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Research paper

Euler–Maruyama scheme for delay-type stochastic McKean–Vlasov equations driven by fractional Brownian motion

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ABSTRACT

This paper focuses on the Euler–Maruyama (EM) scheme for delay-type stochastic McKean–Vlasov equations (DSMVEs) driven by fractional Brownian motion with Hurst parameter $H \in (0, 1/2) \cup (1/2, 1)$. The existence and uniqueness of the solutions to such DSMVEs whose drift coefficients contain polynomial delay terms are proved by exploiting the Banach fixed point theorem. Then the propagation of chaos between interacting particle system and non-interacting system in $\mathcal{L}^p$ sense is shown. We find that even if the delay term satisfies the polynomial growth condition, the unmodified classical EM scheme still can approximate the corresponding interacting particle system without the particle corruption. The convergence rates are revealed for $H \in (0, 1/2) \cup (1/2, 1)$. Finally, as an example that closely fits the original equation, a stochastic opinion dynamics model with both extrinsic memory and intrinsic memory is simulated to illustrate the plausibility of the theoretical result.

1. Introduction

The fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ is a natural generalization of the usual Brownian motion. In light of the extrinsic memory impact of fBm on system, the theories about stochastic differential equations (SDEs) driven by fBm have been systematically studied in [1,2]. Since the analytical solution to SDE driven by fBm cannot be expressed explicitly in many scenarios, the investigation of numerical schemes becomes crucial. Many types of SDEs driven by fBm have been approximated by EM scheme [2], backward EM scheme [3–5], θ-EM scheme [6,7], Milstein-type scheme [8,9], Crank–Nicolson scheme [10], tamed EM scheme [11], truncated EM scheme [12] and so on.

When the coefficients of SDEs are related to the laws of state variables, the equations are called stochastic McKean–Vlasov equations (SMVEs), also called mean-field SDEs or distribution-dependent SDEs. For SMVEs driven by fBm with Hurst parameter $H \in (0, 1/2) \cup (1/2, 1)$, the wellposedness and the Bismut formula for the Lions derivative were presented in [13]. Then EM scheme was exploited to approximate such SMVEs driven by fBm in [14]. In reality, there are already many papers analyzing the numerical schemes of SMVEs driven by standard Brownian motion (fBm with $H = 1/2$) via stochastic particle method, such as implicit EM scheme and tamed EM scheme in [15], tamed Milstein scheme in [16,17], multi-level Monte-Carlo scheme in [18], adaptive EM and Milstein schemes in [19], split-step EM scheme in [20] and so on.

When the influence of the intrinsic memory on system is also taken into account, the delay-type SDEs driven by fBm have been investigated in [21–27]. However, there are few work on the delay-type stochastic McKean–Vlasov equations (DSMVEs) driven by fBm yet. To fill this gap, this paper focuses on a class of DSMVEs driven by fBm with $H \in (0, 1/2) \cup (1/2, 1)$ and their numerical

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schemes via stochastic particle method. We first give the existence and uniqueness of solution to DSMVE driven by fBm by exploiting the Banach fixed point theorem. Then the propagation of chaos in $\mathcal{L}^p$ sense between non-interacting particle system and interacting particle system is presented so that the unmodified classical EM scheme can be established. The convergence rates of classical EM schemes are given for $H \in (0,1/2)\cup(1/2,1)$.

Undoubtedly, the unmodified classical EM scheme fails to approximate the SDEs driven by standard Brownian motion if the coefficients are superlinear [28]. However, we stress that the present state variable in the drift coefficient satisfies the global Lipschitz condition but the past state variable can grow polynomially, and the unmodified classical EM scheme still works in this case, which means that the particle corruption shown in [15] does not occur.

Stochastic opinion dynamics model (SODM), which can reflect changes in people's online or offline opinions in the social scenarios, is an crucial tool to formulate promotional plans and information campaigns, so SODM has been extensively studied [29–31]. Compared to the standard Brownian motion, SODM driven by fBm better reflects the influence of extrinsic memory due to the long-range (or short-range) dependence of fBm. Additionally, the impact of delay state variable and its distribution information on SODM should certainly be taken into account, as communications could be delayed in both online and offline between individuals in real-life scenarios [32]. Then we will find that our target DSMVEs driven by fBm can well match the SODM with both extrinsic memory and intrinsic memory. Therefore, the numerical analysis of DSMVEs driven by fBm is an important task for further research on SODM.

We now highlight the contribution of this paper. The Banach fixed point theorem is exploited to show the wellposedness of solution in Theorem 2.6, where we employ the segmentation technique by dividing the time interval $[0,T]$ into several equal-length segments $[0,\rho]$, where ρ is the finite-time delay. This kind of segmentation is always achievable, i.e., $[0,T]\subset\big[0,(\lfloor T/\rho\rfloor+1)\rho\big]$. Firstly, we prove the wellposedness of the solution on $[0,\rho]$. Within $[0,\rho]$, we apply the segmentation technique again by selecting a sufficiently small ρ_* that satisfies certain constraints with $[0,\rho]\subset\big[0,(\lfloor\rho/\rho_*\rfloor+1)\rho_*\big]$. Then the wellposedness is obtained on $[0,\rho_*]$. Repeat this procedure about ρ_* to get the wellposedness on $[0,\rho]$. Secondly, based on the result on $[0,\rho]$, we can prove the wellposedness on $[0,2\rho]$. Then iterating about the time-delay segment can give the wellposedness on $[0,T]$. As for the numerical analysis, if there are superlinear state variables in the coefficients, scholars often adopt the modified EM scheme or implicit EM scheme to avoid the explosion of numerical solution or particle corruption. However, in this paper, when the delay term is superlinear and the current state variable grows linearly, we use the unmodified classical EM scheme to approximate the corresponding interacting particle system in $\mathcal{L}^p$ sense without the particle corruption.

The rest of the paper is structured as follows. In Section 2, some notations and important lemmas are introduced, then we give the wellposedness of DSMVE driven by fBm. Section 3 aims to reveal the propagation of chaos in $\mathcal{L}^p$ sense. For $H\in(0,1/2)\cup(1/2,1)$, the convergence rates of classical EM scheme for interacting particle system are shown in Section 4. Some numerical simulations for the SODM are performed in Section 5.

2. Preliminaries

Let $|\cdot|$ be the Euclidean norm for vector and the trace norm for matrix. Denote $a_1\vee a_2=\max\{a_1,a_2\}$, $a_1\wedge a_2=\min\{a_1,a_2\}$ for real numbers a_1,a_2. Let $B^H=\{B^H_t,t\ge 0\}$ be a fBm with Hurst parameter $H\in(0,1)$ defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$, i.e., B^H is a centered Gaussian process with covariance function $R_H(t,s)=\frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H})$ for any $s,t\ge 0$. Let $\mathbb{I}_S$ be the indicator function for a set S, i.e., $\mathbb{I}_S(x)=1$ if $x\in S$, otherwise, $\mathbb{I}_S(x)=0$. Let $\lfloor a\rfloor$ be the largest integer which does not exceed a. Denote by $\mathcal{L}^p=\mathcal{L}^p\big(\Omega,\mathcal{F},\mathbb{P}\big)$ the set of random variables X with expectation $\mathbb{E}|X|^p<\infty$ for $p\ge 1$. Let $\mathcal{C}=\mathcal{C}([-\rho,0];\mathbb{R}^d)$ be the family of all continuous functions φ from $[-\rho,0]$ to $\mathbb{R}^d$ with the norm $\|\varphi\|=\sup_{-\rho\le\theta\le 0}|\varphi(\theta)|$. For a positive integer N, let $\mathbb{S}_N=\{1,2,\dots,N\}$.

Denote by $\delta_y(\cdot)$ the Dirac measure at point $y\in\mathbb{R}^d$. Denote by $\mathcal{P}(\mathbb{R}^d)$ the family of all probability measures on $\mathbb{R}^d$. For $q\ge 1$, define

$$\mathcal{P}_q(\mathbb{R}^d)=\left\{\mu\in\mathcal{P}(\mathbb{R}^d):\left(\int_{\mathbb{R}^d}|y|^q\mu(dy)\right)^{1/q}<\infty\right\},$$

and set $\mathcal{W}_q(\mu)=\left(\int_{\mathbb{R}^d}|y|^q\mu(dy)\right)^{1/q}$ for any $\mu\in\mathcal{P}_q(\mathbb{R}^d)$. For $q\ge 1$, let $\mathcal{C}([-\rho,T];\mathcal{P}_q(\mathbb{R}^d))$ be the family of all continuous measures μ from $[-\rho,T]$ to $\mathcal{P}_q(\mathbb{R}^d)$. For $q\ge 1$, the Wasserstein distance of $\mu,\nu\in\mathcal{P}_q(\mathbb{R}^d)$ is defined by

$$\mathbb{W}_q(\mu,\nu)=\inf_{\pi\in\mathfrak{C}(\mu,\nu)}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}|y_1-y_2|^q\pi(dy_1,dy_2)\right)^{1/q},$$

where $\mathfrak{C}(\mu,\nu)$ is the family of all couplings for μ,ν.

In this paper, consider DSMVE driven by fBm of the form

$$dY(t)=\alpha_t\left(Y(t),Y(t-\rho),\mathbb{L}_{Y(t)},\mathbb{L}_{Y(t-\rho)}\right)dt+\beta_t\left(\mathbb{L}_{Y(t)},\mathbb{L}_{Y(t-\rho)}\right)dB^H_t,\quad t\in[0,T],\tag{2.1}$$

with the initial value $\{Y(\theta):-\rho\le\theta\le 0\}=\xi$, which is an $\mathcal{F}_0$-measurable $\mathcal{C}$-valued random variable with $\mathbb{E}\|\xi\|^{\check{p}}<\infty$ for any $\check{p}>0$. Here, $\mathbb{L}_{Y(\cdot)}$ is the distribution of $Y(\cdot)$. Moreover, $\alpha:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^d$, $\beta:[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^{d\times d}$ are Borel measurable functions, and B^H_t is a d-dimensional fBm with $H\in(0,1/2)\cup(1/2,1)$. Note that $\int_0^T\beta_t\left(\mathbb{L}_{Y(t)},\mathbb{L}_{Y(t-\rho)}\right)dB^H_t$ is treated as a Wiener integral with respect to (w.r.t.) fBm since the diffusion coefficient is a deterministic function.

To get the wellposedness of solution to (2.1) by exploiting the Banach fixed point theorem, we consider the distribution-independent delay-type SDE driven by fBm of the form

$$d\hat{Y}(t)=\hat{\alpha}_t\left(\hat{Y}(t),\hat{Y}(t-\rho)\right)dt+\hat{\beta}_t dB^H_t,\quad t\in[0,T],\tag{2.2}$$

with the same initial value ξ for (2.1). Moreover, $\hat{\alpha} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, and $\hat{\beta} : [0,T] \to \mathbb{R}^{d\times d}$ are Borel measurable functions. Assume that $\hat{\beta}_t$ is bounded for any $t \in [0,T]$. We give the following important maximal inequality about fBm for the case $H \in (0,1/2) \cup (1/2,1)$, which has been proved by Theorem 1.2 in [33] and Theorem 2.1 in [34].

Lemma 2.1. *For any $\check{p} > 0$ and $H \in (0,1/2) \cup (1/2,1)$, there exist two constants $c_{H,\check{p}}, C_{H,\check{p}} > 0$ such that*

$$c_{H,\check{p}}\mathbb{E}(\tau^{\check{p}H}) \le \mathbb{E}\big(\sup_{t\in[0,\tau]} |B_t^H|^{\check{p}}\big) \le C_{H,\check{p}}\mathbb{E}(\tau^{\check{p}H}),$$

for any stopping time τ of B_t^H.

Assumption 2.2. There exist two constants $\bar{K}_1 > 0$ and $l \ge 1$ such that

$$|\hat{\alpha}_t(x_1,x_2) - \hat{\alpha}_t(y_1,y_2)| \le \bar{K}_1[|x_1 - y_1| + (1 + |x_2|^l + |y_2|^l)|x_2 - y_2|],$$

for any $t \in [0,T]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$.

By Assumption 2.2, one can see that there exists a $\bar{C}_1 > 0$ such that

$$\left|\hat{\alpha}_t(x_1,x_2)\right| \le \bar{C}_1 \left(1 + |x_1| + |x_2|^{l+1}\right),$$

for any $t \in [0,T]$ and $x_1, x_2 \in \mathbb{R}^d$. The following lemma reveals that there exists a unique global solution to (2.2) under Assumption 2.2.

