

Stochastic inflation as a superfluid

Gianmassimo Tasinato^{a,b}

^a *Physics Department, Swansea University, SA28PP, United Kingdom*

^b *Dipartimento di Fisica e Astronomia, Università di Bologna,
INFN, Sezione di Bologna, viale B. Pichat 6/2, 40127 Bologna, Italy*

Abstract

We point out that inflationary superhorizon fluctuations can be effectively described by a set of equations analogous to those governing a superfluid. This is achieved through a functional Schrödinger approach to the evolution of the inflationary wavefunction, combined with a suitable coarse-graining procedure to capture large-scale dynamics. The irrotational fluid velocity is proportional to the gradient of the wavefunction phase. Marginalizing over short superhorizon modes introduces an external force acting on the fluid velocity. The quantum pressure characteristic of the superfluid plays a role in scenarios involving an ultra-slow-roll phase of inflation. Our superfluid framework is consistent with the standard Starobinsky approach to stochastic inflation while offering complementary insights, particularly by providing more precise information on the phase of the inflationary wavefunction. We also discuss a heuristic approach to include dissipative effects in this description.

1 Introduction

One of the aims of stochastic inflation is to determine an effective description for the dynamics of inflationary fluctuations at superhorizon scales. See e.g. [1–5]. Although born as quantum fields at short distances, a process of classicalization converts small-scale modes into classical stochastic variables at scales well larger than the Hubble horizon [6–13]. Long wavelength inflationary modes sample large super-Hubble regions, in principle assuming different values in different horizon-size patches. Focusing on a long-wavelength, coarse-grained field controlling scalar fluctuations, and assigning to it a probability density, the latter obeys a Fokker-Planck diffusion equation, with a noise induced by short wavelength modes in the process of crossing the horizon. The approach of stochastic inflation is helpful in dealing with quantum divergences of light scalar fields in de Sitter space, and allows one to obtain a full non linear probability distribution for the inflationary scalar fluctuations at superhorizon scales (see e.g. [14–16] for recent reviews).

The Fokker-Planck diffusion equation for the field probability density can be obtained from a functional Schrödinger approach [17, 18], see [10, 19]. Here we propose a complementary viewpoint based on Madelung approach to Schrödinger equation [20], accompanied by an appropriate coarse-graining procedure. We find that the system obeys coarse-grained equations

corresponding to the ones of a (super)fluid, including a contribution of quantum pressure to the Euler equation. The irrotational fluid velocity is proportional to the gradient of the wavefunction phase. The fluid propagates through an abstract coarse-grained scalar space, with density and velocity depending on amplitude and phase of wavefunction. An external force acting on the fluid emerges, being due to short wavelength modes integrated out from the description. The advantage of this perspective is the alternative viewpoint on the physics involved, which can shed new light on known results, and can indicate avenues for possible generalizations.

For slow-roll inflation the process of classicalisation is realized in terms of ‘decoherence without decoherence’ [6], since the contribution of quantum pressure to the Euler equation is very rapidly damped by the universe expansion. Interestingly, this phenomenon does not occur in an ultra-slow-roll regime, where the quantum pressure term is important and contribute to the fluid dynamics. Hence our approach provides an alternative perspective on the quantum-to-classical transition during inflation. We also propose an heuristic approach to include effects of dissipation in this description. Our superfluid framework aligns with the conventional Starobinsky formulation of stochastic inflation, while also delivering complementary perspectives – most notably offering insight into the phase dynamics of the inflationary wavefunction.

2 Set-up

We make use of a Schrödinger functional approach to analyse the dynamics of scalar fluctuations during cosmological inflation [17,18]. We find it convenient for formulating a stochastic approach to the system, and for addressing from a novel viewpoint the process of quantum-to-classical transition in inflationary cosmology.

Our starting point is the quadratic action for free scalar Fourier modes in quasi-de Sitter space (we assume $\varphi_{-k} = \varphi_k^*$)

$$S = \frac{1}{2} \int d^3k d\tau z^2(\tau) (\varphi'_k \varphi'_{-k} - k^2 \varphi_k \varphi_{-k}) . \quad (2.1)$$

The function $z(\tau)$, known as the pump field, is model-dependent, and it characterizes the dynamics of the fluctuations under consideration. We focus on free massless fields for simplicity, being their analysis sufficient for our purposes.

The quadratic expression (2.1) has the generic structure of an action for free fluctuations in single-field inflation. It describes the evolution of the Mukhanov-Sasaki variable ζ_k , which governs curvature fluctuations in standard slow-roll inflationary models, where $z \propto a(\tau)$ with $a(\tau)$ the scale factor expressed in terms of the conformal time. Action (2.1) can also describe massless spin-2 (tensor) or spin-0 (scalar) fluctuations in pure de Sitter space, again with $z \propto a$. More general scenarios are possible, and warrant further exploration. Interestingly, also ultra-slow roll inflation [21–23] can be studied starting from an action (2.1): in this case, a violation of slow-roll conditions leads to a rapid decrease of the pump field, $z \propto 1/a^2$. This scenario is important in models producing primordial black holes: it is interesting to analyse it from a new perspective, in a framework of stochastic inflation. In the examples that follow, we plan to study both the case of SR ($z \propto a$) and USR ($z \propto 1/a^2$) evolution.

The equation of motion for φ_k obtained from eq (2.1) results

$$\frac{1}{z(\tau)} \partial_\tau^2 [z(\tau) \varphi_k] + \left(k^2 - \frac{z''}{z} \right) \varphi_k = 0 . \quad (2.2)$$

We assume that the scalar fluctuations φ_k satisfy a Wronskian normalization condition $\varphi'_k \varphi_{-k} - \varphi_k \varphi'_{-k} = i/z^2$ for $k \neq 0$, as well as Bunch Davies conditions at $\tau \rightarrow -\infty$. This requirement is motivated by the underlying quantum behavior of fluctuations at very small scales. The zero mode φ_0 requires a special treatment. The Lagrangian density \mathcal{L}_k in Fourier space is the integrand of action (2.1). The corresponding conjugate momentum is

$$\pi_k = \frac{\delta \mathcal{L}_k}{\delta \varphi'_k} = z^2(\tau) \varphi'_{-k}, \quad (2.3)$$

which allows us to build the quadratic Hamiltonian for the system:

$$\mathcal{H}_k = \frac{\pi_k \pi_{-k}}{z^2(\tau)} + z^2(\tau) \varphi_k \varphi_{-k}. \quad (2.4)$$

The functional Schrödinger picture promotes the fields φ_k, π_k to operators, equipping them with a hat. A quantum mechanical wave-function $\Psi_k(\varphi_k, \tau)$ for the system is introduced, depending on the c-number quantity φ_k evaluated at conformal time τ . The operators $\hat{\varphi}_k, \hat{\pi}_k$ act on the wavefunction as

$$\hat{\varphi}_k \Psi_k = \varphi_k \Psi_k \quad (2.5)$$

$$\hat{\pi}_k \Psi_k = \frac{\hbar}{i} \frac{\partial \Psi_k}{\partial \varphi_k}. \quad (2.6)$$

Such rules allow us to express the corresponding Schrödinger equation

$$i \hbar \frac{\partial \Psi_k}{\partial \tau} = \mathcal{H}_k \Psi_k, \quad (2.7)$$

with Hamiltonian

$$\mathcal{H}_k = -\frac{\hbar^2}{z^2(\tau)} \frac{\partial^2}{\partial \varphi_k \partial \varphi_{-k}} + z^2(\tau) k^2 \varphi_k \varphi_{-k}. \quad (2.8)$$

Eqs (2.7), (2.8) are the starting point of our treatment. Following Madelung [20, 24] (see [25–30] for applications of this approach to large scale structures and dark matter scenarios), we decompose the wavefunction Ψ_k in an amplitude and a phase

$$\psi_k = \sqrt{\rho_k} e^{i z^2(\tau) \theta_k / \hbar}. \quad (2.9)$$

Assuming that ρ_k and θ_k are real functions of φ_k and τ , we can plug Ansatz (2.9) in the Schrödinger equation (2.7). Its real and imaginary parts lead to a system of two coupled equations

$$0 = \frac{\partial \rho_k}{\partial \tau} + 2\rho_k \frac{\partial^2 \theta_k}{\partial \varphi_k \partial \varphi_{-k}} + \frac{\partial \rho_k}{\partial \varphi_k} \frac{\partial \theta_k}{\partial \varphi_{-k}} + \frac{\partial \rho_k}{\partial \varphi_{-k}} \frac{\partial \theta_k}{\partial \varphi_k}, \quad (2.10)$$

$$0 = \frac{\partial_\tau (z^2 \theta_k)}{z^2(\tau)} + k^2 \varphi_k \varphi_{-k} + \frac{1}{4 z^4(\tau) \rho_k^2} \frac{\partial \rho_k}{\partial \varphi_k} \frac{\partial \rho_k}{\partial \varphi_{-k}} + \frac{\partial \theta_k}{\partial \varphi_k} \frac{\partial \theta_k}{\partial \varphi_{-k}} - \frac{\hbar^2}{2 z^4(\tau) \rho_k} \frac{\partial^2 \rho_k}{\partial \varphi_k \partial \varphi_{-k}}, \quad (2.11)$$

which resemble the continuity and Euler equations of fluid dynamics. One of our aims is to follow the evolution of the wavefunction phase, and study its consequences for the system. An appropriate coarse-graining procedure, which we discuss next, allows us to combine equations (2.10) and (2.11) in a way that makes more manifest the connection with fluid dynamics at superhorizon scales, and clarify the nature of external forces acting on the fluid.

