcal{P}(\mathbb{C})$ and $z \in (-\sigma/\alpha, +\infty)$,

$$(\mathcal{S}p)(z) = \int_{X_{\alpha,\beta}} p\, d\mu_{\alpha,\beta,\sigma+\alpha z}. \tag{4.2}$$

Note that $(\mathcal{S}p)(z)$ can be extended to an entire function of $z \in \mathbb{C}$.

Lemma 4.1. Let $p \in \mathcal{P}(\mathbb{C})$. Then the function $\mathcal{D}_{\alpha,\beta,\sigma} \ni z \mapsto \int_{X_{\alpha,\beta}} p\, d\mu_{\alpha,\beta,\sigma+\alpha z}$ is well-defined and analytic.

Proof. Let $\alpha = \beta$. For $z \in \mathcal{D}_{\alpha,\beta,\sigma}$, we have $\Re(\sigma + \alpha z) > \sigma/2$. Hence, it is sufficient to prove that, for each $k \in \mathbb{N}_0$, the function

$$\zeta \mapsto \int_0^\infty x^k\, \mu_{\alpha,\alpha,\zeta}(dx) = \frac{1}{\Gamma\left(\frac{\zeta}{\eta}\right)}\, \alpha^{-\frac{\zeta}{\eta}} \int_0^\infty x^{k-1+\frac{\zeta}{\eta}}\, e^{-\frac{x}{\alpha}}\, dx$$

is well-defined and analytic on the domain $\{\zeta \in \mathbb{C} \mid \Re(\zeta) > 0\}$. To this end, it is sufficient to check the analyticity of the function

$$\zeta \mapsto \int_0^\infty x^{k-1+\frac{\zeta}{\eta}}\, e^{-\frac{x}{\alpha}}\, dx. \tag{4.3}$$

We have

$$\int_0^\infty \left| x^{k-1+\frac{\zeta}{\eta}}\, e^{-\frac{x}{\alpha}} \right| dx = \int_0^\infty x^{k-1+\frac{\Re(\zeta)}{\eta}}\, e^{-\frac{x}{\alpha}}\, dx < \infty,$$

hence the function in (4.3) is well defined. Furthermore,

$$\left| \frac{d}{d\zeta}\, x^{k-1+\frac{\zeta}{\eta}}\, e^{-\frac{x}{\alpha}} \right| = \frac{1}{\eta}\, |\log(x)|\, x^{k-1+\frac{\Re(\zeta)}{\eta}}\, e^{-\frac{x}{\alpha}}. \tag{4.4}$$

Note that $\log(x) \le x$ for $x \ge 1$ and, for each $\varepsilon > 0$ there exists $C_1 > 0$ such that $|\log(x)| \le C_1 x^{-\varepsilon}$ for $x \in (0,1)$. Hence, formula (4.4) easily implies that the function in (4.3) is indeed analytic on $\{\zeta \in \mathbb{C} \mid \Re(\zeta) > 0\}$.

Next, let $\alpha > \beta$. It is sufficient to prove that, for each $k \in \mathbb{N}_0$, the function

$$\zeta \mapsto \int_{(\alpha-\beta)\mathbb{N}_0} x^k\, \mu_{\alpha,\beta,\zeta}(dx) = \left(1-\frac{\beta}{\alpha}\right)^{\frac{\zeta}{\eta}} \sum_{n=0}^\infty \left(\frac{\beta}{\alpha}\right)^n \frac{1}{n!} \left(\frac{\zeta}{\eta}\right)^{(n)} ((\alpha-\beta)n)^k \tag{4.5}$$

is entire. We have

$$\sum_{n=0}^\infty \left(\frac{\beta}{\alpha}\right)^n \frac{1}{n!} \left| \left(\frac{\zeta}{\eta}\right)^{(n)} \right| ((\alpha-\beta)n)^k \le \sum_{n=0}^\infty \left(\frac{\beta}{\alpha}\right)^n \frac{1}{n!} \left(\frac{|\zeta|}{\eta}\right)^{(n)} ((\alpha-\beta)n)^k < \infty,$$

because each monomial x^k is integrable with respect to the negative binomial distribution $\mu_{\alpha,\beta,|\zeta|}$. Hence, the series in (4.5) converges uniformly on compact sets in $\mathbb{C}$, which implies that the function in (4.5) is entire. □

Now formula (4.2), Lemma 4.1, and the identity theorem for analytic functions imply

$$(\mathcal{S}p)(z) = \int_{X_{\alpha,\beta}} p\, d\mu_{\alpha,\beta,\sigma+\alpha z}, \quad p \in \mathcal{P}(\mathbb{C}),\ z \in \mathcal{D}_{\alpha,\beta,\sigma}. \tag{4.6}$$

Step 2. Let $\alpha = \beta$, let $f \in L^2(\mathbb{R}_+, \mu_{\alpha,\alpha,\sigma})$ and let $-\frac{\sigma}{2\alpha} < \delta < \Delta < +\infty$. Then, for each $z \in \mathbb{C}$ with $\delta \le \Re(z) \le \Delta$, we have

$$\begin{aligned} \int_0^\infty |f(x)| \left| x^{-1+\frac{\alpha z+\sigma}{\eta}} \right| e^{-\frac{x}{\alpha}}\, dx &= \int_0^\infty |f(x)|\, x^{-1+\frac{\alpha\Re(z)+\sigma}{\eta}}\, e^{-\frac{x}{\alpha}}\, dx \\ &\le \left(\int_0^\infty |f(x)|^2\, x^{-1+\frac{\sigma}{\eta}}\, e^{-\frac{x}{\alpha}}\, dx \right)^{\frac{1}{2}} \left(\int_0^\infty x^{-1+\frac{\sigma+2\alpha\Re(z)}{\eta}}\, e^{-\frac{x}{\alpha}}\, dx \right)^{\frac{1}{2}} \\ &\le C_2 \|f\|_{L^2(\mu_{\alpha,\alpha,\sigma})} \end{aligned} \tag{4.7}$$

for a constant $C_2 > 0$ that depends on δ and Δ. Now write $f(x) = \sum_{n=0}^\infty f_n s_n(x)$ and define, for $N \in \mathbb{N}$, $p_N(x) = \sum_{n=0}^N f_n s_n(x)$. Formulas (4.6) and (4.7) imply that $(\mathcal{S}p_N)(z) = \sum_{n=0}^N f_n(z \mid \alpha)_n$ converges uniformly on compact sets in $\mathcal{D}_{\alpha,\alpha,\sigma}$ to an analytic function and

$$(\mathcal{S}f)(z) = \sum_{n=0}^\infty f_n (z \mid \alpha)_n = \int_0^\infty f\, d\mu_{\alpha,\alpha,\alpha z+\sigma}. \tag{4.8}$$

Step 3. Let $\alpha > \beta$ and let $f \in L^2((\alpha-\beta)\mathbb{N}_0, \mu_{\alpha,\beta,\sigma})$. We have

$$\sum_{n=0}^\infty |f((\alpha-\beta)n)| \left(\frac{\beta}{\alpha}\right)^n \frac{1}{n!} \left| \left(\frac{\sigma+\alpha z}{\eta}\right)^{(n)} \right| \le \left(1-\frac{\beta}{\alpha}\right)^{-\frac{\sigma}{\eta}} \|f\|_{L^2(\mu_{\alpha,\beta,\sigma})} \left(\sum_{n=0}^\infty \left(\frac{\beta}{\alpha}\right)^n \frac{\left[\left(\frac{\sigma+\alpha|z|}{\eta}\right)^{(n)}\right]^2}{n! \left(\frac{\sigma}{\eta}\right)^{(n)}} \right)^{\frac{1}{2}}. \tag{4.9}$$

Lemma 4.2. *For any $a_1 > a_2 > 0$ and $0 < q < 1$,*

$$\sum_{n=0}^\infty q^n \frac{\left[(a_1)^{(n)}\right]^2}{n!\, (a_2)^{(n)}} < \infty.$$

Proof. It follows from the construction of a negative binomial distribution that, for each $q \in (0,1)$ and $a_1 > 0$, $\sum_{n=0}^{\infty} q^n \frac{1}{n!} (a_1)^{(n)} < \infty$. Therefore, for each $\varepsilon > 0$, we have $(a_1)^{(n)} \le C_3(1+\varepsilon)^n\, n!$, where the constant $C_3 > 0$ depends only on a_1 and ε. Next, for any $a_2 > 0$, $(a_2)^{(n)} \ge a_2(n-1)!$. Therefore, for any $a_1 > a_2 > 0$ and $\varepsilon > 0$, $(a_1)^{(n)}/(a_2)^{(n)} \le C_4(1+\varepsilon)^n$, where $C_4 > 0$ depends on a_1, a_2 and ε. Hence,

$$\sum_{n=0}^{\infty} q^n \frac{[(a_1)^{(n)}]^2}{n!\,(a_2)^{(n)}} \le C_3 C_4 \sum_{n=0}^{\infty} \left(q(1+\varepsilon)^2\right)^n < \infty,$$

if we choose $\varepsilon > 0$ such that $(1+\varepsilon)^2 < 1/q$. □

Using estimate (4.9) and Lemma 4.2, we now show similarly to Step 2 that, for $f(x) = \sum_{n=0}^{\infty} f_n s_n(x) \in L^2((\alpha-\beta)\mathbb{N}_0, \mu_{\alpha,\beta,\sigma})$,

$$(\mathcal{S}f)(z) = \sum_{n=0}^{\infty} f_n (z \mid \beta)_n = \int_0^{\infty} f \, d\mu_{\alpha,\beta,\alpha z+\sigma}, \tag{4.10}$$

the series in (4.10) converges uniformly on compact sets in $\mathbb{C}$, and hence, $(\mathcal{S}f)(z)$ is an entire function.

