THE ZIEGLER SPECTRUM FOR ENRICHED RINGOIDS AND SCHEMES

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ABSTRACT. The Ziegler spectrum for categories enriched in closed symmetric monoidal Grothendieck categories is defined and studied in this paper. It recovers the classical Ziegler spectrum of a ring. As an application, the Ziegler spectrum as well as the category of generalised quasi-coherent sheaves of a reasonable scheme is introduced and studied. It is shown that there is a closed embedding of the injective spectrum of a coherent scheme endowed with the tensor fl-topology (respectively of a noetherian scheme endowed with the dual Zariski topology) into its Ziegler spectrum. It is also shown that quasi-coherent sheaves and generalised quasi-coherent sheaves are related to each other by a recollement.

1. INTRODUCTION

The Ziegler spectrum of a ring R, defined by Ziegler in [40], associates a topological space ${}_R\mathrm{Zg}$ to R whose points are the isomorphism classes of indecomposable pure-injective (left) R-modules. A basis of quasi-compact open subsets for the Ziegler topology is given by sets

$$(\varphi/\psi) = \{Q \in {}_R\mathrm{Zg} \mid \varphi(Q) > \psi(Q)\}$$

as φ/ψ ranges over pp-pairs (these are pairs of pp-formulas in the same free variables such that $\psi \to \varphi$). Ziegler proved in [40] that the closed sets correspond to definable subcategories of R-modules (a definable subcategory $\mathscr{D}$ is sent to the closed set $\mathscr{D} \cap {}_R\mathrm{Zg}$).

Herzog [24] and Krause [32] defined the Ziegler spectrum for locally coherent Grothendieck categories in the 90s. In this language, ${}_R\mathrm{Zg}$ is recovered from the Ziegler spectrum of the category of generalised R-modules ${}_R\mathscr{C} = (\operatorname{mod} R, \mathrm{Ab})$ (it is a locally coherent Grothendieck category and consists of additive functors from finitely presented right R-modules $\operatorname{mod} R$ to Abelian groups Ab). We also refer the reader to books by Prest [33, 34].

Grothendieck categories of enriched functors were introduced in [1]. Homological algebra associated to these categories was developed in [17, 18]. Applications of [1, 17, 18] have recently been given by Bonart [4, 5] in motivic homotopy theory for reconstructing triangulated categories of Voevodsky's big motives as well as motivic connected/very effective spectra with rational coefficients out of associated Grothendieck categories of enriched functors.

Let $\mathscr{V}$ be a closed symmetric monoidal Grothendieck category. A Grothendieck category of enriched functors is the category $[\mathscr{A}, \mathscr{V}]$ of enriched functors from a (skeletally) small $\mathscr{V}$-category $\mathscr{A}$ to $\mathscr{V}$. In this paper we refer to $\mathscr{A}$ as an enriched ringoid similarly to the terminology for ringoids, which are also called pre-additive categories in the literature (i.e. categories enriched over Abelian groups Ab). A typical example of an enriched ringoid is a DG-category, i.e. a category enriched over the Grothendieck category of chain complexes $\mathscr{V} = \mathrm{Ch}(\mathrm{Ab})$. In this paper we also assume $\mathscr{V}$ to be locally finitely presented having a family of dualizable generators $\mathscr{G} = \{g_i\}_{i \in I}$. In this case we refer to $[\mathscr{A}, \mathscr{V}]$ as the category of left $\mathscr{A}$-modules and denote it by

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$\mathscr{A}$ Mod. In Section 4 we study basic properties of $\mathscr{A}$-modules and, more importantly, enriched version for Serre's localization theory.

We introduce and study in Section 5 the enriched counterpart for the category of generalised modules. It is the $\mathscr{V}$-category ${}_{\mathscr{A}}\mathscr{C} := [\operatorname{mod}\mathscr{A},\mathscr{V}]$ of enriched functors from the $\mathscr{V}$-category of finitely presented right $\mathscr{A}$-modules $\operatorname{mod}\mathscr{A}$ to $\mathscr{V}$. The $\mathscr{V}$-categories ${}_{\mathscr{A}}\mathscr{C}$ and $\mathscr{A}$ Mod are related to each other by a recollement (see Theorem 5.4 for details). Also, the $\mathscr{V}$-fully faithful functor $M \in \mathscr{A}\,\mathrm{Mod} \to -\otimes_{\mathscr{A}} M \in {}_{\mathscr{A}}\mathscr{C}$ identifies $\mathscr{A}$-modules with right exact functors (see Theorem 6.2). As a consequence, one shows in Section 6 that this functor identifies pure-injective $\mathscr{A}$-modules introduced in Section 6 with injective objects of ${}_{\mathscr{A}}\mathscr{C}$.

Although ${}_{\mathscr{A}}\mathscr{C}$ is locally coherent by Theorem 5.2, and therefore the Ziegler spectrum $\mathsf{Zg}\,{}_{\mathscr{A}}\mathscr{C}$ in the sense of [24, 32] applies to ${}_{\mathscr{A}}\mathscr{C}$, this is actually not what we are going to investigate in this paper as $\mathsf{Zg}\,{}_{\mathscr{A}}\mathscr{C}$ does not capture the enriched category information of ${}_{\mathscr{A}}\mathscr{C}$. In Section 7 we define the Ziegler spectrum ${}_{\mathscr{A}}\mathsf{Zg}$ of $\mathscr{A}$ that captures both the enriched category information of ${}_{\mathscr{A}}\mathscr{C}$ and the machinery of [24, 32]. The points of $\mathsf{Zg}\,{}_{\mathscr{A}}\mathscr{C}$ and ${}_{\mathscr{A}}\mathsf{Zg}$ are the same but the topology on ${}_{\mathscr{A}}\mathsf{Zg}$ is coarser than the usual topology on $\mathsf{Zg}\,{}_{\mathscr{A}}\mathscr{C}$. It is worth mentioning that similar topologies on injective spectra of quasi-coherent sheaves which are defined in terms of enriched functors occur in [15, 20]. They play a key role for theorems reconstructing schemes out of $\operatorname{Qcoh}(X)$ (see [15] for details). In the case when $\mathscr{V} = \mathrm{Ab}$ and $\mathscr{A} = R$ is a ring, ${}_{\mathscr{A}}\mathsf{Zg}$ coincides with the classical Ziegler spectrum ${}_R\mathsf{Zg}$ of R.

The Ziegler topology for locally coherent Grothendieck categories defined in [24, 32] was extended to locally finitely presented Grothendieck categories in [15, Theorem 11]. In Section 9 we define the enriched version of this topology on the injective spectrum $\mathsf{Sp}\mathscr{A}$ of the enriched ringoid $\mathscr{A}$. If $\mathscr{V} = \operatorname{Qcoh}(X)$ and $\mathscr{A} = \{\mathscr{O}_X\}$, where X is a reasonable scheme, $\mathsf{Sp}\mathscr{A}$ recovers the topological space $\mathsf{Sp}_{\mathrm{fl},\otimes}(X)$ in the sense of [15, Theorem 19], which is equipped with the tensor fl-topology.

We finish the paper by introducing and studying the Ziegler spectrum Zg_X of a "nice" scheme X. It is realised in the category $\mathscr{C}_X$ of generalised quasi-coherent sheaves defined in Section 10. In this case purity for $\mathscr{A}$-modules is the same with geometric purity studied in [12]. If X is a nice coherent scheme, then we show in Theorem 10.7 that there is a closed embedding $\mathsf{Sp}(X) \hookrightarrow \mathsf{Zg}_X$ of the injective spectrum of X into its Ziegler spectrum. If, moreover, X is noetherian then there is a natural closed embedding of topological spaces $X^* \hookrightarrow \mathsf{Zg}_X$, where X^* is equipped with the dual Zariski topology (see Corollary 10.9).

2. Enriched Category Theory

In this section we collect basic facts about enriched categories we shall need later. We refer the reader to [7, 31] for details. Throughout this paper $(\mathscr{V},\otimes,\underline{\mathrm{Hom}},e)$ is a closed symmetric monoidal category with monoidal product $\otimes$, internal Hom-object $\underline{\mathrm{Hom}}$ and monoidal unit e. We sometimes write $[a,b]$ to denote $\underline{\mathrm{Hom}}(a,b)$, where $a,b \in \operatorname{Ob}\mathscr{V}$. We have structure isomorphisms

$$a_{abc} : (a\otimes b)\otimes c \to a\otimes(b\otimes c), \quad l_a : e\otimes a \to a, \quad r_a : a\otimes e \to a$$

in $\mathscr{V}$ with $a,b,c \in \operatorname{Ob}\mathscr{V}$.

Definition 2.1. A *$\mathscr{V}$-category $\mathscr{C}$*, or *a category enriched over $\mathscr{V}$*, consists of the following data:

(1) a class $\operatorname{Ob}(\mathscr{C})$ of objects;
(2) for every pair $a,b \in \operatorname{Ob}(\mathscr{C})$ of objects, an object $\mathscr{V}_{\mathscr{C}}(a,b)$ of $\mathscr{V}$;

(3) for every triple $a,b,c \in \mathrm{Ob}(\mathscr{C})$ of objects, a composition morphism in $\mathscr{V}$,

$$c_{abc} : \mathscr{V}_{\mathscr{C}}(a,b) \otimes \mathscr{V}_{\mathscr{C}}(b,c) \to \mathscr{V}_{\mathscr{C}}(a,c);$$

(4) for every object $a \in \mathscr{C}$, a unit morphism $u_a : e \to \mathscr{V}_{\mathscr{C}}(a,a)$ in $\mathscr{V}$.

These data must satisfy the following conditions:

- ◇ given objects $a,b,c,d \in \mathscr{C}$, diagram (1) below is commutative (associativity axiom);
- ◇ given objects $a,b \in \mathscr{C}$, diagram (2) below is commutative (unit axiom).

$$\begin{array}{ccc}
(\mathscr{V}_{\mathscr{C}}(a,b) \otimes \mathscr{V}_{\mathscr{C}}(b,c)) \otimes \mathscr{V}_{\mathscr{C}}(c,d) & \xrightarrow{c_{abc}\otimes 1} & \mathscr{V}_{\mathscr{C}}(a,c) \otimes \mathscr{V}_{\mathscr{C}}(c,d) \\
\downarrow{\scriptstyle a_{\mathscr{V}_{\mathscr{C}}(a,b)\mathscr{V}_{\mathscr{C}}(b,c)\mathscr{V}_{\mathscr{C}}(c,d)}} & & \\
\mathscr{V}_{\mathscr{C}}(a,b) \otimes (\mathscr{V}_{\mathscr{C}}(b,c) \otimes \mathscr{V}_{\mathscr{C}}(c,d)) & & \downarrow{\scriptstyle c_{acd}} \\
\downarrow{\scriptstyle 1\otimes c_{bcd}} & & \\
\mathscr{V}_{\mathscr{C}}(a,b) \otimes \mathscr{V}_{\mathscr{C}}(b,d) & \xrightarrow[c_{abd}]{} & \mathscr{V}_{\mathscr{C}}(a,d)
\end{array} \tag{1}$$

$$\begin{array}{ccccc}
e \otimes \mathscr{V}_{\mathscr{C}}(a,b) & \xrightarrow{l_{\mathscr{V}_{\mathscr{C}}(a,b)}} & \mathscr{V}_{\mathscr{C}}(a,b) & \xleftarrow{r_{\mathscr{V}_{\mathscr{C}}(a,b)}} & \mathscr{V}_{\mathscr{C}}(a,b) \otimes e \\
\downarrow{\scriptstyle u_a\otimes 1} & & \downarrow{\scriptstyle 1_{\mathscr{V}_{\mathscr{C}}(a,b)}} & & \downarrow{\scriptstyle 1\otimes u_b} \\
\mathscr{V}_{\mathscr{C}}(a,a) \otimes \mathscr{V}_{\mathscr{C}}(a,b) & \xrightarrow{c_{aab}} & \mathscr{V}_{\mathscr{C}}(a,b) & \xleftarrow{c_{abb}} & \mathscr{V}_{\mathscr{C}}(a,b) \otimes \mathscr{V}_{\mathscr{C}}(b,b)
\end{array} \tag{2}$$

When $\mathrm{Ob}\,\mathscr{C}$ is a set, the $\mathscr{V}$-category $\mathscr{C}$ is called a *small $\mathscr{V}$-category*.

If there is no likelihood of confusion, we will often write $[a,b]$ for the $\mathscr{V}$-object $\mathscr{V}_{\mathscr{C}}(a,b)$.

Definition 2.2. Given $\mathscr{V}$-categories $\mathscr{A},\mathscr{B}$, a *$\mathscr{V}$-functor* or an *enriched functor* $F : \mathscr{A} \to \mathscr{B}$ consists in giving:

(1) for every object $a \in \mathscr{A}$, an object $F(a) \in \mathscr{B}$;

(2) for every pair $a,b \in \mathscr{A}$ of objects, a morphism in $\mathscr{V}$,

$$F_{ab} : \mathscr{V}_{\mathscr{A}}(a,b) \to \mathscr{V}_{\mathscr{B}}(F(a),F(b))$$

in such a way that the following axioms hold:

- ◇ for all objects $a,a',a'' \in \mathscr{A}$, diagram (3) below commutes (composition axiom);
- ◇ for every object $a \in \mathscr{A}$, diagram (4) below commutes (unit axiom).

$$\begin{array}{ccc}
\mathscr{V}_{\mathscr{A}}(a,a') \otimes \mathscr{V}_{\mathscr{A}}(a',a'') & \xrightarrow{c_{aa'a''}} & \mathscr{V}_{\mathscr{A}}(a,a'') \\
\downarrow{\scriptstyle F_{aa'}\otimes F_{a'a''}} & & \downarrow{\scriptstyle F_{aa''}} \\
\mathscr{V}_{\mathscr{B}}(Fa,Fa') \otimes \mathscr{V}_{\mathscr{B}}(Fa',Fa'') & \xrightarrow[c_{Fa,Fa',Fa''}]{} & \mathscr{V}_{\mathscr{B}}(Fa,Fa'')
\end{array} \tag{3}$$

$$\begin{array}{ccc}
e & \xrightarrow{u_a} & \mathscr{V}_{\mathscr{A}}(a,a) \\
& {\scriptstyle u_{Fa}}\searrow & \downarrow{\scriptstyle F_{aa}} \\
& & \mathscr{V}_{\mathscr{B}}(Fa,Fa)
\end{array} \tag{4}$$

Definition 2.3. Let $\mathscr{A}, \mathscr{B}$ be two $\mathscr{V}$-categories and $F, G : \mathscr{A} \to \mathscr{B}$ two $\mathscr{V}$-functors. A *$\mathscr{V}$-natural transformation* $\alpha : F \Rightarrow G$ consists in giving, for every object $a \in \mathscr{A}$, a morphism

$$\alpha_a : e \to \mathscr{V}_{\mathscr{B}}(F(a), G(a))$$

in $\mathscr{V}$ such that diagram (5) below commutes, for all objects $a, a' \in \mathscr{A}$.

$$\begin{array}{ccccc} & & \mathscr{V}_{\mathscr{A}}(a,a') & & \\ & \swarrow^{l^{-1}_{\mathscr{V}_{\mathscr{A}}(a,a')}} & & \searrow^{r^{-1}_{\mathscr{V}_{\mathscr{A}}(a,a')}} & \\ e \otimes \mathscr{V}_{\mathscr{A}}(a,a') & & & & \mathscr{V}_{\mathscr{A}}(a,a') \otimes e \\ \downarrow{\scriptstyle \alpha_a \otimes G_{aa'}} & & & & \downarrow{\scriptstyle F_{aa'} \otimes \alpha_{a'}} \\ \mathscr{V}_{\mathscr{B}}(Fa,Ga) \otimes \mathscr{V}_{\mathscr{B}}(Ga,Ga') & & & & \mathscr{V}_{\mathscr{B}}(Fa,Fa') \otimes \mathscr{V}_{\mathscr{B}}(Fa',Ga') \\ & \searrow_{c_{FaGaGa'}} & & \swarrow_{c_{FaFa'Ga'}} & \\ & & \mathscr{V}_{\mathscr{B}}(Fa,Ga') & & \end{array} \tag{5}$$

By **Set** we shall mean the closed symmetric monoidal category of sets. Categories in the usual sense are **Set**-categories (categories enriched over **Set**). If $\mathscr{A}$ is a category, let $\mathbf{Set}_{\mathscr{A}}(a,b)$ denote the set of maps in $\mathscr{A}$ from a to b. The closed symmetric monoidal category $\mathscr{V}$ is a $\mathscr{V}$-category due to its internal Hom-objects. Any $\mathscr{V}$-category $\mathscr{C}$ defines a **Set**-category $\mathscr{C}_0$, also called the *underlying category*. Its class of objects is $\mathrm{Ob}\mathscr{C}$, the morphism sets are $\mathbf{Set}_{\mathscr{C}_0}(a,b) = \mathbf{Set}_{\mathscr{V}}(e, \mathscr{V}_{\mathscr{C}}(a,b))$ (see [7, p. 316]).