3 Coarse-grained equations

The dynamics of superhorizon quantities is determined by a set of stochastic equations, obtained by a coarse-graining procedure aimed at marginalising over sub-horizon modes. In section 3.1 and 3.2 we develop a coarse-grained version of eqs (2.10) and (2.11), showing that they reduce to the equations governing a superfluid. After analysing specific applications in section 4, in section 5 we further discuss physical consequences of our findings.

3.1 Coarse-graining procedure

We coarse-grain marginalising over sub-Hubble modes, focusing only on large-scale, super-Hubble fields [31–33]. In fact, the subhorizon modes do not directly couple to the superhorizon ones: the former contribute to the dynamics of the latter only through their effects at horizon crossing. For complementary perspectives to stochastic inflation, see also [34–40].

We formally introduce coarse-grained quantities

$$\bar{\rho} = \Pi_k \rho_k, \quad (3.1)$$

$$\bar{\theta} = \sum_k \theta_k, \quad (3.2)$$

where the product and the sum are limited to Fourier modes $k \leq aH$. Correspondingly, the superhorizon wavefunction for the system is $\bar{\Psi} = \Pi_k \psi_k$. Physically, we identify $\bar{\rho}$ and $\bar{\theta}$ as the fluid energy density and the velocity potential in the space of scalar field configurations. (We will later discuss how the velocity potential, related with the wavefunction phase, is connected to the fluid velocity.) The coarse-grained scalar is a real quantity obtained by summing over Fourier modes,

$$\bar{\varphi}(\mathbf{x}) = \frac{1}{\sqrt{2}} \sum_{|\mathbf{k}| < aH} \left(\varphi_k e^{i\mathbf{k}\cdot\mathbf{x}} + \varphi_{-k} e^{-i\mathbf{k}\cdot\mathbf{x}} \right). \quad (3.3)$$

The corresponding gradient along the scalar field direction is defined as

$$\nabla \equiv \frac{\partial}{\partial \bar{\varphi}} = \frac{1}{\sqrt{2}} \sum_{|\mathbf{k}| < aH} \left(e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{\partial}{\partial \varphi_k} + e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\partial}{\partial \varphi_{-k}} \right). \quad (3.4)$$

When applying the previous coarse-graining definitions to the evolution equations (2.10) and (2.11), we further integrate over space, so to track the relevant contributions which are approximately constant over a particular Hubble volume, $\mathcal{V} \sim H^{-3}$, centered say at position $\vec{x} = 0$. This procedure leads to simplifications. For example,

$$\begin{aligned} \vec{\nabla}^2 \theta &= \frac{1}{2} \int_{\mathcal{V}} d^3x \left[\sum_q \left(e^{-iqx} \frac{\partial}{\partial \varphi_q} + e^{iqx} \frac{\partial}{\partial \varphi_{-q}} \right) \right] \left[\sum_k \left(e^{-ikx} \frac{\partial \bar{\theta}}{\partial \varphi_k} + e^{ikx} \frac{\partial \bar{\theta}}{\partial \varphi_{-k}} \right) \right], \\ &= \left(\frac{\partial}{\partial \varphi_k} \frac{\partial \theta_k}{\partial \varphi_{-k}} + \frac{\partial}{\partial \varphi_{-k}} \frac{\partial \theta_k}{\partial \varphi_k} \right). \end{aligned} \quad (3.5)$$

Hence the spatial integration allows us to neglect rapidly oscillating pieces which average to zero. Analogously,

$$\nabla \bar{\theta} \cdot \nabla \bar{\rho} = \frac{\partial \bar{\rho}}{\partial \varphi_k} \frac{\partial \theta_k}{\partial \varphi_{-k}} + \frac{\partial \bar{\rho}}{\partial \varphi_{-k}} \frac{\partial \theta_k}{\partial \varphi_k}. \quad (3.6)$$

In what follows, we also define $\bar{\varphi}^2 \equiv \sum_k |\varphi_k|^2$.

3.2 Continuity and Euler equations

We focus on eq (2.10) for a given mode k , and multiply it by the factors ρ_{k-2} , ρ_{k-1} , ρ_{k+1} etc. We get

$$0 = \left(\dots \rho_{k-1} \frac{\partial \rho_k}{\partial \tau} \rho_{k+1} \dots \right) + \bar{\rho} \left(\frac{\partial}{\partial \varphi_k} \frac{\partial \theta_k}{\partial \varphi_{-k}} + \frac{\partial}{\partial \varphi_{-k}} \frac{\partial \theta_k}{\partial \varphi_k} \right) + \frac{\partial \bar{\rho}}{\partial \varphi_k} \frac{\partial \theta_k}{\partial \varphi_{-k}} + \frac{\partial \bar{\rho}}{\partial \varphi_{-k}} \frac{\partial \theta_k}{\partial \varphi_k}. \quad (3.7)$$

We substitute $\partial \theta_k / \partial \varphi_k = \partial \bar{\theta} / \partial \varphi_k$ in the previous expression, since θ_k depends on φ_k only. We sum over momentum modes k , and we integrate over a volume \mathcal{V} , using the definition of coarse-grained quantities, and relations as eq (3.5). We obtain the expected structure of a fluid continuity equation

$$\frac{\partial \bar{\rho}}{\partial \tau} + \bar{\rho} \nabla^2 \bar{\theta} + \nabla \bar{\rho} \cdot \nabla \bar{\theta} = 0. \quad (3.8)$$

In writing this equation, we assume that the coarse grained quantities $\bar{\rho}$ and $\bar{\theta}$ depend on conformal time τ , and on the coarse-grained scalar $\bar{\varphi}$. The latter plays the role of spatial coordinate along which the fluid propagates.

A bit more work is needed to obtain an equation which resembles Euler's. We re-assemble eq (2.11) as

$$0 = \frac{1}{z^2} \frac{\partial(z^2 \theta_k)}{\partial \tau} + k^2 \varphi_k \varphi_{-k} - \frac{1}{4 z^4} \frac{\partial(\ln \rho_k)}{\partial \varphi_k} \frac{\partial(\ln \rho_k)}{\partial \varphi_{-k}} + \frac{\partial \theta_k}{\partial \varphi_k} \frac{\partial \theta_k}{\partial \varphi_{-k}} - \frac{\hbar^2}{2 z^4} \frac{\partial}{\partial \varphi_k} \left(\frac{\partial(\ln \rho_k)}{\partial \varphi_{-k}} \right). \quad (3.9)$$

Since each ρ_k , θ_k depend on the a single-mode φ_k only, we directly substitute the bar quantities $\bar{\rho}$ and $\bar{\theta}$ in all the terms of eq (3.9) containing derivatives along φ_k . We sum over k , and integrate over a volume \mathcal{V} . We obtain a relation which corresponds to Euler equation in fluid dynamics expressed in terms of velocity potential $\bar{\theta}$:

$$0 = \frac{1}{z^2(\tau)} \frac{\partial(z^2(\tau) \bar{\theta})}{\partial \tau} + \left(\sum_k k^2 \varphi_k \varphi_{-k} \right) + \frac{1}{2} \bar{\nabla} \bar{\theta} \cdot \bar{\nabla} \bar{\theta} - \frac{\hbar^2 \bar{\nabla}^2 (\bar{\rho}^{1/2})}{2 z^4 \bar{\rho}^{1/2}}. \quad (3.10)$$

The sum within square parenthesis is over momentum modes $k < aH$. In order to discuss the physical consequences of the coarse-grained equations, we find convenient to pass from conformal time τ to number of e-folds of expansion, $dn = aH d\tau$ [41, 42]. We rescale the velocity potential – i.e. the phase of the wavefunction – as

$$\bar{\theta}(n, \bar{\Phi}) \equiv a(n) H \bar{\Theta}(n, \bar{\Phi}). \quad (3.11)$$

We find the two coupled equations

$$0 = \frac{\partial \bar{\rho}}{\partial n} + \bar{\rho} \nabla^2 \bar{\Theta} + (\nabla \bar{\rho}) \cdot (\nabla \bar{\Theta}) \quad (3.12)$$

$$0 = \frac{\partial \bar{\Theta}}{\partial n} + \frac{\bar{\Theta}}{aH z^2} \frac{d(aH z^2)}{dn} - K(n) \bar{\varphi}^2 + \frac{1}{2} (\nabla \bar{\Theta})^2 - \frac{\hbar^2 \nabla^2 \bar{\rho}^{1/2}}{2 H^2 a^2 z^4 \bar{\rho}^{1/2}}, \quad (3.13)$$

with

$$K(n) \equiv \frac{1}{a^2 H^2} \frac{\int_0^{aH} k^2 |\varphi_k|^2 d^3 k}{\int_0^{aH} |\varphi_k|^2 d^3 k}. \quad (3.14)$$

In this definition we substitute the sum with an integral, and we adopt the convention to integrate over superhorizon modes from horizon exit $k = aH$ to large scales $k \rightarrow 0$. The resulting quantity depends on the e-fold number n .