Thus, Lemma 3.9 (i) is proven.

Step 4. To finish the Proof of Lemma 3.7, we only need to prove formula (3.12). In fact, the first equality in (3.12) is an immediate consequence of the fact that D_β is the lowering operator for the polynomial sequence $((z \mid \beta)_n)_{n=0}^{\infty}$. The second equality in (3.12) is a well-known identity for the nth difference operator, see e.g., formula (6.2) in Ref. 35.

Step 5. Let $z \in \mathcal{D}_{\alpha,\beta,\sigma}$. It follows from Steps 3 and 4 that there exists a constant $C_5 > 0$ such that, for all $f \in L^2(X_{\alpha,\beta},\ \mu_{\alpha,\beta,\sigma})$, we have $|(\mathcal{S}f)(z)| \le C_5 \|f\|_{L^2(\mu_{\alpha,\beta,\sigma})}$. Hence, by the Riesz representation theorem, there exists $\mathcal{K}_z \in L^2(X_{\alpha,\beta}, \mu_{\alpha,\beta,\sigma})$ such that

$$(\mathcal{S}f)(z) = \int_{X_{\alpha,\beta}} f(x)\mathcal{K}_z(x)\, \mu_{\alpha,\beta,\sigma}(dx) \qquad \text{for all } f \in L^2(X_{\alpha,\beta}, \mu_{\alpha,\beta,\sigma}). \tag{4.11}$$

By (3.18) and (4.11), we conclude that $\mathcal{K}_z(x) = \mathcal{E}(x,z)$, where $\mathcal{E}(x,z)$ is given by (3.13), and $\mathcal{K}_z(x)$ is the Radon–Nykodim derivative $\frac{d\mu_{\alpha,\beta,\alpha z+\sigma}}{d\mu_{\alpha,\beta,\sigma}}(x)$. This easily implies Lemma 3.8.

Step 6. In view of Proposition A.6, to prove Lemma 3.10, we only need to check that the operator $\mathcal{R}$ with domain $\mathcal{P}(\mathbb{C})$ is essentially self-adjoint in $\mathcal{F}_{\alpha,\beta,\sigma}$. But this can be easily shown by using Nelson's analytic vector criterium, see e.g., Ref. 36, Sec. X.6.

2. The case of the Meixner distribution

Now we consider the case $\Re(\alpha) \ge 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$. We again divide the proof into several steps.

Step 1. Let $K > 1$ be fixed. We state that there exists a constant $C_1 > 0$ such that

$$|\Gamma(ix+y)| \le C_1 \exp\left(-\frac{\pi}{2}|x|\right)(1+|x|)^K, \quad x \in \mathbb{R},\ y \in [1/K, K]. \tag{4.12}$$

Indeed, by e.g., Ref. 27, p. 15, the following asymptotic formula holds, for all $x, y \in \mathbb{R}$, $-ix - y \notin \mathbb{N}_0$:

$$|\Gamma(ix+y)| = \sqrt{2\pi} \exp\left(-\frac{\pi}{2}|x|\right) |x|^{y-\frac{1}{2}} (1 + E(x,y)),$$

where the function $E(x,y)$ satisfies, for each fixed $R > 0$,

$$\lim_{|x|\to\infty} \sup_{y \in [-R,R]} |E(x,y)| = 0.$$

Hence, formula (4.12) easily follows if we take into account that the function

$$\mathbb{R} \times [1/K, K] \ni (x,y) \mapsto |\Gamma(ix+y)| \in \mathbb{R}$$

is continuous, hence bounded on $[-L,L] \times [1/K,K]$ for each $L > 0$.

Step 2. Recall the domain $\mathfrak{D}_{\alpha,\beta}$ defined in Sec. III. We state that, for each $p \in \mathcal{P}(\mathbb{C})$, the function $\mathfrak{D}_{\alpha,\beta} \ni \zeta \mapsto \int_{\mathbb{R}} p(x)\, \mu_{\alpha,\beta,\zeta}(dx) \in \mathbb{C}$ is well-defined and analytic.

Indeed, since $\mathrm{Arg}(\alpha) \in (0, \pi/2]$, we have $\cos\left(\frac{\pi}{2} - \mathrm{Arg}\,(\alpha)\right) > 0$. Therefore, the function

$$\zeta \mapsto C_{\alpha,\beta,\zeta} = \frac{\left(2\cos\left(\frac{\pi}{2} - \mathrm{Arg}\,(\alpha)\right)\right)^{\frac{\zeta}{\eta}}}{4\Im(\alpha)\pi\,\Gamma\left(\frac{\zeta}{\eta}\right)} \exp\left(\frac{\left(\frac{\pi}{2} - \mathrm{Arg}\,(\alpha)\right)\zeta\Re(\alpha)}{\Im(\alpha)\eta}\right) \in \mathbb{C}$$

is analytic on the domain $\{\zeta \in \mathbb{C} \mid \Re(\zeta) > 0\}$. Hence, it is sufficient to prove that, for each $n \in \mathbb{N}_0$, the following function is well-defined and analytic:

$$\mathfrak{D}_{\alpha,\beta} \ni \zeta \mapsto \int_{\mathbb{R}} x^n \exp\left((\pi/2 - \operatorname{Arg}(\alpha))x/\Im(\alpha)\right) g_{\alpha,\beta,\zeta}(x)\, dx \in \mathbb{C}, \tag{4.13}$$

where

$$\begin{aligned} g_{\alpha,\beta,\zeta}(x) &= \Gamma\left(\frac{ix}{2\Im(\alpha)} + \frac{i\zeta}{2\alpha\Im(\alpha)}\right)\Gamma\left(-\frac{ix}{2\Im(\alpha)} - \frac{i\zeta}{2\beta\Im(\alpha)}\right) \\ &= \Gamma(d_1(\zeta) + i(l_1(\zeta) + x/(2\Im(\alpha)))\Gamma(d_2(\zeta) + i(l_2(\zeta) - x/(2\Im(\alpha))). \end{aligned} \tag{4.14}$$

Here

$$\begin{aligned} d_1(\zeta) &= \frac{\Re(\zeta)\Im(\alpha) - \Im(\zeta)\Re(\alpha)}{2\eta\Im(\alpha)}, \quad l_1(\zeta) = \frac{\Re(\zeta)\Re(\alpha) + \Im(\zeta)\Im(\alpha)}{2\eta\Im(\alpha)}, \\ d_2(\zeta) &= \frac{\Re(\zeta)\Im(\alpha) + \Im(\zeta)\Re(\alpha)}{2\eta\Im(\alpha)}, \quad l_2(\zeta) = \frac{-\Re(\zeta)\Re(\alpha) + \Im(\zeta)\Im(\alpha)}{2\eta\Im(\alpha)}. \end{aligned} \tag{4.15}$$

For each $\zeta \in \mathfrak{D}_{\alpha,\beta}$, we have $d_1(\zeta) > 0$ and $d_2(\zeta) > 0$. Therefore, for a fixed $x \in \mathbb{R}$, the function $\mathfrak{D}_{\alpha,\beta} \ni \zeta \mapsto g_{\alpha,\beta,\zeta}(x) \in \mathbb{C}$ is analytic.