Proposition 2.4. *Let $\mathscr{V}$ be a symmetric monoidal closed category. If $\mathscr{A}$ is a $\mathscr{V}$-category and $F, G : \mathscr{A} \rightrightarrows \mathscr{V}$ are $\mathscr{V}$-functors, giving a $\mathscr{V}$-natural transformation $\alpha : F \Rightarrow G$ is equivalent to giving a family of morphisms $\alpha : F(a) \to G(a)$ in $\mathscr{V}$, for $a \in \mathscr{A}$, in such a way that the following diagram commutes for all $a, a' \in \mathscr{A}$*

$$\begin{array}{ccc} \mathscr{V}_{\mathscr{A}}(a,a') & \xrightarrow{F_{aa'}} & [F(a), F(a')] \\ {\scriptstyle G_{aa'}}\downarrow & & \downarrow{\scriptstyle [1,\alpha_{a'}]} \\ [G(a), G(a')] & \xrightarrow[{[\alpha_a, 1]}]{} & [F(a), G(a')] \end{array}$$

Proof. See [7, Proposition 6.2.8]. □

Corollary 2.5. *Let $\mathscr{V}$ be a symmetric monoidal closed category. If $\mathscr{A}$ is a $\mathscr{V}$-category and $F, G : \mathscr{A} \rightrightarrows \mathscr{V}$ are $\mathscr{V}$-functors, giving a $\mathscr{V}$-natural transformation $\alpha : F \Rightarrow G$ is equivalent to giving a family of morphisms $\alpha : F(a) \to G(a)$ in $\mathscr{V}$, for $a \in \mathscr{A}$, in such a way that the following* *diagram commutes for all $a, a' \in \mathscr{A}$*

$$\begin{array}{ccc}
\mathscr{V}_{\mathscr{A}}(a,a') \otimes F(a) & \xrightarrow{\eta_F} & F(a') \\
{\scriptstyle 1\otimes\alpha_a}\downarrow & & \downarrow{\scriptstyle \alpha_{a'}} \\
\mathscr{V}_{\mathscr{A}}(a,a') \otimes G(a) & \xrightarrow[\eta_G]{} & G(a'),
\end{array}$$

where η_F, η_G are the morphisms corresponding to the structure morphisms $F_{aa'}$ and $G_{aa'}$ respectively.

Corollary 2.6. *Let $\mathscr{V}$ be a symmetric monoidal closed category. If $\mathscr{A}$ is a small $\mathscr{V}$-category and $F, G : \mathscr{A} \rightrightarrows \mathscr{V}$ are $\mathscr{V}$-functors, suppose $\alpha : F \Rightarrow G$ is a $\mathscr{V}$-natural transformation such that each $\alpha_a : F(a) \to G(a)$, $a \in \operatorname{Ob}\mathscr{A}$, is an isomorphism in $\mathscr{V}$. Then α is an isomorphism in $[\mathscr{A}, \mathscr{V}]$.*

Proof. See [1, Corollary 2.6]. □

Let $\mathscr{C}, \mathscr{D}$ be two $\mathscr{V}$-categories. The *monoidal product* $\mathscr{C} \otimes \mathscr{D}$ is the $\mathscr{V}$-category, where

$$\operatorname{Ob}(\mathscr{C} \otimes \mathscr{D}) := \operatorname{Ob}\mathscr{C} \times \operatorname{Ob}\mathscr{D}$$

and

$$\mathscr{V}_{\mathscr{C}\otimes\mathscr{D}}((a,x),(b,y)) := \mathscr{V}_{\mathscr{C}}(a,b) \otimes \mathscr{V}_{\mathscr{D}}(x,y), \quad a,b \in \mathscr{C}, x,y \in \mathscr{D}.$$

Definition 2.7. A $\mathscr{V}$-category $\mathscr{C}$ is a *right $\mathscr{V}$-module* if there is a $\mathscr{V}$-functor $\operatorname{act} : \mathscr{C} \otimes \mathscr{V} \to \mathscr{C}$, denoted $(c, A) \mapsto c \oslash A$ and a $\mathscr{V}$-natural unit isomorphism $r_c : \operatorname{act}(c, e) \to c$ subject to the following conditions:

(1) there are coherent natural associativity isomorphisms $c \oslash (A \otimes B) \to (c \oslash A) \otimes B$;
(2) the isomorphisms $c \oslash (e \otimes A) \rightrightarrows c \oslash A$ coincide.

A right $\mathscr{V}$-module is *closed* if there is a $\mathscr{V}$-functor

$$\operatorname{coact} : \mathscr{V}^{\mathrm{op}} \otimes \mathscr{C} \to \mathscr{C}$$

such that for all $A \in \operatorname{Ob}\mathscr{V}$, and $c \in \operatorname{Ob}\mathscr{C}$, the $\mathscr{V}$-functor $\operatorname{act}(-, A) : \mathscr{C} \to \mathscr{C}$ is left $\mathscr{V}$-adjoint to $\operatorname{coact}(A, -)$ and $\operatorname{act}(c, -) : \mathscr{V} \to \mathscr{C}$ is left $\mathscr{V}$-adjoint to $\mathscr{V}_{\mathscr{C}}(c, -)$.

If $\mathscr{C}$ is a small $\mathscr{V}$-category, $\mathscr{V}$-functors from $\mathscr{C}$ to $\mathscr{V}$ and their $\mathscr{V}$-natural transformations form the category $[\mathscr{C}, \mathscr{V}]$ of $\mathscr{V}$-functors from $\mathscr{C}$ to $\mathscr{V}$. If $\mathscr{V}$ is complete, then $[\mathscr{C}, \mathscr{V}]$ is also a $\mathscr{V}$-category. We also denote this $\mathscr{V}$-category by $\mathscr{F}(\mathscr{C})$, or $\mathscr{F}$ if no confusion can arise. The morphism $\mathscr{V}$-object $\mathscr{V}_{\mathscr{F}}(X, Y)$ is the end

$$\int_{\operatorname{Ob}\mathscr{C}} \mathscr{V}(X(c), Y(c)). \tag{6}$$

Note that the underlying category $\mathscr{F}_0$ of the $\mathscr{V}$-category $\mathscr{F}$ is $[\mathscr{C}, \mathscr{V}]$.

Given $c \in \operatorname{Ob}\mathscr{C}$, $X \mapsto X(c)$ defines the $\mathscr{V}$-functor $\operatorname{Ev}_c : \mathscr{F} \to \mathscr{V}$ called *evaluation at c*. The assignment $c \mapsto \mathscr{V}_{\mathscr{C}}(c, -)$ from $\mathscr{C}$ to $\mathscr{F}$ is again a $\mathscr{V}$-functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{F}$, called the *$\mathscr{V}$-Yoneda embedding*. $\mathscr{V}_{\mathscr{C}}(c, -)$ is a representable functor, represented by c.

Lemma 2.8 (The Enriched Yoneda Lemma)**.** *Let $\mathscr{V}$ be a complete closed symmetric monoidal category and $\mathscr{C}$ a small $\mathscr{V}$-category. For every $\mathscr{V}$-functor $X : \mathscr{C} \to \mathscr{V}$ and every $c \in \operatorname{Ob}\mathscr{C}$, there is a $\mathscr{V}$-natural isomorphism $X(c) \cong \mathscr{V}_{\mathscr{F}}(\mathscr{V}_{\mathscr{C}}(c, -), X)$.*

Lemma 2.9. *If $\mathscr{V}$ is a bicomplete closed symmetric monoidal category and $\mathscr{C}$ is a small $\mathscr{V}$-category, then $[\mathscr{C},\mathscr{V}]$ is bicomplete. (Co)limits are formed pointwise. Moreover, $\mathscr{F}$ is a closed $\mathscr{V}$-module.*

Proof. See [7, Proposition 6.6.17]. □

Theorem 2.10. *Assume $\mathscr{V}$ is bicomplete, and let $\mathscr{C}$ be a small $\mathscr{V}$-category. Then any $\mathscr{V}$-functor $X : \mathscr{C} \to \mathscr{V}$ is $\mathscr{V}$-naturally isomorphic to the coend*

$$X \cong \int^{\mathrm{Ob}\mathscr{C}} \mathscr{V}_{\mathscr{C}}(c,-) \oslash X(c).$$

Proof. See [7, Theorem 6.6.18]. □

A *monoidal $\mathscr{V}$-category* is a $\mathscr{V}$-category $\mathscr{C}$ together with a $\mathscr{V}$-functor $\diamond : \mathscr{C} \otimes \mathscr{C} \to \mathscr{C}$, a unit $u \in \mathrm{Ob}\mathscr{C}$, a $\mathscr{V}$-natural associativity isomorphism and two $\mathscr{V}$-natural unit isomorphisms. Symmetric monoidal and closed symmetric monoidal $\mathscr{V}$-categories are defined similarly.

Suppose $(\mathscr{C},\diamond,u)$ is a small symmetric monoidal $\mathscr{V}$-category, where $\mathscr{V}$ is bicomplete. In [10], a closed symmetric monoidal product was constructed on the category $[\mathscr{C},\mathscr{V}]$ of $\mathscr{V}$-functors from $\mathscr{C}$ to $\mathscr{V}$. For $X,Y \in \mathrm{Ob}[\mathscr{C},\mathscr{V}]$, the monoidal product $X \odot Y \in \mathrm{Ob}[\mathscr{C},\mathscr{V}]$ is the coend

$$X \odot Y := \int^{\mathrm{Ob}(\mathscr{C}\otimes\mathscr{C})} \mathscr{V}_{\mathscr{C}}(c \diamond d,-) \otimes (X(c) \otimes Y(d)) : \mathscr{C} \to \mathscr{V}. \tag{7}$$

The following theorem is due to Day [10] and plays an important role in our analysis.

Theorem 2.11 (Day [10])**.** *Let $(\mathscr{V},\otimes,e)$ be a bicomplete closed symmetric monoidal category and $(\mathscr{C},\diamond,u)$ a small symmetric monoidal $\mathscr{V}$-category. Then $([\mathscr{C},\mathscr{V}],\odot,\mathscr{V}_{\mathscr{C}}(u,-))$ is a closed symmetric monoidal category. The internal* Hom*-functor in $[\mathscr{C},\mathscr{V}]$ is given by the end*

$$\mathscr{F}(X,Y)(c) = \mathscr{V}_{\mathscr{F}}(X,Y(c\diamond -)) = \int_{d\in\mathrm{Ob}\mathscr{C}} \mathscr{V}(X(d),Y(c\diamond d)). \tag{8}$$

The next lemma computes the tensor product of representable $\mathscr{V}$-functors.

Lemma 2.12. *The tensor product of representable functors is again representable. Precisely, there is a natural isomorphism*

$$\mathscr{V}_{\mathscr{C}}(c,-) \odot \mathscr{V}_{\mathscr{C}}(d,-) \cong \mathscr{V}_{\mathscr{C}}(c\diamond d,-).$$

3. GROTHENDIECK CATEGORIES

In this section we collect basic facts about Grothendieck categories. We mostly follow Herzog [24] and Stenström [38].

Definition 3.1. A family $\{U_i\}_I$ of objects of an Abelian category $\mathscr{A}$ is a *family of generators* for $\mathscr{A}$ if for each non-zero morphism $\alpha : B \to C$ in $\mathscr{A}$ there exist a morphism $\beta : U_i \to B$ for some $i \in I$, such that $\alpha\beta \neq 0$.

Recall that an Abelian category is *cocomplete* or an *Ab3-category* if it has arbitrary direct sums. The cocomplete Abelian category $\mathscr{C}$ is said to be an *Ab5-category* if for any directed family $\{A_i\}_{i\in I}$ if subobjects of A and for any subobject B of A, one has

$$(\sum_{i\in I} A_i) \cap B = \sum_{i\in I}(A_i \cap B).$$

The condition $Ab3$ is equivalent to the existance of arbitrary direct limits. Also $Ab5$ is equivalent to the fact that there exist inductive limits and the inductive limits over directed families of indices are exact, i.e. if I is a directed set and

$$0 \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow 0$$

is an exact sequence for any $i \in I$, then

$$0 \longrightarrow \varinjlim A_i \longrightarrow \varinjlim B_i \longrightarrow \varinjlim C_i \longrightarrow 0$$

is an exact sequence.

An Abelian category which satisfies the condition $Ab5$ and which possesses a family of generators is called a *Grothendieck category*.

Definition 3.2. Recall that an object $A \in \mathscr{C}$ is *finitely generated* if whenever there are subobjects $A_i \subseteq A$ for $i \in I$ satisfying $A = \sum_{i\in I} A_i$, then there is a finite subset $J \subset I$ such that $A = \sum_{i\in J} A_i$. The category of finitely generated subobjects of $\mathscr{C}$ is denoted by $\mathrm{fg}\mathscr{C}$. The category is *locally finitely generated* provided that every object $X \in \mathscr{C}$ is a directed sum $X = \sum_{i\in I} X_i$ of finitely generated subobjects X_i, or equivalently, $\mathscr{C}$ possesses a family of finitely generated generators.

Definition 3.3. A finitely generated object $B \in \mathscr{C}$ is *finitely presented* provided that every epimorphism $\eta : A \to B$ with A finitely generated has a finitely generated kernel $\operatorname{Ker}\eta$. The subcategory of finitely presented objects of $\mathscr{C}$ is denoted by $\mathrm{fp}\mathscr{C}$. Note that the subcategory $\mathrm{fp}\mathscr{C}$ of $\mathscr{C}$ is closed under extensions. Moreover, if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in $\mathscr{C}$ with B finitely presented, then C is finitely presented if and only if A is finitely generated. The category is *locally finitely presented* provided that it is locally finitely generated and every object $X \in \mathscr{C}$ is a direct limit $X = \varinjlim_{i\in I} X_i$ of finitely presented objects X_i, or equivalently, $\mathscr{C}$ possesses a family of finitely presented generators.

Definition 3.4. A finitely presented object C of a locally finitely presented Grothendieck category $\mathscr{C}$ is *coherent* if every finitely generated subobject B of C is finitely presented. Equivalently, every epimorphism $h : C \to A$ with A finitely presented has a finitely presented kernel. Evidently, a finitely generated subobject of a coherent object is also coherent. The subcategory of coherent objects of $\mathscr{C}$ is denoted by $\mathrm{coh}\mathscr{C}$. The category $\mathscr{C}$ is *locally coherent* provided that it is locally finitely presented and every object $X \in \mathscr{C}$ is a direct limit $X = \varinjlim_{i\in I} X_i$ of coherent objects X_i, or equivalently, $\mathscr{C}$ possesses a family of coherent generators.

The subcategories consisting of finitely generated, finitely presented and coherent objects are ordered by inclusion as follows:

$$\mathscr{C} \supseteq \mathrm{fg}\mathscr{C} \supseteq \mathrm{fp}\mathscr{C} \supseteq \mathrm{coh}\mathscr{C}.$$

Suppose $\mathscr{V}$ is a closed symmetric monoidal Grothendieck category. Here are some examples of such categories.

Examples 3.5. (1) Given any commutative ring R, the triple $(\mathrm{Mod}R, \otimes_R, R)$ is a closed symmetric monoidal locally finitely presented Grothendieck category.

(2) More generally, let X be a quasi-compact quasi-separated scheme. Consider the category $\mathrm{Qcoh}(X)$ of quasi-coherent $\mathscr{O}_X$-modules. By [27, Lemma 3.1] $\mathrm{Qcoh}(X)$ is a Grothendieck category, where quasi-coherent $\mathscr{O}_X$-modules of finite type form generators. It is locally finitely

presented by [15, Proposition 7]. The tensor product of $\mathscr{O}_X$-modules preserves quasi-coherence, and induces a closed symmetric monoidal structure on $\mathrm{Qcoh}(X)$.

(3) By [1, Proposition 3.4] the category of unbounded chain complexes $\mathrm{Ch}(\mathscr{V})$ of a Grothendieck category $\mathscr{V}$ is again a Grothendieck category. If, in addition, $\mathscr{V}$ is closed symmetric monoidal, $\mathrm{Ch}(\mathscr{V})$ is with respect to the usual monoidal product and internal Hom-object by [17, Theorem 3.2]. Moreover, if $\mathscr{V}$ is locally finitely presented, then $\mathrm{Ch}(\mathscr{V})$ is.

(4) $(\mathrm{Mod}\,kG, \otimes_k, k)$ is a closed symmetric monoidal locally finitely presented Grothendieck category, where k is a field and G is a finite group.