The term proportional to $K(n)$ describes a stochastic external force acting on the fluid velocity potential, caused by the small-scale modes crossing the horizon. Its origin is analog to the ‘noise’ term in Starobinsky description of stochastic inflation (see section 3.3). From our perspective, the long wavelength modes forming the fluid can be interpreted as an open system coupled to the environment of small-scale modes. Equations (3.12) and (3.13) are written in terms of velocity potential θ : defining

$$\mathbf{v} \equiv \bar{\nabla} \bar{\Theta} \quad (3.15)$$

we can re-express them in a form which is more recognizable in terms of a fluid dynamics description:

$$0 = \frac{\partial \bar{\rho}}{\partial n} + \bar{\nabla} \cdot (\bar{\rho} \mathbf{v}), \quad (3.16)$$

$$0 = \frac{\partial \mathbf{v}}{\partial n} + \frac{\mathbf{v}}{aH z^2} \frac{d(aH z^2)}{dn} - 2K \bar{\varphi} + \mathbf{v} \cdot \bar{\nabla} \mathbf{v} - \frac{\hbar^2}{2H^2 a^2 z^4} \bar{\nabla} \cdot \left(\frac{\bar{\nabla}^2 \bar{\rho}^{1/2}}{\bar{\rho}^{1/2}} \right) \quad (3.17)$$

These equations describe a (super)fluid flowing in one spatial dimension, represented by the scalar manifold $\bar{\varphi}$. The contributions proportional to \hbar^2 in the previous expressions correspond to the so-called quantum pressure. As we will see, they play a role in systems including phases of ultra-slow-roll. Notice that eq (3.17) does *not* contain the contribution $(\nabla \mathcal{P})/\bar{\rho}$ depending on the internal fluid pressure \mathcal{P} [73]. In a sense, our fluid has no internal ‘classical’ pressure, at least within the superhorizon framework we are adopting. We will return to this point in section 5.

3.3 The perspective of Starobinsky diffusion equation

There is a relation between our discussion and the usual stochastic approach to inflation based on the Fokker-Planck equation. The square of the wave function $\Psi^* \Psi = \rho$ – we adopt the Ansatz (2.9) – can be interpret as the probability for a coarse-grained scalar profile to acquire a configuration φ at superhorizon scales. Formally, this quantity is the same thing as the ρ appearing in eqs (3.12) and (3.13), but apparently with a distinct physical interpretation. (More on this in section 5.)

Since the scalar system we start from in eq (2.1) is free, we can assume that such probability $\Psi^* \Psi$ follows a Gaussian distribution, normalised to one upon integration over $\bar{\varphi}$. Notice that, although the equations governing fluctuations are free, nevertheless the coarse-grained system has a non-trivial evolution thanks to the noise terms induced by short wavelength modes crossing the horizon during inflation. This phenomenon leads to an open system where long modes interact with the short mode environment. Under our hypothesis, it is possible to show [10,

19, 43] (see also Appendix A) that the free scalar-field configuration satisfies a diffusion-like, Starobinsky [3] equation

$$\frac{\partial \rho(n, \bar{\varphi})}{\partial n} = \frac{H^2 \mathcal{N}(n)}{8\pi^2} \bar{\nabla}^2 \rho(n, \bar{\varphi}) + \mathcal{D}(n) \bar{\nabla} (\bar{\varphi} \rho(n, \bar{\varphi})) , \quad (3.18)$$

with noise and drift coefficients are given by a combination of momentum modes

$$\mathcal{N}(n) = \frac{2|\varphi_0|^2}{aH^3} \int_{aH}^0 k^2 dk \partial_\tau \left(\frac{|\varphi_k|^2}{|\varphi_0|^2} \right) \quad (3.19)$$

$$\mathcal{D}(n) = -\frac{1}{2aH} \partial_\tau \ln (|\varphi_0|^2) . \quad (3.20)$$

The stochastic noise \mathcal{N} is caused by short modes crossing the horizon during inflation; the drift is driven by the zero mode φ_0 . Eq (3.18) is complementary to equations (3.12) and (3.13): it actually provides useful information in dealing with the fluid evolution, as we will learn through the examples of section 4.

In fact, the three equations (3.12), (3.13), and (3.18) fully characterize the components of the wavefunction. In short, equations (3.18), (3.12) determine the amplitude $\bar{\rho}$, and the part of the phase $\bar{\Theta}$ that depends on the position on the scalar-field space $\bar{\varphi}$. The Euler equation (3.13), then, fully determines the time-dependent part of the phase that does not depend explicitly on $\bar{\varphi}$. We will expand on this in section 5.

4 Examples: slow-roll and ultra slow-roll inflation

We consider two concrete examples as applications of the previous results. We focus on evolution in quasi-de Sitter space, with scale factor well approximated by an exponential $a(n) \simeq e^n$, in terms of the e-fold number n , up to small slow-roll corrections which we neglect. We are interested in determining the late-time solutions for the fluid energy density $\bar{\rho}$ and velocity potential $\bar{\Theta}$, neglecting contributions that decay faster than $1/n$ in order to tackle late-time superhorizon dynamics only. We discuss two possible inflationary regimes: slow-roll (SR) and ultra-slow-roll (USR) epochs, the latter being relevant for scenarios leading to primordial black hole formation. We will learn that while the SR evolution is controlled by classical stochastic equations – thanks to a phenomenon related to decoherence without decoherence [6] – the USR equations receive quantum contributions depending on \hbar . (See also [44–57] for interesting perspectives on quantum-to-classical transition during inflation.)

4.1 Slow-roll

Slow-roll inflation is the leading paradigm for explaining the initial conditions of the observed universe. In this case we are able to follow in detail the dynamics of the fluid potential velocity $\bar{\Theta}$. The pump field reads $z(n) = z_0 a(n)$, with z_0 a constant depending on the physics we wish to describe. The solution of the mode equation satisfying the requested boundary conditions is

$$\varphi_k = \frac{H}{\sqrt{2} k^{3/2}} (1 + ik\tau) e^{-ik\tau} . \quad (4.1)$$

At sufficiently late times, $n \gg 1$, the coefficient K appearing in eq (3.13) is easily evaluated using eq (3.14), resulting

$$K = \frac{3}{4n} + \mathcal{O}(1/n^2). \quad (4.2)$$

The large n limit implies we focus on late time dynamics as discussed above. The fluid equations become

$$0 = \frac{\partial \bar{\Theta}}{\partial n} + 3\bar{\Theta} - \frac{3}{4n} \bar{\Phi}^2 + \frac{1}{2} \left(\frac{\partial \bar{\Theta}}{\partial \bar{\varphi}} \right)^2 - \frac{\hbar^2 e^{-6n}}{2z_0 \bar{\rho}^{1/2}} \frac{\partial^2 (\bar{\rho}^{1/2})}{\partial \bar{\varphi}^2}, \quad (4.3)$$

$$0 = \frac{\partial \bar{\rho}}{\partial n} + \bar{\rho} \frac{\partial^2 \bar{\Theta}}{\partial \bar{\varphi}^2} + \frac{\partial \bar{\rho}}{\partial \bar{\varphi}} \frac{\partial \bar{\Theta}}{\partial \bar{\varphi}}. \quad (4.4)$$

The contribution of the external force to the Euler equation (4.3) corresponds to the term $-3\bar{\Theta}^2/(4n)$: it contributes to a force on the fluid velocity at superhorizon scales, induced by the short wavelength scalar modes crossing the horizon. (A contribution associated with a classical fluid pressure can also be expected (see section 5) but it scales as $1/n^2$, hence we neglect it in our discussion.) The diffusion equation results

$$\frac{\partial \bar{\rho}}{\partial n} = \frac{H^2}{8\pi^2} \frac{\partial^2 \bar{\rho}}{\partial \bar{\varphi}^2}. \quad (4.5)$$

We can neglect quantum pressure in eq (4.3), proportional to \hbar^2 , since its contribution is exponentially suppressed in terms of e-fold number. Including it would be inconsistent in the regime we are interested in. Neglecting such quantum effects is related with the phenomenon of decoherence without decoherence, as discussed in [6]. The rapid expansion of the universe is responsible for erasing quantum contributions. In this case, then, the system is described by classical evolution equations, including stochastic effects associated with noise (in the diffusion equation) and external force (in the Euler equation). Quantum effects proportional to \hbar do not play a role on the late-time superhorizon evolution of the fluid system.

The solutions to the previous set of equations, imposing that the fluid density is initially concentrated at the origin for $n = 0$, result:

$$\bar{\rho}(n, \bar{\varphi}) = \frac{\sqrt{2\pi}}{H \sqrt{n}} \exp \left\{ \left[-\frac{2\pi^2 \bar{\varphi}^2}{H^2 n} \right] \right\}, \quad (4.6)$$

$$\bar{\Theta}(n, \bar{\varphi}) = \frac{\bar{\varphi}^2}{4n}. \quad (4.7)$$

The fluid density is described by a Gaussian whose width depends on time, which spreads in the scalar field space. The solution is the same as in the usual stochastic formulation to the probability density for the superhorizon scalar field. The fluid velocity $\bar{v} = \bar{\varphi}/(2n)$ increases in magnitude as we move away from the origin in field space, while its amplitude decreases with time at fixed position in the scalar field space. Within our approximations, the velocity potential $\bar{\Theta}$ has no contributions independent from the fluid position $\bar{\varphi}$.

