Timelike entanglement entropy: A top-down approach

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We investigate the concept of timelike entanglement entropy (tEE) within the framework of holography. We introduce a robust top-down prescription for computing tEE using the holographic duals to higher-dimensional quantum field theories—both conformal and confining—eliminating the ambiguities typically associated with analytic continuation from Euclidean to Lorentzian signatures. We present accurate analytic approximations for tEE and timelike separations in slab geometries. We establish a clear stability criterion for bulk embeddings and demonstrate that tEE serves as a powerful tool for computing conformal field theory central charges, extending and strengthening previous results. Finally, we apply our framework to holographic confining backgrounds, revealing distinctive behaviors like phase transitions.

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I. INTRODUCTION AND GENERAL IDEA

Maldacena's conjecture [1], along with its refinements [2,3], introduced the idea that space can emerge from the strongly coupled dynamics of a quantum field theory (QFT). One compelling approach to understanding this emergence of space is through entanglement entropy (EE), computed holographically as proposed by Ryu and Takayanagi in [4,5]; see also [6]. In particular, the works [7,8] clearly suggest how spacelike coordinates may emerge from a QFT. This naturally raises the question of whether a timelike coordinate could also emerge from strong dynamics.

With this motivation (among others), the concept of timelike entanglement entropy (tEE) was introduced in [9,10]. From the holographic point of view, the notion of tEE was originally introduced in order to understand the emergence of time from quantum entanglement in timelike subsystems in QFTs [9,10], very similar in spirit to the original Ryu and Takayanagi prescription [4,5] where space emerges from quantum entanglement among spacelike subsystems. Unlike the usual EE for spacelike separated intervals, the analog of the reduced density matrix (also known as the transition matrix) for timelike separated events is non-Hermitian. Although the computation of tEE in higher dimensional conformal field theories (CFTs) is a nontrivial task, for 2D CFT this has been obtained following a Wick rotation of the subregion (A) associated with the Hilbert space ($\mathcal{H}_A$) [10]. The topic has since become an active area of research, especially within the holographic context. Among the works most influential to our study are [11–20].

In the original studies and subsequent developments, the holographic computation of timelike entanglement entropy involves extremizing a surface in the Euclidean regime (where it coincides with the standard RT-EE), followed by analytic continuation to Lorentzian time. However, this analytic continuation remains conceptually challenging for higher-dimensional QFT_d ($d \geq 3$), particularly in slab or spherical regions.

One of the goals of the present work is to address this challenge and offer an alternative prescription for computing timelike entanglement entropy using the holographic prescription for duals to higher-dimensional QFTs—whether conformal or not. Our approach provides a natural transition from Euclidean to Lorentzian (or timelike) entanglement entropy and establishes a framework for mapping QFT features in a real-time formalism. Another achievement of this work is the derivation of analytic expressions that approximate the timelike EE and timelike separation for slab regions, while also offering a criterion for stability and a computable method for predicting potential phase transitions.

We further present formulas using tEE to compute central charges of CFTs, following the approach of Liu and Mezei [21,22]. In addition, we explore two holographic duals of confining QFTs and analyze the tEE and its associated phase transitions in these models.

In the remainder of this section, we recall well-established formulas for the strip timelike entanglement

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entropy, closely following those derived for Wilson loops (see, for instance, [23]). We also provide analytic expressions that approximate the timelike EE and the timelike separation, and we formulate a criterion for the stability of the entangling surface in the bulk. Consider a string background (the extension to eleven dimensions is immediate), holographically dual to a QFT in d space-time dimensions. The string frame metric and dilaton read (there are Ramond and Neveu-Schwarz fields that we do not quote) as

$$ds^2_{st} = f(u,\vec{y})[\lambda dt^2 + d\vec{x}^2_{d-2} + dv^2 + g(u)du^2] + g_{ij,(9-d)}(u,\vec{y})dy^i dy^j, \qquad \Phi(u,\vec{y}). \tag{1}$$

We introduced the parameter $\lambda = \pm 1$ to indicate Euclidean metrics, with isometry $SO(d)$ or Lorentzian ones, with isometry $SO(1,d-1)$. This assumption can be relaxed and is discussed in the confining models studied below. We embed an eight-surface Σ_8 parametrized by the coordinates $[\vec{x}_{d-2},\vec{y}_{9-d},u]$, with $v=$ constant and $t=t(u)$. The induced metric of the eight manifold is

$$\begin{aligned} ds^2_{\Sigma_8} &= f(u,\vec{y})[d\vec{x}^2_{d-2} + (g(u)+\lambda t'^2)du^2] + g_{ij,(9-d)}dy^i dy^j. \\ \det[g_{\Sigma_8}] &= f(u,\vec{y})^{d-1}\det[g_{9-d}](g(u)+\lambda t'^2). \end{aligned} \tag{2}$$

We calculate the timelike entanglement entropy for a strip (or slab) region as

$$\begin{aligned} S_{tEE} &= \frac{1}{4G_N}\int d^{9-d}y d^{d-2}x du \sqrt{e^{-4\Phi}\det g_{\Sigma_8}}. \\ S_{tEE} &= \frac{\int d^{d-2}x}{4G_N}\int du \int d^{9-d}y \sqrt{e^{-4\Phi(u,\vec{y})} f(u,\vec{y})^{d-1}\det[g_{9-d}](g(u)+\lambda t'^2)}. \end{aligned} \tag{3}$$

The evaluation of the integral over the $(d-2)$ coordinates represented by $\vec{x}$ is immediate (as they are isometric coordinates). Less trivial could be the evaluation over the $(9-d)$ coordinates represented by $\vec{y}$. It usually occurs that the expression $e^{-4\Phi(u,\vec{y})}f(u,\vec{y})^{d-1}\det[g_{9-d}]$ factorizes into a function of $\vec{y}$ times a function of the radial coordinate u. This is the case in the examples discussed below and in those studied in the forthcoming work [24]. After evaluating the integrals over the seven coordinates $(\vec{y},\vec{x})$ we arrive at a generic expression for the timelike entropy on slab regions,

$$S_{tEE} = \frac{\mathcal{N}}{4G_{10}}\int_{u_0}^{\infty} du \sqrt{G^2 + F^2 t'^2(u)}. \tag{4}$$

Here $\mathcal{N}$ is a constant, and F, G are functions of the radial coordinate u.