Lemma 2.3. *Let Assumption 2.2 hold and $H \in (0,1/2)\cup(1/2,1)$. Then there exists a unique global solution $\hat{Y}(t)$ to (2.2), and it satisfies, for any $\bar{p} \ge 2$,*

$$\mathbb{E}\big(\sup_{t\in[0,T]} |\hat{Y}(t)|^{\bar{p}}\big) \le C_{\bar{p},T,H,\bar{K}_1,\|\xi\|,l,\rho}.$$

Proof. In view of [35], one can see that (2.2) admits a unique local solution when the drift coefficient is local Lipschitz continuous and the diffusion coefficient is a function of t. So, to achieve the goal of this lemma, we just need to prove

$$\mathbb{E}\big(\sup_{t\in[0,T]} |\hat{Y}(t)|^{\bar{p}}\big) \le C, \quad \text{for any } \bar{p} \ge 2.$$

From (2.2), using Hölder's inequality, Assumption 2.2 and Lemma 2.1 leads to

$$\begin{aligned}
&\mathbb{E}\big(\sup_{t\in[0,T]} |\hat{Y}(t)|^{\bar{p}}\big)\\
\le& C_{\bar{p},T,H,\|\xi\|} + C_{\bar{p},T}\mathbb{E}\int_0^T \left|\alpha_s(\hat{Y}(s),\hat{Y}(s-\rho))\right|^{\bar{p}} ds + \mathbb{E}\left(\sup_{t\in[0,T]} \Big|\int_0^t \hat{\beta}_s dB_s^H\Big|^{\bar{p}}\right)\\
\le& C_{\bar{p},T,H,\|\xi\|} + C_{\bar{p},T,\bar{K}_1}\mathbb{E}\int_0^T \left[1 + |\hat{Y}(s)|^{\bar{p}} + |\hat{Y}(s-\rho)|^{\bar{p}(l+1)}\right] ds\\
\le& C_{\bar{p},T,H,\bar{K}_1,\|\xi\|}\left[1 + \int_0^T \mathbb{E}\big(\sup_{s\in[0,t]} |\hat{Y}(s)|^{\bar{p}}\big) dt + \int_0^T \mathbb{E}|\hat{Y}(s-\rho)|^{\bar{p}(l+1)} ds\right].
\end{aligned}$$

The Gronwall inequality means that

$$\mathbb{E}\big(\sup_{t\in[0,T]} |\hat{Y}(t)|^{\bar{p}}\big) \le C_{\bar{p},T,H,\bar{K}_1,\|\xi\|}\big[1 + \mathbb{E}\big(\sup_{t\in[0,T]} |\hat{Y}(t-\rho)|^{\bar{p}(l+1)}\big)\big]. \tag{2.3}$$

Define a sequence as

$$\bar{p}_m = (2 - m + \lfloor\frac{T}{\rho}\rfloor)\bar{p}(l+1)^{1-m+\lfloor\frac{T}{\rho}\rfloor}, \quad m = 1,2\cdots,\lfloor\frac{T}{\rho}\rfloor + 1.$$

Obviously, it holds that $\bar{p}_{\lfloor\frac{T}{\rho}\rfloor+1} = \bar{p}$ and $\bar{p}_{m+1}(l+1) < \bar{p}_m$ for $m = 1,2\cdots,\lfloor\frac{T}{\rho}\rfloor$. For $t \in [0,\rho]$, we get from (2.3) that

$$\begin{aligned}
\mathbb{E}\big(\sup_{t\in[0,\rho]} |\hat{Y}(t)|^{\bar{p}_1}\big) \le& C_{\bar{p},T,H,\bar{K}_1,\|\xi\|}\big[1 + \mathbb{E}\big(\sup_{t\in[0,\rho]} |\hat{Y}(t-\rho)|^{\bar{p}_1(l+1)}\big)\big]\\
\le& C_{\bar{p},T,H,\bar{K}_1,\|\xi\|,l,\bar{p}_1}.
\end{aligned}$$

Next, for $t \in [0,2\rho]$, using Hölder's inequality, (2.3) and $\bar{p}_{m+1}(l+1) < \bar{p}_m$ leads to

$$\begin{aligned}
\mathbb{E}\big(\sup_{t\in[0,2\rho]} |\hat{Y}(t)|^{\bar{p}_2}\big) \le& C_{\bar{p},T,H,\bar{K}_1,\|\xi\|}\left[1 + \left[\mathbb{E}\big(\sup_{t\in[0,2\rho]} |\hat{Y}(t-\rho)|^{\bar{p}_1}\big)\right]^{\frac{\bar{p}_2(l+1)}{\bar{p}_1}}\right]\\
\le& C_{\bar{p},T,H,\bar{K}_1,\|\xi\|,l,\bar{p}_1,\bar{p}_2}.
\end{aligned}$$

Repeating this procedure gives

$$\mathbb{E}\left(\sup_{t\in\left[0,(\lfloor\frac{T}{\rho}\rfloor+1)\rho\right]}|\hat{Y}(t)|^{\bar{p}}\right)\leq C_{\bar{p},T,H,\bar{K}_1,\|\xi\|,l,\rho}.\quad\square$$

When $H=1/2$, B_t^H becomes a standard Brownian motion, and the results have been presented in [36]. Using the following condition, we shall investigate the wellposedness of (2.1).

Assumption 2.4. There exist three constants $K_1,K_2>0$ and $l\geq 1$ such that

$$|\alpha_t(x_1,x_2,\mu_1,\mu_2)-\alpha_t(y_1,y_2,\nu_1,\nu_2)|\leq K_1[|x_1-y_1|+(1+|x_2|^l+|y_2|^l)|x_2-y_2|+\mathbb{W}_2(\mu_1,\nu_1)+\mathbb{W}_2(\mu_2,\nu_2)],$$

$$|\beta_t(\mu_1,\mu_2)-\beta_t(\nu_1,\nu_2)|\leq K_2(\mathbb{W}_2(\mu_1,\nu_1)+\mathbb{W}_2(\mu_2,\nu_2)),$$

for any $t\in[0,T]$, $x_1,x_2,y_1,y_2\in\mathbb{R}^d$ and $\mu_1,\mu_2,\nu_1,\nu_2\in\mathcal{P}_2(\mathbb{R}^d)$.

Lemma 2.5. *For any $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, we have $\mathbb{W}_2(\mu,\delta_0)=\mathcal{W}_2(\mu)$.*

The proof of above lemma can be found in Lemma 2.3 of [37]. By Assumption 2.4 and Lemma 2.5, one can see that there exist constants $\bar{C}_2,\bar{C}_3>0$ such that

$$\left|\alpha_t(x_1,x_2,\mu_1,\mu_2)\right|\leq\bar{C}_2\left(1+|x_1|+|x_2|^{l+1}+\mathcal{W}_2(\mu_1)+\mathcal{W}_2(\mu_2)\right),$$

$$|\beta_t(\mu_1,\mu_2)|\leq\bar{C}_3(1+\mathcal{W}_2(\mu_1)+\mathcal{W}_2(\mu_2)),$$

for any $t\in[0,T]$, $x_1,x_2\in\mathbb{R}^d$ and $\mu_1,\mu_2\in\mathcal{P}_2(\mathbb{R}^d)$. By borrowing the proof ideas of Theorem 3.3 in [37], Theorem 1.4 in [38] and Theorem 1.12.1 in [2], we give the following theorem.

Theorem 2.6. *The DSMVE (2.1) admits a unique global solution $Y(t)$ satisfying for any $\bar{p}\geq 2$ and $T>0$,*

$$\mathbb{E}\big(\sup_{t\in[0,T]}|Y(t)|^{\bar{p}}\big)\leq C_{\bar{p},T,H,K_1,K_2,\|\xi\|,l,\rho},$$

if one of the following conditions hold:

(i) $H\in(1/2,1)$ and Assumption 2.4 holds.

(ii) $H\in(0,1/2)$, α satisfies Assumption 2.4 and β only depends on t (i.e., β does not depend on the distribution).

Proof. We first show the assertion for $H\in(1/2,1)$. For $x,y\in\mathbb{R}^d$ and $\mu_\cdot\in C\left([-\rho,T];\mathcal{P}_2(\mathbb{R}^d)\right)$, let $\alpha_{t,\rho}^{\mu}(x,y)=\alpha_t(x,y,\mu_t,\mu_{t-\rho})$ and $\beta_{t,\rho}^{\mu}=\beta_t(\mu_t,\mu_{t-\rho})$. Consider the auxiliary SDE of the form

$$dY^{\mu}(t)=\alpha_{t,\rho}^{\mu}(Y^{\mu}(t),Y^{\mu}(t-\rho))\,dt+\beta_{t,\rho}^{\mu}dB_t^H,\quad t\in[0,T],\tag{2.4}$$

with the initial value $Y_0^{\mu}=\xi$. Under Assumption 2.4, we know from Lemma 2.3 that SDE (2.4) admits a unique global solution in a strong sense and it satisfies

$$\mathbb{E}\big(\sup_{t\in[0,T]}|Y^{\mu}(t)|^{\bar{p}}\big)\leq C_{\bar{p},T,H,\bar{K}_1,\|\xi\|,l,\rho},\tag{2.5}$$

for any $\bar{p}\geq 2$. Define an operator

$$\Phi_t:C\left([-\rho,T];\mathcal{P}_2(\mathbb{R}^d)\right)\to C\left([-\rho,T];\mathcal{P}_2(\mathbb{R}^d)\right)$$

by $\Phi_t(\mu)=\mathbb{L}_{Y^{\mu}(t)}$, where $\mathbb{L}_{Y^{\mu}(t)}$ is the distribution of $Y^{\mu}(t)$. Next, we will show Φ is strictly contractive. Using the same techniques of getting (3.5) in the proof of Theorem 3.1 in [13] gives that, for $\bar{p}\geq 2$,

$$\mathbb{E}\big(\sup_{t\in[0,T]}\Big|\int_0^t\beta_s(\mu_s,\mu_{s-\rho})dB_s^H\Big|^{\bar{p}}\big)\leq C_{\bar{p},T,H}\mathbb{E}\int_0^T\left|\beta_t(\mu_t,\mu_{t-\rho})\right|^{\bar{p}}dt.\tag{2.6}$$

For any $t \in [0,T]$ and $\hat{p} \in [2, \frac{\bar{p}}{2l}]$, we get from Hölder's inequality, (2.6) and Assumption 2.4 that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{s\in[0,t]} |Y^{\mu}(s)-Y^{\nu}(s)|^{\hat{p}}\Big)\\
\le& 2^{\hat{p}-1}\mathbb{E}\Big(\sup_{s\in[0,t]}\Big|\int_0^s \big[\alpha^{\mu}_{r,\rho}(Y^{\mu}(r),Y^{\mu}(r-\rho))-\alpha^{\nu}_{r,\rho}(Y^{\nu}(r),Y^{\nu}(r-\rho))\big]dr\Big|^{\hat{p}}\Big)+2^{\hat{p}-1}\mathbb{E}\Big(\sup_{s\in[0,t]}\Big|\int_0^s \big[\beta^{\mu}_{r,\rho}-\beta^{\nu}_{r,\rho}\big]dB^H_r\Big|^{\hat{p}}\Big)\\
\le& C_{\hat{p},T,K_1}\mathbb{E}\int_0^t \Big[\,|Y^{\mu}(r)-Y^{\nu}(r)|^{\hat{p}}+\big(1+|Y^{\mu}(r-\rho)|^{l}+|Y^{\nu}(r-\rho)|^{l}\big)^{\hat{p}}|Y^{\mu}(r-\rho)-Y^{\nu}(r-\rho)|^{\hat{p}}\\
&+\mathbb{W}_2^{\hat{p}}(\mu_r,\nu_r)+\mathbb{W}_2^{\hat{p}}(\mu_{r-\rho},\nu_{r-\rho})\Big]dr + C_{\hat{p},T,H,K_2}\mathbb{E}\int_0^t\Big|\beta^{\mu}_{r,\rho}-\beta^{\nu}_{r,\rho}\Big|^{\hat{p}}dr\\
\le& c_*\int_0^t \Big[\,\mathbb{E}|Y^{\mu}(r)-Y^{\nu}(r)|^{\hat{p}}+\big[\mathbb{E}\big(1+|Y^{\mu}(r-\rho)|^{2\hat{p}l}+|Y^{\nu}(r-\rho)|^{2\hat{p}l}\big)\big]^{\frac{1}{2}}\\
&\cdot\big[\mathbb{E}|Y^{\mu}(r-\rho)-Y^{\nu}(r-\rho)|^{2\hat{p}}\big]^{\frac{1}{2}}+\mathbb{W}_2^{\hat{p}}(\mu_r,\nu_r)+\mathbb{W}_2^{\hat{p}}(\mu_{r-\rho},\nu_{r-\rho})\Big]dr,
\end{aligned}$$

where c_* is a constant depending on $\hat{p},T,H,K_1,K_2$. So, applying (2.5) with $\hat{p}\le\frac{\bar{p}}{2l}$ means that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{s\in[0,t]} |Y^{\mu}(s)-Y^{\nu}(s)|^{\hat{p}}\Big)\\
\le& c_*\int_0^t \mathbb{E}\Big(\sup_{s\in[0,r]}|Y^{\mu}(s)-Y^{\nu}(s)|^{\hat{p}}\Big)dr\\
&+c_*\int_0^t\Big[\mathbb{E}\Big(\sup_{s\in[0,r]}|Y^{\mu}(s-\rho)-Y^{\nu}(s-\rho)|^{2\hat{p}}\Big)\Big]^{\frac{1}{2}}dr+c_*\int_0^t\sup_{s\in[0,r]}\mathbb{W}_2^{\hat{p}}(\mu_s,\nu_s)dr.
\end{aligned}\tag{2.7}$$

