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Stochastic differential equations with critically low regularity growing drift

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Abstract

We obtain the existence and uniqueness of stochastic flows of homeomorphisms for a class of stochastic differential equations driven by the low regularity growing drift in critical Lorentz-Hölder space $L^{q,1}([0,T];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))$ $(q\in(1,2))$. Moreover, we prove that the unique stochastic flow of homeomorphisms is weakly differentiable with respect to the spatial variables.

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Keywords: Low regularity growing drift; Stochastic flow; Lorentz-Hölder space

1. Introduction

Let $T>0$ be a given real number. We are interested in the following stochastic differential equation (SDE for short) in $\mathbb{R}^d$:

$$dX_{s,t}(x)=b(t,X_{s,t}(x))dt+dW_t,\ t\in(s,T],\ X_{s,t}(x)|_{t=s}=x\in\mathbb{R}^d, \tag{1.1}$$

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where $s \in [0, T)$, $\{W_t\}_{0\leqslant t\leqslant T} = \{(W_{1,t}, \dots, W_{d,t})^\top\}_{0\leqslant t\leqslant T}$ is a d-dimensional standard Wiener process defined on a given stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T})$ and $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a Borel measurable function.

The unique strong solvability for SDE (1.1) was first established by Itô [11] for Lipschitz continuous b, and then extended by Veretennikov [33] to bounded Borel measurable ones. Since then, Veretennikov's result was strengthened in different forms, see e.g. [6,27,31,35]. Krylov and Röckner [17] made a breakthrough by establishing the strong well-posedness of SDE (1.1) under the following subcritical Ladyzhenskaya-Prodi-Serrin (LPS for short) condition

$$b \in L^q([0, T]; L^p(\mathbb{R}^d; \mathbb{R}^d)), \quad p, q \in [2, +\infty] \text{ and } \frac{2}{q} + \frac{d}{p} < 1. \tag{1.2}$$

Some further extensions for non-constant diffusion coefficients can be found in Zhang [41,42], Zhang and Yuan [40], Xia, Xie, Zhang and Zhao [39].

However, from the viewpoint of Navier-Stokes equations, b can be taken in the critical case, i.e., the less-than sign in (1.2) is replaced by the equals sign, see [5,26], and in the critical LPS condition, the strong well-posedness of (1.1) is a long-standing open problem since the work of Krylov and Röckner [17]. Recently, this problem was solved by Röckner and Zhao [30, Theorem 1.1] for $d \geqslant 3$ (see also [12–14,29] for weak solutions) in the following cases:

$$b \in L^q([0, T]; L^p(\mathbb{R}^d; \mathbb{R}^d)), \ p, q \in (2, +\infty) \text{ and } \frac{2}{q} + \frac{d}{p} = 1 \text{ or } b \in \mathcal{C}([0, T]; L^d(\mathbb{R}^d; \mathbb{R}^d)). \tag{1.3}$$

When $q = 2,\ p = +\infty$, the strong existence and pathwise uniqueness were also proved by Beck, Flandoli, Gubinelli and Maurelli [2] for almost every starting point $x \in \mathbb{R}^d$; also see [15] for form-bounded drifts. This result was later extended by Wei, Wang, Lv, and Duan [38] to every starting point $x \in \mathbb{R}^d$ under the assumption that b is locally Dini continuous in the spatial variables. More recently, Krylov [23] proved strong uniqueness for Morrey drift and VMO (vanishing mean oscillation) diffusion coefficients, notably covering cases such as $q = 2$ and $p = +\infty$. However, the problem of strong existence remains open. We also refer to [7,18–22,28,36] for more details.

On the other hand, from classical SDE theory, equation (1.1) remains strongly well-posed even if $b(t, x)$ is merely integrable with respect to t and satisfies a Lipschitz condition in x. Inspired by classical SDE theory and the result in [2], there exist some function spaces, which are intermediate ones between $L^1([0, T]; Lip(\mathbb{R}^d))$ and $L^2([0, T]; L^\infty(\mathbb{R}^d))$ such that SDE (1.1) is strongly well-posed if the drift belongs to one of these working spaces. The natural choice for the working spaces is $L^q([0, T]; \mathcal{C}^\alpha_l(\mathbb{R}^d))$ with $q \in (1, 2)$ and $\alpha \in (0, 1)$, where $\mathcal{C}^\alpha_l(\mathbb{R}^d)$ denotes the set of all locally Hölder continuous functions with Hölder index α. By the scaling transformation, we also get an analogue of LPS condition for b:

$$b \in L^q([0, T]; \mathcal{C}^\alpha_l(\mathbb{R}^d; \mathbb{R}^d)), \quad q \in (1, 2), \quad \alpha \in (0, 1) \text{ and } \frac{2}{q} - \alpha \leqslant 1. \tag{1.4}$$

The unique strong solvability for (1.1) with condition (1.4) seems to be as important and difficult as (1.1) with LPS condition.

When $b \in L^\infty([0, T]; \mathcal{C}^\alpha_b(\mathbb{R}^d; \mathbb{R}^d))$ with $\alpha \in (0, 1)$, the unique strong solvability as well as the existence of stochastic flows of diffeomorphisms for SDE (1.1) has been established by

Flandoli, Gubinelli and Priola [8], and then generalized by Wei, Duan, Gao and Lv [34] to $b \in L^q([0,T]; \mathcal{C}_b^\alpha(\mathbb{R}^d;\mathbb{R}^d))$ with $q > 2/\alpha$. When $b \in L^2([0,T]; \mathcal{C}_b^\alpha(\mathbb{R}^d;\mathbb{R}^d))$, the strong well-posedness was also proved by Tian, Ding and Wei [32]. Recently, Galeati and Gerencsér [9] studied (1.1) and (1.4), in which b is bounded in the spatial variables and $2/q - \alpha < 1$, and they derived the existence and uniqueness for stochastic flows of diffeomorphisms. Galeati and Gerencsér's result was then generalized by Wei, Hu and Yuan [37] to the unbounded Hölder continuous drift, and the authors also obtained the stability for the gradient of solutions through the Itô-Tanaka trick.

All above mentioned works are concerned with the subcritical regime $2/q - \alpha < 1$. For the supercritical regime $2/q - \alpha > 1$, SDE (1.1) is not strongly well-posed. In fact, for $q \in (1,2)$, $\alpha \in (0,1)$ such that $2/q - \alpha > 1$, if one takes $d = 1$ and

$$b(t,x) = t^{-\frac{1}{\bar{q}}} \operatorname{sign}(x)|x|^\alpha, \quad \bar{q} \in (q,2) \text{ and } \frac{2}{\bar{q}} - \alpha > 1, \tag{1.5}$$

then $b \in L^q([0,T]; \mathcal{C}_l^\alpha(\mathbb{R}^d;\mathbb{R}^d))$ and the corresponding SDE (1.1) with the initial data $X_{s,s} = 0$ has two different solutions (see [9, Section 1.3]). Thus, the strong well-posedness of SDE (1.1) depends delicately on the integral index q and the Hölder continuous index α.

However, it is still unknown whether the unique strong solvability is true or not for the drift in the critical regime $2/q - \alpha = 1$. In this paper, we suppose b belongs to the critical Lorentz-Hölder space $L^{q,1}([0,T]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d;\mathbb{R}^d))$ with $q \in (1,2)$, and prove the strong well-posedness as well as the existence of stochastic flows of homeomorphisms for SDE (1.1).

1.1. Lebesgue-Hölder and Lorentz-Hölder spaces

Firstly, let us introduce our working spaces.

Definition 1.1 *(Hölder and Lebesgue-Hölder spaces).* Let $\alpha \in (0,1)$ and $d \in \mathbb{N}$. $\mathcal{C}_l^\alpha(\mathbb{R}^d)$ is the set consisting of all uniformly locally Hölder continuous functions $h : \mathbb{R}^d \to \mathbb{R}$ for which

$$[h]_\alpha = \sup_{x,y\in\mathbb{R}^d, 0<|x-y|\leqslant 1} \frac{|h(x)-h(y)|}{|x-y|^\alpha} < +\infty.$$

For $x_0 \in \mathbb{R}^d$, if $|x_0| > 1$, then there exist $x_1, x_2, \ldots, x_\beta$ (β is the integer part of $|x_0|$) such that $|x_0 - x_1| = |x_1 - x_2| = \ldots = |x_{\beta-1} - x_\beta| = 1$ and $|x_\beta| < 1$. Thus,

$$\begin{aligned} |h(x_0)| &\leqslant |h(x_0) - h(0)| + |h(0)| \\ &\leqslant \sum_{i=0}^{\beta-1} |h(x_i) - h(x_{i+1})| + |h(x_\beta) - h(0)| + |h(0)| \\ &\leqslant \beta[h]_\alpha + |x_\beta|^\alpha [h]_\alpha + |h(0)| \leqslant ([h]_\alpha + |h(0)|)(1+|x_0|). \end{aligned}$$

Therefore, the set $\mathcal{C}_l^\alpha(\mathbb{R}^d)$ becomes a Banach space with respect to the norm

$$\|h\|_{\mathcal{C}_l^\alpha(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \frac{|h(x)|}{1+|x|} + [h]_\alpha =: \|(1+|\cdot|)^{-1} h(\cdot)\|_0 + [h]_\alpha.$$

We then define $\mathcal{C}_b^\alpha(\mathbb{R}^d)$ as the subset of $\mathcal{C}_l^\alpha(\mathbb{R}^d)$ consisting all bounded elements; for $h \in \mathcal{C}_b^\alpha(\mathbb{R}^d)$, we set

$$\|h\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} |h(x)| + \sup_{x,y\in\mathbb{R}^d, 0<|x-y|\leqslant 1} \frac{|h(x)-h(y)|}{|x-y|^\alpha} =: \|h\|_0 + [h]_\alpha.$$

Moreover, if $h \in \mathcal{C}_l^\alpha(\mathbb{R}^d)$ and its i-th order spatial gradient $\nabla^i h$ ($i = 1, 2, \ldots, k \in \mathbb{N}$) are bounded and continuous, and the Hölder seminorm $[\nabla^k h]_\alpha$ is finite, then we say $h \in \mathcal{C}_l^{k+\alpha}(\mathbb{R}^d)$. For $h \in \mathcal{C}_l^{k+\alpha}(\mathbb{R}^d)$, we set

$$\|h\|_{\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d)} = \|(1+|\cdot|)^{-1}h(\cdot)\|_0 + \sum_{i=1}^{k} \|\nabla^i h\|_0 + [\nabla^k h]_\alpha.$$

For $q \in (1, +\infty)$ and $k \in \mathbb{N}_0$, we define the Lebesgue-Hölder space $L^q([0,T]; \mathcal{C}_l^{k+\alpha}(\mathbb{R}^d))$ as the set consisting of all elements f that belong to $L^q([0,T])$ as $\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d)$-valued functions, where

$$\|f\|_{L^q([0,T];\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d))} = \left[\int_0^T \|f(t,\cdot)\|^q_{\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d)} dt\right]^{\frac{1}{q}} < +\infty.$$

The definition of the analogues spaces for $\mathbb{R}^d$ or $\mathbb{R}^{d\times d}$-valued functions is simply understood coordinate-wise. For simplicity, if $f \in L^q([0,T]; \mathcal{C}_l^{k+\alpha}(\mathbb{R}^d; \mathbb{R}^d))$, we also use $\|f\|_{L^q([0,T];\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d))}$ instead of $\|f\|_{L^q([0,T];\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d;\mathbb{R}^d))}$ for its norm (and similarly for other vector-valued functions).

Remark 1.2. Let $f \in L^q([0,T]; \mathcal{C}_l^\alpha(\mathbb{R}^d))$. For $x, y \in \mathbb{R}^d$, if $|x-y| > 1$, then there exist $x_1, x_2, \ldots, x_\gamma$ (γ is the integer part of $|x-y|$) such that $|x-x_1| = |x_1-x_2| = \ldots = |x_{\gamma-1} - x_\gamma| = 1$ and $|x_\gamma - y| < 1$. Therefore,

$$\begin{aligned} &|f(t,x)-f(t,y)| \\ &\leqslant |f(t,x)-f(t,x_1)| + \Big[\sum_{i=1}^{\gamma-1} |f(t,x_i)-f(t,x_{i+1})|\Big] + |f(t,x_\gamma)-f(t,y)| \\ &\leqslant \gamma[f(t,\cdot)]_\alpha + |x_\gamma - y|^\alpha [f(t,\cdot)]_\alpha \leqslant 2[f(t,\cdot)]_\alpha |x-y|. \end{aligned} \tag{1.6}$$

Definition 1.3 *(Lorentz and Lorentz-Hölder spaces).* Let g be a Borel measurable function on $[0,T]$. The distribution function δ_g corresponding to g on $[0,T]$ is given by

$$\delta_g(\varsigma) = Leb\{t \in [0,T] : |g(t)| > \varsigma\}$$

and is nonincreasing on $[0, +\infty)$, where Leb denotes the Lebesgue measure. The equimeasurable decreasing rearrangement of g is the function g^* defined by

$$g^*(\tau) = \inf\{\varsigma \geqslant 0 : \delta_g(\varsigma) \leqslant \tau\}.$$

Let

$$g^{**}(r) = \frac{1}{r}\int_0^r g^*(\tau)d\tau.$$

For $p \in [1, +\infty]$ and $q \in (1, +\infty)$, we define the functional

$$\|g\|_{L^{q,p}([0,T])} = \begin{cases} \left[\int_0^\infty \left(r^{\frac{1}{q}} g^{**}(r)\right)^p \frac{dr}{r}\right]^{\frac{1}{p}}, & \text{if } p \in [1, +\infty), \\ \sup_{r>0} \left(r^{\frac{1}{q}} g^{**}(r)\right), & \text{if } p = +\infty. \end{cases} \tag{1.7}$$

The Lorentz space $L^{q,p}([0, T])$ consists of those measurable functions g on $[0, T]$ for which $\|g\|_{L^{q,p}([0,T])} < +\infty$ (see [1, Sec. 7.24 and 7.25]). Suppose $k \in \mathbb{N}_0$ and $\alpha \in (0, 1)$, we then define the Lorentz-Hölder space $L^{q,p}([0, T]; \mathcal{C}_l^{k+\alpha}(\mathbb{R}^d))$ as the set consisting of all elements f belonging to $L^{q,p}([0, T])$ as $\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d)$-valued functions, where

$$\|f\|_{L^{q,p}([0,T];\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d))} = \begin{cases} \left[\int_0^\infty \left(r^{\frac{1}{q}} \|f(r,\cdot)\|^{**}_{\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d)}\right)^p \frac{dr}{r}\right]^{\frac{1}{p}} < +\infty, & \text{if } p \in [1, +\infty), \\ \sup_{r>0} \left(r^{\frac{1}{q}} \|f(r,\cdot)\|^{**}_{\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d)}\right) < +\infty, & \text{if } p = +\infty, \end{cases} \tag{1.8}$$

and

$$\begin{cases} \|f(r,\cdot)\|^{**}_{\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d)} = \frac{1}{r}\int_0^r \|f(\tau,\cdot)\|^{*}_{\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d)} d\tau, \\ \|f(\tau,\cdot)\|^{*}_{\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d)} = \inf\{\varsigma \geqslant 0 : \delta_{\|f(t,\cdot)\|_{\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d)}}(\varsigma) \leqslant \tau\}, \\ \delta_{\|f(t,\cdot)\|_{\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d)}}(\varsigma) = Leb\{t \in [0, T];\ \|f(t,\cdot)\|_{\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d)} > \varsigma\}. \end{cases}$$

Now, let us first illustrate the relationship between Lebesgue-Hölder and Lorentz-Hölder spaces. We then remark on a fundamental property of Lorentz-Hölder functions. Since the functions in these spaces exhibit the same spatial regularity, we focus on space-independent cases. For simplicity, we assume $p = 1$ and $q \in (1, 2)$, which simplifies the technical analysis while preserving key characteristics.