Let $\zeta \in \mathfrak{D}_{\alpha,\beta}$ be fixed. For $R > 0$, denote $B(\zeta, R) = \{z \in \mathbb{C} \mid |z - \zeta| \le R\}$. Choose $R > 0$ such that $B(\zeta, R) \subset \mathfrak{D}_{\alpha,\beta}$. To prove the differentiability of the map in (4.13) at point ζ, it is sufficient to prove that

$$\int_{\mathbb{R}} |x|^n \exp\left((\pi/2 - \operatorname{Arg}(\alpha))|x|/\Im(\alpha)\right) \sup_{z \in B(\zeta, R/2)} \left|\frac{\partial}{\partial z} g_{\alpha,\beta,z}(x)\right| dx < \infty. \tag{4.16}$$

[We used the inequality $\pi/2 - \operatorname{Arg}(\alpha) \ge 0$.] By Cauchy's integral formula, (4.16) would follow from

$$\int_{\mathbb{R}} |x|^n \exp\left((\pi/2 - \operatorname{Arg}(\alpha))|x|/\Im(\alpha)\right) \sup_{z \in B(\zeta, R)} |g_{\alpha,\beta,z}(x)|\, dx < \infty. \tag{4.17}$$

Choose $K > 0$ such that, for all $z \in B(\zeta, R)$, both $d_1(z)$ and $d_2(z)$ belong to $[\frac{1}{K}, K]$. Denote $L_1 = \max_{z \in B(\zeta,R)} |l_1(z)|$ and $L_2 = \max_{z \in B(\zeta,R)} |l_2(z)|$. Then, by (4.12), there exists a constant $C_2 > 0$ such that

$$\sup_{z \in B(\zeta, R)} |g_{\alpha,\beta,z}(x)| \le C_2 \exp\left(-\frac{\pi\,|x|}{2\Im(\alpha)}\right)(1 + |x|)^{2K}. \tag{4.18}$$

Hence, the integral in (4.16) is bounded by the following integral

$$C_2 \int_{\mathbb{R}} |x|^n \exp\left(-\operatorname{Arg}(\alpha)|x|/\Im(\alpha)\right)(1 + |x|)^{2K} < \infty,$$

where we used that $\operatorname{Arg}(\alpha) > 0$.

Step 3. Let $(p_n)_{n=0}^{\infty}$ be the monic polynomial sequence over $\mathbb{C}$ satisfying the recurrence formula (A6). Thus, for $x \in \mathbb{R}$, we have $s_n(x) = p_n(x + \sigma/\alpha)$. Let $\mathcal{I}$ be the linear operator in $\mathcal{P}(\mathbb{C})$ as defined in Proposition A.6. Define $\widetilde{\mathcal{S}} = \mathcal{I}^{-1}$. Thus, $(\widetilde{\mathcal{S}}p_n)(z) = (z \mid \beta)_n$. Similarly to (4.1), we conclude from Corollary A.5 and Proposition A.6 that

$$(\widetilde{\mathcal{S}}\zeta^n)(z) = \sum_{k=1}^{n} (\alpha - \beta)^{n-k} S(n,k)\, (z + \sigma/\alpha \mid -\beta)_k, \quad n \in \mathbb{N}. \tag{4.19}$$

On the other hand, for each $r \in (-\sigma/\eta, +\infty)$ and $z = \beta r$, we have $\sigma + \alpha z = \sigma + \eta r \in (0, \infty)$. Hence, by (A14),

$$\int_{\mathbb{R}} (x + z + \sigma/\alpha)^n\, \mu_{\alpha,\beta,\sigma+\alpha z}(dx) = \sum_{k=1}^{n} (\alpha - \beta)^{n-k} S(n,k)(z + \sigma/\alpha \mid -\beta)_k, \quad n \in \mathbb{N}. \tag{4.20}$$

By (4.19) and (4.20), we have, for each $p \in \mathcal{P}(\mathbb{C})$,

$$(\widetilde{\mathcal{S}}p)(z) = \int_{\mathbb{R}} p(x + z + \sigma/\alpha)\, \mu_{\alpha,\beta,\sigma+\alpha z}(dx), \quad z = \beta r,\ r \in (-\sigma/\eta, +\infty). \tag{4.21}$$

Setting $p = p_n$ into (4.21) gives

$$\int_{\mathbb{R}} s_n(x+z)\ \mu_{\alpha,\beta,\sigma+\alpha z}(dx) = \int_{\mathbb{R}} p_n(x+z+\sigma/\alpha)\ \mu_{\alpha,\beta,\sigma+\alpha z}(dx)$$
$$= (\widetilde{\mathcal{S}}p_n)(z) = (z \mid \beta)_n = (\mathcal{S}s_n)(z), \quad z = \beta r,\ r \in (-\sigma/\eta, +\infty),\ n \in \mathbb{N}_0.$$

Therefore, for each $p \in \mathcal{P}(\mathbb{C})$,

$$(\mathcal{S}p)(z) = \int_{\mathbb{R}} p(x+z)\ \mu_{\alpha,\beta,\sigma+\alpha z}(dx), \quad z = \beta r,\ r \in (-\sigma/\eta, +\infty). \tag{4.22}$$

Recall the open domain $\Psi_{\alpha,\beta,\sigma}$ defined by (3.20). Obviously,

$$\{z = \beta r \mid r \in (-\sigma/\eta, +\infty)\} \subset \Psi_{\alpha,\beta,\sigma}.$$

It follows from Step 2 that, for each $p \in \mathcal{P}(\mathbb{C})$, the function

$$\Psi_{\alpha,\beta,\sigma} \ni z \mapsto \int_{\mathbb{R}} p(x+z)\ \mu_{\alpha,\beta,\sigma+\alpha z}(dx) \in \mathbb{C}$$

is analytic. On the other hand, $(\mathcal{S}p)(z)$ is an entire function. Hence, by (4.22) and the identity theorem for analytic functions,

$$(\mathcal{S}p)(z) = \int_{\mathbb{R}} p(x+z)\ \mu_{\alpha,\beta,\sigma+\alpha z}(dx), \quad z \in \Psi_{\alpha,\beta,\sigma}. \tag{4.23}$$

Step 4. A direct calculation shows that, if $\Re(\alpha) = 0$ then $\mathbb{R} \subset \Psi_{\alpha,\beta,\sigma}$, and if $\Re(\alpha) > 0$ then $(-\sigma/(2\Re(\alpha)), \infty) \subset \Psi_{\alpha,\beta,\sigma}$. Below we will use the notation $-\sigma/(2\Re(\alpha))$ even if $\Re(\alpha) = 0$, meaning that $-\sigma/(2\Re(\alpha)) = -\infty$. Hence, by (4.23),

$$(\mathcal{S}p)(z) = \int_{\mathbb{R}} p(x+z)\ \mu_{\alpha,\beta,\sigma+\alpha z}(dx), \quad z \in (-\sigma/(2\Re(\alpha)), \infty). \tag{4.24}$$

The change of variable $x' = x + z$ in the integral in (4.24) implies

$$(\mathcal{S}p)(z) = \int_{\mathbb{R}} p(x)\ \mathcal{G}(x,z)\ dx, \quad z \in (-\sigma/(2\Re(\alpha)), \infty), \tag{4.25}$$

where for $x \in \mathbb{R}$ and $z \in (-\sigma/(2\Re(\alpha)), \infty)$,

$$\mathcal{G}(x,z) = \frac{\left(2\cos\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\right)^{\frac{\alpha z+\sigma}{\eta}}}{4\Im(\alpha)\pi\Gamma\left(\frac{\alpha z+\sigma}{\eta}\right)} \exp\left(\frac{\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)(\alpha z+\sigma)\Re(\alpha)}{\Im(\alpha)\eta}\right)$$
$$\times \exp\left(\frac{\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)(x-z)}{\Im(\alpha)}\right)\Gamma\left(\frac{ix}{2\Im(\alpha)} + \frac{i\sigma\beta}{2\eta\Im(\alpha)}\right)\Gamma\left(\frac{-ix}{2\Im(\alpha)} - \frac{i\sigma\alpha}{2\eta\Im(\alpha)} + \frac{z\alpha}{\eta}\right). \tag{4.26}$$

[Note that the real part of the argument of each of the gamma functions in (4.26) is positive.]