Define a sequence as

$$\hat{p}_m=(2-m+\lfloor\frac{T}{\rho}\rfloor)\hat{p}2^{1-m+\lfloor\frac{T}{\rho}\rfloor},\quad m=1,2,\dots,\lfloor\frac{T}{\rho}\rfloor+1.$$

Obviously, it holds that $\hat{p}_{\lfloor\frac{T}{\rho}\rfloor+1}=\hat{p}$ and $2\hat{p}_{m+1}<\hat{p}_m$ for $m=1,2,\dots,\lfloor\frac{T}{\rho}\rfloor$.

For $s\in[0,\rho]$, since the initial values are the same, $\int_0^t\big[\mathbb{E}\big(\sup_{s\in[0,r]}|Y^{\mu}(s-\rho)-Y^{\nu}(s-\rho)|^{2\hat{p}}\big)\big]^{\frac{1}{2}}dr=0$. The Gronwall inequality with (2.7) gives that

$$\begin{aligned}
\sup_{s\in[0,\rho]}\mathbb{W}_2^{\hat{p}_1}(\Phi_s(\mu),\Phi_s(\nu))\le&\mathbb{E}\Big(\sup_{s\in[0,\rho]}|Y^{\mu}(s)-Y^{\nu}(s)|^{\hat{p}_1}\Big)\\
\le& c_*e^{c_*\rho}\int_0^\rho\sup_{s\in[0,r]}\mathbb{W}_2^{\hat{p}_1}(\mu_s,\nu_s)dr.
\end{aligned}\tag{2.8}$$

For $\rho_*>0$ (which is independent of the initial data) such that $c_*e^{c_*\rho_*}\rho_*\le\frac{1}{2}$, let

$$\tilde{S}_{\rho_*}=\big\{\mu_\cdot\in C\big([0,\rho_*];\mathcal{P}_2(\mathbb{R}^d)\big):\tilde{d}(\mu_\cdot,\mu_0)<\infty,\mu_0=\mathbb{L}_{X(0)}\big\}$$

equipped with the uniform metric

$$\tilde{d}(\mu,\nu):=\sup_{s\in[0,\rho_*]}\mathbb{W}_2(\mu_s,\nu_s).$$

We see that $(\tilde{S}_{\rho_*},\tilde{d})$ is a complete metric space. Then using (2.8) implies

$$\tilde{d}^{\hat{p}_1}(\Phi(\mu),\Phi(\nu))\le\frac{1}{2}\tilde{d}^{\hat{p}_1}(\mu,\nu),$$

which means that Φ is strictly contractive in the complete metric space $(\tilde{S}_{\rho_*},\tilde{d})$. Hence, the Banach fixed point theorem with the definition of Φ reveals that there exists a unique $\mu\in\tilde{S}_{\rho_*}$ such that

$$\Phi_t(\mu)=\mu_t=\mathbb{L}_{Y^{\mu}(t)}$$

on $t\in[0,\rho_*]$. Thus, the strong wellposedness of (2.1) is obtained on $[0,\rho_*]$. Repeating this procedure with the initial value $Y_{n\rho_*}$ (and the initial time $n\rho_*$) for $1\le n\le\lfloor\frac{\rho}{\rho_*}\rfloor+1$ gives the strong wellposedness of (2.1) on $[0,\rho]$.

For $s \in [0, 2\rho]$, since $\int_0^t \big[\mathbb{E}\big(\sup_{s\in[0,r]} |Y^{\mu}(s-\rho) - Y^{\nu}(s-\rho)|^{2\hat{p}}\big)\big]^{\frac{1}{2}} dr$ may not be vanished, the exponents of both sides of (2.7) are different, we need to estimate $\hat{p}_2$−moment in the following. The Hölder inequality with (2.7) and (2.8) gives that

$$\begin{aligned}
&\sup_{s\in[0,2\rho]} \mathbb{W}_2^{\hat{p}_2}(\Phi_s(\mu), \Phi_s(\nu)) \\
\le&\mathbb{E}\Big(\sup_{s\in[0,2\rho]} |Y^{\mu}(s) - Y^{\nu}(s)|^{\hat{p}_2}\Big) \\
\le&2c_* e^{2c_*\rho}\rho \left[\mathbb{E}\Big(\sup_{s\in[0,\rho]} |Y^{\mu}(s) - Y^{\nu}(s)|^{\hat{p}_1}\Big)\right]^{\frac{\hat{p}_2}{\hat{p}_1}} + 2c_* e^{2c_*\rho}\rho \sup_{s\in[0,2\rho]} \mathbb{W}_2^{\hat{p}_2}(\mu_s, \nu_s) \\
\le&2c_* e^{2c_*\rho}\rho \left[c_* e^{c_*\rho}\rho \sup_{s\in[0,\rho]} \mathbb{W}_2^{\hat{p}_1}(\mu_s, \nu_s)\right]^{\frac{\hat{p}_2}{\hat{p}_1}} + 2c_* e^{2c_*\rho}\rho \sup_{s\in[0,2\rho]} \mathbb{W}_2^{\hat{p}_2}(\mu_s, \nu_s) \\
\le&2c_* e^{2c_*\rho}\rho \left[1 \vee (c_* e^{c_*\rho}\rho)^{\frac{\hat{p}_2}{\hat{p}_1}}\right] \sup_{s\in[0,2\rho]} \mathbb{W}_2^{\hat{p}_2}(\mu_s, \nu_s)
\end{aligned} \tag{2.9}$$

For $\rho_* > 0$ such that $2c_* e^{2c_*\rho_*}\rho_*[1 \vee (c_* e^{c_*\rho_*}\rho_*)^{\hat{p}_2/\hat{p}_1}] \le \frac{1}{2}$, we get from (2.9) that

$$\tilde{d}^{\hat{p}_2}(\Phi(\mu), \Phi(\nu)) \le \frac{1}{2}\tilde{d}^{\hat{p}_2}(\mu, \nu),$$

which means that Φ is strictly contractive in the complete metric space $(\tilde{S}_{\rho_*}, \tilde{d})$. Again, the Banach fixed point theorem gives that there exists a unique $\mu \in \tilde{S}_{\rho_*}$ such that

$$\Phi_t(\mu) = \mu_t = \mathbb{L}_{Y^{\mu}(t)}$$

on $t \in [0, \rho_*]$. Then the strong wellposedness of (2.1) is achieved on $[0, \rho_*]$. Repeating this procedure with the initial value $Y_{n\rho_*}$ for $1 \le n \le \lfloor \frac{2\rho}{\rho_*} \rfloor + 1$ gives the strong wellposedness of (2.1) on $[0, 2\rho]$.

Repeating this procedure above, we can establish the strong wellposedness of (2.1) on $[0, 3\rho], [0, 4\rho]$, etc. The proof of strong wellposedness of (2.1) on $[0, T]$ is therefore completed.

We now give the detailed moment estimation of the solution to (2.1). For any $\bar{p} \ge 2$ and $t \in [0, T]$, the Hölder inequality leads to

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{r\in[0,t]} |Y(r)|^{\bar{p}}\Big) \\
\le&C_{\bar{p}}\mathbb{E}\|\xi\|^{\bar{p}} + C_{\bar{p},T}\mathbb{E}\int_0^t \Big|\alpha_r(Y(r), Y(r-\rho), \mathbb{L}_{Y(r)}, \mathbb{L}_{Y(r-\rho)})\Big|^{\bar{p}} dr \\
&+ C_{\bar{p}}\mathbb{E}\Big(\sup_{r\in[0,t]} \Big| \int_0^r \beta_u(\mathbb{L}_{Y(u)}, \mathbb{L}_{Y(u-\rho)}) dB_u^H \Big|^{\bar{p}}\Big).
\end{aligned}$$

By Assumption 2.4, Lemma 2.5 and (2.6), we find that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{r\in[0,t]} |Y(r)|^{\bar{p}}\Big) \\
\le&C_{\bar{p}}\mathbb{E}\|\xi\|^{\bar{p}} + C_{\bar{p},T,K_1}\mathbb{E}\int_0^t \Big[\, 1 + |Y(r)|^{\bar{p}} + |Y(r-\rho)|^{\bar{p}(l+1)} + \mathcal{W}_2^{\bar{p}}(\mathbb{L}_{Y(r)}) + \mathcal{W}_2^{\bar{p}}(\mathbb{L}_{Y(r-\rho)})\,\Big]\, dr \\
&+ C_{\bar{p},T,H,K_2}\mathbb{E}\int_0^t \big[1 + \mathcal{W}_2^{\bar{p}}(\mathbb{L}_{Y(r)}) + \mathcal{W}_2^{\bar{p}}(\mathbb{L}_{Y(r-\rho)})\big] dr \\
\le&C_{\bar{p},T,H,K_1,K_2,\|\xi\|}\Big[1 + \int_0^t \mathbb{E}\Big(\sup_{u\in[0,r]} |Y(u)|^{\bar{p}}\Big) dr + \mathbb{E}\int_0^t |Y(r-\rho)|^{\bar{p}(l+1)} dr\Big].
\end{aligned}$$

Thanks to the Gronwall inequality,

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |Y(t)|^{\bar{p}}\Big) \le C_{\bar{p},T,H,K_1,K_2,\|\xi\|}\Big[1 + \mathbb{E}\Big(\sup_{t\in[0,T]} |Y(t-\rho)|^{\bar{p}(l+1)}\Big)\Big].$$

Constructing a sequence $\bar{p}_m = (2 - m + \lfloor \frac{T}{\rho} \rfloor)\bar{p}(l+1)^{1-m+\lfloor \frac{T}{\rho} \rfloor}$ and using the same iteration technique about $\bar{p}_m$ as that in Lemma 2.3 yield that

$$\mathbb{E}\Big(\sup_{t\in[0,((\lfloor \frac{T}{\rho} \rfloor+1)\rho)]} |Y(t)|^{\bar{p}}\Big) \le C_{\bar{p},T,H,K_1,K_2,\|\xi\|,l,\rho}.$$

So the desired result holds.

For the case $H \in (0, 1/2)$, since β does not depend on the distribution, the assertion can be obtained analogically by Lemma 2.3. □

In the rest of this paper, the DSMVEs are all autonomous for convenience. Moreover, the relevant settings of the equations are stated for the case $H \in (1/2, 1)$. As for $H \in (0, 1/2)$, we assume that the coefficient $\beta(\cdot, \cdot)$ is a constant.