Example 1.4. Let $q \in (1, 2)$, $\vartheta \geqslant 1$ and $T = e^{-q\vartheta}$. If one takes

$$g(t) = t^{-\frac{1}{q}} |\log(t)|^{-\vartheta} = t^{-\frac{1}{q}} (-\log(t))^{-\vartheta},$$

then

$$g \in L^q([0, T]), \quad \text{but} \quad g \notin L^{\check{q}}([0, T]), \text{ for every } \check{q} \in (q, 2). \tag{1.9}$$

Observe that

$$g'(t) = -t^{-\frac{1}{q}-1}(-\log(t))^{-\vartheta-1}\Big[-\frac{\log(t)}{q}-\vartheta\Big] < 0, \quad \forall\, t \in (0,T),$$

then g is strictly monotonically decreasing on $[0,T]$. Thus,

$$g^*(t) = \begin{cases} \inf\{\varsigma \geqslant 0 : g^{-1}(\varsigma) \leqslant t\} = g(t), & \text{if } t \in [0,T], \\ 0, & \text{if } t > T, \end{cases}$$

and

$$g^{**}(t) = \begin{cases} \frac{1}{t}\int_0^t g^*(r)dr = \frac{1}{t}\int_0^t g(r)dr, & \text{if } t \in [0,T], \\ \frac{1}{t}\int_0^T g^*(r)dr = \frac{1}{t}\int_0^T g(r)dr, & \text{if } t > T. \end{cases}$$

For $\vartheta \geqslant 1$, one calculates that

$$\begin{aligned} \|g\|_{L^{q,1}([0,T])} &= \int_0^\infty t^{\frac{1}{q}-1} g^{**}(t)dt = \int_0^T t^{\frac{1}{q}-2}dt \int_0^t g(r)dr + \int_T^\infty t^{\frac{1}{q}-2}dt \int_0^T g(r)dr \\ &= \int_0^T g(r)dr \int_r^T t^{\frac{1}{q}-2}dt + \int_T^\infty t^{\frac{1}{q}-2}dt \int_0^T g(r)dr \\ &= \frac{q}{q-1}\int_0^T g(r) r^{\frac{1}{q}-1}dr = \frac{q}{q-1}\int_0^T \frac{1}{r|\log(r)|^\vartheta}dr. \end{aligned}$$

Therefore $g \in L^{q,1}([0,T])$ when $\vartheta > 1$ and $g \notin L^{q,1}([0,T])$ when $\vartheta = 1$. On the other hand, in view of Calderón's inequality (see (2.3)), one has $L^{q,1}([0,T]) \subset L^q([0,T])$. Thus $L^{q,1}([0,T])$ is a proper subset of $L^q([0,T])$, i.e.

$$L^{q,1}([0,T]) \subsetneqq L^q([0,T]). \tag{1.10}$$

Notice that, for $h \in L^{\check{q}}([0,T])$ with $\check{q} \in (q,2)$, the following estimate holds:

$$\begin{aligned} \|h\|_{L^{q,1}([0,T])} &= \int_0^T r^{\frac{1}{q}-1} h^{**}(r)dr + \int_T^\infty r^{\frac{1}{q}-2}dr \int_0^T h^*(\tau)d\tau \\ &\leqslant \Big[\int_0^T (h^{**}(r))^{\check{q}}dr\Big]^{\frac{1}{\check{q}}} \Big[\int_0^T r^{-\frac{\check{q}(q-1)}{q(\check{q}-1)}}dr\Big]^{\frac{\check{q}-1}{\check{q}}} + \frac{q}{q-1}T^{\frac{\check{q}-q}{q\check{q}}}\Big[\int_0^T (h^*(\tau))^{\check{q}}d\tau\Big]^{\frac{1}{\check{q}}} \\ &\leqslant C(T,q,\check{q})\big[\|h\|_{L^{\check{q},\check{q}}([0,T])} + \|h\|_{L^{\check{q}}([0,T])}\big] \leqslant C(T,q,\check{q})\|h\|_{L^{\check{q}}([0,T])}, \end{aligned} \tag{1.11}$$

where in the last inequality we have used (2.2). Combining (1.9)–(1.11), we conclude

$$L^{\check{q}}([0,T]) \subsetneqq L^{q,1}([0,T]) \subsetneqq L^{q}([0,T]). \tag{1.12}$$

Remark 1.5. Let $q \in (1,2)$ and $g \in L^{q,1}([0,T])$. For every $0 \leqslant \hat{T} < \tilde{T} \leqslant T$, then $g \in L^{q,1}([\hat{T},\tilde{T}])$ and by [1, Corollary 7.28],

$$[g]_{L^{q,1}([\hat{T},\tilde{T}])} \leqslant \|g\|_{L^{q,1}([\hat{T},\tilde{T}])} \leqslant \frac{q}{q-1}[g]_{L^{q,1}([\hat{T},\tilde{T}])},$$

where

$$[g]_{L^{q,1}([\hat{T},\tilde{T}])} = \int_0^\infty r^{\frac{1}{q}-1} \inf\{\varsigma \geqslant 0 : Leb\{t \in [\hat{T},\tilde{T}] : |g(t)| > \varsigma\} \leqslant r\} dr.$$

Notice that

$$Leb\{t \in [\hat{T},\tilde{T}] : |g(t)| > \varsigma\} \leqslant (\hat{T}-\tilde{T}), \quad \forall\, \varsigma \geqslant 0,$$

then

$$\inf\{\varsigma \geqslant 0 : Leb\{t \in [\hat{T},\tilde{T}] : |g(t)| > \varsigma\} \leqslant r\} = 0, \quad \forall\, r \geqslant \tilde{T}-\hat{T}.$$

Therefore,

$$\begin{aligned} \|g\|_{L^{q,1}([\hat{T},\tilde{T}])} &\leqslant \frac{q}{q-1} \int_0^{\tilde{T}-\hat{T}} r^{\frac{1}{q}-1} \inf\{\varsigma \geqslant 0 : Leb\{t \in [\hat{T},\tilde{T}] : |g(t)| > \varsigma\} \leqslant r\} dr \\ &\leqslant \frac{q}{q-1} \int_0^{\tilde{T}-\hat{T}} r^{\frac{1}{q}-1} \inf\{\varsigma \geqslant 0 : Leb\{t \in [0,T] : |g(t)| > \varsigma\} \leqslant r\} dr. \end{aligned} \tag{1.13}$$

Observe that the integrand in the last integral of (1.13) is integrable on $[0,T]$, by the absolute continuity of the Lebesgue integral, we conclude

$$\lim_{\hat{T}-\tilde{T}\to 0} \|g\|_{L^{q,1}([\hat{T},\tilde{T}])} = 0, \quad \forall\, 0 \leqslant \hat{T} < \tilde{T} \leqslant T. \tag{1.14}$$

1.2. Main result

Before giving the main result, let us give another notion.

Definition 1.6. ([25, p. 114]) A stochastic flow of homeomorphisms on a given stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T})$ associated to SDE (1.1) is a map $(s,t,x,\omega) \to X_{s,t}(x,\omega)$, defined for $0 \leqslant s \leqslant t \leqslant T,\ x \in \mathbb{R}^d,\ \omega \in \Omega$ with values in $\mathbb{R}^d$, such that

(i) the process $\{X_{s,\cdot}(x)\} = \{X_{s,t}(x),\ t \in [s,T]\}$ is a continuous $\{\mathcal{F}_{s,t}\}_{s\leqslant t\leqslant T}$-adapted solution of SDE (1.1) for every $s \in [0,T]$ and $x \in \mathbb{R}^d$;

(ii) $\mathbb{P}$-a.s., $X_{s,t}(\cdot)$ is a homeomorphism, for all $0 \leqslant s \leqslant t \leqslant T$, and the functions $X_{s,t}(x)$ and $X_{s,t}^{-1}(x)$ are continuous in (s,t,x), where $X_{s,t}^{-1}(\cdot)$ is the inverse of $X_{s,t}(\cdot)$;

(iii) $\mathbb{P}$-a.s., $X_{s,t}(x) = X_{r,t}(X_{s,r}(x))$ for all $0 \leqslant s \leqslant r \leqslant t \leqslant T$, $x \in \mathbb{R}^d$ and $X_{s,s}(x) = x$.

We are now state our main result.

Theorem 1.7. *Let $b \in L^{q,1}([0,T]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d;\mathbb{R}^d))$ with $q \in (1,2)$. Then there exists a unique stochastic flow of homeomorphisms $\{X_{s,t}(x),\ 0 \leqslant s \leqslant t \leqslant T,\ x \in \mathbb{R}^d\}$ to SDE (1.1). Moreover, $X_{s,t}(\cdot)$ and $X_{s,t}^{-1}(\cdot)$ are weakly differentiable, a.s., and for every $p \in [2,+\infty)$,*

$$\sup_{x\in\mathbb{R}^d}\sup_{0\leqslant s\leqslant t\leqslant T} \mathbb{E}\left[\|\nabla X_{s,t}(x)\|^p + \|\nabla X_{s,t}^{-1}(x)\|^p\right] < +\infty. \tag{1.15}$$

Remark 1.8. Let $\breve{q} \in (q,2)$ and $\bar{q} \in (1,q)$. By (1.12), we have the following inclusion relationship:

$$\begin{aligned} L^{\breve{q}}([0,T]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d)) &\subsetneqq L^{q,1}([0,T]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d)) \subsetneqq L^{q}([0,T]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d)) \\ &\subsetneqq L^{\bar{q}}([0,T]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d)). \end{aligned}$$

We expect the SDE (1.1) to be strongly well-posed when b belongs to the Lebesgue-Hölder space $L^q([0,T]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d;\mathbb{R}^d)) = L^{q,q}([0,T]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d;\mathbb{R}^d))$. However, we cannot provide a definitive answer to whether this condition is sharp. Notably, if the drift exhibits low regularity growth in the supercritical Lebesgue-Hölder space, i.e. $b \in L^{\bar{q}}([0,T]; \mathcal{C}_l^{\frac{2}{\bar{q}}-1}(\mathbb{R}^d;\mathbb{R}^d))$, the corresponding SDE is strongly ill-posed (see (1.5)). Conversely, when b lies in the subcritical Lebesgue-Hölder space $L^{\breve{q}}([0,T]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d;\mathbb{R}^d))$, a unique strong solution exists by adapting the argument analogues to [37, Theorem 1.2]. Therefore, our result is nearly optimal, despite the drift being only Lorentz integrable in the time variable with the first integrability index q.

1.3. Outline of the proof strategy for Theorem 1.7

We now provide a schematic overview of the proof methodology for our main result. To streamline the presentation, we restrict our attention to the case $s = 0$. The resolution hinges on the systematic application of Itô-Tanaka's trick (or Zvonkin's transformation). Consequently, the central challenge revolves around establishing rigorous a priori bounds for solutions to the following backward nonhomogeneous Kolmogorov equation:

$$\begin{cases} \partial_t U(t,x) + \frac{1}{2}\Delta U(t,x) + \nabla U(t,x)\cdot b(t,x) - \lambda U(t,x) \\ \qquad\qquad = -b(t,x), \quad (t,x) \in [0,T) \times \mathbb{R}^d, \\ U(t,x)|_{t=T} = 0, \ x \in \mathbb{R}^d. \end{cases} \tag{1.16}$$

When the drift lies in the subcritical Lebesgue-Hölder space, i.e. $b \in L^q([0,T]; \mathcal{C}_l^{\alpha}(\mathbb{R}^d;\mathbb{R}^d))$ with $\alpha \in (0,1)$ and $q \in (1, 2/(1+\alpha))$, by extending the analytical methodology in parallel to the framework established in [37, Theorem 2.3] and using the Sobolev embedding, we claim that $U \in B([0,T]; \mathcal{C}^1(\mathbb{R}^d;\mathbb{R}^d))$. Moreover,

$$\sup_{0\leqslant t\leqslant T} \|\nabla U(t,\cdot)\|_0 \leqslant C(\alpha, d, q, \|b\|_{L^q([0,T];\mathcal{C}_l^\alpha(\mathbb{R}^d))})\lambda^{-\epsilon_0}, \quad \epsilon_0 \in (0, 1+\alpha-2/q). \tag{1.17}$$

If one takes λ large enough, then $\sup_{0\leqslant t\leqslant T} \|\nabla U(t,\cdot)\|_0 < 1/2$. Thus, $\Phi(t,x) = x + U(t,x)$ forms a $\mathcal{C}^1$ diffeomorphism uniformly in $t \in [0,T]$. Suppose X_t satisfies SDE (1.1), by applying Itô's formula to the process $Y_t := X_t + U(t, X_t)$, we find that Y_t satisfies

$$\begin{cases} dY_t = \lambda U(t, \Psi(t,Y_t))dt + [I + \nabla U(t, \Psi(t,Y_t))]dW_t, \quad t \in (0,T], \\ Y_t|_{t=0} = y = x + U(0,x), \end{cases} \tag{1.18}$$

and vice versa, where $\Psi(t,\cdot) = \Phi^{-1}(t,\cdot)$. By the equivalence between (1.1) and (1.18), the present authors in [37] established the strong well-posedness for SDE (1.1).

When dealing with the critical drift, i.e. $b \in L^q([0,T]; \mathcal{C}_l^{\frac{2}{q}}(\mathbb{R}^d;\mathbb{R}^d))$ with $q \in (1,2)$, we encounter two principal difficulties. First, the Sobolev embedding theorem ceases to hold, precluding us from obtaining $\mathcal{C}^1$-regularity estimates for U; second, the parameter ϵ_0 on the right-hand side of equation (1.17) vanishes, thereby preventing the construction of a diffeomorphism. To resolve the first challenge, we impose Lorentz regularity condition on the temporal component, while to address the second, we restrict the time domain to a sufficiently small interval. Specifically, we consider the Kolmogorov equation (1.16) with $\lambda = 0$ and small enough T, construct an equivalent SDE via the diffeomorphism $\Phi(t,x) = x + U(t,x)$, thereby establishing strong well-posedness for SDE (1.1).

For arbitrary $T > 0$, we partition the time interval $[0,T]$ into subintervals $[T_i, T_{i+1}]$, $0 = T_0 < T_1 < \ldots < T_{k-1} < T_k = T$. On each subinterval $[T_i, T_{i+1}]$, the absolute continuity of the integral (see (1.14)) ensures that the Lorentz-Hölder norm $\|b\|_{L^{q,1}([T_i,T_{i+1}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))}$ is sufficiently small. If one considers a sequence of Kolmogorov equations:

$$\begin{cases} \partial_t U^i(t,x) + \frac{1}{2}\Delta U^i(t,x) + \nabla U^i(t,x)\cdot b(t,x) = -b(t,x), \quad (t,x) \in [T_i, T_{i+1}) \times \mathbb{R}^d, \\ U^i(t,x)|_{t=T_{i+1}} = 0, \ x \in \mathbb{R}^d, \end{cases} \tag{1.19}$$

then $\max_{0\leqslant i\leqslant k-1} \sup_{T_i\leqslant t\leqslant T_{i+1}} \|\nabla U^i(t,\cdot)\|_0 < 1/2$. Let $U(t,x) = U^i(t,x)$ when $t \in (T_i, T_{i+1}]$, $i = 0, 1, \ldots, k-1$. We set $\Phi(t,x) = x + U(t,x)$ and $\Psi(t,x) = \Phi^{-1}(t,x)$. Then there is a unique strong solution to the following SDE:

$$dY_t = (I + \nabla U(t, \Psi(t,Y_t))dW_t, \ t \in (0,T], \ Y_t|_{t=0} = y = x + U^0(0,x). \tag{1.20}$$

Thus SDE (1.1) is strongly well-posed and the unique strong solution is given by $X_t = \Psi(t, Y_t)$.