One can define a function $\mathcal{V}$ in terms of which separation between the two regions is expressed. The function and the timelike separation in the case of a strip are

$$\begin{aligned} \mathcal{V} &= \frac{F(u)}{G(u)F(u_0)}\sqrt{F^2(u)-F^2(u_0)}, \\ T &= 2\int_{u_0}^{\infty}\frac{du}{\mathcal{V}}. \end{aligned} \tag{5}$$

Using a first integral of the equations of motion derived from Eq. (4), the timelike entanglement entropy reads as

$$\begin{aligned} \frac{4G_{10}}{\mathcal{N}}S_{tEE} &= F(u_0)T + 2\int_{u_0}^{\infty} du \frac{G(u)}{F(u)}\sqrt{F^2(u)-F^2(u_0)} - 2\int_{u_*}^{\infty} G(u)du \\ &= 2\int_{u_0}^{\infty} du \frac{F(u)G(u)}{\sqrt{F^2(u)-F^2(u_0)}} - 2\int_{u_*}^{\infty} G(u)du. \end{aligned} \tag{6}$$

Usually, analytically evaluating the integrals in Eqs. (5) and (6) is not possible. There are approximate expressions that are very useful. For the time separation T in the case of the strip the expression proposed in [25] reads as

$$T_{app} = \frac{\pi G(u)}{F'(u)}\bigg|_{u=u_0}. \tag{7}$$

A useful quantity indicating whether the embedding is stable under fluctuations was proposed in [26]. This quantity is

$$Z(u_0) = \frac{d}{du}\left(\frac{\pi G(u)}{F'(u)}\right)\bigg|_{u=u_0}. \tag{8}$$

In fact, stability of the embedding is implied by $Z(u_0) < 0$; on the other hand for positive $Z(u_0)$ the embedding is unstable [26]. This criterion for stability can be proven using the formal similarity between the action for a generic Wilson loop and that of the timelike EE in Eq. (4). The proof is presented in [24]. Intuitively, it can be understood as follows: if the time separation is not monotonically increasing toward smaller values of u_0, there is a value of u_0 for which there are two possible time separations. Characteristically, this leads to a competition between embeddings and a phase transition. This is encoded by the derivative of the time separation turning positive $Z(u_0) > 0$. We discuss examples of this phase transition in the confining models discussed below.

For the approximate timelike entanglement entropy of a strip we present an expression that relies on Eq. (7). The derivation is presented in [24],

$$\begin{aligned}\frac{4G_{10}}{\mathcal{N}} S_{tEE,app}(u_0) &= \int^{u_0} dz\, F(z) T_{app}{}'(z) \\ &= \int^{u_0} dz \frac{F(z)}{F'(z)^2}[G'(z)F'(z) - G(z)F''(z))]. \quad (9)\end{aligned}$$

Using these expressions, let us study the timelike entanglement entropy for generic CFTs in d dimensions.

II. TIMELIKE ENTANGLEMENT ENTROPY FOR GENERIC CFTS

We develop a top-down approach to derive expressions for the timelike entanglement entropy and for the time separation. We reproduce and generalize expressions already derived using a bottom-up approach [9,10] and give a nice interpretation to the result in [10]. We present expressions for the timelike entanglement entropy and the time separation, both for the cases of slabs and for spherical entangling regions. Let us consider a family of backgrounds dual to generic CFTs in d dimensions:

$$ds_{10}^2 = f_1(y) ds^2_{\mathrm{AdS}_{d+1}} + g_{ij}(y) dy^i dy^j, \quad (10)$$

where g_{ij} is the metric of the internal manifold and

$$ds^2_{\mathrm{AdS}_{d+1}}|_{\Sigma^{(\lambda)}} = u^2(\lambda dt^2 + dv^2 + d\mathbf{x}_{d-2}) + \frac{du^2}{u^2}, \quad (11)$$

that is used to compute the tEE in strip regions. For spherical/hyperbolic regions we write the anti–de Sitter (AdS) metric as

$$ds^2_{\mathrm{AdS}_{d+1}}|_{\hat{\Sigma}^{(\lambda)}} = u^2(\lambda dt^2 + t^2 d\Omega^{(\lambda)}_{d-2} + dv^2) + \frac{du^2}{u^2}. \quad (12)$$

We use the parameter $\lambda = \pm 1$ with (+) to indicate the Euclidean case and (−) the Minkowski signature. This parameter keeps track of the character of the calculation. For $\lambda = +1$ $\Omega^{(\lambda)}$ denotes a sphere whilst for $\lambda = -1$, it denotes the compact part of a hyperbolic space.

In each of the above cases the eight manifold needed to calculate the timelike entanglement entropy is parametrized by setting $v = 0$ in Eqs. (10)–(12), and considering an embedding of the form $t = t(u)$. The line elements and expressions for the separation T and the tEE are studied below in each case.