Theorem 3.6 (see [1]). *Let $\mathscr{V}$ be a closed symmetric monoidal Grothendieck category with a set of generators $\{g_i\}_I$. If $\mathscr{C}$ is a small $\mathscr{V}$-category, then the category of enriched functors $[\mathscr{C},\mathscr{V}]$ is a Grothendieck $\mathscr{V}$-category with the set of generators $\{\mathscr{V}(c,-)\oslash g_i \mid c \in \mathrm{Ob}\,\mathscr{C}, i \in I\}$. Moreover, if $\mathscr{C}$ is a small symmetric monoidal $\mathscr{V}$-category, $[\mathscr{C},\mathscr{V}]$ is closed symmetric monoidal with tensor product and internal* Hom-*object computed by the formulas* (7) *and* (8) *of Day.*

We say that a full subcategory $\mathscr{S}$ of an Abelian category $\mathscr{C}$ is a *Serre subcategory* if for any short exact sequence

$$0 \to X \to Y \to Z \to 0$$

in $\mathscr{C}$ an object $Y \in \mathscr{S}$ if and only if X, $Z \in \mathscr{S}$. A Serre subcategory $\mathscr{S}$ of a Grothendieck category $\mathscr{C}$ is *localizing* if it is closed under taking direct limits. Equivalently, the inclusion functor $i : \mathscr{S} \to \mathscr{C}$ admits the right adjoint $t = t_{\mathscr{S}} : \mathscr{C} \to \mathscr{S}$ which takes every object $X \in \mathscr{C}$ to the maximal subobject $t(X)$ of X belonging to $\mathscr{S}$. The functor t we call the *torsion functor*. An object C of $\mathscr{C}$ is said to be *$\mathscr{S}$-torsionfree* if $t(C) = 0$. Given a localizing subcategory $\mathscr{S}$ of $\mathscr{C}$ the *quotient category* $\mathscr{C}/\mathscr{S}$ consists of $C \in \mathscr{C}$ such that $t(C) = t^1(C) = 0$, where t^1 stands for the first derived functor associated with t. The objects from $\mathscr{C}/\mathscr{S}$ we call *$\mathscr{S}$-closed objects*. Given $C \in \mathscr{C}$ there exists a canonical exact sequence

$$0 \to A' \to C \xrightarrow{\lambda_C} C_{\mathscr{S}} \to A'' \to 0$$

with $A' = t(C)$, $A'' \in \mathscr{S}$, and where $C_{\mathscr{S}} \in \mathscr{C}/\mathscr{S}$ is the maximal essential extension of $\widetilde{C} = C/t(C)$ such that $C_{\mathscr{S}}/\widetilde{C} \in \mathscr{S}$. The object $C_{\mathscr{S}}$ is uniquely defined up to a canonical isomorphism and is called the *$\mathscr{S}$-envelope* of C. Moreover, the inclusion functor $\iota : \mathscr{C}/\mathscr{S} \to \mathscr{C}$ has the left adjoint *localizing functor* $(-)_{\mathscr{S}} : \mathscr{C} \to \mathscr{C}/\mathscr{S}$, which is also exact. It takes each $C \in \mathscr{C}$ to $C_{\mathscr{S}} \in \mathscr{C}/\mathscr{S}$. Then,

$$\mathrm{Hom}_{\mathscr{C}}(X,Y) \cong \mathrm{Hom}_{\mathscr{C}/\mathscr{S}}(X_{\mathscr{S}},Y)$$

for all $X \in \mathscr{C}$ and $Y \in \mathscr{C}/\mathscr{S}$.

4. The category of $\mathscr{A}$-modules

From now on we fix a closed symmetric monoidal Grothendieck category $\mathscr{V}$ and assume that it has a family of dualizable generators $\mathscr{G} = \{g_i\}_{i \in I}$ such that the monoidal unit e is finitely presented. Under these assumptions, each $g \in \mathscr{G}$ is finitely presented as well. In particular, $\mathscr{V}$ is locally finitely presented.

Recall from [26, 6.5] that the following conditions are equivalent for an object $p \in \mathscr{V}$:

(1) The canonical morphism $p^\vee \otimes p \to [p,p]$ is an isomorphism, where $p^\vee = [p,e]$.
(2) The canonical morphism $p^\vee \otimes z \to [p,z]$ is an isomorphism for all $z \in \mathscr{V}$.
(3) $[p,y] \otimes z \to [p, y \otimes z]$ is an isomorphism for all $y,z \in \mathscr{V}$.

Such an object $p \in \mathscr{V}$ is said to be *dualizable*. By [26, Lemma 6.7] if $p, q \in \mathscr{V}$ are dualizable, then so are $p \oplus q$, $p \otimes q$, and $[p, q]$.

Examples 4.1. (1) If R is a commutative ring, then R is a dualizable generator for $\mathrm{Mod}\,R$.

(2) By Example 3.5(2) $\mathrm{Qcoh}(X)$ is a closed symmetric monoidal locally finitely presented Grothendieck category over a quasi-compact quasi-separated scheme X. By [8, Proposition 4.7.5] dualizable objects of $\mathrm{Qcoh}(X)$ are precisely locally free of finite rank. We say that X satisfies the *strong resolution property* if they also generate $\mathrm{Qcoh}(X)$. It is worth noticing that Corollary 4.3 below and [36, Main Theorem] imply X must be semiseparated.

(3) By Example 3.5(3) $\mathrm{Ch}(\mathscr{V})$ is a closed symmetric monoidal locally finitely presented Grothendieck category. It is directly verified that $\mathrm{Ch}(\mathscr{V})$ has dualizable generators. Precisely, they are dual to the standard generators of $\mathrm{Ch}(\mathscr{V})$.

(4) By Example 3.5(4) $\mathrm{Mod}\,kG$ is a closed symmetric monoidal locally finitely presented Grothendieck category, where k is a field and G is a finite group. It is well known that finite dimensional kG-modules are dualizable (see, e.g., [28, §2.14]).

In what follows we assume that the set of dualizable generators $\mathscr{G}$ is closed under tensor products $g \otimes g'$ and dual objects $g^\vee$. $\mathscr{G}$ becomes a symmetric monoidal $\mathscr{V}$-category if we assume $e \in \mathscr{G}$. The category $[\mathscr{G}^{\mathrm{op}}, \mathscr{V}]$ is then closed symmetric monoidal with respect to the Day monoidal product $\odot$ of Theorem 2.11.

The following statement is reminiscent of the fact saying that the category of modules $\mathrm{Mod}\,R$ over a commutative ring R is equivalent to the category of additive functors $(\mathscr{P}^{\mathrm{op}}, \mathrm{Mod}\,R)$, where $\mathscr{P}$ is the full subcategory of finitely generated projective R-modules (note that each $P \in \mathscr{P}$ is dualizable in $\mathrm{Mod}\,R$).

Proposition 4.2. *Suppose $e \in \mathscr{G}$. The adjoint pair $- \otimes [-, e] : \mathscr{V} \leftrightarrows [\mathscr{G}^{\mathrm{op}}, \mathscr{V}] : Ev_e$ is an equivalence of categories. Likewise, the adjoint pair $[e, -] \otimes - : \mathscr{V} \leftrightarrows [\mathscr{G}, \mathscr{V}] : Ev_e$ is an equivalence of categories.*

Proof. If we identify $\mathscr{V}$ with $[e, \mathscr{V}]$, the functor $- \otimes [-, e]$ is nothing but the enriched left Kan extension sending $X \in \mathscr{V}$ to $[-, X]$, hence it is fully faithful. One also has $(X \otimes [-, e])(g) = X \otimes g^\vee \cong [g, X]$.

Suppose $Y \in [\mathscr{G}^{\mathrm{op}}, \mathscr{V}]$, then there is a canonical isomorphism for all $g \in \mathscr{G}$:

$$Ev_e(Y \odot (g^\vee \otimes [-, e])) \cong Ev_e((Y \odot [-, e]) \otimes g^\vee) \cong Ev_e(Y) \otimes g^\vee = Y(e) \otimes g^\vee.$$

One has,

$$Ev_e(Y \odot (g^\vee \otimes [-, e])) \cong Ev_e(Y \odot [-, g^\vee]) \cong Ev_e\Big(\int^{g' \in \mathscr{G}} (Y(g') \otimes [-, g'] \odot [-, g^\vee])\Big) =$$
$$= \int^{g' \in \mathscr{G}} Y(g') \otimes [e, g' \otimes g^\vee] \cong \int^{g' \in \mathscr{G}} Y(g') \otimes [g, g'] \cong Y(g).$$

We use Lemma 2.12 here. We see that there is a canonical isomorphism $Y(g) \cong Y(e) \otimes g^\vee \cong [g, Y(e)]$. It follows that the adjunction unit morphism $Y \to Ev_e(Y) \otimes [-, e]$ is an isomorphism at every $g \in \mathscr{G}$, and hence it is an isomorphism by Corollary 2.6. □

Corollary 4.3. *For any dualizable generator $g \in \mathscr{G}$ the functors $g \otimes -, [g, -] : \mathscr{V} \to \mathscr{V}$ are exact. In particular, the generator $G := \oplus_{g \in \mathscr{G}} g$ of $\mathscr{V}$ is flat in the sense that $G \otimes - : \mathscr{V} \to \mathscr{V}$ is an exact functor.*

Proof. Without loss of generality we may assume $e \in \mathscr{G}$, because e is dualizable and $\mathscr{G} \cup \{e\}$ remains a family of generators of $\mathscr{V}$. The proof of the preceding proposition shows that the equivalence $- \otimes [-,e] : \mathscr{V} \to [\mathscr{G}^{\mathrm{op}}, \mathscr{V}]$ is isomorphic to $X \mapsto [-,X]$. As it preserves exact sequences, the functor $[g,-]$ is exact. Likewise, the functor $g \otimes -$ is exact. □

Recall that a preadditive category is a category enriched over Ab. It is also called a *ring with several objects* or a *ringoid* in the literature. Likewise, we refer to a (skeletally) small $\mathscr{V}$-category $\mathscr{A}$ as an *enriched ring with several objects* or as an *enriched ringoid*. A typical example of an enriched ringoid is a DG-category, in which case $\mathscr{V} = \mathrm{Ch}(\mathrm{Ab})$. In what follows the category of contravariant (respectively covariant) $\mathscr{V}$-functors $[\mathscr{A}^{\mathrm{op}}, \mathscr{V}]$ (respectively $[\mathscr{A}, \mathscr{V}]$) will be denoted by $\mathrm{Mod}\,\mathscr{A}$ (respectively $\mathscr{A}\,\mathrm{Mod}$). If there is no likelihood of confusion, we refer to objects of $\mathrm{Mod}\,\mathscr{A}$ (respectively $\mathscr{A}\,\mathrm{Mod}$) as *right $\mathscr{A}$-modules* (respectively *left $\mathscr{A}$-modules*).

Lemma 4.4. *The Grothendieck $\mathscr{V}$-categories* $\mathrm{Mod}\,\mathscr{A}$ *and* $\mathscr{A}\,\mathrm{Mod}$ *are locally finitely presented with finitely presented generators* $\{g \oslash [-,a] \mid a \in \mathscr{A}, g \in \mathscr{G}\}$ *and* $\{[a,-] \oslash g \mid a \in \mathscr{A}, g \in \mathscr{G}\}$ *respectively.*

Proof. This follows from Theorem 3.6 and the fact that

$$\mathrm{Hom}_{\mathscr{A}}([a,-] \oslash g, \varinjlim_I M_i) = \mathrm{Hom}_{\mathscr{V}}(g, \varinjlim_I M_i(a)) = \\ = \varinjlim_I \mathrm{Hom}_{\mathscr{V}}(g, M(a)) = \varinjlim_I \mathrm{Hom}_{\mathscr{A}}([a,-] \oslash g, M_i)$$

for any $a \in \mathscr{A}$ and $g \in \mathscr{G}$. □

The proof of the preceding lemma also leads to the following useful lemma.

Lemma 4.5. *A left or right $\mathscr{A}$-module M is finitely presented if and only if the canonical morphism $\varinjlim_I [M, X_i] \to [M, \varinjlim_I X_i]$ is an isomorphism in $\mathscr{V}$ for any direct limit of $\mathscr{A}$-modules.*

Proof. Without loss of generality we may assume $e \in \mathscr{G}$, because otherwise we add e to $\mathscr{G}$. As each generator $g \in \mathscr{G}$ is finitely presented and $\mathscr{V}$ is equivalent to $[\mathscr{G}^{\mathrm{op}}, \mathscr{V}]$ by Proposition 4.2 by means of the functor $X \mapsto [-,X]$, every direct limit in $\mathscr{V}$ is mapped to a direct limit in $[\mathscr{G}^{\mathrm{op}}, \mathscr{V}]$. Therefore $\varinjlim_I [g, X_i] \to [g, \varinjlim_I X_i]$ is an isomorphism in $\mathscr{V}$ for any $g \in \mathscr{G}$.

Let $M \in \mathrm{Mod}\,\mathscr{A}$ be finitely presented. There is an exact sequence in $\mathrm{Mod}\,\mathscr{A}$

$$\oplus_{s=1}^m g_s \oslash [-,a_s] \to \oplus_{t=1}^n g_t \oslash [-,a_t] \to M \to 0, \quad g_s, g_t \in \mathscr{G}, \quad a_s, a_t \in \mathscr{A}.$$

It induces an exact sequence

$$0 \to [M, \varinjlim_I X_i] \to \varinjlim_I \oplus_{t=1}^n [g_t, X_i(a_t)] \to \varinjlim_I \oplus_{s=1}^m [g_s, X_i(a_s)].$$

It follows that the canonical morphism $\varinjlim_I [M, X_i] \to [M, \varinjlim_I X_i]$ is an isomorphism in $\mathscr{V}$. The converse is straightforward if we apply $\mathrm{Hom}_{\mathscr{V}}(e,-)$ to the isomorphism $\varinjlim_I [M, X_i] \cong [M, \varinjlim_I X_i]$ and use the assumption that $e \in \mathrm{fp}(\mathscr{V})$. □

As $\mathrm{Mod}\,\mathscr{A}$ and $\mathscr{A}\,\mathrm{Mod}$ are closed $\mathscr{V}$-modules by Lemma 2.9, the functors $[X,-]$ and $[-,X]$ are left exact for any $\mathscr{A}$-module X. It will be useful to have the following

Lemma 4.6. *If E is an injective left or right $\mathscr{A}$-module, then the functor $[-,E]$ is exact.*

Proof. A sequence of $\mathscr{A}$-modules $0 \to A \to B \to C \to 0$ is short exact if and only if the sequence $0 \to A(a) \to B(a) \to C(a) \to 0$ is short exact in $\mathscr{V}$ for any $a \in \mathscr{A}$. It follows from Corollary 4.3 that $0 \to G \oslash A \to G \oslash B \to G \oslash C \to 0$ is short exact, where $G = \oplus_{g \in \mathscr{G}} g$ is a generator of $\mathscr{V}$. As the functor $\mathrm{Hom}_{\mathscr{V}}(G, -) : \mathscr{V} \to \mathrm{Ab}$ is faithful, it reflects epimorphisms by [30, Proposition 1.2.12].

So $[B,E] \to [A,E]$ is an epimorphism in $\mathscr{V}$ if $\mathrm{Hom}_{\mathscr{V}}(G,[B,E]) \to \mathrm{Hom}_{\mathscr{V}}(G,[A,E])$ is an epimorphism. The latter arrow is an epimorphism in Ab as it is isomorphic to the epimorphism $\mathrm{Hom}_{\mathscr{V}}(G \oslash B, E) \to \mathrm{Hom}_{\mathscr{V}}(G \oslash A, E)$ (we use here our assumption that E is injective). □

Lemma 4.7. *If E is an injective left or right $\mathscr{A}$-module and $g \in \mathscr{G}$, then $[g,E]$ and $g \oslash E$ are injective $\mathscr{A}$-modules.*

Proof. As $[g,E] \cong g^{\vee} \oslash E$, it is enough to show that $[g,E]$ is injective. But this immediately follows from isomorphisms of the form $\mathrm{Hom}_{\mathscr{A}}(A,[g,E]) \cong \mathrm{Hom}_{\mathscr{A}}(g \oslash A, E)$, where A is an $\mathscr{A}$-module, and Corollary 4.3. □

A localising subcategory $\mathscr{P} \subseteq \mathrm{Mod}\,\mathscr{A}$ is said to be *enriched* if $M \oslash V \in \mathscr{P}$ for any $V \in \mathscr{V}$ and $M \in \mathscr{P}$. The quotient category $\mathrm{Mod}\,\mathscr{A}/\mathscr{P}$ will also be regarded as a full $\mathscr{V}$-subcategory of the $\mathscr{V}$-category $\mathrm{Mod}\,\mathscr{A}$. Denote by $\underline{\mathrm{Ext}}^1(P,-)$ the first derived functor associated with the left exact functor $[P,-] : \mathrm{Mod}\,\mathscr{A} \to \mathscr{V}$.

Theorem 4.8. *The following statements are true for an enriched localising subcategory $\mathscr{P}$:*

1. *for any $\mathscr{P}$-closed $\mathscr{A}$-module $M \in \mathrm{Mod}\,\mathscr{A}/\mathscr{P}$ and any $V \in \mathscr{V}$, the $\mathscr{A}$-module $[V,M]$ is $\mathscr{P}$-closed;*
2. *the localisation functor $(-)_{\mathscr{P}} : \mathrm{Mod}\,\mathscr{A} \to \mathrm{Mod}\,\mathscr{A}/\mathscr{P}$ is a $\mathscr{V}$-functor;*
3. *the quotient category $\mathrm{Mod}\,\mathscr{A}/\mathscr{P}$ is a closed $\mathscr{V}$-module, where tensor (respectively cotensor) objects are defined as $(M \oslash V)_{\mathscr{P}}$ (respectively $[V,M]$) for any $\mathscr{P}$-closed $\mathscr{A}$-module M and $V \in \mathscr{V}$;*
4. *an $\mathscr{A}$-module M is $\mathscr{P}$-closed if and only if $[P,M] = \underline{\mathrm{Ext}}^1(P,M) = 0$ for every $P \in \mathscr{P}$.*

Proof. (1). Recall that an injective $\mathscr{A}$-module E is $\mathscr{P}$-closed if and only if it is $\mathscr{P}$-torsionfree. By Lemma 4.7 $[g,E]$ is injective for all $g \in \mathscr{G}$. As $\mathscr{P}$ is enriched, $[g,E]$ is $\mathscr{P}$-torsionfree, and hence $\mathscr{P}$-closed for any $\mathscr{P}$-closed injective E. Suppose $V \in \mathscr{V}$. There is an exact sequence

$$\oplus_I g_i \to \oplus_J g_j \to V \to 0$$

in $\mathscr{V}$. It induces an exact sequence in $\mathrm{Mod}\,\mathscr{A}$

$$0 \to [V,E] \to \prod_J [g_j,E] \to \prod_I [g_i,E].$$

As $[V,E]$ is the kernel of a map between $\mathscr{P}$-closed modules, it is $\mathscr{P}$-closed itself. Next, each $\mathscr{P}$-closed module M fits into an exact sequence

$$0 \to M \to E_1 \to E_2$$

with E_1, E_2 injective $\mathscr{P}$-closed modules. It induces an exact sequence in $\mathrm{Mod}\,\mathscr{A}$

$$0 \to [V,M] \to [V,E_1] \to [V,E_2].$$

Since $[V,E_1],[V,E_2]$ are $\mathscr{P}$-closed, $[V,M]$ is.