Recall the domain $\mathcal{D}_{\alpha,\beta,\sigma}$ defined by (3.10). It is straightforward to see that, for each fixed $x \in \mathbb{R}$, the function $\mathcal{G}(x,\cdot)$ admits a unique extension to an analytic function on $\mathcal{D}_{\alpha,\beta,\sigma}$, and this extension is still given by formula (4.26). Similarly to Step 3, we show that, for each $p \in \mathcal{P}(\mathbb{C})$, the function $\mathcal{D}_{\alpha,\beta,\sigma} \ni z \mapsto \int_{\mathbb{R}} p(x)\ \mathcal{G}(x,z)\ dx \in \mathbb{C}$ is analytic. Therefore, formula (4.25) implies

$$(\mathcal{S}p)(z) = \int_{\mathbb{R}} p(x)\ \mathcal{G}(x,z)\ dx, \quad z \in \mathcal{D}_{\alpha,\beta,\sigma}. \tag{4.27}$$

Step 5. Let $z_0 \in \mathcal{D}_{\alpha,\beta,\sigma}$. Choose $R > 0$ such that the closed ball $B(z_0, R)$ is a subset of $\mathcal{D}_{\alpha,\beta,\sigma}$ We state that there exists a constant $C_3 > 0$ such that, for all $f \in L^2(\mathbb{R}, \mu_{\alpha,\beta,\sigma})$,

$$\sup_{z \in B(z_0,R)} \int_{\mathbb{R}} |f(x)|\ |\mathcal{G}(x,z)|\ dx \le C_3 \|f\|_{L^2(\mu_{\alpha,\beta,\sigma})}. \tag{4.28}$$

Indeed, we have, by the Cauchy inequality, for each $z \in B(z_0, R)$,

$$\int_{\mathbb{R}} |f(x)|\ |\mathcal{G}(x,z)|\ dx \le \|f\|_{L^2(\mu_{\alpha,\beta,\sigma})} \left(\int_{\mathbb{R}} \frac{|\mathcal{G}(x,z)|^2}{G_{\alpha,\beta,\sigma}(x)}\ dx\right)^{\frac{1}{2}}. \tag{4.29}$$

Here $G_{\alpha,\beta,\sigma}(x)$ is the density of the measure $\mu_{\alpha,\beta,\sigma}$ with respect to the Lebesgue measure:

$$G_{\alpha,\beta,\sigma}(x) = C_{\alpha,\beta,\sigma}\ \exp\left((\pi/2 - \operatorname{Arg}(\alpha))x/\Im(\alpha)\right) g_{\alpha,\beta,\sigma}(x).$$

Using (1.9), (4.14), (4.26), and the equality $|\Gamma(\zeta)| = |\Gamma(\bar{\zeta})|$, we find

$$\frac{|\mathcal{G}(x,z)|^2}{G_{\alpha,\beta,\sigma}(x)} = \left| \frac{\left(2\cos\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\right)^{\frac{2\alpha z+\sigma}{\eta}}}{4\Im(\alpha)\pi} \cdot \frac{\Gamma\left(\frac{\sigma}{\eta}\right)}{\Gamma^2\left(\frac{\alpha z+\sigma}{\eta}\right)} \exp\left[\frac{\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)\Re(\alpha)\,(2\alpha z+\sigma)}{\Im(\alpha)\eta} + \frac{\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)(x-2z)}{\Im(\alpha)} \right] \Gamma^2\left(-\frac{ix}{2\Im(\alpha)} - \frac{i\sigma\alpha}{2\eta\,\Im(\alpha)} + \frac{z\alpha}{\eta} \right) \right|. \tag{4.30}$$

Thus, by (4.29) and (4.30), to prove (4.28), it is sufficient to show that

$$\sup_{z\in B(z_0,R)} \int_{\mathbb{R}} \exp\left[\frac{\left(\frac{\pi}{2} - \operatorname{Arg}(\alpha)\right)|x|}{\Im(\alpha)} \right] \cdot \left| \Gamma\left(-\frac{ix}{2\Im(\alpha)} - \frac{i\sigma\alpha}{2\eta\Im(\alpha)} + \frac{z\alpha}{\eta} \right) \right|^2 dx < \infty. \tag{4.31}$$

We note that, for each $z \in B(z_0,R) \subset \mathcal{D}_{\alpha,\beta,\sigma}$,

$$\Re\left(-\frac{i\sigma\alpha}{2\eta\Im(\alpha)} + \frac{z\alpha}{\eta} \right) = \frac{\sigma}{2\eta} + \frac{\Re(\alpha z)}{\eta} > 0.$$

Hence, there exists $K > 1$ such that

$$\Re\left(-\frac{i\sigma\alpha}{2\eta\Im(\alpha)} + \frac{z\alpha}{\eta} \right) \in [1/K, K] \quad \forall z \in B(z_0,R).$$

Since $\Re(\alpha z)$ is bounded on $B(z_0,R)$, estimate (4.12) easily implies (4.31). (Compare with Step 2.)

Step 6. Similarly to Step 2 in Subsection IV A 1, we conclude from (4.28) that, for each $f(x) = \sum_{n=0}^{\infty} f_n s_n(x) \in L^2(\mathbb{R}, \mu_{\alpha,\beta,\sigma})$, the series $\sum_{n=0}^{\infty} f_n (z \mid \beta)_n$ converges uniformly on compact sets in $\mathcal{D}_{\alpha,\beta,\sigma}$ to an analytic function and

$$(\mathcal{S}f)(z) = \sum_{n=0}^{\infty} f_n (z \mid \beta)_= \int_{\mathbb{R}} f(x)\,\mathcal{G}(x,z)\,dx. \tag{4.32}$$

Similarly to Step 4 in Subsection IV A 1, this proves Lemma 3.7. Next, using (1.9), (3.9), (4.26), and (4.32), we find that formulas (3.14) and (3.17) hold for each $f \in L^2(\mathbb{R}, \mu_{\alpha,\beta,\sigma})$. Similarly to Step 5 in Subsection IV A 1, we see that $\mathcal{E}(\cdot, z) \in L^2(\mathbb{R}, \mu_{\alpha,\beta,\sigma})$ for each $z \in \mathcal{D}_{\alpha,\beta,\sigma}$ and formula (3.13) holds. This proves Lemma 3.8.

Step 7. Note that $((z \mid \beta)_n)_{n=0}^{\infty}$ is Sheffer sequence with generating function (1.4) in which $A(t) = 0$ and $B(t) = \frac{1}{\beta}\log(1+\beta t)$. Therefore, by Theorem 2.5, the linear operator $\mathcal{S}$, acting in $\mathcal{P}(\mathbb{C})$ and satisfying $\mathcal{S}s_n = (\cdot \mid \beta)_n$ $(n \in \mathbb{N}_0)$, extends to a continuous linear operator in $\mathcal{E}^1_{\min}(\mathbb{C})$.

Let $f(z) = \sum_{n=0}^{\infty} f_n s_n(z) \in \mathcal{E}^1_{\min}(\mathbb{C})$ and define $p_N(z) = \sum_{n=0}^{N} f_n s_n(z) \in \mathcal{P}(\mathbb{C})$ $(N \in \mathbb{N})$. Then $p_N \to f$ and $\mathcal{S}p_N \to \mathcal{S}f$ in $\mathcal{E}^1_{\min}(\mathbb{C})$. In particular, for each fixed $z \in \mathbb{C}$, we have $p_N(z) \to f(z)$ and $(\mathcal{S}p_N)(z) \to (\mathcal{S}f)(z)$ as $N \to \infty$.

In view of (4.23), to prove Lemma 3.9 (ii), it is sufficient to show that, for each $\zeta \in \mathfrak{D}_{\alpha,\beta}$ and $z \in \mathbb{C}$, we have

$$\lim_{N\to\infty} \int_{\mathbb{R}} p_N(x+z)\,\mu_{\alpha,\beta,\zeta}(dx) = \int_{\mathbb{R}} f(x+z)\,\mu_{\alpha,\beta,\zeta}(dx),$$

which is equivalent to

$$\lim_{N\to\infty} \int_{\mathbb{R}} p_N(x+z) \exp\left((\pi/2 - \operatorname{Arg}(\alpha))x/\Im(\alpha)\right) g_{\alpha,\beta,\zeta}(x)\,dx = \int_{\mathbb{R}} f(x+z) \exp\left((\pi/2 - \operatorname{Arg}(\alpha))x/\Im(\alpha)\right) g_{\alpha,\beta,\zeta}(x)\,dx. \tag{4.33}$$

It follows from (4.18) that, for each $\zeta \in \mathfrak{D}_{\alpha,\beta}$, there exist constants $C_4 > 0$ and $K > 0$ such that

$$|g_{\alpha,\beta,\zeta}(x)| \le C_4 \exp\left(-\frac{\pi\,|x|}{2\Im(\alpha)} \right) (1+|x|)^{2K}. \tag{4.34}$$

Since the sequence $(p_N)_{N=1}^{\infty}$ converges in $\mathcal{E}^1_{\min}(\mathbb{C})$, for each $t > 0$, there exists a constant $C_t > 0$ (depending on the fixed $z \in \mathbb{C}$) such that

$$\sup_{x\in\mathbb{R}} |p_N(x+z)| \le C_t \exp(t|x|). \tag{4.35}$$

Choosing $t \in (0, \operatorname{Arg}(\alpha)/\Im(\alpha))$, we conclude (4.33) from (4.34), (4.35), and the dominated convergence theorem.

