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Logical models of mathematical texts II: Legality conventions for division by zero in inconsistent contexts

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Abstract

To avoid the risk of problems to do with division by zero (DbZ), arithmetical texts involving division use what may be called *traditional conventions* on DbZ. Earlier, we developed a method for exploring these conventions using informal notions of legal and illegal texts, which are used to analyse simple fragments of arithmetical texts. We showed how these texts can be transformed into logical formulae over special total algebras, called *common meadows*, that are able to approximate partiality but in a total world. The subtleties of the legal/illegal distinction call for further development of these mathematical methods. Here we examine a more complex type of text, namely proof by contradiction, in which inconsistent assumptions can coexist with DbZ. We formulate more advanced criteria of legality for this case. We introduce a three-valued logic to capture the resulting semiformal conventions that is based on a notion we call *frugal equality* for partial operators. We apply the method to a proof of the Bayes-Price Theorem in probability theory, whose proof has DbZ issues.

Keywords Division by zero · Traditional conventions for writing arithmetic · Legal texts · Illegal texts · Proof by contradiction · Bayes-Price Theorem · Common meadows · Signed common meadows.

1 Introduction

Partiality is a common property of mathematical formulae, not least because a few basic operations, such as division and logarithms, occur throughout science and technology

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and are partial operations.[1] In practice, writing mathematical texts ought to adhere to traditional conventions that try to guard against formulae being undefined. At work seems to be this general precept:

Avoidance Principle. *Do not write down mathematical expressions unless you know they denote something.*

But what are these traditional conventions? How subtle can the issues surrounding partiality be? How meaningful, consistent and comprehensive are the conventions? To try to answer these questions we need to cast light on some current mathematical practices regarding partiality. Actually, we consider the elucidation and specification of traditional conventions on partiality to be a problem worthy of research. In Bergstra and Tucker (2004), we proposed a methodology and some mathematical tools to begin to undertake this investigation, starting with division: division x/y is a partial operation that is used everywhere, and on the rational, real and complex numbers it is undefined when $y = 0$.[2] Thus, we have the special case:

Avoidance of Division by Zero Principle. *Do not write down a division x/y unless you know that $y \neq 0$.*

From experience, writing down $1/0$ is sufficiently rare and provocative to suggest that traditional conventions that guard against division by zero (DbZ) are effective. However, our examples in Bergstra and Tucker (2004) suggest that this is *not* the case. Whilst in Bergstra and Tucker (2004) we examined fragments of text in pursuit of conventions and precise criteria, here we examine whole proofs with division by zero issues, and resolve some issues we left open.

1.1 Our methodology

Mathematical texts are highly structured: they declare notations, definitions, assumptions, assertions, deductions, proofs, exercises, etc., and other syntactic units. To model aspects of mathematical texts, we employ a simplified semiformal language *EAT*, for *elementary arithmetical texts*, containing division; a simpler version of *EAT* was introduced in our earlier Bergstra and Tucker (2004).

A basic aim is to characterise a text belonging to *EAT* to be *legal* if a DbZ is not allowed by the text, and *illegal* if the text allows *some* circumstance in which a DbZ could occur. So, we seek to isolate rules that can separate the legal texts from the illegal texts in the language *EAT*. From this perspective, texts ought to be legal, but need not be true.

The method developed in Bergstra and Tucker (2004) begins with examining many examples of fragments of text containing divisions, seeking to test if there are intuitive conventions that are obvious, understandable and adequate. In simple cases we found ambiguities and lacunae that suggested a need for new working definitions of legal/illegal texts and possibly conventions.

[1] For example, division and logarithms play a role in probability and information theory and their many applications.

[2] We will vary the notation for division, $\frac{x}{y}$, $x \div y$, etc., without ceremony.

The second stage of the method turns to logic(s) to try to formalise semantic conventions for the legality of texts in *EAT*. Now, logics are well understood and far easier to work with when applied to total algebras rather than to partial algebras. Our idea is to develop a formal model of legal texts using first order logical formulae over total arithmetical algebras called *common meadows*. Meadows are fields with explicit division operators added (Bergstra and Tucker 2007); common meadows are meadows in which division is made total by adding an absorptive value $\perp$ in order to define $1/0 = \perp$, Bergstra and Ponse (2015); Bergstra and Tucker (2023c, 2025). A signed common meadow is a common meadow with a sign function s that mimics an ordering on the meadow. A standard example of a signed common meadow is an algebra built from a field of rational numbers having the form:

$$(\mathbb{Q} \cup \{\perp\} | 0, 1, \perp, +, -, \cdot, \div, \mathsf{s}).$$

We have in mind mathematical texts involving division and so our *EAT* will focus on common meadows.

Method. *Choose a logic L over a common meadow M. On transforming a mathematical text T from the text language EAT into a formula* ϕ_T *in the logic L,*

$$T \in EAT \to \phi_T \in L,$$

we examine the correspondence and classify the legal texts and illegal texts of EAT by analysing the well-formed formulae in L representing the texts.

Thus, this method searches for old and new logics that applied to common meadows yield more faithful formal approximations to legal texts. We expect that the well-formed formulae we characterise to be a superset of the (translations of the) legal texts. In this way, we seek to uncover and approximate formally the underlying logic of writing conventions.

The subtlety of the legal/illegal distinction we found in Bergstra and Tucker (2004) calls for further development of these mathematical methods. Our treatment in Bergstra and Tucker (2004) discussed the legal/illegal distinctions in the context of inconsistent assumptions in a text, but only for very simple cases; our analysis was inconclusive, or rather incomplete. Here we examine a more complex type of text that is very common, namely *proof by contradiction*, in which seemingly essential inconsistent assumptions can coexist in a text with DbZ. We examine some easy proofs by contradiction in order to make some subtle issues plainly visible.

The problem of inconsistency for legality is this. Suppose two inconsistent statements appear in a text prior to a division, say $\frac{x}{y}$. Then the inconsistencies mean anything could now be proved. In particular, $y = 0$ could be deduced and a DbZ would become possible, i.e., the text would not be legal. This seems to be a feature of proofs by contradiction that needs investigating.

Specifically, we employ the algebra of our main tools for a case study: we consider proofs by contradiction of this basic property of signed common meadows (cf. Propositions 2.1 and 2.2 below):

$$\forall x (x \neq \perp \ \&\ \mathsf{s}(x) = 0 \to x = 0).$$

We formulate a more general, advanced semiformal criterion of legality to cover proof by contradiction in a text.

For the next stage, we introduce a three-valued logic to capture the semiformal conventions. The logic is tailored to a three-valued notion of equality for partial operators that we call *frugal equality*. Formally, frugal equality corresponds to the partial equality, the logic of which has been developed in detail, which is based on short-circuit logic of Bergstra and Ponse (2025). Frugal equality corresponds to partial equality in the following manner: frugal equality assumes the presence of absorptive element $\perp$ which is then take to represent "being undefined". Frugal equality is just one of a number of semantic options for equality between $\perp$-totalized partial functions that merits to be studied in the context of common meadows (Bergstra and Tucker 2023b).