3. Propagation of chaos

In this section, based on the stochastic particle method in [39,40], the corresponding interacting particle system is used to approximate the original DSMVE. For any $i \in \mathbb{S}_N$, consider the non-interacting particle system

$$dY^i(t) =\alpha\left(Y^i(t), Y^i(t-\rho), \mathbb{L}_{Y^i(t)}, \mathbb{L}_{Y^i(t-\rho)}\right) dt + \beta\left(\mathbb{L}_{Y^i(t)}, \mathbb{L}_{Y^i(t-\rho)}\right) dB_t^{H,i}, \quad t \in [0,T], \tag{3.1}$$

with the initial value $X_0^i = \xi^i$, which is an $\mathcal{F}_0$-measurable C-valued random variable with $\mathbb{E}\|\xi^i\|^{\check{p}} < \infty$ for any $\check{p} > 0$, where $\mathbb{L}_{Y^i(\cdot)}$ is distribution of $Y^i(\cdot)$. Here, $(\xi^i, B^{H,i})$ are the independent copies of (ξ, B^H), and all $(\xi^i, B^{H,i})$ are independent and identically distributed. Moreover, it holds that $\mathbb{L}_{Y^i(t)} = \mathbb{L}_{Y(t)}$ for any $t \in [0,T]$ and $i \in \mathbb{S}_N$. The corresponding interacting particle system driven by fBm is

$$dY^{i,N}(t) =\alpha\left(Y^{i,N}(t), Y^{i,N}(t-\rho), \mathscr{L}_{Y^N(t)}, \mathscr{L}_{Y^N(t-\rho)}\right) dt + \beta\left(\mathscr{L}_{Y^N(t)}, \mathscr{L}_{Y^N(t-\rho)}\right) dB_t^{H,i}, \quad t \in [0,T], \tag{3.2}$$

with the initial value $X_0^i = \xi^i$, where $\mathscr{L}_{Y^N(\cdot)} := \frac{1}{N}\sum_{j=1}^N \delta_{Y^{j,N}(\cdot)}$. Under Assumption 2.4, one can get the wellposedness of (3.1) and (3.2).

Lemma 3.1. *For $H \in (0,1/2)\cup(1/2,1)$, let Assumption 2.4 hold. Then for any $\bar{p} \geq 2$ and $T > 0$,*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |Y^i(t)|^{\bar{p}}\Big) \vee \mathbb{E}\Big(\sup_{t\in[0,T]} |Y^{i,N}(t)|^{\bar{p}}\Big) \leq C_{\bar{p},T,H,K_1,K_2,\|\xi\|,l,\rho}.$$

Theorem 3.2. *Let $H \in (0,1/2)\cup(1/2,1)$ and Assumption 2.4 hold. Fix any $\bar{p} > 4$ in Lemma 3.1. Then for any $i \in \mathbb{S}_N$, $p \in [2, \bar{p}/2)$ and $\varepsilon \in (0,1]$, we have*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |Y^i(t) - Y^{i,N}(t)|^p\Big) \leq C\begin{cases}(N^{-1/2})^{\lambda_{p,T,\rho}}, & \text{if } p > d/2,\\ [N^{-1/2}\log(1+N)]^{\lambda_{p,T,\rho}}, & \text{if } p = d/2,\\ (N^{-p/d})^{\lambda_{p,T,\rho}}, & \text{if } 2 \leq p < d/2,\end{cases}$$

where $\lambda_{p,T,\rho} = (\frac{p-\varepsilon}{p})^{\lfloor \frac{T}{\rho} \rfloor}$ and C is a positive constant dependent of d, p, T, H, K_1, K_2, $\|\xi\|$, l, ρ but independent of N.

Proof. We begin with the case $H \in (1/2,1)$. For any $i \in \mathbb{S}_N$, $p \geq 2$ and $t \in [0,T]$, by Hölder's inequality and (2.6), it follows that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{s\in[0,t]} |Y^i(s) - Y^{i,N}(s)|^p\Big)\\
\leq& 2^{p-1}\mathbb{E}\left(\sup_{s\in[0,t]}\Big|\int_0^s \Big[\alpha\big(Y^i(r), Y^i(r-\rho), \mathbb{L}_{Y^i(r)}, \mathbb{L}_{Y^i(r-\rho)}\big) - \alpha\big(Y^{i,N}(r), Y^{i,N}(r-\rho), \mathscr{L}_{Y^N(r)}, \mathscr{L}_{Y^N(r-\rho)}\big)\Big] dr\Big|^p\right)\\
&+ 2^{p-1}\mathbb{E}\Big(\sup_{s\in[0,t]}\Big|\int_0^s \big[\beta\big(\mathbb{L}_{Y^i(r)}, \mathbb{L}_{Y^i(r-\rho)}\big) - \beta\big(\mathscr{L}_{Y^N(r)}, \mathscr{L}_{Y^N(r-\rho)}\big)\big] dB_r^H\Big|^{\hat{p}}\Big)\\
\leq& C_{p,T,K_1}\mathbb{E}\int_0^t \Big[|Y^i(r) - Y^{i,N}(r)| + \big(1 + |Y^i(r-\rho)|^l + |Y^{i,N}(r-\rho)|^l\big)|Y^i(r-\rho) - Y^{i,N}(r-\rho)|\\
&+ \mathbb{W}_2(\mathbb{L}_{Y^i(r)}, \mathscr{L}_{Y^N(r)}) + \mathbb{W}_2(\mathbb{L}_{Y^i(r-\rho)}, \mathscr{L}_{Y^N(r-\rho)})\Big]^p dr\\
&+ C_{p,T,H,K_2}\mathbb{E}\int_0^t \Big[\mathbb{W}_2(\mathbb{L}_{Y^i(r)}, \mathscr{L}_{Y^N(r)}) + \mathbb{W}_2(\mathbb{L}_{Y^i(r-\rho)}, \mathscr{L}_{Y^N(r-\rho)})\Big]^p dr\\
\leq& C_{p,T,H,K_1,K_2}\mathbb{E}\int_0^t \Big[|Y^i(r) - Y^{i,N}(r)|^p + \big(1 + |Y^i(r-\rho)|^l + |Y^{i,N}(r-\rho)|^l\big)^p|Y^i(r-\rho) - Y^{i,N}(r-\rho)|^p\\
&+ \mathbb{W}_p^p(\mathbb{L}_{Y^i(r)}, \mathscr{L}_{Y^N(r)}) + \mathbb{W}_p^p(\mathbb{L}_{Y^i(r-\rho)}, \mathscr{L}_{Y^N(r-\rho)})\Big] dr.
\end{aligned}$$

Due to Hölder's inequality and 3.1, for $\varepsilon \in (0,1]$, we see that

$$\begin{aligned}
&\mathbb{E}\big[\big(1 + |Y^i(r-\rho)|^l + |Y^{i,N}(r-\rho)|^l\big)^p|Y^i(r-\rho) - Y^{i,N}(r-\rho)|^p\big]\\
\leq& C_p\big[\mathbb{E}\big(1 + |Y^i(r-\rho)|^{lp+\varepsilon} + |Y^{i,N}(r-\rho)|^{lp+\varepsilon}\big)^{\frac{p}{\varepsilon}}\big]^{\frac{\varepsilon}{p}}\big[\mathbb{E}|Y^i(r-\rho) - Y^{i,N}(r-\rho)|^p\big]^{\frac{p-\varepsilon}{p}}\\
\leq& C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\big[\mathbb{E}|Y^i(r-\rho) - Y^{i,N}(r-\rho)|^p\big]^{\frac{p-\varepsilon}{p}}.
\end{aligned}$$

The Gronwall inequality leads to

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{s\in[0,t]} |Y^i(s) - Y^{i,N}(s)|^p\Big)\\
\leq& C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\left(\Big[\mathbb{E}\Big(\sup_{s\in[0,t]} |Y^i(s-\rho) - Y^{i,N}(s-\rho)|^p\Big)\Big]^{\frac{p-\varepsilon}{p}}\right.\\
&\left.+ \mathbb{E}\int_0^t \big(\mathbb{W}_p^p(\mathbb{L}_{Y^i(r)}, \mathscr{L}_{Y^N(r)}) + \mathbb{W}_p^p(\mathbb{L}_{Y^i(r-\rho)}, \mathscr{L}_{Y^N(r-\rho)})\big) dr\right).
\end{aligned} \tag{3.3}$$

For $s \in [0,\rho]$, we know from (3.3) that

$$\mathbb{E}\big(\sup_{s\in[0,\rho]}|Y^i(s)-Y^{i,N}(s)|^p\big) \le C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\mathbb{E}\int_0^\rho \big(\mathbb{W}_p^p(\mathbb{L}_{Y^i(r)},\mathscr{L}_{Y^N(r)}) + \mathbb{W}_p^p(\mathbb{L}_{Y^i(r-\rho)},\mathscr{L}_{Y^N(r-\rho)})\big)dr.$$

Define the empirical measure $\overline{\mathbb{L}}_{Y^N(\cdot)} = \frac{1}{N}\sum_{j=1}^N \delta_{Y^j(\cdot)}$. For any $i\in\mathbb{S}_N$,

$$\begin{aligned}\mathbb{W}_p^p(\mathbb{L}_{Y^i(\cdot)},\mathscr{L}_{Y^N(\cdot)}) \le& C_p\mathbb{W}_p^p(\mathbb{L}_{Y^i(\cdot)},\overline{\mathbb{L}}_{Y^N(\cdot)}) + C_p\mathbb{W}_p^p(\overline{\mathbb{L}}_{Y^N(\cdot)},\mathscr{L}_{Y^N(\cdot)})\\ \le& C_p\mathbb{W}_p^p(\mathbb{L}_{Y^i(\cdot)},\overline{\mathbb{L}}_{Y^N(\cdot)}) + C_p\frac{1}{N}\sum_{j=1}^N|Y^j(\cdot)-Y^{j,N}(\cdot)|^p.\end{aligned}$$

Since the distributions of all j are identical, we obtain

$$\begin{aligned}&\mathbb{E}\big(\sup_{s\in[0,\rho]}|Y^i(s)-Y^{i,N}(s)|^p\big)\\ \le& C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\mathbb{E}\int_0^\rho\Big[|Y^i(r)-Y^{i,N}(r)|^p + |Y^i(r-\rho)-Y^{i,N}(r-\rho)|^p + \mathbb{W}_p^p(\mathbb{L}_{Y^i(r)},\overline{\mathbb{L}}_{Y^N(r)}) + \mathbb{W}_p^p(\mathbb{L}_{Y^i(r-\rho)},\overline{\mathbb{L}}_{Y^N(r-\rho)})\Big]dr\\ \le& C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\int_0^\rho \mathbb{E}\big(\sup_{s\in[0,r]}|Y^i(s)-Y^{i,N}(s)|^p\big)dr\\ &+ C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\mathbb{E}\int_0^\rho\big[\mathbb{W}_p^p(\mathbb{L}_{Y^i(r)},\overline{\mathbb{L}}_{Y^N(r)}) + \mathbb{W}_p^p(\mathbb{L}_{Y^i(r-\rho)},\overline{\mathbb{L}}_{Y^N(r-\rho)})\big]dr.\end{aligned}$$

In view of Theorem 1 in [41] and Gronwall's inequality, we get

$$\mathbb{E}\big(\sup_{s\in[0,\rho]}|Y^i(s)-Y^{i,N}(s)|^p\big) \le C_{d,p,T,H,K_1,K_2,\|\xi\|,l,\rho}\begin{cases} N^{-\frac{1}{2}} + N^{-\frac{\bar p-p}{\bar p}}, & \text{if } p>\frac{d}{2}, \bar p\neq 2p,\\ N^{-\frac{1}{2}}\log(1+N) + N^{-\frac{\bar p-p}{\bar p}}, & \text{if } p=\frac{d}{2}, \bar p\neq 2p,\\ N^{-\frac{p}{d}} + N^{-\frac{\bar p-p}{\bar p}}, & \text{if } 2\le p<\frac{d}{2},\end{cases}$$

where $\bar p$ is the same as $\bar p$ in Lemma 3.1. Then for $p\in[2,\frac{\bar p}{2})$, the above inequality becomes

$$\mathbb{E}\big(\sup_{s\in[0,\rho]}|Y^i(s)-Y^{i,N}(s)|^p\big) \le C_{d,p,T,H,K_1,K_2,\|\xi\|,l,\rho}\begin{cases} N^{-\frac{1}{2}}, & \text{if } p>\frac{d}{2},\\ N^{-\frac{1}{2}}\log(1+N), & \text{if } p=\frac{d}{2},\\ N^{-\frac{p}{d}}, & \text{if } 2\le p<\frac{d}{2}.\end{cases} \tag{3.4}$$

