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Applications of Chern-Simons Theories to Integrable Sigma Models

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Abstract

This thesis studies the applications of Chern-Simons theories to integrable sigma models. Until recently, finding new examples of integrability was challenging, and the origins of the Lax formalism remained mysterious. However, the recent discovery of four-dimensional and six-dimensional Chern-Simons theories offers a new perspective on these topics, and allows for the systematic construction of novel integrable sigma models. This thesis discusses the extension of these formalisms to a wider set of the known integrable theories. We incorporate sectors of gravity, integrable deformations, and gauged models into the scope of six-dimensional Chern-Simons theory, leading to generalisations of four-dimensional Chern-Simons theory and novel integrable field theories. In a separate line of study, we consider the three-dimensional Chern-Simons approach to chiral bosons, showing how it is related to various two-dimensional approaches. This allows us to generalise the known two-dimensional approaches to include both non-Abelian fields and twisted chirality.

Declarations

- This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.
- This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by explicit references, and a bibliography is appended. In particular, this thesis includes results from the following publications, which were written during the candidature period.
  - Alex S. Arvanitakis, Lewis T. Cole, Ondrej Hulik, Alexander Sevrin, Daniel C. Thompson. "Unifying approaches to chiral bosons". (2023) [Arv+23]
  - Lewis T. Cole, Ryan A. Cullinan, Ben Hoare, Joaquin Liniado, Daniel C. Thompson. "Integrable Deformations from Twistor Space". (2024) [Col+24b]
  - Lewis T. Cole, Ryan A. Cullinan, Ben Hoare, Joaquin Liniado, Daniel C. Thompson. "Gauging The Diamond: Integrable Coset Models from Twistor Space". (2024) [Col+24b]
  - Lewis T. Cole, Peter Weck. "Integrability in Gravity from Chern-Simons Theory". (2024) [CW24]
- I hereby give consent for my thesis, if accepted, to be available for electronic sharing.
- The University's ethical procedures have been followed and, where appropriate, that ethical approval has been granted.

Signed: Lewis T. Cole

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Acknowledgements

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Informal introduction

The views and opinions expressed in this section are those of the author and do not necessarily reflect the official policy or position of Swansea University.

Science is about understanding the nature of reality. Unfortunately, this turns out to be pretty hard. The universe is a hot, wet, messy place and things have an awful habit of moving around and bumping into one another. Fortunately, many generations of intrepid explorers have ventured out into this wilderness and returned with insights into the way things work, extending the map of human knowledge into previously uncharted territory. Remarkable progress has been made, especially in the field of theoretical physics.

There is some sense in which physics is the most fundamental of the sciences. In the relay race of our quest for understanding, physics is the final runner. Sociologists eventually pass the baton to psychologists, biologists eventually pass to chemists, but physicists refuse to let go, even as the finish line recedes indefinitely into the distance. Whilst this might be worn as a badge of honour, in practice it makes things a whole lot easier. By focusing on the microscopic or the cosmic, the immediate or the distant future, the ultra-cold or the ultra-hot, many complex systems become simpler in these extreme regimes. This distinction, between those problems which are unreasonably challenging and those which are surmountable, often comes down to a distinction between linear and non-linear differential equations.

The laws of physics describe how a system in a given state evolves as we move forwards in time. In mathematics, these types of equations are known as differential equations, and solving them amounts to "integrating" them. Looking at a map describing the landscape of differential equations (figure 1), there is a small garden of serenity in the lower left corner. Inside this garden lies the region of linear differential equations, where techniques such as separation of variables and Fourier transforms can often be used to find explicit solutions. Outside of the garden, however, lies the arid plains of non-linear differential equations. In these lands, phenomena such as shock waves, turbulence, and chaotic behaviour make explicit solutions rare and difficult to obtain.

Many of the successes of theoretical physics come down to turning non-linear differential equations into linear ones. By taking approximations, considering extreme regimes, or leveraging symmetries, complicated non-linear systems can be reduced to tractable linear problems. Indeed, one of the most prolific techniques in modern theoretical physics is perturbation theory, where solutions to a simpler linear equation are gradually corrected so that they approximate solutions to a more challenging non-linear equation. Moreover, some of the most important challenges appear in the contexts where this approach breaks down.

In the year 2000, the Clay Mathematics Institute announced a $1 million prize for each solution to one of the millenium prize problems. These were seven of the most difficult and important open problems at the turn of the second millennium, and so far only one of them has been solved. Two of the remaining unsolved problems are of central interest to the physics community, and have a (slightly tenuous) connection to the contents of this thesis.

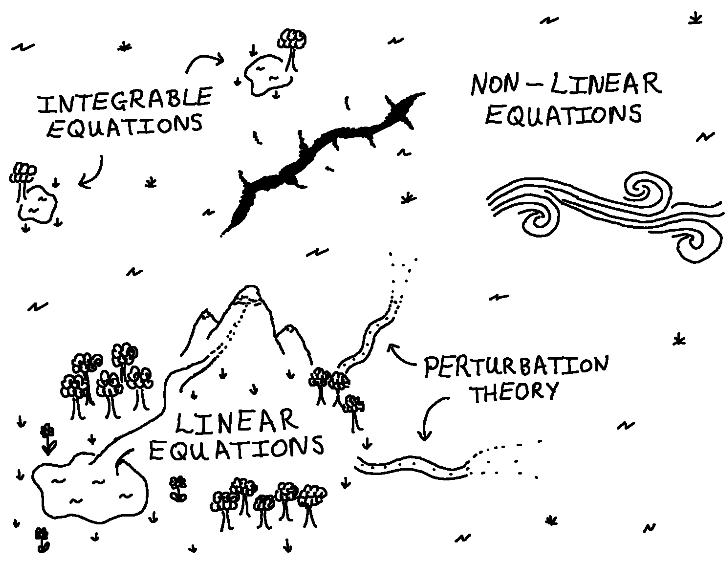


Figure 1: The landscape of differential equations.

The first concerns the Navier-Stokes equation: a non-linear differential equation which governs the flow of fluids such as water and air. The first questions that mathematicians ask about a differential equations are whether solutions exist, and whether they are unique. This incredibly basic and fundamental question about the Navier-Stokes equation is one of the millenium prize problems. It is a testament to the difficulties of non-linear differential equations that this problem comes with a $1 million prize, and that this prize remains unclaimed.

The second concerns Yang-Mills theory: a non-Abelian gauge theory which plays a central role in the standard model of particle physics. A simpler Abelian version of Yang-Mills theory describes electromagnetism which governs the dynamics of electric fields (E) and magnetic fields (B). Whilst the millenium problem asks for a proof of the "mass gap" in non-Abelian Yang-Mills theory, there is another aspect of this theory which is relevant to this discussion. The non-Abelian Yang-Mills equations are non-linear differential equations which are manifested in the standard model through the theory of quantum chromodynamics. In this context, the preferred tool of perturbation theory proves ineffective because the theory is strongly coupled. This means that the simpler linear equation does not give a good approximation of the non-linear system, and the methodology of perturbation theory is invalidated.

Returning to the map of differential equations (figure 1), these problems are hard to address because they lie very far away from the garden of linear equations. Perturbation theory allows us to take small steps, gradually extending our reach into the arid plains, but this progress appears fundamentally infinitesimal. Questions like the existence of solutions to the Navier-Stokes equation, and the strong coupling dynamics of Yang-Mills theory, require more dramatic leaps away from the charted territory. One might then wonder which tools are available to make this ambitious progress.

Looking closer at the map, a few small oases are distributed across the landscape. Far out in the dangerous lands of non-linear differential equations, these pockets of relative safety might offer us a foothold from which to launch further expeditions. These oases represent to the minority of non-linear differential equations which can be solved exactly. Since solving a differential equation amounts to integrating it, these special equations as referred to as integrable. It is possible to solve these systems due to the existence of a large number of hidden symmetries. If the number of symmetries is equal to the number of variables in the system, then one can transform the non-linear equation into a system of linear equations, a result known as the Liouville-Arnold theorem in Hamiltonian dynamics.

Whilst neither the Navier-Stokes equation nor the Yang-Mills equation is integrable, they each have integrable sectors which at least offer us a place to start. In the case of the Navier-Stokes equation, one can make a series of assumptions and approximations to derive the Korteweg-de Vries (KdV) equation which describes waves on shallow water surfaces. This is the prototypical integrable equation, and its history starts in 1834 with John Scott Russell. While conducting experiments on Edinburgh's Union Canal to determine the most efficient boat design, Russell observed an impressive phenomenon which we describe here in his own words:

> *I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped — not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.*

It wasn't until much later that the KdV equation was discovered and solutions reproducing this behaviour were found — these solutions are known as solitons, and they are a typical feature of integrable systems.

Turning to Yang-Mills theory, in this case the integrable sector doesn't require any assumptions or approximations, it is just a special class of solutions which can be found exactly. These solutions satisfy a technical property known as self-duality, which corresponds to $E = B$ in the context of electromagnetism (anti-self-duality is just as good with $E = -B$). These configurations play a privileged role in the theory earning the title of instantons (or anti-instantons). One especially intriguing feature of the self-dual Yang-Mills equation is that many other known integrable systems can be derived as reductions of this single equation. By imposing additional constraints on our solutions, lower dimensional models emerge as distinguished sectors of self-dual Yang-Mills theory. In fact, Richard Ward once conjectured that all integrable equations might arise as reductions of this single master equation, though this is no longer believed to be the case.

Another active research area in modern theoretical physics is string theory: a framework in which point particles are replaced by one-dimensional extended objects called strings. This has been one of the most extensive and popular fields over the last quarter of a century. Of the 92,113 papers published in high energy theoretical physics (hep-th) since I was born (in Nov 1997), eight of the ten most cited papers are on the topic of string theory, with 21,525 papers citing at least one of these eight publications. Just like the Navier-Stokes equation and Yang-Mills theory, there are certain sectors of string theory which are integrable, and these go under the name of integrable sigma models.

Having observed that integrable systems appear in many important areas of physics, one might ask why these equations exist in the first place. After all, non-linear differential equations are supposed to be hard, so hard that it is impossible to find an algorithm which solves a generic non-linear differential equation. This is not just a practical statement, it has been mathematically proven that no such algorithm may exist. It is therefore very surprising that these special integrable equations do exist and that they can often be solved exactly.

One step towards answering the question of why integrable equations exist might be to provide a common origin of the many known examples. This could also enable the construction of further examples of integrability, provided that the mechanism for deriving known models was somewhat systematic. On the one hand, a candidate for this common origin is self-dual Yang-Mills, and there has been a long and successful history of studying integrable equations as reductions of self-dual Yang-Mills. On the other hand, in recent years another candidate theory has emerged under the name of four-dimensional Chern-Simons theory. The standard Chern-Simons theory is a three-dimensional non-Abelian gauge theory, just like Yang-Mills, which has applications ranging from the mathematical field of topology to the fractional quantum Hall effect in condensed-matter physics. Its recently-discovered four-dimensional counterpart manifests a certain integrable structure known as the Lax formalism. In fact, both four-dimensional Chern-Simons and self-dual Yang-Mills theory have a common origin in a six-dimensional Chern-Simons theory on twistor space. This correspondence of theories is summarised in figure 2.

This exciting recent development in the field of integrability might well offer profound and far-reaching insights, relevant to many areas of theoretical physics. However, it is still early days for this burgeoning topic — one can count the number of papers addressing this correspondence of theories on two hands. Much of the work discussed in this thesis concerns the extension of this correspondence to a wider subset of the known integrable models. We have taken a particular focus on integrable sigma models, due to their popular application in string theory.

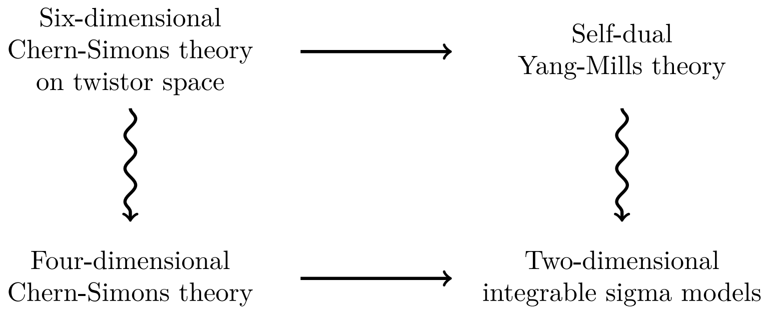


Figure 2: A correspondence of integrable models. The arrows in this diagram represent systematic procedures which take you from one model to the next.

Outline

Let us now outline the contents of this thesis.

The first two chapters of this thesis review background material, and are intended to provide pedagogical introductions to the relevant topics. In chapter 1, we describe the relationship between two-dimensional integrable sigma models and four-dimensional Chern-Simons theory. This includes a brief introduction to both sigma models and three-dimensional Chern-Simons theory. In chapter 2, we review self-dual Yang-Mills theory and its relationship to integrability. In addition, we discuss the Penrose-Ward transform, a method of finding solutions to self-dual Yang-Mills theory via twistor space. This culminates in the introduction of six-dimensional Chern-Simons theory, a gauge theory on twistor space which encodes the Penrose-Ward transform.

The remaining chapters present results from publications written during the candidate period. These results are not presented in chronological order. Chapter 3 describes a method for studying solutions to four-dimensional general relativity via the correspondence of integrable models described above. We consider spacetime metrics with two commuting Killing vectors, such that the problem reduces to that of a two-dimensional sigma model. The associated four-dimensional Chern-Simons theory is constructed as a reduction of six-dimensional Chern-Simons theory.

In chapter 4, we extend the integrable sigma models accessible via six-dimensional Chern-Simons theory to include continuous families of integrable deformations. We study the specific example of the λ-model, a well-studied integrable sigma model with a deformation parameter commonly denoted by λ. Technical obstacles to the recovery of this model from six-dimensional Chern-Simons theory are highlighted and overcome, leading to the generalisation of certain results in four-dimensional Chern-Simons theory. A novel four-dimensional integrable field theory is constructed which descends to the λ-model via reduction.