Notations. The letter C denotes a positive constant, whose values may change in different places. For a parameter or a function $\tilde{\nu}$, $C(\tilde{\nu})$ means the constant is only dependent on $\tilde{\nu}$, and we also write it as C if there is no confusion. We use ∇ to denote the gradient of a function with respect to the spatial variables. As usual, $\mathbb{N}$ is the set of positive natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. a.s. is the abbreviation of almost surely. For every $R > 0$, $B_R := \{x \in \mathbb{R}^d : |x| \leqslant R\}$. For a given $\mathbb{R}^{n\times n}$ matrix-valued function Ξ with $n \in \mathbb{N}$, $\|\Xi\|$ represents its Hilbert-Schmidt norm.

2. Useful lemmas

Initially, we recall some useful properties about Lorentz-Hölder spaces. The first one is a generalization of Marcinkiewicz's interpolation theorem for Lebesgue spaces (see [3, Theorems 5.3.1 and 5.3.2]), where the functions were assumed to take real or complex values. We need a slight extension for this theorem to the case of functions taking values in a Banach space. Since the proof is the same except that we use the norm of the function instead of its modulus, we give the interpolation theorem without the proof. We then list some elementary properties for functions in Lorentz-Hölder spaces. In the following, we assume $\hat{T}$ is a given positive real number.

Lemma 2.1. *([3, Theorems 5.3.1 and 5.3.2]) Let $\theta \in (0,1)$ and $1 < p_1 < q < p_2 \leqslant 2$ such that $1/q = (1-\theta)/p_1 + \theta/p_2$.*

(i) For every $\alpha \in (0,1)$ and $k \in \mathbb{N}_0$, the following interpolation holds:

$$[L^{p_1}([0,\hat{T}];\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d)), L^{p_2}([0,\hat{T}];\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d))]_{\theta,1} = L^{q,1}([0,\hat{T}];\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d)). \tag{2.1}$$

(ii) Let $\mathcal{T}$ be a linear operator from

$$L^{p_1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d)) + L^{p_2}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))$$

to

$$L^{p_1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}+1}(\mathbb{R}^d)) + L^{p_2}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}+1}(\mathbb{R}^d)).$$

If

$$\mathcal{T}:\ L^{p_i}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d)) \to L^{p_i}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}+1}(\mathbb{R}^d)),\ \ i=1,2,$$

are bounded, then $\mathcal{T}$ is also bounded from

$$[L^{p_1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d)), L^{p_2}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))]_{\theta,1}$$

to

$$[L^{p_1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}+1}(\mathbb{R}^d)), L^{p_2}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}+1}(\mathbb{R}^d))]_{\theta,1}.$$

Lemma 2.2. *Let $q \in (1,+\infty)$, $\alpha \in (0,1)$ and $k \in \mathbb{N}_0$. Let g be Borel measurable in $[0,T]\times\mathbb{R}^d$.*

(i) ([4, Lemma 6.13]) Then g belongs to $L^q([0,\hat{T}];\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d))$ if and only if it lies in $L^{q,q}([0,\hat{T}];\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d))$. Moreover,

$$\|g\|_{L^q([0,\hat{T}];\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d))} \leqslant \|g\|_{L^{q,q}([0,\hat{T}];\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d))} \leqslant \frac{q}{q-1}\|g\|_{L^q([0,\hat{T}];\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d))}. \tag{2.2}$$

(ii) ([4, Lemma 6.17 (Calderón)]) Let $1 \leqslant q_1 < q_2 < +\infty$. If $g \in L^{q,q_1}([0,\hat{T}];\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d))$, then $g \in L^{q,q_2}([0,\hat{T}];\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d))$ and

$$\|g\|_{L^{q,q_2}([0,\hat{T}];\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d))} \leqslant \left(\frac{q_1}{q}\right)^{\frac{1}{q_1}-\frac{1}{q_2}} \|g\|_{L^{q,q_1}([0,\hat{T}];\mathcal{C}_l^{k+\alpha}(\mathbb{R}^d))}. \tag{2.3}$$

(iii) ([4, Theorems 6.9 and 6.14]) Let $q_1, q_1' \in (1,+\infty)$ and $q_2, q_2' \in [1,+\infty]$ such that $1/q_1 + 1/q_1' = 1$ and $1/q_2 + 1/q_2' = 1$. If $h_1 \in L^{q_1,q_2}([0,\hat{T}])$ and $h_2 \in L^{q_1',q_2'}([0,\hat{T}])$, then $h_1h_2 \in L^1([0,\hat{T}])$ and

$$\|h_1h_2\|_{L^1([0,\hat{T}])} \leqslant \|h_1\|_{L^{q_1,q_2}([0,\hat{T}])}\|h_2\|_{L^{q_1',q_2'}([0,\hat{T}])}. \tag{2.4}$$

We now introduce another useful lemma.

Lemma 2.3. *([10, Theorem 4.5.3]) (Hardy-Littlewood-Sobolev's convolution inequality) Suppose $\phi(r) = |r|^{-\frac{1}{a}}$. Let $1 < p_1 < p_2 < +\infty$ such that $1/p_1 + 1/a = 1 + 1/p_2$. If $h \in L^{p_1}(\mathbb{R})$, then $h * \phi \in L^{p_2}(\mathbb{R})$ and there is a positive constant $C(p_1, a)$ such that*

$$\|h * \phi\|_{L^{p_2}(\mathbb{R})} \leqslant C(p_1,a)\|h\|_{L^{p_1}(\mathbb{R})}. \tag{2.5}$$

Let $b : [0,\hat{T}] \times \mathbb{R}^d \to \mathbb{R}^d$ and $f : [0,\hat{T}] \times \mathbb{R}^d \to \mathbb{R}$ be Borel measurable functions. We discuss the following Kolmogorov equation for $u : [0,\hat{T}] \times \mathbb{R}^d \to \mathbb{R}$:

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \nabla u(t,x)\cdot b(t,x) + f(t,x), \ (t,x) \in (0,\hat{T}] \times \mathbb{R}^d, \\ u(t,x)|_{t=0} = 0, \ x \in \mathbb{R}^d. \end{cases} \tag{2.6}$$

If $u \in L^1([0,\hat{T}]; W^{2,1}_{loc}(\mathbb{R}^d)) \cap W^{1,1}([0,\hat{T}]; L^1_{loc}(\mathbb{R}^d))$ such that (2.6) holds true for almost all $(t,x) \in [0,\hat{T}] \times \mathbb{R}^d$, then the unknown function u is said to be a strong solution of (2.6). Let us establish the Lebesgue-Hölder maximum regularity estimates of solutions for the Cauchy problem (2.6).

Lemma 2.4. *(Lebesgue-Hölder maximum regularity estimates) We assume that*

$$b \in L^\infty([0,\hat{T}];\mathcal{C}_b^{\frac{2}{q}-1}(\mathbb{R}^d;\mathbb{R}^d)), \ f \in L^p([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d)), \ q \in (1,2) \text{ and } p \in (1,+\infty).$$

Then there is a unique $u \in L^p([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}+1}(\mathbb{R}^d)) \cap W^{1,p}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))$ solving the Cauchy problem (2.6). Moreover, there is some positive constant C such that

$$\|\partial_t u\|_{L^p([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))} + \|u\|_{L^p([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}+1}(\mathbb{R}^d))} \leqslant C\|f\|_{L^p([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))}. \tag{2.7}$$

Proof. By the method of continuity, it is sufficient to show the case of $b \equiv 0$, and now u can be represented by

$$u(t,x)=\int_0^t dr\int_{\mathbb{R}^d}K(r,z)f(t-r,x-z)dz=\int_0^t dr\int_{\mathbb{R}^d}K(t-r,x-z)f(r,z)dz, \tag{2.8}$$

where $K(r,z)=(2\pi r)^{-\frac{d}{2}}e^{-\frac{|z|^2}{2r}}$.

If u satisfies

$$\|u\|_{L^p([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}+1}(\mathbb{R}^d))}\leqslant C\|f\|_{L^p([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))}, \tag{2.9}$$

then (2.7) holds mutatis mutandis. On the other hand, we obtain by (2.8) that

$$\begin{aligned}\sup_{x\in\mathbb{R}^d}\frac{|u(t,x)|}{1+|x|}&\leqslant\int_0^t dr\int_{\mathbb{R}^d}K(r,z)\sup_{x\in\mathbb{R}^d}\Big[\frac{|f(t-r,x-z)|}{1+|x-z|}\frac{1+|x-z|}{1+|x|}\Big]dz\\&\leqslant\int_0^t dr\int_{\mathbb{R}^d}K(r,z)\sup_{x\in\mathbb{R}^d}\|(1+|\cdot|)^{-1}f(t-r,\cdot)\|_0[1+|z|]dz\\&\leqslant C\int_0^t(1+r^{\frac{1}{2}})\|(1+|\cdot|)^{-1}f(t-r,\cdot)\|_0dr,\end{aligned}$$

which suggests

$$\Big[\int_0^{\hat{T}}\|(1+|\cdot|)^{-1}u(t,\cdot)\|_0^p dt\Big]^{\frac{1}{p}}\leqslant C\Big[\int_0^{\hat{T}}\|(1+|\cdot|)^{-1}f(t,\cdot)\|_0^p dt\Big]^{\frac{1}{p}}. \tag{2.10}$$

For the gradient, we have

$$\nabla u(t,x)=\int_0^t dr\int_{\mathbb{R}^d}\nabla K(t-r,x-z)[f(r,z)-f(r,x)]dz.$$

Thus,

$$\begin{aligned}&\int_0^{\hat{T}}\sup_{x\in\mathbb{R}^d}|\nabla u(t,x)|^p dt\leqslant 2^p\int_0^{\hat{T}}\Big|\int_0^t dr\int_{\mathbb{R}^d}|\nabla K(t-r,z)|[f(r,\cdot)]_{\frac{2}{q}-1}(|z|^{\frac{2}{q}-1}+|z|)dz\Big|^p dt\\&\leqslant C^p\int_0^{\hat{T}}\Big|\int_0^t(1+|t-r|^{\frac{1}{q}-1})[f(r,\cdot)]_{\frac{2}{q}-1}dr\Big|^p dt\leqslant C^p\int_0^{\hat{T}}[f(t,\cdot)]^p_{\frac{2}{q}-1}dt,\end{aligned} \tag{2.11}$$

where in the first and third inequalities we have used (1.6) and Young's convolution inequality, respectively.

Similar calculations also imply

$$\|\nabla^2 u\|_{L^p([0,\hat{T}];L^\infty(\mathbb{R}^d))} \leqslant C\big\|[f(t,\cdot)]_{\frac{2}{q}-1}\big\|_{L^p([0,\hat{T}])}. \tag{2.12}$$

For every $x, y \in \mathbb{R}^d$ such that $|x-y| \leqslant 1$, by repeating all calculations of the proof for (3.10) in [16, Theorem 3.3], it yields to

$$\left(\int_0^{\hat{T}} \sup_{0<|x-y|\leqslant 1} \frac{|\nabla^2 u(t,x) - \nabla^2 u(t,y)|^p}{|x-y|^{\frac{2p}{q}-p}} dt\right)^{\frac{1}{p}} \leqslant C\left(\int_0^{\hat{T}} \sup_{0<|x-y|\leqslant 1} \frac{|f(t,x)-f(t,y)|^p}{|x-y|^{\frac{2p}{q}-p}} dt\right)^{\frac{1}{p}},$$

i.e.

$$\left(\int_0^{\hat{T}} [\nabla^2 u(t,\cdot)]^p_{\frac{2}{q}-1} dt\right)^{\frac{1}{p}} \leqslant C\left(\int_0^{\hat{T}} [f(t,\cdot)]^p_{\frac{2}{q}-1} dt\right)^{\frac{1}{p}}. \tag{2.13}$$

In view of (2.10)–(2.13), (2.9) holds true. □

Remark 2.5. Let p and q be stated in Lemma 2.4. In [16, Theorems 3.1 and 3.3], by using a Banach-space version of the Calderón-Zygmund theorem, Krylov proved the following estimate:

$$\left(\int_{\mathbb{R}} \sup_{x\neq y} \frac{|\nabla^2 \tilde{u}(t,x) - \nabla^2 \tilde{u}(t,y)|^p}{|x-y|^{\frac{2p}{q}-p}} dt\right)^{\frac{1}{p}} \leqslant C\left(\int_{\mathbb{R}} \sup_{x\neq y} \frac{|\tilde{f}(t,x)-\tilde{f}(t,y)|^p}{|x-y|^{\frac{2p}{q}-p}} dt\right)^{\frac{1}{p}}, \tag{2.14}$$

where $\tilde{f} \in \mathcal{C}_0^\infty(\mathbb{R}^{d+1})$ and $\tilde{u}$ satisfies

$$\partial_t \tilde{u}(t,x) = \frac{1}{2}\Delta \tilde{u}(t,x) - \lambda \tilde{u}(t,x) + \tilde{f}(t,x),\ (t,x) \in \mathbb{R}\times\mathbb{R}^d,\ \ \lambda > 0.$$

If one restricts the compact support of $\tilde{f}$, as a function of the time variable, to $[0,\hat{T}]$, estimate (2.14) remains valid even for $\lambda = 0$. On the other hand, $\mathcal{C}_0^\infty([0,T]\times\mathbb{R}^d)$ is dense in $L^p([0,\hat{T}];\mathcal{C}_b^{\frac{2}{q}-1}(\mathbb{R}^d))$, by the extension, (2.14) holds ad hoc for every $\tilde{f} \in L^p([0,\hat{T}];\mathcal{C}_b^{\frac{2}{q}-1}(\mathbb{R}^d))$. In particular, if $\tilde{u}$ vanishes when $t = 0$, then

$$\left(\int_0^{\hat{T}} \sup_{x\neq y} \frac{|\nabla^2 \tilde{u}(t,x) - \nabla^2 \tilde{u}(t,y)|^p}{|x-y|^{\frac{2p}{q}-p}} dt\right)^{\frac{1}{p}} \leqslant C\left(\int_0^{\hat{T}} \sup_{x\neq y} \frac{|\tilde{f}(t,x)-\tilde{f}(t,y)|^p}{|x-y|^{\frac{2p}{q}-p}} dt\right)^{\frac{1}{p}}, \tag{2.15}$$

where $\tilde{f} \in L^p([0,\hat{T}];\mathcal{C}_b^{\frac{2}{q}-1}(\mathbb{R}^d))$ and $\tilde{u}$ satisfies

$$\partial_t \tilde{u}(t,x) = \frac{1}{2}\Delta \tilde{u}(t,x) + \tilde{f}(t,x),\ (t,x) \in (0,\hat{T}] \times \mathbb{R}^d,\ \tilde{u}(t,x)|_{t=0} = 0,\ x \in \mathbb{R}^d.$$

We now use $\sup_{0<|x-y|\leqslant 1}$ instead of $\sup_{x \neq y}$ in (2.15) and repeat the same manipulations as in the proof of (2.15), then get (2.13). Since the proof is trivial, we omit the details.

3. Lorentz-Schauder estimates

In this section, let us establish the Lorentz-Schauder estimates for the Kolmogorov equation (2.6).