For $s\in[0,2\rho]$, the Hölder inequality with (3.3) leads to

$$\begin{aligned}&\mathbb{E}\big(\sup_{s\in[0,2\rho]}|Y^i(s)-Y^{i,N}(s)|^p\big)\\ \le& C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\big[\mathbb{E}\big(\sup_{s\in[0,2\rho]}|Y^i(s-\rho)-Y^{i,N}(s-\rho)|^p\big)\big]^{\frac{p-\varepsilon}{p}}\\ &+ C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\mathbb{E}\int_0^{2\rho}\Big[\mathbb{W}_p^p(\mathbb{L}_{Y^i(r)},\overline{\mathbb{L}}_{Y^N(r)}) + \mathbb{W}_p^p(\overline{\mathbb{L}}_{Y^N(r)},\mathscr{L}_{Y^N(r)}) + \mathbb{W}_p^p(\mathbb{L}_{Y^i(r-\rho)},\overline{\mathbb{L}}_{Y^N(r-\rho)}) + \mathbb{W}_p^p(\overline{\mathbb{L}}_{Y^N(r-\rho)},\mathscr{L}_{Y^N(r-\rho)})\Big]dr\\ \le& C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\big[\mathbb{E}\big(\sup_{s\in[0,\rho]}|Y^i(s)-Y^{i,N}(s)|^p\big)\big]^{\frac{p-\varepsilon}{p}}\\ &+ C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\mathbb{E}\int_0^{2\rho}\Big[\mathbb{W}_p^p(\mathbb{L}_{Y^i(r)},\overline{\mathbb{L}}_{Y^N(r)}) + \mathbb{W}_p^p(\mathbb{L}_{Y^i(r-\rho)},\overline{\mathbb{L}}_{Y^N(r-\rho)})\Big]dr\\ &+ C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\int_0^{2\rho}\mathbb{E}\big(\sup_{s\in[0,r]}|Y^i(s)-Y^{i,N}(s)|^p\big)dr.\end{aligned}$$

Using Gronwall's inequality with (3.4) on $s\in[0,\rho]$ gives that

$$\begin{aligned}&\mathbb{E}\big(\sup_{s\in[0,2\rho]}|Y^i(s)-Y^{i,N}(s)|^p\big)\\ \le& C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\big[\mathbb{E}\big(\sup_{s\in[0,\rho]}|Y^i(s)-Y^{i,N}(s)|^p\big)\big]^{\frac{p-\varepsilon}{p}}\\ &+ C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\mathbb{E}\int_0^{2\rho}\Big[\mathbb{W}_p^p(\mathbb{L}_{Y^i(r)},\overline{\mathbb{L}}_{Y^N(r)}) + \mathbb{W}_p^p(\mathbb{L}_{Y^i(r-\rho)},\overline{\mathbb{L}}_{Y^N(r-\rho)})\Big]dr\\ \le& C_{d,p,T,H,K_1,K_2,\|\xi\|,l,\rho}\begin{cases}(N^{-\frac{1}{2}})^{\frac{p-\varepsilon}{p}}, & \text{if } p>\frac{d}{2},\\ \big(N^{-\frac{1}{2}}\log(1+N)\big)^{\frac{p-\varepsilon}{p}}, & \text{if } p=\frac{d}{2},\\ (N^{-\frac{p}{d}})^{\frac{p-\varepsilon}{p}}, & \text{if } 2\le p<\frac{d}{2}.\end{cases}\end{aligned}$$

For $s \in [0, 3\rho]$, it is similar to see

$$\begin{aligned}
&\mathbb{E}\big(\sup_{s\in[0,3\rho]} |Y^i(s) - Y^{i,N}(s)|^p\big)\\
\leq &C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\big[\mathbb{E}\big(\sup_{s\in[0,2\rho]} |Y^i(s) - Y^{i,N}(s)|^p\big)\big]^{\frac{p-\varepsilon}{p}}\\
&+ C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\mathbb{E}\int_0^{3\rho}\Big[\mathbb{W}_p^p(\mathbb{L}_{Y^i(r)}, \overline{\mathbb{L}}_{Y^N(r)}) + \mathbb{W}_p^p(\mathbb{L}_{Y^i(r-\rho)}, \overline{\mathbb{L}}_{Y^N(r-\rho)})\Big]dr\\
\leq &C_{d,p,T,H,K_1,K_2,\|\xi\|,l,\rho}\begin{cases}(N^{-\frac{1}{2}})^{(\frac{p-\varepsilon}{p})^2}, & \text{if } \ p > \frac{d}{2},\\ \big(N^{-\frac{1}{2}}\log(1+N)\big)^{(\frac{p-\varepsilon}{p})^2}, & \text{if } \ p = \frac{d}{2},\\ (N^{-\frac{p}{d}})^{(\frac{p-\varepsilon}{p})^2}, & \text{if } \ 2 \leq p < \frac{d}{2}.\end{cases}
\end{aligned}$$

The desired result follows by the iteration about the time segment generated by the delay ρ.

As for the case $H \in (0, 1/2)$, the stochastic integral in $Y^i(s) - Y^{i,N}(s)$ vanishes since it is an additive noise, then the assertion can be achieved through the similar procedure. □

4. Numerical scheme

In this section, the classical EM scheme, which is not modified, is established for interacting particle system (3.2) whose delay term is superlinear. We will theoretically prove the convergence of classical EM scheme and that particle corruption will not occur. Let $\Delta = \frac{\rho}{M} = \frac{T}{M_T}$ for some positive integers M and M_T. Set $t_k = k\Delta$ for $k = -M, \dots, 0, \dots, M_T$ and define the EM scheme as

$$\begin{aligned}
Z^{i,N}(t_{k+1}) = Z^{i,N}(t_k) &+ \alpha\big(Z^{i,N}(t_k), Z^{i,N}(t_{k-M}), \mathscr{L}_{Z^N(t_k)}, \mathscr{L}_{Z^N(t_{k-M})}\big)\Delta\\
&+ \beta\big(\mathscr{L}_{Z^N(t_k)}, \mathscr{L}_{Z^N(t_{k-M})}\big)\Delta B_k^{H,i}, \quad k = 0, 1, \dots, M_T - 1,
\end{aligned} \tag{4.1}$$

with the initial value $Z^{i,N}(t_k) = \xi^i(t_k)$, $k = -M, \dots, 1, 0$, where $\Delta B_k^{H,i} = B_{t_{k+1}}^{H,i} - B_{t_k}^{H,i}$ and $\mathscr{L}_{Z^N(\cdot)} = \frac{1}{N}\sum_{j=1}^N \delta_{Z^{j,N}(\cdot)}$.

For $t \in [0, T]$, the continuous-sample numerical scheme is

$$\tilde{Z}^{i,N}(t) = \xi^i(0) + \int_0^t \alpha\big(Z^{i,N}(s), Z^{i,N}(s-\rho), \mathscr{L}_{Z^N(s)}, \mathscr{L}_{Z^N(s-\rho)}\big)ds + \int_0^t \beta\big(\mathscr{L}_{Z^N(s)}, \mathscr{L}_{Z^N(s-\rho)}\big)dB_s^{H,i}, \tag{4.2}$$

with the step process $Z^{i,N}(t) := \sum_{k=-M}^{M_T} Z^{i,N}(t_k)\mathbb{I}_{[t_k,t_{k+1})}(t)$ and the empirical measure $\mathscr{L}_{Z^N(t)} = \frac{1}{N}\sum_{j=1}^N \delta_{Z^{j,N}(t)}$. Obviously, $\tilde{Z}^{i,N}(t_k) = Z^{i,N}(t_k) = Z^{i,N}(t)$ for $t \in [t_k, t_{k+1})$.

Lemma 4.1. *For $H \in (0, 1/2) \cup (1/2, 1)$, let Assumption 2.4 hold. Then, for $\bar{p} \geq 2$, one has*

$$\max_{i\in\mathbb{S}_N}\sup_{\Delta\in(0,1]}\mathbb{E}\big(\sup_{t\in[0,T]}|\tilde{Z}^{i,N}(t)|^{\bar{p}}\big) \leq C_{\bar{p},T,H,K_1,K_2,\|\xi\|,l,\rho}.$$

Proof. We first discuss the case $H \in (1/2, 1)$. For any $t \in [0, T]$ and $i \in \mathbb{S}_N$, the Hölder inequality with Assumption 2.4 and (2.6) yields that

$$\begin{aligned}
&\mathbb{E}\big(\sup_{s\in[0,t]}|\tilde{Z}^{i,N}(s)|^{\bar{p}}\big)\\
\leq &3^{\bar{p}-1}\mathbb{E}\|\xi^i(0)\|^{\bar{p}} + 3^{\bar{p}-1}\mathbb{E}\Big(\sup_{s\in[0,t]}\Big|\int_0^s \alpha\big(Z^{i,N}(r), Z^{i,N}(r-\rho), \mathscr{L}_{Z^N(r)}, \mathscr{L}_{Z^N(r-\rho)}\big)dr\Big|^{\bar{p}}\Big)\\
&+ 3^{\bar{p}-1}\mathbb{E}\Big(\sup_{s\in[0,t]}\Big|\int_0^s \beta\big(\mathscr{L}_{Z^N(r)}, \mathscr{L}_{Z^N(r-\rho)}\big)dB_r^{H,i}\Big|^{\bar{p}}\Big)\\
\leq &3^{\bar{p}-1}\mathbb{E}\|\xi\|^{\bar{p}} + C_{\bar{p},T,H,K_1,K_2}\mathbb{E}\int_0^t \big[\,1 + |Z^{i,N}(r)|^{\bar{p}} + |Z^{i,N}(r-\rho)|^{\bar{p}(l+1)} + \mathcal{W}_2^{\bar{p}}(\mathscr{L}_{Z^N(r)}) + \mathcal{W}_2^{\bar{p}}(\mathscr{L}_{Z^N(r-\rho)})\,\big]\,dr\\
\leq &C_{\bar{p},T,H,K_1,K_2,\|\xi\|}\int_0^t\big(1 + \mathbb{E}\big(\sup_{r\in[0,s]}|\tilde{Z}^{i,N}(r)|^{\bar{p}}\big)\big)ds + C_{\bar{p},T,H,K_1,K_2}\mathbb{E}\big(\sup_{s\in[0,t]}|\tilde{Z}^{i,N}(s-\rho)|^{\bar{p}(l+1)}\big),
\end{aligned}$$

Thanks to Gronwall's inequality, we see

$$\mathbb{E}\big(\sup_{s\in[0,t]}|\tilde{Z}^{i,N}(s)|^{\bar{p}}\big) \leq C_{\bar{p},T,H,K_1,K_2,\|\xi\|}\big[1 + \mathbb{E}\big(\sup_{s\in[0,t]}|\tilde{Z}^{i,N}(s-\rho)|^{\bar{p}(l+1)}\big)\big].$$

Construct a sequence: $\bar{p}_m = (2 - m + \lfloor\frac{T}{\rho}\rfloor)\bar{p}(l+1)^{1-m+\lfloor\frac{T}{\rho}\rfloor}$ for $m = 1, 2, \dots, \lfloor\frac{T}{\rho}\rfloor + 1$. For any $i \in \mathbb{S}_N$, then using the same iteration technique as that in Lemma 2.3 leads to

$$\mathbb{E}\big(\sup_{s\in[0,t]}|\tilde{Z}^{i,N}(s)|^{\bar{p}}\big) \leq C_{\bar{p},T,H,K_1,K_2,\|\xi\|,l,\rho}.$$

As for the case $H \in (0, 1/2)$, the proof process is similar after using Lemma 2.1. □

Lemma 4.2. *For $H \in (1/2, 1)$, let Assumption 2.4 hold. For any $i \in \mathbb{S}_N$ and $\hat{p} \geq 2$, we find that*

$$\mathbb{E}\big(\sup_{t\in[0,T]} |\tilde{Z}^{i,N}(t) - Z^{i,N}(t)|^{\hat{p}}\big) \leq C_{\hat{p},T,H,K_1,K_2,\|\xi\|,l,\rho}\Delta^{\hat{p}H}.$$

Proof. For any $t \in [t_k, t_{k+1})$ and $i \in \mathbb{S}_N$, we obtain from (4.2) that

$$\begin{aligned}&\tilde{Z}^{i,N}(t) - Z^{i,N}(t)\\ =&\int_{t_k}^t \alpha\big(Z^{i,N}(s), Z^{i,N}(s-\rho), \mathcal{L}_{Z^N(s)}, \mathcal{L}_{Z^N(s-\rho)}\big)ds + \int_{t_k}^t \beta\big(\mathcal{L}_{Z^N(s)}, \mathcal{L}_{Z^N(s-\rho)}\big)dB_s^{H,i}\\ =:&J_1(t) + J_2(t).\end{aligned}$$