4.2 Ultra-slow-roll

Stochastic effects in regime of ultra-slow-roll (USR) evolution have received much attention in the recent literature – see e.g. [58–66] – given their importance for discussing the production

of primordial black holes, see e.g. [67] for a review. We discuss this topic within our superfluid perspective. We express the pump field as $z = z_0/a^2(n)$, while maintaining a de Sitter evolution for the scale factor. In this case, the amplitude of the would-be decaying mode actually increases exponentially with the e-fold number upon crossing the horizon. The role of the nearly constant mode, then, is much suppressed relatively to such would-be decaying mode (which actually increases in size). Let us see explicitly how our equations describe these phenomena. The fluid equations become

$$0 = \frac{\partial \bar{\Theta}}{\partial n} - 3 \bar{\Theta} - \frac{3}{4n} \bar{\Phi}^2 + \frac{1}{2} \left(\frac{\partial \bar{\Theta}}{\partial \bar{\varphi}} \right)^2 - \frac{\hbar^2 e^{6n}}{2 z_0 \bar{\rho}^{1/2}} \frac{\partial^2 (\bar{\rho}^{1/2})}{\partial \bar{\varphi}^2}, \quad (4.8)$$

$$0 = \frac{\partial \bar{\rho}}{\partial n} + \bar{\rho} \frac{\partial^2 \bar{\Theta}}{\partial \bar{\varphi}^2} + \frac{\partial \bar{\rho}}{\partial \bar{\varphi}} \frac{\partial \bar{\Theta}}{\partial \bar{\varphi}}. \quad (4.9)$$

Importantly, notice that the contribution of quantum pressure, proportional to \hbar^2 , increases exponentially with the e-fold number in eq (4.9). The diffusion equation results

$$\frac{\partial \bar{\rho}}{\partial n} = \frac{H^2 e^{6n}}{8\pi^2} \frac{\partial^2 \bar{\rho}}{\partial \bar{\varphi}^2} - 3 \frac{\partial (\bar{\Phi} \bar{\rho})}{\partial \bar{\varphi}}. \quad (4.10)$$

The drift term is due to the contribution of the zero mode (see section 3.3). The noise term is exponentially enhanced with respect to the SR case of eq (4.5). The solutions to the previous set of equations, describing a fluid system with energy density localized at the origin for $n = 0$, result

$$\bar{\rho}(n, \varphi) = \frac{\sqrt{2\pi}}{H e^{3n} \sqrt{n}} \exp \left\{ \left[-\frac{2\pi^2 \bar{\varphi}^2}{H^2 e^{6n} n} \right] \right\}, \quad (4.11)$$

$$\bar{\Theta}(n, \varphi) = \frac{(1 + 6n)}{4n} \bar{\varphi}^2 + \frac{\hbar^2 \pi^2}{3 H^2 z_0} \frac{1}{n}, \quad (4.12)$$

up to corrections that scale faster than $1/n$. The fluid density is again described by a Gaussian, whose width is exponentially enhanced in terms of e-fold number. The fluid velocity potential $\bar{\Theta}$ again depends on the position in field space; interestingly, it also receives a position-independent contribution depending on \hbar^2 , associated with the quantum pressure in the Euler equation (4.9). Hence, in this context even in a late-time limit, quantum effects play a role for determining the super-horizon evolution of the phase of the waveform. By making use of the approach developed in [68, 69], we also studied the case of a brief phase of USR sandwiched between two long phases of slow-roll, without finding qualitative differences with respect to our discussion above.

To conclude, these two examples demonstrate that even free, Gaussian open systems can have a rich interesting dynamics from the viewpoint of the superfluid equations we derived – thanks to effects of the environment constituted by the modes crossing the horizon.

5 Physical implications

After having discussed the coarse-grained equations and their solutions in representative cases, in this section we analyse more general physical implications of our approach, and of the information we gain about the wavefunction phase.

We interpret the continuity and Euler equations (3.12) and (3.13) as describing the dynamics of a pressureless (super)fluid flowing along the single dimension corresponding to the coarse grained scalar $\bar{\varphi}$. The fluid flows over super-horizon patches with irrotational velocity of size $|\mathbf{v}| = |\nabla\bar{\Theta}|$. In this perspective for stochastic inflation the solution of the fluid density $\rho(n, \varphi)$ is normalised to one, once integrated over the $\bar{\varphi}$ coordinate. The Euler equation (3.13) for the velocity potential includes a term, $-K(n)\bar{\varphi}^2$, which accounts for the effects of short-wavelength modes over which we marginalize. It acts as external stochastic force for the fluid system. The coefficient $K(n)$ can be explicitly computed from the classical solution of Eq. (2.2) (see Section 4 for examples). Importantly, eq (3.13) also contains a ‘quantum pressure’ contribution through its last term. Its role is relevant in certain contexts as in ultra-slow-roll, see section 4.2. In such example, in fact, the solutions of coarse-grained equations contain explicitly a quantum contribution proportional to \hbar^2 , affecting the velocity potential. Our approach can then help in characterizing the quantum-to-classical transition of fluctuations during inflation.

The three equations (3.12), (3.13), (3.18) can be solved together, fully determining a solution for the fluid energy density and velocity potential. The solution of the equations for the system allows us to track in detail all the components of the inflationary wavefunction discussed in section 3: not only its amplitude, but also its phase. In the remaining part of this section, we discuss physical instances where the phase of the wavefunction is important for understanding the physics involved.

We work with dimensionless variables, and we rescale the coarse-grained value of the super-horizon scalar $\bar{\varphi}$ as

$$\bar{\varphi} \equiv \frac{H x}{2\pi}. \quad (5.1)$$

Assuming a normalized Gaussian Ansatz for the solution of the fluid density (appropriate for our free system (2.1))

$$\rho(n, x) = \frac{1}{\sqrt{\pi g(n)}} e^{-x^2/g(n)}, \quad (5.2)$$

the diffusion equation (3.18) implies that the function $g(n)$ of eq (5.2) satisfies the following equation

$$g'(n) - 4\mathcal{N}(n) + 2\mathcal{D}(n)g(n) = 0, \quad (5.3)$$

whose solution determines the time-dependent evolution of the Gaussian width $g(n)$ in terms of noise and drift functions. The continuity equation (3.12) then requires that the velocity potential Θ assumes the form

$$\Theta(n, x) = \left[\frac{\mathcal{N}(n)}{g(n)} - \frac{\mathcal{D}(n)}{2} \right] x^2 + c(n), \quad (5.4)$$

with a function $c(n)$ finally determined by Euler equation (3.13). The quantity $\Theta(n, x)$ is proportional to the phase of the wavefunction, as we learned in section 3.

These general results can be assembled and used for reconsidering the coarse-grained wavefunction of the system. Using equations (5.2), (5.4), we find that its structure can be expressed as (from now on, set $\hbar = 1$):

$$\bar{\Psi}(n, x) = \frac{1}{(\pi g)^{1/4}} e^{-[1+iH a z^2 (2\mathcal{N}-\mathcal{D}g)] x^2/(2g)+iH a z^2 c}. \quad (5.5)$$

All functions of n entering the expression (5.5) are determined by the equations of the system, and their solutions. The phase of the wavefunction, in particular, controls the off-diagonal terms of the density matrix associated with this system. The complete wavefunction is also useful for obtaining a distribution in phase space of the fluid elements, which can be used to study its properties and how they depend on the wavefunction phase. We do so here, focusing on the physical implications of the Wigner distribution [6, 70] which allows us to describe the system in a statistical sense. See also [71].

5.1 Statistical description

Denoting with p the momentum conjugate to the ‘position’ x of the fluid element in scalar field space (see eq (3)), the Wigner function gives a normalized, Gaussian phase-space distribution $f(x, p)$:

$$\begin{aligned} f(n, x, p) &\equiv \int \frac{dr}{\pi} \Psi^*(x + r, n) \Psi(x - r, n) e^{2irp} \\ &= \frac{1}{\pi} e^{-\frac{x^2 + [pg + Hax(\mathcal{D}g - 2\mathcal{N})z^2]^2}{g}}, \end{aligned} \quad (5.6)$$

which can be interpreted as describing the distribution of position and momenta of fluid elements at superhorizon scales. (For simplicity, we assume the Hubble parameter to be constant.) Notice that the phase part of the wavefunction (5.5) controls the coupling between the variable x and its conjugate momentum p in the exponent of eq (5.6).

Then, the fluid system is characterized by position x and its conjugate momentum p in phase space. The coarse-grained fluid density $\rho(n, x)$ gives the marginal probability to find a fluid element at position x : it is obtained by integrating over conjugate momenta

$$\begin{aligned} \rho(n, x) &\equiv \int dp f(n, x, p) \\ &= \frac{1}{\sqrt{\pi g(n)}} e^{-\frac{x^2}{g(n)}}, \end{aligned} \quad (5.7)$$

matching the results of eq (5.2). The normalized distribution of conjugate momenta is obtained integrating over the coordinate x

$$q(n, p) \equiv \int dx f(n, x, p) = \frac{\sqrt{g} \exp\left(-\frac{p^2 g}{H^2 a^2 z^4 (\mathcal{D}g - 2\mathcal{N})^2 + 1}\right)}{\sqrt{\pi} \sqrt{H^2 a^2 z^4 (\mathcal{D}g - 2\mathcal{N}^2 + 1)}}. \quad (5.8)$$

Using the distribution $f(x, p)$ we can also compute the fluid pressure \mathcal{P} , as

$$\begin{aligned} \mathcal{P}(n, x) &\equiv \rho(n, x) \left[\int f(n, x, p) p^2 dp - \left(\int f(n, x, p) p dp \right)^2 \right], \\ &= \frac{\rho(n, x)}{2g(n)}, \end{aligned} \quad (5.9)$$

hence the pressure is proportional to the energy density, times a time-dependent factor. We might expect that the fluid pressure above contributes to the Euler equation (3.13) through a

contribution scaling as $\nabla\mathcal{P}/\rho \propto 1/g^2(n)$. But notice that such term typically gives a subleading contribution which can be neglected in solving our equations¹: this can explain why we did not find the classical pressure contributions in our analysis.