Theorem 3.1. *Let $\hat{T} \in (0,1)$ be small enough, and let*

$$f \in L^{q,1}([0,\hat{T}]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d)),\ \ b \in L^{q,1}([0,\hat{T}]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d;\mathbb{R}^d)),\ \ q \in (1,2).$$

Then it holds that:

(i) ***(Existence and uniqueness)*** *There is a unique strong solution u to (2.6). Moreover, $u \in \mathcal{H}^{2,\infty}_{q,0,\hat{T}}$ with*

$$\mathcal{H}^{2,\infty}_{q,0,\hat{T}} = \{v \in B([0,\hat{T}];\mathcal{C}(\mathbb{R}^d));\ \nabla v \in B([0,\hat{T}];\mathcal{C}_b(\mathbb{R}^d;\mathbb{R}^d)),\ |\nabla^2 v| \in L^2([0,\hat{T}];L^\infty(\mathbb{R}^d)),$$
$$\partial_t v \in L^q([0,\hat{T}];L^\infty_{loc}(\mathbb{R}^d)) \text{ and } \|(1+|\cdot|)^{-1}\partial_t v(t,\cdot)\|_{L^q([0,\hat{T}];L^\infty(\mathbb{R}^d))} < +\infty\}, \quad (3.1)$$

where $\mathcal{C}(\mathbb{R}^d)$ denotes all continuous functions on $\mathbb{R}^d$, $\mathcal{C}_b(\mathbb{R}^d;\mathbb{R}^d)$ denotes all bounded continuous functions from $\mathbb{R}^d$ to $\mathbb{R}^d$, and $B([0,\hat{T}])$ is the set of all bounded functions on $[0,\hat{T}]$. Furthermore, there is a positive constant $C(d,q)$ such that

$$\|(1+|\cdot|)^{-1}\partial_t u(t,\cdot)\|_{L^q([0,\hat{T}];L^\infty(\mathbb{R}^d))} + \|\nabla^2 u\|_{L^2([0,\hat{T}];L^\infty(\mathbb{R}^d))} \leqslant C(d,q)\|f\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))} \quad (3.2)$$

and

$$\sup_{0\leqslant t\leqslant \hat{T}} \|\nabla u(t,\cdot)\|_0 = \sup_{(t,x)\in[0,\hat{T}]\times\mathbb{R}^d} |\nabla u(t,x)| \leqslant \min\Big\{C(d,q)\big\|[f(t,\cdot)]_{\frac{2}{q}-1}\big\|_{L^{q,1}([0,\hat{T}])}, \frac{1}{2}\Big\}. \quad (3.3)$$

(ii) ***(Stability)*** *Let ρ be a symmetric regularizing kernel, that is*

$$\rho(x) = \rho(-x) \ \text{ with } \ 0 \leqslant \rho \in \mathcal{C}_0^\infty(\mathbb{R}^d),\ \ \mathrm{supp}(\rho) \subset B_1,\ \int_{\mathbb{R}^d} \rho(x)dx = 1.$$

For $m \in \mathbb{N}$, we set $\rho_m(x) = m^d \rho(mx)$ and

$$\begin{cases} b_m(t,x) = b * \rho_m(t,x) = \int\limits_{\mathbb{R}^d} b(t,x-y)\rho_m(y)dy, \\ f_m(t,x) = f * \rho_m(t,x) = \int\limits_{\mathbb{R}^d} f(t,x-y)\rho_m(y)dy. \end{cases} \tag{3.4}$$

Let u_m be the unique strong solution of (2.6) with b and f replaced by b_m and f_m, respectively. Then $u_m \in \mathcal{H}^{2,\infty}_{q,0,\hat{T}}$ and satisfies (3.2)–(3.3) uniformly in m. Moreover, $u_m - u \in B([0,\hat{T}];\mathcal{C}_b^1(\mathbb{R}^d))$ and for every $p \in [2,+\infty)$, we have

$$\lim_{m\to+\infty} \Big[\sup_{0\leqslant t\leqslant \hat{T}} \|u_m(t,\cdot) - u(t,\cdot)\|_0 + \|\nabla u_m - \nabla u\|_{L^p([0,\hat{T}];\mathcal{C}_b(\mathbb{R}^d))} \Big] = 0. \tag{3.5}$$

Proof. We extend b from $[0,\hat{T}]$ to $(-\infty,\hat{T}]$ and define it by 0 for $t<0$. Let ϱ be a regularizing kernel:

$$0 \leqslant \varrho \in \mathcal{C}_0^\infty(\mathbb{R}), \quad \mathrm{supp}(\varrho) \subset [0,1], \int\limits_{\mathbb{R}} \varrho(r)dr = 1.$$

For $n \in \mathbb{N}$, we set $\varrho_n(r) = n\varrho(nr)$ and smooth b in the time variable by ϱ_n:

$$b^n(t,x) = (b(\cdot,x) * \varrho_n)(t) = \int\limits_{\mathbb{R}} b(t-r,x)\varrho_n(r)dr.$$

For $R>0$, we set $b^n_R(t,x) = b^n(t, x\chi_R(x))$, where $\chi_R(x) = \chi(x/R)$ and

$$\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d), \quad 0 \leqslant \chi \leqslant 1, \quad |\chi'| \leqslant 2 \ \text{ and } \chi(x) = \begin{cases} 1, & \text{if } x \in B_1, \\ 0, & \text{if } x \in \mathbb{R}^d \setminus B_2. \end{cases} \tag{3.6}$$

Then $b^n_R \in L^\infty([0,\hat{T}];\mathcal{C}_b^{\frac{2}{q}-1}(\mathbb{R}^d;\mathbb{R}^d))$ and there exists a (unlabelled) subsequence such that

$$\lim_{n\to+\infty} \lim_{R\to+\infty} |b^n_R(t,x) - b(t,x)| = 0, \quad a.e.\ (t,x) \in [0,\hat{T}] \times \mathbb{R}^d. \tag{3.7}$$

Moreover,

$$\|(1+|\cdot|)^{-1} b^n_R(t,\cdot)\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_b(\mathbb{R}^d))} \leqslant \|(1+|\cdot|)^{-1} b(t,\cdot)\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_b(\mathbb{R}^d))}. \tag{3.8}$$

For every $x,y \in \mathbb{R}^d$ such that $|x-y| \leqslant 1$ and every $R \geqslant 1$, by (3.6) it follows that

$$\begin{aligned} |x\chi_R(x) - y\chi_R(y)| &\leqslant \big[|x-y| + |y||\chi_R(x) - \chi_R(y)|\big] \\ &\leqslant |x-y|\big[1 + \sup_{\tau\in[0,1]} |\chi_R'(\tau x + (1-\tau)y)|\big] \leqslant 3|x-y|, \end{aligned} \tag{3.9}$$

which also implies

$$
\begin{aligned}
|x\chi_R(x) - y\chi_R(y)| &= |x\chi_R(x) - y\chi_R(y)|^{2-\frac{2}{q}} |x\chi_R(x) - y\chi_R(y)|^{\frac{2}{q}-1} \\
&\leqslant 3|x\chi_R(x) - y\chi_R(y)|^{\frac{2}{q}-1}.
\end{aligned} \tag{3.10}
$$

Due to (1.6), (3.9) and (3.10), then

$$
\begin{aligned}
\left\| [b_R^n(t,\cdot)]_{\frac{2}{q}-1} \right\|_{L^{q,1}([0,\hat{T}])} &= \left\| \sup_{0<|x-y|\leqslant 1} \frac{|b_R^n(t,x) - b_R^n(t,y)|}{|x-y|^{\frac{2}{q}-1}} \right\|_{L^{q,1}([0,\hat{T}])} \\
&\leqslant 6 \left\| [b(t,\cdot)]_{\frac{2}{q}-1} \right\|_{L^{q,1}([0,\hat{T}])} \sup_{0<|x-y|\leqslant 1} \frac{|x\chi_R(x) - y\chi_R(y)|^{\frac{2}{q}-1}}{|x-y|^{\frac{2}{q}-1}} \\
&\leqslant 18 \left\| [b(t,\cdot)]_{\frac{2}{q}-1} \right\|_{L^{q,1}([0,\hat{T}])}.
\end{aligned} \tag{3.11}
$$

Let $\tilde{f} \in L^p([0,\hat{T}]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))$ with $p \in (1,2]$. Consider the following Kolmogorov equation:

$$
\begin{cases}
\partial_t u_R^n(t,x) = \frac{1}{2}\Delta u_R^n(t,x) + \nabla u_R^n(t,x) \cdot b_R^n(t,x) + \tilde{f}(t,x), & (t,x) \in (0,\hat{T}] \times \mathbb{R}^d, \\
u_R^n(t,x)|_{t=0} = 0, \quad x \in \mathbb{R}^d.
\end{cases} \tag{3.12}
$$

In view of Lemma 2.4, there exists a unique

$$
u_R^n \in L^p([0,\hat{T}]; \mathcal{C}_l^{\frac{2}{q}+1}(\mathbb{R}^d)) \cap W^{1,p}([0,\hat{T}]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))
$$

solving the Cauchy problem (3.12), and there is some positive constant $C(n,R)$ such that

$$
\|\partial_t u_R^n\|_{L^p([0,\hat{T}]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))} + \|u_R^n\|_{L^p([0,\hat{T}]; \mathcal{C}_l^{\frac{2}{q}+1}(\mathbb{R}^d))} \leqslant C(n,R) \|\tilde{f}\|_{L^p([0,\hat{T}]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))}. \tag{3.13}
$$

We define the solution mapping $\mathcal{T}$ by $\mathcal{T}\tilde{f} = u_R^n$. By (3.13),

$$
\mathcal{T}: \ L^p([0,\hat{T}]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d)) \to L^p([0,\hat{T}]; \mathcal{C}_l^{\frac{2}{q}+1}(\mathbb{R}^d)), \quad \forall\, p \in (1,2],
$$

is linear and bounded. Let $\theta \in (0,1)$ and $1 < p_1 < q < p_2 \leqslant 2$ such that $1/q = (1-\theta)/p_1 + \theta/p_2$. With the help of Lemma 2.1 (i) and (ii), $\mathcal{T}$ is also bounded from $L^{q,1}([0,\hat{T}]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))$ to $L^{q,1}([0,\hat{T}]; \mathcal{C}_l^{\frac{2}{q}+1}(\mathbb{R}^d))$. In view of Lemma 2.2 (i) and (ii), we have the following inclusion:

$$
L^{q,1}([0,\hat{T}]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d)) \subset L^q([0,\hat{T}]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d)).
$$

Thus there is a unique

$$
u_R^n \in L^{q,1}([0,\hat{T}]; \mathcal{C}_l^{\frac{2}{q}+1}(\mathbb{R}^d)) \cap W^{1,q}([0,\hat{T}]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))
$$

solving the following Cauchy problem:

$$\begin{cases} \partial_t u_R^n(t,x) = \frac{1}{2}\Delta u_R^n(t,x) + \nabla u_R^n(t,x)\cdot b_R^n(t,x) + f(t,x), \quad (t,x)\in(0,\hat{T}]\times\mathbb{R}^d, \\ u_R^n(t,x)|_{t=0} = 0, \quad x\in\mathbb{R}^d, \end{cases} \tag{3.14}$$

which satisfies

$$\|u_R^n\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_l^{1+\frac{2}{q}}(\mathbb{R}^d))} \leqslant C(n,R)\|f\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))}.$$

Therefore, $u_R^n \in L^{q,1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}+1}(\mathbb{R}^d))$, $\partial_t u_R^n \in L^{q,1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))$ and

$$\|\partial_t u_R^n\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))} + \|u_R^n\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_l^{1+\frac{2}{q}}(\mathbb{R}^d))} \leqslant C(n,R)\|f\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))}. \tag{3.15}$$

On the other hand, by the heat kernel representation, the unique strong solution has the following equivalent form:

$$u_R^n(t,x) = \int_0^t dr \int_{\mathbb{R}^d} K(t-r,x-y)\big[\nabla u_R^n(r,y)\cdot b_R^n(r,y) + f(r,y)\big]dy. \tag{3.16}$$

Let $\hat{x}\in\mathbb{R}^d$. Consider the following ordinary differential equation (ODE for short):

$$\dot{x}_t = -b_R^n(t,\hat{x}+x_t), \quad x_t|_{t=0} = 0. \tag{3.17}$$

There is a unique solution to (3.17). By setting $\hat{u}_R^n(t,x) := u_R^n(t,x+\hat{x}+x_t)$, $\hat{f}(t,x) := f(t,x+\hat{x}+x_t)$ and $\hat{b}_R^n(t,x) := b_R^n(t,x+\hat{x}+x_t) - b_R^n(t,\hat{x}+x_t)$, one derives

$$\hat{u}_R^n(t,x) = \int_0^t dr \int_{\mathbb{R}^d} K(t-r,x-y)\big[\nabla \hat{u}_R^n(r,y)\cdot \hat{b}_R^n(r,y) + \hat{f}(r,y)\big]dy. \tag{3.18}$$

Consequently,

$$\begin{aligned} &|\nabla \hat{u}_R^n(t,0)| \\ &= \left|\int_0^t dr \int_{\mathbb{R}^d} \nabla K(t-r,y)\big[\nabla \hat{u}_R^n(r,y)\cdot \hat{b}_R^n(r,y) + \hat{f}(r,y) - \hat{f}(r,0)\big]dy\right| \\ &\leqslant 2\int_0^t dr \int_{\mathbb{R}^d} |\nabla K(t-r,y)|\Big\{[b_R^n(r,\cdot)]_{\frac{2}{q}-1}\|\nabla u_R^n(r,\cdot)\|_0 + [f(r,\cdot)]_{\frac{2}{q}-1}\Big\} \\ &\quad \times\big[1_{|y|\leqslant 1}|y|^{\frac{2}{q}-1} + |y|1_{|y|>1}\big]dy \\ &\leqslant C(d,q)\int_0^t \Big\{[b_R^n(r,\cdot)]_{\frac{2}{q}-1}\|\nabla u_R^n(r,\cdot)\|_0 + [f(r,\cdot)]_{\frac{2}{q}-1}\Big\}\big[1+(t-r)^{\frac{1}{q}-1}\big]dr \end{aligned}$$

$$\leqslant C(d,q)\Big\{\big\|[b_R^n(t,\cdot)]_{\frac{2}{q}-1}\big\|_{L^\infty([0,\hat{T}])}\|\nabla u_R^n\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_b(\mathbb{R}^d))}+\big\|[f(t,\cdot)]_{\frac{2}{q}-1}\big\|_{L^{q,1}([0,\hat{T}])}\Big\}, \tag{3.19}$$

where in the last inequality we have used (2.3) and (2.4) since $(t-\cdot)^{\frac{1}{q}-1}\in L^{\frac{q}{q-1},\infty}([0,t])$ and $\|(t-\cdot)^{\frac{1}{q}-1}\|_{L^{\frac{q}{q-1},\infty}([0,t])}=1$.

Observe that $\hat{x}\in\mathbb{R}^d$ is arbitrary, we conclude from (3.19) and (3.15) that $|\nabla u_R^n|\in L^\infty([0,\hat{T}]\times\mathbb{R}^d)$. This, together with (3.11) and the second inequality in (3.19), also implies

$$\begin{aligned}\sup_{0\leqslant t\leqslant\hat{T}}\|\nabla u_R^n(t,\cdot)\|_0\leqslant C(d,q)\Big\{&\big\|[b(t,\cdot)]_{\frac{2}{q}-1}\big\|_{L^{q,1}([0,\hat{T}])}\sup_{0\leqslant t\leqslant\hat{T}}\|\nabla u_R^n(t,\cdot)\|_0\\&+\big\|[f(t,\cdot)]_{\frac{2}{q}-1}\big\|_{L^{q,1}([0,\hat{T}])}\Big\}.\end{aligned} \tag{3.20}$$

Notice that $\hat{T}$ is small enough, by (1.14) one has

$$\max\{C(d,q),1\}\times\max\Big\{\|b\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))},\|f\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))}\Big\}\leqslant\frac{1}{3}, \tag{3.21}$$

which, together with (3.20), yields that there exists a new constant $C(d,q)$ such that

$$\sup_{0\leqslant t\leqslant\hat{T}}\|\nabla u_R^n(t,\cdot)\|_0\leqslant\min\Big\{C(d,q)\big\|[f(t,\cdot)]_{\frac{2}{q}-1}\big\|_{L^{q,1}([0,\hat{T}])},\frac{1}{2}\Big\}. \tag{3.22}$$

By (3.16), we get

$$\begin{aligned}|u_R^n(t,x)|\leqslant\int_0^t dr\int_{\mathbb{R}^d}K(t-r,y)\big[1+|x|+|y|\big]\Big[&\|(1+|\cdot|)^{-1}f(r,\cdot)\|_0\\&+\|(1+|\cdot|)^{-1}b_R^n(r,\cdot)\|_0\sup_{0\leqslant r\leqslant\hat{T}}\|\nabla u_R^n(r,\cdot)\|_0\Big]dy.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sup_{0\leqslant t\leqslant\hat{T}}\sup_{x\in\mathbb{R}^d}\frac{|u_R^n(t,x)|}{1+|x|}\\&\leqslant C(d)\int_0^{\hat{T}}\big[1+(\hat{T}-r)^{\frac{1}{2}}\big]\Big[\|(1+|\cdot|)^{-1}b_R^n(r,\cdot)\|_0+\|(1+|\cdot|)^{-1}f(r,\cdot)\|_0\Big]dr\\&\leqslant C(d,q)\Big[\|b\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))}+\|f\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))}\Big],\end{aligned} \tag{3.23}$$

where in the last inequality we have used Hölder's inequality, (2.2), (2.3) and (3.8).