A. The case of the strip

The line element of the eight manifold is given by

$$\begin{aligned}ds_8^2\Big|_{\Sigma_8^{(\lambda)}} &= f_1(y)(1 + \lambda u^4 t'^2(u)) \frac{du^2}{u^2} + f_1(y) u^2 d\mathbf{x}_{d-2} \\ &\quad + g_{ij}(y) dy^i dy^j. \quad (13)\end{aligned}$$

The timelike entanglement entropy reads as

$$\begin{aligned}S_{EE}^{(\lambda)}\left[\Sigma_8^{(\lambda)}\right] &= \frac{1}{4G_{10}} \int d^8x \sqrt{e^{-4\Phi} \det g_8} \\ &= \frac{\mathcal{N}}{4G_{10}} \int_{u_0}^{\infty} du u^{d-3} \sqrt{1 + \lambda u^4 t'^2(u)}. \quad (14)\end{aligned}$$

The timelike entanglement entropy arises when $\lambda = -1$. For $\lambda = 1$, we have the usual Ryu-Takayanagi entanglement entropy [4]. In Eq. (14) we denoted

$$\begin{aligned}\mathcal{N} &= L^{(d-2)} \int d^{9-d}y \sqrt{e^{-4\Phi} \det g_{ij}} f_1^{\frac{d-1}{2}}(y) \\ L^{(d-2)} &= \int d^{d-2}x. \quad (15)\end{aligned}$$

The expression in Eq. (14) has a generic structure of the form in Eq. (4), with

$$G = u^{d-3} \quad \text{and} \quad F = \sqrt{\lambda} u^{d-1}. \quad (16)$$

Following the expressions in Eq. (5), we find

$$\mathcal{V} = \frac{\sqrt{\lambda} u^2}{u_0^{d-1}} \sqrt{u^{2d-2} - u_0^{2d-2}}. \quad (17)$$

The time separation is

$$\begin{aligned}T &= \frac{2}{\sqrt{\lambda}} u_0^{d-1} \int_{u_0}^{\infty} \frac{du}{u^2} \frac{1}{\sqrt{u^{2d-2} - u_0^{2d-2}}} \\ &= \frac{2\sqrt{\pi}\Gamma(\frac{d}{2d-2})}{\sqrt{\lambda}\Gamma(\frac{1}{2d-2})} \times \frac{1}{u_0}. \quad (18)\end{aligned}$$

Note that the time separation is purely imaginary for the Minkowski signature ($\lambda = -1$). In contrast to [10], we *do not* continue $T \to iT$; all the effect of calculating with the Lorentzian signature is encoded by λ. The approximate expression in Eq. (7) for T_{app} captures the behavior of the function $T(u_0)$. In fact, using Eq. (4),

$$T_{app} = \pi \frac{G}{F'}\Big|_{u=u_0} = \frac{\pi}{\sqrt{\lambda}(d-1)u_0}. \quad (19)$$

Notice that $Z(u_0)$ defined in Eq. (8) is always negative indicating that all these embeddings are stable.

We compute the tEE and its approximate expression, using Eqs. (4) and (9). We find

$$\begin{aligned}\frac{4G_{10}}{\mathcal{N}} S^{(\lambda)}_{EE}[\Sigma^{(\lambda)}_8] &= \left[\frac{2}{(2-d)}\,{}_2F_1\left(\frac{1}{2},\frac{2-d}{2d-2};\frac{d}{2(d-1)};1\right)\right]\times u_0^{d-2}\\ &= \frac{1}{(2-d)}\left[2\sqrt{\pi}\frac{\Gamma(\frac{d}{2d-2})}{\Gamma(\frac{1}{2d-2})}\right]^{(d-1)}\frac{1}{\sqrt{\lambda^{d-2}}|T|^{d-2}}. \quad (20)\end{aligned}$$

We have used Eq. (18) to replace u_0 in terms of T. The expression of Eq. (20) is an exact and regularized expression, valid for $3 \leq d$. We compare this expression with those in equations (4.36)–(4.37) in the paper [10], obtaining agreement. Note that depending on the spacetime dimension d, one may have imaginary values for the tEE when written in terms of the time separation T. We believe this imaginary value should be considered physical. In special circumstances this imaginary value can be understood in field theoretic terms; see [18,19].

In [10] the Newton constant in $(d+1)$ dimensions is used; here this appears in the quotient $\frac{1}{G_{d+1}} = \frac{\mathcal{N}}{G_{10}}$. As we explain below, the effective Newton constant in lower dimensions depends on the particular dual CFT, and it is related to the central charge of such dual CFT.

For Lorentzian signature ($\lambda=-1$), the quantities T, T_{app} are purely imaginary. The timelike entanglement entropy in terms of T is real for even d and purely imaginary for odd d (we remind the reader that d is the dimension of the holographically dual QFT). We calculate the approximate tEE using Eq. (9). We find

$$S_{app} = \int^{u_0} dz F(z) T'_{app}(z) = -\frac{\pi}{(d^2-3d+2)} u_0^{d-2}. \quad (21)$$

This expression should be supplemented by an integration constant, that we omitted above. The same logic for the Lorentzian result applies.

The result in Eq. (21) agrees with the exact behavior in terms of the turning point u_0. Combining Eqs. (19) and (21) we find

$$S_{\rm app} = -\frac{\pi^{d-1}}{\sqrt{\lambda^{(d-2)}}(d-1)^{d-2}(d^2-3d+2)}\frac{1}{|T_{\rm app}|^{d-2}}. \quad (22)$$

Again, we note that for the Minkowski signature ($\lambda=-1$) the approximate timelike entanglement in terms of the approximate time separation is real for even dimensions and purely imaginary in odd dimensions.

Let us now replace the slab/strip by a sphere or a hyperboloid.