Defining the quantity

$$\sigma_{x^n p^q} = \int x^n p^q f(n, x, p) dx dp, \quad (5.10)$$

we find that the means σ_x and σ_p associated with the distribution $f(x, p)$ vanish. The variances and the covariance instead read

$$\sigma_{x^2} = \frac{g(n)}{2}, \quad (5.11)$$

$$\sigma_{p^2} = \left[\frac{1}{2g(n)} + \frac{H^2 a^2(n) z^4(n) (\mathcal{D}(n)g(n) - 2\mathcal{N}(n))^2}{2g(n)} \right], \quad (5.12)$$

$$\sigma_{xp} = -\frac{H a(n) z^2(n)}{2} [\mathcal{D}(n)g(n) - 2\mathcal{N}(n)]. \quad (5.13)$$

The covariance σ_{xp} is entirely controlled by the phase of the wavefunction. Heisenberg uncertainty relation can be expressed as

$$\sqrt{\sigma_{x^2}\sigma_{p^2}} = \frac{1}{2}\sqrt{1 + \sigma_{xp}^2} \geq \frac{1}{2}. \quad (5.14)$$

The covariance contribution allows to satisfy the previous inequality in a strict sense. For the examples of section 4, we parameterize the pump field $z(n) = z_0 a^{1-3c_0}(n)$, with z_0 a constant, and $c_0 = 0, 1$ depending on whether we are in SR or USR phases. We find

$$\sigma_{xp}^2 = \frac{z_0^4 H^6}{16\pi^4} (1 + 6c_0 n)^2 e^{6n}, \quad (5.15)$$

hence

$$\sqrt{\sigma_{x^2}\sigma_{p^2}} = \frac{1}{2} \left[1 + \frac{z_0^4 H^6}{16\pi^4} (1 + 6c_0 n)^2 e^{6n} \right]^{1/2}. \quad (5.16)$$

The right-hand-side of this equation, again depending on the wave-function phase, is an exponentially increasing function scaling as e^{3n} . Heisenberg inequality gets more and more satisfied as time flows, and the system rapidly classicalises, both in the SR [6] and USR phases. We find it particularly interesting that our approach demonstrates that both cases behave quite similarly for what concerns the process of classicalization from a perspective of Wigner statistical distribution.

Entropies: As further application of our results, we compute the entropies associated with the distributions $\rho(n, x)$, $q(n, p)$, $f(n, x, p)$ of eqs (5.7), (5.8), and (5.6) in the phase space (x, q) of position and conjugate momentum. Such entropies control subspaces of the full phase space (x, p) , hence we can expect entropy inequalities to hold [72].

¹In fact, for the examples of SR and USR discussed in section 4, such pressure term scale at least as $1/n^2$ and is neglected in the considerations of those systems where we focus on contributions scaling at most as $1/n$ at late times.

The entropy for the distribution $\rho(n, x)$ along the coordinate x results (we set the Boltzmann constant to one)

$$S_x(n) = - \int dx \rho(n, x) \ln \rho(n, x) = \frac{1}{2} (1 + \ln[\pi g(n)]) . \quad (5.17)$$

In the examples of section 4 the function $g(n)$ increases with time, hence the entropy $S_x(n)$ as well. The entropy $S_p(n)$ for the distribution of $q(n, p)$ is

$$S_p(n) = - \int dp q(n, p) \ln q(n, p) = \frac{1}{2} (1 + \ln(\pi/g(n)) + \ln [1 + \sigma_{xp}^2(n)]) . \quad (5.18)$$

Notice that it is an increasing function of n thanks to the contribution of the covariance σ_{xp} of the distribution. Instead, the entropy associated to the entire Gaussian distribution $f(n, x, p)$ is constant:

$$S_{\text{tot}} = - \int dx dp f(n, x, p) \ln f(n, x, p) = (1 + \ln \pi) . \quad (5.19)$$

Hence, the system satisfies the subadditivity condition

$$S_x(n) + S_p(n) - S_{\text{tot}}(n) = \ln [1 + \sigma_{xp}^2(n)] \geq 0 , \quad (5.20)$$

with σ_{xp}^2 given in eq (5.15). The entropy of the sum of the subsystems (marginalizing over x or over p) is, as expected, much larger than the entropy of the total system, in the limit of large n . Finally, the so-called mutual entropy controls the mutual information among the subsystems in coordinates x and p . It results

$$S_{\text{mut}}(n) \equiv - \int dx dp f(n, x, p) \ln \left(\frac{\rho(n, x) q(n, p)}{f(n, x, p)} \right) = \frac{1}{2} \ln [1 + \sigma_{xp}^2(n)] . \quad (5.21)$$

As expected, it increases with time thanks to σ_{xp}^2 , controlled by the phase of the coarse-grained wave-function.

Hence the results of this section demonstrate the important role of the phase of the wave-function in characterizing the physics of superhorizon modes, and how the perspective of fluid dynamics allows to track its evolution.

6 A heuristic approach to the interacting system

In the previous sections, we derived from first principles evolution equations for super-horizon coarse grained fields, associated with amplitude and phase of the coarse-grained wavefunction during inflation. For simplicity, we focused on a free system, with action given in eq (2.1). The resulting equations correspond to Euler and continuity equations of fluid dynamics. Interestingly, the free-field approach can be applied to SR but also to USR scenarios, the two cases differing by the behaviour of the pump field.

In this final section, we discuss an alternative, heuristic method for capturing the evolution of super-horizon fluctuations in an interacting scalar-field set-up. Our aim is to propose a phenomenological approach based on fluid dynamics, which explicitly include the effects of external forces, and can then be extended to include phenomena as dissipation. Also in this case, we wish to organize the evolution equations as the ones of a (super)-fluid. The difference

with what done above is that we do not proceed starting from first principles – as in the functional Schrödinger approach of section 2 – but from the heuristic manipulation of coarse-grained stochastic equations, whose structure we assume.

We consider stochastic, large-scale fluctuations of a scalar field during inflation, characterized by self-interactions controlled by a potential $U(\varphi)$. In this section, we understand the bars, and we express the coarse-grained version of superhorizon scalar fluctuations as φ . We assume the density ρ satisfies a diffusion equation

$$\partial_n \rho(n, \varphi) = \frac{H^2 \mathcal{N}(n)}{8\pi^2} \partial_\varphi^2 \rho(n, \varphi) + \frac{1}{3H^2} \partial_\varphi [(\partial_\varphi U(\varphi) + 3H^2 \mathcal{D}(n) \varphi) \rho(n, \varphi)] , \quad (6.1)$$

with H the constant Hubble parameter during inflation. The (assumed) structure of the previous equation, which should be valid deep in a superhorizon regime, is our starting point. When $\mathcal{N} = 1$ and $\mathcal{D} = 0$ we obtain the standard Starobinsky equation for stochastic evolution in de Sitter space. More general noise \mathcal{N} and drift \mathcal{D} contributions allow us to catch in principle deviations from slow-roll evolution, as the USR phase.

As in the previous sections, we can interpret $\rho(n, \varphi)$ in eq (6.1) as the fluid energy density, and assume it satisfies a continuity eq as (6.2)

$$\frac{\partial \rho}{\partial n} + \rho \nabla^2 \Theta + (\nabla \rho) (\nabla \Theta) = 0 , \quad (6.2)$$

which involves a velocity potential $\Theta(n, \varphi)$, with $\mathbf{v} = \nabla \Theta(n, \varphi)$ the fluid velocity.

Combining eqs (6.1) and (6.2), and integrating along the scalar direction φ , we find the following expression for the velocity potential

$$\Theta(n, \varphi) = -\frac{H^2 \mathcal{N}(n)}{8\pi^2} \ln \left(\frac{\rho(n, \varphi)}{\rho_0(n)} \right) - \frac{[2U(\varphi) + 3H^2 \mathcal{D}(n) \varphi^2]}{6H^2} , \quad (6.3)$$

with $\rho_0(n)$ an arbitrary function of n . Interestingly, defining the fluid pressure as

$$\mathcal{P}(n, \varphi) = \frac{3H^2}{8\pi^2} \left(\mathcal{N} + \frac{\partial_n \mathcal{N}}{3} \right) \rho(n, \varphi) , \quad (6.4)$$

we notice that the velocity potential satisfies an Euler-type equation

$$\begin{aligned} 0 = & \partial_n \Theta(n, \varphi) + 3\Theta(n, \varphi) + \frac{1}{2} (\partial_\varphi \Theta(n, \varphi))^2 + \frac{H^4 \mathcal{N}^2(n)}{32\pi^4} \left(\frac{\partial_\varphi^2 \sqrt{\rho(n, \varphi)}}{\sqrt{\rho(n, \varphi)}} \right) \\ & + \int d\varphi \frac{\partial_\varphi \mathcal{P}(n, \varphi)}{\rho(n, \varphi)} + W(n, \varphi) , \end{aligned} \quad (6.5)$$

with

$$\begin{aligned} W(n, \varphi) = & \frac{U(\varphi)}{H^2} - \frac{(U'(\varphi))^2}{18H^4} + \frac{\mathcal{N}(n) U''(\varphi)}{24\pi^2} - \frac{\mathcal{D}(n) \varphi U'(\varphi)}{3H^2} \\ & + (3\mathcal{D}(n) - \mathcal{D}^2(n) + \mathcal{D}'(n)) \frac{\varphi^2}{2} , \end{aligned} \quad (6.6)$$

representing the effect of external forces. Some comments are in order

- The arguments leading to eqs (6.2) and (6.5) aims to determine a system of two coupled equations for the density and velocity of the fluid. After the derivation above has been carried out, we can take these two equations as the *fundamental relations* to solve for determining the system dynamics, without considering any more the diffusion-like equation (6.1) from which they originate.
- The structure of the two equations is different from what found in the previous sections. This being due to the fact that they have not been derived from first principles, but from a manipulation of more phenomenological expressions, by identifying – through educated guesses – the role of each contribution to the Euler equation.
- Eqs (6.2) and (6.5) are classical stochastic equations independent from quantum effects (there is no \hbar). They aim to be valid at super-horizon scales. The ‘quantum pressure’-like term of Euler equation proportional to H^4 depends on the noise parameter \mathcal{N} , and the classical pressure \mathcal{P} is proportional to the energy density through relation (6.4). The fluid system feels an external force controlled by the function $W(n, \varphi)$ depending on the scalar potential $U(\varphi)$ as well as on the drift term $\mathcal{D}(n)$.