Finally, to illustrate the issues and logical tools further, we apply them to proofs of the Bayes-Price Theorem in probability theory that contain DbZ and so have illegality issues.

1.2 Structure of the paper

In Section 2, we recall the essentials of common meadows and the sign function; we give axiomatisations and number of propositions relating signed common meadows to ordered fields. In Section 3, we introduce the language *EAT* and consider division by zero in two proofs by contradiction; we propose a general notion of legality for such situations where inconsistencies can arise. Section 4 introduces notions of equality that make sense for partial operations and an accompanying 3-valued logic to formalise our texts. In Section 5, we derive the Bayes-Price Theorem which involves potential divisions by zero. Finally, we offer a few reflections in Section 6. We assume the reader is familiar with some very elementary field theory (van der Waerden 1970; Lang 1965). At a few points reference to our earlier (Bergstra and Tucker 2004) will be needed.

2 Tools: common meadows with a sign function

In this section we recall from Bergstra and Tucker (2004), and elsewhere, the main tool of common meadows. Then we extend common meadows with a sign function that encodes an ordering relation as an operator on a common meadow (Bergstra and Tucker 2004).

2.1 Common meadows

The algebras we use are fields. A field is an algebra of the form

$$(G \mid 0, 1, x + y, -x, x \cdot y)$$

in which all operations are total functions, van der Waerden (1970).

A field G is enlarged with a division operator $\div$ to form a *meadow* $G(\div)$, which is an algebra of the form

$$(G \mid 0, 1, x + y, -x, x \cdot y, x \div y)$$

in which division is partial at $y = 0$, Bergstra and Tucker (2007).

To a meadow is added a new element $\perp \notin G$, called a peripheral element. The peripheral element $\perp$ is *absorptive*, i.e., for all $x \in G$,

$$x + \perp = \perp + y = \perp, \; -\perp = \perp, \; x \cdot \perp = \perp \cdot y = \perp, \; x \div \perp = \perp \div y = \perp.$$

In particular, when $x \div y$ is not defined in G then we define $x \div 0 = \perp$. A meadow whose division is made total in this way is called a *common meadow*, Bergstra and Ponse (2015).

Actually, this technique of adding an absorptive $\perp$ to totalise the partial operations of an algebraic structure A is quite general and has been studied in Bergstra and Tucker (2022), where the process is specified by a general operator $\mathsf{Enl}_\perp(A)$ on partial algebras A.

The total algebras we will use for our investigation are common meadows: using the general method we define:

Definition 2.1 For G a field,

$$\mathsf{Enl}_\perp(G(\div))$$

is a *common meadow*. Thus, in common meadows, for all x,

$$x \div 0 = \perp.$$

The use of a partial division operator on G will be faithfully reflected by our total methods on $\mathsf{Enl}_\perp(G(\div))$. A simple but important point is that the legality of terms and formulae over a total algebra $\mathsf{Enl}_\perp(G(\div))$ is not an issue as they are defined by standard rules of first order logic, i.e., all well-formed first order formulae are legal relative to $\mathsf{Enl}_\perp(G(\div))$.

Secondly, note the important role of equality. Now, the underlying field G and the enlargement $\mathsf{Enl}_\perp(G(\div))$ are total algebras and therefore both come with a native notion of equality, written $=$ in both cases, from first order logic. However, whilst equality in the common meadow $\mathsf{Enl}_\perp(G(\div))$ has an unambiguous standard notion, in the partial meadow $G(\div)$ there are several options as what to do when terms are undefined, and so equality becomes a complicated matter; we will note four options in Section 4 below. These semantic options for equality can be replicated in $\mathsf{Enl}_\perp(G(\div))$.

Native equality in the common meadow roughly *approximates* the use of $=$ in elementary arithmetic with $\div$ under traditional conventions: e.g., this well-formed and valid equation over $\mathsf{Enl}_\perp(G(\div))$

$$\mathsf{Enl}_\perp(G(\div)) \models 1 \div 0 = (-1) \div 0$$

is legal in the first order logical language, but would be rejected by traditional conventions because of its illegal division by zero.

So, common meadows have the general form

$$(G \cup \{\bot\} \mid 0, 1, \bot, x + y, -x, x \cdot y, x \div y)$$

where G is a field and $\bot$ is an absorptive element that behaves like an error value. The idea of a common meadow was introduced in Bergstra and Ponse (2015).

The signatures of rings and fields are the same and are denoted Σ_{r}. The signature of a meadow is denoted Σ_{m}. The signature Σ_{cm} of a common meadow extends the signature Σ_{m} of meadows with the constant $\bot$. The addition of $\div$ introduces the all important fractions which in the setting of meadows can be defined unambiguously:

Definition 2.2 Any term over a signature containing Σ_{m} that contains a division operator as its leading function symbol is called a *fracterm*.

The terminology of fracterms is due to Bergstra and Ponse (2016) and has been further developed in Bergstra (2020).

2.2 Equational axioms for common meadows

Just as rings and fields are defined axiomatically we have found axiomatisations of common meadows. The importance of division led us to call these equation-based formal systems *fracterm calculi* (after Definition 2.2). Following the investigation of a number of formalisations, we settled upon sets of axioms *equivalent* to a set of equations in Bergstra and Tucker (2023c) and usually named as $E_{\mathsf{ftc-cm}}$ in earlier work, where the E emphasises the axioms are equational.

The axioms we use here are a new formulation, using a little more of the expressive power of first order logic than mere equational logic. In particular, use is made of propositional connectives including negation. These axioms together are stronger than any of the equational axiomatizations of common meadows given in our earlier work for the simple reason that an equational axiom system cannot prove any inequality, in particular not $0 \neq 1$. These new axioms are built up in stages, given by Tables 1 and 2. In the presence of propositional connectives axioms for division become more concise.

In Table 3 we list some equations derivable from Tables 1 and 2 that we commonly use in arguments, many of which have appeared as axioms in earlier work.

2.3 The sign function

An *ordered field* $G_{\leq}$ is a field G with an ordering relation that is

(i) reflexive, antisymmetric, and transitive;
(ii) total: for all $x, y \in G_{\leq}$, either $x \leq y$ or $y \leq x$; and
(iii) compatible with the operations: for all $x, y, a \in G_{\leq}$, if $x \leq y$ then $x + a \leq y + a$; and if $0 \leq x$ and $0 \leq y$ then $0 \leq x \cdot y$.