We first estimate $J_2(t)$. For $\hat{p} \geq 2$, choose η to satisfy $1 - H < \eta < 1 - \frac{1}{\hat{p}}$. Denote $\varphi_\eta(t) = \int_s^t (t-u)^{-\eta}(u-s)^{\eta-1}du$. Applying stochastic Fubini's theorem and Hölder's inequality yields that

$$\begin{aligned}&\mathbb{E}\big(\sup_{t\in[t_k,t_{k+1}]} |J_2(t)|^{\hat{p}}\big)\\ \leq&(\varphi_\eta(t))^{-\hat{p}}\mathbb{E}\Big(\sup_{t\in[t_k,t_{k+1}]}\Big|\int_{t_k}^t\Big(\int_s^t (t-u)^{-\eta}(u-s)^{\eta-1}du\Big)\beta\big(\mathcal{L}_{Z^N(s)}, \mathcal{L}_{Z^N(s-\rho)}\big)dB_s^{H,i}\Big|^{\hat{p}}\Big)\\ \leq&(\varphi_\eta(t))^{-\hat{p}}\mathbb{E}\Big(\sup_{t\in[t_k,t_{k+1}]}\Big|\int_{t_k}^t (t-u)^{-\eta}\Big(\int_0^u (u-s)^{\eta-1}\beta\big(\mathcal{L}_{Z^N(s)}, \mathcal{L}_{Z^N(s-\rho)}\big)dB_s^{H,i}\Big)du\Big|^{\hat{p}}\Big)\\ \leq&(\varphi_\eta(t))^{-\hat{p}}\mathbb{E}\Big(\sup_{t\in[t_k,t_{k+1}]}\Big[\Big(\int_{t_k}^t (t-u)^{-\frac{\hat{p}\eta}{\hat{p}-1}}du\Big)^{\hat{p}-1}\Big(\int_{t_k}^t\Big|\int_0^u (u-s)^{\eta-1}\beta\big(\mathcal{L}_{Z^N(s)}, \mathcal{L}_{Z^N(s-\rho)}\big)dB_s^{H,i}\Big|^{\hat{p}}du\Big)\Big]\Big)\\ \leq&\frac{(\varphi_\eta(t))^{-\hat{p}}(\hat{p}-1)}{(\hat{p}-1-\hat{p}\eta)^{\hat{p}-1}}\Delta^{\hat{p}-1-\hat{p}\eta}\int_{t_k}^{t_{k+1}}\mathbb{E}\Big|\int_0^u (u-s)^{\eta-1}\beta\big(\mathcal{L}_{Z^N(s)}, \mathcal{L}_{Z^N(s-\rho)}\big)dB_s^{H,i}\Big|^{\hat{p}}du.\end{aligned}$$

By Theorem 1.1 in [42], we see that

$$\mathbb{E}\Big|\int_0^u (u-s)^{\eta-1}\beta\big(\mathcal{L}_{Z^N(s)}, \mathcal{L}_{Z^N(s-\rho)}\big)dB_s^{H,i}\Big|^{\hat{p}} \leq C_{\hat{p},H}\Big[\int_0^u (u-s)^{\frac{\eta-1}{H}}\Big|\beta\big(\mathcal{L}_{Z^N(s)}, \mathcal{L}_{Z^N(s-\rho)}\big)\Big|^{\frac{1}{H}}ds\Big]^{\hat{p}H}.$$

By choosing $\tilde{q} = \hat{p}H$, $\alpha = \frac{H-1-\eta}{H}$, $\tilde{p} = \frac{\hat{p}H}{\hat{p}(\eta+H-1)+1}$ in Lemma 3.2 in [13], we know from Assumption 2.4 and Lemma 4.1 that

$$\begin{aligned}&\mathbb{E}\big(\sup_{t\in[t_k,t_{k+1}]} |J_2(t)|^{\hat{p}}\big)\\ \leq&\frac{(\varphi_\eta(t))^{-\hat{p}}(\hat{p}-1)}{(\hat{p}-1-\hat{p}\eta)^{\hat{p}-1}}\Delta^{\hat{p}-1-\hat{p}\eta}\mathbb{E}\Big[\int_{t_k}^{t_{k+1}}\Big|\beta\big(\mathcal{L}_{Z^N(s)}, \mathcal{L}_{Z^N(s-\rho)}\big)\Big|^{\frac{\hat{p}}{\hat{p}(\eta+H-1)+1}}ds\Big]^{\hat{p}(\eta+H-1)+1}\\ \leq&C_{\hat{p},H,\eta}\Delta^{\hat{p}-1-\hat{p}\eta}\Delta^{\hat{p}(\eta+H-1)}\int_{t_k}^{t_{k+1}}\mathbb{E}\Big|\beta\big(\mathcal{L}_{Z^N(s)}, \mathcal{L}_{Z^N(s-\rho)}\big)\Big|^{\hat{p}}ds\\ \leq&C_{\hat{p},H,\eta,K_2}\Delta^{\hat{p}H-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\big[1 + \mathcal{W}_2^{\hat{p}}(\mathcal{L}_{Z^N(s)}) + \mathcal{W}_2^{\hat{p}}(\mathcal{L}_{Z^N(s-\rho)})\big]ds\\ \leq&C_{\hat{p},H,\eta}\Delta^{\hat{p}H-1}\int_{t_k}^{t_{k+1}}\Big(\mathbb{E}|\tilde{Z}^{i,N}(s)|^{\hat{p}} + \mathbb{E}|\tilde{Z}^{i,N}(s-\rho)|^{\hat{p}}\Big)ds\\ \leq&C_{\hat{p},H,\eta}\Delta^{\hat{p}H}.\end{aligned}$$

For $J_1(t)$, using the Hölder inequality and Lemma 4.1 implies that

$$\begin{aligned}&\mathbb{E}\big(\sup_{t\in[t_k,t_{k+1}]} |J_1(t)|^{\hat{p}}\big)\\ \leq&\Delta^{\hat{p}-1}\mathbb{E}\int_{t_k}^{t_{k+1}}\Big|\alpha\big(Z^{i,N}(s), Z^{i,N}(s-\rho), \mathcal{L}_{Z^N(s)}, \mathcal{L}_{Z^N(s-\rho)}\big)\Big|^{\hat{p}}ds\\ \leq&C_{\hat{p},K_1}\Delta^{\hat{p}-1}\mathbb{E}\int_{t_k}^{t_{k+1}}\big[1 + |Z^{i,N}(s)|^{\hat{p}} + |Z^{i,N}(s-\rho)|^{\hat{p}(l+1)} + \mathcal{W}_2^{\hat{p}}(\mathcal{L}_{Z^N(s)}) + \mathcal{W}_2^{\hat{p}}(\mathcal{L}_{Z^N(s-\rho)})\big]ds\\ \leq&C_{\hat{p},T,H,K_1,K_2,\|\xi\|,l,\rho}\Delta^{\hat{p}}.\end{aligned}$$

Thus,

$$\begin{aligned}&\mathbb{E}\big(\sup_{t\in[t_k,t_{k+1}]} |\tilde{Z}^{i,N}(t) - Z^{i,N}(t)|^{\hat{p}}\big)\\ \leq&2^{\hat{p}-1}\mathbb{E}\big(\sup_{t\in[t_k,t_{k+1}]} |J_1(t)|^{\hat{p}}\big) + 2^{\hat{p}-1}\mathbb{E}\big(\sup_{t\in[t_k,t_{k+1}]} |J_2(t)|^{\hat{p}}\big)\\ \leq&C_{\hat{p},T,H,K_1,K_2,\|\xi\|,l,\rho}\Delta^{\hat{p}H}.\end{aligned}$$

So the desired assertion holds. □

Assumption 4.3. There exist constants $K_3 > 0$ and $\vartheta \in (0,1]$ such that, for any $\check{p} > 0$,

$$\mathbb{E}\left(\sup_{t,s\in[-\rho,0]} |\xi(t)-\xi(s)|^{\check{p}}\right) \le K_3|t-s|^{\vartheta\check{p}}.$$

And ξ^i in the particle systems also satisfies this condition.

The following theorem reveals the convergence rate of EM scheme when $H \in (1/2,1)$.

Theorem 4.4. *For $H \in (1/2,1)$, let Assumptions 2.4 and 4.3 hold. Then for any $i \in \mathbb{S}_N$ and $p \ge 2$, one has*

$$\mathbb{E}\big(\sup_{t\in[0,T]} |\tilde{Z}^{i,N}(t) - Y^{i,N}(t)|^p\big) \le C_{p,T,H,K_1,K_2,K_3,\|\xi\|,l,\rho}\Delta^{(\vartheta\wedge H)p}.$$

Proof. For any $t\in[0,T]$ and $i \in \mathbb{S}_N$, applying Hölder's inequality, Assumption 2.4, (2.6) on (3.2) and (4.2) gives that

$$\begin{aligned}
&\mathbb{E}\big(\sup_{s\in[0,t]} |\tilde{Z}^{i,N}(s) - Y^{i,N}(s)|^p\big)\\
\le&2^{p-1}\mathbb{E}\left(\sup_{s\in[0,t]}\Big|\int_0^s \big[\,\alpha\big(Z^{i,N}(u), Z^{i,N}(u-\rho), \mathscr{L}_{Z^N(u)}, \mathscr{L}_{Z^N(u-\rho)}\big) - \alpha\big(Y^{i,N}(u), Y^{i,N}(u-\rho), \mathscr{L}_{Y^N(u)}, \mathscr{L}_{Y^N(u-\rho)}\big)\,\big]\,du\Big|^p\right)\\
&+2^{p-1}\mathbb{E}\Big(\sup_{s\in[0,t]}\Big|\int_0^s \big[\beta\big(\mathscr{L}_{Z^N(u)}, \mathscr{L}_{Z^N(u-\rho)}\big) - \beta\big(\mathscr{L}_{Y^N(u)}, \mathscr{L}_{Y^N(u-\rho)}\big)\big]dB_u^{H,i}\Big|^p\Big)\\
\le&C_{p,T,H,K_1,K_2}\mathbb{E}\int_0^t \Big[\,|Z^{i,N}(u)-Y^{i,N}(u)|^p + \big(1+|Z^{i,N}(u-\rho)|^l + |Y^{i,N}(u-\rho)|^l\big)^p|Z^{i,N}(u-\rho)-Y^{i,N}(u-\rho)|^p\\
&+\mathbb{W}_p^p(\mathscr{L}_{Z^N(u)}, \mathscr{L}_{Y^N(u)}) + \mathbb{W}_p^p(\mathscr{L}_{Z^N(u-\rho)}, \mathscr{L}_{Y^N(u-\rho)})\,\Big]\,du.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\mathbb{E}\big[\big(1+|Z^{i,N}(u-\rho)|^l + |Y^{i,N}(u-\rho)|^l\big)^p|Z^{i,N}(u-\rho)-Y^{i,N}(u-\rho)|^p\big]\\
\le&C_p\big[\mathbb{E}\big(1+|Z^{i,N}(u-\rho)|^{2lp} + |Y^{i,N}(u-\rho)|^{2lp}\big)\big]^{\frac12}\big[\mathbb{E}|Z^{i,N}(u-\rho)-Y^{i,N}(u-\rho)|^{2p}\big]^{\frac12}\\
\le&C_p\cdot C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\big[\mathbb{E}|Z^{i,N}(u-\rho)-Y^{i,N}(u-\rho)|^{2p}\big]^{\frac12}\\
=&C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\Big(\big[\mathbb{E}|Z^{i,N}(u-\rho)-\tilde{Z}^{i,N}(u-\rho)+\tilde{Z}^{i,N}(u-\rho)-Y^{i,N}(u-\rho)|^{2p}\big]^{\frac12}\Big)\\
\le&C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\Big(\big[\mathbb{E}|Z^{i,N}(u-\rho)-\tilde{Z}^{i,N}(u-\rho)|^{2p}\big]^{\frac12} + \big[\mathbb{E}|\tilde{Z}^{i,N}(u-\rho)-Y^{i,N}(u-\rho)|^{2p}\big]^{\frac12}\Big),
\end{aligned}$$