For the second derivatives of $u_R^n(t,x)$, we get an analogue of (3.19) that

$$
\begin{aligned}
|\nabla^2 \hat{u}_R^n(t,0)| \leqslant C(d) \int_0^t dr \int_{\mathbb{R}^d} |\nabla^2 K(t-r,y)| \Big\{ [b_R^n(r,\cdot)]_{\frac{2}{q}-1} + [f(r,\cdot)]_{\frac{2}{q}-1} \Big\} \\
\times \big[1_{|y|\leqslant 1} |y|^{\frac{2}{q}-1} + |y| 1_{|y|>1} \big] dy \\
\leqslant C(d) \int_0^t \Big\{ [b_R^n(r,\cdot)]_{\frac{2}{q}-1} + [f(r,\cdot)]_{\frac{2}{q}-1} \Big\} \big[(t-r)^{\frac{1}{q}-\frac{3}{2}} + (t-r)^{-\frac{1}{2}} \big] dr,
\end{aligned}
$$

which suggests that

$$
\|\nabla^2 u_R^n\|_{L^2([0,\hat{T}];L^\infty(\mathbb{R}^d))} \leqslant C(d,q) \Big[\|b\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))} + \|f\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))} \Big], \quad (3.24)
$$

if one uses (2.5) first, (2.2) next and (2.3) third.

Further, by (3.14) and (3.22)–(3.24), then $\partial_t u_R^n \in L^q([0,\hat{T}]; L^\infty_{loc}(\mathbb{R}^d))$ and there is a positive constant $C(d,q)$ such that

$$
\begin{aligned}
& \|(1+|\cdot|)^{-1} \partial_t u_R^n(t,\cdot)\|_{L^q([0,\hat{T}];L^\infty(\mathbb{R}^d))} \\
& \leqslant C(d,q) \Big[\|\nabla^2 u_R^n\|_{L^2([0,\hat{T}];L^\infty(\mathbb{R}^d))} + \|(1+|\cdot|)^{-1} f(t,\cdot)\|_{L^q([0,\hat{T}];\mathcal{C}_b(\mathbb{R}^d))} \\
& \quad + \sup_{0\leqslant t\leqslant \hat{T}} \|\nabla u_R^n(t,\cdot)\|_0 \|(1+|\cdot|)^{-1} b_R^n(t,\cdot)\|_{L^q([0,\hat{T}];\mathcal{C}_b(\mathbb{R}^d))} \Big] \\
& \leqslant C(d,q) \Big[\|b\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))} + \|f\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))} \Big]. \qquad (3.25)
\end{aligned}
$$

On account of (3.22)–(3.25) and (3.14), there exists a (unlabelled) subsequence u_R^n and a measurable function $u \in \tilde{\mathcal{H}}^{2,\infty}_{q,0,\hat{T}}$:

$$
\begin{aligned}
\tilde{\mathcal{H}}^{2,\infty}_{q,0,\hat{T}} = \{ & v \in B([0,\hat{T}]; L^\infty_{loc}(\mathbb{R}^d));\ |\nabla v| \in B([0,\hat{T}]\times\mathbb{R}^d),\ |\nabla^2 v| \in L^2([0,\hat{T}]; L^\infty(\mathbb{R}^d)), \\
& \partial_t v \in L^q([0,\hat{T}]; L^\infty_{loc}(\mathbb{R}^d)) \text{ and } \|(1+|\cdot|)^{-1} \partial_t v(t,\cdot)\|_{L^q([0,\hat{T}];L^\infty(\mathbb{R}^d))} < +\infty \},
\end{aligned}
$$

such that $u_R^n(t,x) \to u(t,x)$ for every $(t,x) \in [0,\hat{T}]\times\mathbb{R}^d$ and $\nabla u_R^n(t,x) \to \nabla u(t,x)$ for a.e. $(t,x) \in [0,\hat{T}]\times\mathbb{R}^d$, as R and n tend to infinity in turn. Moreover, (3.2)–(3.3) hold for u, and by (3.7) and (3.16), u satisfies

$$
u(t,x) = \int_0^t dr \int_{\mathbb{R}^d} K(t-r,x-y) \big[\nabla u(r,y) \cdot b(r,y) + f(r,y) \big] dy.
$$

which suggests u satisfies (2.6). Therefore, u is a strong solution. On the other hand, $|\nabla^2 u| \in L^2([0,\hat{T}]; L^\infty(\mathbb{R}^d))$ and $|\nabla u| \in B([0,\hat{T}] \times \mathbb{R}^d)$, we get $\nabla u \in B([0,\hat{T}]; \mathcal{C}_b(\mathbb{R}^d; \mathbb{R}^d))$. Whence, $u \in \mathcal{H}^{2,\infty}_{q,0,\hat{T}}$.

Now let us prove the uniqueness. Observing that the equation is linear, it suffices to prove that $u \equiv 0$ for the following homogeneous Cauchy problem:

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \nabla u(t,x) \cdot b(t,x), & (t,x) \in (0,\hat{T}] \times \mathbb{R}^d, \\ u(t,x)|_{t=0} = 0, & x \in \mathbb{R}^d. \end{cases}$$

For the above Cauchy problem, by using analogues of (3.20) and (3.21) for ∇u, we deduce

$$\sup_{0 \leqslant t \leqslant \hat{T}} \|\nabla u(t,\cdot)\|_0 \leqslant \frac{1}{3} \sup_{0 \leqslant t \leqslant \hat{T}} \|\nabla u(t,\cdot)\|_0,$$

and then get $\nabla u \equiv 0$, which leads to $u \equiv 0$ by a similar argument as in (3.23).

(ii) Let u_m be the unique strong solution of (2.6) with b and f replaced by b_m and f_m, respectively. Then $u_m \in \mathcal{H}^{2,\infty}_{q,0,\hat{T}}$ and satisfies (3.2)–(3.3) uniformly in m. It remains to check $u_m - u \in B([0,\hat{T}] \times \mathbb{R}^d)$ and (3.5) holds.

Let $v_m = u_m - u$. Then v_m satisfies

$$\begin{cases} \partial_t v_m(t,x) = \frac{1}{2}\Delta v_m(t,x) + \nabla v_m(t,x) \cdot b_m(t,x) + F_m(t,x), \ (t,x) \in (0,\hat{T}] \times \mathbb{R}^d, \\ v_m(t,x)|_{t=0} = 0, \ x \in \mathbb{R}^d, \end{cases}$$

where $F_m(t,x) = f_m(t,x) - f(t,x) + \nabla u(t,x) \cdot (b_m(t,x) - b(t,x))$.

Observe that

$$\begin{cases} \left\|[f_m(t,\cdot) - f(t,\cdot)]_{\frac{2}{q}-1}\right\|_{L^{q,1}([0,\hat{T}])} \leqslant 2\left\|[f(t,\cdot)]_{\frac{2}{q}-1}\right\|_{L^{q,1}([0,\hat{T}])}, \\ \left\|[b_m(t,\cdot) - b(t,\cdot)]_{\frac{2}{q}-1}\right\|_{L^{q,1}([0,\hat{T}])} \leqslant 2\left\|[b(t,\cdot)]_{\frac{2}{q}-1}\right\|_{L^{q,1}([0,\hat{T}])}, \end{cases}$$

and

$$\begin{cases} \left\|\|f_m(t,\cdot) - f(t,\cdot)\|_0\right\|_{L^{q,1}([0,\hat{T}])} \leqslant \left\|[f(t,\cdot)]_{\frac{2}{q}-1}\right\|_{L^{q,1}([0,\hat{T}])} \int_{\mathbb{R}^d} |z|^{\frac{2}{q}-1} \rho_m(z)dz, \\ \left\|\|b_m(t,\cdot) - b(t,\cdot)\|_0\right\|_{L^{q,1}([0,\hat{T}])} \leqslant \left\|[b(t,\cdot)]_{\frac{2}{q}-1}\right\|_{L^{q,1}([0,\hat{T}])} \int_{\mathbb{R}^d} |z|^{\frac{2}{q}-1} \rho_m(z)dz, \end{cases}$$

then

$$f_m - f \in L^{q,1}([0,\hat{T}]; \mathcal{C}_b^{\frac{2}{q}-1}(\mathbb{R}^d)) \ \text{ and } \ b_m - b \in L^{q,1}([0,\hat{T}]; \mathcal{C}_b^{\frac{2}{q}-1}(\mathbb{R}^d; \mathbb{R}^d)).$$

Moreover, for every $0 < \epsilon < 1/q - 1/2$,

$$\lim_{m \to +\infty} \|f_m - f\|_{L^{q,1}([0,\hat{T}]; \mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))} = \lim_{m \to +\infty} \|b_m - b\|_{L^{q,1}([0,\hat{T}]; \mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))} = 0. \quad (3.26)$$

Let $\hat{x} \in \mathbb{R}^d$ and x_t^m be the unique solution of the ODE $\dot{x}_t^m = -b_m(t, \hat{x} + x_t^m)$ starting from zero. We set $\hat{v}_m(t,x) := v_m(t, x + \hat{x} + x_t^m)$, $\hat{b}_m(t,x) := b_m(t, x + \hat{x} + x_t^m) - b_m(t, \hat{x} + x_t^m)$ and $\hat{F}_m(t,x) := F_m(t, x + \hat{x} + x_t^m)$, then

$$\hat{v}_m(t,x) = \int_0^t dr \int_{\mathbb{R}^d} K(t-r, x-y)\big[\nabla \hat{v}_m(r,y) \cdot \hat{b}_m(r,y) + \hat{F}_m(r,y)\big] dy. \tag{3.27}$$

We then get an analogue of (3.19) that

$$\begin{aligned}
&|\nabla \hat{v}_m(t,0)| \\
&\leqslant \int_0^t dr \int_{\mathbb{R}^d} |\nabla K(t-r,y)| \Big\{ [b_m(r,\cdot)]_{\frac{2}{q}-1} \|\nabla v_m(r,\cdot)\|_0 [1_{|y|\leqslant 1}|y|^{\frac{2}{q}-1} + |y| 1_{|y|>1}] \\
&+ [F_m(r,\cdot)]_{\frac{2}{q}-1-2\epsilon} |y|^{\frac{2}{q}-1-2\epsilon} \Big\} dy \\
&\leqslant C \int_0^t [b(r,\cdot)]_{\frac{2}{q}-1} \|\nabla v_m(r,\cdot)\|_0 [1 + (t-r)^{\frac{1}{q}-1}] dr + C \int_0^t \Big\{ [f_m(r,\cdot) - f(r,\cdot)]_{\frac{2}{q}-1-2\epsilon} \\
&+ \|b_m(r,\cdot) - b(r,\cdot)\|_{\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}} \|\nabla u(r,\cdot)\|_{\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}} \Big\} (t-r)^{\frac{1}{q}-1-\epsilon} dr =: I_1(t) + I_2(t).
\end{aligned} \tag{3.28}$$

With the help of (2.5), Hölder's inequality, (2.2) and (2.3), we get

$$\begin{aligned}
\|I_2\|_{L^2([0,\hat{T}])} &\leqslant C \Big\| [f_m(t,\cdot) - f(t,\cdot)]_{\frac{2}{q}-1-2\epsilon} \Big\|_{L^{\hat{q}}([0,\hat{T}])} \\
&\quad + C \Big\| \|b_m(t,\cdot) - b(t,\cdot)\|_{\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}} \|\nabla u(t,\cdot)\|_{\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}} \Big\|_{L^{\hat{q}}([0,\hat{T}])} \\
&\leqslant C \|f_m - f\|_{L^q([0,\hat{T}];\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))} \\
&\quad + C \|b_m - b\|_{L^q([0,\hat{T}];\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))} \|\nabla u\|_{L^{\tilde{q}}([0,\hat{T}];\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))} \\
&\leqslant C \|f_m - f\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))} \\
&\quad + C \|b_m - b\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))} \|\nabla u\|_{L^{\tilde{q}}([0,\hat{T}];\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))},
\end{aligned} \tag{3.29}$$

where $\hat{q} = 2q/(q + 2 - 2q\epsilon)$ and $\tilde{q} = 2/(1-2\epsilon)$.

Notice that

$$\nabla u \in B([0,\hat{T}]; \mathcal{C}_b(\mathbb{R}^d;\mathbb{R}^d)) \cap L^2([0,\hat{T}]; W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)),$$

by (3.2) and (3.3), we obtain

$$
\begin{aligned}
&\|\nabla u\|_{L^{\tilde q}([0,\hat T];\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))}\\
&=\|\nabla u\|_{L^{\tilde q}([0,\hat T];\mathcal{C}_b(\mathbb{R}^d))}+\big\|[\nabla u(t,\cdot)]_{\frac{2}{q}-1-2\epsilon}\big\|_{L^{\tilde q}([0,\hat T])}\\
&\leqslant C\|\nabla u\|_{L^\infty([0,\hat T];\mathcal{C}_b(\mathbb{R}^d))}+C\|\nabla^2 u\|^{\frac{2-q-2\epsilon q}{q}}_{L^2([0,\hat T];L^\infty(\mathbb{R}^d))}\|\nabla u\|^{\frac{2q+2\epsilon q-2}{q}}_{L^\infty([0,\hat T];\mathcal{C}_b(\mathbb{R}^d))}\\
&\leqslant C\|f\|_{L^{q,1}([0,\hat T];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))}.
\end{aligned}
$$

This, together with (3.29), yields that

$$
\|I_2\|_{L^2([0,\hat T])}\leqslant C\Big[\|f_m-f\|_{L^{q,1}([0,\hat T];\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))}+\|b_m-b\|_{L^{q,1}([0,\hat T];\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))}\Big]. \tag{3.30}
$$

For I_1, we get an analogue of (3.30) that

$$
\begin{aligned}
\|I_1\|_{L^2([0,\hat T])}&\leqslant C(d,q)\|b\|_{L^{q,1}([0,\hat T];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))}\|\nabla v_m\|_{L^2([0,\hat T];\mathcal{C}_b(\mathbb{R}^d))}\\
&\leqslant \frac{1}{3}\|\nabla v_m\|_{L^2([0,\hat T];\mathcal{C}_b(\mathbb{R}^d))},
\end{aligned}\tag{3.31}
$$

where in the last inequality we have used the fact $\hat T$ is sufficiently small (see (3.21)).

Combining (3.28), (3.30) and (3.31), one arrives at

$$
\begin{aligned}
&\|\nabla v_m\|_{L^2([0,\hat T];\mathcal{C}_b(\mathbb{R}^d))}\\
&\leqslant C\Big[\|f_m-f\|_{L^{q,1}([0,\hat T];\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))}+\|b_m-b\|_{L^{q,1}([0,\hat T];\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))}\Big],
\end{aligned}\tag{3.32}
$$

which also implies

$$
\begin{aligned}
&\lim_{m\to+\infty}\|\nabla v_m\|_{L^p([0,\hat T];\mathcal{C}_b(\mathbb{R}^d))}\\
&\leqslant \lim_{m\to+\infty}\Big[\|\nabla v_m\|^{\frac{2}{p}}_{L^2([0,\hat T];\mathcal{C}_b(\mathbb{R}^d))}\|\nabla v_m\|^{1-\frac{2}{p}}_{L^\infty([0,\hat T];\mathcal{C}_b(\mathbb{R}^d))}\Big]\\
&\leqslant C\lim_{m\to+\infty}\Big[\|f_m-f\|_{L^{q,1}([0,\hat T];\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))}+\|b_m-b\|_{L^{q,1}([0,\hat T];\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))}\Big]^{\frac{2}{p}}=0
\end{aligned}\tag{3.33}
$$

for every $p>2$, where in the second line we have used an interpolation inequality for Lebesgue spaces, and in the third line we have used (3.32) and the fact that ∇u_m satisfies (3.3) uniformly in m, and in the last identity we used (3.26).