Chapter 5 also looks to extend the reach of six-dimensional Chern-Simons theory, this time in the direction of gauged integrable sigma models. We consider a theory built from the difference of two six-dimensional Chern-Simons theories, leading to the recovery of certain coset CFTs and their deformations. In particular, when attempting to construct gauged integrable field theories in four-dimensions, we present constraints that the gauging must satisfy in order to preserve integrability.

Another approach to studying integrable sigma models is via the various “doubled” formalisms which involve considering two dual frames of the theory at once. Since this amounts to working with twice as many coordinates on target space, these doubled theories exhibit chiral dynamics which are notoriously hard to quantise. In chapter 6, we provide a Chern-Simons origin for some popular approaches to studying chiral bosons. We review numerous two-dimensional approaches which introduce auxiliary fields to simplify the dynamics and symmetries of the system. Then, we show that each of these approaches can be recovered from three-dimensional Chern-Simons theory, and that each auxiliary field can be understood as an additional component of the three-dimensional gauge field which has yet to be integrated out. This allows us to extend the known approaches to include both non-Abelian fields and twisted chirality.

Chapter 1

Integrable sigma models and Chern-Simons theories

1.1 Integrable sigma models

The fundamental field of a sigma model is a map from some manifold to another manifold. For our purposes, the domain is either $\mathbb{R}^4$, in which case we refer to it as spacetime, or some Riemann surface Σ, in which case we refer to it as the worldsheet. We will mostly consider the case where the fundamental field is valued in a Lie group G which we will refer to as the target space. These models were first introduced in an attempt to study charged pion decay [GL60], where a scalar meson was coupled to nucleons and pions in the action. This scalar meson was denoted by σ leading to the name of "sigma model".

Over the last quarter of a century, these models have more often appeared in the context of string theory, where the domain is the string worldsheet. The two-dimensional worldsheet Σ describes the surface traversed by a moving string, and this is embedded into some target space by the embedding maps $\{X^I\}$ which also act as coordinates on the target space. In this simplest case, where the target space is flat, worldsheet string theory is described by the Polakov action [Pol81],

$$S_{\mathrm{P}}[X,\gamma] = \frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2x\,\sqrt{-\det(\gamma)}\,\gamma^{\mu\nu}\partial_\mu X^I\partial_\nu X^J\eta_{IJ}\ . \tag{1.1}$$

In this expression, γ is a dynamical worldsheet metric, η is the Minkowski target space metric, and α' is a coupling constant related to the tension of the string. The dynamical worldsheet metric γ is often gauge fixed to the flat Lorentzian metric $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2$ where $\{t,x\}$ are coordinates on Σ. Adopting lightcone coordinates $\partial_\pm = \partial_t \pm \partial_x$, the Polyakov action may be written in this gauge as

$$S[X] = \frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2x\ \partial_+X^I\,\eta_{IJ}\,\partial_-X^J\ . \tag{1.2}$$

If we interpret this theory as a gauge-fixed Polyakov action, we should retain the equations of motion associated with γ as constraints — namely the Virasoro constraints. On the other hand, this action may be taken to define a sigma model as a two-dimensional quantum field theory in its own right. Whether or not one imposes the Virasoro constraints determines whether one is studying the sigma model as a quantum field theory or as string theory. In addition, a complete description of string theory should also include a sum over topologies of the worldsheet Σ.

The massless excitations of closed string theory include three geometric structures on target space. The first is a symmetric tensor G_{IJ} which plays the role of a target space metric. Secondly, there is an anti-symmetric tensor B_{IJ} which is known as the (Kalb-Ramond) B-field. This field is a 2-form on target space which comes with a gauge symmetry acting as $B \mapsto B + \mathrm{d}\Lambda$. The physical information contained in B is captured by its field strength $H = \mathrm{d}B$ which is a 3-form on target space. Finally, there is a scalar excitation Φ known as the dilaton, but we will neglect this field in our discussions.

One can then consider string theory in a coherent state of these massless excitations, which has the effect of modifying the target space geometry. In practice, this amounts to considering the sigma model action

$$S[X] = \frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2x \left(G_{IJ} + B_{IJ}\right) \partial_+ X^I \partial_- X^J \ . \tag{1.3}$$

This action always makes sense as a two-dimensional quantum field theory, but we might also want to interpret it as a gauge-fixed string theory. In order for this to be consistent, the string theory action with this non-trivial metric and B-field must have the same gauge symmetries as the Polyakov action. In particular, the worldsheet string theory action must be Weyl-invariant, which amounts to the vanishing of the beta-functions for the metric, B-field, and dilaton.

In the context of quantum field theory, beta-functions encode the dependence of coupling parameters on the energy scale. The metric and B-field appear in the action as matrices of coupling parameters, so we can compute how these objects will change as we change the energy scale. For example, the beta-function of the metric is given by

$$\beta^{(G)}_{IJ} = \alpha' R_{IJ} - \frac{\alpha'}{4} H_{IKL} H_J{}^{KL} + O(\alpha'^2) \ . \tag{1.4}$$

If the beta-functions of a theory vanish, often at some fixed point in the coupling parameters, then we say that the theory is scale-invariant. This feature is required in order to interpret the sigma model as a gauge-fixed string theory.

Setting aside the string theory motivations, we can also study sigma models which are not scale-invariant. These might play the role of a single component in a larger scale-invariant background, or we might study sigma models as interesting examples of two-dimensional quantum field theories. With either of these approaches in mind, a noteworthy class to consider are the integrable sigma models. The adjective 'integrable' implies that these models have a large number of symmetries (infinitely many), and this allows for the computation of exact results, far beyond what one can hope for in a usual quantum field theory. For example, the exact scattering matrix of these quantum field theories can often be computed by virtue of a factorisation property [ZZ79]. Whilst quantum aspects of these models will be beyond the scope of this thesis, they are an exciting active area of modern research.

1.1.1 Principal chiral model

The prototypical example of an integrable sigma model is the principal chiral model (PCM) [Poh76; ZM78]. In this case, the target space is a Lie group G and we take the B-field to vanish, leaving the sigma model action

$$S_{\mathrm{PCM}}[X] = \int_\Sigma \mathrm{d}^2 x \; G_{IJ} \, \partial_+ X^I \partial_- X^J \ . \tag{1.5}$$

There is a canonical metric on the group manifold which is built from the Killing form on its Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathsf{G})$. To build this metric, we start by considering the left-invariant Maurer-Cartan forms $j = g^{-1}\mathrm{d}g$ which are algebra-valued 1-forms on the group manifold, defined in terms of a group element $g \in \mathsf{G}$. We will pair two of these 1-forms together using the Killing form $\langle \cdot\,, \cdot \rangle$, which is a bilinear form on the Lie algebra $\mathfrak{g}$,

$$\langle \cdot\,, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \ . \tag{1.6}$$

We will mostly consider matrix groups, in which case the Killing form is proportional to the matrix trace in some representation. The metric on the group manifold can then be written as

$$G_{IJ} = r^2 \ \mathrm{tr}\big(g^{-1} \partial_I g \cdot g^{-1} \partial_J g \big) \ , \tag{1.7}$$

where we have included an overall coefficient $r \in \mathbb{R}$ measuring the radius of the target space. For example, if we specialise to $\mathsf{G} = \mathsf{SU}(2)$, then this metric may be written as

$$\mathrm{d}X^I G_{IJ} \, \mathrm{d}X^J = r^2 \big[\mathrm{d}\psi^2 + \sin^2(\psi) \big(\mathrm{d}\theta^2 + \sin^2(\theta) \, \mathrm{d}\varphi^2 \big) \big] \ . \tag{1.8}$$

This is the usual round metric on the three-sphere $S^3 \cong \mathsf{SU}(2)$.

Under renormalisation group flow (changing the energy scale), the metric changes according to the beta-function

$$\beta^{(G)}_{IJ} = R_{IJ} \ . \tag{1.9}$$

This is the specialisation of the expression given above for vanishing B-field, and it is known as Ricci flow [Fri80; Fri85]. In fact, this differential equation on the space of Riemannian metrics played a foundational role in proving the Poincaré conjecture — the only millennium prize problem which has been solved to date. In the context of the PCM, it tells us that theory is asymptotically free (the radius gets large at high energies making the geometry appear flat), whilst it is strongly coupled at low energies. These properties are shared by quantum chromodynamics (QCD) in the real world, making the PCM an enticing toy model for this challenging theory.

The metric on G is bi-invariant, meaning that it is invariant under the $\mathsf{G} \times \mathsf{G}$ action

$$g \mapsto h_\ell^{-1} \cdot g \cdot h_r \ . \tag{1.10}$$

These are global symmetries of the PCM, and the associated conserved currents are the Maurer-Cartan forms. We can make these symmetries manifest in the action by writing it in terms of a map $g : \Sigma \to \mathsf{G}$, instead of the target space coordinates $\{X^I\}$. The PCM action is written in terms of this group-valued map as

$$S_{\text{PCM}} = r^2 \int_\Sigma \mathrm{d}^2 x \ \mathrm{tr}\big(g^{-1}\partial_+ g \cdot g^{-1}\partial_- g\big) \ . \tag{1.11}$$

This expression is written in terms of the pullback of the Maurer-Cartan form to the worldsheet, whose components are given by $j_\pm = g^{-1}\partial_\pm g$. Varying this action gives the equations of motion

$$\partial_+ j_- + \partial_- j_+ = 0 \ . \tag{1.12}$$

This characterises the dynamics of the PCM, together with the Maurer-Cartan identity $\mathrm{d}j + j \wedge j = 0$ which can be verified using $\mathrm{d}(g^{-1}) = -g^{-1}\mathrm{d}g g^{-1}$.

As stated earlier, the PCM is the prototypical example of an integrable sigma model. Whilst integrable models share many common features, such as a large number of symmetries and some susceptibility to exact methods, it is hard to give a precise definition of integrability which holds across all known examples. That being said, most approaches to integrable models start with the introduction of a linear system which captures the dynamics of the theory. In the context of integrable sigma models, this linear system is provided by a Lax connection [Lax68; ZM78], a $\mathfrak{g}^{\mathbb{C}}$-valued 1-form on Σ which depends on an auxiliary spectral parameter $\zeta \in \mathbb{C}$. This Lax connection should be built from the fundamental fields of the theory, and flatness of the Lax for all values of the spectral parameter should be equivalent to the equations of motion.

For example, the Lax connection associated with the PCM is given by

$$L_\pm = \frac{g^{-1}\partial_\pm g}{1 \pm \zeta} \ . \tag{1.13}$$

The field strength of the Lax is defined by $F = \mathrm{d}L + L \wedge L$, and (since the worldsheet Σ is two-dimensional) this only has one independent component, namely F_{+-}. A connection is flat when its field strength vanishes ($F = 0$), and for the PCM Lax this condition reads

$$F_{+-} = \frac{1}{1-\zeta^2}\big(\partial_+ j_- - \partial_- j_+ + [j_+, j_-]\big) + \frac{\zeta}{1-\zeta^2}\big(\partial_+ j_- + \partial_- j_+\big) = 0 \ . \tag{1.14}$$

In order for this to vanish identically in $\zeta \in \mathbb{C}$, both terms must vanish independently. The first term is zero due to the Maurer-Cartan identity, and the second term coincides with the equations of motion of the PCM. This demonstrates the required equivalence between the flatness of the Lax and the dynamics of the integrable model,

$$F_{+-} = 0 \qquad \forall \zeta \in \mathbb{C} \qquad \Longleftrightarrow \qquad \text{EOM} \ . \tag{1.15}$$

Once a Lax connection has been found, there is a recipe for constructing the infinite set of conserved charges. We define the monodromy matrix as the path-ordered exponential of the Lax connection integrated over some curve $\gamma \subset \Sigma$,

$$M(\gamma) = \text{Pexp} \int_\gamma L \ . \tag{1.16}$$

Performing a Laurent expansion of the monodromy matrix in the spectral parameter $\zeta \in \mathbb{C}$ produces an infinite tower of conserved charges which are associated with an infinite dimensional symmetry algebra. Higher orders terms in this expansion are progressively more non-local, as they contain more and more nested integrals over the curve $\gamma \subset \Sigma$. This means that the infinitely many symmetries are also non-local and very hard to see at the level of the action, they are often referred to as hidden symmetries for this reason.

In the context of integrable sigma models, a theory is said to have 'weak integrability' if one can show that its equations of motion are equivalent to the flatness of some Lax connection. This establishes the existence of an infinite set of conserved charges, but does not tell you about any relations amongst them. For a more complete treatment, one should also show that the Poisson brackets of the Lax take the Maillet form [Mai86a; Mai86b], thereby demonstrating 'strong integrability' of the theory. This ensures that the infinite set of conserved charges are in involution, meaning that their Poisson brackets with one another vanish. In this thesis, we will focus our attention on the condition of weak integrability.

1.1.2 PCM plus WZ term

The target space of the principal chiral model (PCM) is a Lie group G equipped with a bi-invariant metric G_{IJ}. This theory has a vanishing B-field, and one might wonder whether one can add a non-trivial B-field to the action which preserves the global $\mathsf{G} \times \mathsf{G}$ symmetry. In fact, since the B-field changes under gauge transformations, we should be aiming for the 3-form field strength $H = \mathrm{d}B$ which is an invariant geometric structure on target space.