Table 1 $AX_{\mathsf{wcr},\bot}$: Axioms for weak commutative rings with $\bot$

$(x+y)+z = x+(y+z)$	(1)
$x+y = y+x$	(2)
$x+0 = x$	(3)
$x+(-x) = 0\cdot x$	(4)
$x\cdot(y\cdot z) = (x\cdot y)\cdot z$	(5)
$x\cdot y = y\cdot x$	(6)
$1\cdot x = x$	(7)
$x\cdot(y+z) = (x\cdot y)+(x\cdot z)$	(8)
$-(-x) = x$	(9)
$0\cdot(x\cdot x) = 0\cdot x$	(10)
$x+\bot = \bot$	(11)
$0 \neq 1$	(12)
$x \neq \bot \rightarrow 0\cdot x = 0$	(13)

Table 2 $AX_{\mathsf{ftc-cm}}$: Axioms for fracterm calculus for common meadows

`import:` $AX_{\mathsf{wcr},\bot}$	
$\frac{x}{y} = x\cdot\frac{1}{y}$	(14)
$\frac{1}{x}\cdot\frac{1}{y} = \frac{1}{x\cdot y}$	(15)
$x \neq 0 \wedge x \neq \bot \rightarrow \frac{x}{x} = 1$	(16)
$\bot = \frac{1}{0}$	(17)

Table 3 Equations and inequations derivable from $AX_{\mathsf{ftc-cm}}$

$0 \neq \bot$	(18)
$1 \neq \bot$	(19)
$-\frac{x}{y} = \frac{-x}{y}$	(20)
$\frac{x}{y}\cdot\frac{u}{v} = \frac{x\cdot u}{y\cdot v}$	(21)
$\frac{x}{y}+\frac{u}{v} = \frac{(x\cdot v)+(y\cdot u)}{y\cdot v}$	(22)
$\frac{x}{(\frac{u}{v})} = x\cdot\frac{v\cdot v}{u\cdot v}$	(23)
$\frac{x}{y+0\cdot z} = \frac{x+0\cdot z}{y}$	(24)

The basic theory of ordered fields can be found in many algebra text-books, starting with standard works such as van der Waerden (1970); Lang (1965).

To work with orderings in common meadows we can employ a sign operator:

Definition 2.3 A *sign operator* $\mathsf{s}(-)$ is defined for $x \in G_{\leq}$,

$$\text{if } x > 0 \text{ then } \mathsf{s}(x) = 1; \text{ if } x < 0 \text{ then } \mathsf{s}(x) = -1;\ \ \mathsf{s}(0) = 0;\ \ \mathsf{s}(\bot) = \bot.$$

However, we will work more generally and build on the characterisation $AX_{\mathsf{ftc-cm}}$ of common meadows and provide axioms for common meadows equipped with sign functions.

Definition 2.4 A *signed common meadow* is a common meadow that is equipped with a sign function $\mathsf{s}(-)$ which satisfies the axioms of Table 4.

For this definition, we do *not* assume the meadow is ordered (though it can be shown that an ordering exists derived from a sign function).

So, signed common meadows have the general form

$$G_{\perp,\mathsf{s}} = (G \cup \{\perp\} \mid 0, 1, \perp, x + y, -x, x \cdot y, x \div y, \mathsf{s}).$$

We recall that, following Bergstra and Tucker (2004), a signed common meadow is an expansion of a common meadow with a sign function $\mathsf{s}(-)$ which satisfies the axioms of Table 4. We note that the approach in Bergstra and Tucker (2004) is more general: first weakly signed meadows are introduced and then 4-signed meadows are found as a special case, satisfying axiom 29.

In particular, the common meadow of complex numbers can be equipped with a weak sign function which takes all points on the unit circle as its values; thus, in this way, the complex numbers becomes a weakly signed common meadow which is not 4-signed. For the purposes of the present paper, 4-signed is the relevant notion and providing additional generality is of no use.

Thus, in a 4-signed common meadow the sign function s takes precisely the four different values $0, 1, -1$, and $\perp$.

Lemma 2.1 *The property of 4-signedness cannot be expressed with equations or conditional equations.*

Proof One may assume, for an argument by contradiction, that some set of conditional equations E has the property that

(i) the signed common meadow of rationals satisfies E, and
(ii) only 4-signed structures satisfy E.

Then let c be a new constant, and let G_c be an initial algebra of E in the extended signature. If G_c is 4-signed then $\mathsf{s}(c)$ takes one of the values $0, 1, -1, \perp$ in G_c. Suppose, for instance, $G_c \models \mathsf{s}(c) = 0$ then, because of the initiality of G_c it must be the case that $E_c \vdash \mathsf{s}(c) = 0$; this is not true because expanding the common meadow of rationals by interpreting c as, say, 1 provides a model of E just as well. The other three cases ($1, -1$ and, $\perp$) work similarly. We have included the 4-signed requirement as axiom 29 in Table 4. □

We notice that, whilst $\mathsf{s}(\frac{1}{x}) = \frac{1}{\mathsf{s}(x)}$, it is not in general the case that

$$\mathsf{s}(\frac{1}{x}) = \mathsf{s}(x) \text{ because } \mathsf{s}(\frac{1}{0}) = \mathsf{s}(\perp) = \perp \neq 0 = \mathsf{s}(0).$$

Here are two basic properties of the sign operator.

Proposition 2.1 $\mathsf{s}(x) = \perp \rightarrow x = \perp$

Table 4 $AX_{\text{ftc-scm}}$: Axioms for signed common meadows

import: $AX_{\text{ftc-cm}}$	
$\mathsf{s}(0) = 0$	(25)
$\mathsf{s}(1) = 1$	(26)
$\mathsf{s}(-1) = -1$	(27)
$\mathsf{s}(\bot) = \bot$	(28)
$\mathsf{s}(x) \neq 0 \wedge \mathsf{s}(x) \neq 1 \wedge \mathsf{s}(x) \neq -1 \to \mathsf{s}(x) = \bot$	(29)
$\mathsf{s}(x \cdot y) = \mathsf{s}(x) \cdot \mathsf{s}(y)$	(30)
$\mathsf{s}(x) = \mathsf{s}(y) \to \mathsf{s}(x + y) = \mathsf{s}(y)$	(31)
$\mathsf{s}(\frac{1}{x}) = \frac{1}{\mathsf{s}(x)}$	(32)

Proof Given $\mathsf{s}(x) = \bot$, now if $x = 0$, $\mathsf{s}(x) = 0$, so that $x \neq 0$. Then, by axiom 16, if $x \neq \bot$ we have $\frac{x}{x} = 1$ and

$$1 = \mathsf{s}(1) = \mathsf{s}(\frac{x}{x}) = \mathsf{s}(x) \cdot \mathsf{s}(\frac{1}{x}) = \bot,$$

which is false so that the assumption $x \neq \bot$ has been refuted. □

Proposition 2.2 $\mathsf{s}(x) = 0 \to x = 0$

Proof Given $\mathsf{s}(x) = 0$, now $x \neq \bot$ as otherwise $\mathsf{s}(x) = \bot \neq 0$. Then, if $x \neq 0$, using axiom 16, one obtains $\frac{x}{x} = 1$ and

$$1 = \mathsf{s}(\frac{x}{x}) = \mathsf{s}(x) \cdot \mathsf{s}(\frac{1}{x}) = 0 \cdot \frac{1}{\mathsf{s}(x)} = 0 \cdot \frac{1}{0} = \bot,$$

which is false so that the assumption $x \neq 0$ has been refuted. □

Signed meadows can be ordered; the following is taken from Bergstra and Tucker (2004) with minor modifications.