(2). Suppose $A,B \in \mathrm{Mod}\,\mathscr{A}$, then a morphism of $\mathscr{V}$-objects $[A,B] \to [A_{\mathscr{P}},B_{\mathscr{P}}]$ is determined by a morphism of representable functors $(-,[A,B]) \to (-,[A_{\mathscr{P}},B_{\mathscr{P}}])$ if we use Yoneda's lemma [6, Theorem 1.3.3]. Given $V \in \mathscr{V}$, the definition of an arrow $(V,[A,B]) \to (V,[A_{\mathscr{P}},B_{\mathscr{P}}])$

is equivalent to defining an arrow $(A,[V,B]) \to (A,[V,B_{\mathscr{P}}])$ due to the fact that $[V,B_{\mathscr{P}}]$ is $\mathscr{P}$-closed by the previous statement. The latter arrow is induced by the canonical map $\lambda_B : B \to B_{\mathscr{P}}$. This defines a canonical map $u_{A,B} : [A,B] \to [A_{\mathscr{P}},B_{\mathscr{P}}]$. Commutativity of diagrams (3) and (4) associated with maps $u_{A,B}$-s is directly verified. We see that the localisation functor $(-)_{\mathscr{P}} : \operatorname{Mod}\mathscr{A} \to \operatorname{Mod}\mathscr{A}/\mathscr{P}$ is a $\mathscr{V}$-functor.

(3). By Lemma 2.9 $\operatorname{Mod}\mathscr{A}$ is a closed $\mathscr{V}$-module. The inclusion functor $\operatorname{Mod}\mathscr{A}/\mathscr{P} \hookrightarrow \operatorname{Mod}\mathscr{A}$ is a $\mathscr{V}$-functor. It follows that the composite functor

$$(\operatorname{Mod}\mathscr{A}/\mathscr{P}) \otimes \mathscr{V} \to (\operatorname{Mod}\mathscr{A}) \otimes \mathscr{V} \to \operatorname{Mod}\mathscr{A} \xrightarrow{(-)_{\mathscr{P}}} \operatorname{Mod}\mathscr{A}/\mathscr{P}$$

is a $\mathscr{V}$-functor (we tacitly use the previous statement here). The conditions of Definition 2.7 are plainly satisfied defining a right $\mathscr{V}$-module structure on $\operatorname{Mod}\mathscr{A}/\mathscr{P}$. It is also a closed $\mathscr{V}$-module with the coaction $\mathscr{V}$-functor defined by the composition

$$\mathscr{V}^{\mathrm{op}} \otimes (\operatorname{Mod}\mathscr{A}/\mathscr{P}) \to \mathscr{V}^{\mathrm{op}} \otimes (\operatorname{Mod}\mathscr{A}) \to \operatorname{Mod}\mathscr{A} \xrightarrow{(-)_{\mathscr{P}}} \operatorname{Mod}\mathscr{A}/\mathscr{P}.$$

The adjointness conditions for the action and coaction functors of Definition 2.7 are straightforward, as was to be shown.

(4). Suppose M is $\mathscr{P}$-closed. Note that $[P,M] = 0$ if and only if $\operatorname{Hom}_{\mathscr{V}}(G,[P,M]) = 0$, where $G = \oplus_{g\in\mathscr{G}}\, g$ is the generator of $\mathscr{V}$. As $\mathscr{P}$ is enriched and M is $\mathscr{P}$-torsionfree, $\operatorname{Hom}_{\mathscr{V}}(G,[P,M]) = \operatorname{Hom}_{\mathscr{V}}(P \oslash G, M) = 0$. There is an exact sequence

$$0 \to M \to E_0 \to E_1$$

with E_0, E_1 injective $\mathscr{P}$-torsionfree. One has a short exact sequence

$$0 \to M \to E_0 \to E_0/M \to 0$$

with E_0/M being $\mathscr{P}$-torsionfree. As $\mathscr{P}$ is enriched, it follows that $[P,E_0/M] = 0$ for all $P \in \mathscr{P}$. The long exact sequence

$$0 \to [P,M] \to [P,E_0] \to [P,E_0/M] \to \underline{\operatorname{Ext}}^1(P,M) \to \underline{\operatorname{Ext}}^1(P,E_0) = 0$$

implies $[P,M] = \underline{\operatorname{Ext}}^1(P,M) = 0$.

Conversely, suppose $[P,M] = \underline{\operatorname{Ext}}^1(P,M) = 0$. Then M is $\mathscr{P}$-torsionfree, and hence the injective envelope $E(M)$ of M is. As above, $[P,E(M)] = 0$ for all $P \in \mathscr{P}$. The long exact sequence

$$0 \to [P,M] \to [P,E(M)] \to [P,E(M)/M] \to \underline{\operatorname{Ext}}^1(P,M) = 0$$

implies $[P,E(M)/M] = 0$. Therefore $E(M)/M$ is $\mathscr{P}$-torsionfree, and hence its injective envelope $E(E(M)/M)$ is. We see that M is the kernel of a morphism between $\mathscr{P}$-closed objects, hence it is $\mathscr{P}$-closed itself. □

Note that $\mathscr{V}$ can be regarded as the category of $\mathscr{A}$-modules with $\mathscr{A} = \{e\}$. In this case enriched localising subcategories of $\mathscr{V}$ are also referred to as *tensor* in the literature.

Corollary 4.9 (Jeremías López, López López and Villanueva Nóvoa [29]). *For any tensor localising subcategory $\mathscr{P}$ of $\mathscr{V}$, the quotient $\mathscr{V}$-category $\mathscr{V}/\mathscr{P}$ is closed symmetric monoidal if we define the monoidal unit by $e_{\mathscr{P}}$, monoidal product by $V \boxtimes W := (V \otimes W)_{\mathscr{P}}$ and the internal Hom-object by $\underline{\operatorname{Hom}}_{\mathscr{V}/\mathscr{P}}(V,W) := [V,W]$, $V,W \in \mathscr{V}/\mathscr{P}$. In particular, the quotient $\mathscr{V}$-functor $\mathscr{V} \to \mathscr{V}/\mathscr{P}$ is strong symmetric monoidal. Moreover, $\mathscr{G}_{\mathscr{P}} = \{g_{\mathscr{P}}\}_{g\in\mathscr{G}}$ is a set of dualizable generators of $\mathscr{V}/\mathscr{P}$.*

Proof. This follows from Theorem 4.8. In more detail, the associativity, left/right unit isomorphisms for the monoidal structure on $\mathscr{V}/\mathscr{P}$ are, by definition, $\mathscr{P}$-localisations of the associativity, left/right unit isomorphisms for the monoidal structure on $\mathscr{V}$ (see [7, Section 6.1] for the corresponding definitions). We also use here the fact that the canonical morphism

$$(\lambda_A \otimes B)_{\mathscr{P}} : (A \otimes B)_{\mathscr{P}} \to (A_{\mathscr{P}} \otimes B)_{\mathscr{P}}$$

is an isomorphism in $\mathscr{V}/\mathscr{P}$ (for this one uses Theorem 4.8(1)). This also implies a natural isomorphism $[A,X] \cong [A_{\mathscr{P}},X]$ for any $A \in \mathscr{V}$, $X \in \mathscr{V}/\mathscr{P}$.

It remains to show that each $g_{\mathscr{P}}$ is dualizable. Consider the following exact sequence

$$0 \to P \to e \xrightarrow{\lambda_e} e_{\mathscr{P}} \to P' \to 0,$$

where $P, P' \in \mathscr{P}$. By Corollary 4.3 the sequence

$$0 \to [g,P] \to g^{\vee} \xrightarrow{[g,\lambda_e]} [g,e_{\mathscr{P}}] \to [g,P'] \to 0$$

is exact in $\mathscr{V}$ and $[g,P],[g,P'] \in \mathscr{P}$. As $[g,e_{\mathscr{P}}] \cong [g_{\mathscr{P}},e_{\mathscr{P}}]$ is $\mathscr{P}$-closed by Theorem 4.8(1), we have that $(g^{\vee})_{\mathscr{P}} \cong (g_{\mathscr{P}})^{\vee}$ in $\mathscr{V}/\mathscr{P}$. Therefore one has for any $V \in \mathscr{V}/\mathscr{P}$

$$(g_{\mathscr{P}})^{\vee} \boxtimes V \cong (g^{\vee} \otimes V)_{\mathscr{P}} \cong [g,V]_{\mathscr{P}} = [g,V] \cong [g_{\mathscr{P}},V].$$

We see that $g_{\mathscr{P}}$ is dualizable in $\mathscr{V}/\mathscr{P}$. ☐

Example 4.10. Suppose R is a commutative ring. Every localising subcategory $\mathscr{P} \subseteq \operatorname{Mod} R$ is tensor. Corollary 4.9 implies that $\operatorname{Mod} R/\mathscr{P}$ is the category of right $\mathscr{A}$-modules $[\mathscr{A}^{\mathrm{op}},\mathscr{V}]$, where $\mathscr{A} = \{R_{\mathscr{P}}\}$ and $\mathscr{V} = \operatorname{Mod} R/\mathscr{P}$. It is worth mentioning that, in general, $\operatorname{Mod} R/\mathscr{P}$ is rarely equivalent to the category $\operatorname{Mod} R_{\mathscr{P}} = [R_{\mathscr{P}}, \mathrm{Ab}]$ (see [38, Proposition XI.3.4]), in which case we forget the enriched information and regard $R_{\mathscr{P}}$ as an Ab-enriched ring and $\mathscr{V} = \mathrm{Ab}$.

5. The category of enriched generalised modules

The full subcategories of finitely presented right and left $\mathscr{A}$-modules will be denoted by $\operatorname{mod}\mathscr{A}$ and $\mathscr{A}\operatorname{mod}$ respectively.

Definition 5.1. The *category of right (left) enriched generalised $\mathscr{A}$-modules* $\mathscr{C}_{\mathscr{A}}$ (${}_{\mathscr{A}}\mathscr{C}$) is defined as the Grothendieck category of $\mathscr{V}$-functors $[\mathscr{A}\operatorname{mod},\mathscr{V}]$ ($[\operatorname{mod}\mathscr{A},\mathscr{V}]$), where $\operatorname{mod}\mathscr{A}$ ($\mathscr{A}\operatorname{mod}$) is regarded as a $\mathscr{V}$-full subcategory of $\operatorname{Mod}\mathscr{A}$ ($\mathscr{A}\operatorname{Mod}$). By definition, ${}_{\mathscr{A}}\mathscr{C}$ ($\mathscr{C}_{\mathscr{A}}$) is the category of left $\operatorname{mod}\mathscr{A}$-modules (left $\mathscr{A}\operatorname{mod}$-modules).

Note that if $\mathscr{V} = \mathrm{Ab}$ then $\mathscr{C}_{\mathscr{A}}$ is the category of generalised $\mathscr{A}$-modules of [14, §6]. If, moreover, $\mathscr{A}$ has one object, in which case $\mathscr{A}$ is a ring R, $\mathscr{C}_{\mathscr{A}}$ coincides with the category of generalised modules $\mathscr{C}_R$ of [24].

Theorem 5.2. *The category of generalised $\mathscr{A}$-modules $\mathscr{C}_{\mathscr{A}}$ is a locally coherent Grothendieck $\mathscr{V}$-category with $\{[M,-] \mid M \in \mathscr{A}\operatorname{mod}\}$ being a family of coherent generators.*

Proof. By Theorem 3.6 $\mathscr{C}_{\mathscr{A}}$ is a Grothendieck $\mathscr{V}$-category with $\{[M,-] \oslash g \mid M \in \mathscr{A}\operatorname{mod}, g \in \mathscr{G}\}$ being a family of generators. As $\mathscr{V}$ is locally finitely presented and each object $g \in \mathscr{G}$ is finitely presented, $[M,-] \oslash g$ is finitely presented as well due to isomorphisms

$$\operatorname{Hom}_{\mathscr{C}_{\mathscr{A}}}([M,-]\oslash g, \varinjlim_{i\in I} B_i) \cong \operatorname{Hom}_{\mathscr{V}}(g, \varinjlim_{i\in I} B_i(M)) = \varinjlim_{i\in I} \operatorname{Hom}_{\mathscr{V}}(g, B_i(M)) \cong \varinjlim_{i\in I}([M,-]\oslash g, B_i).$$

Since g is dualizable, it follows that

$$[M,-]\oslash g \cong [M\oslash g^\vee,-].$$

We claim that $M\oslash g^\vee \in \mathscr{A}\,\mathrm{mod}$. Indeed, M fits into an exact sequence

$$\oplus_{i=1}^m [a_i,-]\oslash g_i \to \oplus_{i=1}^n [a_j,-]\oslash g_j \to M \to 0$$

for some $a_i, a_j \in \mathscr{A}$, $g_i, g_j \in \mathscr{G}$. As the sequence

$$\oplus_{i=1}^m [a_i,-]\oslash (g_i\otimes g^\vee) \to \oplus_{i=1}^n [a_j,-]\oslash (g_j\otimes g^\vee) \to M\oslash g^\vee \to 0$$

is exact and $\mathscr{G}$ is closed under tensor products and dual objects, our claim follows.

We see that $\mathscr{C}_{\mathscr{A}}$ is generated by $\{[M,-] \mid M\in\mathscr{A}\,\mathrm{mod}\}$. It remains to show that each generator $[M,-]$ is coherent. Let A be a finitely generated subobject of $[M,-]$. There is $N\in\mathscr{A}\,\mathrm{mod}$ and an epimorphism $\eta : [N,-] \twoheadrightarrow A$. Composing it with the embedding $A \hookrightarrow [M,-]$, one gets a morphism $\tau : [N,-]\to[M,-]$. Since

$$\mathrm{Hom}_{\mathscr{C}_{\mathscr{A}}}([N,-],[M,-]) = \mathrm{Hom}_{\mathscr{V}}(e,[[N,-],[M,-]]) \cong \mathrm{Hom}_{\mathscr{V}}(e,[M,N]) = \mathrm{Hom}_{\mathscr{A}}(M,N),$$

it follows that $\tau = [f,-]$ for some $f\in\mathrm{Hom}_{\mathscr{A}}(M,N)$. Let $K = \mathrm{Coker}\, f$, then $K\in\mathscr{A}\,\mathrm{mod}$ and $\mathrm{Ker}\,\eta = [K,-]$ is finitely presented. We see that A is finitely presented, and hence $[M,-]$ is coherent. This completes the proof. □

Let $M\in\mathrm{Mod}\,\mathscr{A}$ and $N\in\mathscr{A}\,\mathrm{Mod}$. By Theorem 2.10 one has

$$M \cong \int^{\mathrm{Ob}\,\mathscr{A}} M(a)\oslash[-,a] \quad\text{and}\quad N\cong\int^{\mathrm{Ob}\,\mathscr{A}} [b,-]\oslash N(b).$$

By definition,

$$M\otimes_{\mathscr{A}} N := \int^{a\in\mathrm{Ob}\,\mathscr{A}}\int^{b\in\mathrm{Ob}\,\mathscr{A}} M(a)\oslash[b,a]\oslash N(b)\in\mathrm{Ob}\,\mathscr{V}.$$

In particular, $[-,a]\otimes_{\mathscr{A}}[b,-]=[b,a]$, $M\otimes_{\mathscr{A}}[b,-]=M(b)$ and $[-,a]\otimes_{\mathscr{A}} N = N(a)$. As coends in $\mathscr{V}$ are coequalizers, the latter three equalities uniquely determine $M\otimes_{\mathscr{A}} N$. In this way we arrive at a bifunctor

$$-\otimes_{\mathscr{A}} - : \mathrm{Mod}\,\mathscr{A}\times\mathscr{A}\,\mathrm{Mod}\to\mathscr{V}.$$

If $\gamma\in\mathrm{Hom}_{\mathscr{A}}(M,M')$, $\delta\in\mathrm{Hom}_{\mathscr{A}}(N,N')$, then

$$\gamma\otimes\delta := \int^{a\in\mathrm{Ob}\,\mathscr{A}}\int^{b\in\mathrm{Ob}\,\mathscr{A}} \gamma_a\oslash[b,a]\oslash\delta_b\in\mathrm{Mor}\,\mathscr{V}.$$

We use Corollary 2.5 here as well.