Fortunately, there is a canonical 3-form on the group manifold which is invariant under the $\mathsf{G} \times \mathsf{G}$ action. We will refer to this 3-form as the Wess-Zumino (WZ) form and denote it by

$$\text{WZ}[g] = \frac{1}{3} \,\text{tr}\big(g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g\big) \ . \tag{1.17}$$

This term is manifestly invariant under the $\mathsf{G} \times \mathsf{G}$ action, but we run into a problem when trying to include it in the action. The worldsheet Σ is only two-dimensional, so it doesn't make sense to integrate a 3-form over this manifold. However, it is possible to overcome this obstruction by introducing an extension of the worldsheet which we will denote by $\widetilde{\Sigma} = \Sigma \times [0,1]$. In tandem with this extended manifold, we also introduce a smooth homotopy from the trivial map to our fundamental field g which we denote by $\tilde{g} : \widetilde{\Sigma} \to \mathsf{G}$. This extended field agrees with the trivial map at one end of the interval ($\tilde{g}|_0 = \mathrm{id}$) and with our fundamental field at the other ($\tilde{g}|_1 = g$).

We can then construct the PCM plus WZ term [Nov82; Wit84] whose action is given by

$$S_{\mathrm{PCMWZ}} = r^2 \int_\Sigma \mathrm{d}^2 x \ \mathrm{tr}\big(g^{-1}\partial_+ g \cdot g^{-1}\partial_- g\big) + k \int_{\widetilde{\Sigma}} \mathrm{WZ}[\tilde{g}] \ . \tag{1.18}$$

This sigma model has the same bi-invariant metric G_{IJ}, but now the WZ term sources a non-trivial field strength $H = k\,\mathrm{WZ}[\tilde{g}]$ for the B-field. We would like this sigma model to describe a two-dimensional quantum field theory, but the action appears to care about the details of the extended field $\tilde{g} : \widetilde{\Sigma} \to \mathsf{G}$. If we choose two different extensions, we would like the theory to be insensitive to this choice so that the true degrees of freedom are still captured by the original fundamental field $g : \Sigma \to \mathsf{G}$. We will now argue that this requires the coefficient k to be quantised $k \in \mathbb{Z}$. This coefficient is known as the level of the WZ term.

Let us denote two such extensions by $\tilde{g}$ and $\tilde{g}'$. The images of these maps are three-dimensional submanifolds of the Lie group G which we will denote by B_3 and B_3' respectively. In the action above, we wrote the WZ term as an integral over $\widetilde{\Sigma}$ after pulling back the 3-form via the map $\tilde{g} : \widetilde{\Sigma} \to \mathsf{G}$, but it is equally valid to write this term as an integral over B_3 viewed as a submanifold of the Lie group G. The difference of these integrals for the two different choices of extension may then be written as

$$\Delta = \int_{B_3} \mathrm{WZ}[g] - \int_{B_3'} \mathrm{WZ}[g] \ . \tag{1.19}$$

Since the two maps $\tilde{g}$ and $\tilde{g}'$ agree on the boundary of $\widetilde{\Sigma}$, we know that $\partial B_3 = \partial B_3'$. This means that we can glue these two submanifolds together (with opposite orientations) to form a submanifold of G which is isomorphic to $S^3 \cong B_3 \cup B_3'$. The difference Δ is then a topological quantity [Wit84] measuring the winding number of a map from S^3 to the Lie group G. These maps are classified by the third homotopy group $\pi_3(\mathsf{G}) \cong \mathbb{Z}$ which is isomorphic to the integers for all simple Lie groups.

In practice, if we properly normalise the trace, we can ensure that the change in the action produced by choosing two different extensions is quantised, $\Delta \in 2\pi\mathbb{Z}$. Whilst the action of the theory will not be invariant, our quantum description depends on the action through the path integral, and we can arrange for this to be invariant. Since the WZ term appears in the action against the coefficient k, the path integral will be invariant so long as this parameter is quantised,

$$\exp\big(\mathrm{i}S + \mathrm{i}k\Delta\big) = \exp\big(\mathrm{i}S\big) \qquad \Longleftrightarrow \qquad k \in \mathbb{Z} \ . \tag{1.20}$$

Despite appearances at first glance, this means that the PCM plus WZ term is still a two-dimensional quantum field theory.

What's more, it is also an integrable sigma model. For generic values of the parameters $r \in \mathbb{R}$ and $k \in \mathbb{Z}$, the Lax connection of the PCM plus WZ term is given by

$$L_{\pm} = \left(1 \pm \frac{k}{r^2}\right) \frac{g^{-1}\partial_{\pm} g}{1 \pm \zeta} \ . \tag{1.21}$$

One can check that the flatness of this connection for all values of the spectral parameter is equivalent to the equations of motion of the PCM plus WZ term, demonstrating the weak integrability of this theory. Something peculiar happens when the two parameters satisfy $r^2 = k$, as this causes one component of the Lax connection to vanish. This means that the spectral parameter dependence of the Lax can be removed by an overall rescaling, seemingly nullifying our previous justification of integrability (without spectral parameter dependence, the Laurent expansion of the monodromy matrix truncates).

On the other hand, when the parameters satisfy $r^2 = k$, one can show that the global $\mathsf{G} \times \mathsf{G}$ symmetry is enhanced to a pair of semi-local symmetries satisfying

$$g \mapsto h_\ell^{-1} \cdot g \cdot h_r \ , \qquad \partial_- h_\ell = 0 \ , \qquad \partial_+ h_r = 0 \ . \tag{1.22}$$

The associated conserved currents become holomorphic and anti-holomorphic respectively, and they can be Laurent expanded to provide an infinite tower of conserved charges. The theory at this point is known as the conformal Wess-Zumino-Witten (WZW) model. Furthermore, in the beta-functions for the metric and B-field, the contribution from the Ricci curvature cancels against the H-flux term, meaning that the model is scale-invariant for this special value of the parameters.

This is not a typical integrable sigma model, in the sense that its putative Lax connection degenerates and the infinite set of charges has an alternative origin, however it shares many other properties with integrable sigma models: an infinite dimensional symmetry algebra; the possibility of constructing exact solutions; extraordinary control over the quantum theory. In this case, these properties originate from the fact that the WZW model is a two-dimensional conformal field theory (CFT).

1.1.3 Gauged WZW models

Gauging a theory entails promoting a global symmetry to a local symmetry by coupling to a dynamical gauge field. Generically, this procedure will not preserve the integrability of a theory, but there is a notable exception to this statement. Gauged Wess-Zumino-Witten (gWZW) models are conformal field theories (CFTs) associated with the coset manifolds G/H. Whilst these are not the typical integrable sigma models which are amenable to the Lax formalism, they share many of the features of integrability by virtue of the infinite-dimensional conformal algebra. They also play an important role in the study of integrable deformations, where new integrable models are constructed as deformations of known ones. The gauging of a WZW model [Wit92] is bit more involved than the usual case, so we will devote this section to briefly reviewing this construction.

The WZW model is defined by the action

$$S_{\text{WZW}} = k \int_{\Sigma} \mathrm{d}^2x \ \mathrm{tr}\big(g^{-1}\partial_+ g \cdot g^{-1}\partial_- g\big) + k \int_{\widetilde{\Sigma}} \mathrm{WZ}[g] \ , \tag{1.23}$$

which is invariant under a global $\mathsf{G} \times \mathsf{G}$ symmetry acting as

$$g \mapsto h_\ell^{-1} \cdot g \cdot h_r \ , \qquad (h_\ell, h_r) \in \mathsf{G} \times \mathsf{G} \ . \tag{1.24}$$

In fact, these symmetries are enhanced to semi-local symmetries satisfying $\partial_- h_\ell = 0$ and $\partial_+ h_r = 0$ respectively, but we will focus on the global component for this discussion. We would like to gauge a subgroup of this global symmetry which we will denote by $\mathsf{H} \subset \mathsf{G} \times \mathsf{G}$. At a technical level, we will specify this subgroup with the following data. First, we will provide a Lie group H along with its associated Lie algebra $\mathfrak{h} = \mathrm{Lie}(\mathsf{H})$. In addition to this, we will provide two embedding maps,

$$\ell : \mathfrak{h} \to \mathfrak{g} \ , \qquad r : \mathfrak{h} \to \mathfrak{g} \ , \tag{1.25}$$

which we require to be Lie algebra homomorphisms. Denoting the associated Lie group homomorphisms with the same symbols, the action of H is then given by

$$g \mapsto \ell(h)^{-1} \cdot g \cdot r(h) \ , \qquad h \in \mathsf{H} \ . \tag{1.26}$$

Infinitesimally, this action looks like

$$\delta_\epsilon g = g \cdot r(\epsilon) - \ell(\epsilon) \cdot g \ , \qquad \epsilon \in \mathfrak{h} \ . \tag{1.27}$$

Let us now see how to gauge this symmetry in the action.

Gauging the quadratic kinetic term is quite typical: we couple the theory to a dynamical $\mathfrak{h}$-valued gauge field A via the covariant derivative operators

$$\nabla g = \mathrm{d}g + \ell(A) \cdot g - g \cdot r(A) \ . \tag{1.28}$$

Replacing the partial derivatives in the kinetic term by these operators, one can explicitly check that it is invariant under the transformation of g above, together with the transformation of the gauge field

$$\delta_\epsilon A = \mathrm{d}\epsilon + [A, \epsilon]_{\mathfrak{h}} \ . \tag{1.29}$$

If we neglect the WZ term, this provides a gauging of the kinetic term, which is also a gauging of the PCM since this is simply the WZW model with no WZ term.

Turning to the cubic WZ term, one might now considering taking the same approach and replacing the derivative operators with covariant derivatives. Whilst this would provide a gauge invariant 3-form, the problem is that it would depend on the extension of the gauge field A over the 3-manifold $\widetilde{\Sigma}$. If we want to preserve the interpretation of our theory as a two-dimensional model, we also need the gauged model to be insensitive to such a choice of extension. Formally, this requirement is equivalent to the WZ 3-form being closed, meaning it is annihilated by the exterior derivative ($\mathrm{d}\,\mathrm{WZ}[g]=0$). At least locally, this implies that it can be written as the total derivative of a 2-form and therefore sources two-dimensional dynamics. To preserve this property, we would like to find a 3-form, built from the group-valued field g and the gauge field A, which is both gauge invariant and closed. Questions of this type are addressed in the field of equivariant cohomology.

For brevity, and because it is not central to our perspective, we will simply import the result from equivariant cohomology (see the appendix of [Wit92] or the papers [FS94b; FM05] for more details). Firstly, it turns out that not every choice of subgroup H is compatible with gauging the WZ term. In order for a gauge invariant and closed 3-form to exist, we require the embedding maps to satisfy the isotropy condition

$$\mathrm{tr}\big(\ell(X)\cdot\ell(Y)\big)-\mathrm{tr}\big(r(X)\cdot r(Y)\big)=0 \qquad \forall\, X,Y\in\mathfrak{h}\ . \tag{1.30}$$

Provided that this condition is satisfied, the gauge invariant extension of the WZ 3-form is

$$\mathrm{WZ}^{\#}[g,A]=\mathrm{WZ}[g]+\mathrm{d}\,\mathrm{tr}\big(\mathrm{d}gg^{-1}\wedge\ell(A)+g^{-1}\mathrm{d}g\wedge r(A)+g^{-1}\ell(A)g\wedge r(A)\big)\ . \tag{1.31}$$

This 3-form is closed because the WZ 3-form is closed and the second term is a total derivative. We can now implement this gauging in the full action.

The gauged Wess-Zumino-Witten (gWZW) model action is given by

$$S_{\mathrm{gWZW}}=k\int_{\Sigma}\mathrm{d}^2x\ \mathrm{tr}\big(g^{-1}\nabla_+g\cdot g^{-1}\nabla_-g\big)+k\int_{\widetilde{\Sigma}}\mathrm{WZ}^{\#}[g,A]\ . \tag{1.32}$$

The quadratic term and the WZ term are independently gauge invariant which is manifest in this presentation. It is also helpful to write this action as the usual WZW model action plus some terms which depend on the gauge field. In this presentation, the action is written as

$$\begin{aligned} S_{\mathrm{gWZW}}=S_{\mathrm{WZW}}+2\,k\int_{\Sigma}\mathrm{d}^2x\ \mathrm{tr}\Big[&\partial_-gg^{-1}\cdot\ell(A_+)-g^{-1}\partial_+g\cdot r(A_-)\\ &-g^{-1}\ell(A_+)g\cdot r(A_-)+\frac{1}{2}\,\ell(A_+)\cdot\ell(A_-)+\frac{1}{2}\,r(A_+)\cdot r(A_-)\Big]\ . \end{aligned} \tag{1.33}$$

The last two terms can be combined using the isotropy condition on the embedding maps. Also, notice that the left embedding $\ell(A)$ only couples to g via the $\mathrm{d}x^+$ component, whilst the right embedding $r(A)$ only couples via the $\mathrm{d}x^-$ component.

It is important to highlight that the gauge field does not come with a kinetic term, meaning that it is an auxiliary field which can be integrated out. Doing this, one finds a sigma model action for the group-valued field $g \in \mathsf{G}$ with a modified metric and B-field. In particular, the metric and B-field are invariant under the local H-action, meaning that they can be interpreted as a metric and B-field on the coset manifold G/H. Indeed, one may also explicitly fix this gauge symmetry in the action by choosing a representative of each equivalence class in G/H. Either way, one derives a coset CFT with target space G/H.

1.1.4 Integrable deformations

We have reviewed some of the most well-known integrable field theories (IFTs) and conformal field theories (CFTs): the PCM, the PCM plus WZ term, the WZW model, and the gWZW model. Beyond these prototypical examples, there exists a wider class of sigma models with some amenability to exact methods. A certain subset of these, known as integrable deformations, are derived by taking a known theory and deforming it in some manner. We will now turn our attention to these models.