Thus, Lemma 3.9 (ii) is proven. Finally, the Proof of Lemma 3.10 is similar to Step 6 in Subsection IV A 1.

B. The remaining proofs

Proof of Lemma 3.11. We state that, for each $z \in \mathbb{C}$,

$$\int_{\mathbb{N}_0} (\xi)_n \, \pi_z(d\xi) = z^n, \quad n \in \mathbb{N}. \tag{4.36}$$

For $z > 0$, equality (4.36) is well-known. [To show it, one can use the equality $\int_{\mathbb{N}_0} \xi^n \, \pi_\sigma(d\xi) = \sum_{k=1}^n S(n,k)\sigma^k$ and formula (A1).] As easily seen, the function $\mathbb{C} \ni z \mapsto \int_{\mathbb{N}_0} (\xi)_n \, \pi_z(d\xi)$ is entire. Hence, formula (4.36) holds for all $z \in \mathbb{C}$ by the identity theorem for analytic functions.

Formula (4.36) implies $\int_{\mathbb{N}_0} (\beta\xi \mid \beta)_n \, \pi_{\frac{z}{\beta}}(d\xi) = z^n$. Therefore, formula (1.18) holds for $f \in \mathcal{P}(\mathbb{C})$.

Let $f(z) = \sum_{n=0}^\infty f_n (z \mid \beta)_n \in \mathcal{F}_{\alpha,\beta,\sigma}$. Using (4.36), we have, for $z \in \mathbb{C}$,

$$\begin{aligned}
\sum_{k=0}^\infty |f(\beta k)| \, \frac{1}{k!} \left|\frac{z}{\beta}\right|^k &\le \sum_{k=0}^\infty \sum_{n=0}^\infty |f_n| \, |(\beta k \mid \beta)_n| \, \frac{1}{k!} \left|\frac{z}{\beta}\right|^k \\
&= \sum_{n=0}^\infty |f_n| \, |\beta|^n \sum_{k=0}^\infty (k)_n \, \frac{1}{k!} \left|\frac{z}{\beta}\right|^k = \exp(|z|/\beta) \sum_{n=0}^\infty |f_n| \, |\beta|^n \int_{\mathbb{N}_0} (\xi)_n \, \pi_{|\frac{z}{\beta}|}(d\xi) \\
&= \exp(|z|/|\beta|) \sum_{n=0}^\infty |f_n| \, |z|^n \le \exp(|z|/|\beta|) \left(\sum_{n=0}^\infty \frac{|z|^{2n}}{n! \, (\sigma \mid -\eta)_n} \right)^{1/2} \|f\|_{\mathcal{F}_{\alpha,\beta,\sigma}}.
\end{aligned}$$

Hence, the integral on the right-hand side of formula (1.18) is well-defined and formula (1.18) holds. □

Proof of Theorem 3.1. By Lemmas 3.8 and 3.11, we have, for each $f \in L^2(X_{\alpha,\beta}, \, \mu_{\alpha,\beta,\sigma})$ and $z \in \mathbb{C}$,

$$(\mathbb{S}f)(z) = \int_{\mathbb{N}_0} \int_{X_{\alpha,\beta}} f(x) \, \mathcal{E}(x, \beta\xi) \, \mu_{\alpha,\beta,\sigma}(dx) \, \pi_{\frac{z}{\beta}}(d\xi). \tag{4.37}$$

By (3.13), we have, for $x \in X_{\alpha,\beta}$ and $\xi \in \mathbb{N}_0$,

$$|\mathcal{E}(x, \beta\xi)| \le \sum_{n=0}^\infty \frac{|(\beta\xi \mid \beta)_n|}{n! \, (\sigma \mid -\eta)_n} \, |s_n(x)| = \sum_{n=0}^\infty \frac{|\beta|^n}{n! \, (\sigma \mid -\eta)_n} \, (\xi)_n |s_n(x)|.$$

Therefore, using (4.36), we obtain, for $f \in L^2(X_{\alpha,\beta}, \, \mu_{\alpha,\beta,\sigma})$ and $z \in \mathbb{C}$,

$$\begin{aligned}
\int_{\mathbb{N}_0} \int_{X_{\alpha,\beta}} |f(x)| \, |\mathcal{E}(x, \beta\xi)| \, \mu_{\alpha,\beta,\sigma}(dx) \, \pi_{|\frac{z}{\beta}|}(d\xi) &\le \sum_{n=0}^\infty \frac{|\beta|^n}{n! \, (\sigma \mid -\eta)_n} \int_{X_{\alpha,\beta}} |f(x) s_n(x)| \left(\int_{\mathbb{N}_0} (\xi)_n \, \pi_{|\frac{z}{\beta}|}(d\xi) \right) \mu_{\alpha,\beta,\sigma}(dx) \\
&= \sum_{n=0}^\infty \frac{|z|^n}{n! \, (\sigma \mid -\eta)_n} \int_{X_{\alpha,\beta}} |f(x) s_n(x)| \, \mu_{\alpha,\beta,\sigma}(dx) \\
&\le \|f\|_{L^2(\mu_{\alpha,\beta,\sigma})} \sum_{n=0}^\infty \frac{|z|^n}{\sqrt{n! \, (\sigma \mid -\eta)_n}} < \infty.
\end{aligned} \tag{4.38}$$

Formulas (4.37) and (4.38) imply formula (3.2) in which

$$\mathbb{E}(x, z) = \int_{\mathbb{N}_0} \mathcal{E}(x, \beta\xi) \, \pi_{\frac{z}{\beta}}(d\xi). \tag{4.39}$$

Formulas (3.2), (4.39), and Lemma 3.8 imply formulas (3.3)–(3.5). For each $z \in \mathbb{C}$, the map $L^2(X_{\alpha,\beta}, \mu_{\alpha,\beta,\sigma}) \ni z \mapsto (\mathbb{S}f)(z) \in \mathbb{C}$ is continuous, see e.g., Refs. (4.37) and (4.38). Hence, $\mathbb{E}(\cdot, z) \in L^2(X_{\alpha,\beta}, \mu_{\alpha,\beta,\sigma})$ for each $z \in \mathbb{C}$. Formula (3.1) is then also obvious. □

Proof of Corollary 3.2. The equality $(\partial^+ p, q)_{L^2(\mu_{\alpha,\beta,\sigma})} = (p, A^- q)_{L^2(\mu_{\alpha,\beta,\sigma})}$ for $p, q \in \mathcal{P}(\mathbb{C})$ follows from (1.5). Since the adjoint of the operator A^- is densely defined, the operator A^- is closable. For $N \in \mathbb{N}$, define $\mathbb{E}_N(x, z) = \sum_{n=0}^N \frac{z^n}{n! \, (\sigma \mid -\eta)_n} \, s_n(x)$. Then, for each $z \in \mathbb{C}$, $\mathbb{E}_N(\cdot, z) \to \mathbb{E}(\cdot, z)$ in $L^2(X_{\alpha,\beta}, \, \mu_{\alpha,\beta,\sigma})$ as $N \to \infty$, and $A^- \mathbb{E}_N(\cdot, z) = z\mathbb{E}_{N-1}(\cdot, z) \to z\mathbb{E}(\cdot, z)$ in $L^2(X_{\alpha,\beta}, \, \mu_{\alpha,\beta,\sigma})$ as $N \to \infty$. Hence, $\mathbb{E}(\cdot, z)$ belongs to the domain of A^- and $A^- \mathbb{E}(\cdot, z) = z\mathbb{E}(\cdot, z)$. □

Proof of Theorem 3.3. By Lemma 3.9 (i) and Lemma 3.11, we have, for $f \in L^2(X_{\alpha,\beta}, \, \mu_{\alpha,\beta,\sigma})$ and $z \in \mathbb{C}$:

$$(\mathbb{S}f)(z) = \int_{\mathbb{N}_0} \int_{X_{\alpha,\beta}} f(x) \, \mu_{\alpha,\beta,\eta\xi+\sigma}(dx) \, \pi_{\frac{z}{\beta}}(d\xi). \tag{4.40}$$

To conclude from (4.40) that formulas (3.6) and (3.7) hold, it is sufficient to show that

$$\int_{\mathbb{N}_0}\int_{X_{\alpha,\beta}} |f(x)|\,\mu_{\alpha,\beta,\eta\xi+\sigma}(dx)\,\pi_{\frac{|z|}{\beta}}(d\xi) < \infty. \tag{4.41}$$