Proposition 2.3 *Each ordered field $G_{\leq}$ can be enlarged to a signed common meadow A. Conversely, each signed common meadow A is an enlargement of an ordered field $G_{\leq}$.*

Proof First, enlarge the ordered field G to a common ordered meadow $G_{\bot}$, then enrich $G_{\bot}$ with a sign function as in Definition 2.3.

Conversely, the domain of the required field G is taken to be $G = A - \{\bot\}$. Consider the ordering $<_{\mathsf{s}}$ on $A - \{\bot\}$ given by

(i) $a <_{\mathsf{s}} b \iff \mathsf{s}(b - a) = 1$,
(ii) $a =_{\mathsf{s}} b \iff \mathsf{s}(a - b) = 0$ and,
(iii) $a \leq_{\mathsf{s}} b \iff \mathsf{s}(b - a)^2 = \mathsf{s}(b - a)$.

From Proposition 2.2, we find that $a =_{\mathsf{s}} b$ implies $a = b$. It follows from the definition of a 4-signed common meadow that for all non-$\bot$ a and b in G either $a <_{\mathsf{s}} b$ or $a =_{\mathsf{s}} b$ or $b <_{\mathsf{s}} a$.

Table 5 $AX_{\mathsf{ftc-ascm}}$: Axioms for signed common meadows with absolute value or modulus function

`import:` $AX_{\mathsf{ftc-scm}}$	(33)
$\lvert x \rvert = x \cdot \mathsf{s}(x)$	(34)

Reflexivity of $=_{\mathsf{s}}$ is immediate, and so is antisymmetry of $<_{\mathsf{s}}$. For transitivity assume $a <_{\mathsf{s}} b$ and $b <_{\mathsf{s}} c$, then $\mathsf{s}(b-a) = \mathsf{s}(c-b) = 1$ so that $\mathsf{s}(c-a) = \mathsf{s}((b-a)+(c-b)) = \mathsf{s}(b-a) = 1$ (with equation 31) whence $a <_{\mathsf{s}} c$. □

On a *weakly* signed common meadow one may define an absolute value or modulus function $\lvert - \rvert$ (see Table 5). Interestingly, this operator makes sense on complex numbers where orderings do not. We need this operator when discussing probability.

3 Legality

As in our Bergstra and Tucker (2004), the search for intuitions about legality that can be made precise is driven by examples of texts. In this section we introduce the simple language EAT and examine two texts containing proofs of Proposition 2.2. We argue that the first proof does not qualify as a legal text, whilst the second can. Then we formalise the underlying intuition on legality, which suffices to capture the distinctions made concerning the two proofs.

For our general purposes it is enough to work with arithmetic texts about the natural numbers, integers and rational numbers.

An *elementary arithmetic text (EAT)* is a sequence of statements composed of the following options:

1. introduce $x : \pi$ with π in nat, int, rat
2. assume ϕ
3. claim ϕ
4. assume for PBC ϕ
5. known ϕ
6. infer ϕ
7. infer contradiction $0 = 1$ and drop the PBC assumption in line K, [where line K contains a PBC assumption made in connection with a proof of the claim in line L]
8. confirm claim in line L.

The types nat, int, rat stand for natural, integer and rational numbers. PBC stands for 'proof by contraction'. Note the statements 4 and 7. In a text, we require that statements are separated by ';'. We also require that variables must be introduced before they are used and formulae must be type-correct in the usual manner.

3.1 Example of text: Proposition 2.2 and Proof$_1$

The following text from EAT begins in line 1 with an adaptation of the Proposition 2.2, and is followed by its proof in 15 statements; we refer to this text as Proof$_1$.

The proof does not comply with what ought to be required in connection with textual legality. We then revise the proof to obtain Proof$_2$ which complies with our intuition on legality.

1. claim: $\forall x.(x \neq \bot \mathbin{\&} \mathsf{s}(x) = 0 \rightarrow x = 0)$;
2. introduce $a \in \mathbb{Q}$;
3. then we know $a \neq \bot$;
4. assume $\mathsf{s}(a) = 0$;
5. assume for a PBC $a \neq 0$;
6. known $\forall x.(x \neq \bot \mathbin{\&} x \neq 0 \rightarrow \frac{x}{x} = 1)$ (axiom 16);
7. infer $1 = \frac{a}{a}$ (using line 6 and line 5);
8. known $1 = \mathsf{s}(1)$ (from axiom 26);
9. infer $\mathsf{s}(1) = \mathsf{s}(\frac{a}{a})$ (from line 7);
10. infer $\mathsf{s}(\frac{a}{a}) = \mathsf{s}(a \cdot \frac{1}{a})$ (via axiom 14);
11. infer $\mathsf{s}(a \cdot \frac{1}{a}) = \mathsf{s}(a) \cdot \mathsf{s}(\frac{1}{a})$ (from the axiom 30);
12. infer $\mathsf{s}(a) \cdot \mathsf{s}(\frac{1}{a}) = \mathsf{s}(a) \cdot \frac{1}{\mathsf{s}(a)}$ (from the axiom 32);
13. claim $\frac{1}{\mathsf{s}(a)} \neq \bot$;
14. to prove the claim in line 13 assume for PBC that $\frac{1}{\mathsf{s}(a)} = \bot$;
15. now using $1 = \mathsf{s}(a) \cdot \frac{1}{a} = 1 \cdot \bot = \bot$, infer $1 = \bot$;
16. then $0 = 0 \cdot 1 = 0 \cdot \bot = \bot + (-\bot) = \bot$;
17. combining the above two lines: $1 = 0$, a contradiction is found and the PBC assumption in line 15 can be dropped and the claim in line 13 is validated;
18. infer $1 = \mathsf{s}(a) \cdot \frac{1}{\mathsf{s}(a)} = 0 \cdot \frac{1}{\mathsf{s}(a)}$ (using lines 4, 7, 8, 9, 10, and 11);
19. known $\forall x.(x \neq \bot \rightarrow 0 \cdot x = 0)$, from axiom 13;
20. infer $0 \cdot \frac{1}{\mathsf{s}(a)} = 0$ (using line 19 and Claim 13);
21. infer the contradiction $1 = 0$ (from line 8 onwards) and drop the PBC assumption 5;
22. confirm claim 1.

Now, the above proof may be criticised because the fracterm $\frac{1}{\mathsf{s}(a)}$, which appears in line 12, is problematic. This is because given the assumption on line 4, it is immediate from that assumption that line 12 contains a *potential* instance of division by zero; this violates conventional intuitions of legality. We write potential because, there might be some issue with the insight that the denominator of the fracterm $\frac{1}{\mathsf{s}(a)}$ in line 12 equals zero, but in fact there is no such issue as the assumption in line 4 is not a PBC assumption.