When the prototypical examples above have a group manifold G as their target space, they possess a $\mathsf{G} \times \mathsf{G}$ symmetry which leaves the associated target space metric and B-field invariant. The deformations we will consider generically break some of this symmetry, meaning that the resulting target space geometry will not respect the $\mathsf{G} \times \mathsf{G}$ group action. Nonetheless, we will require a much more stringent condition on these deformations — the deformed theory must be integrable. Surprisingly, there are non-trivial deformations, controlled by continuous parameters, which are integrable for a range of values of the deformation parameters. We will now review some key examples of integrable deformations (see [Tho19; Hoa22] for recent reviews).

Let us start with the action of the PCM, and deform it by introducing some endomorphism of the Lie algebra $\mathcal{O} : \mathfrak{g} \to \mathfrak{g}$. We can consider inserting this operator into the action to give the deformed model

$$S_{\mathcal{O}} = r^2 \int_\Sigma \mathrm{d}^2x \, \mathrm{tr}\big(g^{-1}\partial_+ g \cdot \mathcal{O} \cdot g^{-1}\partial_- g\big) \ . \tag{1.34}$$

If we assume that this operator is constant, meaning it is independent of the group element $g \in \mathsf{G}$, then this action is still invariant under the left G action,

$$g \mapsto h_\ell^{-1} \cdot g \ , \qquad h_\ell \in \mathsf{G} \ . \tag{1.35}$$

This is because we have chosen to insert the deforming operator between two left-invariant Maurer-Cartan forms, had we written the action with right-invariant Maurer-Cartan forms, the right action would always be preserved. By comparison, in the present case the right action is generically broken, only the subset which commutes with the deforming operator survives,

$$g \mapsto g \cdot h_r \ , \qquad \mathrm{Ad}_{h_r} \cdot \mathcal{O} \cdot \mathrm{Ad}_{h_r}^{-1} = \mathcal{O} \ . \tag{1.36}$$

In this expression, we are denoting the adjoint action by $\mathrm{Ad}_g X = gXg^{-1}$.

Asking for this deformed model to be integrable is a tall order, and generically hard to prove in either direction. However, one integrable deformation known in the literature is the Yang-Baxter deformation [Kli02; Kli09]. Let us assume that the Lie algebra $\mathfrak{g}$ admits a solution $\mathcal{R}$ to the modified classical Yang-Baxter equation (mcYBe),

$$[\mathcal{R}X, \mathcal{R}Y] - \mathcal{R}[X, \mathcal{R}Y] - \mathcal{R}[\mathcal{R}X, Y] + c^2[X, Y] = 0 \qquad \forall\, X, Y \in \mathfrak{g} \ . \tag{1.37}$$

Solutions to this equation fall into three categories based on the value of $c \in \{0, \mathrm{i}, 1\}$ and have been completely classified for simple Lie algebras [BD82; BD98]. When a solution to this equation exists, we can define the Yang-Baxter deformation of the PCM by the action

$$S_{\mathrm{YB}} = r^2 \int_\Sigma \mathrm{d}^2x \ \mathrm{tr}\big(g^{-1}\partial_+ g \cdot \frac{1}{1 - \eta\mathcal{R}} \cdot g^{-1}\partial_- g\big) \ . \tag{1.38}$$

Whilst the global left-symmetry is manifest in this presentation, we previously stated that the right-acting symmetry will be broken by the deforming operator. In fact, a closer analysis of this theory reveals that the right-acting symmetry is modified to a so-called q-deformed symmetry where $q = q(\eta)$ is a function of the deformation parameter [DMV13].

In the action, the fraction should be understood as a matrix inverse applied to the operator in the denominator. This only makes sense when the denominator is invertible, which means that $\eta\mathcal{R}$ cannot have any $+1$ eigenvalues. Assuming that the Yang-Baxter operator $\mathcal{R}$ is normalised to have unit norm eigenvalues, we can ensure that $(1 - \eta\mathcal{R})$ is invertible by restricting the deformation parameter to lie in the range $\eta \in [0, 1)$. In the limit $\eta \to 0$, we recover the undeformed PCM. The integrability of this theory is exhibited by the Lax connection

$$L_\pm = \frac{1 - c^2\eta^2}{1 \pm \eta\mathcal{R}} \left(\frac{g^{-1}\partial_\pm g}{1 \pm \zeta} \right) \ . \tag{1.39}$$

One can explicitly check that the flatness of this Lax for all values of $\zeta \in \mathbb{CP}^1$ is equivalent to the equations of motion of the Yang-Baxter deformation. The interest in these models grew substantially when it was understood that they could be used to construct integrable deformations of string theory backgrounds [DMV13; DMV14].

Another non-trivial integrable deformation can be found by starting with the gWZW model. If we restrict our attention to the diagonal G/G model, that is the WZW model on G gauged by the diagonal G action, the action for this theory is

$$\begin{aligned} S_{\mathrm{gWZW}} = S_{\mathrm{WZW}} + 2\,k \int_\Sigma \mathrm{d}^2x \ \mathrm{tr}\Big(&\partial_- g g^{-1} \cdot A_+ - g^{-1}\partial_+ g \cdot A_- \\ &- g^{-1} A_+ g \cdot A_- + A_+ \cdot A_- \Big) \ . \end{aligned} \tag{1.40}$$

In the language of the previous section, this corresponds to taking the embedding maps to be $\ell = \mathrm{id}$ and $r = \mathrm{id}$. By construction, this theory is invariant under a diagonal G gauge symmetry acting as

$$g \mapsto h^{-1} \cdot g \cdot h \ , \qquad A \mapsto h^{-1} A h + h^{-1} \mathrm{d} h \ , \qquad h \in \mathsf{G} \ . \tag{1.41}$$

The λ-deformation [Sfe14], named after the deformation parameter, corresponds to changing the coefficient of the final term in the action,

$$\begin{aligned} S_\lambda = S_{\mathrm{WZW}} + 2\,k \int_\Sigma \mathrm{d}^2 x \ \mathrm{tr}\Big(& \partial_- g g^{-1} \cdot A_+ - g^{-1} \partial_+ g \cdot A_- \\ & - g^{-1} A_+ g \cdot A_- + \lambda^{-1} A_+ \cdot A_- \Big) \ . \end{aligned} \tag{1.42}$$

There are a number of ways of thinking about this deformation. In the original derivation [Sfe14], this modification of the action was found by coupling the theory to a gauged PCM model with the same gauge field. Fixing the PCM field to the trivial map means that the PCM action only contributes a quadratic term in the gauge field, culminating in the deformation given above. Alternatively, we can think of this term as modifying the mass of the gauge field.

In either case, this deformation of the action breaks the gauge symmetry, though it preserves the global diagonal G action. This is because the deformed term $A_+ A_-$ is independently invariant under the global diagonal action, meaning the rest of the action is also invariant. We can now integrate out the non-dynamical gauge field A to find an action for a sigma model on G,

$$S_\lambda = S_{\mathrm{WZW}} + 2\,k \int_\Sigma \mathrm{d}^2 x \ \mathrm{tr}\Big(g^{-1} \partial_+ g \cdot \frac{\lambda}{1 - \lambda\, \mathrm{Ad}_g} \cdot \partial_- g g^{-1} \Big) \ . \tag{1.43}$$

As before the fraction should be understood as a matrix inverse which requires the denominator to be invertible. In this case, because the adjoint action is orthogonal ($Ad_g^T = Ad_g^{-1}$), its eigenvalues will be ± 1. We will therefore restrict the deformation parameter to the range $\lambda \in [0, 1)$, where the limit $\lambda \to 0$ corresponds to the WZW model. In fact, for small values of the deformation parameter the action above is a current-current deformation of the WZW model (see [Bor24] for a recent review). The full λ-model can then be understood as an all-orders completion of this infinitesimal deformation.

It can also be helpful to write this action in the alternative form

$$S_\lambda = k \int_\Sigma \mathrm{d}^2 x \ \mathrm{tr}\Big(g^{-1} \partial_+ g \cdot \frac{1 + \lambda\, \mathrm{Ad}_g}{1 - \lambda\, \mathrm{Ad}_g} \cdot g^{-1} \partial_- g \Big) + k \int_{\widetilde{\Sigma}} \mathrm{WZ}[g] \ . \tag{1.44}$$

This model has a global diagonal G symmetry acting as $g \mapsto h^{-1} g h$ which is inherited from the derivation above. It is also integrable for all values of the deformation parameter, and this is demonstrated by the Lax connection

$$L_+ = \Big(\frac{1}{1 + \lambda} \Big) \frac{1}{\lambda^{-1} - \mathrm{Ad}_g^{-1}} \Big(\frac{-\partial_+ g g^{-1}}{1 + \zeta} \Big) \ , \qquad L_- = \Big(\frac{1}{1 + \lambda} \Big) \frac{1}{\lambda^{-1} - \mathrm{Ad}_g} \Big(\frac{g^{-1} \partial_- g}{1 - \zeta} \Big) \ . \tag{1.45}$$

As with the Yang-Baxter deformation, one reason for the interest in this model is that it can be used to construct string theory backgrounds [HMS14b; HMS14a].

1.2 Three-dimensional Chern-Simons theory

Some people[1] would argue that Chern-Simons theory is the first and most important example of a topological quantum field theory. It is a three-dimensional gauge theory whose action is defined to be the integral of the Chern-Simons 3-form, first discovered in 1974 by Chern and Simons during their work on characteristic classes in differential geometry. A foundational paper [Wit89b] on this topic was written[2] by Witten in 1988, in which Atiyah's question "What is the physical interpretation of the Jones polynomial?" was answered. The Jones polynomial is a topological invariant which appears in the context of knot theory, and Witten showed that it was related to certain observables in three-dimensional Chern-Simons theory. In 1990, Witten was awarded the Fields medal, one of the highest honours in mathematics, in large part due to his work on Chern-Simons theory.

1.2.1 Action and gauge invariance

Three-dimensional Chern-Simons (3dCS) theory is defined by the action

$$S_{3\text{dCS}}[A] = k\int_{M_3} \text{CS}[A] \ , \qquad \text{CS}[A] = \text{tr}\Big(A\wedge \text{d}A + \frac{2}{3}A\wedge A\wedge A\Big) \ . \tag{1.46}$$

It is surprising that this theory exists. In order for a gauge theory to make sense, the action must be gauge-invariant. This is because the kinetic term vanishes on gauge-trivial connections, and the propagator of the quantum theory is built by inverting the quadratic piece of the action. If the quadratic piece vanishes on some field configurations, it must have a kernel meaning it is non-invertible. Supposing that the action is gauge-invariant, we can circumvent this problem by interpreting it as a functional over the equivalence classes of field configurations modulo gauge transformations, rather than over all field configurations. On this quotient space, the quadratic part of the action is invertible and we can define the quantum propagator.

The connection A transforms under gauge transformations as

$$A \mapsto A^g \equiv g^{-1}Ag + g^{-1}\text{d}g \ , \tag{1.47}$$

whilst the field strength $F[A] \equiv \text{d}A + A\wedge A$ transforms as $F \mapsto g^{-1}Fg$. It is therefore much harder to build invariant functionals of A when compared to F. Any polynomial of F will be

[1]This is a quote from Greg Moore's extensive lecture notes, currently available at this link (see also the accompanying video lectures on YouTube).

[2]In the interest of some local history, it is inspiring to note that this seminal result was conceived in Swansea. At a conference in July 1988, talking to Atiyah and Segal, Witten realised that the correct theory to describe the Jones polynomial was three-dimensional Chern-Simons theory. The story goes that this conversation took place in Annie's restaurant on St. Helen's road, which has since closed down. A plaque to commemorate this dinner apparently hung in the restaurant for many years and is now in the possession of the mathematics department.

gauge-invariant within a trace, leading to two well-studied examples,

$$\mathrm{tr}\big(F \wedge \star F\big) \ , \qquad \mathrm{tr}\big(F \wedge F\big) \ . \tag{1.48}$$

The first of these is the Yang-Mills Lagrangian, which depends on the metric through the Hodge star operator $\star$, whilst the second is known as the topological theta term. Unlike these functionals of the field strength, a generic functional of A will certainly not be gauge invariant. The fact that Chern-Simons theory makes sense is therefore rather surprising.

In fact, the Chern-Simons functional is closely related to the second quadratic polynomial in F given above. The topological theta term is a gauge-invariant 4-form which also happens to be closed under the exterior derivative operator. It turns out that the Chern-Simons 3-form provides a local potential for the theta term,

$$\mathrm{d}\,\mathrm{CS}[A] = \mathrm{tr}\big(F \wedge F\big) \ . \tag{1.49}$$

It is important to highlight that this expression only holds locally, integrating the right hand side over a closed 4-manifold gives a topological invariant which measures the failure of this expression to extend globally.

This origin of the Chern-Simons 3-form explains why it is gauge-invariant, at least up to terms which are closed under the exterior derivative. To be more explicit, the gauge transformation of the Chern-Simons 3-form is given by

$$\mathrm{CS}[A] \mapsto \mathrm{CS}[A^g] = \mathrm{CS}[A] + \mathrm{d}\,\mathrm{tr}\big(A \wedge \mathrm{d}g g^{-1}\big) + \frac{1}{3}\mathrm{tr}\big(g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g\big) \ . \tag{1.50}$$

Let us first consider the case where the 3-manifold M_3 has no boundary, such that the second term will not contribute to the action. The third term, however, will generically modify the action, but it will do so in a controlled way, allowing us make sense of the Chern-Simons functional. One can show that the contribution of the third term is quantised,

$$\frac{1}{3}\int_{M_3} \mathrm{tr}\big(g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g\big) \in 2\pi\mathbb{Z} \ , \tag{1.51}$$

and it measures a winding number of the map $g : M_3 \to \mathsf{G}$. This means that, whilst the action will transform non-trivially, its contribution to the path integral will be gauge-invariant if the level k is also quantised,

$$\exp\big(\mathrm{i}\, S_{3\mathrm{dCS}}[A^g]\big) = \exp\big(\mathrm{i}\, S_{3\mathrm{dCS}}[A]\big) \qquad \Longleftrightarrow \qquad k \in \mathbb{Z} \ . \tag{1.52}$$

Having assured ourselves that the theory makes sense, let us study some of its basic features. The equations of motion of 3dCS theory are given by

$$F[A] \equiv \mathrm{d}A + A \wedge A = 0 \ . \tag{1.53}$$

This theory describes flat gauge fields, which may locally be written as $A = g^{-1}\mathrm{d}g$. The existence of a group-valued field $g : M_3 \to \mathsf{G}$ solving this equation is not guaranteed globally, but flatness ensures that it at least exists locally. A connection of this form is gauge-trivial, and can be fixed to $A = 0$ by a gauge transformation. This means that 3dCS theory has no local dynamics, all solutions to the equations of motion vanish up to a gauge transformation.