B. The case of the sphere/hyperboloid

After setting the coordinate $v=0$ and $t=t(u)$ in Eqs. (10) and (12), the corresponding line element of the eight manifold needed to calculate tEE is

$$\begin{aligned} ds^2_8|_{\hat{\Sigma}^{(\lambda)}_8} = f_1(y)(1+\lambda u^4 t'^2(u))\frac{du^2}{u^2} + f_1(y)u^2t^2 d\Omega^{(\lambda)}_{d-2}\\ + g_{ij}(y)dy^i dy^j. \quad (23)\end{aligned}$$

The tEE is given by

$$\begin{aligned} S^{(\lambda)}_{EE}\left[\hat{\Sigma}^{(\lambda)}_8\right] &= \frac{1}{4G_{10}}\int d^8x\sqrt{e^{-4\Phi}\det g_8}\\ &= \frac{\hat{\mathcal{N}}}{4G_{10}}\int_{u_0}^{\infty} du u^{d-3} t^{d-2}\sqrt{1+\lambda u^4 t'^2(u)}, \quad (24)\end{aligned}$$

where we denote

$$\hat{\mathcal{N}} = \mathrm{Vol}\left(\Omega^{(\lambda)}_{d-2}\right)\int d^{9-d}y\sqrt{e^{-4\Phi}\det g_{ij}} f_1^{\frac{d-1}{2}}(y). \quad (25)$$

The equation of motion that follows from the "action" in Eq. (24) is solved by

$$t(u) = \frac{\sqrt{R^2u^2-\lambda}}{u}. \quad (26)$$

We introduce a small parameter ϵ to regulate UV divergencies and change variables according to $u = \frac{\sqrt{\lambda}}{R}x$. Using Eq. (26), the tEE in Eq. (24) reads as

$$\frac{4G_{10}S^{(\lambda)}_{EE}\left[\hat{\Sigma}^{(\lambda)}_8\right]}{\hat{\mathcal{N}}\lambda^{(d-2)/2}} = \int_1^{\frac{R}{\sqrt{\lambda}\epsilon}} dx (x^2-1)^{\frac{d-3}{2}}. \quad (27)$$

Let us evaluate this integral. For odd dimension d the result is (see [27])

$$\begin{aligned}\frac{4G_{10}S^{(\lambda)}_{EE}\left[\hat{\Sigma}^{(\lambda)}_8\right]}{\hat{\mathcal{N}}\lambda^{(d-2)/2}} = \sum_{j=0}^{[\frac{d-3}{2}]}\frac{(\frac{3-d}{2})_j}{j!(d-2j-2)}\left(\frac{R}{\sqrt{\lambda}\epsilon}\right)^{d-2j-2}\\ -(-1)^{\frac{d+1}{2}}\frac{\sqrt{\pi}\Gamma\left(\frac{(d-1)}{2}\right)}{2\Gamma(\frac{d}{2})}. \quad (28)\end{aligned}$$

We used the Pochhammer symbol $(\frac{3-d}{2})_j = \frac{\Gamma(\frac{3-d}{2}+j)}{\Gamma(\frac{3-d}{2})}$. For even d, we follow [27] to obtain

$$\frac{4G_{10}S_{EE}^{(\lambda)}\left[\hat{\Sigma}_8^{(\lambda)}\right]}{\hat{\mathcal{N}}\lambda^{(d-2)/2}} = \sum_{j=0}^{[\frac{d-3}{2}]}\frac{(\frac{3-d}{2})_j}{j!(d-2j-2)}\left(\frac{R}{\sqrt{\lambda}\epsilon}\right)^{d-2j-2} - \frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})}\frac{(-1)^{d/2}}{\sqrt{\pi}}\left(\log\left(\frac{2R}{\epsilon\sqrt{\lambda}}\right)+\frac{1}{2}\mathcal{H}_{\frac{d-2}{2}}\right). \tag{29}$$

We denoted by $\mathcal{H}_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ the harmonic numbers.

For $\lambda = +1$, this result coincides with that written in equations (4.5) and (4.6) of Ref. [10]. For $\lambda = -1$, this reproduces the bottom-up result of equations (4.9)–(4.10) of [10].

Notice that we bypass the important problem of finding the timelike or spacelike surface homologous to the timelike subregion. This problem was carefully discussed in [17].

In summary, our approach uses an eight manifold if the holographic background is in Type II and a nine manifold if it were written in 11-dimensional supergravity. It clarifies the relation between the ten-dimensional Newton constant G_{10} and the one on lower dimensional supergravity of the bottom-up approach $G_{d+1} = \frac{G_{10}\mathrm{Vol}\Omega_{d-2}}{\hat{\mathcal{N}}}$. We now display a relation between the central charge of the dual CFT and $\hat{\mathcal{N}}$.

C. A Liu-Mezei central charge

It is natural, given the results in Eqs. (28) and (29), to apply the Liu-Mezei [21] formalism to define a central charge. In particular, we define for dimension d odd

$$(d-2)!!c_{LM,\mathrm{odd}} = (R\partial_R - 1)\ldots(R\partial_R - d + 2)S_{EE}^{(\lambda)}\left[\hat{\Sigma}_8^{(\lambda)}\right]. \tag{30}$$

For dimension d even, we have

$$(d-2)!!c_{LM,\mathrm{even}} = R\partial_R\ldots(R\partial_R - d + 2)S_{EE}^{(\lambda)}\left[\hat{\Sigma}_8^{(\lambda)}\right]. \tag{31}$$

In both cases, we should take *the absolute value of the result* to guarantee a positive number. We can also define a central charge using the slab timelike entanglement entropy by calculating

$$c_{\mathrm{slab}} \propto \frac{T^{d-2}}{L^{d-2}}T\partial_T S_{EE}^{(\lambda)}. \tag{32}$$

For related treatment of central charge using the tEE, see Ref. [28]. As an illustration of the previous results, we test all the expressions derived in this section for an infinite family of four-dimensional $N = 2$ superconformal field theories (SCFTs). These checks can be generalized; see [24].