The Hölder inequality is used in the first inequality, Lemma 4.1 is used in the second inequality and the elementary inequality is used in the last inequality. Then by $\mathbb{E}\mathbb{W}_p^p(\mathscr{L}_{Z^N(\cdot)}, \mathscr{L}_{Y^N(\cdot)}) \le \frac{1}{N}\sum_{j=1}^N \mathbb{E}|Z^{j,N}(\cdot)-Y^{j,N}(\cdot)|^p$, we derive from Lemma 4.2 that

$$\begin{aligned}
&\mathbb{E}\big(\sup_{s\in[0,t]} |\tilde{Z}^{i,N}(s) - Y^{i,N}(s)|^p\big)\\
\le&C_{p,T,H,K_1,K_2,\|\xi\|,l,\rho}\int_0^t \Big[\,\mathbb{E}\big(\sup_{u\in[0,s]}|Z^{i,N}(u)-\tilde{Z}^{i,N}(u)|^p\big) + \mathbb{E}\big(\sup_{u\in[0,s]}|\tilde{Z}^{i,N}(u)-Y^{i,N}(u)|^p\big) + \mathbb{E}\big(\sup_{u\in[-\rho,0]}|\xi(u)-\xi(\lfloor\frac{u}{\Delta}\rfloor\Delta)|^p\big)\\
&+\big(\mathbb{E}|\tilde{Z}^{i,N}(s-\rho)-Y^{i,N}(s-\rho)|^{2p}\big)^{\frac12} + \Delta^{pH}\,\Big]\,ds.
\end{aligned}$$

The Gronwall inequality with Assumption 4.3 and Lemma 4.2 leads to

$$\begin{aligned}
&\mathbb{E}\big(\sup_{s\in[0,t]} |\tilde{Z}^{i,N}(s) - Y^{i,N}(s)|^p\big)\\
\le&C_{p,T,H,K_1,K_2,K_3,\|\xi\|,l,\rho}\Big(\Delta^{(\vartheta\wedge H)p} + \big[\mathbb{E}\big(\sup_{s\in[0,t]}|\tilde{Z}^{i,N}(s-\rho)-Y^{i,N}(s-\rho)|^{2p}\big)\big]^{\frac12}\Big).
\end{aligned}\tag{4.3}$$

Define a sequence as

$$p_m = (2-m+\lfloor\frac{T}{\rho}\rfloor)p2^{1-m+\lfloor\frac{T}{\rho}\rfloor}, \qquad m=1,2,\ldots,\lfloor\frac{T}{\rho}\rfloor+1.$$

One can see $p_{\lfloor\frac{T}{\rho}\rfloor+1} = p$ and $2p_{m+1} < p_m$ for $m=1,2,\ldots,\lfloor\frac{T}{\rho}\rfloor$. For $s\in[0,\rho]$, we get from (4.3) that

$$\mathbb{E}\big(\sup_{s\in[0,\rho]} |\tilde{Z}^{i,N}(s) - Y^{i,N}(s)|^{p_1}\big) \le C_{p_1,T,H,K_1,K_2,K_3,\|\xi\|,l,\rho}\Delta^{(\vartheta\wedge H)p_1}.\tag{4.4}$$

For $s \in [0, 2\rho]$, the Hölder inequality with (4.3) and (4.4) yields that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{s\in[0,2\rho]}|\tilde{Z}^{i,N}(s)-Y^{i,N}(s)|^{p_2}\Big)\\
\le& C_{p_1,p_2,T,H,K_1,K_2,K_3,\|\xi\|,l,\rho}\Big(\Delta^{(\vartheta\wedge H)p_2}+\big[\mathbb{E}\big(\sup_{s\in[0,2\rho]}|\tilde{Z}^{i,N}(s-\rho)-Y^{i,N}(s-\rho)|^{2p_2}\big)\big]^{\frac{1}{2}}\Big)\\
\le& C_{p_1,p_2,T,H,K_1,K_2,K_3,\|\xi\|,l,\rho}\Big(\Delta^{(\vartheta\wedge H)p_2}+\big[\mathbb{E}\big(\sup_{s\in[0,2\rho]}|\tilde{Z}^{i,N}(s-\rho)-Y^{i,N}(s-\rho)|^{p_1}\big)\big]^{\frac{p_2}{p_1}}\Big)\\
\le& C_{p_1,p_2,T,H,K_1,K_2,K_3,\|\xi\|,l,\rho}\Delta^{(\vartheta\wedge H)p_2}.
\end{aligned}$$

The induction about the time segment generated by time-delay ρ gives that

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|\tilde{Z}^{i,N}(t)-Y^{i,N}(t)|^{p}\Big)\le C_{p,T,H,K_1,K_2,K_3,\|\xi\|,l,\rho}\Delta^{(\vartheta\wedge H)p}. \quad \square$$

As for the convergence rate of EM scheme in the case $H \in (0, 1/2)$, the Corollary 5.5.3 in [1] plays a key role, so we quote it as the following lemma.

Lemma 4.5. *For each $\check{p} > 1$, there is a $C_{\check{p}} < \infty$ such that*

$$\mathbb{E}|B^H_t - B^H_s|^{\check{p}} \le C_{\check{p}}|t-s|^{\check{p}H}.$$

Lemma 4.6. *For $H \in (0, 1/2)$, let Assumption 2.4 hold. For any $i \in \mathbb{S}_N$, $t \in [0, T]$ and $\hat{p} \ge 2$, we have*

$$\mathbb{E}|\tilde{Z}^{i,N}(t)-Z^{i,N}(t)|^{\hat{p}} \le C_{\hat{p},T,H,K_1,\beta,\|\xi\|,l,\rho}\Delta^{\hat{p}H}.$$

Proof. For any $t \in [t_k, t_{k+1})$ and $i \in \mathbb{S}_N$, we see

$$\begin{aligned}
&\mathbb{E}|\tilde{Z}^{i,N}(t)-Z^{i,N}(t)|^{\hat{p}}\\
\le& 2^{\hat{p}-1}\mathbb{E}\left|\int_{t_k}^{t}\alpha\big(Z^{i,N}(s),Z^{i,N}(s-\rho),\mathscr{L}_{Z^N(s)},\mathscr{L}_{Z^N(s-\rho)}\big)ds\right|^{\hat{p}}+2^{\hat{p}-1}\mathbb{E}\left|\int_{t_k}^{t}\beta dB^{H,i}_s\right|^{\hat{p}}\\
\le& C_{\hat{p},T,H,K_1,\beta,\|\xi\|,l,\rho}\Delta^{\hat{p}H},
\end{aligned}$$

where we have used Lemma 4.5 and the estimation of J_1 in the proof of Lemma 4.2. $\square$

Theorem 4.7. *For $H \in (0, 1/2)$, let Assumptions 2.4 and 4.3 hold, and the diffusion coefficient be a constant β. Then, for any $i \in \mathbb{S}_N$ and $p \ge 2$, we have*

$$\mathbb{E}|\tilde{Z}^{i,N}(T)-Y^{i,N}(T)|^{p} \le C_{p,T,H,K_1,\beta,K_3,\|\xi\|,l,\rho}\Delta^{(\vartheta\wedge H)p}.$$

Thanks to Lemma 4.6, the proof of Theorem 4.7 is similar to Theorem 4.4, since the stochastic integral vanishes in $\tilde{Z}^{i,N}(t)-Y^{i,N}(t)$. Then the following main conclusion is reached by exploiting the triangle inequality for Theorems 3.2, 4.4 and 4.7.

Theorem 4.8. *For $H \in (1/2, 1)$, let all conditions in Theorems 3.2 and 4.4 be satisfied. Then we have*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|Y^i(t)-\tilde{Z}^{i,N}(t)|^p\Big)\le C\begin{cases}(N^{-1/2})^{\lambda_{p,T,\rho}}+\Delta^{(\vartheta\wedge H)p}, & \text{if } p>d/2,\\ [N^{-1/2}\log(1+N)]^{\lambda_{p,T,\rho}}+\Delta^{(\vartheta\wedge H)p}, & \text{if } p=d/2,\\ (N^{-p/d})^{\lambda_{p,T,\rho}}+\Delta^{(\vartheta\wedge H)p}, & \text{if } 2\le p<d/2,\end{cases}$$

where $\lambda_{p,T,\rho} = (\frac{p-\varepsilon}{p})^{\lfloor \frac{T}{\rho}\rfloor}$ and C is a positive real constant dependent of d, p, T, H, K_1, K_2, K_3, $\|\xi\|$, l, ρ but independent of N, Δ.

For $H \in (0, 1/2)$, let all conditions in Theorems 3.2 and 4.7 be satisfied. Then we have

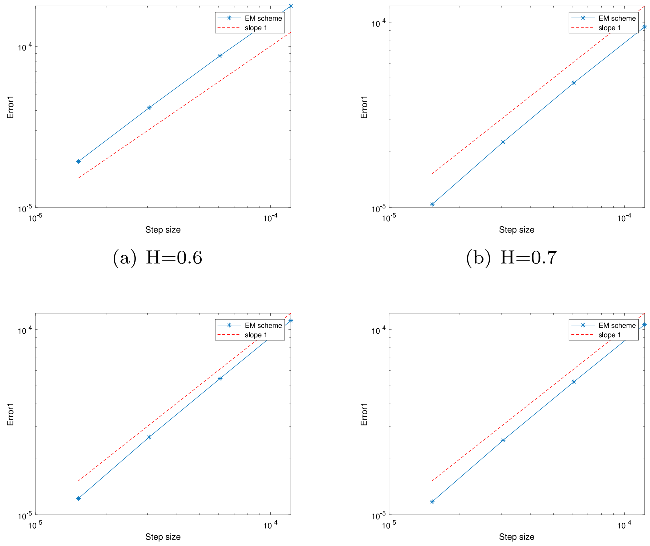
$$\mathbb{E}|Y^i(T)-\tilde{Z}^{i,N}(T)|^p\le C\begin{cases}(N^{-1/2})^{\lambda_{p,T,\rho}}+\Delta^{(\vartheta\wedge H)p}, & \text{if } p>d/2,\\ [N^{-1/2}\log(1+N)]^{\lambda_{p,T,\rho}}+\Delta^{(\vartheta\wedge H)p}, & \text{if } p=d/2,\\ (N^{-p/d})^{\lambda_{p,T,\rho}}+\Delta^{(\vartheta\wedge H)p}, & \text{if } 2\le p<d/2,\end{cases}$$

where $\lambda_{p,T,\rho} = (\frac{p-\varepsilon}{p})^{\lfloor \frac{T}{\rho}\rfloor}$ and C is a positive real constant dependent of d, p, T, H, K_1, β, K_3, $\|\xi\|$, l, ρ but independent of N and Δ.

5. Numerical experiment

In this section, we perform some numerical simulations for the SODM whose delay variable grows polynomially. Consider the following SODM

$$dY(t)=\Big[\,a_1\int_{\mathbb{R}}\Psi(|Y(t)-x|)(Y(t)-x)\mathbb{L}_Y(dx)+a_2Y(t)+a_3\psi(Y(t-\rho))+a_4\mathbb{E}Y(t-\rho)\,\Big]\,dt+a_5dB^H_t,\quad t\in[0,T], \tag{5.1}$$



(a) H=0.6 (b) H=0.7

(c) H=0.8 (d) H=0.9

Fig. 1. Convergence rates of $Z^{i,N}_{(\varsigma_r)}$ for (5.1).

with the initial value $\xi(\theta) = |\theta|, \theta \in [-\rho, 0]$, where the interaction kernel $\Psi(\cdot)$ represents the affect of an individual to each other, and $\psi(Y(t-\rho))$ is the function of $Y(t-\rho)$. Obviously, this SODM can reflect the effects of both intrinsic memory (delay state variable and its law) and extrinsic memory (fractional noise). For more details about such SODM, please refer to [14,43,44]. Let $a_1 = a_2 = a_4 = a_5 = 1$, $a_3 = -1$, $\rho = \frac{1}{8}$ and $\psi(Y(t-\rho)) = Y^3(t-\rho)$,

$$\Psi(|Y(t) - x|) = \begin{cases} \sin(|Y(t) - x| - 0.5), & \text{if } |Y(t) - x| < 0.5, \\ \cos(|Y(t) - x| + 0.5), & \text{if } |Y(t) - x| \geq 0.5. \end{cases}$$

The assumptions are satisfied. In the subsequent numerical simulations, the values utilized are those that are initially presented in this section, unless otherwise specified.