One might say that the text of the proof contains in line 12 an instance of a division by zero error, found on substituting from line 4. This problem need not indicate a failure of reasoning in the text, i.e., making a wrong inference, but our concerns are general conventions for guarding against DbZ errors and characterising legality.

The fracterm $\frac{a}{a}$ which occurs in line 7 merits attention: as it is shown later on in the proof, given the PBC assumption, $1 = 0$ so that also $a = 0$ can be inferred. However, we adopt the following position: in the text of the proof the occurrence of the fracterm $\frac{a}{a}$ in line 7 constitutes no problematic instance of DbZ because the proof that the numerator is nonzero *easily* follows from the PBC assumption in line 5, while, in contrast, the proof that said numerator equals zero is *significantly longer*. We will make the above guideline on legality involving lengths of proof in Definition 3.1 below.

3.2 Revised example of text: Proposition 2.2 and Proof$_2$

We can revise Proof$_1$ to try to avoid the division by zero problem. We keep the first 11 lines of the proof unchanged, and modify the subsequent lines of Proof$_1$ to obtain Proof$_2$ as follows:

11. infer $\mathsf{s}(a \cdot \frac{1}{a}) = \mathsf{s}(a) \cdot \mathsf{s}(\frac{1}{a})$ (copy of line 11 of Proof$_1$);

12b. claim $\mathsf{s}(\frac{1}{a}) \neq \bot$;

13b. to prove the claim in line 12b assume for PBC that $\mathsf{s}(\frac{1}{a}) = \bot$;

14b. now using $1 = \mathsf{s}(a) \cdot \frac{1}{a} = 1 \cdot \bot = \bot$ infer $1 = \bot$;

15b. then $0 = 0 \cdot 1 = 0 \cdot \bot = \bot + (-\bot) = \bot$;

16b. combining the above two lines: $1 = 0$, a contradiction is found and the PBC assumption in line 13b can be dropped whereby the claim in line 12b is validated;

17b. infer $\mathsf{s}(a) \cdot \mathsf{s}(\frac{1}{a}) = 0 \cdot \mathsf{s}(\frac{1}{a})$ (using line 4);

18b. known $\forall x.(x \neq \bot \rightarrow 0 \cdot x = 0)$, from axiom 13;

19b. infer $0 \cdot \mathsf{s}(\frac{1}{a}) = 0$ (using the claim of line 12b and line 18b);

20b. infer $1 = 0$ (from line 8 onwards with line 17b and line 19b) and drop assumption 5;

21b. confirm claim 1.

Just as Proof$_1$, also Proof$_2$ may be criticised because after all $a = 0$, given the assumption in line 5, so that arguably there is an instance of division by zero in Proof$_2$ in Proof$_2$. The reason for dismissing this case of a potential DbZ error, is precisely the same as for Proof$_1$. The legality of the fracterm occurrence in line 5 of Proof$_2$ is an instance of what is called the "legal backward" criterion in item 4 of Definition 4.3 in Bergstra and Tucker (2004). We notice that $a \neq 0$ can be derived from a consistent trail (i.e., looking backward from the fracterm occurrence in focus) of assumptions as these occur in Proof$_2$, where assumption 5 is included while assumption 4 is not included. The legal backward criterion suffices for a justification for the use of the expressions in the RHS of line 10 (in Proof$_1$ as well as in Proof$_2$).

3.3 A contradiction compatible legality criterion

In Bergstra and Tucker (2004) our analysis was inconclusive, or rather incomplete concerning cases where the combination of assumptions in the text is inconsistent.

The difficulty is this: suppose line n of a text contains a fracterm $\frac{P}{Q}$ and the combination of assumptions preceding line n (and which have not yet been dropped) is inconsistent. Then it is always possible to prove $Q = 0$ from these assumptions. If inconsistent statements are present then opportunities for errors in division are present.

In view of criterion 4.4 (1) in Bergstra and Tucker (2004) where the criterion illegal_A is detailed, condition (b) of illegal_A would be satisfied, but condition (a) about consistency of the assumptions is not satisfied. Precisely such cases are left unanalysed in Bergstra and Tucker (2004) and, as we have seen, such cases arise *unavoidably* in any case of a proof by contradiction that makes mention of any fracterm.

We will now formulate our new legality criterion for a mathematical text T on elementary arithmetic.

Notice that the order in which statements appear in a text is technically important for legality.

For an assessment of textual legality for texts containing fracterm occurrences following an inconsistent combination of assumptions due to the presence of preceding, though a not yet undone/retracted proof by contradiction (PBC) assumptions, the following criterion for legality is plausible.

Definition 3.1 *Contradiction compatible legality criterion (CCLC).* A text T is *legal* if for each occurrence F_{occ} of a fracterm $F \equiv \frac{P}{Q}$ in T either condition (I) or (II) below is the case:

Condition I: Safe: It can be shown that, on the basis of a consistent combination $A_1, \ldots, A_k$ of assumptions in T such that

1. each A_i precedes the occurrence of F_{occ}, and
2. no A_i is a PBC assumption,

that $Q \neq 0$.

Or, if (I) is not the case, then this condition (II) is the case:

Condition II: Unsafe: There is a consistent combination $A_1, \ldots, A_k$ of assumptions in T such that

1. each A_i precedes the occurrence F_{occ},
2. at least one A_i is a PBC assumption,
3. no A_i has been undone/retracted in a line preceding F_{occ}, and so that
4. $A_1, \ldots, A_k$ implies, say by way of an argument π_1, $Q \neq 0$, and, moreover,
5. there is no consistent combination $B_1, \ldots, B_l$ of assumptions in T (also written in the order of their occurrence), none of which have been undone in a line preceding F_{occ}, from which it follows that $Q = 0$ by means of an argument, say π_2, which is shorter (or less involved, or simpler) than the proof π_1 of $Q \neq 0$ from $A_1, \ldots, A_k$ as mentioned under assumption 4 above.

Lemma 3.1 *Proof_2 satisfies the CCLC and is a legal text.*

Proof Proof_2 satisfies condition II. In this case the only fracterm which occurs in the text is $F \equiv \frac{1}{a}$, and using the unique PBC-condition,

"assume for PBC: $a \neq 0$",

serving as A_1 with $k = 1$, provides a straightforward option for obtaining that $a \neq 0$. Using assumption 4,

"assume: $\mathsf{s}(a) = 0$",

serving as B_1 with $l = 1$, it is also possible to obtain $a = 0$ but that proof is longer/harder. □

Lemma 3.2 *Proof_1 does not satisfy the CCLC and is not a legal text.*

Proof As in Proof_2, the use of the fracterms $\frac{a}{a}$ and $\frac{1}{a}$ is unproblematic. Problematic, however, is the first occurrence of $F \equiv \frac{1}{\mathsf{s}(a)}$, in line 12. Using the PBC-assumption,

"assume for PBC: $a \neq 0$",

serving as A_1 with $k = 1$, it may be inferred in two steps that $\mathsf{s}(a) \neq 0$, while using assumption 4,

"assume: $\mathsf{s}(a) = 0$",

serving as B_1 with $l = 1$, it is immediate (and thereby simpler than the two step demonstration just mentioned) to see that the numerator of $\frac{1}{\mathsf{s}(a)}$ vanishes. □

Admittedly Proof_1 involves an unnecessary detour, when compared with Proof_2 (the use of a fracterm with denominator $\mathsf{s}(a)$), which causes the DbZ problem of Proof_1. It would be nice to have a more convincing instance of a DbZ problem in a setting of inconsistent assumptions.