But this is immediate since $f \in L^2(X_{\alpha,\beta},\,\mu_{\alpha,\beta,\sigma})$ and the left-hand side of (4.41) is equal to $(\mathbb{S}|f|)(|z|)$. □

Proof of Theorem 3.4. By Theorem 2.5, the operator $\mathbb{S}$ acts continuously in $\mathcal{E}^1_{\min}(\mathbb{C})$. Now the theorem follows from Lemma 3.9 (ii) and Lemma 3.11 [note that $\mathcal{E}^1_{\min}(\mathbb{C})$ can be naturally embedded into $\mathcal{F}_{\alpha,\beta,\sigma}$.] Indeed, the only fact that needs to be checked is that, for each $\xi \in \mathbb{N}_0$, we have $\beta\xi \in \Psi_{\alpha,\beta,\sigma}$. But this is immediate since $\alpha\beta\xi + \sigma = \eta\xi + \sigma > 0$ and so $\alpha\beta\xi + \sigma \in \mathfrak{D}_{\alpha,\beta}$. □

Proof of Proposition 3.6. The proposition follows immediately from Lemma 3.10. □

Proof of Propositions 3.12 and 3.13. We divide the proof into several steps.

Step 1. The operator $1 - \alpha D_{\beta-\alpha}$ maps a monic polynomial sequence to a monic polynomial sequence. Hence, it is bijective as a map in $\mathcal{P}(\mathbb{C})$.

The equality

$$V = \alpha\partial^- + 1 = (1 - \alpha D_{\beta-\alpha})^{-1} \quad \text{on } \mathcal{P}(\mathbb{C}) \tag{4.42}$$

easily follows from umbral calculus. Indeed, $\partial^- = B(D)$, where D is the differentiation operator and the function B is as in formula (1.4), see e.g., Ref. 26, Sec. 4.4. By Ref. 30, if $\alpha \neq \beta$, we have

$$B(t) = \frac{e^{(\beta-\alpha)t} - 1}{\beta - \alpha e^{(\beta-\alpha)t}} = \frac{\frac{e^{(\beta-\alpha)t}-1}{\beta-\alpha}}{1 - \alpha\frac{e^{(\beta-\alpha)t}-1}{\beta-\alpha}}. \tag{4.43}$$

By Boole's formula (e.g., Ref. 26, Sec. 4.3.1), for $h \in \mathbb{C}$, the h-derivative has the representation $D_h = \frac{e^{hD}-1}{h}$. Hence, by (4.43), $\partial^- = D_{\beta-\alpha}(1 - \alpha D_{\beta-\alpha})^{-1}$, which implies (4.42). In the case $\alpha = \beta$, we have $B(t) = \frac{t}{1-\alpha t}$, which similarly implies (4.42).

Recall that, in the case $\alpha \geq \beta > 0$, we have $Z = UV$ on $\mathcal{P}(\mathbb{C})$, and in the case $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$, we have $Z + \frac{\sigma}{\alpha} = UV$ on $\mathcal{P}(\mathbb{C})$. Since $V^{-1} = 1 - \alpha D_{\beta-\alpha}$, this immediately implies that $U = Z(1 - \alpha D_{\beta-\alpha})$ in the former case, and $U = (Z + \frac{\sigma}{\alpha})(1 - \alpha D_{\beta-\alpha})$ in the latter case.

Step 2. Similarly to Step 1, we easily find that $\partial^- = D_{\alpha-\beta}(1 - \beta D_{\alpha-\beta})^{-1} = \sum_{k=0}^{\infty} \beta^k D_{\alpha-\beta}^{k+1}$. Since $D_{\alpha-\beta}$ is the lowering operator for the polynomial sequence $((z \mid \alpha-\beta)_n)_{n=0}^{\infty}$, we get

$$\big(\partial^-(\cdot \mid \alpha-\beta)_n\big)(z) = \sum_{k=0}^{n-1} \frac{n!}{k!}\,\beta^{n-k-1}\,(z \mid \alpha-\beta)_k, \quad n \in \mathbb{N}. \tag{4.44}$$

Step 3. We state that, when $\alpha \geq \beta > 0$,

$$\int_{X_{\alpha,\beta}} (x \mid \alpha-\beta)_n\,\mu_{\alpha,\beta,\sigma}(dx) = \beta^n(\sigma/\eta)^{(n)}, \tag{4.45}$$

and when $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$,

$$\int_{\mathbb{R}} (x + \sigma/\alpha \mid \alpha-\beta)_n\,\mu_{\alpha,\beta,\sigma}(dx) = \beta^n(\sigma/\eta)^{(n)}. \tag{4.46}$$

Note that, when $\alpha = \beta > 0$, formula (4.45) is just (A13). We will prove formula (4.45) when $\alpha > \beta > 0$, the proof of (4.46) being similar. By (A1) and (A13), we obtain

$$\begin{aligned}
\int_{(\alpha-\beta)\mathbb{N}_0} (x \mid \alpha-\beta)_n\,\mu_{\alpha,\beta,\sigma}(dx) &= (\alpha-\beta)^n \int_{(\alpha-\beta)\mathbb{N}_0} (x/(\alpha-\beta))_n\,\mu_{\alpha,\beta,\sigma}(dx)\\
&= \sum_{k=1}^{n} s(n,k)(\alpha-\beta)^{n-k} \int_{(\alpha-\beta)\mathbb{N}_0} x^k\,\mu_{\alpha,\beta,\sigma}(dx)\\
&= \sum_{k=1}^{n} s(n,k)(\alpha-\beta)^{n-k} \sum_{i=1}^{k} (\alpha-\beta)^{k-i} S(k,i)(\sigma/\alpha \mid -\beta)_i\\
&= \sum_{i=1}^{n} (\alpha-\beta)^{n-i}(\sigma/\alpha \mid -\beta)_i \sum_{k=i}^{n} s(n,k)S(k,i)\\
&= (\sigma/\alpha \mid -\beta)_n = \beta^n(\sigma/\eta)^{(n)}.
\end{aligned}$$

Step 4. Let $p \in \mathcal{P}(\mathbb{C})$. We state that, if $\alpha \geq \beta > 0$,

$$(\partial^- p)(z) = \int_{X_{\alpha,\beta}} (p(z+x) - p(z))\beta^{-1}\, \mu_{\alpha,\beta,\eta}(dx), \tag{4.47}$$

and if $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$,

$$(\partial^- p)(z) = \int_{X_{\alpha,\beta}} (p(z+x+\beta) - p(z))\beta^{-1}\, \mu_{\alpha,\beta,\eta}(dx), \tag{4.48}$$

To prove formula (4.47), it is sufficient to show that it holds for $p(z) = (z \mid \alpha - \beta)_n$ ($n \in \mathbb{N}$). Then, by (4.44) and (4.45),

$$\begin{aligned}\int_{X_{\alpha,\beta}} ((z+x \mid \alpha-\beta)_n - (z \mid \alpha-\beta)_n)\beta^{-1}\, \mu_{\alpha,\beta,\eta}(dx) &= \sum_{k=0}^{n-1} \binom{n}{k} (z \mid \alpha-\beta)_k\, \beta^{-1} \int_{X_{\alpha,\beta}} (x \mid \alpha-\beta)_{n-k}\, \mu_{\alpha,\beta,\eta}(dx) \\ &= \sum_{k=0}^{n-1} \binom{n}{k} (z \mid \alpha-\beta)_k\, \beta^{n-k-1}(n-k)! = (\partial^- p)(z).\end{aligned}$$

The proof of (4.48) is similar. We only need to note that $\eta/\alpha = \beta$.

Since $V = \alpha\partial^- + 1$, formulas (4.47) and (4.48) imply that formulas (3.21)–(3.23) hold for $f(z) = p(z) \in \mathcal{P}(\mathbb{C})$.

Step 5. Using Theorem 2.5, one can easily show that the operators ∂^+, ∂^-, Z and $D_{\beta-\alpha}$ admit a (unique) extension to continuous linear operators in $\mathcal{E}^1_{\min}(\mathbb{C})$. Hence, U and V also admit a continuous extension, $Z = UV$, respectively $Z + \sigma/\alpha = UV$, and $U = Z(1 - \alpha D_{\beta-\alpha})$, respectively $U = (Z + \sigma/\alpha)(1 - \alpha D_{\beta-\alpha})$.