One might then suppose that this is quite a boring theory, and not worthy of very much attention. On the contrary, whilst the usual difficulties of quantum field theory are absent due to the lack of local dynamics, these are exchanged with subtle questions in topology which are both intriguing and tractable. For example, the natural observables in this theory are not local operators corresponding to point-like excitations, but rather Wilson lines defined over a 1-dimensional curve $\gamma \subset M_3$,

$$W(\gamma) = \mathrm{Pexp} \int_\gamma A \ . \tag{1.54}$$

For flat gauge fields these line operators are topological, meaning that they are invariant under smooth deformations of the curve γ. In particular, this means that the Wilson line of a contractible curve is trivial, but the gauge field may have a non-trivial holonomy around non-trivial classes in $\pi_1(M_3)$. Witten's seminal paper [Wit89b] on 3dCS theory related correlation functions of these Wilson lines to a topological knot invariant known as the Jones polynomial.

1.2.2 Boundaries and edge modes

What about when the 3-manifold M_3 has a non-empty boundary? Denoting the boundary by $\partial M_3 = \Sigma$, the gauge transformation of the action will include a boundary term,

$$\int_{M_3} \mathrm{d}\,\mathrm{tr}\big(A \wedge \mathrm{d}g g^{-1}\big) = \int_\Sigma \mathrm{tr}\big(A \wedge \mathrm{d}g g^{-1}\big) \ . \tag{1.55}$$

Gauge transformations which act trivially at the boundary (meaning $g|_\Sigma = \mathrm{id}$) will still leave the action invariant, and we will refer to these as bulk gauge transformations. There may also be some residual symmetries on the boundary, but to detect these we must first revisit the action.

Varying the 3dCS action produces a boundary term of the form

$$\theta = \int_\Sigma \mathrm{tr}\big(\delta A \wedge A\big) \ . \tag{1.56}$$

As a general statement, boundary conditions should be imposed on quantum field theories such that asking the action to be stationary is a well-posed problem. In practice, this means that the boundary variation of the action should vanish, which corresponds to a choice of maximal isotropic subspace in the language of symplectic geometry. We will take the time to elaborate on this perspective here since it highlights some important aspects of Chern-Simons theory.

Let us start by briefly reviewing the story for free scalar fields. Consider the action of a free scalar field in two dimensions,

$$S[\phi] = \frac{1}{2}\int_{M_2} \mathrm{d}^2x \left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right] . \tag{1.57}$$

We will consider the case where the 2-manifold is $M_2 = \mathbb{R}\times\mathbb{R}_{\geq 0}$ which has a spatial boundary at $x = 0$. The variation of this action gives the bulk equations of motion, and a boundary component which looks like

$$\int_{\mathbb{R}} \mathrm{d}t \left(\delta\phi\,\partial_x\phi\right) . \tag{1.58}$$

This expression involves an infinitesimal variation of the field configuration denoted by $\delta\phi$, and the normal derivative of the field to the boundary which is $\partial_x\phi$.

In this case, we have two distinguished choices of boundary conditions: Dirichlet conditions, where the value of the field configuration is fixed at the boundary; and Neumann conditions, where the normal derivative of the configuration is fixed at the boundary.

$$\begin{aligned} \text{Dirichlet :} \qquad \phi|_{x=0} = \phi_0 \quad &\Longrightarrow \quad \delta\phi|_{x=0} = 0 \ , \\ \text{Neumann :} \qquad \partial_x\phi|_{x=0} = 0 \quad &\Longrightarrow \quad \partial_x(\delta\phi)|_{x=0} = 0 \ . \end{aligned} \tag{1.59}$$

It is important to highlight[3] that, once boundary conditions have been imposed on the field configurations, we should also restrict to variations that preserve these boundary conditions. For example, in the first line we note that it is equivalent to specify a fixed boundary configuration for the field ϕ_0, or to demand that its variation at the boundary vanishes. In the second line, we emphasise that demanding that the normal derivative of the field vanishes at the boundary also implies that the normal derivative of the variation must vanish. In principle, Neumann boundary conditions can also specify a fixed non-zero value for the normal derivative at the boundary.

There are also other types of boundary conditions that one could impose on free scalar fields. For example, the Robin boundary conditions are given in terms of a linear combination of the value of the field and its normal derivative. The essential requirement is that the boundary variation of the action must vanish, which we will shortly rephrase in the language of symplectic geometry. One final comment is that the distinction between Dirichlet and Neumann boundary conditions appears in the context of second order differential equations, where the normal derivative of the field appears in the boundary variation. We will argue that this distinction becomes less meaningful in the context of first order differential equations.

The boundary variation of the action provides a symplectic potential for the symplectic form on phase space. In the context of the Hamiltonian approach, we should consider a spatial slice at

[3] One potential source of confusion here is that $\partial_x\phi$ and $\delta\phi$ both measure a change in the field configuration ϕ, but imposing $\phi|_{x=0} = 0$ does not imply $\partial_x\phi|_{x=0} = 0$, even though it implies $\delta\phi|_{x=0} = 0$. This is because $\partial_x\phi$ measures the change of a given field configuration as we move along the line parameterised by $x \in \mathbb{R}_{\geq 0}$, whereas $\delta\phi$ is an infinitesimal change from one field configuration to the next ($\phi \mapsto \phi + \delta\phi$). In order for the new configuration to obey the boundary conditions whenever the old configuration does, we must impose $\delta\phi|_{x=0} = 0$.

fixed time, in which case the boundary variation would read

$$\theta = \int_{\mathbb{R}} \mathrm{d}x \left(\delta\phi\, p\right) \ , \qquad p = \partial_t \phi \ . \tag{1.60}$$

We have written this expression in terms of the canonical phase space variables, namely the field configuration ϕ and its conjugate momentum $p = \partial_t \phi$. This boundary variation can be interpreted as a symplectic potential on phase space, associated with the symplectic form

$$\Omega = \delta\theta = \int_{\mathbb{R}} \mathrm{d}x \left(\delta p \wedge \delta\phi\right) \ . \tag{1.61}$$

Here, we are treating ϕ and p as phase space coordinates such that $\delta\phi$ and δp provide a basis for 1-forms on phase space. The symplectic form Ω is then a 2-form on phase space which is both closed and non-degenerate.

In this context, the boundary conditions are the initial conditions for our field configuration from which we will evolve our system. We can either fix the value of the field ϕ with Dirichlet conditions, or the normal derivative (which is the momentum p) with Neumann conditions. Each of these choices corresponds to a maximal isotropic subspace of phase space with respect to the symplectic form Ω. These subspaces, also known as Lagrangian subspaces, are half-dimensional subspaces on which the restriction of the symplectic form Ω must vanish. Fixing the field configuration corresponds to imposing $\delta\phi = 0$, whilst fixing the normal derivative corresponds to $\delta p = 0$. Both of these conditions ensure that the symplectic form vanishes on these subspaces, and they each define half-dimensional subspaces of phase space. This concludes our review of the symplectic perspective on free scalar fields.

Returning to Chern-Simons theory, there are some similarities and some differences. The action for Chern-Simons theory is first order in time derivatives, which is quite a significant difference. For example, let us specialise to the case $M_3 = \Sigma \times \mathbb{R}_{\geq 0}$ where we denote the radial direction by $r \in \mathbb{R}_{\geq 0}$. We will equip the boundary 2-manifold Σ with the flat Lorentzian metric $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2$ and occasionally work in light-cone coordinates $\partial_\pm = \partial_t \pm \partial_x$. Writing the action in these coordinates, we notice that we can perform a couple of integration by parts to put the action in the form [Eli+89]

$$S_{\text{3dCS}}[A] = k \int_{M_3} \mathrm{d}^3x \ \mathrm{tr}\big(2\, A_t F_{xr} + A_r \partial_t A_x - A_x \partial_t A_r\big) + k \int_{\Sigma} \mathrm{d}^2x \ \mathrm{tr}\big(A_x A_t\big) \ . \tag{1.62}$$

Focusing on the bulk terms, we see that A_t appears as a Lagrange multiplier enforcing the constraint $F_{xr} = 0$. The other two components of the gauge field are dynamical, but they are also the conjugate momenta to one another,

$$P_x = A_r \ , \qquad P_r = -A_x \ . \tag{1.63}$$

This means that, on a fixed time slice, A_x and A_r provide a complete set of coordinates on phase space. This is in contrast to the second order action of a free scalar field, where the field configuration and its conjugate momenta are independent degrees of freedom.

Having made this observation, we should interpret the boundary variation of Chern-Simons theory as a symplectic potential on phase space,

$$\theta = \int_\Sigma \mathrm{tr}\big(\delta A \wedge A\big) \ . \tag{1.64}$$

In particular, all boundary conditions in this theory will involve fixing the field configuration on the boundary, which will also impose constraints on its variation. This is analogous to the Dirichlet boundary conditions of the free scalar field, but there is no analog of the Neumann conditions as the derivative of the field does not appear as a coordinate on phase space. This is fundamentally a difference between first order and second order systems.

One choice of boundary conditions which cause this boundary variation to vanish are given by

$$A_+|_\Sigma = 0 \qquad \Longrightarrow \qquad \delta A_+|_\Sigma = 0 \ . \tag{1.65}$$

These define a half-dimensional subspace of phase space which is isotropic with respect to the symplectic form $\Omega = \delta\theta$. Let us consider the gauge invariance of the action with these boundary conditions. Applying a gauge transformation to the action, the boundary term vanishes if we impose the constraint

$$\partial_+ g|_\Sigma = 0 \ . \tag{1.66}$$

This can also be derived by considering transformations which preserve the boundary condition on the gauge field. If a given configuration satisfies $A_+|_\Sigma = 0$, then the transformed field $A^g = g^{-1}Ag + g^{-1}\mathrm{d}g$ will also satisfy this condition if the transformation parameter obeys the boundary conditions $\partial_+ g|_\Sigma = 0$.

An important conceptual point is the physical status of these symmetries. Whilst we have presented them as the gauge symmetries which preserve the boundary conditions, their status as gauge symmetries is not necessarily preserved after imposing these constraints. One should compute the associated Noether charge and check whether or not it vanishes on a generic field configuration. If the charge vanishes, then these residual boundary symmetries are gauge symmetries, but if the charge is non-vanishing then they parameterise an emergent physical symmetry of the system.

The explicit introduction of a metric in these boundary conditions seems like it might break the topological nature of 3dCS theory, so let us revisit the local degrees of freedom in this theory. The equations of motion are still flatness, meaning that any solution may locally be written as $A = g^{-1}\mathrm{d}g$. Earlier, we argued that a connection of this type may be gauge fixed to $A = 0$ meaning that there are no local dynamics in the theory. However, there is now an additional subtlety, the gauge transformation required to bring a given configuration to $A = 0$ may not satisfy the boundary conditions we have just imposed. In this case, some degrees of freedom in A

will survive the gauge fixing and source local dynamics on the boundary. We will refer to these physical degrees of freedom as edge modes.

We can see how this works at the level of the action [Eli+89]. If we write the 3dCS action in terms of the coordinates $\{t, x, r\}$ on M_3, we have already see that it is given by

$$S_{3\mathrm{dCS}}[A] = k\int_{M_3} \mathrm{d}^3x\ \mathrm{tr}\big(2\,A_t F_{xr} + A_r\partial_t A_x - A_x\partial_t A_r\big) + k\int_{\Sigma} \mathrm{d}^2x\ \mathrm{tr}\big(A_x A_t\big)\ . \tag{1.67}$$

Making use of the boundary condition (which may be written as $A_t|_\Sigma = -A_x|_\Sigma$) we can eliminate A_t in the boundary term in favour of A_x. Then, the component A_t only appears in the action as a Lagrange multiplier enforcing $F_{xr} = 0$, and we will integrate out this component of the gauge field, solving the constraint with $A_x = g^{-1}\partial_x g$ and $A_r = g^{-1}\partial_r g$. Substituting this back into the action, we can write it in terms of the Maurer-Cartan form $j = g^{-1}\mathrm{d}g$ as

$$S[g] = k\int_{M_3} \mathrm{d}^3x\ \mathrm{tr}\big(j_r\partial_t j_x - j_x\partial_t j_r\big) - k\int_{\Sigma} \mathrm{d}^2x\ \mathrm{tr}\big(j_x\, j_x\big)\ . \tag{1.68}$$

The Maurer-Cartan equation is $\mathrm{d}j + j\wedge j = 0$, and applying this identity together with an integration by parts gives the action

$$S_{\mathrm{cWZW}}[g] = -k\int_{\Sigma} \mathrm{d}^2x\ \mathrm{tr}\big(j_x\cdot j_+\big) - \frac{k}{3}\int_{M_3} \mathrm{tr}\big(j\wedge j\wedge j\big)\ . \tag{1.69}$$

This is the "chiral" WZW action which has an unusual kinetic term, first order in time derivatives. This theory only has half of the usual semi-local symmetries, and these can be directly identified with the residual symmetries preserving the boundary conditions of 3dCS theory. In essence, this theory encodes half of the usual degrees of freedom in a WZW model.