Observe that the restriction of $-\otimes_{\mathscr{A}} N$, $N\in\mathscr{A}\,\mathrm{Mod}$, to the full $\mathscr{V}$-subcategory $\mathrm{mod}\,\mathscr{A}$ of $\mathrm{Mod}\,\mathscr{A}$ is recovered as the enriched left Kan extension of the $\mathscr{V}$-functor $N:\mathscr{A}\to\mathscr{V}$ along the full embedding $\mathscr{A}\hookrightarrow\mathrm{mod}\,\mathscr{A}$ due to the fact that $\mathrm{mod}\,\mathscr{A}$ is (skeletally) small (see [7, Theorem 6.7.7] and [31, Proposition 4.33]).

We see that $-\otimes_{\mathscr{A}} N$ belongs to ${}_{\mathscr{A}}\mathscr{C}$. Though this fact will be enough for our purposes, nevertheless we want to show that the functor

$$-\otimes_{\mathscr{A}} N : \mathrm{Mod}\,\mathscr{A}\to\mathscr{V}$$

is actually a $\mathscr{V}$-functor between "large" $\mathscr{V}$-categories.

To this end, we have to construct morphisms in $\mathscr{V}$

$$\alpha_{M,M'} : [M,M']\to[M\otimes_{\mathscr{A}} N, M'\otimes_{\mathscr{A}} N],\quad M,M'\in\mathrm{Mod}\,\mathscr{A}.$$

By using Yoneda's lemma [6, Theorem 1.3.3], it is equivalent to constructing a natural transformation of functors

$$\mathrm{Hom}_{\mathscr{V}}(-,[M,M']) \to \mathrm{Hom}_{\mathscr{V}}(-,[M\otimes_{\mathscr{A}} N, M'\otimes_{\mathscr{A}} N]). \tag{9}$$

For any $U\in\mathscr{V}$ and any $f : U \oslash M \to M'$, we get a morphism $f\otimes \mathrm{id} : U\oslash M\otimes_{\mathscr{A}} N \to M'\otimes_{\mathscr{A}} N$, which uniquely corresponds to an element of $\mathrm{Hom}_{\mathscr{V}}(U,[M\otimes_{\mathscr{A}} N, M'\otimes_{\mathscr{A}} N])$. As this correspondence is plainly natural in U, the desired natural transformation (9) and morphisms $\alpha_{M,M'}$ are now constructed.

Next, for any $U,V\in\mathscr{V}$ one has bimorphisms

$$b(U,V) : (U,[M,M'])\otimes_{\mathbb{Z}}(V,[M',M'']) \to (U\otimes V,[M,M''])$$

associated to the composition map $c_{M,M',M''} : [M,M']\otimes[M',M''] \to [M,M'']$. One also has bimorphisms

$$b_N(U,V) : (U,[M\otimes_{\mathscr{A}} N,M'\otimes_{\mathscr{A}} N])\otimes_{\mathbb{Z}}(V,[M'\otimes_{\mathscr{A}} N,M''\otimes_{\mathscr{A}} N]) \to (U\otimes V,[M\otimes_{\mathscr{A}} N,M''\otimes_{\mathscr{A}} N])$$

associated to the composition map $c_{M\otimes N,M'\otimes N,M''\otimes N} : [M\otimes_{\mathscr{A}} N, M'\otimes_{\mathscr{A}} N]\otimes[M'\otimes_{\mathscr{A}} N, M'\otimes_{\mathscr{A}} N'] \to [M\otimes_{\mathscr{A}} N, M''\otimes_{\mathscr{A}} N]$. By construction, both bimorphisms guarantee commutativity of (3). Commutativity of (4) is obvious.

Thus we get the following statement:

Proposition 5.3. *For any left $\mathscr{A}$-module N, the functor $-\otimes_{\mathscr{A}} N : \mathrm{Mod}\,\mathscr{A} \to \mathscr{V}$ is a $\mathscr{V}$-functor between $\mathscr{V}$-categories. Moreover, the $\mathscr{V}$-functor*

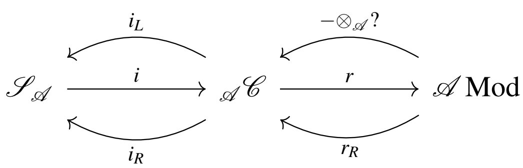
$$\mathscr{A}\,\mathrm{Mod} \to {}_{\mathscr{A}}\mathscr{C}, \quad N \mapsto -\otimes_{\mathscr{A}} N,$$

is $\mathscr{V}$-fully faithful.

Proof. The first part has been verified above. We have also mentioned that the restriction of $-\otimes_{\mathscr{A}} N$, $N\in\mathscr{A}\,\mathrm{Mod}$, to the full $\mathscr{V}$-subcategory $\mathrm{mod}\,\mathscr{A}$ of $\mathrm{Mod}\,\mathscr{A}$ is recovered as the enriched left Kan extension of the $\mathscr{V}$-functor $N : \mathscr{A} \to \mathscr{V}$ along the full embedding $\mathscr{A} \hookrightarrow \mathrm{mod}\,\mathscr{A}$, see [7, Theorem 6.7.7]. As the enriched Kan extension is $\mathscr{V}$-fully faithful by the proof of [7, Theorem 6.7.7], the second part follows as well. □

The following result relating $\mathscr{A}$-modules and generalised $\mathscr{A}$-modules is an enriched version for [14, Proposition 7.1] and [16, Corollary 2.2].

Theorem 5.4. *Define an enriched localizing subcategory $\mathscr{S}_{\mathscr{A}} := \{Y\in {}_{\mathscr{A}}\mathscr{C} \mid Y(a)=0 \text{ for all } a\in\mathscr{A}\} \subset {}_{\mathscr{A}}\mathscr{C}$. There is a recollement*



with functors i,r being the canonical inclusion and restriction functors respectively. The functor r_R is the enriched right Kan extension, i_R is the torsion functor associated with the localizing subcategory $\mathscr{S}_{\mathscr{A}}$. Furthermore, if ${}_{\mathscr{A}}\mathscr{C}/\mathscr{S}_{\mathscr{A}}$ is the quotient category of ${}_{\mathscr{A}}\mathscr{C}$ with respect to $\mathscr{S}_{\mathscr{A}}$, the functor $\mathscr{A}\,\mathrm{Mod} \to {}_{\mathscr{A}}\mathscr{C}/\mathscr{S}_{\mathscr{A}}$ sending M to $(-\otimes_{\mathscr{A}} M)_{\mathscr{S}_{\mathscr{A}}}$ is an equivalence of categories.

Proof. The theorem follows from [18, Theorems 3.4-3.5] and [1, Theorem 5.3] if we observe that the left Kan extension functor $r_L : \mathscr{A}\,\mathrm{Mod} \to {}_{\mathscr{A}}\mathscr{C}$ equals the functor $M \mapsto -\otimes_{\mathscr{A}} M$. □

The Ziegler spectrum can be defined for arbitrary locally coherent Grothendieck categories (see [24, 32]). Although $\mathscr{C}_{\mathscr{A}}$ is locally coherent by Theorem 5.2, and therefore the Ziegler spectrum $\mathrm{Zg}\,\mathscr{C}_{\mathscr{A}}$ in the sense of [24, 32] applies to ${}_{\mathscr{A}}\mathscr{C}$, this is actually not what we are going to investigate as $\mathrm{Zg}\,\mathscr{C}_{\mathscr{A}}$ does not capture the enriched category information of $\mathscr{C}_{\mathscr{A}}$. Below we will define the Ziegler spectrum $\mathrm{Zg}_{\mathscr{A}}$ associated with $\mathscr{C}_{\mathscr{A}}$ that captures both the enriched category information of $\mathscr{C}_{\mathscr{A}}$ and the machinery of [24, 32] (the points of $\mathrm{Zg}\,\mathscr{C}_{\mathscr{A}}$ and $\mathrm{Zg}_{\mathscr{A}}$ are the same but topologies are different). To this end, we need to introduce and study pure-injective $\mathscr{A}$-modules and $\underline{\mathrm{coh}}$-injective objects.

6. Pure-injective modules and $\underline{\mathrm{coh}}$-injective objects

Definition 6.1. A generalised $\mathscr{A}$-module $X \in {}_{\mathscr{A}}\mathscr{C}$ is said to be $\underline{\mathrm{coh}}$-*injective* if the functor $[-,X] : \mathrm{coh}\,{}_{\mathscr{A}}\mathscr{C} \to \mathscr{V}^{\mathrm{op}}$ is exact.

Theorem 6.2. *The following conditions are equivalent for $X \in {}_{\mathscr{A}}\mathscr{C}$:*

(1) *X is $\underline{\mathrm{coh}}$-injective;*
(2) *X is right exact on $\mathrm{mod}\,\mathscr{A}$;*
(3) *X is isomorphic to $-\otimes_{\mathscr{A}} N$ for some $N \in \mathscr{A}\,\mathrm{Mod}$.*

Proof. (1) $\Rightarrow$ (2). Given an exact sequence $K \xrightarrow{f} L \xrightarrow{g} M \to 0$ in $\mathrm{mod}\,\mathscr{A}$, one has a long exact sequence in the Abelian $\mathscr{V}$-category $\mathrm{coh}\,{}_{\mathscr{A}}\mathscr{C}$

$$0 \to [M,-] \xrightarrow{[f,-]} [L,-] \xrightarrow{[g,-]} [K,-] \xrightarrow{h} C \to 0. \tag{10}$$

As X is $\underline{\mathrm{coh}}$-injective by assumption, the functor $[-,X]$ is exact on $\mathrm{coh}\,{}_{\mathscr{A}}\mathscr{C}$. Therefore one has a long exact sequence in $\mathscr{V}$

$$0 \to [C,X] \xrightarrow{h^*} X(K) \xrightarrow{[g,X]} X(L) \xrightarrow{[f,X]} X(M) \to 0.$$

We see that X is right exact on $\mathrm{mod}\,\mathscr{A}$.

(2) $\Rightarrow$ (3). Suppose X is right exact. Denote by N the restriction of X to the full $\mathscr{V}$-subcategory $\mathscr{A}$ of $\mathrm{mod}\,\mathscr{A}$. Then $N \in \mathscr{A}\,\mathrm{Mod}$ and the counit of the adjunction gives a morphism $\varepsilon : -\otimes_{\mathscr{A}} N \to X$ in ${}_{\mathscr{A}}\mathscr{C}$. For any generator $g \oslash [-,a]$ of $\mathrm{Mod}\,\mathscr{A}$ one has

$$X(g \oslash [-,a]) = [[g \oslash [-,a],-],X] = [[g,[[-,a],-]],X] = [g^\vee \oslash [[-,a],-],X] =$$
$$= [g^\vee, X([-,a])] = g \oslash X([-,a]) = g \oslash N(a).$$

Any $M \in \mathrm{mod}\,\mathscr{A}$ fits into an exact sequence

$$\oplus_{i=1}^n g_i \oslash [-,a_i] \to \oplus_{j=1}^m g_j \oslash [-,a_j] \to M \to 0.$$

One gets a commutative diagramin $\mathscr{V}$

$$\begin{array}{ccccccc}
\oplus_{i=1}^n X(g_i \oslash [-,a_i]) & \longrightarrow & \oplus_{j=1}^m X(g_j \oslash [-,a_j]) & \longrightarrow & X(M) & \longrightarrow & 0 \\
\| & & \| & & \uparrow{\scriptstyle \varepsilon} & & \\
\oplus_{i=1}^n g_i \oslash N(a_i) & \longrightarrow & \oplus_{j=1}^m g_j \oslash N(a_j) & \longrightarrow & M \otimes_{\mathscr{A}} N & \longrightarrow & 0
\end{array}$$

It follows that $\varepsilon : -\otimes_{\mathscr{A}} N \to X$ is an isomorphism.

(3) $\Rightarrow$ (2). As the functor $\mathrm{Mod}\,\mathscr{A} \to \mathscr{C}_{\mathscr{A}}$, $M \mapsto M \otimes_{\mathscr{A}} -$, is left adjoint to the restriction functor, it is right exact and preserves arbitrary colimits. If $N \in \mathscr{A}\,\mathrm{mod}$, it follows that $-\otimes_{\mathscr{A}} N$ is right exact. If N is any left $\mathscr{A}$-module, then $N = \varinjlim_I N_i$ with $N_i \in \mathscr{A}\,\mathrm{mod}$. As the direct

limit functor is exact in $\mathscr{V}$ and $-\otimes_{\mathscr{A}} N \cong \varinjlim_I(-\otimes_{\mathscr{A}} N_i)$ is a direct limit of right exact functors, $-\otimes_{\mathscr{A}} N$ is right exact itself.

(2) $\Rightarrow$ (1). Let $C \in \operatorname{coh}_{\mathscr{A}}\mathscr{C}$ and let X be right exact. Then it fits into an exact sequence of the form (10). One has a diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow [M,-] & \xrightarrow{[f,-]} & [L,-] & & \xrightarrow{[g,-]} & & [K,-] & \xrightarrow{h} & C \longrightarrow 0 \\ & & & \overset{\ell}{\searrow} & & \overset{k}{\nearrow} & & & \\ & & & & \operatorname{Ker} h & & & & \end{array}$$

It induces an exact sequence in $\mathscr{V}$

$$0 \to [C,X] \xrightarrow{h^*} X(K) \xrightarrow{k^*} [\operatorname{Ker} h, X].$$

Since X is right exact, the row of the diagram

$$\begin{array}{ccccccc} X(K) & & \xrightarrow{X(g)} & & X(L) & \xrightarrow{X(f)} & X(M) \longrightarrow 0 \\ & \overset{k^*}{\searrow} & & \overset{\ell^*}{\nearrow} & & & \\ & & [\operatorname{Ker} h, X] & & & & \end{array}$$

is exact, and hence $\operatorname{Im} X(g) = \operatorname{Coker}(\ker X(g)) \cong \operatorname{Ker}(\operatorname{coker} X(g)) = \operatorname{Ker} X(f) = [\operatorname{Ker} h, X]$. We see that k^* is an epimorphism.

The functor between Abelian categories $[-,X] : \operatorname{coh}_{\mathscr{A}}\mathscr{C} \to \mathscr{V}^{\mathrm{op}}$ is right exact. The above arguments yield the following properties:

(1) the collection of objects $\mathscr{R} = \{[M,-] \mid M \in \operatorname{mod}\mathscr{A}\}$ is adapted to the functor $[-,X]$ in the sense of [22, Section III.6.3], and hence it induces a triangulated functor between derived categories $L[-,X] : D^-(\operatorname{coh}_{\mathscr{A}}\mathscr{C}) \to D^-(\mathscr{V}^{\mathrm{op}}) = D^+(\mathscr{V})$ — see [22, Section III.6.7];

(2) $[-,X]$ takes any resolution $R_\bullet \to C$ of $C \in \operatorname{coh}_{\mathscr{A}}\mathscr{C}$ by objects in $\mathscr{R}$ to an acyclic complex $[C,X] \to [R_\bullet, X]$ in $\mathrm{Ch}^+(\mathscr{V})$;

A short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\operatorname{coh}_{\mathscr{A}}\mathscr{C}$ gives rise to a triangle $A_3[-1] \to A_1 \to A_2 \to A_3$ in $D^-(\operatorname{coh}_{\mathscr{A}}\mathscr{C})$. Similarly to constructing projective resolutions there is a resolution $R^i_\bullet : [M_i^2,-] \hookrightarrow [M_i^1,-] \to [M_i^0,-]$ of each A_i, $i = 1,2,3$, such that the short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ fits to a commutative diagram of complexes

$$\begin{array}{ccccc} R^1_\bullet & \xrightarrow{\alpha} & R^2_\bullet & \xrightarrow{\beta} & R^3_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ A_1 & \rightarrowtail & A_2 & \twoheadrightarrow & A_3 \end{array}$$

The canonical functor $K^-(\mathscr{R})[S_{\mathscr{R}}^{-1}] \to D^-(\operatorname{coh}_{\mathscr{A}}\mathscr{C})$ is an equivalence of triangulated categories by [22, Proposition III.6.4], where $S_{\mathscr{R}}^{-1}$ is the class of quasi-isomorphisms in $K^-(\mathscr{R})$. It follows that the sequence of resolutions above fits to a triangle

$$R^3_\bullet[-1] \to R^1_\bullet \xrightarrow{\alpha} R^2_\bullet \xrightarrow{\beta} R^3_\bullet$$

in $K^-(\mathscr{R})[S_{\mathscr{R}}^{-1}]$. After applying the functor $L[-,X]$ to it, one gets a triangle in $D^+(\mathscr{V})$

$$[R^3_\bullet, X] \xrightarrow{\beta^*} [R^2_\bullet, X] \xrightarrow{\alpha^*} [R^1_\bullet, X] \to [R^3_\bullet, X][1].$$

This triangle induces a long exact sequence in $\mathscr{V}$ of cohomology objects

$$0 \to H^0([R^3_\bullet, X]) = [A_3, X] \to H^0([R^2_\bullet, X]) = [A_2, X] \to H^0([R^1_\bullet, X]) = [A_1, X] \to H^1([R^3_\bullet, X]) = 0.$$

Thus $[-, X]$ is an exact functor, as was to be shown. □

Definition 6.3. We say that an $\mathscr{A}$-homomorphism $f : K \to L$ in $\mathscr{A}\,\mathrm{Mod}$ is a *pure-monomorphism* if for any $M \in \mathrm{mod}\,\mathscr{A}$ the induced morphism $M \otimes f : M \otimes_{\mathscr{A}} K \to M \otimes_{\mathscr{A}} L$ in $\mathscr{V}$ is a monomorphism or, equivalently, $- \otimes f : - \otimes_{\mathscr{A}} K \to - \otimes_{\mathscr{A}} L$ is a monomorphism in ${}_{\mathscr{A}}\mathscr{C}$. A left $\mathscr{A}$-module Q is said to be *pure-injectiye* if every pure-monomorphism $p : Q \to N$ is a split monomorphism.