It is straightforward to show that the equations admit the expected solutions in special cases. In absence of potential, $U(\varphi) = 0$, we recover the solutions of section 4. If we turn on the potential, and set $\mathcal{N} = 1$ and $\mathcal{D} = 0$, we find the correct equilibrium solution [3]

$$\rho(\varphi) = \rho_0 e^{-\frac{8\pi^2 U(\varphi)}{3H^4}}, \quad (6.7)$$

with ρ_0 a constant, fixed accordingly to normalization as $\rho_0^{-1} = \int_{-\infty}^{\infty} d\varphi \exp\left\{-\frac{8\pi^2 U(\varphi)}{3H^4}\right\}$, when this integral converges.

What is interesting of this approach is that we can phenomenologically extend it, by adding dissipative contributions to the Euler equation. We do so using a standard textbook approach [73]. In fluid dynamics, dissipation is associated with the viscosity stress tensor, which adds contributions to the Euler equation depending on second spatial derivatives acting on the fluid velocity. Accordingly, in our one-dimensional example the (gradient of the) Euler equation (6.5) is expected to become

$$\begin{aligned} 0 = & \partial_n \mathbf{v}(n, \varphi) + 3\mathbf{v}(n, \varphi) + \mathbf{v}(n, \varphi) \partial_\varphi \mathbf{v}(n, \varphi) + \frac{H^4 \mathcal{N}^2(n)}{32\pi^4} \partial_\varphi \left(\frac{\partial_\varphi^2 \sqrt{\rho(n, \varphi)}}{\sqrt{\rho(n, \varphi)}} \right) \\ & + \frac{\partial_\varphi \mathcal{P}(n, \varphi) - \partial_\varphi (\eta(n, \varphi) \partial_\varphi \mathbf{v}(n, \varphi))}{\rho(n, \varphi)} + \partial_\varphi W(n, \varphi), \end{aligned} \quad (6.8)$$

with $\eta(n, \varphi)$ a viscosity coefficient. This equation can be seen as generalization of Navier-Stokes equation in this of fluid in an expanding universe context.

It would be nice to find quantitative ways to estimate the structure of $\eta(n, \varphi)$ – from a purely phenomenological perspectives, or using specific physical arguments. We leave this question open. But if η is proportional to the fluid density, say $\eta(n, \varphi) = \eta_0(n) \rho(n, \varphi)$, then the equations can be solved analytically, at least in the simplest case $U = 0$, $\mathcal{N} = 1$, and $\mathcal{D} = 0$ of evolution in de Sitter space. The solution of eqs (6.2) and (6.8), at leading order in $1/n$,

results

$$\rho(n, \varphi) = \frac{1}{\sqrt{\pi g(n)}} e^{-\varphi^2/g(n)} \quad (6.9)$$

$$\Theta(n, \varphi) = \frac{g'}{4g} \phi^2 \quad (6.10)$$

with

$$g(n) = \left(\frac{H^2}{2\pi^2} - \frac{2}{3n} \eta_0(n) \right) n \quad (6.11)$$

Hence the viscous dissipative term can affect the variance of the Gaussian distribution of the fluid density.

The study of dissipative effects during inflation is an interesting topic, which is currently developed through many fronts, also using an open effective field theory approach [16, 19, 74–82]. It will be interesting to understand whether our approach based on a fluid description of superhorizon fluctuations can help to further develop and investigate this subject.

7 Outlook

We developed a superfluid approach to describe the physics of long wavelength fluctuations in a framework related with stochastic inflation. We did so by making use of the Madelung approach to the functional Schrödinger equation for the inflationary wavefunction. We shown that our method allows to consistently control the evolution of the inflationary wavefunction, and discussed physical implications of our approach. We pointed out that the quantum pressure characterizing the Euler equation for the superfluid can have an important role during an ultra-slow-roll phase of inflationary dynamics. Hence, our approach can provide an alternative perspective on the quantum-to-classical transition during inflation. By implementing some phenomenological considerations, we also proposed how to include dissipative effects in our description.

Much work is left for the future. It would be interesting consider the case of multiple fields, and look for vortex solutions of the fluid equations exhibiting the phenomenon of conservation of vorticity. It would be important to better implement effects due to interactions and nonlinearities, also possibly through a Gross-Pitaevskii version of the Schrödinger equation. Finally, it would be very interesting to include dissipative effects from first principles.

Acknowledgments

It is a pleasure to thank Thomas Colas for discussions. GT is partially funded by the STFC grants ST/T000813/1 and ST/X000648/1. For the purpose of open access, the author has applied a Creative Commons Attribution licence to any Author Accepted Manuscript version arising.

A Standard stochastic formulas

Since ours is a free field, we assume the following Gaussian Ansatz for the wavefunction of eq (2.7):

$$\Psi_k = \Omega_k(\tau) e^{-z^2(\tau)(\alpha_k(\tau)\varphi_k\varphi_{-k} - \alpha_0(k)\delta_{k0}\varphi_k)} \quad (\text{A.1})$$

The Schroedinger equation (2.7) imposes the conditions

$$0 = \Omega'_k + i\alpha_k\Omega_k \quad (\text{A.2})$$

$$0 = \alpha'_k + i\alpha_k^2 + \frac{2z'(\tau)}{z(\tau)}\alpha_k - ik^2 \quad (\text{A.3})$$

By defining

$$\alpha_k(\tau) = -i\partial_\tau \ln[u_k^*] \quad (\text{A.4})$$

we find that the second condition above is equivalent to the equation (2.2) for the mode φ_k :

$$(zu_k)'' + \left(k^2 - \frac{z''}{z}\right) zu_k = 0 \quad (\text{A.5})$$

We impose the Wronskian normalization $u'_k u_{-k} - u'_{-k} u_k = i/z^2$, and Bunch-Davies boundary conditions at early times. Hence, from now on, we identify $u_k = \varphi_k$.

$$\alpha_k + \alpha_{-k} = -\frac{1}{z^2 \varphi_k \varphi_{-k}} \quad (\text{A.6})$$

$$\alpha_k - \alpha_{-k} = -i\partial_\tau \ln[\varphi_k \varphi_{-k}] \quad (\text{A.7})$$

Defining $\rho_k = \Psi_k^* \Psi_k$, it satisfies the relation

$$\frac{\partial \rho_k}{\partial \tau} = \omega_k \frac{\partial^2 \rho_k}{\partial \varphi_k \partial \varphi_{-k}} + \omega_0 \left[\frac{\partial}{\partial \varphi_k} (\varphi_k \rho_k) + \frac{\partial}{\partial \varphi_{-k}} (\varphi_{-k} \rho_{-k}) \right] \quad (\text{A.8})$$

with

$$\omega_k = \frac{i}{z^2} \frac{\alpha_k - \alpha_0 - \alpha_k^* + \alpha_0^*}{\alpha_k + \alpha_k^*} = -|\varphi_0|^2 \partial_\tau \left(\frac{|\varphi_k|^2}{|\varphi_0|^2} \right) \quad (\text{A.9})$$

$$\omega_0 = -\frac{i}{2}(\alpha_0 - \alpha_0^*) = -\frac{1}{2} \partial_\tau \ln(|\varphi_0|^2) \quad (\text{A.10})$$

Coarse graining as above, we get a Starobinsky diffusion equation

$$\frac{\partial \bar{\rho}}{\partial n} = \mathcal{N} \frac{\partial^2 \bar{\rho}}{\partial \bar{\Phi}^2} + \mathcal{D} \frac{\partial (\bar{\Phi} \bar{\rho})}{\partial \bar{\Phi}} \quad (\text{A.11})$$

with noise and drift terms given by

$$\mathcal{N} = \frac{|\varphi_0|^2}{4\pi^2 a H} \int_{aH}^0 k^2 dk \partial_\tau \left(\frac{|\varphi_k|^2}{|\varphi_0|^2} \right) \quad (\text{A.12})$$

$$\mathcal{D} = -\frac{1}{2aH} \partial_\tau \ln(|\varphi_0|^2) \quad (\text{A.13})$$

reproducing eqs (3.19) and (3.20).