Problem 3.1 Is there an example of a proof by contradiction the legal validity of which depends arguably on the presence of condition (II) in the description of CCLC above, and which cannot be so easily repaired as Proof_1?

3.4 Comments on CCLC

Let us reflect on the plausibility of criterion CCLC. Using the notation of Definition 3.1 we consider both cases in description of CCLC.

If condition (I) obtains then there is a consistent sequence of assumptions, none of which are problematic by having been introduced as PBC assumptions, from which $Q \neq 0$ can be derived. In this situation there is little objection against the fracterm occurrence $\frac{P}{Q}$. Implicit in this condition is the idea that deriving $Q \neq 0$ involves at most a simple detour relative to the arguments occurring in the text at hand. An important aspect of this case is that $Q \neq 0$ may correspond to mathematical intuition, inspection of graphs, or pictures and so on, as only consistent assumptions come into play so that it can be claimed with confidence that the meaning of asserting $Q \neq 0$ is known to mathematical practice.

Condition II covers the case that one or more PBC assumptions are needed (or used) to see that $Q \neq 0$. Now one is working with an inconsistent set of assumptions and the mere derivation of $Q \neq 0$ is uninformative, because the available collection of inconsistent assumptions allow the reader to arrive at *any* conclusion including $Q = 0$. However, in this situation, it is also required that *it is 'simpler' to demonstrate* $Q \neq 0$ *than to demonstrate* $Q = 0$.

Admittedly, it may be very hard to check the requirements of condition II in practice, as it may require the comparison of the length or complexity of many different proofs, most of which (in particular, the arguments deriving $Q = 0$ from $B_1, \ldots, B_n$) may not at all have counterparts in the text under investigation.

One may require in a formal setting that, in condition I as well as in condition II, a proof π_1 that $Q \neq 0$ from $A_1, \ldots, A_k$ is included in the text. Now, condition II is decidable, as indeed one may list all of the finitely many formal proofs π_2 based on the yet non-retracted preceding assumptions $B_1, \ldots, B_l$ which are shorter than π_1 and see if π_2 proves $Q = 0$; if so, the occurrence of $\frac{?}{Q}$ at hand cannot be justified by means of CCLC (at least not without replacing π_1 with a simpler proof).

Having argued that CCLC is sound in the sense that its validity seems to be plausible, CCLC may not be complete, i.e., it may not cover all cases of texts one may wish to consider legal. For instance consider the text T:

introduce: $a \in \mathbb{Q}$;

known: $0 = 1 \rightarrow \dfrac{a}{a} = 1$.

Following Bergstra and Tucker (2004) we consider T to be legal, while it fails to comply with CCLC because there is no way to prove $a \neq 0$ from any assumptions in T.

We notice that with T'
introduce: $a \in \mathbb{Q}$;
known: $0 = 1 \rightarrow \dfrac{0}{0} = \mathrm{a}$

we find that, following Bergstra and Tucker (2004), T' is non-legal because it can be shown (trivially) that the numerator of the only fracterm in T' equals 0 and there is no way to prove it non-zero, let alone a shorter way.

Now T is somehow unusual for an efficient mathematical text will *not* contain any assertion of the form $0 = 1 \rightarrow \phi$. We expect that all "conventional" mathematical texts on elementary arithmetic can be written, i.e., the intended content can be conveyed, in a textual form that complies with CCLC.

We have taken care not to base CCLC on any formalised notion of proof which would move it to the realm of pure formalism. Instead, CCLC is accessible for an author or reader who thinks in terms of a Platonistic ontology where truth is validity in an imagined world. The consistency criterion then coincides with the existence of such worlds and the phrase "can be shown" may be understood informally as including all forms of reasoning and intuition which the person involved considers admissible or convincing. Nevertheless the comparison mechanism for such proofs introduces an informal aspect which seems hard to remove.

4 Logical formalisations of proofs: frugal equality and a 3-valued logic

By studying partial operations through the lens of common meadows, we access all the benefits of native equality $=$ and reasoning with classical first order logics. However, the partiality invested in the presence of $\perp$ introduces options for equality that are of technical interest and utility:

native equality for common meadows,

eager equality, where $a = \perp$ is true for all a, and

cautious equality where $a = \perp$ is false for all a.

Both eager equality and cautious equality are studied in detail in Bergstra and Tucker (2023b).

There is yet another equality and a non-classical first order logic that makes sense as a tool for formalising legality/illegality.

4.1 Frugal equality and its logic

Frugal equality differs from ordinary equality in that $a =_{\mathsf{f}} b$ is an atom in a three-valued logic with third truth value U. Let $(a =_{\mathsf{f}} b)$ denote the truth-value evaluation of $a =_{\mathsf{f}} b$. Then

$$(a =_{\mathsf{f}} \perp) = (\perp =_{\mathsf{f}} \perp) = \mathsf{U}$$

while for non-$\perp$ a and b,

$$(a =_{\mathsf{f}} b) = \mathsf{T} \text{ if } a = b \text{ and } (a =_{\mathsf{f}} b) = \mathsf{F} \text{ if } a \neq b.$$

In the specific case of common meadows we find for instance that

$$(\frac{1}{0} =_{\mathsf{f}} \frac{1}{0}) = (\frac{1}{0} =_{\mathsf{f}} 0) = (\frac{1}{0} =_{\mathsf{f}} 1) = \mathsf{U}$$

while

$$(0 =_{\mathsf{f}} 1 + (-1)) = (1 =_{\mathsf{f}} 1 + 0) = \mathsf{T} \quad \text{and} \quad (0 =_{\mathsf{f}} 1) = \mathsf{F}.$$

To create a logic we need to accommodate these three values, and so logical connectives need adaptation. Furthermore, to reflect the role of the ordering of statements in texts, which influence legality, we adopt a "short-circuit logic" with the basic truth values for conjunction and negation defined by operators

$$\mathsf{T} \mathbin{\circ\!\!\wedge} X = X, \ \mathsf{F} \mathbin{\circ\!\!\wedge} X = \mathsf{F}, \ \mathsf{U} \mathbin{\circ\!\!\wedge} X = \mathsf{U}, \ \neg \mathsf{U} = \mathsf{U}.$$

The short-circuit property is apparent from $\mathsf{F} \mathbin{\circ\!\!\wedge} X = \mathsf{F}$ which indicates that evaluation of X plays no role in the evaluation of $\mathsf{F} \mathbin{\circ\!\!\wedge} X = \mathsf{F}$.