Chiral theories of this form also appear in the context of duality-invariant approaches to string theory, and they are notoriously hard to study as the action explicitly breaks Lorentz invariance. There are various attempts in the literature to construct Lorentz-invariant actions for chiral fields in exchange for introducing additional auxiliary fields into the theory [PST97; Mkr19]. On the other hand, the approach advocated for in [Wit97; BM06] is to define the chiral boundary theory through its Chern-Simons description. In chapter 6, we see how these two approaches are related by following a generalisation of the derivation presented above.

The full WZW model can also be recovered from 3dCS theory [Eli+89]. Consider the 3-manifold $M_3 = \Sigma\times[0,1]$ and impose the boundary conditions

$$A_+|_{\Sigma\times\{0\}} = 0\ ,\qquad A_-|_{\Sigma\times\{1\}} = 0\ . \tag{1.70}$$

In effect, this leads to two chiral WZW models, one on each boundary. The residual symmetries preserving these boundary conditions provide the chiral and anti-chiral symmetries of the WZW model. The fact that these two sectors do not talk to one another is manifest in the 3dCS description as they live on disconnected boundaries.

1.3 Four-dimensional Chern-Simons theory

The integrability of sigma models is characterised by the existence of a flat Lax connection, and Chern-Simons theory is a gauge theory which describes flat connections. One might therefore hope to construct a Chern-Simons theory which describes integrable sigma models. The missing ingredient in 3dCS theory is the complex spectral parameter $\zeta \in \mathbb{C}$ which plays a crucial role in the Lax formalism. This prompts us to search for a four-dimensional Chern-Simons (4dCS) theory defined over $M_4 = C \times \Sigma$ where C is some complex curve. In this thesis, the complex curve will almost always be identified with $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, though other canonical examples are provided by $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $T^2 = S^1 \times S^1$. These correspond to rational, trigonometric, and elliptic integrable models respectively. This 4dCS theory was recently introduced in a series of papers [CWY18a; CWY18b; CY19] and the relationship to integrable deformations was developed in [Del+20]. We will primarily focus on the third paper in the original series [CY19] as well as the subsequent paper [Del+20] which address two-dimensional field theories rather than lattice models (see [Lac22] for a recent review).

Our first problem comes in defining an action for 4dCS theory. The Lagrangian must be a 4-form to be integrated over M_4, but the Chern-Simons functional is only a 3-form. We must therefore introduce a 1-form ω to define the action

$$S_{4\mathrm{dCS}}[A] = \frac{1}{2\pi \mathrm{i}} \int_{M_4} \omega \wedge \mathrm{CS}[A] \ , \qquad \mathrm{CS}[A] \equiv \mathrm{tr}\Big(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \Big) \ . \tag{1.71}$$

We will assume[4] that ω is (the pullback of) a meromorphic $(1,0)$-form on C. Examples of such 1-forms on $\mathbb{CP}^1$ are given by

$$\mathrm{d}\zeta \ , \qquad \frac{\mathrm{d}\zeta}{\zeta} \ , \qquad \frac{1-\zeta^2}{\zeta^2}\,\mathrm{d}\zeta \ . \tag{1.72}$$

The first example $\omega = \mathrm{d}\zeta$ is nowhere vanishing and has a second order pole at $\zeta = \infty$. This can be seen by performing the coordinate transformation $\zeta = \tilde{\zeta}^{-1}$ which moves to the other patch covering $\mathbb{CP}^1$. The second example $\omega = \mathrm{d}\zeta/\zeta$ is also nowhere vanishing and has two simple poles, one at $\zeta = 0$ and the other at $\zeta = \infty$. The third example has two simple zeroes at $\zeta = \pm 1$ and two double poles at $\zeta = 0$ and $\zeta = \infty$. In all of these examples, the number of poles minus the number of zeros is two (counted with multiplicity), which is guaranteed by the Riemann-Roch theorem. The essential data in the 1-form ω is the location of any poles and zeroes, each of which play a distinguished role in 4dCS theory.

The significance of poles in ω can be understood by varying the 4dCS action,

$$\delta S_{4\mathrm{dCS}}[A] = \frac{2}{2\pi \mathrm{i}} \int_{M_4} \omega \wedge \mathrm{tr}\big(\delta A \wedge F\big) + \frac{1}{2\pi \mathrm{i}} \int_{M_4} \mathrm{d}\omega \wedge \mathrm{tr}\big(\delta A \wedge A\big) \ . \tag{1.73}$$

The first term provides the bulk equations of motion $\omega \wedge F = 0$ which we will return to later. For

[4] The necessary and sufficient constraints on ω are not currently known. One proposal could be the following. There must exist a complex curve C and a projection $q : M_4 \to C$ such that $\omega = q^*(\alpha)$ for some meromorphic $(1,0)$-form $\alpha \in \Omega^{1,0}(C)$. In particular, the identification of M_4 with the product manifold $C \times \Sigma$ might be relaxed.

the moment, let us focus on the second term which involves the 2-form $\mathrm{d}\omega$. Consider the example $\omega = \mathrm{d}\zeta/\zeta$. Since this only depends on ζ and the $\mathrm{d}\zeta$ leg is already saturated, one might think that $\mathrm{d}\omega = 0$. This is true almost everywhere, but not at the poles of ω where we must use the identity from complex analysis

$$\partial_{\bar{\zeta}}\left(\frac{1}{\zeta}\right) = -2\pi\mathrm{i}\,\delta(\zeta)\ , \qquad \int_{\mathbb{CP}^1} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}\,\delta(\zeta)\,f(\zeta) = f(0)\ . \tag{1.74}$$

For the example $\omega = \mathrm{d}\zeta/\zeta$, this means that $\mathrm{d}\omega$ will have two contributions, one localised at $\zeta = 0$ and the other at $\zeta = \infty$. Higher order poles will lead to ∂_ζ derivatives of delta-functions in the evaluation of $\mathrm{d}\omega$.

In this example, the second term in the variation is explicitly given by[5]

$$\frac{1}{2\pi\mathrm{i}}\int_{M_4} \mathrm{d}\left(\frac{\mathrm{d}\zeta}{\zeta}\right) \wedge \mathrm{tr}\big(\delta A \wedge A\big) = \int_\Sigma \Big[\ \mathrm{tr}\big(\delta A \wedge A\big)|_{\zeta=0} - \mathrm{tr}\big(\delta A \wedge A\big)|_{\zeta=\infty}\Big]\ . \tag{1.75}$$

This plays the role of a boundary term in our theory, and we must impose boundary conditions on the gauge field A at $\zeta = 0$ and $\zeta = \infty$ such that it vanishes. In general, the poles of ω will source similar contributions to the variation and we will impose boundary conditions at these points.

1.3.1 Wess-Zumino-Witten model

Sticking with the example $\omega = \mathrm{d}\zeta/\zeta$, one valid choice of boundary conditions are provided by $A_+|_0 = 0$ and $A_-|_\infty = 0$. Generic gauge transformations will not leave the action invariant as it will transform by a boundary term similar to that in the variation. For this choice of boundary condition, only those transformations obeying $\partial_+ g|_0 = 0$ and $\partial_- g|_\infty = 0$ will leave the action invariant, and these can be understood as the transformations which preserve the boundary conditions. It is important to highlight that the transformations which act non-trivially at the poles of ω are not necessarily gauge transformations. One should compute the associated Noether charge to check whether they are gauge or physical symmetries of the theory.

As in 3dCS theory, boundary conditions on the gauge field A break some gauge symmetry and give rise to physical degrees of freedom living at the boundary which are known as edge modes. Let us derive an action for this two-dimensional theory in the example with $\omega = \mathrm{d}\zeta/\zeta$. First, we will implement a field redefinition to separate the bulk gauge field from the edge modes,

$$A = L^{\hat{g}} \equiv \hat{g}^{-1}L\hat{g} + \hat{g}^{-1}\mathrm{d}\hat{g}\ . \tag{1.76}$$

Whilst this field redefinition has the same formal expression as a gauge transformation, it is important to emphasise that we are treating L and $\hat{g}$ as the new dynamical field content of our theory. In particular, $\hat{g}$ is not required to obey the boundary conditions imposed on gauge

[5]More precisely, the boundary term for $\omega = \mathrm{d}\zeta/\zeta$ includes an integral over $\{0\}\times\Sigma$ and $\{\infty\}\times\Sigma$. We are formally identifying these two disjoint subspaces of M_4 to write the boundary term as a single integral over Σ.

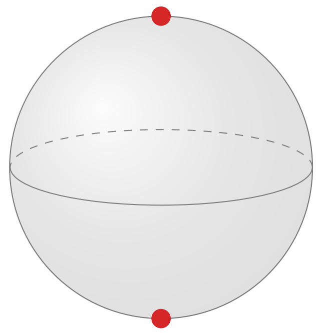


Figure 1.1: The meromorphic 1-form $\omega = \mathrm{d}\zeta/\zeta$ associated with the WZW model has two simple poles (red dots), one at $\zeta = 0$ and another at $\zeta = \infty$.

transformations. As hinted by the choice of symbol, we will see that the gauge field L generically[6] becomes a Lax connection for the boundary theory.

The 4dCS action is written in these new variables as

$$S_{4\mathrm{dCS}}[L, \hat{g}] = \frac{1}{2\pi \mathrm{i}} \int_{M_4} \Big[\, \omega \wedge \mathrm{CS}[L] + \mathrm{d}\omega \wedge \mathrm{tr}\big(L \wedge \mathrm{d}\hat{g}\hat{g}^{-1}\big) + \omega \wedge \mathrm{CS}[\hat{g}^{-1}\mathrm{d}\hat{g}] \, \Big] \, . \tag{1.77}$$

The first term in this action is a bulk Chern-Simons term for the gauge field L, whilst the second term will localise to the poles of ω due to delta-functions in the distribution $\mathrm{d}\omega$. Explicitly computing the Chern-Simons 3-form for $\hat{g}^{-1}\mathrm{d}\hat{g}$ appearing in the third term, we find

$$\mathrm{CS}[\hat{g}^{-1}\mathrm{d}\hat{g}] = -\frac{1}{3}\mathrm{tr}\big(\hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g}\big) \equiv -\mathrm{WZ}[\hat{g}] \, . \tag{1.78}$$

This is the Wess-Zumino (WZ) 3-form which also appears in the context of two-dimensional Wess-Zumino-Witten (WZW) models. In that context, this term only sources two-dimensional boundary dynamics, despite being a 3-form defined in terms of an extension of the fundamental field. Following this analogy, something similar happens for the third term in our 4dCS action. Let us introduce the 5-manifold $M_5 = M_4 \times [0, 1]$ and an associated extension $\hat{g} : M_5 \to \mathsf{G}$ which we denote with the same symbol in an abuse of notation. We require this extension to be a smooth homotopy from the trivial map at $M_4 \times \{0\}$ to the edge mode $\hat{g}$ at $M_4 \times \{1\}$ such that the third term in the action may be written as

$$\int_{M_4} \omega \wedge \mathrm{WZ}[\hat{g}] = \int_{M_5} \mathrm{d}\Big(\omega \wedge \mathrm{WZ}[\hat{g}]\Big) \, . \tag{1.79}$$

The WZ 3-form is closed ($\mathrm{d}\,\mathrm{WZ}[\hat{g}] = 0$) so the exterior derivative in this expression can only act on the 1-form ω. This produces the distribution $\mathrm{d}\omega$ which localises the integral to the poles of ω.

[6] In the present example, the boundary theory will turn out to be the WZW model whose Lax famously degenerates: one of the components of the Lax vanishes and any spectral parameter dependence may be removed. The gauge field L will also exhibit these properties, but will become a familiar Lax in other more typical examples.

In summary, the 4dCS action may be written as

$$S_{\text{4dCS}}[L,\hat{g}] = \frac{1}{2\pi\mathrm{i}}\int_{M_4} \omega\wedge \mathrm{CS}[L] + \frac{1}{2\pi\mathrm{i}}\int_{M_4} \mathrm{d}\omega \wedge \left[\mathrm{tr}\big(L\wedge \mathrm{d}\hat{g}\hat{g}^{-1}\big) - \int_{[0,1]} \mathrm{WZ}[\hat{g}] \right] . \tag{1.80}$$

From this expression, we can see that the action only depends on $\hat{g}$ through its values at the poles of ω, justifying the title of "edge mode".

Let us take stock of the physical degrees of freedom and symmetries of this theory. While the action depends on the whole field configuration L, it only depends on the boundary values of $\hat{g}$. We will denote the relevant degrees of freedom in $\hat{g}$ by

$$\hat{g}|_{\zeta=0} = g \ , \qquad \hat{g}|_{\zeta=\infty} = \tilde{g} \ . \tag{1.81}$$

There are then two types of symmetries in the theory: transformations which acted on the original field variable A and descend to the new field content — we refer to these as external symmetries; and redundancies in the new parameterisation which we refer to as internal symmetries. Whilst all of the internal symmetries are gauge symmetries, one must be more careful about determining the status of the external symmetries.

The external symmetries acting as $A\mapsto A^{\hat{h}}$ act on the new variables as $L^{\hat{g}}\mapsto (L^{\hat{g}})^{\hat{h}} = L^{\hat{g}\hat{h}}$ so we can interpret these as right-acting symmetries $\hat{g}\mapsto \hat{g}\cdot\hat{h}$ on the edge mode. Denoting the boundary values of the external symmetry as $\hat{h}|_0 = h$ and $\hat{h}|_\infty = \tilde{h}$, the action on the boundary values of the edge mode is

$$g\mapsto g\cdot h \ , \qquad \tilde{g}\mapsto \tilde{g}\cdot\tilde{h} \ , \qquad \partial_+ h = 0 \ , \qquad \partial_-\tilde{h} = 0 \ , \tag{1.82}$$

where the differential constraints ensure that they leave the action invariant. In principle, there is also a bulk component to the external symmetries but this is indistinguishable from the bulk internal symmetries precisely because the value of the edge mode in the bulk plays no role in the theory. For this reason, we focus our attention on the action of the external symmetries at the poles of ω.