5.1. Convergence rates

In this subsection, we show the convergence rates of EM scheme in two ways. The numerical solution with $\Delta = 2^{-19}$ is used as the true solution since it cannot be represented explicitly. Then the numerical simulation for the error which is defined by

$$error1 = \left[\frac{1}{N} \sum_{i=1}^{N} \max_{1 \leq k \leq M_T} |Y^{i,N}_{(\varsigma)}(t_k) - Z^{i,N}_{(\varsigma_r)}(t_k)|^2 \right]^{\frac{1}{2}}, \tag{5.2}$$

with $T = 1, N = 200, \rho = \frac{1}{8}$ is implemented, where ς and $\varsigma_r (r \in \{1,2,3,4\})$ mean the level of the time discretization. Here, $Y^{i,N}_{(\varsigma)}$ is regarded as the solution to interacting particle system (3.2) with $\Delta = 2^{-19}$, and $Z^{i,N}_{(\varsigma_r)}$ is the numerical solution to classical EM scheme (4.1) with $\varsigma_r \in \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ matching $\Delta \in \{2^{-13}, 2^{-14}, 2^{-15}, 2^{-16}\}$. The errors between interacting particle system and numerical solution are depicted in Fig. 1, from which we observe that the convergence rate is approximately 1.

To demonstrate that the convergence rate is related to the Hurst parameter H, we introduce the following piecewise linear interpolation

$$Z^{i,N}_{\Delta}(t) = Z^{i,N}(t_k) + \frac{t - t_k}{\Delta}(Z^{i,N}(t_{k+1}) - Z^{i,N}(t_k)), \tag{5.3}$$

for any $t \in (t_k, t_{k+1}], k = 0, 1, \dots, M_T - 1$, where $Z^{i,N}(t_k)$ is defined in (4.1). We also simulate the true solution by using the small step size $\Delta_* = 2^{-19}$. Then $\Delta = 2^{-15}, 2^{-16}, 2^{-17}, 2^{-18}$ are step sizes of the numerical solutions. The piecewise linear interpolation (5.3) is used to get the numerical solutions $Z^{i,N}_{\Delta}(k\Delta_*)$. The error w.r.t. $Z^{i,N}_{\Delta}$ by

$$error2 = \left[\frac{1}{N} \sum_{i=1}^{N} \max_{k=1,\dots,2^{19}} |Y^{i,N}(k\Delta_*) - Z^{i,N}_{\Delta}(k\Delta_*)|^2 \right]^{\frac{1}{2}}, \tag{5.4}$$

is calculated for Fig. 2, where $Y^{i,N}$ is regarded as the true solution with $\Delta_* = 2^{-19}$, and $Z^{i,N}_{\Delta}$ is the numerical solution. From Fig. 2, we can observe that the slope of these lines are H.

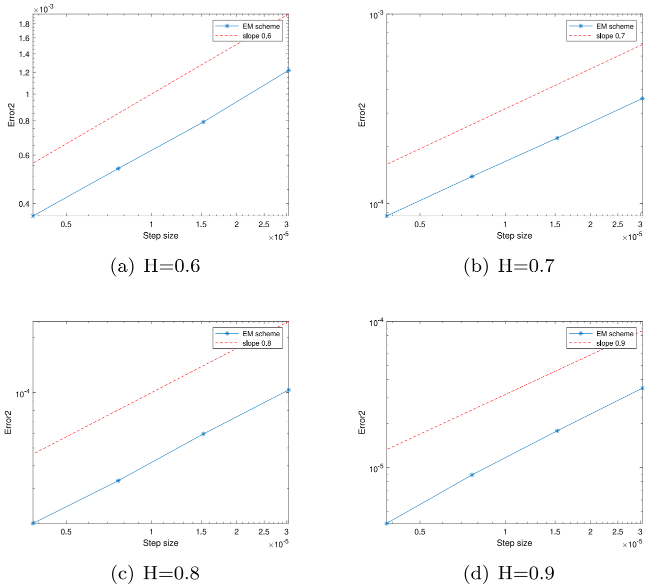


Fig. 2. Convergence rates of $Z_{\Delta}^{i,N}$ for (5.1).

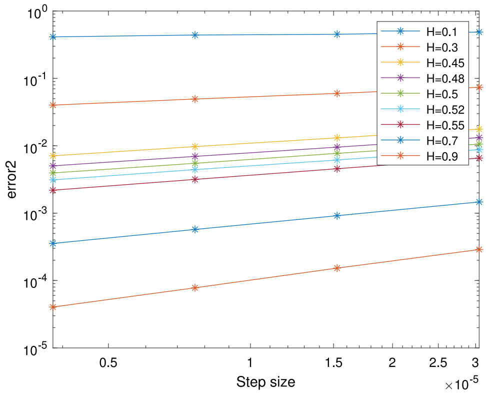


Fig. 3. Convergence rates for $H = 0.1, 0.3,\ 0.45, 0.48,\ 0.5, 0.52, 0.55, 0.7, 0.9$.

The reason why the convergence rates shown in Figs. 1 and 2 are different is that the Hölder regularity of fBm is reflected through piecewise linear interpolation. For more theoretical analysis on the Hölder regularity of fBm, please refer to [3,45].

5.2. Convergence rates when H approaches 0.5

To visually demonstrate the convergence of the numerical solution obtained by piecewise linear interpolation when the Hurst parameter H approaches 0.5, we plot the convergence rates corresponding to different parameters $H = 0.1, 0.3,\ 0.45, 0.48,$ $0.5, 0.52, 0.55, 0.7, 0.9$ on a figure, Fig. 3. In the simulation, the values of the parameters are the same as those for Fig. 2, so we omit these descriptions. From Fig. 3, we can infer that the convergence rate of the EM scheme remains consistent when fBm transitions to standard Brownian motion. Then Fig. 4 presents the corresponding pathwise behaviors of the numerical solutions for these nine cases $H = 0.1, 0.3,\ 0.45, 0.48,\ 0.5, 0.52, 0.55, 0.7, 0.9$.

5.3. Sensitivity analysis

Sensitivity analysis for the Hurst parameter H for the solution to SDE is important in many applications [46,47]. So we provide the sensitivity analysis w.r.t. H for SODM (5.1) via numerical simulation. When $H = 0.5$, SODM (5.1) becomes

$$d\tilde{Y}(t) = \Big[\, a_1 \int_{\mathbb{R}} \Psi(|\tilde{Y}(t) - x|)(\tilde{Y}(t) - x)\mathbb{L}_{\tilde{Y}}(dx) + a_2\tilde{Y}(t) + a_3\psi(\tilde{Y}(t-\rho)) + a_4\mathbb{E}\tilde{Y}(t-\rho) \,\Big]\, dt + a_5 dB_t, \quad t \in [0,T], \tag{5.5}$$

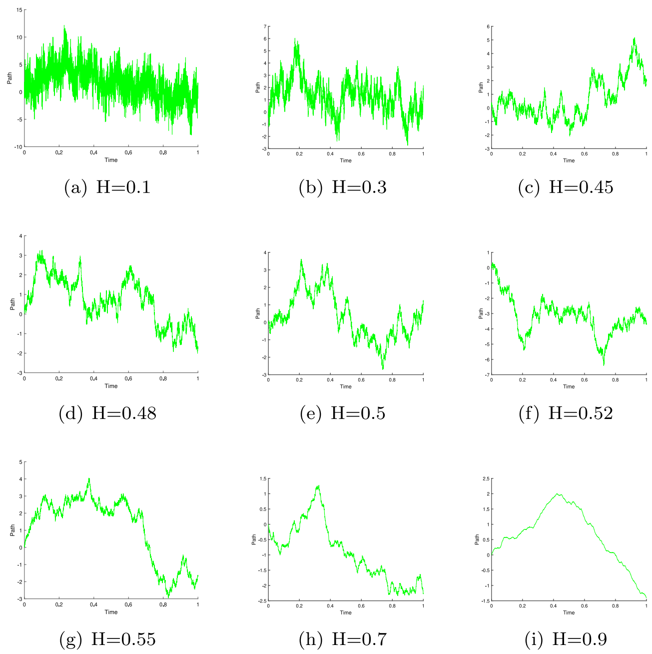


Fig. 4. Paths of the numerical solutions for $H = 0.1, 0.3,\ 0.45, 0.48,\ 0.5, 0.52, 0.55, 0.7, 0.9$.

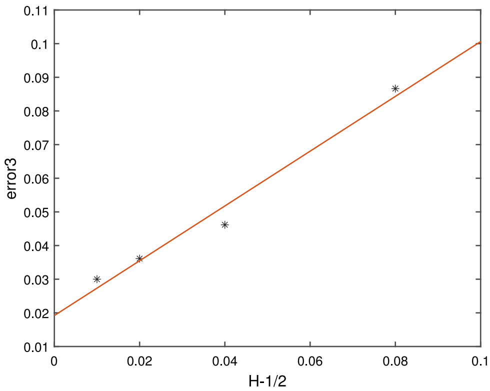


Fig. 5. Regression of (5.6) against $H - 1/2$.

with the initial value $\xi(\theta)$, where B_t is standard Brownian motion. Then the error is defined by

$$error3 = \left| \frac{1}{N} \sum_{i=1}^{N} |Y_H^{i,N}(T)|^2 - \frac{1}{N} \sum_{i=1}^{N} |\tilde{Y}^{i,N}(T)|^2 \right|, \tag{5.6}$$

with $T = 1, N = 200$, where $Y_H^{i,N}$ is the solution to (5.1) used to emphasize different H. Let the numerical solutions with $\Delta = 2^{-10}$ be regarded as the true solutions to (5.1) and (5.5), and make other parameters be the same as before. The regression line is plotted in Fig. 5, from which we can observe that the error (5.6) is linear w.r.t. $H - 0.5$.

5.4. Running time

In this subsection, the numerical simulations are all performed on the computer with Intel(R) Xeon(R) CPU E5-1630 v4 3.70 GHz. And all time units are in seconds. By [15], we know that the particle corruption means that one particle diverging can cause the whole system to diverge, and the more particles the more likely a divergence is to occur. In order to provide numerical evidence that the unmodified EM scheme does not experience particle corruption, we rerun each case 1000 times for the same SODM (5.1), but with $N = 500, 1000, 2000, 3000, 5000, 8000$ and record the average running time, shortest running time, longest running time in

Table 1
Running times in each 1000 times for different number of particles.

Number of particles	500	1000	2000	3000	5000	8000
Average running time	2.187	8.801	35.005	76.263	212.366	574.867
Shortest running time	1.692	7.519	31.040	70.447	191.270	527.285
Longest running time	3.392	9.605	38.084	85.683	239.309	619.714

Table 2
Running times for $T = 1, N = 50$.

	1	2	3	4	5
Tamed EM	3.953	3.609	3.593	3.684	3.712
EM	2.627	2.608	2.719	2.623	2.689

Table 3
Running times for $T = 1, N = 100$.

	1	2	3	4	5
Tamed EM	5.500	5.388	5.298	5.432	5.391
EM	4.590	4.388	4.318	4.044	4.365

Table 4
Running times for $T = 2, N = 50$.

	1	2	3	4	5
Tamed EM	3.671	3.595	3.699	3.650	3.614
EM	2.655	2.608	2.776	2.674	2.559

Table 5
Running times for $T = 2, N = 100$.

	1	2	3	4	5
Tamed EM	5.298	5.199	5.582	5.443	5.312
EM	4.519	4.452	4.436	4.605	4.208

Table 1. Let $T = 2, \Delta = 0.05$ and other parameters are the same as before. In the simulation, if the particle corruption occurs, the simulation will stop and report an error; otherwise, it runs until $T = 2$. From Table 1, we can infer that no particle corruption occurs in 6000 numerical simulations, even if the delay variable grows polynomially.

Next, we use the tamed EM scheme investigated in [15] as a comparison object to illustrate the efficiency advantage of the unmodified EM scheme. The tamed drift coefficient is defined as $\alpha_\Delta(x_1, x_2, \mu_1, \mu_2) = \frac{\alpha(x_1,x_2,\mu_1,\mu_2)}{1+\Delta|\alpha(x_1,x_2,\mu_1,\mu_2)|}$. Let $\Delta = 2^{-10}, T = 1, 2, N = 50, 100$ and other parameters are the same as before. The five simulations are performed for each case and the corresponding running times are shown in Tables 2–5. We can find that the unmodified EM scheme requires less time than the tamed one.

CRediT authorship contribution statement

Shuaibin Gao: Writing – original draft. **Qian Guo:** Writing – review & editing. **Zhuoqi Liu:** Writing – review & editing. **Chenggui Yuan:** Writing – review & editing.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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