D. An explicit example

To illustrate our expressions, let us discuss the tEE for $N = 2$ SCFTs in four dimensions. The SCFTs of choice for this example are long linear quivers (of length $P \to \infty$). See Refs. [29–35] for a summary of field theory aspects and to set the notation. The background consists of a metric, a dilaton, NS H_3 field, and Ramond F_2, F_4 fields. For our calculation, only the metric and dilaton are needed. These read (in a convention where $g_s = \alpha' = 1$) as

$$ds_{10}^2 = \sqrt{\tilde{f}_1^3\tilde{f}_5}\Big[4ds_{\mathrm{AdS}_5}^2 + \tilde{f}_2 d\Omega_2(\theta,\phi) + \tilde{f}_3 d\chi^2 + \tilde{f}_4(d\sigma^2 + d\eta^2)\Big],$$
$$e^{-4\Phi} = (\tilde{f}_1\tilde{f}_5)^{-3}. \tag{33}$$

The functions $\tilde{f}_i(\sigma,\eta)$ are expressed in terms of a single function $V(\sigma,\eta)$ as [34,35]

$$\tilde{f}_1^3 = \frac{\dot{V}\Delta}{2V''}, \quad \tilde{f}_2 = \frac{2V''\dot{V}}{\Delta}, \quad \tilde{f}_3 = \frac{4\sigma^2 V''}{2\dot{V}-\ddot{V}}, \quad \tilde{f}_4 = \frac{2V''}{\dot{V}},$$
$$\tilde{f}_5 = \frac{2(2\dot{V}-\ddot{V})}{\dot{V}\Delta}, \quad \Delta = (2\dot{V}-\ddot{V})V'' + (\dot{V}')^2,$$
$$\dot{V} = \sigma\partial_\sigma V, \quad \ddot{V} = \sigma\partial_\sigma\dot{V}, \quad V'' = \partial_\eta^2 V. \tag{34}$$

In turn, the function $V(\sigma,\eta)$ can be written in terms of the CFT data, encoded in a rank function $R(\eta)$, with Fourier decomposition coefficients R_k,

$$V(\sigma,\eta) = -\sum_{k=1}^{\infty} R_k \sin\left(\frac{k\pi\eta}{P}\right)K_0\left(\frac{k\pi\sigma}{P}\right)$$
$$\text{where } R_k = \frac{2}{P}\int_0^P \mathcal{R}(\eta)\sin\left(\frac{k\pi\eta}{P}\right)d\eta. \tag{35}$$

In order to compute tEE, we identify the quantities in Eqs. (10)–(12),

$$d=4, \quad f_1(y) = 4(\tilde{f}_1^3\tilde{f}_3)^{1/2},$$
$$g_{ij}dy^i dy^j = \sqrt{\tilde{f}_1^3\tilde{f}_5}[\tilde{f}_2 d\Omega_2(\theta,\phi) + \tilde{f}_3 d\chi^2 + \tilde{f}_4(d\sigma^2 + d\eta^2)]. \tag{36}$$

Using Eqs. (15) and (25) we find

$$\mathcal{N} = 256\pi^2 L^2\int_0^\infty d\sigma\int_0^P d\eta\,\dot{V}V''\sigma, \tag{37}$$

$$\hat{\mathcal{N}} = 256\pi^2\mathrm{Vol}S_\lambda^2\int_0^\infty d\sigma\int_0^P d\eta\,\dot{V}V''\sigma. \tag{38}$$

Using Eq. (35) it follows that

$$\int_0^\infty d\sigma \int_0^P d\eta\, \dot{V} V'' \sigma = \frac{P}{4} \sum_{k=1}^{\infty} R_k^2; \tag{39}$$

from this we find

$$\mathcal{N} = 64\pi^2 L^2 P \sum_{k=1}^{\infty} R_k^2, \quad \hat{\mathcal{N}} = 64\pi^2 \mathrm{Vol} S_\lambda^2 P \sum_{k=1}^{\infty} R_k^2. \tag{40}$$

The coefficients in Eqs. (37)–(40) appear when computing the free energy of any member of the family of 4D SCFTs; see for example [33,35]. In other words the tEE captures (by intermediate of the coefficients $\mathcal{N}$ and $\hat{\mathcal{N}}$) the central charge of the dual CFT. This result is in agreement with that obtained via the localization-matrix model approach; see [36].

The expressions for the timelike entanglement entropy and the time separation T in the case of the slab are

$$T = \frac{\zeta}{\sqrt{\lambda} u_0}, \qquad \frac{4G_{10}}{\mathcal{N}} S_{EE}^{(\lambda)}\left[\Sigma_8^{(\lambda)}\right] = -\frac{\zeta}{2} u_0^2,$$
$$\zeta = 2\sqrt{\pi} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{6} p)}. \qquad S_{EE}^{(\lambda)}\left[\Sigma_8^{(\lambda)}\right] = -\frac{\mathcal{N}}{8G_{10}} \frac{\zeta^3}{\lambda T^2}. \tag{41}$$

Using Eq. (32), we find

$$c_{\mathrm{slab}} \propto \frac{\mathcal{N} \zeta^3}{4G_{10} L^2 \lambda}. \tag{42}$$

Note that $\frac{\pi}{2}\zeta^3 \approx 1$, and $\frac{\mathcal{N}}{L^2}$ is proportional to the volume of the internal manifold that appears in Eq. (15). Hence c_{slab} is proportional to the free energy of the dual CFT obtained by purely field theoretical means. For the tEE on the sphere/hyperbolic space we find

$$S_{EE}^{(\lambda)}[\hat{\Sigma}_8^{(\lambda)}] = \frac{\hat{\mathcal{N}} \lambda}{4G_{10}} \left(\frac{R^2}{2\lambda \epsilon^2} - \frac{1}{2} \log\left(\frac{2R}{\epsilon \sqrt{\lambda}} \right) - \frac{1}{4} \right). \tag{43}$$

Using Eq. (31), the Liu-Mezei central charge is

$$c_{\mathrm{LM}} = \frac{\lambda}{8G_{10}} \hat{\mathcal{N}}. \tag{44}$$

The Euclidean case ($\lambda = +1$) gives the usual central charge. The Lorentzian case suggests that we should take the *absolute value* of the result (as we anticipated), in order to interpret this as a number of degrees of freedom. Note also that c_{slab} and c_{LM} are measuring the same physical observable, related to the free energy of the dual CFT.