References

- [1] A. A. Starobinsky, “STOCHASTIC DE SITTER (INFLATIONARY) STAGE IN THE EARLY UNIVERSE,” *Lect. Notes Phys.* **246** (1986) 107–126.
- [2] Y. Nambu and M. Sasaki, “Stochastic Stage of an Inflationary Universe Model,” *Phys. Lett. B* **205** (1988) 441–446.
- [3] A. A. Starobinsky and J. Yokoyama, “Equilibrium state of a selfinteracting scalar field in the De Sitter background,” *Phys. Rev. D* **50** (1994) 6357–6368, [arXiv:astro-ph/9407016](#).
- [4] K. Enqvist, S. Nurmi, D. Podolsky, and G. I. Rigopoulos, “On the divergences of inflationary superhorizon perturbations,” *JCAP* **04** (2008) 025, [arXiv:0802.0395 \[astro-ph\]](#).
- [5] F. Finelli, G. Marozzi, A. A. Starobinsky, G. P. Vacca, and G. Venturi, “Generation of fluctuations during inflation: Comparison of stochastic and field-theoretic approaches,” *Phys. Rev. D* **79** (2009) 044007, [arXiv:0808.1786 \[hep-th\]](#).
- [6] D. Polarski and A. A. Starobinsky, “Semiclassicality and decoherence of cosmological perturbations,” *Classical and Quantum Gravity* **13** no. 3, (Mar., 1996) 377–391. <http://dx.doi.org/10.1088/0264-9381/13/3/006>.
- [7] C. Kiefer, D. Polarski, and A. A. Starobinsky, “Quantum-to-classical transition for fluctuations in the early universe,” *International Journal of Modern Physics D* **07** no. 03, (June, 1998) 455–462. <http://dx.doi.org/10.1142/S0218271898000292>.
- [8] C. Kiefer and D. Polarski, “Why do cosmological perturbations look classical to us?,” *Adv. Sci. Lett.* **2** (2009) 164–173, [arXiv:0810.0087 \[astro-ph\]](#).
- [9] D. Sudarsky, “Shortcomings in the understanding of why cosmological perturbations look classical,” *International Journal of Modern Physics D* **20** no. 04, (Apr., 2011) 509–552. <http://dx.doi.org/10.1142/S0218271811018937>.
- [10] C. P. Burgess, R. Holman, G. Tasinato, and M. Williams, “EFT Beyond the Horizon: Stochastic Inflation and How Primordial Quantum Fluctuations Go Classical,” *JHEP* **03** (2015) 090, [arXiv:1408.5002 \[hep-th\]](#).
- [11] J. Martin and V. Vennin, “Quantum discord of cosmic inflation: Can we show that cmb anisotropies are of quantum-mechanical origin?” *Physical Review D* **93** no. 2, (Jan., 2016) . <http://dx.doi.org/10.1103/PhysRevD.93.023505>.
- [12] J. Martin, A. Micheli, and V. Vennin, “Discord and decoherence,” *Journal of Cosmology and Astroparticle Physics* **2022** no. 04, (Apr., 2022) 051. <http://dx.doi.org/10.1088/1475-7516/2022/04/051>.

- [13] S. M. Chandran, K. Rajeev, and S. Shankaranarayanan, “Real-space quantum-to-classical transition of time dependent background fluctuations,” *Physical Review D* **109** no. 2, (Jan., 2024) .
<http://dx.doi.org/10.1103/PhysRevD.109.023503>.
- [14] V. Vennin, *Stochastic inflation and primordial black holes*. PhD thesis, U. Paris-Saclay, 6, 2020. [arXiv:2009.08715](https://arxiv.org/abs/2009.08715) [[astro-ph.CO](#)].
- [15] D. Green, *EFT for de Sitter Space*. 2023. [arXiv:2210.05820](https://arxiv.org/abs/2210.05820) [[hep-th](#)].
- [16] C. P. Burgess and G. Kaplanek, “Gravity, Horizons and Open EFTs,” [arXiv:2212.09157](https://arxiv.org/abs/2212.09157) [[hep-th](#)].
- [17] A. H. Guth and S.-Y. Pi, “The Quantum Mechanics of the Scalar Field in the New Inflationary Universe,” *Phys. Rev. D* **32** (1985) 1899–1920.
- [18] J. Guven, B. Lieberman, and C. T. Hill, “Schrodinger Picture Field Theory in Robertson-walker Flat Space-times,” *Phys. Rev. D* **39** (1989) 438.
- [19] C. P. Burgess, R. Holman, and G. Tasinato, “Open EFTs, IR effects & late-time resummations: systematic corrections in stochastic inflation,” *JHEP* **01** (2016) 153, [arXiv:1512.00169](https://arxiv.org/abs/1512.00169) [[gr-qc](#)].
- [20] E. Madelung, “Quantentheorie in hydrodynamischer Form,” *Z. Phys.* **40** no. 3, (1927) 322–326.
- [21] W. H. Kinney, “Horizon crossing and inflation with large eta,” *Phys. Rev. D* **72** (2005) 023515, [arXiv:gr-qc/0503017](https://arxiv.org/abs/gr-qc/0503017).
- [22] S. M. Leach and A. R. Liddle, “Inflationary perturbations near horizon crossing,” *Phys. Rev. D* **63** (2001) 043508, [arXiv:astro-ph/0010082](https://arxiv.org/abs/astro-ph/0010082).
- [23] S. M. Leach, M. Sasaki, D. Wands, and A. R. Liddle, “Enhancement of superhorizon scale inflationary curvature perturbations,” *Phys. Rev. D* **64** (2001) 023512, [arXiv:astro-ph/0101406](https://arxiv.org/abs/astro-ph/0101406).
- [24] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman lectures on physics; New millennium ed.* Basic Books, New York, NY, 2010.
<https://cds.cern.ch/record/1494701>. Originally published 1963-1965.
- [25] L. M. Widrow and N. Kaiser, “Using the Schrodinger equation to simulate collisionless matter,” *Astrophys. J. Lett.* **416** (1993) L71–L74.
- [26] C. Uhlemann, M. Kopp, and T. Haugg, “Schrödinger method as N -body double and UV completion of dust,” *Phys. Rev. D* **90** no. 2, (2014) 023517, [arXiv:1403.5567](https://arxiv.org/abs/1403.5567) [[astro-ph.CO](#)].
- [27] L. Hui, J. P. Ostriker, S. Tremaine, and E. Witten, “Ultralight scalars as cosmological dark matter,” *Phys. Rev. D* **95** no. 4, (2017) 043541, [arXiv:1610.08297](https://arxiv.org/abs/1610.08297) [[astro-ph.CO](#)].

- [28] M. Garny, T. Konstandin, and H. Rubira, “The Schrödinger-Poisson method for Large-Scale Structure,” *JCAP* **04** (2020) 003, [arXiv:1911.04505 \[astro-ph.CO\]](#).
- [29] L. Hui, “Wave Dark Matter,” *Ann. Rev. Astron. Astrophys.* **59** (2021) 247–289, [arXiv:2101.11735 \[astro-ph.CO\]](#).
- [30] E. G. M. Ferreira, “Ultra-light dark matter,” *Astron. Astrophys. Rev.* **29** no. 1, (2021) 7, [arXiv:2005.03254 \[astro-ph.CO\]](#).
- [31] D. Wands, K. A. Malik, D. H. Lyth, and A. R. Liddle, “A New approach to the evolution of cosmological perturbations on large scales,” *Phys. Rev. D* **62** (2000) 043527, [arXiv:astro-ph/0003278](#).
- [32] D. S. Salopek and J. R. Bond, “Nonlinear evolution of long wavelength metric fluctuations in inflationary models,” *Phys. Rev. D* **42** (1990) 3936–3962.
- [33] M. Sasaki and E. D. Stewart, “A General analytic formula for the spectral index of the density perturbations produced during inflation,” *Prog. Theor. Phys.* **95** (1996) 71–78, [arXiv:astro-ph/9507001](#).
- [34] G. I. Rigopoulos and E. P. S. Shellard, “The separate universe approach and the evolution of nonlinear superhorizon cosmological perturbations,” *Phys. Rev. D* **68** (2003) 123518, [arXiv:astro-ph/0306620](#).
- [35] Y. Tanaka and M. Sasaki, “Gradient expansion approach to nonlinear superhorizon perturbations. II. A Single scalar field,” *Prog. Theor. Phys.* **118** (2007) 455–473, [arXiv:0706.0678 \[gr-qc\]](#).
- [36] A. J. Tolley and M. Wyman, “Stochastic Inflation Revisited: Non-Slow Roll Statistics and DBI Inflation,” *JCAP* **04** (2008) 028, [arXiv:0801.1854 \[hep-th\]](#).
- [37] C. Agon, V. Balasubramanian, S. Kasko, and A. Lawrence, “Coarse Grained Quantum Dynamics,” *Phys. Rev. D* **98** no. 2, (2018) 025019, [arXiv:1412.3148 \[hep-th\]](#).
- [38] J. Grain and V. Vennin, “Stochastic inflation in phase space: Is slow roll a stochastic attractor?,” *JCAP* **05** (2017) 045, [arXiv:1703.00447 \[gr-qc\]](#).
- [39] S. Céspedes, A.-C. Davis, and D.-G. Wang, “On the IR divergences in de Sitter space: loops, resummation and the semi-classical wavefunction,” *JHEP* **04** (2024) 004, [arXiv:2311.17990 \[hep-th\]](#).
- [40] Y. L. Launay, G. I. Rigopoulos, and E. P. S. Shellard, “Stochastic inflation in general relativity,” *Phys. Rev. D* **109** no. 12, (2024) 123523, [arXiv:2401.08530 \[gr-qc\]](#).
- [41] D. S. Salopek and J. R. Bond, “Stochastic inflation and nonlinear gravity,” *Phys. Rev. D* **43** (1991) 1005–1031.
- [42] V. Vennin and A. A. Starobinsky, “Correlation Functions in Stochastic Inflation,” *Eur. Phys. J. C* **75** (2015) 413, [arXiv:1506.04732 \[hep-th\]](#).
- [43] G. Tasinato, “Stochastic approach to gravitational waves from inflation,” *Phys. Rev. D* **105** no. 2, (2022) 023521, [arXiv:2201.10333 \[hep-th\]](#).