To determine whether these symmetries are gauge symmetries (representing a redundancy which should be fixed) or physical symmetries (acting non-trivially on the space of solutions) one should compute the associated Noether charges. This is an important aspect of the analysis of boundary symmetries, but also a detour from the main path we are following. For that reason, we will simply state the result here, and provide more details in a later example. Due to the differential constraints on the external gauge parameters, these are semi-local physical symmetries with non-vanishing Noether charges. This means that we should not attempt to impose gauge fixing constraints with these symmetries, and instead they will descend to physical symmetries of the boundary theory.

In addition to these symmetries, we also have the internal symmetries coming from redundancies in the parameterisation $A = L^{\hat{g}}$. The original gauge field A is invariant under the simultaneous transformation

$$L \mapsto L^{\check{h}} \ , \qquad \hat{g} \mapsto \check{h}^{-1} \cdot \hat{g} \ . \tag{1.83}$$

These redundancies represent gauge symmetries which we should fix by imposing constraints on the new field content. Let us start with the gauge field L. Notice that the $\mathrm{d}\zeta$ component of L drops out of the action immediately since ω saturates this leg. If we are to interpret this gauge field as a Lax for the boundary theory, we expect it to only have legs along the worldsheet Σ. Fortunately, there is sufficient freedom in the internal symmetries to fix the $\mathrm{d}\bar{\zeta}$ component of L to zero,

$$L_{\bar{\zeta}} = 0 \ . \tag{1.84}$$

Whenever one imposes a gauge fixing condition, one should check whether there are any residual gauge symmetries preserving this constraint. In this case, we are still free to perform internal gauge transformations which are independent of the spectral parameter. These are sufficient to fix the value of the edge mode $\hat{g}$ at one point in $\mathbb{CP}^1$, so we will also impose the gauge fixing condition

$$\hat{g}|_{\zeta=\infty} = \mathrm{id} \ . \tag{1.85}$$

How does this constraint interact with the external symmetry acting at $\zeta = \infty$? The external symmetry parameterised by $\tilde{h}$ must now be accompanied by a simultaneous internal gauge symmetry $\check{h} = \tilde{h}$ such that the condition $\hat{g}|_\infty = \mathrm{id}$ is preserved. The surviving edge mode degree of freedom now transforms under both external symmetries as

$$g \mapsto \tilde{h}^{-1} \cdot g \cdot h \ , \qquad \partial_+ h = 0 \ , \qquad \partial_- \tilde{h} = 0 \ . \tag{1.86}$$

It is no coincidence that these mimic the semi-local symmetries of the conformal WZW model.

In order to localise the 4dCS action to the boundary, we solve some of the bulk equations of motion for L. Having imposed the gauge fixing $L_{\bar{\zeta}} = 0$, two of the equations of motion become holomorphicity conditions on the remaining components,

$$F_{\bar{\zeta}\pm} = \partial_{\bar{\zeta}} L_\pm = 0 \ . \tag{1.87}$$

We will solve these equations explicitly, and use the boundary conditions to write $L_\pm$ in terms of the surviving edge mode $g : \Sigma \to \mathsf{G}$. Substituting this back into the 4dCS action, we can compute the integral over $\mathbb{CP}^1$ to find an action for the two-dimensional boundary theory.

The two equations of motion above say that the components $L_\pm$ must be holomorphic functions of ζ. Since $\mathbb{CP}^1$ is a compact manifold, Liouville's theorem[7] states that any holomorphic function of ζ is constant, so the components $L_\pm$ must be independent of the spectral parameter. Turning to the boundary conditions, we can translate the conditions on A into conditions on L and $\hat{g}$ using the field redefinition. Making use of the gauge fixing conditions above and the fact that $L_\pm$

[7]Liouville's theorem applies to bounded holomorphic functions on compact manifolds. In other examples, we will relax the condition that these components are bounded and allow for poles in the spectral plane.

are constant along $\mathbb{CP}^1$, we find the boundary conditions are solved by

$$L_+ = -\partial_+ g g^{-1} \ , \qquad L_- = 0 \ . \tag{1.88}$$

This allows us to eliminate the gauge field L from the action and find a theory which only depends on the group-valued field g.

Returning to the 4dCS action, the Chern-Simons 3-form for L vanishes (nothing can contribute the $\mathrm{d}\bar{\zeta}$ leg) and the boundary terms localise to the poles of ω. Only the $\zeta = 0$ contributions to the boundary terms will survive as we have gauge fixed $\hat{g}|_\infty = \mathrm{id}$. These contributions are given by

$$S_{4\mathrm{dCS}}[L, \hat{g}] = \int_\Sigma \left[\mathrm{tr}\big(L \wedge \mathrm{d}\hat{g}\hat{g}^{-1}\big) - \int_{[0,1]} \mathrm{WZ}[\hat{g}] \right]\Bigg|_{\zeta=0} \ . \tag{1.89}$$

The final step is to substitute in the solution for L and evaluate each term at $\zeta = 0$ explicitly. This gives the action of the WZW model,

$$S_{\mathrm{WZW}}[g] = -\int_\Sigma \mathrm{d}^2 x \ \mathrm{tr}\big(g^{-1}\partial_+ g \cdot g^{-1}\partial_- g\big) - \int_{\widetilde{\Sigma}} \mathrm{WZ}[g] \ . \tag{1.90}$$

We denote the extension of the 2-manifold Σ by $\widetilde{\Sigma} = \Sigma \times [0,1]$. This concludes our first example of deriving a two-dimensional field theory from 4dCS theory. The residual boundary symmetries acting on the edge mode descend to the semi-local symmetries of the WZW model. The level k can also be introduced as an overall coefficient in the action, but it is not clear that it should be quantised from the 4dCS perspective.

Compactification of 4dCS to 3dCS. We have noted that the two-dimensional WZW model can also be recovered from 3dCS theory, and this is closely related to the 4dCS description we have just described [Yam19]. Let us return to the 4dCS action with $\omega = \mathrm{d}\zeta/\zeta$ which we can write in polar coordinates $\zeta = r\exp(\mathrm{i}\,\theta)$ as

$$S_{4\mathrm{dCS}}[A] = \frac{1}{2\pi\mathrm{i}} \int_{M_4} \left(\frac{\mathrm{d}r}{r} + \mathrm{i}\,\mathrm{d}\theta \right) \wedge \mathrm{CS}[A] \ . \tag{1.91}$$

We would like to perform a dimensional reduction along the $\mathsf{U}(1)$ action generated by ∂_θ. We will only keep the zero modes in the Fourier expansion (meaning the fields are independent of θ) and compute the integral over $\mathrm{d}\theta$. In preparation, we can make use of a trivial shift symmetry in 4dCS theory acting as

$$A \mapsto A + C_\zeta \, \mathrm{d}\zeta \ . \tag{1.92}$$

This leaves the action invariant because ω saturates the $\mathrm{d}\zeta$ leg, and we usually fix it by setting $A_\zeta = 0$. In this context, it more helpful to impose the constraint $A_\theta = 0$ so that (combined with

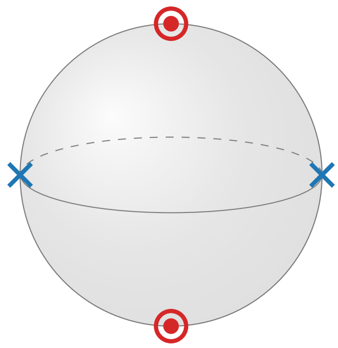


Figure 1.2: The meromorphic 1-form associated with the PCM plus WZ term has double poles (red dots) at $\zeta = 0$ and $\zeta = \infty$. It also has simple zeroes (blue crosses) at $\zeta = a$ and $\zeta = b$. This figure shows the PCM point where $a = +1$ and $b = -1$.

the constraint that the fields are independent of θ) the action becomes

$$S_{4\mathrm{dCS}}[A] = \frac{1}{2\pi} \int_{M_4} \mathrm{d}\theta \wedge \mathrm{CS}[A] \ . \tag{1.93}$$

Since the whole integrand is independent of θ, we can explicitly compute this integral to find

$$S_{3\mathrm{dCS}}[A] = \int_{\Sigma \times [0,1]} \mathrm{CS}[A] \ . \tag{1.94}$$

The endpoints of the interval are identified with $\zeta = 0$ and $\zeta = \infty$, so the 3dCS gauge field inherits the 4dCS boundary conditions $A_+|_0 = 0$ and $A_-|_1 = 0$. This is the 3dCS setup which is known to recover the two-dimensional WZW model [Eli+89].

1.3.2 PCM plus WZ term

Whilst the setup with $\omega = \mathrm{d}\zeta/\zeta$ provided a nice introduction to 4dCS theory, the WZW model is hardly a typical integrable model. For one, the Lax connection is independent of the spectral parameter, and there are very few 1-forms ω which are nowhere vanishing. With this in mind, let us turn to a different example where the meromorphic 1-form is given by

$$\omega = \frac{(\zeta - a)(\zeta - b)}{\zeta^2} \, \mathrm{d}\zeta \ . \tag{1.95}$$

This 1-form has two second order poles at $\zeta = 0$ and $\zeta = \infty$ which will play the role of boundaries in 4dCS theory. Since these poles are second order, the boundary variation will depend on both the value of the gauge field at these points and also its ∂_ζ derivative. Similarly, the relevant degrees of freedom in the edge modes will include both its values and its ∂_ζ derivatives.

There are also simple zeroes in ω at $\zeta = a$ and $\zeta = b$ where $a, b \in \mathbb{C}$ are input parameters. These zeroes are a new ingredient which we must treat carefully. We will allow the gauge field A

to have specified singularities at the zeros of ω which can be justified by the following argument. Whilst we will primarily focus on classical physics in this thesis, we might still aspire for 4dCS theory to make sense as a quantum theory. Defining the quantum theory would entail constructing a propagator for the gauge field by inverting the quadratic term in the action. As it stands, the quadratic term vanishes at the zeroes of ω, meaning it would be non-invertible at these points. To compensate for this, we will prescribe singular behaviour in our field configurations at the zeroes of ω. In the present example, we will allow a simple pole in A_+ at $\zeta = a$ and a simple pole in A_- at $\zeta = b$. Since each of these components appears only once in the cubic term, this part of the action also remains finite. The prescription of singularities in the gauge field is known as disorder defects in the literature [CY19]. These singularities in the spacetime components of the gauge field will lead to the typical meromorphic dependence of the Lax connection.

As before, the poles of ω play the role of boundaries in 4dCS theory and we should impose boundary conditions such that the boundary variation of the action vanishes. The boundary term in the variation receives two contributions, one from $\zeta = 0$ and another from $\zeta = \infty$. Since the analysis at these two poles is identical, let us restrict our attention to the contribution at $\zeta = 0$ which is given by

$$\int_\Sigma \Big[\, (ab)\, \partial_\zeta \mathrm{tr}\big(\delta A \wedge A\big)|_{\zeta=0} - (a+b)\, \mathrm{tr}\big(\delta A \wedge A\big)|_{\zeta=0} \Big] \ . \tag{1.96}$$

The first term involves a ∂_ζ derivative[8] because the pole at $\zeta = 0$ is second order. The second term indicates the presence of a first order pole hiding inside the second order pole of ω, which can be seen explicitly by writing the meromorphic 1-form as

$$\omega = \left(\frac{ab}{\zeta^2} - \frac{a+b}{\zeta} + 1 \right) \mathrm{d}\zeta \ . \tag{1.97}$$

It is possible to fine tune the parameters $a, b \in \mathbb{C}$ such that the residue of this simple pole vanishes by setting $b = -a$. The coefficient of the first order pole is called the residue of ω at this point, so this constraint on the parameters would amount to setting the residue to zero. Importantly, if the second term is absent from the boundary variation there is a wider class of admissible boundary conditions. Later, we will comment on the consequences of this observation.

For the time being, we will impose the boundary conditions (valid for any $a, b \in \mathbb{C}$) given by

$$A_\pm|_{\zeta=0} = 0 \ , \qquad A_\pm|_{\zeta=\infty} = 0 \ . \tag{1.98}$$

Gauge transformations acting on A are also required to obey similar boundary conditions at the poles, namely $\partial_\pm g|_0 = 0$ and $\partial_\pm g|_\infty = 0$. Since these residual symmetries must be constant along Σ, they will generate to a global $\mathsf{G} \times \mathsf{G}$ symmetry in the boundary theory.

As a general comment about 4dCS theory, the choice of 1-form ω, the prescribed singular

[8] Terms such as $\partial_{\bar{\zeta}}(1/\zeta^2)$ can be computed by first using $\partial_\zeta(1/\zeta) = -1/\zeta^2$ and then commuting the derivatives and applying the usual identity for simple poles. This can be applied iteratively to higher order poles.

behaviour at its zeroes, and the boundary conditions at its poles, should all be thought of as input data. From here, the derivation of the integrable sigma model and its associated Lax connection is a systematic procedure which takes more or less the same form regardless of the input data. Varying the input data then leads to different two-dimensional theories, providing a mechanism for exploring the landscape of integrable models.