We use (3.27) again to get an upper bound for $\|v_m(t,\cdot)\|_0$ that

$$
C\int_0^t dr\int_{\mathbb{R}^d}K(t-r,y)\Big\{[b_m(r,\cdot)]_{\frac{2}{q}-1}\|\nabla v_m(r,\cdot)\|_0[1_{|y|\leqslant 1}|y|^{\frac{2}{q}-1}+|y|1_{|y|>1}]+[F_m(r,\cdot)]_0\Big\}dy.
$$

This, together with the estimate (3.32) and an interpolation inequality for Lebesgue spaces, yields that

$$\begin{aligned}\|v_m(t,\cdot)\|_0 \leqslant\, & C\Big[\|b\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))}\|\nabla v_m\|_{L^{\frac{q}{q-1}}([0,\hat{T}];\mathcal{C}_b(\mathbb{R}^d))}+\|f_m-f\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_b(\mathbb{R}^d))}\\ &+\|\nabla u\|_{L^{\frac{q}{q-1}}([0,\hat{T}];\mathcal{C}_b(\mathbb{R}^d))}\|b_m-b\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_b(\mathbb{R}^d))}\Big]\\ \leqslant\, & C\Big[\|f_m-f\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))}+\|b_m-b\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_b^{\frac{2}{q}-1-2\epsilon}(\mathbb{R}^d))}\Big]^{2-\frac{2}{q}}\\ &+C\Big[\|f_m-f\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_b(\mathbb{R}^d))}+\|b_m-b\|_{L^{q,1}([0,\hat{T}];\mathcal{C}_b(\mathbb{R}^d))}\Big],\end{aligned}$$

which suggests $v_m \in B([0,\hat{T}];\mathcal{C}_b(\mathbb{R}^d))$ and

$$\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant\hat{T}}\|v_m(t,\cdot)\|_0=0. \tag{3.34}$$

Combining (3.33) and (3.34), we complete the proof. □

4. Proof of Theorem 1.7

Let $C(d,q)$ be given in Theorem 3.1 (i). By (1.14), there exists a positive integer $k\in\mathbb{N}$, which is greater than T, such that

$$C(d,q)\sup_{0\leqslant i\leqslant k-1}\|b\|_{L^{q,1}([T_i,T_{i+1}];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))}\leqslant\frac{1}{4}, \tag{4.1}$$

where $T_i=iT/k$, $i=0,1,\dots,k$. Firstly, we assume $s\in[0,T_1)$ and prove the conclusions on $[0,T_1]$.

Consider the following vector-valued Cauchy problem:

$$\begin{cases}\partial_t U^0(t,x)+\frac{1}{2}\Delta U^0(t,x)+\nabla U^0(t,x)\cdot b(t,x)=-b(t,x), & (t,x)\in[0,T_1)\times\mathbb{R}^d,\\ U^0(t,x)|_{t=T_1}=0, \ x\in\mathbb{R}^d.\end{cases} \tag{4.2}$$

Since $b\in L^{q,1}([0,T];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d;\mathbb{R}^d))$ and (4.1) holds, in view of Theorem 3.1, there is a unique $U^0\in(\mathcal{H}^{2,\infty}_{q,0,T_1})^d$ (see (3.1)) solving the Cauchy problem (4.2), which satisfies

$$\sup_{0\leqslant t\leqslant T_1}\|\nabla U^0(t,\cdot)\|_0\leqslant\frac{1}{4} \quad\text{and}\quad \|\nabla^2U^0\|_{L^2([0,T_1];L^\infty(\mathbb{R}^d))}\leqslant\frac{1}{4}. \tag{4.3}$$

Let $\Phi^0(t,x)=x+U^0(t,x)$. By (4.3), Φ forms a diffeomorphism of class $\mathcal{C}^1$ uniformly in $t\in[0,T_1]$. In addition,

$$\frac{3}{4} \leqslant \sup_{0 \leqslant t \leqslant T_1} \|\nabla \Phi^0(t,\cdot)\|_0 \leqslant \frac{5}{4} \quad \text{and} \quad \frac{4}{5} \leqslant \sup_{0 \leqslant t \leqslant T_1} \|\nabla \Psi^0(t,\cdot)\|_0 \leqslant \frac{4}{3}, \tag{4.4}$$

where $\Psi^0(t,\cdot) = [\Phi^0(t,\cdot)]^{-1} =: \Phi^{0,-1}(t,\cdot)$.

For $0 < \varepsilon < 1$ and $t \in [0, T_1]$, define

$$U_\varepsilon^0(t,x) = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} U^0(r,x)dr = \int_0^1 U^0(t + r\varepsilon, x)dr,$$

then for every $x \in \mathbb{R}^d$, as ε tends to 0,

$$\begin{cases} U_\varepsilon^0(t,x) \longrightarrow U^0(t,x), & \forall\, t \in [0,T_1], \\ (\partial_t U_\varepsilon^0(t,x), \nabla U_\varepsilon^0(t,x), \nabla^2 U_\varepsilon^0(t,x)) & \\ \longrightarrow (\partial_t U^0(t,x), \nabla U^0(t,x), \nabla^2 U^0(t,x)), & a.e.\ t \in [0,T_1]. \end{cases} \tag{4.5}$$

We set $\Phi_\varepsilon^0(t,x) = x + U_\varepsilon^0(t,x)$, where $U^0(t,x) := U^0(T_1,x) = 0$ when $t > T_1$. Suppose $X_{s,t}^0$ is a strong solution of the following SDE:

$$dX_{s,t}^0(x) = b(t, X_{s,t}^0(x))dt + dW_t,\ t \in (s, T_1],\ X_{s,t}^0(x)|_{t=s} = x, \tag{4.6}$$

in light of Itô's formula (see [17, Theorem 3.7]), then

$$\begin{aligned} \Phi_\varepsilon^0(t, X_{s,t}^0(x)) = \Phi_\varepsilon^0(s,x) &+ \int_s^t \partial_r U_\varepsilon^0(r, X_{s,r}^0(x))dr + \frac{1}{2}\int_s^t \Delta U_\varepsilon^0(r, X_{s,r}^0(x))dr \\ &+ \int_s^t \nabla U_\varepsilon^0(r, X_{s,r}^0(x)) \cdot b(r, X_{s,r}^0(x))dr + \int_s^t b(r, X_{s,r}^0(x))dr \\ &+ \int_s^t [I + \nabla U_\varepsilon^0(r, X_{s,r}^0(x))]dW_r. \end{aligned} \tag{4.7}$$

By (4.5), (4.7) and Lebesgue's dominated convergence theorem, one has

$$\begin{aligned} &\Phi^0(t, X_{s,t}^0(x)) = \Phi^0(s,x) + \int_s^t \partial_r U^0(r, X_{s,r}^0(x))dr + \int_s^t \nabla U^0(r, X_{s,r}^0(x)) \cdot b(r, X_{s,r}^0(x))dr \\ &+ \frac{1}{2}\int_s^t \Delta U^0(r, X_{s,r}^0(x))dr + \int_s^t b(r, X_{s,r}^0(x))dr + \int_s^t [I + \nabla U^0(r, X_{s,r}^0(x))]dW_r \\ &= \Phi^0(s,x) + \int_s^t [I + \nabla U^0(r, X_{s,r}^0(x))]dW_r, \end{aligned} \tag{4.8}$$

where in the second identity we have used the fact that $U^0(t,x)$ satisfies the Cauchy problem (4.2).

Denote $Y^0_{s,t}(y) = \Phi^0(t, X^0_{s,t}(x))$, it follows from (4.8) that

$$\begin{cases} dY^0_{s,t}(y) = [I + \nabla U^0(t, \Psi^0(t, Y^0_{s,t}(y)))]dW_t =: \tilde{\sigma}^0(t, Y^0_{s,t}(y))dW_t, \quad t \in (s, T_1], \\ Y^0_{s,t}(y)|_{t=s} = y = \Phi^0(s,x). \end{cases} \tag{4.9}$$

Conversely, suppose $Y^0_{s,t}$ is a strong solution of SDE (4.9), then $X^0_{s,t} = \Psi^0(t, Y^0_{s,t})$ satisfies SDE (4.6). Therefore, SDEs (4.6) and (4.9) are equivalent. We first show the conclusions for SDE (4.9).

By the regularity of U^0, we have $\tilde{\sigma}^0 \in L^2([0,T_1]; W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^{d\times d}))$. Let $\{Z_{s,t}(y)\}_{s\leqslant t\leqslant T_1}$ be a continuous $\{\mathcal{F}_{s,t}\}_{s\leqslant t\leqslant T_1}$-adapted process starting from $y \in \mathbb{R}^d$. We define a mapping $\mathcal{S}$ by

$$\mathcal{S}(Z_{s,t}(y)) = y + \int_s^t \tilde{\sigma}^0(r, Z_{s,r}(y))dW_r.$$

Then $\{\mathcal{S}(Z_{s,t}(y))\}_{s\leqslant t\leqslant T_1}$ is also a continuous $\{\mathcal{F}_{s,t}\}_{s\leqslant t\leqslant T_1}$-adapted process starting from $y \in \mathbb{R}^d$. Moreover, for such processes $Z^1_{s,t}(y)$ and $Z^2_{s,t}(y)$, we have

$$\begin{aligned}
&\mathbb{E}\sup_{s\leqslant t\leqslant T_1}|\mathcal{S}(Z^1_{s,t}(y)) - \mathcal{S}(Z^2_{s,t}(y))|^2 \\
&= \mathbb{E}\sup_{s\leqslant t\leqslant T_1}\left|\int_s^t [\tilde{\sigma}^0(r, Z^1_{s,r}(y)) - \tilde{\sigma}^0(r, Z^2_{s,r}(y))]dW_r\right|^2 \\
&\leqslant 4\mathbb{E}\int_s^{T_1} \|\tilde{\sigma}^0(r, Z^1_{s,r}(y)) - \tilde{\sigma}^0(r, Z^2_{s,r}(y))\|^2 dr \\
&\leqslant 4\mathbb{E}\sup_{s\leqslant t\leqslant T_1}|Z^1_{s,t}(y) - Z^2_{s,t}(y)|^2 \|\nabla^2 U^0\|^2_{L^2([0,T_1];L^\infty(\mathbb{R}^d))} \sup_{s\leqslant r\leqslant T_1}\|\nabla\Psi^0(r,\cdot)\|_0^2 \\
&\leqslant \frac{4}{9}\mathbb{E}\sup_{s\leqslant t\leqslant T_1}|Z^1_{s,t}(y) - Z^2_{s,t}(y)|^2,
\end{aligned}$$

where in the first inequality we have used Doob's inequality and in the last inequality we used (4.3) and (4.4).

By Banach's contraction mapping principle, there exists a unique strong solution $Y^0_{s,t}$ to (4.9), which also belongs to $L^2(\Omega; \mathcal{C}([s,T_1]))$. On the other hand, $\tilde{\sigma}^0 \in L^\infty([0,T_1]\times\mathbb{R}^d; \mathbb{R}^{d\times d})$, by virtue of Itô's formula, we get

$$d|Y^0_{s,t}(y)|^p \leqslant C[1 + |Y^0_{s,t}(y)|^p]dt + p|Y^0_{s,t}(y)|^{p-2}\langle Y^0_{s,t}(y), \tilde{\sigma}^0(t, Y^0_{s,t}(y))dW_t\rangle, \quad \forall\, p \geqslant 2.$$

Since for every $t > s$,

$$\int_s^t |Y^0_{s,r}(y)|^{p-2}\langle Y^0_{s,r}(y), \tilde{\sigma}^0(r, Y^0_{s,r}(y))dW_r\rangle$$

is a martingale, by Grönwall's inequality, one has

$$\sup_{s\leqslant t\leqslant T_1} \mathbb{E}|Y^0_{s,t}(y)|^p \leqslant C(1+|y|^p). \tag{4.10}$$

Next, we verify the homeomorphisms. Applying the Itô formula again, for every $\xi < 0$, we derive

$$\begin{aligned}
d\big[1+|Y^0_{s,t}(y)|^2\big]^{\frac{\xi}{2}} &\leqslant \frac{\xi}{2}\big[1+|Y^0_{s,t}(y)|^2\big]^{\frac{\xi}{2}-1}\big[2\langle Y^0_{s,t}(y), \tilde{\sigma}(t, Y^0_{s,t}(y))dW_t\rangle + \|\tilde{\sigma}^0(t, Y^0_{s,t}(y))\|^2 dt\big] \\
&\quad + \frac{\xi(\xi-2)}{2}\big[1+|Y^0_{s,t}(y)|^2\big]^{\frac{\xi}{2}-2}|Y^0_{s,t}(y)|^2\|\tilde{\sigma}^0(t, Y^0_{s,t}(y))\|^2 dt \\
&\leqslant \xi\big[1+|Y^0_{s,t}(y)|^2\big]^{\frac{\xi}{2}-1}\langle Y^0_{s,t}(y), \tilde{\sigma}(t, Y^0_{s,t}(y))dW_t\rangle \\
&\quad + \frac{\xi(\xi-2)}{2}\big[1+|Y^0_{s,t}(y)|^2\big]^{\frac{\xi}{2}}\|\tilde{\sigma}^0\|^2_{L^\infty([0,T_1]\times\mathbb{R}^d)}dt.
\end{aligned}$$

By using Grönwall's inequality again, we obtain

$$\sup_{s\leqslant t\leqslant T_1} \mathbb{E}[1+|Y^0_{s,t}(y)|^2\big]^{\frac{\xi}{2}} \leqslant C(1+|y|^2)^{\frac{\xi}{2}}, \quad \forall\, \xi < 0.$$

This, together with the elementary inequalities $(1+a)^\xi \leqslant (1+a^2)^{\xi/2} \leqslant 2^{-\xi/2}(1+a)^\xi$ ($\forall\, a > 0$ and $\xi < 0$), implies

$$\sup_{s\leqslant t\leqslant T_1} \mathbb{E}(1+|Y^0_{s,t}(y)|)^\xi \leqslant C(1+|y|)^\xi, \quad \forall\, \xi < 0.$$

Thanks to [24, Lemmas II.2.4, II.4.1 and II.4.2], it is sufficient to prove that for every $y, y' \in \mathbb{R}^d$ ($y \neq y'$) and every $s, t, s', t' \in [0, T_1]$ ($s < t,\ s' < t'$),

$$\sup_{s\leqslant t\leqslant T_1} \mathbb{E}|Y^0_{s,t}(y) - Y^0_{s,t}(y')|^{2\zeta} \leqslant C|y-y'|^{2\zeta}, \quad \forall\, \zeta < 0, \tag{4.11}$$

and

$$\mathbb{E}|Y^0_{s,t}(y) - Y^0_{s',t'}(y')|^p \leqslant C\Big[|y-y'|^p + |s-s'|^{\frac{p}{2}} + |t-t'|^{\frac{p}{2}}\Big], \quad \forall\, p \geqslant 2. \tag{4.12}$$

For $\mu > 0$, we choose $f_\mu(x) = (\mu + |x|^2)$ and set $Y^0_{s,t}(y, y') := Y^0_{s,t}(y) - Y^0_{s,t}(y')$. In view of Itô's formula, for every $\zeta < 0$, then

$$f_\mu^\zeta(Y_{s,t}^0(y,y')) \leqslant 2\zeta \int_s^t f_\mu^{\zeta-1}(Y_{s,r}^0(y,y'))\langle Y_{s,r}^0(y,y'), (\tilde{\sigma}^0(r, Y_{s,r}^0(y)) - \tilde{\sigma}^0(r, Y_{s,r}^0(y')))dW_r\rangle$$
$$+ f_\mu^\zeta(y-y') + C|\zeta(\zeta-1)| \int_s^t \kappa_0^2(r) f_\mu^\zeta(Y_{s,r}^0(y,y'))dr, \tag{4.13}$$

where $\kappa_0(r) := \|\tilde{\sigma}^0(r,\cdot)\|_{L^\infty(\mathbb{R}^d)} \in L^2([0,T_1])$ for $\tilde{\sigma}^0 = \nabla U^0 \in L^2([0,T_1]; W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^{d\times d}))$.