Finally, using the definition of the space $\mathcal{E}^1_{\min}(\mathbb{C})$, we easily see that the integrals on the right-hand side of formulas (3.21)–(3.23) are well defined for each $f \in \mathcal{E}^1_{\min}(\mathbb{C})$, and furthermore, the right-hand side of each of the formulas (3.21)–(3.23) determines a continuous linear operator in $\mathcal{E}^1_{\min}(\mathbb{C})$. Hence, formulas (3.21)–(3.23) hold for $f \in \mathcal{E}^1_{\min}(\mathbb{C})$. Since $(1 - \alpha D_{\beta-\alpha})V = V(1 - \alpha D_{\beta-\alpha}) = 1$ in $\mathcal{E}^1_{\min}(\mathbb{C})$, the operator $1 - \alpha D_{\beta-\alpha}$ is invertible in $\mathcal{E}^1_{\min}(\mathbb{C})$ and $V = (1 - \alpha D_{\beta-\alpha})^{-1}$. □

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Chadaphorn Kodsueb: Conceptualization (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal).
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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: NORMAL ORDERING IN A CLASS OF GENERALIZED WEYL ALGEBRAS AND ITS CONNECTION TO ORTHOGONAL SHEFFER SEQUENCES

We consider a special class of generalized Weyl algebras. For $a, b \in \mathbb{C}$, we are interested in the complex free algebra in two generators $\mathcal{U}$ and $\mathcal{V}$ satisfying the commutation relation $[\mathcal{V}, \mathcal{U}] = a\mathcal{V} + b$.

Recall that the Stirling numbers of the first kind, $s(n,k)$, and of the second kind, $S(n,k)$, are defined as the coefficients of the expansions $(z)_n = \sum_{k=1}^n s(n,k)\, z^k$ and $z^n = \sum_{k=1}^n S(n,k)(z)_k$, respectively. This definition immediately implies the orthogonality property of the Stirling numbers:

$$\sum_{k=i}^{n} S(n,k)s(k,i) = \sum_{k=i}^{n} s(n,k)S(k,i) = \delta_{n,i}, \quad 1 \leq i \leq n. \tag{A1}$$

Proposition A.3. Assume that the generators $\mathcal{U}$, $\mathcal{V}$ satisfy $[\mathcal{V},\mathcal{U}] = a\mathcal{V} + b$. Then, for each $n \in \mathbb{N}$, we have

$$(\mathcal{U}\mathcal{V})^n = \sum_{k=1}^{n} b^{n-k} S(n,k)\mathcal{U}(\mathcal{U}+a)(\mathcal{U}+2a)\cdots(\mathcal{U}+(k-1)a)\mathcal{V}^k$$
$$= \sum_{k=1}^{n} b^{n-k} S(n,k)\ (\mathcal{U} \mid -a)_k\ \mathcal{V}^k. \tag{A2}$$

Remark A.4. Note that, in the existent literature, one would usually consider the normal ordering of $(\mathcal{V}\mathcal{U})^n$ in which all operators $\mathcal{V}$ are to the left of the operators $\mathcal{U}$ (see e.g., Ref. 29, Sec. 8.5 the references therein), while we are interested in the opposite situation. The reader is advised to compare Proposition A.3 with Ref. 38.

Proof of Proposition A.3. First, we state that

$$\mathcal{V}^n\mathcal{U} = (\mathcal{U}+na)\mathcal{V}^n + nb\mathcal{V}^{n-1}. \tag{A3}$$

This formula follows immediately from Ref. 22. Nevertheless, an interested reader can prove formula (A3) directly by induction.

Now we prove (A2) by induction. For $n = 1$, (A2) becomes the tautology $\mathcal{U}\mathcal{V} = \mathcal{U}\mathcal{V}$. Assume that (A2) holds for n and let us prove it for $n+1$. We have, by (A3),

$$(\mathcal{U}\mathcal{V})^{n+1} = \sum_{k=1}^{n} b^{n-k}S(n,k)\mathcal{U}(\mathcal{U}+a)(\mathcal{U}+2a)\cdots(\mathcal{U}+(k-1)a)\mathcal{V}^k\mathcal{U}\mathcal{V}$$
$$= \sum_{k=1}^{n} b^{n-k}S(n,k)\ \mathcal{U}(\mathcal{U}+a)(\mathcal{U}+2a)\cdots(\mathcal{U}+(k-1)a)\Big[(\mathcal{U}+ka)\mathcal{V}^k + \mathrm{kb}\mathcal{V}^{k-1}\Big]\mathcal{V}$$
$$= \sum_{k=1}^{n} b^{n-k}S(n,k)\mathcal{U}(\mathcal{U}+a)(\mathcal{U}+2a)\cdots(\mathcal{U}+ka)\mathcal{V}^{k+1}$$
$$+ \sum_{k=1}^{n} kb^{n-k+1}S(n,k)\mathcal{U}(\mathcal{U}+a)(\mathcal{U}+2a)\cdots(\mathcal{U}+(k-1)a)\mathcal{V}^k. \tag{A4}$$

Setting $S(n,0) = S(n,n+1) = 0$, we continue (A4) as follows:

$$= \sum_{k=1}^{n+1} S(n,k-1)b^{n-k+1}\mathcal{U}(\mathcal{U}+a)(\mathcal{U}+2a)\cdots(\mathcal{U}+(k-1)a)\mathcal{V}^k$$
$$+ \sum_{k=1}^{n+1} S(n,k)kb^{n-k+1}\ \mathcal{U}(\mathcal{U}+a)(\mathcal{U}+2a)\cdots(\mathcal{U}+(k-1)a)\mathcal{V}^k$$
$$= \sum_{k=1}^{n+1} (S(n,k-1)+kS(n,k))b^{n+1-k}\ \mathcal{U}(\mathcal{U}+a)(\mathcal{U}+2a)\cdots(\mathcal{U}+(k-1)a)\mathcal{V}^k$$
$$= \sum_{k=1}^{n+1} b^{n+1-k}S(n+1,k)\mathcal{U}(\mathcal{U}+a)(\mathcal{U}+2a)\cdots(\mathcal{U}+(k-1)a)\mathcal{V}^k,$$

where we used the well known recurrence formula $S(n+1,k) = S(n,k-1) + kS(n,k)$. □

Let now $\sigma > 0$, $\alpha,\beta \in \mathbb{C}\backslash\{0\}$. Define linear operators $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{P}(\mathbb{C})$ by (1.17). It is straightforward to see that the operators $\mathcal{U}$, $\mathcal{V}$ generate a generalized Weyl algebra as discussed above with $a = \beta$ and $b = \alpha - \beta$. Let $\mathcal{R} = \mathcal{U}\mathcal{V}$.

Since $\mathcal{V}1 = 1$ and $\mathcal{U} = Z + \frac{\sigma}{\alpha}$, Proposition A.3 immediately implies

Corollary A.5. We have

$$(\mathcal{R}^n 1)(z) = \sum_{k=1}^{n} (\alpha-\beta)^{n-k}S(n,k)\ (z+\sigma/\alpha \mid -\beta)_k. \tag{A5}$$

The following proposition explains a connection between the generalized Weyl algebra generated by $\mathcal{U}$ and $\mathcal{V}$ and an orthogonal Sheffer sequence.

Proposition A.6. Let $\sigma > 0$ and $\alpha,\beta \in \mathbb{C}\backslash\{0\}$. Let $(p_n(z))_{n=0}^{\infty}$ be the monic polynomial sequence satisfying by the recurrence formula

$$zp_n(z) = p_{n+1}(z) + (\lambda n + \sigma/\alpha)p_n(z) + (\sigma n + \eta n(n-1))p_{n-1}(z), \quad n \in \mathbb{N}_0, \tag{A6}$$

where $\lambda = \alpha + \beta$ and $\eta = \alpha\beta$. In particular, for $\alpha \geq \beta > 0$, we have $s_n(z) = p_n(z)$, and for $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$ and $\beta = \overline{\alpha}$, we have $s_n(z) = p_n(z + \frac{\sigma}{\alpha})$. Define a linear bijective operator $\mathcal{I}$ in $\mathcal{P}(\mathbb{C})$ by setting $\mathcal{I}(\cdot \mid \beta)_n = p_n$ for $n \in \mathbb{N}_0$. Then $Z = \mathcal{I}\mathcal{R}\mathcal{I}^{-1}$.