Corollary 6.4. *An object $E \in {}_{\mathscr{A}}\mathscr{C}$ is injective if and only if it is isomorphic to one of the functors $- \otimes_{\mathscr{A}} Q$, where Q is a pure-injective left $\mathscr{A}$-module.*

Proof. The proof literally repeats that of [24, Proposition 4.1]. In detail, if $E \in {}_{\mathscr{A}}\mathscr{C}$ is injective, then it is coh-injective by Lemma 4.6. Theorem 6.2 implies it is isomorphic to one of the functors $- \otimes_{\mathscr{A}} Q$ with $Q \in \mathscr{A}\,\mathrm{Mod}$. The injective hypothesis implies that Q must be pure-injective (we implicitly use Proposition 5.3 here).

Conversely, suppose Q is pure-injective and $\alpha : - \otimes_{\mathscr{A}} Q \to X$ is a monomorphism in ${}_{\mathscr{A}}\mathscr{C}$. Let E be the injective envelope of X. By the first part of the proof $E \cong - \otimes_{\mathscr{A}} N$ for some pure-injective $N \in \mathscr{A}\,\mathrm{Mod}$. Then the composite monomorphism $- \otimes_{\mathscr{A}} Q \overset{\alpha}{\hookrightarrow} X \hookrightarrow - \otimes_{\mathscr{A}} N$ splits in ${}_{\mathscr{A}}\mathscr{C}$ as the pure-monomorphism $Q \hookrightarrow N$ splits in $\mathscr{A}\,\mathrm{Mod}$ (we implicitly use Proposition 5.3 here). It follows that α is a split monomorphism. □

We have now collected all the necessary information to pass to the definition of the Ziegler spectrum of $\mathscr{A}$.

7. The Ziegler spectrum of an enriched ringoid

As ${}_{\mathscr{A}}\mathscr{C}$ is locally coherent, the full $\mathscr{V}$-subcategory of coherent objects $\mathrm{coh}\,{}_{\mathscr{A}}\mathscr{C}$ is Abelian. A full $\mathscr{V}$-subcategory $\mathscr{S}$ of $\mathrm{coh}\,{}_{\mathscr{A}}\mathscr{C}$ is called an *enriched Serre subcategory* if it is a Serre subcategory after forgetting the $\mathscr{V}$-structure and $m \oslash A \in \mathscr{S}$ for any finitely presented object m of $\mathscr{V}$ and any $A \in \mathscr{S}$. Recall from [24, 32] that any localizing subcategory of finite type in ${}_{\mathscr{A}}\mathscr{C}$ is of the form $\vec{\mathscr{S}}$, where $\mathscr{S}$ is a Serre subcategory and every object of $\vec{\mathscr{S}}$ is a direct limit of objects from $\mathscr{S}$.

Proposition 7.1. *The following statements are equivalent for a localising subcategory of finite type $\vec{\mathscr{S}}$ of ${}_{\mathscr{A}}\mathscr{C}$:*

1. *$\vec{\mathscr{S}}$ is enriched;*
2. *$\mathscr{S}$ is an enriched Serre subcategory;*
3. *$g \oslash A \in \mathscr{S}$ for any generator $g \in \mathscr{G}$ and $A \in \mathscr{S}$.*

Proof. (1) $\Rightarrow$ (2). The only thing to verify here is to show that $m \oslash A \in \mathrm{coh}\,{}_{\mathscr{A}}\mathscr{C}$ for any $m \in \mathrm{fp}(\mathscr{V})$ and $A \in \mathscr{S}$ as $m \oslash A \in \vec{\mathscr{S}}$ by assumption. Using Lemma 4.5, one has

$$(m \oslash A, \varinjlim_I X_i) = (m, [A, \varinjlim_I X_i]) = \varinjlim_I (m, [A, X_i]) = \varinjlim_I (m \oslash A, X_i).$$

(2) $\Rightarrow$ (3). This is straightforward.

(3) $\Rightarrow$ (1). Let $M \in \mathscr{V}$ and $X \in \vec{\mathscr{S}}$. Then $M = \varinjlim_I m_i$ and $X = \varinjlim_J X_j$ for some $m_i \in \mathrm{fp}(\mathscr{V})$ and $X_j \in \mathscr{S}$. It follows that $M \oslash X \cong \varinjlim_{I,J} m_i \oslash X_j$. It is enough to check that each $m_i \oslash X_j$ is in

$\mathscr{S}$. Each m_i fits in an exact sequence

$$\oplus_{s=1}^m g_s \to \oplus_{t=1}^n g_t \to m_i \to 0, \quad g_s, g_t \in \mathscr{G}.$$

It induces an epimorphism $\oplus_{t=1}^n (g_t \oslash X_j) \twoheadrightarrow m_i \oslash X_j$. Since each $g_t \oslash X_j \in \mathscr{S}$ by assumption and $\mathscr{S}$ is a Serre subcategory, we see that $m_i \oslash X_j \in \mathscr{S}$, as required. $\square$

By theorems of Herzog [24] and Krause [32] there is a bijective correspondences between Serre subcategories in $\operatorname{coh}\mathscr{C}$ and localising subcategories $\mathscr{T}$ of $\mathscr{C}$ of finite type, where $\mathscr{C}$ is a locally coherent Grothendieck category. Combining this correspondence with the preceding proposition, one gets the following statement.

Corollary 7.2. *There is an inclusion-preserving bijective correspondence between enriched Serre subcategories $\mathscr{S}$ of $\operatorname{coh}_{\mathscr{A}}\mathscr{C}$ and enriched localising subcategories $\mathscr{T}$ of ${}_{\mathscr{A}}\mathscr{C}$ of finite type. This correspondence is given by the functions*

$$\mathscr{S} \mapsto \vec{\mathscr{S}}, \quad \mathscr{T} \mapsto \mathscr{T} \cap \operatorname{coh}_{\mathscr{A}}\mathscr{C},$$

which are mutual inverses.

Denote by ${}_{\mathscr{A}}\mathrm{Zg}$ (respectively $\mathrm{Zg}_{\mathscr{A}}$) the set of the isomorphism classes of indecomposable pure-injective modules of $\mathscr{A}\,\mathrm{Mod}$ (respectively $\mathrm{Mod}\,\mathscr{A}$). To an arbitrary $\mathscr{V}$-subcategory $\mathscr{X}$ of $\operatorname{coh}_{\mathscr{A}}\mathscr{C}$, we associate the subset of ${}_{\mathscr{A}}\mathrm{Zg}$,

$$\mathscr{O}(\mathscr{X}) = \{Q \in {}_{\mathscr{A}}\mathrm{Zg} \mid \text{for some } C \text{ in } \mathscr{X}, [C, - \otimes_{\mathscr{A}} Q] \neq 0\}.$$

If $\mathscr{X} = \{C\}$ is a singleton, we write $\mathscr{O}(C)$ to denote $\mathscr{O}(\mathscr{X})$. Thus $\mathscr{O}(\mathscr{X}) = \bigcup_{C \in \mathscr{X}} \mathscr{O}(C)$. Denote by $\sqrt{\mathscr{X}}$ (respectively $\langle \mathscr{X} \rangle$) the smallest Serre subcategory (respectively the smallest enriched Serre subcategory) of $\operatorname{coh}_{\mathscr{A}}\mathscr{C}$ containing $\mathscr{X}$.

Recall that an object $A \in \operatorname{coh}_{\mathscr{A}}\mathscr{C}$ is a *subquotient* of $B \in \operatorname{coh}_{\mathscr{A}}\mathscr{C}$, if there is a filtration of B by coherent subobjects $B = B_0 \geqslant B_1 \geqslant B_2 \geqslant 0$ such that $A \cong B_1/B_2$.

Lemma 7.3. *$\langle \mathscr{X} \rangle = \sqrt{(\mathscr{G} \oslash \mathscr{X})}$, where $\mathscr{G} \oslash \mathscr{X} = \{g \oslash X \mid g \in \mathscr{G}, X \in \mathscr{X}\}$.*

Proof. Clearly, $\langle \mathscr{X} \rangle$ contains $\sqrt{(\mathscr{G} \oslash \mathscr{X})}$. By [24, Proposition 3.1] a coherent object $C \in \sqrt{(\mathscr{G} \oslash \mathscr{X})}$ if and only if there are a finite filtration of C by coherent subobjects

$$C = C_0 \supseteq C_1 \supseteq \cdots \supseteq C_n = 0$$

and, for every $i < n$, $A_i \in \mathscr{G} \oslash \mathscr{X}$ such that C_i/C_{i+1} is a subquotient of A_i. It follows from Corollary 4.3 that the functor $g \oslash - : \operatorname{coh}_{\mathscr{A}}\mathscr{C} \to \operatorname{coh}_{\mathscr{A}}\mathscr{C}$ is exact. In particular, it respects filtrations as above. Therefore $\sqrt{(\mathscr{G} \oslash \mathscr{X})}$ is an enriched Serre subcategory of $\operatorname{coh}_{\mathscr{A}}\mathscr{C}$ by Proposition 7.1, and hence $\sqrt{(\mathscr{G} \oslash \mathscr{X})}$ contains $\langle \mathscr{X} \rangle$. $\square$

Lemma 7.4. *If $A, B \in \operatorname{coh}_{\mathscr{A}}\mathscr{C}$ and A is a subquotient of B, then $\mathscr{O}(A) \subset \mathscr{O}(B)$. If $0 \to A \to B \to C \to 0$ is a short exact sequence in $\operatorname{coh}_{\mathscr{A}}\mathscr{C}$, then $\mathscr{O}(B) = \mathscr{O}(A) \cup \mathscr{O}(C)$.*

Proof. This immediately follows from Lemma 4.6 saying that $[-, E] : \operatorname{coh}_{\mathscr{A}}\mathscr{C} \to \mathscr{V}$ is an exact functor for every injective object of ${}_{\mathscr{A}}\mathscr{C}$. $\square$

Corollary 7.5. *For any $\mathscr{V}$-subcategory $\mathscr{X} \subseteq \operatorname{coh}_{\mathscr{A}}\mathscr{C}$, $\mathscr{O}(\mathscr{X}) = \mathscr{O}(\langle \mathscr{X} \rangle)$.*

Proof. It follows from Lemma 7.4 and the proof of Lemma 7.3 that $\mathscr{O}(\mathscr{G} \oslash \mathscr{X}) = \mathscr{O}(\langle \mathscr{X} \rangle)$. Suppose $[g \oslash C, E] \neq 0$ for some $g \in \mathscr{G}$ and $C \in \mathscr{X}$. As $\mathscr{G}$ is a family of generators, there is $g' \in \mathscr{G}$ such that $\operatorname{Hom}_{\mathscr{V}}(g', [g \oslash C, E]) \neq 0$. Then $\operatorname{Hom}_{\mathscr{V}}(g' \otimes g, [C, E]) \neq 0$, and hence $[C, E] \neq 0$, because $g' \otimes g \in \mathscr{G}$. We see that $E \in \mathscr{O}(\mathscr{X})$. $\square$

Lemma 7.6. *If $C, E \in {}_{\mathscr{A}}\mathscr{C}$, then $[C,E] \neq 0$ if and only if there is $g \in \mathscr{G}$ such that* $\mathrm{Hom}_{{}_{\mathscr{A}}\mathscr{C}}(g \oslash C, E) \neq 0$.

Proof. Suppose $[C,E] \neq 0$. Since $\mathscr{G}$ is a family of generators of $\mathscr{V}$, there is $g \in \mathscr{G}$ such that $0 \neq \mathrm{Hom}_{{}_{\mathscr{A}}\mathscr{C}}(g, [C,E]) \cong \mathrm{Hom}_{{}_{\mathscr{A}}\mathscr{C}}(g \oslash C, E)$. Conversely, if $\mathrm{Hom}_{{}_{\mathscr{A}}\mathscr{C}}(g \oslash C, E) \neq 0$ for some $g \in \mathscr{G}$ then $[C,E] \neq 0$. □

If $\vec{\mathscr{S}}$ is a localising subcategory of finite type in ${}_{\mathscr{A}}\mathscr{C}$, denote by $t_{\mathscr{S}} : {}_{\mathscr{A}}\mathscr{C} \to \vec{\mathscr{S}}$ the torsion functor associated to $\vec{\mathscr{S}}$. Without loss of generality (for this use Lemma 7.6) we can restrict our discussion to subsets of the form

$$\mathscr{O}(\mathscr{S}) = \{Q \in {}_{\mathscr{A}}\mathrm{Zg} \mid t_{\mathscr{S}}(- \otimes_{\mathscr{A}} Q) \neq 0\},$$

where $\mathscr{S}$ is an enriched Serre subcategory of $\mathrm{coh}\,{}_{\mathscr{A}}\mathscr{C}$.

The following result is the enriched version of the celebrated Ziegler Theorem [40, Theore 4.9] that associates to a ring R a topological space ${}_R\mathrm{Zg}$, which we refer to as the *Ziegler spectrum of the ring R*, whose points are the isomorphism classes of the pure-injective indecomposable left R-modules. This theorem has been treated by Herzog [24, Section 4] in terms of the category of generalised modules ${}_R\mathscr{C} = (\mathrm{mod}\,R, \mathrm{Ab})$. We also refer the reader to books by Prest [33, 34].

Theorem 7.7 (Ziegler)**.** *The following statements are true:*

(1) *The collection of subsets of ${}_{\mathscr{A}}\mathrm{Zg}$,*

$$\{\mathscr{O}(\mathscr{S}) \mid \mathscr{S} \subseteq \mathrm{coh}\,{}_{\mathscr{A}}\mathscr{C} \text{ is an enriched Serre subcategory}\},$$

satisfies the axioms for the open sets of a topology on ${}_{\mathscr{A}}\mathrm{Zg}$. This topological space is called the Ziegler spectrum of the enriched ringoid $\mathscr{A}$.

(2) *The collection of open subsets $\{\mathscr{O}(C) \mid C \in \mathrm{coh}\,{}_{\mathscr{A}}\mathscr{C}\}$ satisfies the axioms for a basis of open subsets of the Ziegler spectrum. Furthermore, $\mathscr{O}(C) = \emptyset$ if and only if $C = 0$.*

(3) *An open subset $\mathscr{O}$ of ${}_{\mathscr{A}}\mathrm{Zg}$ is quasi-compact if and only if it is one of the basic open subsets $\mathscr{O}(C)$ with $C \in \mathrm{coh}\,{}_{\mathscr{A}}\mathscr{C}$.*

Proof. (1). Clearly, $\mathscr{O}(0) = \emptyset$ and $\mathscr{O}(\mathrm{coh}\,{}_{\mathscr{A}}\mathscr{C}) = {}_{\mathscr{A}}\mathrm{Zg}$. By Corollary 7.5 one has $\bigcup_{i\in I}\mathscr{O}(\mathscr{S}_i) = \mathscr{O}(\bigcup_{i\in I}\mathscr{S}_i) = \mathscr{O}(\langle\bigcup_{i\in I}\mathscr{S}_i\rangle)$. The fact that $\mathscr{O}(\mathscr{S}_1)\cap\mathscr{O}(\mathscr{S}_2) = \mathscr{O}(\mathscr{S}_1\cap\mathscr{S}_2)$ literally repeats the proof of [24, Theorem 3.4] if we observe that the intersection of two enriched Serre subcategories is an enriched Serre subcategory.

(2). The first part of this statement follows from the fact that $\mathscr{O}(\mathscr{X}) = \bigcup_{C\in\mathscr{X}}\mathscr{O}(C)$ for any $\mathscr{V}$-subcategory $\mathscr{X}$ of $\mathrm{coh}\,{}_{\mathscr{A}}\mathscr{C}$. Suppose $[C, -\otimes_{\mathscr{A}} Q] = 0$ for any $Q \in {}_{\mathscr{A}}\mathrm{Zg}$. It follows that $\mathrm{Hom}_{{}_{\mathscr{A}}\mathscr{C}}(e, [C, -\otimes_{\mathscr{A}} Q]) = \mathrm{Hom}_{{}_{\mathscr{A}}\mathscr{C}}(C, -\otimes_{\mathscr{A}} Q) = 0$. Then $C = 0$ by [24, Corollary 3.5].