Due to the Grönwall inequality, we obtain from (4.13) that

$$\sup_{s\leqslant t\leqslant T_1} \mathbb{E}\big[\mu + |Y_{s,t}^0(y) - Y_{s,t}^0(y')|^2\big]^\zeta \leqslant C\big[\mu + |y-y'|^2\big]^\zeta.$$

Then (4.11) follows by letting $\mu \downarrow 0$.

To prove (4.12), we assume without loss of generality that $s < s' < t < t'$, then

$$Y_{s,t}^0(y) - Y_{s',t'}^0(y') = y - y' + \int_s^{s'} \tilde{\sigma}^0(r, Y_{s,r}^0(y))dW_r + \int_t^{t'} \tilde{\sigma}^0(r, Y_{s',r}^0(y'))dW_r$$
$$+ \int_{s'}^t \big[\tilde{\sigma}^0(r, Y_{s,r}^0(y)) - \tilde{\sigma}^0(r, Y_{s',r}^0(y))\big]dW_r + \int_{s'}^t \big[\tilde{\sigma}^0(r, Y_{s',r}^0(y)) - \tilde{\sigma}^0(r, Y_{s',r}^0(y'))\big]dW_r.$$

On account of the Burkholder-Davis-Gundy (BDG for short) inequality,

$$\begin{aligned}
&\mathbb{E}|Y_{s,t}^0(y) - Y_{s',t'}^0(y')|^p \\
&\leqslant C|y-y'|^p + C\mathbb{E}\left[\int_s^{s'} \|\tilde{\sigma}^0(r, Y_{s,r}^0(y))\|^2 dr\right]^{\frac{p}{2}} + C\mathbb{E}\left[\int_t^{t'} \|\tilde{\sigma}^0(r, Y_{s',r}^0(y'))\|^2 dr\right]^{\frac{p}{2}} \\
&\quad + C\mathbb{E}\left[\int_{s'}^t \kappa_0^2(r)|Y_{s,r}^0(y) - Y_{s',r}^0(y)|^2 dr\right]^{\frac{p}{2}} + C\mathbb{E}\left[\int_{s'}^t \kappa_0^2(r)|Y_{s',r}^0(y) - Y_{s',r}^0(y')|^2 dr\right]^{\frac{p}{2}} \\
&\leqslant C\big[|y-y'|^p + |s-s'|^{\frac{p}{2}} + |t-t'|^{\frac{p}{2}}\big] \\
&\quad + C\mathbb{E}\sup_{s'\leqslant r\leqslant T_1} |Y_{s,r}^0(y) - Y_{s',r}^0(y)|^p + C\mathbb{E}\sup_{s'\leqslant r\leqslant T_1} |Y_{s',r}^0(y) - Y_{s',r}^0(y')|^p.
\end{aligned} \tag{4.14}$$

We use Itô's formula to $|Y_{s,r}^0(y) - Y_{s',r}(y)|^{2p}$ first, Grönwall's inequality next, to get

$$\begin{aligned}
&\sup_{s'\leqslant r\leqslant T_1} \mathbb{E}|Y_{s,r}^0(y) - Y_{s',r}^0(y)|^{2p} \\
&\leqslant C\mathbb{E}|Y_{s,s'}^0(y) - y|^{2p} = C\mathbb{E}\left|\int_s^{s'} \tilde{\sigma}^0(\tau, Y_{s,\tau}^0(y))dW_\tau\right|^{2p} \leqslant C|s-s'|^p.
\end{aligned}$$

The Itô formula is again used, leading to

$$\begin{aligned}
&\mathbb{E}\sup_{s'\leqslant r\leqslant T_1}|Y^0_{s,r}(y)-Y^0_{s',r}(y)|^p\\
&\leqslant \mathbb{E}|Y^0_{s,s'}(y)-y|^p+p\mathbb{E}\sup_{s'\leqslant r\leqslant T_1}\int_{s'}^{r}|Y^0_{s,r}(y)-Y^0_{s',r}(y)|^{p-2}\langle Y^0_{s,r}(y)-Y^0_{s',r}(y),\\
&\quad[\tilde\sigma^0(r,Y^0_{s,r}(y))-\tilde\sigma^0(r,Y^0_{s',r}(y))]dW_r\rangle\\
&\quad+\frac{p(p-1)}{2}\mathbb{E}\int_{s'}^{T_1}|Y^0_{s,r}(y)-Y^0_{s',r}(y)|^{p-2}\|\tilde\sigma^0(r,Y^0_{s,r}(y))-\tilde\sigma^0(r,Y^0_{s',r}(y))\|^2dr.
\end{aligned}$$

By BDG's inequality, one gets

$$\begin{aligned}
\mathbb{E}\sup_{s'\leqslant r\leqslant T_1}|Y^0_{s,r}(y)-Y^0_{s',r}(y)|^p&\leqslant C|s-s'|^{\frac{p}{2}}+C\mathbb{E}\Bigg[\int_{s'}^{T_1}\kappa_0^2(r)|Y^0_{s,r}(y)-Y^0_{s',r}(y)|^{2p}dr\Bigg]^{\frac12}\\
&\quad+C\sup_{s'\leqslant r\leqslant T_1}\mathbb{E}|Y^0_{s,r}(y)-Y^0_{s',r}(y)|^p\leqslant C|s-s'|^{\frac{p}{2}}. \qquad (4.15)
\end{aligned}$$

Similarly, by applying the Itô formula to $|Y_{s',r}(y)-Y_{s',r}(y')|^{2p}$, it follows that

$$\sup_{s'\leqslant r\leqslant T_1}\mathbb{E}|Y^0_{s',r}(y)-Y^0_{s',r}(y')|^{2p}\leqslant C|y-y'|^{2p}.$$

This, together with the BDG and Grönwall inequalities, leads to

$$\mathbb{E}\sup_{s'\leqslant r\leqslant T_1}|Y_{s',r}(y)-Y_{s',r}(y')|^p\leqslant C|y-y'|^p. \qquad (4.16)$$

Summing over (4.14)–(4.16), we obtain (4.12). Thus $Y^0_{s,t}(\cdot)$ forms a homeomorphism. Since $Y^0_{s,t}$ satisfies equation (4.9), then

$$Y^0_{s,t}(Y^{0,-1}_{s,t}(y))=Y^{0,-1}_{s,t}(y)+\int_s^t\tilde\sigma^0(r,Y^0_{s,r}(Y^{0,-1}_{s,t}(y)))dW_r,$$

where $Y^{0,-1}_{s,t}(y):=(Y^0_{s,t}(\cdot))^{-1}(y)$.

Noting that $Y^0_{s,r}(Y^{0,-1}_{s,t}(y))=Y^{0,-1}_{r,t}(y)$, one gets

$$Y^{0,-1}_{s,t}(y)=y-\int_s^t\tilde\sigma^0(r,Y^{0,-1}_{r,t}(y))dW_r. \qquad (4.17)$$

We can get an analogue of (4.12) for $Y_{s,t}^{0,-1}(y)$ once taken into account the backward character of the equation (4.17). In view of the Kolmogorov-Chentsov continuity criterion, $Y_{s,t}^{0}(y)$ and $Y_{s,t}^{0,-1}(y)$ are continuous in (s,t,y) almost surely in ω, and $\{Y_{s,t}^{0}(y),\ 0 \leqslant s \leqslant t \leqslant T_1,\ y \in \mathbb{R}^d\}$ forms a stochastic flow of homeomorphisms. Therefore, there is a unique stochastic flow of homeomorphisms $\{X_{s,t}^{0}(x),\ 0 \leqslant s \leqslant t \leqslant T_1,\ x \in \mathbb{R}^d\}$ to SDE (4.6).

We then consider the vector-valued Cauchy problem on $[T_1, T_2]$:

$$\begin{cases} \partial_t U^1(t,x) + \frac{1}{2}\Delta U^1(t,x) + \nabla U^1(t,x)\cdot b(t,x) = -b(t,x), & (t,x) \in [T_1,T_2)\times\mathbb{R}^d, \\ U^1(t,x)|_{t=T_2} = 0, \ x \in \mathbb{R}^d. \end{cases} \tag{4.18}$$

With the aid of Theorem 3.1, there is a unique $U^1 \in (\mathcal{H}_{q,T_1,T_2}^{2,\infty})^d$ solving the Cauchy problem (4.18). If one sets $\Phi^1(t,x) = x + U^1(t,x)$, then Φ^1 forms a diffeomorphism of class $\mathcal{C}^1$ uniformly in $t \in [T_1, T_2]$. Moreover, if $X_{T_1,t}^1$ solves the following SDE:

$$dX_{T_1,t}^1(x) = b(t, X_{T_1,t}^1(x))dt + dW_t,\ t \in (T_1,T_2],\ X_{T_1,t}^1(x)|_{t=T_1} = x \in \mathbb{R}^d, \tag{4.19}$$

then $Y_{T_1,t}^1(y) = \Phi^1(t, X_{T_1,t}^1(x))$ satisfies

$$\begin{cases} dY_{T_1,t}^1(y) = [I + \nabla U^1(t, \Psi^1(t, Y_{T_1,t}^1(y)))]dW_t =: \tilde{\sigma}^1(t, Y_{T_1,t}^1(y))dW_t,\ t \in (T_1,T_2], \\ Y_{T_1,t}^1(y)|_{t=T_1} = y = \Phi^1(T_1,x), \end{cases} \tag{4.20}$$

and vice versa, where $\Psi^1(t,\cdot) = [\Phi^1(t,\cdot)]^{-1}$.

On the other hand, by the regularity of $\tilde{\sigma}^1$, there exists a unique stochastic flow of homeomorphisms $\{Y_{T_1,t}^1(y),\ T_1 \leqslant t \leqslant T_2,\ y \in \mathbb{R}^d\}$ to SDE (4.20), thus there is a unique stochastic flow of homeomorphisms $\{X_{T_1,t}^1(x),\ T_1 \leqslant t \leqslant T_2,\ x \in \mathbb{R}^d\}$ to SDE (4.19). We define $Y_{s,t}(y)$ and $X_{s,t}(x)$ for $t \in (s, T_2]$ by

$$Y_{s,t}(y) = \begin{cases} Y_{s,t}^0(y), & \text{if } t \in [s,T_1], \\ Y_{T_1,t}^1(Y_{s,T_1}^0(y)), & \text{if } t \in [T_1,T_2], \end{cases}$$

and

$$X_{s,t}(x) = \begin{cases} X_{s,t}^0(x), & \text{if } t \in [s,T_1], \\ X_{T_1,t}^1(X_{s,T_1}^0(x)), & \text{if } t \in [T_1,T_2], \end{cases}$$

respectively, then $Y_{s,t}(y)$ and $X_{s,t}(x)$ solve

$$dY_{s,t}(y) = \tilde{\sigma}(t, Y_{s,t}(y))dW_t,\ t \in (s,T_2],\ Y_{s,t}(y)|_{t=s} = y = \Phi^0(s,x), \tag{4.21}$$

and

$$dX_{s,t}(x) = b(t, X_{s,t}(x))dt + dW_t,\ t \in (s,T_2],\ X_{s,t}(x)|_{t=s} = x, \tag{4.22}$$

respectively, where

$$\tilde{\sigma}(t,y)=\begin{cases}\tilde{\sigma}^0(t,y), & \text{if } t\in[s,T_1],\\ \tilde{\sigma}^1(t,y), & \text{if } t\in(T_1,T_2].\end{cases}$$

Moreover,

$$\{Y_{s,t}(y),\ 0\leqslant s\leqslant T_1,\ s\leqslant t\leqslant T_2,\ y\in\mathbb{R}^d\} \text{ and } \{X_{s,t}(x),\ 0\leqslant s\leqslant T_1,\ s\leqslant t\leqslant T_2,\ x\in\mathbb{R}^d\}$$

form the unique stochastic flow of homeomorphisms to SDEs (4.21) and (4.22), respectively. We then repeat the preceding arguments to extend the solution to the time interval $[T_2,T_3]$. Continuing this procedure with finitely many steps, we construct a unique stochastic flow of homeomorphisms $\{Y_{s,t}(y),\ 0\leqslant s\leqslant T_1,\ s\leqslant t\leqslant T,\ y\in\mathbb{R}^d\}$ to

$$dY_{s,t}(y)=\tilde{\sigma}(t,Y_{s,t}(y))dW_t,\ t\in(s,T],\ Y_{s,t}(y)|_{t=s}=y=\Phi^0(s,x), \tag{4.23}$$

and a unique stochastic flow of homeomorphisms $\{X_{s,t}(x),\ 0\leqslant s\leqslant T_1,\ s\leqslant t\leqslant T,\ x\in\mathbb{R}^d\}$ to SDE (1.1), where $\tilde{\sigma}(t,x)=I+\nabla U(t,\Psi(t,x))$, $\Psi(t,x)=\Phi^{-1}(t,x)$, $\Phi(t,x)=x+U(t,x)$ and

$$U(t,x)=\begin{cases}U^0(t,x), & \text{if } t\in[0,T_1],\\ U^i(t,x), & \text{if } t\in(T_i,T_{i+1}],\quad i=1,2,\dots,k-1,\end{cases} \tag{4.24}$$

with $U^i(t,x)$, $i=0,1,\dots,k-1$, satisfying

$$\begin{cases}\partial_t U^i(t,x)+\frac{1}{2}\Delta U^i(t,x)+\nabla U^i(t,x)\cdot b(t,x)=-b(t,x), & (t,x)\in[T_i,T_{i+1})\times\mathbb{R}^d,\\ U^i(t,x)|_{t=T_{i+1}}=0,\ x\in\mathbb{R}^d.\end{cases} \tag{4.25}$$

For general $T_{i-1}\leqslant s<T_i$ with $1\leqslant i\leqslant k$, we define

$$Y_{s,t}(y)=\begin{cases}Y^{i-1}_{s,t}(y), & \text{if } t\in[s,T_i],\\ Y^i_{T_i,t}(\cdot)\circ Y^{i-1}_{s,T_i}(\cdot)(y), & \text{if } t\in[T_i,T_{i+1}],\\ Y^j_{T_j,t}(\cdot)\circ Y^{j-1}_{T_{j-1},T_j}(\cdot)\circ\cdots\circ Y^{i-1}_{s,T_i}(\cdot)(y), & \text{if } t\in[T_j,T_{j+1}] \text{ and } i<j\leqslant k-1,\end{cases}$$

and

$$X_{s,t}(x)=\begin{cases}X^{i-1}_{s,t}(x), & \text{if } t\in[s,T_i],\\ X^i_{T_i,t}(\cdot)\circ X^{i-1}_{s,T_i}(\cdot)(x), & \text{if } t\in[T_i,T_{i+1}],\\ X^j_{T_j,t}(\cdot)\circ X^{j-1}_{T_{j-1},T_j}(\cdot)\circ\cdots\circ X^{i-1}_{s,T_i}(\cdot)(x), & \text{if } t\in[T_j,T_{j+1}] \text{ and } i<j\leqslant k-1,\end{cases}$$

respectively, then

$$\{Y_{s,t}(y),\ 0\leqslant s\leqslant t\leqslant T,\ y\in\mathbb{R}^d\} \text{ and } \{X_{s,t}(x),\ 0\leqslant s\leqslant t\leqslant T,\ x\in\mathbb{R}^d\}$$

form the unique stochastic flow of homeomorphisms to (4.23) and (1.1), respectively.