In short-circuit logic conjunction and disjunction are not commutative, for instance $\mathsf{T} \overset{\circ}{\vee} \mathsf{U} = \mathsf{T}$ while $\mathsf{U} \overset{\circ}{\vee} \mathsf{T} = \mathsf{U}$, and $\mathsf{F} \underset{\circ}{\wedge} \mathsf{U} = \mathsf{F}$ while $\mathsf{U} \underset{\circ}{\wedge} \mathsf{F} = \mathsf{U}$. Further $\neg(\neg X) = X$. Implication is as usual: $X \circ\!\!\rightarrow Y = \neg X \overset{\circ}{\vee} Y$. However $X \circ\!\!\rightarrow Y$ is not always equivalent with $\neg Y \circ\!\!\rightarrow \neg X$. Indeed $\neg X \overset{\circ}{\vee} Y$ differs from $\neg(\neg Y) \overset{\circ}{\vee} \neg X = Y \overset{\circ}{\vee} \neg X$ on $X = \mathsf{F}$, $Y = \mathsf{U}$. It follows that proving an implication by way of contradiction is not supported in a short-circuit logic, at least not in the usual manner.

We add the quantifier: $\forall_{\mathsf{f}} x.\phi$ to mean that for all $a \neq \bot$ in the domain of the structure at hand, $[a/x]\phi = \mathsf{T}$. The need for a subscript for the universal quantifier is apparent when considering existential quantification: in order to satisfy $\neg\forall_{\mathsf{f}} x.\phi = \exists_{\mathsf{f}} x.\neg\phi$ the interpretation of $\exists_{\mathsf{f}} x.\neg\phi$ is as follows: there is an $a \neq \bot$ such that $[a/x]\phi = \mathsf{T}$ and, moreover, for all $b \neq \bot$ either $[b/x]\phi = \mathsf{T}$ or $[a/x]\phi = \mathsf{F}$.

As an example consider the following assertion:

$$\frac{x}{2} =_{\mathsf{f}} 1 \circ\!\!\rightarrow \exists_{\mathsf{f}}\, y\, (\frac{x}{y} =_{\mathsf{f}} 1),$$

which seems plausible but fails in a common meadow with frugal equality. However, this property fails because 0 might be substituted for y while the following assertion holds:

$$\frac{x}{2} =_{\mathsf{f}} 1 \circ\!\!\rightarrow \exists_{\mathsf{f}}\, y\, (y \neq_{\mathsf{f}} 0 \underset{\circ}{\wedge} \frac{x}{y} =_{\mathsf{f}} 1)$$

in the following sense:

$$(\forall_{\mathsf{f}} x\, (\frac{x}{2} =_{\mathsf{f}} 1) \circ\!\!\rightarrow \exists_{\mathsf{f}}\, y\, (\frac{x}{y} =_{\mathsf{f}} 1)) = \mathsf{T}.$$

When working with three-valued short-circuit logic we write texts where $t =_{\mathsf{f}} r$ is used as an abbreviation of $(t =_{\mathsf{f}} r) = \mathsf{T}$.

Three-valued logic for frugal equality adopts the notations and style of proposition algebra of Bergstra and Ponse (2011). A full technical account is in Bergstra and Ponse (2025), though using the framework of partial algebras and partial equality.

4.2 Comparing native equality, eager equality and cautious equality

For working with common meadows, the 3-valued frugal equality has three 2-valued variations noted above: common meadow equality, eager equality and cautious equality each assign 2-valued truth values to identities of the form $a = \bot$, though each in a different manner.

The advantage of using three-valued logic with frugal equality can be seen by noticing that the explanation of the validity of

$$\forall_{\mathsf{f}} x.(x \neq_{\mathsf{f}} 0 \circ\!\!\rightarrow \frac{x}{x} =_{\mathsf{f}} 1)$$

will not rest on any, unavoidably questionable, assignment of a two-valued truth value to the identity

$$\frac{0}{0} =_{\mathsf{f}} 1.$$

Notice that substituting $x = 0$ gives

$$0 \neq_{\mathsf{f}} 0 \circ\!\!\rightarrow \frac{0}{0} =_{\mathsf{f}} 1$$

which reduces to

$$\mathsf{F} \circ\!\!\rightarrow \mathsf{U} = (\neg\mathsf{F}) \overset{\circ}{\vee} \mathsf{U} = \mathsf{T} \overset{\circ}{\vee} \mathsf{U} = \mathsf{T}.$$

Compare item (6) in the proof below.

4.3 Revisiting the proofs by contradiction with frugal equality and a 3-valued logic

We will rewrite Proof_1 into Proof_3 making use of the notation of frugal equality. Now the idea is not to make any use of the constant symbol $\perp$, which constitutes a simplification, though at the price of using a three valued logic.

1. claim: $\forall_{\mathsf{f}} x.(\mathsf{s}(x) =_{\mathsf{f}} 0 \circ\!\!\rightarrow x =_{\mathsf{f}} 0)$;
2. introduce: $a : \mathbb{Q}$;
3. assume: $\mathsf{s}(a) =_{\mathsf{f}} 0$;
4. assume for PBC: $a \neq_{\mathsf{f}} 0$;
5. known: $\forall_{\mathsf{f}} x.(x \neq_{\mathsf{f}} 0 \circ\!\!\rightarrow \frac{x}{x} =_{\mathsf{f}} 1)$;
6. infer: $1 =_{\mathsf{f}} \frac{a}{a}$ (using 4 and line 5);
7. known $1 =_{\mathsf{f}} \mathsf{s}(1)$ (from axioms for the sign function);
8. infer $\mathsf{s}(1) =_{\mathsf{f}} \mathsf{s}(\frac{a}{a})$ (from 6, and using the fact that the sign function is total);
9. infer $\mathsf{s}(\frac{a}{a}) =_{\mathsf{f}} \mathsf{s}(a \cdot \frac{1}{a})$ (from the axioms for division);
10. infer $\mathsf{s}(a \cdot \frac{1}{a}) =_{\mathsf{f}} \mathsf{s}(a) \cdot \mathsf{s}(\frac{1}{a})$ (from the axioms for the sign function);
11. infer $\mathsf{s}(a) \cdot \mathsf{s}(\frac{1}{a}) =_{\mathsf{f}} \mathsf{s}(a) \cdot \frac{1}{\mathsf{s}(a)}$ (from the axioms for $\mathsf{s}(-)$);
12. infer $\mathsf{s}(a) \cdot \frac{1}{\mathsf{s}(a)} =_{\mathsf{f}} 0 \cdot \frac{1}{\mathsf{s}(a)}$ (using 4);
13. infer $0 \cdot \frac{1}{\mathsf{s}(a)} =_{\mathsf{f}} 0$ (using informal “knowledge" of $\forall_{\mathsf{f}} x.(0 \cdot x =_{\mathsf{f}} 0)$);
14. infer $1 =_{\mathsf{f}} 0$ and drop the PBC assumption in line 4;
15. infer the negation of the PBC assumption in line 4;
16. confirm claim 1.