Proof. We have $\mathcal{R} = \alpha Z D_\beta + Z + \sigma D_\beta + \frac{\sigma}{\alpha}$. Recall that D_β is the lowering operator for the monic polynomial sequence $((z \mid \beta)_n)_{n=0}^{\infty}$. Furthermore, it is easy to see that $z(z \mid \beta)_n = (z \mid \beta)_{n+1} + n\beta(z \mid \beta)_n$. In view of the recurrence formula (A6), the statement easily follows. □

As a special case of generalized Stirling numbers of Hsu and Shiue,[21] we define, for $0 \leq k \leq n$ and $h, r \in \mathbb{C}$, numbers $S(n,k;\ h,r)$ as the coefficients of the expansion $(z + r \mid h)_n = \sum_{k=0}^{n} S(n,k;h;r)(z \mid -h)_k$.

Recall that the (unsigned) Lah numbers, $L(n,k)$, are defined as the coefficients of the expansion $(z)_n = \sum_{k=1}^{n} (-1)^{n-k} L(n,k)(z)^{(k)}$. Explicitly, $L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!}$. Note that $L(n,k) = (-1)^{n-k} S(n,k;\ 1,0) = S(n,k;-1,0)$.

Lemma A.7. We have $S(n,0;\ h,r) = (r \mid h)_n$ and for $k = 1, \ldots, n$,

$$S(n,k;h,r) = \sum_{j=0}^{n-k} \binom{n}{j} (-h)^{n-j-k} L(n-j,k)(r \mid h)_j. \tag{A7}$$

Proof. Since the h-derivative D_h is the lowering operator for the monic polynomial sequence $((z \mid h)_n)_{n=0}^{\infty}$ and $(0 \mid h)_n = 0$ for all $n \in \mathbb{N}$, $((z \mid h)_n)_{n=0}^{\infty}$ is a polynomial sequence of binomial type, see e.g., Ref. 26, 4.3.3 Theorem. Hence,

$$\begin{aligned}
(z + r \mid h)_n &= \sum_{i=0}^{n} \binom{n}{i} (z \mid h)_i\ (r \mid h)_{n-i} = (r \mid h)_n + \sum_{i=1}^{n} \binom{n}{i} h^i \left(\frac{z}{h}\right)_i (r \mid h)_{n-i} \\
&= (r \mid h)_n + \sum_{i=1}^{n} \binom{n}{i} h^i (r \mid h)_{n-i} \sum_{k=1}^{i} (-1)^{i-k} L(i,k) \left(\frac{z}{h}\right)^{(k)} \\
&= (r \mid h)_n + \sum_{i=1}^{n} \binom{n}{i} h^i (r \mid h)_{n-i} \sum_{k=1}^{i} (-1)^{i-k} L(i,k)\ h^{-k} (z \mid -h)_k \\
&= (r \mid h)_n + \sum_{k=1}^{n} \left(\sum_{i=k}^{n} \binom{n}{i} (-h)^{i-k} (r \mid h)_{n-i}\ L(i,k) \right) (z \mid -h)_k \\
&= (r \mid h)_n + \sum_{k=1}^{n} \left(\sum_{j=0}^{n-k} \binom{n}{n-j} (-h)^{n-j-k} (r \mid h)_j\ L(n-j,k) \right) (z \mid -h)_k. \qquad \square
\end{aligned}$$

The following result can be of independent interest.

Theorem A.8. *Let $(p_n(z))_{n=0}^{\infty}$ be a monic polynomial sequence as in Proposition A.6. We have*

$$z^n = \sum_{k=1}^{n} (\alpha - \beta)^{n-k}\ S(n,k)\ (\sigma/\alpha \mid -\beta)_k + \sum_{i=1}^{n} \left(\sum_{k=i}^{n} (\alpha - \beta)^{n-k} S(n,k) S(k,i;-\beta,\sigma/\alpha) \right) p_i(z) \tag{A8}$$

and

$$p_n(z) = (-\sigma/\alpha \mid \beta)_n + \sum_{i=1}^{n} \left(\sum_{k=i}^{n} S(n,k;\beta,-\sigma/\alpha)(\alpha - \beta)^{k-i} s(k,i) \right) z^i. \tag{A9}$$

Proof. By Corollary A.5, we have

$$\begin{aligned}
(\mathcal{R}^n 1)(z) &= \sum_{k=1}^{n} (\alpha - \beta)^{n-k} S(n,k) \sum_{i=0}^{k} S(k,i;-\beta,\sigma/\alpha)(z \mid \beta)_i \\
&= \sum_{k=1}^{n} (\alpha - \beta)^{n-k} S(n,k)(\sigma/\alpha \mid -\beta)_k + \sum_{i=1}^{n} \sum_{k=i}^{n} (\alpha - \beta)^{n-k} S(n,k)\ S(k,i;-\beta,\sigma/\alpha)(z \mid \beta)_i.
\end{aligned} \tag{A10}$$

Applying the operator $\mathcal{I}$ to (A10) and using Proposition A.6, we obtain (A8).

Recall Corollary A.5. Note that there exists a unique monic polynomial sequence $(q_n(z))_{n=0}^{\infty}$ that satisfies $(\mathcal{R}^n 1)(z) = \sum_{k=1}^{n} (\alpha - \beta)^{n-k} S(n,k) q_k(z)$ for $n \in \mathbb{N}_0$ and $q_n(z) = (z + \sigma/\alpha \mid -\beta)_n$.

Define the monic polynomial sequence $(\tilde{q}_n(z))_{n=0}^{\infty}$ by

$$\tilde{q}_n(z) = \sum_{k=1}^{n} (\alpha - \beta)^{n-k} s(n,k)\ (\mathcal{R}^n 1)(z).$$

We state that $q_n(z) = \tilde{q}_n(z)$, i.e.,

$$(z+\sigma/\alpha \mid -\beta)_n = \sum_{k=1}^{n} (\alpha-\beta)^{n-k} s(n,k)\ (\mathcal{R}^n 1)(z). \tag{A11}$$

Indeed, using formula (A1), we have

$$\begin{aligned}\sum_{k=1}^{n} (\alpha-\beta)^{n-k} S(n,k)\tilde{q}_k(z) &= \sum_{k=1}^{n} (\alpha-\beta)^{n-k} S(n,k) \sum_{i=1}^{k} (\alpha-\beta)^{k-i} s(k,i)(\mathcal{R}^i 1)(z) \\ &= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} S(n,k)s(k,i)\right)(\alpha-\beta)^{n-i}(\mathcal{R}^i 1)(z) = (\mathcal{R}^n 1)(z),\end{aligned}$$

which proves (A11).

By Lemma A.7, (A11) and the definition of the generalized Stirling numbers $S(n,k;\ \beta,-\sigma/\alpha)$, we have

$$\begin{aligned}(z \mid \beta)_n &= \sum_{k=0}^{n} S(n,k;\beta,-\sigma/\alpha)(z+\sigma/\alpha \mid -\beta)_k \\ &= S(n,0;\beta,-\sigma/\alpha) + \sum_{k=1}^{n} S(n,k;\beta,-\sigma/\alpha)\left(\sum_{i=1}^{k} (\alpha-\beta)^{k-i} s(k,i)(\mathcal{R}^i 1)(z)\right) \\ &= (-\sigma/\alpha \mid \beta)_n + \sum_{i=1}^{n} \left(\sum_{k=i}^{n} S(n,k;\beta,-\sigma/\alpha)(\alpha-\beta)^{k-i} s(k,i)\right)(\mathcal{R}^i 1)(z).\end{aligned} \tag{A12}$$

Applying $\mathcal{I}$ to (A12) and using Proposition A.6, we obtain (A9). □

The corollary below follows immediately from formula (A8).

Corollary A.9. Let $(p_n(z))_{n=0}^{\infty}$ be a monic polynomial sequence as in Proposition A.6. Let $\Phi : \mathcal{P}(\mathbb{C}) \to \mathbb{C}$ be a linear functional defined by $\Phi(1) = 1$ and $\Phi(p_n) = 0$ for all $n \in \mathbb{N}$. Then $\Phi(z^n) = \sum_{k=1}^{n} (\alpha-\beta)^{n-k}\ S(n,k)\ (\sigma/\alpha \mid -\beta)_k$. In particular, for $\alpha \geq \beta > 0$,

$$\int_{\mathbb{R}} x^n\ \mu_{\alpha,\beta,\sigma}(dx) = \sum_{k=1}^{n} (\alpha-\beta)^{n-k} S(n,k)(\sigma/\alpha \mid -\beta)_k \tag{A13}$$

and for $\Re(\alpha) \geq 0$, $\Im(\alpha) > 0$, $\beta = \overline{\alpha}$,

$$\int_{\mathbb{R}} (x+\sigma/\alpha)^n\ \mu_{\alpha,\beta,\sigma}(dx) = \sum_{k=1}^{n} (\alpha-\beta)^{n-k} S(n,k)(\sigma/\alpha \mid -\beta)_k. \tag{A14}$$

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