We now turn to show the weak differentiability for $X_{s,t}(\cdot)$ and $X_{s,t}^{-1}(\cdot)$, and the estimate (1.15). From the observation that U defined by (4.24) satisfies (4.25), and (4.1) holds, we can get an analogue of (4.4) for Φ and Ψ on $[T_i, T_{i+1}]$. Since the estimates are independent of i ($i = 0, 1, \dots, k-1$), it leads to

$$\frac{3}{4} \leqslant \sup_{0 \leqslant t \leqslant T} \|\nabla \Phi(t, \cdot)\|_0 \leqslant \frac{5}{4} \quad \text{and} \quad \frac{4}{5} \leqslant \sup_{0 \leqslant t \leqslant T} \|\nabla \Psi(t, \cdot)\|_0 \leqslant \frac{4}{3}, \tag{4.26}$$

where $\Psi(t, \cdot) = \Phi^{-1}(t, \cdot)$. Invoking the following fact

$$\begin{cases} \nabla X_{s,t}(x) = \nabla \Psi(t, Y_{s,t}(\Psi(s, x))) \nabla Y_{s,t}(\Psi(s, x)) \nabla \Psi(s, x), \\ \nabla X_{s,t}^{-1}(x) = \nabla \Psi(t, Y_{s,t}^{-1}(\Psi(s, x))) \nabla Y_{s,t}^{-1}(\Psi(s, x)) \nabla \Psi(s, x), \end{cases}$$

it suffices to show the weak differentiability for $Y_{s,t}(\cdot)$ and $Y_{s,t}^{-1}(\cdot)$. The inverse flow $Y_{s,t}^{-1}$ is observed to satisfy a new SDE, which has the same form as the original one (only the diffusion has an opposite sign), the proof of the weak differentiability for $Y_{s,t}^{-1}$ is similar to that of $Y_{s,t}$ after taking into consideration the backward character of the equation. We only show the conclusions for $Y_{s,t}$.

Let b_m be given in (3.4). For $i = 0, 1, \dots, k-1$, let U_m^i be the unique strong solution of (4.25) with b replaced by b_m. In view of Theorem 3.1 (ii), $U_m^i \in (\mathcal{H}^{2,\infty}_{q,T_i,T_{i+1}})^d$. Moreover, by Theorem 3.1, (4.1) and the following fact

$$\begin{cases} \left\| [b_m(t, \cdot)]_{\frac{2}{q}-1} \right\|_{L^{q,1}([T_i, T_{i+1}])} \leqslant \left\| [b_m(t, \cdot)]_{\frac{2}{q}-1} \right\|_{L^{q,1}([T_i, T_{i+1}])}, \\ \|(1 + |\cdot|)^{-1} b_m(t, \cdot)\|_{L^{q,1}([T_i, T_{i+1}]; \mathcal{C}_b(\mathbb{R}^d))} \leqslant 2 \|(1 + |\cdot|)^{-1} b(t, \cdot)\|_{L^{q,1}([T_i, T_{i+1}]; \mathcal{C}_b(\mathbb{R}^d))}, \end{cases}$$

we obtain

$$\begin{cases} \sup\limits_{m \geqslant 1} \sup\limits_{T_i \leqslant t \leqslant T_{i+1}} \|\nabla U_m^i(t, \cdot)\|_0 \leqslant \frac{1}{4}, \\ \sup\limits_{m \geqslant 1} \|\nabla^2 U_m^i\|_{L^2([T_i, T_{i+1}]; L^\infty(\mathbb{R}^d))} \leqslant C \|b\|_{L^{q,1}([T_i, T_{i+1}]; \mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))}, \end{cases} \tag{4.27}$$

and

$$\begin{cases} \lim\limits_{m \to +\infty} \sup\limits_{T_i \leqslant t \leqslant T_{i+1}} \|U_m^i(t, \cdot) - U^i(t, \cdot)\|_0 = 0, \\ \lim\limits_{m \to +\infty} \|\nabla U_m^i - \nabla U^i\|_{L^p([T_i, T_{i+1}]; \mathcal{C}_b(\mathbb{R}^d))} = 0, \quad \forall\, p \geqslant 2. \end{cases} \tag{4.28}$$

We define

$$U_m(t, x) = \begin{cases} U_m^0(t, x), & \text{if } t \in [0, T_1], \\ U_m^i(t, x), & \text{if } t \in (T_i, T_{i+1}], \quad i = 1, 2, \dots, k-1. \end{cases}$$

By (4.27) and (4.28),

$$\begin{cases} \sup\limits_{m\geqslant 1}\sup\limits_{0\leqslant t\leqslant T}\|\nabla U_m(t,\cdot)\|_0\leqslant \frac{1}{4}, \\ \sup\limits_{m\geqslant 1}\|\nabla^2 U_m\|_{L^2([0,T];L^\infty(\mathbb{R}^d))}\leqslant C\|b\|_{L^{q,1}([0,T];\mathcal{C}_l^{\frac{2}{q}-1}(\mathbb{R}^d))}, \end{cases} \tag{4.29}$$

and

$$\lim_{m\to+\infty}\Big[\sup_{0\leqslant t\leqslant T}\|U_m(t,\cdot)-U(t,\cdot)\|_0+\|\nabla U_m-\nabla U\|_{L^p([0,T];\mathcal{C}_b(\mathbb{R}^d))}\Big]=0,\quad \forall\ p\geqslant 2. \tag{4.30}$$

We set $\Phi_m(t,x)=x+U_m(t,x)$, then $\Phi_m(t,\cdot)$ forms a diffeomorphism of class $\mathcal{C}^1$ uniformly in $t\in[0,T]$. Furthermore, Φ_m and its inverse Ψ_m satisfy (4.26) uniformly in m. Let $X^m_{s,t}(x)$ be the unique strong solution of the following SDE:

$$dX^m_{s,t}(x)=b_m(t,X^m_{s,t}(x))dt+dW_t,\ t\in(s,T],\ X^m_{s,t}(x)|_{t=s}=x\in\mathbb{R}^d. \tag{4.31}$$

Then $X^m_{s,t}(x)$ is weakly differentiable in x. If one defines $Y^m_{s,t}(y)=\Phi_m(t,X^m_{s,t}(x))$, then $Y^m_{s,t}(\cdot)$ is weakly differentiable and satisfies

$$\begin{cases} dY^m_{s,t}(y)=[I+\nabla U_m(t,\Psi_m(t,Y^m_{s,t}(y)))]dW_t=:\tilde\sigma_m(t,Y^m_{s,t}(y))dW_t,\quad t\in(s,T], \\ Y^m_{s,t}(y)|_{t=s}=y=\Phi_m(s,x). \end{cases} \tag{4.32}$$

Additionally, SDEs (4.31) and (4.32) are equivalent.

For every $p\geqslant 2$, by using Itô's formula to $|Y^m_{s,t}(y)-Y_{s,t}(y)|^p$, we obtain

$$\begin{aligned} & d|Y^m_{s,t}(y)-Y_{s,t}(y)|^p \\ =& \frac{p(p-1)}{2}|Y^m_{s,t}(y)-Y_{s,t}(y)|^{p-2}\|\tilde\sigma_m(t,Y^m_{s,t}(y))-\tilde\sigma(t,Y_{s,t}(y))\|^2dt \\ & +p|Y^m_{s,t}(y)-Y_{s,t}(y)|^{p-2}\langle Y^m_{s,t}(y)-Y_{s,t}(y),[\tilde\sigma_m(t,Y^m_{s,t}(y))-\tilde\sigma(t,Y_{s,t}(y))]dW_t\rangle. \end{aligned} \tag{4.33}$$

Invoking the fact that

$$\begin{aligned} \tilde\sigma_m(t,Y^m_{s,t}(y))-\tilde\sigma(t,Y_{s,t}(y)) =& \nabla U_m(t,\Psi_m(t,Y^m_{s,t}(y)))-\nabla U(t,\Psi_m(t,Y^m_{s,t}(y))) \\ & +\nabla U(t,\Psi_m(t,Y^m_{s,t}(y)))-\nabla U(t,\Psi(t,Y^m_{s,t}(y))) \\ & +\nabla U(t,\Psi(t,Y^m_{s,t}(y)))-\nabla U(t,\Psi(t,Y_{s,t}(y))), \end{aligned}$$

we obtain

$$\begin{aligned} & \|\tilde\sigma_m(t,Y^m_{s,t}(y))-\tilde\sigma(t,Y_{s,t}(y))\| \\ \leqslant & \|\nabla U_m(t,\cdot)-\nabla U(t,\cdot)\|_0+\|\nabla^2 U(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}\Big[\|\Psi_m(t,\cdot)-\Psi(t,\cdot)\|_0 \\ & +\|\nabla\Psi\|_{L^\infty([0,T]\times\mathbb{R}^d)}|Y^m_{s,t}(y)-Y_{s,t}(y)|\Big], \end{aligned}$$

and then guarantee

$$
\begin{aligned}
&|Y^m_{s,t}(y) - Y_{s,t}(y)|^{p-2}\|\tilde{\sigma}_m(t, Y^m_{s,t}(y)) - \tilde{\sigma}(t, Y_{s,t}(y))\|^2 \\
&\leqslant C\kappa^2(t)|Y^m_{s,t}(y) - Y_{s,t}(y)|^p + |Y^m_{s,t}(y) - Y_{s,t}(y)|^{p-2}\Big[\|\nabla U_m(t,\cdot) - \nabla U(t,\cdot)\|_0^2 \\
&\quad +\kappa^2(t)\|\Psi_m(t,\cdot) - \Psi(t,\cdot)\|_0^2\Big] \\
&\leqslant C[1+\kappa^2(t)]|Y^m_{s,t}(y) - Y_{s,t}(y)|^p + C\Big[\|\nabla U_m(t,\cdot) - \nabla U(t,\cdot)\|_0^p \\
&\quad +\kappa^2(t)\|\Psi_m(t,\cdot) - \Psi(t,\cdot)\|_0^p\Big],
\end{aligned} \tag{4.34}
$$

where $\kappa(t) := \|\nabla^2 U(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \in L^2([0,T])$, and in the second inequality of (4.34) we have used the Young inequality

$$
a_1 b_1 \leqslant \frac{a_1^{\hat{p}}}{\hat{p}} + \frac{b_1^{\hat{p}'}}{\hat{p}'}, \quad \forall\, a_1, b_1 \geqslant 0,\ \hat{p} > 1 \text{ and } \frac{1}{\hat{p}} + \frac{1}{\hat{p}'} = 1.
$$

By (4.33) and (4.34), we guarantee

$$
\begin{aligned}
&\mathbb{E}|Y^m_{s,t}(y) - Y_{s,t}(y)|^p \leqslant C(p)\int_s^t [1+\kappa^2(r)]|Y^m_{s,r}(y) - Y_{s,r}(y)|^p dr \\
&+C(p)\int_s^t \Big[\|\nabla U_m(r,\cdot) - \nabla U(r,\cdot)\|_0^p + \kappa^2(r)\|\Psi_m(r,\cdot) - \Psi(r,\cdot)\|_0^p\Big]dr,
\end{aligned}
$$

and then get

$$
\begin{aligned}
&\sup_{y\in\mathbb{R}^d}\sup_{s\leqslant t\leqslant T} \mathbb{E}|Y^m_{s,t}(y) - Y_{s,t}(y)|^p \\
&\leqslant C(p)\int_s^T \Big[\|\nabla U_m(r,\cdot) - \nabla U(r,\cdot)\|_0^p + \kappa^2(r)\|\Psi_m(r,\cdot) - \Psi(r,\cdot)\|_0^p\Big]dr,
\end{aligned} \tag{4.35}
$$

if one uses Grönwall's inequality.

Observing that Φ_m and its inverse Ψ_m satisfy (4.26) uniformly in m, one achieves

$$
\begin{aligned}
|\Phi(r,\Psi(r,y)) - \Phi_m(r,\Psi(r,y))| &= |\Phi_m(r,\Psi_m(r,y)) - \Phi_m(r,\Psi(r,y))| \\
&\geqslant \frac{3}{4}|\Psi_m(r,y) - \Psi(r,y)|,
\end{aligned}
$$

which also suggests that

$$
\|\Psi_m(r,\cdot) - \Psi(r,\cdot)\|_0 \leqslant \frac{4}{3}\|\Phi(r,\cdot) - \Phi_m(r,\cdot)\|_0 = \frac{4}{3}\|U_m(r,\cdot) - U(r,\cdot)\|_0. \tag{4.36}
$$

In consequence of (4.30) and (4.36), we conclude from (4.35) that

$$\lim_{m\to+\infty}\sup_{y\in\mathbb{R}^d}\sup_{s\leqslant t\leqslant T}\mathbb{E}|Y^m_{s,t}(y)-Y_{s,t}(y)|^p=0. \tag{4.37}$$

Notice that $Y^m_{s,t}(y)$ is weakly differentiable in y, if one differentiates $Y^m_{s,t}(y)$ with respect to the initial data and denotes the derivative by $\xi^m_{s,t}(y)$, then

$$\begin{cases} d\xi^m_{s,t}(y)=\nabla^2U_m(t,\Psi_m(t,Y^m_{s,t}(y)))\nabla\Psi_m(t,Y^m_{s,t}(y))\xi^m_{s,t}(y)dW_t, \quad t\in(s,T],\\ \xi^m_{s,t}(y)|_{t=s}=I_{d\times d}. \end{cases} \tag{4.38}$$

For every $p\geqslant 2$, by applying the Itô formula to $\|\xi^m_{s,t}(y)\|^p$, we achieve

$$\begin{aligned} &d\|\xi^m_{s,t}(y)\|^p\\ \leqslant\, &C(p)\|\xi^m_{s,t}(y)\|^{p-2}\langle\xi^m_{s,t}(y),\nabla^2U_m(t,\Psi_m(t,Y^m_{s,t}(y)))\nabla\Psi_m(t,Y^m_{s,t}(y))\xi^m_{s,t}(y)dW_t\rangle\\ &+C(p)\|\xi^m_{s,t}(y)\|^p\|\nabla^2U_m(t,\cdot)\|^2_{L^\infty(\mathbb{R}^d)}dt. \end{aligned} \tag{4.39}$$

By (4.29), (4.39) and Grönwall's inequality, then

$$\sup_{m\geqslant 1}\sup_{y\in\mathbb{R}^d}\sup_{0\leqslant s\leqslant t\leqslant T}\mathbb{E}\|\xi^m_{s,t}(y)\|^p<+\infty. \tag{4.40}$$

From (4.40), for every $t\in[s,T]$, there exists a random field $\xi_{s,t}$ such that when m approaches to infinity

$$\xi^m_{s,t}(\cdot)\rightharpoonup\xi_{s,t}(\cdot),\ \ \text{weak}-*\text{ in }L^\infty(\mathbb{R}^d;L^p(\Omega)),\quad \forall\ p\geqslant 2, \tag{4.41}$$

up to choosing an appropriate subsequence.

For any test function $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^d)$ and $\eta\in L^{\frac{p}{p-1}}(\Omega)$, then

$$\begin{aligned} \mathbb{E}\left[\int_{\mathbb{R}^d}\xi_{s,t}(y)\varphi(y)dy\eta\right]&=-\lim_{m\to+\infty}\mathbb{E}\left[\int_{\mathbb{R}^d}Y^m_{s,t}(y)\cdot\nabla\varphi(y)dy\eta\right]\\ &=-\mathbb{E}\left[\int_{\mathbb{R}^d}Y_{s,t}(y)\cdot\nabla\varphi(y)dy\eta\right], \end{aligned} \tag{4.42}$$

where in the first line we have used (4.41) and the integration by parts, in the second line we used (4.37). By (4.42), $Y_{s,t}(y)$ is weakly differentiable in y and $\nabla Y_{s,t}(y)=\xi_{s,t}(y)$. Moreover, for every $p\geqslant 2$,

$$\sup_{y\in\mathbb{R}^d}\sup_{0\leqslant s\leqslant t\leqslant T}\mathbb{E}\|\nabla Y_{s,t}(y)\|^p<+\infty.$$

From this, we complete our proof. □

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