(3). The proof is like that of [24, Corollary 3.9]. Suppose $\mathscr{O}$ is quasi-compact. By the previous statement $\mathscr{O} = \bigcup_{i\in I}\mathscr{O}(C_i)$. Then there is a finite subset J of I such that $\mathscr{O} = \bigcup_{i\in J}\mathscr{O}(C_i) = \mathscr{O}(\sqcup_{i\in J}C_i)$. Conversely, suppose $\mathscr{O}(C) = \bigcup_{i\in I}\mathscr{O}(C_i) = \mathscr{O}(\{C_i \mid i\in I\}) = \mathscr{O}(\langle\{C_i \mid i \in I\}\rangle) = \{Q \in {}_{\mathscr{A}}\mathrm{Zg} \mid t_{\langle\{C_i|i\in I\}\rangle}(-\otimes_{\mathscr{A}} Q) \neq 0\} = \{Q \in {}_{\mathscr{A}}\mathrm{Zg} \mid t_{\langle\{C\}\rangle}(-\otimes_{\mathscr{A}} Q) \neq 0\}$.

By [24, Theorem 3.8] $\langle\{C\}\rangle = \langle\{C_i \mid i\in I\}\rangle$. By Lemma 7.3 $\langle\{C_i \mid i \in I\}\rangle = \sqrt{(\mathscr{G}\oslash\{C_i \mid i\in I\})} = \sqrt{(\mathscr{G}\oslash\{C\})}$. Since $\mathscr{G}$ is a family of generators for $\mathscr{V}$ and $e \in \mathrm{fp}(\mathscr{V})$, there is an epimorphism $\oplus_{i=1}^n g_i \twoheadrightarrow e$ in $\mathscr{V}$. It induces an epimorphism $\oplus_{i=1}^n g_i \oslash C \twoheadrightarrow C$ in ${}_{\mathscr{A}}\mathscr{C}$. Since $\oplus_{i=1}^n g_i \oslash C \in \sqrt{(\mathscr{G}\oslash\{C\})}$, one has $C \in \sqrt{(\mathscr{G}\oslash\{C\})}$.

We see that $C \in \sqrt{(\mathscr{G}\oslash\{C_i \mid i\in I\})}$. By [24, Proposition 3.1] there are a finite filtration of C by coherent subobjects

$$C = D_0 \supseteq D_1 \supseteq \cdots \supseteq D_n = 0$$

and, for every $\ell < n$, $g_\ell \in \mathscr{G}$, $C_\ell \in \{C_i \mid i \in I\}$, such that $D_\ell/D_{\ell+1}$ is a subquotient of $g_\ell \oslash C_\ell$. Since only finitely many of the C_i are needed, there is a finite subset J of I such that $C \in \sqrt{(\mathscr{G} \oslash \{C_i \mid i \in J\})} = \langle \mathscr{G} \oslash \{C_i \mid i \in J\} \rangle$. Using Lemma 7.4, it follows that $\mathscr{O}(C) \subset \bigcup_{i\in J} \mathscr{O}(C_i)$, and hence $\mathscr{O}(C) = \bigcup_{i\in J} \mathscr{O}(C_i)$. □

8. Enriched Auslander–Gruson–Jensen Duality

In his thesis [37] Sorokin constructed various bifunctors between enriched categories. As an application, he gets an enriched version of the Auslander–Gruson–Jensen Duality. In this section we treat this duality in our context and refer the reader to [37] for more general context.

Namely, define a $\mathscr{V}$-functor

$$D : (\operatorname{coh}_{\mathscr{A}}\mathscr{C})^{\mathrm{op}} \to \operatorname{coh}\mathscr{C}_{\mathscr{A}}$$

by the rule

$$D(C)(N) := [C, - \otimes_{\mathscr{A}} N], \quad C \in \operatorname{coh}_{\mathscr{A}}\mathscr{C}, \quad N \in \mathscr{A}\operatorname{mod}.$$

We have to check that $D(C) \in \operatorname{coh}\mathscr{C}_{\mathscr{A}}$. First note that for any $M \in \operatorname{mod}\mathscr{A}$ enriched Yoneda's lemma implies

$$D([M,-])(N) = [[M,-], - \otimes_{\mathscr{A}} N] \cong M \otimes_{\mathscr{A}} N.$$

Therefore $D([M,-]) \cong M \otimes_{\mathscr{A}} -$. Given $C \in \operatorname{coh}_{\mathscr{A}}\mathscr{C}$, there is an exact sequence

$$[L,-] \to [M,-] \to C \to 0$$

in $\operatorname{coh}_{\mathscr{A}}\mathscr{C}$. As $- \otimes_{\mathscr{A}} N$ is $\underline{\text{coh}}$-injective by Theorem 6.2, D is exact and takes the exact sequence to an exact sequence in $\operatorname{coh}\mathscr{C}_{\mathscr{A}}$

$$0 \to D(C) \to M \otimes_{\mathscr{A}} - \to L \otimes_{\mathscr{A}} -.$$

We see that $D(C) \in \operatorname{coh}\mathscr{C}_{\mathscr{A}}$ as claimed.

By construction, $D(- \otimes_{\mathscr{A}} M)(N) = [- \otimes_{\mathscr{A}} M, - \otimes_{\mathscr{A}} N]$. it follows from Proposition 5.3 that $D(- \otimes_{\mathscr{A}} M) \cong [M,-]$. By symmetry, consider

$$D' : (\operatorname{coh}\mathscr{C}_{\mathscr{A}})^{\mathrm{op}} \to \operatorname{coh}_{\mathscr{A}}\mathscr{C}$$

defined similarly to D. Then $D'D$ and DD' are exact functors respecting functors of the form $[M,-]$ (up to isomorphism). Therefore D and D' are mutually inverse $\mathscr{V}$-equivalences between $\mathscr{V}$-categories.

Thus we have proven the following result (cf. [24, Theorem 5.1]).

Theorem 8.1 (Auslander [3], Gruson and Jensen [23]). *The $\mathscr{V}$-functor $D : (\operatorname{coh}_{\mathscr{A}}\mathscr{C})^{\mathrm{op}} \to \operatorname{coh}\mathscr{C}_{\mathscr{A}}$ defined above puts the $\mathscr{V}$-categories $\operatorname{coh}_{\mathscr{A}}\mathscr{C}$ and $\operatorname{coh}\mathscr{C}_{\mathscr{A}}$ in duality. Moreover, if $M \in \operatorname{mod}\mathscr{A}$ and $N \in \mathscr{A}\operatorname{mod}$, we have that $D([M,-]) \cong M \otimes_{\mathscr{A}} -$ and $D(- \otimes_{\mathscr{A}} N) \cong [N,-]$.*

An object X of an Abelian $\mathscr{V}$-category $\mathscr{C}$ is said to be *internally projective* (respectively *internally injective*) if the functor $[X,-] : \mathscr{C} \to \mathscr{V}$ (respectively $[-,X] : \mathscr{C} \to \mathscr{V}^{\mathrm{op}}$) is exact. By construction, $[M,-]$ is internally projective for ${}_{\mathscr{A}}\mathscr{C}$. As every $C \in \operatorname{coh}_{\mathscr{A}}\mathscr{C}$ is covered by an object of the form $[M,-]$, the Abelian $\mathscr{V}$-category $\operatorname{coh}_{\mathscr{A}}\mathscr{C}$ has enough internally projective objects. The following statement follows from Theorems 8.1 and 6.2.

Corollary 8.2. *Every internally injective object of $\operatorname{coh}_{\mathscr{A}}\mathscr{C}$ is isomorphic to one of the $\mathscr{V}$-functors $- \otimes_{\mathscr{A}} N$, where $N \in \mathscr{A}\operatorname{mod}$. The Abelian $\mathscr{V}$-category $\operatorname{coh}_{\mathscr{A}}\mathscr{C}$ has enough internally injective objects, that is, for every $C \in \operatorname{coh}_{\mathscr{A}}\mathscr{C}$, there is a monomorphism $\iota : C \hookrightarrow - \otimes_{\mathscr{A}} N$ with $N \in \mathscr{A}\operatorname{mod}$.*

Given an enriched Serre subcategory $\mathscr{S} \subseteq \operatorname{coh}{}_{\mathscr{A}}\mathscr{C}$, the full $\mathscr{V}$-subcategory $D\mathscr{S} = \{DC \mid C \in \mathscr{S}\}$ is plainly Serre. If $g \in \mathscr{G}$, $C \in \operatorname{coh}{}_{\mathscr{A}}\mathscr{C}$ and $N \in \mathscr{A}\,\mathrm{mod}$, then

$$g \oslash D(C)(N) = g \oslash [C, - \otimes_{\mathscr{A}} N] \cong [g^{\vee} \oslash C, - \otimes_{\mathscr{A}} N] = D(g^{\vee} \oslash C)(N).$$

Since $g^{\vee} \oslash C \in \mathscr{S}$, it follows that $g \oslash D(C) \in D\mathscr{S}$, and hence $D\mathscr{S}$ is an enriched Serre subcategory of $\operatorname{coh}\mathscr{C}_{\mathscr{A}}$ by Proposition 7.1.

We arrive at an enriched version of a theorem of Herzog [24, Theorem 5.5] (the proof of the second half of the theorem literally repeats arguments given in [24, p. 536]).

Theorem 8.3 (Herzog)**.** *There is an inclusion-preserving bijective correspondence between the enriched Serre subcategories of* $\operatorname{coh}{}_{\mathscr{A}}\mathscr{C}$ *and those of* $\operatorname{coh}\mathscr{C}_{\mathscr{A}}$ *given by*

$$\mathscr{S} \mapsto D\mathscr{S}.$$

The induced map $\mathscr{O}(\mathscr{S}) \mapsto \mathscr{O}(D\mathscr{S})$ *is an isomorphism between the topologies, that is, the respective algebras of open sets, of the left and right Ziegler spectra of* $\mathscr{A}$*. Furthermore, the duality* $D : (\operatorname{coh}{}_{\mathscr{A}}\mathscr{C})^{\mathrm{op}} \to \operatorname{coh}\mathscr{C}_{\mathscr{A}}$ *induces dualities between respective* $\mathscr{V}$*-subcategories* $D : \mathscr{S}^{\mathrm{op}} \to D\mathscr{S}$ *and* $D : (\operatorname{coh}{}_{\mathscr{A}}\mathscr{C}/\vec{\mathscr{S}})^{\mathrm{op}} \to \operatorname{coh}\mathscr{C}_{\mathscr{A}}/\vec{D\mathscr{S}}$ *as given by the following commutative diagram of Abelian* $\mathscr{V}$*-categories:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{S} & \longrightarrow & \operatorname{coh}{}_{\mathscr{A}}\mathscr{C} & \longrightarrow & \operatorname{coh}({}_{\mathscr{A}}\mathscr{C}/\vec{\mathscr{S}}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} & & \\ 0 & \longrightarrow & D\mathscr{S} & \longrightarrow & \operatorname{coh}\mathscr{C}_{\mathscr{A}} & \longrightarrow & \operatorname{coh}(\mathscr{C}_{\mathscr{A}}/\vec{D\mathscr{S}}) & \longrightarrow & 0 \end{array}$$

We have collected all the necessary facts about the Ziegler spectrum of an enriched ringoid and (coherent) generalised $\mathscr{A}$-modules in oder to pass to applications.

9. The injective spectrum of an enriched ringoid

Recall that a localizing subcategory $\mathscr{S}$ of a Grothendieck category $\mathscr{C}$ is *of finite type* (respectively *of strictly finite type*) if the functor $i : \mathscr{C}/\mathscr{S} \to \mathscr{C}$ preserves directed sums (respectively direct limits). If $\mathscr{C}$ is a locally finitely generated (respectively, locally finitely presented) Grothendieck category and $\mathscr{S}$ is of finite type (respectively, of strictly finite type), then $\mathscr{C}/\mathscr{S}$ is a locally finitely generated (respectively, locally finitely presented) Grothendieck category and

$$\operatorname{fg}(\mathscr{C}/\mathscr{S}) = \{C_{\mathscr{S}} \mid C \in \operatorname{fg}\mathscr{C}\} \quad (\text{respectively } \operatorname{fp}(\mathscr{C}/\mathscr{S}) = \{C_{\mathscr{S}} \mid C \in \operatorname{fp}\mathscr{C}\}).$$

If $\mathscr{C}$ is a locally coherent Grothendieck category then $\mathscr{S}$ is of finite type if and only if it is of strictly finite type (see, e.g., [14, Theorem 5.14]). In this case $\mathscr{C}/\mathscr{S}$ is locally coherent.

Suppose $\mathscr{C}$ is a locally finitely presented Grothendieck category. Denote by $\mathsf{Sp}\,\mathscr{C}$ the set of the isomorphism classes of indecomposable injective objects in $\mathscr{C}$. The Ziegler topology defined for locally coherent categories in [24, 32] was extended to $\mathsf{Sp}\,\mathscr{C}$ in [15, Theorem 11]. In detail, the collection of open subsets of $\mathsf{Sp}\,\mathscr{C}$ is given by

$$\{O(\mathscr{S}) \mid \mathscr{S} \subset \mathscr{C} \text{ is a localizing subcategory of finite type}\},$$

where

$$O(\mathscr{S}) = \{E \in \mathsf{Sp}\,\mathscr{C} \mid t_{\mathscr{S}}(E) \neq 0\}.$$

Following these results, denote by $\mathsf{Sp}\,\mathscr{A}$, where $\mathscr{A}$ is an enriched ringoid, the set of the isomorphism classes of indecomposable injective objects in $\mathscr{A}\,\mathrm{Mod}$. Given an enriched localizing subcategory of finite type $\mathscr{S} \subseteq \mathscr{A}\,\mathrm{Mod}$, we set

$$\mathscr{O}(\mathscr{S}) = \{E \in \mathsf{Sp}\,\mathscr{A} \mid t_{\mathscr{S}}(E) \neq 0\}.$$

Similarly to Lemma 7.6 and the fact that $\mathscr{S}$ is closed under tensor products with objects from $\mathscr{V}$, one gets that

$$\mathscr{O}(\mathscr{S}) = \{E \in \mathsf{Sp}\,\mathscr{A} \mid [\mathscr{S}, E] \neq 0\}.$$

Theorem 9.1 (after [15])**.** *The collection of subsets of* $\mathsf{Sp}\,\mathscr{A}$,

$$\{\mathscr{O}(\mathscr{S}) \mid \mathscr{S} \subset \mathscr{A}\,\mathrm{Mod} \text{ is an enriched localizing subcategory of finite type}\},$$

satisfies the axioms for the open sets of a topology on the injective spectrum $\mathsf{Sp}\,\mathscr{A}$*. Moreover,*

$$\mathscr{S} \longmapsto \mathscr{O}(\mathscr{S}) \tag{11}$$

is an inclusion-preserving bijection between the enriched localizing $\mathscr{V}$*-subcategories* $\mathscr{S}$ *of finite type in* $\mathscr{A}\,\mathrm{Mod}$ *and the open subsets of* $\mathsf{Sp}\,\mathscr{A}$*.*

Proof. Note that $\mathscr{O}(0) = \emptyset$ and $\mathscr{O}(\mathscr{A}\,\mathrm{Mod}) = \mathsf{Sp}\,\mathscr{A}$. The proof of [15, Theorem 11] shows that $\mathscr{O}(\mathscr{S}_1) \cap \mathscr{O}(\mathscr{S}_2) = \mathscr{O}(\mathscr{S}_1 \cap \mathscr{S}_2)$. This is due to the fact that $\mathscr{S}_1 \cap \mathscr{S}_2$ is enriched of finite type. The proof also shows that $\sqrt{\cup_{i\in I}\mathscr{S}_i}$ is of finite type with each $\mathscr{S}_i$ enriched of finite type.

Suppose $E \in \mathscr{A}\,\mathrm{Mod}$ is injective. It follows from [15, Theorem 11] that $t_{\sqrt{\cup_{i\in I}\mathscr{S}_i}}(E) = 0$ if and only if $t_{\mathscr{S}_i}(E) = 0$ for all $i \in I$. By Lemma 4.7 $[g, E]$ is injective for all $g \in \mathscr{G}$. Since each $\mathscr{S}_i$ is enriched, $[g, E]$ is $\mathscr{S}_i$-torsionfree if E is $\mathscr{S}_i$-torsionfree. So $g \oslash S \in \sqrt{\cup_{i\in I}\mathscr{S}_i}$ for all $g \in \mathscr{G}$ and $S \in \sqrt{\cup_{i\in I}\mathscr{S}_i}$. Since every $V \in \mathscr{V}$ is covered by $\oplus_I g_i$ with each $g_i \in \mathscr{G}$, we see that $V \oslash S$ also belongs to $\sqrt{\cup_{i\in I}\mathscr{S}_i}$. We conclude that $\sqrt{\cup_{i\in I}\mathscr{S}_i}$ is enriched. It follows from [15, Theorem 11] that $\cup_{i\in I}\mathscr{O}(\mathscr{S}_i) = \mathscr{O}(\cup_{i\in I}\mathscr{S}_i) = \mathscr{O}(\sqrt{\cup_{i\in I}\mathscr{S}_i})$ and that the map (11) is bijective. $\square$

The proof of the preceding theorem also implies the following statement.

Corollary 9.2. *A localizing subcategory* $\mathscr{S}$ *of* $\mathscr{A}\,\mathrm{Mod}$ *is enriched if and only if for any* $\mathscr{S}$*-closed injective* $\mathscr{A}$*-module* E *and any* $g \in \mathscr{G}$ *the injective* $\mathscr{A}$*-module* $[g, E]$ *is* $\mathscr{S}$*-closed.*

We say that the enriched ringoid $\mathscr{A}$ is *left coherent* if the category of left $\mathscr{A}$-modules $\mathscr{A}\,\mathrm{Mod}$ is locally coherent or, equivalently, $\mathscr{A}\,\mathrm{mod}$ is an Abelian $\mathscr{V}$-category.