- [44] R. H. Brandenberger, R. Laflamme, and M. Mijic, “Classical Perturbations From Decoherence of Quantum Fluctuations in the Inflationary Universe,” *Mod. Phys. Lett. A* **5** (1990) 2311–2318.
- [45] E. Calzetta and B. L. Hu, “Quantum fluctuations, decoherence of the mean field, and structure formation in the early universe,” *Phys. Rev. D* **52** (1995) 6770–6788, [arXiv:gr-qc/9505046](#).
- [46] J. Lesgourgues, D. Polarski, and A. A. Starobinsky, “Quantum to classical transition of cosmological perturbations for nonvacuum initial states,” *Nucl. Phys. B* **497** (1997) 479–510, [arXiv:gr-qc/9611019](#).
- [47] F. C. Lombardo, “Influence functional approach to decoherence during inflation,” *Braz. J. Phys.* **35** (2005) 391–396, [arXiv:gr-qc/0412069](#).
- [48] C. P. Burgess, R. Holman, and D. Hoover, “Decoherence of inflationary primordial fluctuations,” *Phys. Rev. D* **77** (2008) 063534, [arXiv:astro-ph/0601646](#).
- [49] J. W. Sharman and G. D. Moore, “Decoherence due to the Horizon after Inflation,” *JCAP* **11** (2007) 020, [arXiv:0708.3353 \[gr-qc\]](#).
- [50] E. Nelson, “Quantum Decoherence During Inflation from Gravitational Nonlinearities,” *JCAP* **03** (2016) 022, [arXiv:1601.03734 \[gr-qc\]](#).
- [51] T. J. Hollowood and J. I. McDonald, “Decoherence, discord and the quantum master equation for cosmological perturbations,” *Phys. Rev. D* **95** no. 10, (2017) 103521, [arXiv:1701.02235 \[gr-qc\]](#).
- [52] C. P. Burgess, R. Holman, G. Kaplanek, J. Martin, and V. Vennin, “Minimal decoherence from inflation,” *JCAP* **07** (2023) 022, [arXiv:2211.11046 \[hep-th\]](#).
- [53] J. Martin and V. Vennin, “Observational constraints on quantum decoherence during inflation,” *JCAP* **05** (2018) 063, [arXiv:1801.09949 \[astro-ph.CO\]](#).
- [54] T. Colas, J. Grain, and V. Vennin, “Quantum recoherence in the early universe,” *EPL* **142** no. 6, (2023) 69002, [arXiv:2212.09486 \[gr-qc\]](#).
- [55] A. Daddi Hammou and N. Bartolo, “Cosmic decoherence: primordial power spectra and non-Gaussianities,” *JCAP* **04** (2023) 055, [arXiv:2211.07598 \[astro-ph.CO\]](#).
- [56] C. Burgess, T. Colas, R. Holman, G. Kaplanek, and V. Vennin, “Cosmic purity lost: perturbative and resummed late-time inflationary decoherence,” *Journal of Cosmology and Astroparticle Physics* **2024** no. 08, (Aug., 2024) 042. <http://dx.doi.org/10.1088/1475-7516/2024/08/042>.
- [57] F. Lopez and N. Bartolo, “Quantum signatures and decoherence during inflation from deep subhorizon perturbations,” [arXiv:2503.23150 \[astro-ph.CO\]](#).
- [58] C. Pattison, V. Vennin, H. Assadullahi, and D. Wands, “Quantum diffusion during inflation and primordial black holes,” *JCAP* **10** (2017) 046, [arXiv:1707.00537 \[hep-th\]](#).

- [59] J. M. Ezquiaga and J. García-Bellido, “Quantum diffusion beyond slow-roll: implications for primordial black-hole production,” *JCAP* **08** (2018) 018, [arXiv:1805.06731 \[astro-ph.CO\]](#).
- [60] M. Biagetti, G. Franciolini, A. Kehagias, and A. Riotto, “Primordial Black Holes from Inflation and Quantum Diffusion,” *JCAP* **07** (2018) 032, [arXiv:1804.07124 \[astro-ph.CO\]](#).
- [61] D. G. Figueroa, S. Raatikainen, S. Rasanen, and E. Tomberg, “Non-Gaussian Tail of the Curvature Perturbation in Stochastic Ultraslow-Roll Inflation: Implications for Primordial Black Hole Production,” *Phys. Rev. Lett.* **127** no. 10, (2021) 101302, [arXiv:2012.06551 \[astro-ph.CO\]](#).
- [62] A. Achúcarro, S. Cespedes, A.-C. Davis, and G. A. Palma, “The hand-made tail: non-perturbative tails from multifield inflation,” *JHEP* **05** (2022) 052, [arXiv:2112.14712 \[hep-th\]](#).
- [63] Y.-F. Cai, X.-H. Ma, M. Sasaki, D.-G. Wang, and Z. Zhou, “Highly non-Gaussian tails and primordial black holes from single-field inflation,” *JCAP* **12** (2022) 034, [arXiv:2207.11910 \[astro-ph.CO\]](#).
- [64] C. Animali and V. Vennin, “Primordial black holes from stochastic tunnelling,” *JCAP* **02** (2023) 043, [arXiv:2210.03812 \[astro-ph.CO\]](#).
- [65] S. Hooshangi, M. H. Namjoo, and M. Noorbala, “Tail diversity from inflation,” *JCAP* **09** (2023) 023, [arXiv:2305.19257 \[astro-ph.CO\]](#).
- [66] V. Vennin and D. Wands, *Quantum Diffusion and Large Primordial Perturbations from Inflation*. 2025. [arXiv:2402.12672 \[astro-ph.CO\]](#).
- [67] O. Özsoy and G. Tasinato, “Inflation and Primordial Black Holes,” *Universe* **9** no. 5, (2023) 203, [arXiv:2301.03600 \[astro-ph.CO\]](#).
- [68] G. Tasinato, “An analytic approach to non-slow-roll inflation,” *Phys. Rev. D* **103** no. 2, (2021) 023535, [arXiv:2012.02518 \[hep-th\]](#).
- [69] G. Tasinato, “Large $-\eta$ approach to single field inflation,” *Phys. Rev. D* **108** no. 4, (2023) 043526, [arXiv:2305.11568 \[hep-th\]](#).
- [70] M. Hillery, R. F. O’Connell, M. O. Scully, and E. P. Wigner, “Distribution functions in physics: Fundamentals,” *Phys. Rept.* **106** (1984) 121–167.
- [71] S. Habib, “Stochastic inflation: The Quantum phase space approach,” *Phys. Rev. D* **46** (1992) 2408–2427, [arXiv:gr-qc/9208006](#).
- [72] A. Serafini, F. Illuminati, and S. De Siena, “Von Neumann entropy, mutual information and total correlations of Gaussian states,” *J. Phys. B* **37** (2004) L21, [arXiv:quant-ph/0307073](#).
- [73] L. D. Landau, E. M. Lifshitz, J. B. Sykes, and W. H. Reid, *Fluid Mechanics*. Pergamon Press Oxford, England, 1959.

- [74] H. P. Breuer and F. Petruccione, *The theory of Open Quantum Systems*. Oxford University Press, 2002.
- [75] E. A. Calzetta and B.-L. B. Hu, *Nonequilibrium Quantum Field Theory*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Sept., 2008.
- [76] D. Boyanovsky, “Effective Field Theory out of Equilibrium: Brownian quantum fields,” *New J. Phys.* **17** no. 6, (2015) 063017, [arXiv:1503.00156 \[hep-ph\]](#).
- [77] D. Boyanovsky, “Effective field theory during inflation: Reduced density matrix and its quantum master equation,” *Phys. Rev. D* **92** no. 2, (2015) 023527, [arXiv:1506.07395 \[astro-ph.CO\]](#).
- [78] D. Boyanovsky, “Effective field theory during inflation. II. Stochastic dynamics and power spectrum suppression,” *Phys. Rev. D* **93** (2016) 043501, [arXiv:1511.06649 \[astro-ph.CO\]](#).
- [79] C. P. Burgess, *Introduction to Effective Field Theory*. Cambridge University Press, Dec., 2020.
- [80] T. Colas, *Open Effective Field Theories for primordial cosmology : dissipation, decoherence and late-time resummation of cosmological inhomogeneities*. PhD thesis, Institut d’astrophysique spatiale, France, AstroParticule et Cosmologie, France, APC, Paris, 2023.
- [81] S. A. Salcedo, T. Colas, and E. Pajer, “The open effective field theory of inflation,” *JHEP* **10** (2024) 248, [arXiv:2404.15416 \[hep-th\]](#).
- [82] S. A. Salcedo, T. Colas, and E. Pajer, “An Open Effective Field Theory for light in a medium,” *JHEP* **03** (2025) 138, [arXiv:2412.12299 \[hep-th\]](#).