The example above can be extended to other dimensions. In fact, for any AdS_{d+1} (with $3 \leq d$) it is feasible to study systems testing the calculations we presented in this section. The details are given in [24]. It is interesting to observe that the definition for timelike entanglement entropy used in this work is invariant under generic U duality. For example, we could have considered the 11-dimensional supergravity description of the family of backgrounds in this section. After calculating the induced metric on a nine manifold, the result obtained for the tEE would be the same. In other words, the dimensionality of the space-time $d = 4$ does not change, as it refers to the quantum field theory and does not rely on the string or M-theory embedding. This is a virtue of our definition being U-duality invariant.

Below we discuss an example with nonconformal field theory and discuss the calculation of the tEE for a strip region in the case of a confining QFT.

III. TIMELIKE ENTANGLEMENT ENTROPY FOR CONFINING SYSTEMS

There are various established holographic duals to confining field theories; see for example [37–39]. To begin with, we choose to work with a model proposed by Witten [37]. The model consists of a stack of D4 branes that wrap a circle. Supersymmetry (SUSY) breaking boundary conditions on the circle are imposed for the fields. The second type of model is a $(2+1)$-dimensional and SUSY version of the Witten model discussed above. It consists of D3 branes that wrap a circle with periodic boundary conditions for bosons and antiperiodic for fermions. A twist is performed on this field theory that allows the preservation of four supercharges. The system flows from $N = 4$ SYM in $(3+1)$ dimensions to a SUSY gapped QFT in $(2+1)$ dimensions. The perturbative spectrum of such a theory is discussed in [40,41]. The holographic dual was presented in [42] and further studied in [43,44]. Let us start our study with the system of D4 branes wrapping S^1.

A. Witten's model for (3 + 1) Yang-Mills

The metric and dilaton (there is also a Ramond F_4 that we do not quote) are

$$ds_{10}^2 = f_1(u)(\lambda dt^2 + dx^2 + dy^2 + dv^2) + f_2(u) d\phi^2 + \frac{du^2}{f_2(u)} + f_3(u) d\Omega_4,$$
$$f_1(u) = \frac{u^{3/2}}{R^{3/2}}; \quad f_2(u) = f_1(u) h(u); \quad h(u) = 1 - \frac{u_\Lambda^3}{u^3};$$
$$f_3(u) = \frac{u^2}{f_1(u)}, \qquad e^{4\Phi(u)} = g_s^4 f_1^2(u). \tag{45}$$

The eight manifold used to calculate the tEE on a strip is $\Sigma_8 = [x, y, \Omega_4, \phi, u]$, with $v = 0$ and $t(u)$. We write

$$4G_{10}S_{tEE,\mathrm{strip}} = \int d^8x\sqrt{e^{-4\Phi}\det[g_8]} = \tilde{\mathcal{N}}\int_{u_0}^{\infty}\sqrt{F^2t'^2 + G^2},$$
$$G = u, F = \sqrt{\lambda}uf_1(u)\sqrt{h(u)},$$
$$\tilde{\mathcal{N}} = \frac{R^3}{g_s^2}\mathrm{Vol}[S^4]L_\phi L_x L_y. \tag{46}$$

Using Eqs. (4)–(6) we write the time separation T and timelike entanglement entropy,

$$T = \frac{2R^{3/2}}{\sqrt{\lambda}}u_0\sqrt{u_0^3 - u_\Lambda^3} \times \int_{u_0}^{\infty}\frac{du}{\sqrt{u^3-u_\Lambda^3}}\frac{1}{\sqrt{u^2(u^3-u_\Lambda^3)-u_0^2(u_0^3-u_\Lambda^3)}}. \tag{47}$$

$$\frac{2G_{10}S_{EE}}{\tilde{\mathcal{N}}} = 2\int_{u_0}^{\infty}du\frac{u^2\sqrt{u^3-u_\Lambda^3}}{\sqrt{u^2(u^3-u_\Lambda^3)-u_0^2(u_0^3-u_\Lambda^3)}} - 2\int_{u_\Lambda}^{\infty}du u. \tag{48}$$

Like in the conformal case, the Lorentzian signature ($\lambda = -1$) case gives a purely imaginary time separation. We evaluate the approximate expressions for the time separation and tEE in Eqs. (7)–(9). We find

$$T_{\mathrm{app}} = \frac{2\pi R^{3/2}u_0^{5/2}\sqrt{1-\frac{u_\Lambda^3}{u_0^3}}}{\sqrt{\lambda}(5u_0^3 - 2u_\Lambda^3)}. \tag{49}$$

$$Z(u_0) = \sqrt{\frac{\pi^2R^3}{\lambda(u_0^3 - u_\Lambda^3)}}\frac{(-5u_0^6 + 10u_0^3u_\Lambda^3 + 4u_\Lambda^6)}{(5u_0^3 - 2u_\Lambda^3)^2} \tag{50}$$

$$S_{\mathrm{app}}(u_0) = -\pi\frac{u_0^2(u_0^3 + 2u_\Lambda^3)}{10u_0^3 - 4u_\Lambda^3}. \tag{51}$$

As we mentioned above, there should be an integration constant in S_{app}, which can be needed to compare with the exact expression (that in this case must be obtained numerically). Notice that the double valuedness of T_{app} indicates the possibility of a phase transition. The function $Z(u_0) < 0$ indicates that the embedding is stable for $u_0^3 > u_\Lambda^3(1 + \frac{3}{\sqrt{5}})$, but unstable for values of u_0 closer to u_Λ. These two indications suggest that the phase transition must take place.