A continuous function $f : X \to Y$ between topological spaces is called an *embedding* if in its image factorization

$$f : X \to f(X) \hookrightarrow Y$$

with the image $f(X) \hookrightarrow Y$ equipped with the subspace topology, we have that $X \to f(X)$ is a homeomorphism. This is called a *closed embedding* if the image $f(X) \subset Y$ is a closed subset.

Theorem 9.3. *Let* $\mathscr{A}$ *be left coherent. Then the enriched localising subcategory of Theorem* 5.4 $\mathscr{S}_{\mathscr{A}} := \{Y \in {}_{\mathscr{A}}\mathscr{C} \mid Y(a) = 0 \text{ for all } a \in \mathscr{A}\}$ *is of finite type in* ${}_{\mathscr{A}}\mathscr{C}$ *and the functor* $\mathscr{A}\,\mathrm{Mod} \to {}_{\mathscr{A}}\mathscr{C}$, $M \mapsto - \otimes_{\mathscr{A}} M$*, induces a closed embedding of topological spaces* $\mathsf{Sp}\,\mathscr{A} \hookrightarrow {}_{\mathscr{A}}\,\mathsf{Zg}$*.*

Proof. By Theorem 5.4 the functor $\mathscr{A}\,\mathrm{Mod} \to {}_{\mathscr{A}}\mathscr{C}/\mathscr{S}_{\mathscr{A}}$ sending M to $(- \otimes_{\mathscr{A}} M)_{\mathscr{S}_{\mathscr{A}}}$ is an equivalence of categories. Denote by $\mathscr{P} := \mathscr{S}_{\mathscr{A}} \cap \mathrm{coh}\,{}_{\mathscr{A}}\mathscr{C}$. Then $\mathscr{P}$ is an enriched Serre subcategory of $\mathrm{coh}\,{}_{\mathscr{A}}\mathscr{C}$. If we literally repeat the proof of the implication $(1) \Rightarrow (4)$ of [14, Theorem 7.4], we get that $\mathscr{S}_{\mathscr{A}} = \vec{\mathscr{P}}$, and hence $\mathscr{S}_{\mathscr{A}}$ is of finite type. Since ${}_{\mathscr{A}}\mathscr{C}$ is locally coherent, enriched localising subcategories of finite type and of strictly finite type coincide by [14, Theorem 5.14]. Our theorem now follows from [15, Proposition 12] and Theorems 7.7 and 9.1. $\square$

Consider the case $\mathscr{A} = \{e\}$, where e is the monoidal unit of $\mathscr{V}$. Then $\mathscr{A}\,\mathrm{Mod}$ or $\mathrm{Mod}\,\mathscr{A}$ is identified with $\mathscr{V}$ and $\mathscr{A}\,\mathrm{mod}\,/\,\mathrm{mod}\,\mathscr{A}$ is identified with $\mathrm{fp}(\mathscr{V})$. In this case the category of generalised $\mathscr{A}$-modules coincides with the Grothendieck category of enriched functors $[\mathrm{fp}(\mathscr{V}),\mathscr{V}]$.

Denote by $\mathrm{Zg}_{\mathscr{V}}$ the Ziegler spectrum of the enriched ringoid $\mathscr{A} = \{e\}$. It consists of the isomorphism classes of pure-injective objects of $\mathscr{V}$ equipped with topology of Theorem 7.7. We also refer the reader to [26, Section 6] for further properties of pure-injective objects in $\mathscr{V}$.

Lemma 10.1. *$\mathrm{Zg}_{\mathscr{V}}$ is a quasi-compact topological space.*

Proof. This follows from Theorem 7.7(3) if we observe that $\mathrm{Zg}_{\mathscr{V}} = \mathscr{O}(-\otimes e)$. □

In [9] Darbyshire showed that if R is a commutative ring then $[\mathrm{mod}\,R,\mathrm{Mod}\,R]$ is isomorphic to the category of generalised R-modules ${}_R\mathscr{C} = (\mathrm{mod}\,R,\mathrm{Ab})$. Since $\mathrm{mod}\,R$ is a symmetric monoidal $\mathrm{Mod}\,R$-category, a theorem of Day [10] says that ${}_R\mathscr{C} = (\mathrm{mod}\,R,\mathrm{Ab}) \cong [\mathrm{mod}\,R,\mathrm{Mod}\,R]$ is equipped with a closed symmetric monoidal structure $(\underline{\mathrm{Hom}}_{{}_R\mathscr{C}},\odot,-\otimes_R R)$, where $-\otimes_R R \cong \mathrm{Hom}_R(R,-)$ is a monoidal unit for the Day monoidal product $\odot$. Darbyshire [9] proved that the Auslander–Gruson–Jensen Duality

$$D : (\mathrm{coh}\,{}_R\mathscr{C})^{\mathrm{op}} \to \mathrm{coh}\,\mathscr{C}_R$$

is isomorphic to the internal Hom-functor $\underline{\mathrm{Hom}}_{{}_R\mathscr{C}}(?,-\otimes_R R)$.

Likewise, $\mathrm{fp}(\mathscr{V})$ is a symmetric monoidal $\mathscr{V}$-category, and hence $[\mathrm{fp}(\mathscr{V}),\mathscr{V}]$ is equipped with Day's closed symmetric monoidal structure $(\underline{\mathrm{Hom}}_{[\mathrm{fp}(\mathscr{V}),\mathscr{V}]},\odot,-\otimes e)$ [10], where $-\otimes e \cong [e,-]$ is a monoidal unit with respect to the Day monoidal product $\odot$ — see Theorem 2.11.

Theorem 10.2. *The Auslander–Gruson–Jensen Duality of Theorem 8.1*

$$D : (\mathrm{coh}[\mathrm{fp}(\mathscr{V}),\mathscr{V}])^{\mathrm{op}} \to \mathrm{coh}[\mathrm{fp}(\mathscr{V}),\mathscr{V}]$$

is isomorphic to the internal Hom-functor $\underline{\mathrm{Hom}}_{[\mathrm{fp}(\mathscr{V}),\mathscr{V}]}(?,-\otimes e)$.

Proof. There is an isomorphism of $\mathscr{V}$-functors

$$\underline{\mathrm{Hom}}_{[\mathrm{fp}(\mathscr{V}),\mathscr{V}]}(?,-\otimes e) \cong \underline{\mathrm{Hom}}_{[\mathrm{fp}(\mathscr{V}),\mathscr{V}]}(?,[e,-]).$$

For any $C \in \mathrm{coh}[\mathrm{fp}(\mathscr{V}),\mathscr{V}]$ and $N \in \mathrm{fp}(\mathscr{V})$ one has

$$\underline{\mathrm{Hom}}_{[\mathrm{fp}(\mathscr{V}),\mathscr{V}]}(C,-\otimes e)(N) = [C,[e,-](N\otimes -)] = [C,[e,N\otimes -]] \cong [C,N\otimes -] = D(C)(N).$$

The isomorphism induces a $\mathscr{V}$-natural transformation of $\mathscr{V}$-functors $\underline{\mathrm{Hom}}_{[\mathrm{fp}(\mathscr{V}),\mathscr{V}]}(?,-\otimes e) \to D$, which is objectwise an isomorphism. Therefore it is an equivalence of $\mathscr{V}$-functors. □

Lemma 10.3. *The functor $\mathscr{V} \to [\mathrm{fp}(\mathscr{V}),\mathscr{V}]$, $N \mapsto -\otimes N$, is strong monoidal. In particular, $(-\otimes M)\odot(-\otimes N) \cong -\otimes(M\otimes N)$.*

Proof. By Theorems 2.10, 2.11 one has $\mathscr{V}$-natural isomorphisms

$$(-\otimes M)\odot(-\otimes N) \cong (\int^A [A,-]\otimes A\otimes M)\odot(\int^B [B,-]\otimes B\otimes N) \cong$$

$$(\int^A [A,-]\otimes A\odot\int^B [B,-]\otimes B)\otimes(M\otimes N) \cong (-\otimes e)\odot(-\otimes e)\otimes(M\otimes N) \cong -\otimes(M\otimes N).$$

Our lemma now follows. □

Remark 10.4. It is worth mentioning that generators $[M,-]$, $M \in \mathrm{fp}(\mathscr{V})$, of $[\mathrm{fp}(\mathscr{V}),\mathscr{V}]$ are not dualizable with respect to the tensor product $\odot$ in general. Indeed, following [1, Example 7.5] consider $\mathscr{V} = \mathrm{Ab}$. Then $(\mathbb{Z}_2,-)\odot - \otimes \mathbb{Z}_2 \cong (\mathbb{Z}_2,-)$ but $\underline{\mathrm{Hom}}_{\mathscr{C}_{\mathbb{Z}}}(-\otimes\mathbb{Z}_2, -\otimes\mathbb{Z}_2) \cong -\otimes\mathbb{Z}_2$.

We are now in a position to pass to the construction of the Ziegler spectrum of a "nice" scheme X. We start with the closed symmetric monoidal category $(\mathrm{Qcoh}(X), \otimes_X, \mathscr{H}om_X^{qc})$ of quasi-coherent scheaves over a scheme X. It is a Grothendieck category by [2, Lemma 1.3] (see [11, Corollary 3.5] as well).

Definition 10.5. A quasi-compact quasi-separated scheme X satisfying the strong resolution property is called a *nice scheme*. By Example 4.1(2) if X is nice, $\mathrm{Qcoh}(X)$ has a family of dualizable generators $\mathscr{G} = \{g_i\}_{i\in I}$ such that the monoidal unit $\mathscr{O}_X$ is finitely presented.

Given a nice scheme X, pure-injective objects in $\mathrm{Qcoh}(X)$ in the sense of Definition 6.3 (the enriched ringoid $\mathscr{A}$ is the singleton $\{\mathscr{O}_X\}$ in this case) are the same with pure-injective quasi-coherent sheaves in the sense of [12, Definition 4.1].

Definition 10.6. The *category of generalised quasi-coherent sheaves* $\mathscr{C}_X$ of a nice scheme X is defined as $[\mathrm{fp}(\mathrm{Qcoh}(X)), \mathrm{Qcoh}(X)]$.

The *Ziegler spectrum of a nice scheme* X, denoted by Zg_X, is the Ziegler spectrum $\mathsf{Zg}_{\mathscr{V}}$ associated with $\mathscr{V} = \mathrm{Qcoh}(X)$. By definition, the points of Zg_X are the isomorphism classes of indecomposable pure-injective quasi-coherent sheaves. Zg_X is a topological space equipped with the Ziegler topology of Theorem 7.7.

The *injective spectrum of* X, denoted by $\mathsf{Sp}(X)$, is the injective spectrum associated with $\mathrm{Qcoh}(X)$ equipped with topology of Theorem 9.1 (the enriched ringoid $\mathscr{A}$ is the singleton $\{\mathscr{O}_X\}$ in this case). Observe that the topological space $\mathsf{Sp}(X)$ is nothing but the topological space $\mathsf{Sp}_{\mathrm{fl},\otimes}(X)$ in the sense of [15, Theorem 19] (in this paper we drop these subscripts for the injective spectrum of X to simplify our notation).

Following [15] a scheme X is *locally coherent* if it can be covered by open affine subsets $\mathrm{Spec}\, R_i$, where each R_i is a coherent ring. X is *coherent* if it is locally coherent, quasi-compact and quasi-separated.

The injective spectrum $\mathsf{Sp}(X)$ plays a prominent role for classifying finite localizations of quasicoherent sheaves and for the theorem reconstructing X out of $\mathrm{Qcoh}(X)$ in [15] (see [19, 20, 21] as well). The following theorem relates quasi-coherent sheaves and generalised quasi-coherent sheaves. It also relates injective and Ziegler spectra of X.

Theorem 10.7. *The following statements are true for a nice scheme X:*

(1) Define an enriched localizing subcategory $\mathscr{S}_X := \{Y \in \mathscr{C}_X \mid Y(\mathscr{O}_X) = 0\} \subset \mathscr{C}_X$. Then there is a recollement

$$\mathscr{S}_X \begin{array}{c} \xleftarrow{\;i_L\;} \\ \xrightarrow{\;i\;} \\ \xleftarrow[\;i_R\;]{} \end{array} \mathscr{C}_X \begin{array}{c} \xleftarrow{\;-\otimes_X ?\;} \\ \xrightarrow{\;r\;} \\ \xleftarrow[\;r_R\;]{} \end{array} \mathrm{Qcoh}(X)$$

with functors i, r being the canonical inclusion and restriction functors respectively. The functor r_R is the enriched right Kan extension, i_R is the torsion functor associated with the localizing subcategory $\mathscr{S}_X$. Furthermore, if $\mathscr{C}_X/\mathscr{S}_X$ is the quotient category of $\mathscr{C}_X$ with respect to $\mathscr{S}_X$, the functor $\mathrm{Qcoh}(X) \to \mathscr{C}_X/\mathscr{S}_X$ sending M to $(-\otimes_X M)_{\mathscr{S}_X}$ is an equivalence of categories.

(2) The Ziegler spectrum Zg_X *is a quasi-compact topological space and the Auslander–Gruson–Jensen Duality of Theorem 10.2*

$$D \cong \underline{\mathrm{Hom}}_{\mathscr{C}_X}(?, - \otimes_X \mathscr{O}_X) : (\mathrm{coh}\,\mathscr{C}_X)^{\mathrm{op}} \to \mathrm{coh}\,\mathscr{C}_X$$

induces an isomorphism $\mathscr{O}(\mathscr{S}) \mapsto \mathscr{O}(D\mathscr{S})$ *between open sets of* Zg_X.

(3) If X *is coherent,* $\mathscr{S}_X$ *is of finite type and there is a closed embedding* $\mathsf{Sp}(X) \hookrightarrow \mathsf{Zg}_X$ *of the injective spectrum of* X *into its Ziegler spectrum.*

Proof. This follows from Theorems 5.4, 8.3, 9.3, Lemma 10.1 and [15, Proposition 40]. □

Remark 10.8. In [35] Prest and Slávik study the categorical Ziegler spectrum $\mathsf{Zg}(\mathrm{Qcoh}(X))$. It is not quasi-compact [35, Corollary 5.7] in contrast with Zg_X, but the subset $\mathscr{D}_X$ of the indecomposable geometrically pure-injective quasicoherent sheaves form a closed quasi-compact subset of $\mathsf{Zg}(\mathrm{Qcoh}(X))$ [35, Theorem 4.8]. It would be interesting to compare Zg_X of Theorem 10.7 with $\mathscr{D}_X$ endowed with the subspace topology of $\mathsf{Zg}(\mathrm{Qcoh}(X))$.

Recall from [25] that a topological space is *spectral* if it is T_0, quasi-compact, if the quasi-compact open subsets are closed under finite intersections and form an open basis, and if every non-empty irreducible closed subset has a generic point. Given a spectral topological space, X, Hochster [25] endows the underlying set with a new, "dual", topology, denoted X^*, by taking as open sets those of the form $Y = \bigcup_{i \in \Omega} Y_i$ where Y_i has quasi-compact open complement $X \setminus Y_i$ for all $i \in \Omega$. Then X^* is spectral and $(X^*)^* = X$ (see [25, Proposition 8]). As an example, the underlying topological space (denote it by the same symbol) of a quasi-compact, quasi-separated scheme X is spectral.

Corollary 10.9. *Let* X *be a noetherian nice scheme. Then there is a natural closed embedding of topological spaces* $X^* \hookrightarrow \mathsf{Zg}_X$.

Proof. By a theorem of Gabriel [13, Chapter IV] the map $\mathsf{Sp}(X) \to X^*$ sending $E \in \mathsf{Sp}(X)$ to the generic point of the irreducible subset $\mathrm{Supp}_X(E)$ of X is a homeomorphism. Our statement now follows from Theorem 10.7(3). □

We conclude the paper with mentioning yet another topological space $\mathsf{Zg}_{\mathscr{V},\odot}$ whose points are those of $\mathsf{Zg}_{\mathscr{V}}$ but with open sets being in bijective correspondence with tensor-closed Serre subcategories of $\mathrm{coh}[\mathrm{fp}(\mathscr{V}),\mathscr{V}]$ with respect to the monoidal product $\odot$ on $[\mathrm{fp}(\mathscr{V}),\mathscr{V}]$. This topology is coarser than the Ziegler topology on $\mathsf{Zg}_{\mathscr{V}}$ as every tensor Serre subcategory is enriched. It literally repeats the construction of the tensor fl-topology on $\mathsf{Sp}(X)$ in the sense of [15, Theorem 19]. In [39, § 4.1] Wagstaffe studied the Ziegler topology for closed symmetric monoidal categories of additive functors $(\mathrm{fp}(\mathscr{C}),\mathrm{Ab})$, where $\mathscr{C}$ is a finitely accessible tensor category. It is described in terms of tensor-closed Serre subcategories with respect to Day's tensor product $\odot$. The space $\mathsf{Zg}_{\mathscr{V},\odot}$ should share lots of common properties with [39, § 4.1] and we invite the interested reader to study the topological space $\mathsf{Zg}_{\mathscr{V},\odot}$.

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