Now the above proof may be criticised just as Proof$_1$ because the term $\frac{1}{\mathsf{s}(a)}$ which appears in line 11 is problematic given the assumption under line 4. Just as Proof$_1$, it is easy to adapt Proof$_3$ so that the DbZ error is avoided. We omit the details of the revised proof.

4.4 CCLC for 3-valued logic over frugal equality

The CCLC criterion translates literally to frugal equality with 3-valued logic and sequential connectives. The discussion of the example of a proof by contradiction remains essentially the same.

Thus, to be explicit: as far as we know with the current state of development of the theory of common meadows:

Claim 4.1 We claim that working in a common meadow of rationals and using frugal equality as well as frugal quantifiers, and adopting the legality conventions of Bergstra and Tucker (2004) as amended by CCLC, provides the best detailed approximation of the conventional practices concerning how to deal with division by zero of elementary arithmetic.

Doubtlessly, this is a claim that can attract further 'test and challenge'.

5 Probability calculus

The modern form of the Bayes-Price theorem[3] on inverse probabilities can be formulated as a modest equation involving division and ordering, namely:

$$P(X|Y) = \frac{P(Y|X) \cdot P(X)}{P(Y)}$$

which is often claimed under the constraint that $P(Y) > 0$ to avoid division by zero.

The formula can be derived from some simple postulates about probability. In fact, the derivations of the formula can involve divisions by $P(X)$: thus, requiring the constraint that $P(X) > 0$ as well. This latter point is often left unmentioned. Of course, $P(X) = 0$ and $P(X) \neq 0$ are meaningful and possible. So, in the proof lie legality/illegality issues.

Here we will illustrate the use of signed common meadows in providing

(i) an axiomatisation of probability calculus,
(ii) a formally rigorous statement of the Bayes-Price theorem, and
(iii) a basis and logic for a completely formal and legal proof of the theorem.

The totalisation of division allows us to accommodate conditions on equational formulae, which is needed.

[3] In fact the theorem was published, refined and first applied by Richard Price; see Bayes and Price (1763).

Table 6 E_{sba}: Symmetric equations for Boolean algebra

$(X \cup Y) \cap Y = Y$	(35)
$(X \cap Y) \cup Y = Y$	(36)
$X \cap (Y \cup Z) = (Y \cap Z) \cup (Z \cap X)$	(37)
$X \cup (Y \cap Z) = (Y \cup Z) \cap (Z \cup X)$	(38)
$X \cup \overline{X} = \mathbb{U}$	(39)
$X \cap \overline{X} = \emptyset$	(40)

Table 7 $AX^{P}_{\mathsf{ftc-ascm}}$: Equations for a probability function P into a signed common meadow

`import:` $AX_{\mathsf{ftc-ascm}}$	(41)
`import:` E_{sba}	(42)
$P(\emptyset) =_{\mathsf{f}} 0$	(43)
$P(\mathbb{U}) =_{\mathsf{f}} 1$	(44)
$P(X) =_{\mathsf{f}} \vert P(X)\vert$	(45)
$P(X \cup Y) =_{\mathsf{f}} (P(X) + P(Y)) - P(X \cap Y)$	(46)

Table 8 Conditional probability definition

$P(X\vert Y) =_{\mathsf{f}} \frac{P(X \cap Y)}{P(Y)}$	(47)

5.1 Axioms for a probability function

First, we determine plausible equations for the postulates that define probability. Such axioms are given in Table 7. To do this we define a probability function on a Boolean algebra, requiring it to take its values $P(X)$ in a 4-signed meadow.

The axioms for a Boolean algebra in Table 6 have been taken (with a change of notation) from Padmanabhan (1983), where the completeness of these axioms is shown.

5.2 Reformulating the Bayes-Price theorem

The Bayes-Price theorem, which appears in nearly all introductions to probability theory (and usually without mention of Price), and which is often taken for the mathematical core of Bayesian statistics and probability, can take the form of a conditional formula over signed meadows.

As an illustration of frugal equality we provide a re-formulation of Bayes-Price. The difference with other formulations is in the necessary conditions. In this case two nonzero conditions are required for a legal proof.

Proposition 5.1 *The following form of Bayes-Price's theorem, stated in terms of frugal equality ($=_{\mathsf{f}}$), is formally provable from the axioms of $AX^{P}_{\mathsf{ftc-ascm}}$ and the definition of $=_{\mathsf{f}}$, respectively. $\neq_{\mathsf{f}}$:*

$$(P(X) \neq_{\mathsf{f}} 0 \mathbin{\circ\!\!\wedge} P(Y) \neq_{\mathsf{f}} 0) \mathbin{\circ\!\!\rightarrow} P(X|Y) =_{\mathsf{f}} \frac{P(Y|X) \cdot P(X)}{P(Y)}$$

Proof We understand the fact as a semantic matter, that is, when working in a signed meadow of rationals we find that given a finite Boolean algebra serving as the event space, and a probability mass function P that meets the given axioms for probability functions, under both conditions the conclusion follows immediately from the definition of conditional probability in Table 8 in the usual way. The full formal argument is based on Bergstra and Tucker (2004), though re-stated in terms of frugal equality and 3-valued logic.

We also notice that in the absence of either condition the conclusion fails as $P(X|Y) =_{\mathsf{f}} \frac{P(Y|X) \cdot P(X)}{P(Y)}$ may then take truth value U. □

6 Concluding remarks

The aims, methods and some potential problems and directions of this research programme were introduced in Bergstra and Tucker (2004) and we will not repeat these explanations here. Here we have picked up and addressed some of these ideas and extended our scope from text fragments to whole proofs. In particular, we have considered legality in the light of the common form of argument of proof by contradiction, and chosen and applied a 3-valued logic to the translation; we have also noted and reconsidered a classic theorem in probability for illustration of the tools.

Two points are worth repeating from Bergstra and Tucker (2004). Our ideas are general and, in principle, can be applied to all partial functions. We focus on division because this work belongs to a (now extensive) programme on the semantics of computing with arithmetic structures (which we sketched in Bergstra and Tucker (2004)). In particular, division perfectly displays key semantic issues at the level of elementary mathematical education. Division is everywhere, of course.

Secondly, in Bergstra and Tucker (2004), we showed that the legality/illegality of texts with DbZ is algorithmically undecidable for texts belonging to a simpler mathematical text language. Thus, we know that a finite system of rules that can establish precisely which texts are legal and which are illegal does *not* exist. As we observed: we view this result as intriguing, encouraging a necessary forensic investigation of traditional conventions about partiality by reflecting on examples and intuitions and attempting their precise formalisation.

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