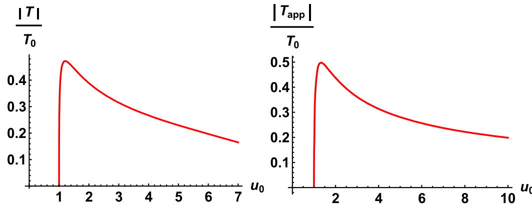

FIG. 1. On the left panel, the exact time separation $|T|$ in Eq. (47) in terms of the turning point u_0. On the right, the approximate $|T_{\mathrm{app}}|$ in Eq. (49) in terms of u_0.

We plot the exact expressions to compare with their analog approximate ones. Figure 1 displays the exact separation T in Eq. (47) in terms of the turning point u_0 and the approximate expression in Eq. (49). In Fig. 2 we show the exact timelike entanglement entropy in Eq. (48) and its approximate expression in Eq. (51) in terms of the turning point u_0. Figure 2 displays the exact timelike entanglement entropy in terms of u_0 (on the left) and the parametric plot of the entropy in terms of the separation. Figure 3 displays the parametric plot of S_{app} in Eq. (51) in terms of the approximate time separation T_{app} in Eq. (49), and the same for the exact quantities in Eqs. (47) and (48).

Notice that the double valuedness of the function $T(u_0)$ in Eq. (47), or in the analog approximate quantity in Eq. (49) indicates the presence of a phase transition. In our case, we believe this is a first order transition, as indicated by the swallow tail in Fig. 2.

The presence of a phase transition on the slab Ryu-Takayanagi entanglement entropy can be put in

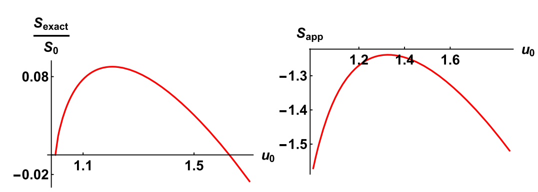

FIG. 2. The exact timelike entanglement entropy in terms of u_0, on the left. The right panel displays the approximate timelike entanglement in terms of u_0. Note that an integration constant can take account of the shift of both plots.

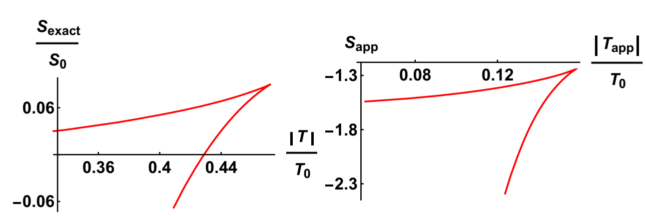

FIG. 3. The exact timelike entanglement entropy in terms $|T|$ on the left, both conveniently normalized. On the right, the approximate entanglement in terms of the approximate separation (both conveniently normalized). These display the signs of a phase transition.

correspondence with the confining behavior of the dual QFT. In fact, this was proposed in [45] and critically analyzed in [25,46]. Based on this we propose that a phase transition in the timelike entanglement entropy corresponds to a confining behavior in the dual QFT. The provisos mentioned in [25,46] do apply also for the timelike entanglement entropy of [9,10]. Similar ideas were put forward in [15]. Let us now study another confining system, with certain similarities but important differences from the one considered above.

B. Anabalón-Ross' model for (2 + 1) gapped theory

In this case the background consists of a metric, a constant dilaton (that we take to vanish), and a Ramond five form that we do not quote here. The metric reads as

$$ds^2 = \frac{u^2}{l^2}[\lambda dt^2 + dx_1^2 + dv^2 + f(u)d\phi^2] + \frac{l^2 du^2}{f(u)u^2} + l^2 d\tilde{\Omega}_5^2,$$
$$d\tilde{\Omega}_5^2 = d\theta^2 + \sin^2\theta d\psi^2 + \sin^2\theta \sin^2\psi (d\varphi_1 - A_1)^2 + \sin^2\theta\cos^2\psi(d\varphi_2 - A_1)^2 + \cos^2\theta(d\varphi_3 - A_1)^2.$$
$$A_1 = Q\left(1 - \frac{l^2Q^2}{u^2}\right)d\phi, \qquad f(u) = 1 - \left(\frac{Ql}{u}\right)^6. \tag{52}$$

For $Q = 0$ the metric is $AdS_5 \times S^5$. The parameter Q corresponds to a vacuum expectation value deformation of $N = 4$ Super-Yang-Mills. The radial coordinate ranges in $QL \leq u < \infty$. The space ends in a smooth way if the period of the ϕ direction is chosen appropriately, $L_\phi = \frac{1}{3Q}$ [40].

There is a qualitative difference between this model and Witten's in Eq. (45). The background in Eq. (52) asymptotes to $AdS_5 \times S^5$; hence the far UV is described in terms of a four-dimensional CFT which is deformed and flows to a (2 + 1) SUSY gapped and confining QFT. On the other hand, Witten's metric asymptotes to that of N_c D4 branes. Hence, the dual field theory is (4 + 1)-dimensional at high energies, which needs to be UV completed in terms of the six-dimensional (0, 2) SCFT. These differences impact some of the physical observables as we mention below.

The eight manifold needed to calculate the timelike entanglement entropy is $\Sigma_8 = [x_1, \phi, u, \theta, \psi, \varphi_1, \varphi_2, \varphi_3]$, with $v = 0$ and $t = t(u)$. The timelike entanglement is

$$S_{EE}^{(\lambda)}\left[\hat{\Sigma}_8^{(\lambda)}\right] = \frac{\hat{\mathcal{N}}}{4G_{10}}\int_{u_0}^{\infty} du\sqrt{G^2(u) + F^2(u)t'^2},$$
$$G^2(u) = \frac{u^2}{l^2}, \qquad F^2(u) = \frac{u^6}{l^6}f(u)\lambda,$$
$$\hat{\mathcal{N}} = L_{x_1}L_\phi l^5 \int d\tilde{\Omega}_5. \tag{53}$$

We start by calculating the approximate time separation T_{app}, the function $Z(u)$ indicating stability of the embedding and the approximate timelike entanglement entropy S_{app}. In terms of a variable,

$$z = \frac{u}{Ql}, \qquad 1 \leq z < \infty. \tag{54}$$

The approximate quantities read as

$$T_{app} = \frac{\pi l\sqrt{\lambda}}{3Qz_0^4}\sqrt{z_0^6 - 1}, \qquad Z = \frac{\pi\sqrt{\lambda}}{3Q^2z_0^5}\frac{(4 - z_0^6)}{\sqrt{z_0^6 - 1}},$$
$$S_{app} = -\frac{\pi l Q^2}{6z_0^4}(2 + z_0^6). \tag{55}$$

Here $z_0 = \frac{u_0}{Ql}$ indicates how much the embedding explores the radial coordinate (large z_0 are small embeddings barely entering the bulk). Qualitative aspects are revealed by these approximate quantities. Notice that T_{app} is double valued, indicating the possibility of a phase transition. Also, note that for z_0 large $T_{app} \sim z_0^{-1}$, which is the behavior observed for CFTs in Eq. (18). For the Witten model we obtain that at large u_0, $T_{app} \sim u_0^{-1/2}$. The quantity $Z(z_0)$ is negative (indicating stability of the embedding) for $z_0 > 2^{\frac{1}{3}}$; hence embeddings that penetrate into the bulk deeper than this value are unstable (indicating a phase transition). The exact expressions for the timelike separation and the timelike entanglement entropy are

$$\frac{QT(z_0)}{2l} = \sqrt{\lambda}\sqrt{z_0^6 - 1}\int_{z_0}^{\infty} dz\frac{z}{\sqrt{(z^6 - 1)(z^6 - z_0^6)}},$$
$$\frac{S_{EE}(z_0)}{2Q^2 l\hat{\mathcal{N}}} = \int_{z_0}^{\infty} dz z\sqrt{\frac{z^6 - 1}{z^6 - z_0^6}} - \int_1^{\infty} z dz. \tag{56}$$

Both integrals can be evaluated exactly in terms of Appel functions. We do not quote the result here. The numerical plots of the separation $T(z_0)$ and the comparison with the plot of $T_{app}(z_0)$ in Eq. (55) are qualitatively very similar to those Fig. 1. The same occurs for the plots of S_{EE} in Eq. (56) compared with S_{app} in Eq. (55). Finally we parametrically plot S_{app} in terms of T_{app} and compare it

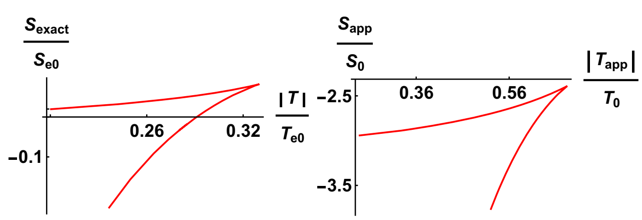

FIG. 4. The exact timelike entanglement entropy for the Anabalón-Ross model in terms $|T|$ on the left, both conveniently normalized. On the right, the approximate entanglement in terms of the approximate separation (both conveniently normalized). These display the signs of a phase transition.

with the parametric plot of S_{EE} in terms of T. Both parametric plots are in Fig. 4. Notice the similarities with the result for Witten's model in Fig. 3.

IV. SUMMARY AND CONCLUSIONS

Let us begin with a summary of the contents of this paper. We propose a top-down approach to computing timelike entanglement entropy, which does *not* rely on analytic continuation. We introduce a parameter $\lambda = \pm 1$ in front of the g_{tt} component of the metric, which governs both the Euclidean and Lorentzian cases. At the conclusion of our calculations, setting $\lambda = +1$ yields the usual Ryu–Takayanagi entanglement entropy, while $\lambda = -1$ gives the result for timelike entanglement entropy. We test our method for CFTs in various dimensions and reproduce the results of [10] in all cases. As a by-product, our analysis establishes a relation between the central charge of the dual CFTs (in all dimensions) and the entanglement entropy. In analogy with the Euclidean case, we propose formulas for the Liu–Mezei central charge for CFTs in various dimensions, derived from timelike entanglement entropy. We also perform this analysis for slab geometries. To illustrate our results, we discuss an infinite family of 4D $\mathcal{N} = 2$ linear quiver SCFTs and their holographic duals, using them to test our expressions.

We also discuss nonconformal field theories with slab entangling regions. In these cases, the computation of timelike entanglement entropy and time separation generally requires a numerical analysis of the extremal surface. We propose analytic expressions that closely approximate the exact results. Additionally, we present a criterion for the stability of the extremal surface. These expressions are particularly useful for analyzing phase transitions in timelike entanglement entropy, especially for nonconformal QFTs and their holographic duals. In fact, using these results, we study a holographic dual to $(3+1)$-dimensional Yang-Mills theory and a supersymmetric version of $(2+1)$-dimensional Yang-Mills theory. Our approach enables the study of infinite family generalizations, which we elaborate on in [24]. We suggest that, as in the case of the Euclidean eight manifold (corresponding to usual RT entanglement), there exists a phase transition in timelike entanglement. Specifically, the extremal surface becomes unstable beyond a certain time separation, leading to a disconnected, lower-energy configuration. In contrast, we show that such a transition does not occur for CFTs. Our approach is not addressing the important issue regarding what is the correct surface (time or spacelike) that minimizes the calculation [11,17].

Our forthcoming paper [24] provides detailed derivations of several results presented in this paper and extends them to infinite families of CFTs and confining models in various dimensions. See, for instance, the models discussed in [39,47–54].

We plan to continue developing this promising line of investigation.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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