We will now applying the localisation procedure to this new example. The first step is to make the field redefinition

$$A = L^{\hat{g}} \equiv \hat{g}^{-1} L \hat{g} + \hat{g}^{-1} \mathrm{d}\hat{g} \ . \tag{1.99}$$

The new field content is a gauge field L, which will be identified with the Lax connection, and an edge mode $\hat{g}$. The action may be written in terms of these variables as

$$S_{4\mathrm{dCS}}[L, \hat{g}] = \frac{1}{2\pi \mathrm{i}} \int_{M_4} \omega \wedge \mathrm{CS}[L] + \frac{1}{2\pi \mathrm{i}} \int_{M_4} \mathrm{d}\omega \wedge \left[\mathrm{tr}\big(L \wedge \mathrm{d}\hat{g}\hat{g}^{-1}\big) - \int_{[0,1]} \mathrm{WZ}[\hat{g}] \right] . \tag{1.100}$$

The edge mode $\hat{g}$ only appears in the action against the 2-form $\mathrm{d}\omega$ which is a distribution with support at the poles of ω. This means that the theory only depends on the edge mode through its boundary values which we will denote by[9]

$$\hat{g}|_{\zeta=0} = g \ , \qquad \hat{g}^{-1}\partial_\zeta \hat{g}|_{\zeta=0} = \phi \ , \qquad \hat{g}|_{\zeta=\infty} = \tilde{g} \ , \qquad \hat{g}^{-1}\partial_\zeta \hat{g}|_{\zeta=\infty} = \tilde{\phi} \ . \tag{1.101}$$

In contrast to the previous example, the ∂_ζ derivatives of the edge mode are also relevant degrees of freedom in this case because ω has second order poles.

At this stage, it is often helpful to examine the symmetries of the theory and impose some convenient choice of gauge fixing to reduce the degrees of freedom. The external symmetries act on the edge mode as $\hat{g} \mapsto \hat{g} \cdot \hat{h}$ and must obey the boundary conditions $\partial_\pm \hat{h}|_0$ and $\partial_\pm \hat{h}|_\infty$. Whilst the value of $\hat{h}$ at each pole must be constant along spacetime, the ∂_ζ derivative of $\hat{h}$ is unconstrained by these boundary conditions. This is relevant because the transformation of the edge mode fields ϕ and $\tilde{\phi}$ will depend on $\partial_\zeta \hat{h}$. Let us denote the boundary values of this transformation parameter by

$$\hat{h}|_{\zeta=0} = h \ , \qquad \hat{h}^{-1}\partial_\zeta \hat{h}|_{\zeta=0} = \epsilon \ , \qquad \hat{h}|_{\zeta=\infty} = \tilde{h} \ , \qquad \hat{h}^{-1}\partial_\zeta \hat{h}|_{\zeta=\infty} = \tilde{\epsilon} \ . \tag{1.102}$$

Then, the edge mode fields transform under the external symmetries as

$$g \mapsto g \cdot h \ , \qquad \phi \mapsto \mathrm{Ad}_h^{-1} \phi + \epsilon \ , \qquad \tilde{g} \mapsto \tilde{g} \cdot \tilde{h} \ , \qquad \tilde{\phi} \mapsto \mathrm{Ad}_{\tilde{h}}^{-1} \tilde{\phi} + \tilde{\epsilon} \ . \tag{1.103}$$

In these expressions, h and $\tilde{h}$ must be constant group elements whilst ϵ and $\tilde{\epsilon}$ are unconstrained by the boundary conditions. Next, we should check whether the symmetries parameterised by these variables are gauge or physical symmetries.

[9] The definition of $\tilde{\phi}$ given above is slightly inaccurate. When moving from between the patches on $\mathbb{CP}^1$, the vector field ∂_ζ transforms non-trivially, so it is more accurate to defined the edge mode by $\tilde{\phi} = \hat{g}^{-1}\partial_{\tilde{\zeta}}\hat{g}|_{\tilde{\zeta}=0}$.

To check if these symmetries are gauge or physical, we can compute their associated Noether charges. For this calculation, it is easiest to work with the original field variable A and consider and infinitesimal transformation of the action. We will retain the same notation for the external gauge parameters, but we will neglect all higher-order terms in the variation of the action. This means that the relevant term in the gauge transformation of the action is given by

$$\frac{1}{2\pi \mathrm{i}} \int_{M_4} \mathrm{d}\omega \wedge \mathrm{tr}\big(A \wedge \mathrm{d}\hat{h}\hat{h}^{-1}\big) \ . \tag{1.104}$$

At this stage, it is helpful to split the external gauge transformations into their component parts. One class of transformations are those which act trivially at the boundary and non-trivially in the bulk. Since the variation in this expression is localised to the poles of ω, these bulk transformations will have a vanishing Noether charge and remain gauge symmetries of the theory. Having understood this class of transformations, we can now restrict our attention to those transformations that only act non-trivially on the boundary, neglecting their action on the bulk.

When considering the boundary symmetries, it is also helpful to partition the transformations using the parameters introduced above. We can consider one class where h is non-trivial whilst the other three parameters take trivial values. Doing this for each component of the external symmetry, we have four transformations whose Noether charges we would like to compute. Explicitly evaluating the integral over $\mathbb{CP}^1$, the $\zeta = 0$ variation of the action becomes

$$(ab) \int_{\Sigma} \mathrm{tr}\big(\partial_\zeta A|_{\zeta=0} \wedge \mathrm{d}hh^{-1}\big) \ . \tag{1.105}$$

Many terms in this expression have dropped due to the boundary conditions on the gauge field A. In particular, we see that ϵ does not appear in this variation, implying that it is a gauge symmetry. By comparison, we also learn that h parameterises a global symmetry of the theory whose Noether current is given by $\partial_\zeta A|_{\zeta=0}$. A similar analysis at $\zeta = \infty$ tells us that $\tilde{\epsilon}$ parameterises a gauge symmetry whilst $\tilde{h}$ parameterises a physical symmetry.

In essence, ϵ and $\tilde{\epsilon}$ represent gauge symmetries of the system because they are fully local transformation parameters with no differential constraints. This allows us to impose the gauge fixing conditions

$$\phi = 0 \ , \qquad \tilde{\phi} = 0 \ . \tag{1.106}$$

By comparison, h and $\tilde{h}$ parameterise physical symmetries of the system because they are subject to differential constraints.

Independent of the choice of ω or boundary conditions, there is always an internal gauge symmetry arising from a redundancy in field redefinition. This leaves $A = L^{\hat{g}}$ invariant and acts on the new field content as

$$L \mapsto L^{\check{h}} \ , \qquad \hat{g} \mapsto \check{h}^{-1} \cdot \hat{g} \ . \tag{1.107}$$

We will always use this to impose the constraint $L_{\bar{\zeta}} = 0$ so that L takes the usual form of a Lax connection with legs only along $\mathrm{d}x^{\pm}$. In addition, there is sufficient residual gauge symmetry after imposing this constraint to fix the value of the edge mode at one pole. In this example, we will gauge fix

$$L_{\bar{\zeta}} = 0 \ , \qquad \hat{g}|_{\zeta=\infty} = \mathrm{id} \ . \tag{1.108}$$

In summary, the degrees of freedom in the edge mode after gauge fixing are

$$\hat{g}|_{\zeta=0} = g \ , \qquad \hat{g}^{-1}\partial_{\zeta}\hat{g}|_{\zeta=0} = 0 \ , \qquad \hat{g}|_{\zeta=\infty} = \mathrm{id} \ , \qquad \hat{g}^{-1}\partial_{\zeta}\hat{g}|_{\zeta=\infty} = 0 \ . \tag{1.109}$$

Having made a field redefinition to extract the edge modes and implemented some helpful gauge fixings, the next step is to solve some of the bulk equations of motion for L. Given that we have gauge fixed $L_{\bar{\zeta}} = 0$, two of the equations of motion become holomorphicity conditions on the remaining components $L_{\pm}$,

$$F_{\bar{\zeta}\pm} = \partial_{\bar{\zeta}} L_{\pm} = 0 \ . \tag{1.110}$$

If the components $L_{\pm}$ were bounded, this would imply there were constant by Liouville's theorem. However, we should recall our earlier discussion on the allowed field configurations in 4dCS theory. Now that ω has zeroes, we should allow certain prescribed singular behaviour in the gauge field at these points. In the present example, we allowed a simple pole in L_{+} at $\zeta = a$ and a simple pole in L_{-} at $\zeta = b$. This means that the most general solution[10] to these two equations of motion is given by

$$L_{+} = V_{+} + \frac{U_{+}}{1-\zeta/a} \ , \qquad L_{-} = V_{-} + \frac{U_{-}}{1-\zeta/b} \ , \tag{1.111}$$

where $V_{\pm}$ and $U_{\pm}$ are independent of the spectral parameter. We can now substitute these solutions into the boundary conditions and solve them in terms of the surviving edge mode g. The boundary condition at infinity $A_{\pm}|_{\infty} = 0$ implies that $V_{\pm} = 0$ after taking into account our choice of gauge fixing. Meanwhile, the boundary condition at zero implies

$$L_{+} = \frac{-\,\partial_{+} g g^{-1}}{1-\zeta/a} \ , \qquad L_{-} = \frac{-\,\partial_{-} g g^{-1}}{1-\zeta/b} \ . \tag{1.112}$$

This should provide a Lax connection for our two-dimensional boundary theory.

The final step in the derivation of the two-dimensional integrable model is to substitute this solution back into the action and compute the integral over $\mathbb{CP}^1$. The Chern-Simons term for L will always vanish as nothing can contribute the $\mathrm{d}\bar{\zeta}$ leg. The remaining terms appear against $\mathrm{d}\omega$ which will localise the integral to the poles of ω in $\mathbb{CP}^1$. Since we have fixed all components of the

[10] One might worry that the meromorphic solutions for $L_{\pm}$ do not actually satisfy the bulk equations of motion as the $\partial_{\bar{\zeta}}$ derivative produces delta-functions at the poles. The resolution is that the true equations of motion are $\omega \wedge F = 0$ and the zeroes in ω compensate for these delta-function contributions.

edge mode at infinity to vanish, this will only receive a contribution from $\zeta = 0$ which is given by

$$
\begin{aligned}
S_{\text{4dCS}}[L,\hat{g}] = \int_{\Sigma} \Bigg[&(ab)\,\partial_\zeta \text{tr}\big(L \wedge \text{d}\hat{g}\hat{g}^{-1}\big) - (a+b)\,\text{tr}\big(L \wedge \text{d}\hat{g}\hat{g}^{-1}\big) \\
&- \int_{[0,1]} \Big((ab)\,\partial_\zeta \text{WZ}[\hat{g}] - (a+b)\,\text{WZ}[\hat{g}] \Big) \Bigg] \Bigg|_{\zeta=0} .
\end{aligned}
\tag{1.113}
$$

Substituting in the expressions for $L_\pm$ above and evaluating at $\zeta = 0$ gives the action of our two-dimensional integrable sigma model. Since we have fixed the ∂_ζ derivatives of the edge mode to vanish, these derivatives may only act on the explicit spectral parameter dependence in the Lax. The second and third terms vanish identically, whilst the first term produces a kinetic term and the fourth gives a WZ term. In total, we find the action

$$
S_{\text{2dIFT}}[g] = (b-a) \int_{\Sigma} \text{d}^2x \ \text{tr}\big(g^{-1}\partial_+ g \cdot g^{-1}\partial_- g\big) + (a+b) \int_{\widetilde{\Sigma}} \text{WZ}[g] \ . \tag{1.114}
$$

We recognise this theory as the family of models known as the PCM plus WZ term. In particular the parameters $a, b \in \mathbb{C}$ are related to the radius of the PCM term and the level of the WZ term by $r^2 = a - b$ and $k = a + b$. Earlier, we highlighted a special point in the parameters where the residue of ω at $\zeta = 0$ vanished. This was given by $b = -a$ which we can now see corresponds to taking the WZ level to zero. Making this choice of parameters recovers the standard PCM.

Another special point occurs when $|\,a-b\,| = |\,a+b\,|$ which corresponds to the conformal WZW model. The 4dCS setup with $\omega = \text{d}\zeta/\zeta$ is recovered by simultaneously sending one zero to $\zeta = 0$ and the other to $\zeta = \infty$ such that they coincide with the second order poles. These limits tend to be subtle and require care, but in this case we can offer an interpretation. The simple zeroes of ω collide with the second order poles resulting is first order poles at $\zeta = 0$ and $\zeta = \infty$. To derive the boundary conditions on A at these poles, it is helpful to think of the conditions $A_\pm|_0 = 0$ and $A_\pm|_\infty = 0$ as simple zeros in the gauge field at these points. On the other hand, these components are allowed to have simple poles at $\zeta = a$ and $\zeta = b$. When we take the limit, these simple poles will cancel against the zeroes from the boundary conditions leaving a regular field configuration. For example, if we allow A_- to have a simple pole at $\zeta = b$ and then take $b \to 0$, we would say that the simple pole in A_- cancels against the simple zero from the boundary condition. The remaining simple pole in ω therefore comes with the boundary condition $A_+|_0 = 0$.

1.3.3 Non-Abelian T-dual of PCM

In order to recover the PCM plus WZ term from 4dCS theory, one imposes the boundary conditions $A_\pm|_0 = 0$ and $A_\pm|_\infty = 0$. If one were to impose different boundary conditions on the gauge field, this would generically lead to different two-dimensional integrable models. Whilst the PCM plus WZ term boundary conditions are valid for the whole family of 1-forms (parameterised by $a, b \in \mathbb{C}$) there is a wider class of allowed boundary conditions if we fix $b = -a$. This choice gives the meromorphic 1-form

$$\omega = \frac{a^2 - \zeta^2}{\zeta^2}\, \mathrm{d}\zeta \ . \tag{1.115}$$

The relevance of this constraint is that we have removed the simple poles hiding inside the double poles of ω as their residue was given by $(a+b)$. In the two-dimensional integrable model, this has the effect of turning off the WZ term leaving the standard PCM.

With this choice of 1-form, the boundary variation at $\zeta = 0$ is given by

$$\int_\Sigma \Big[\, a^2\, \partial_\zeta \mathrm{tr}\big(\delta A \wedge A\big)|_{\zeta=0} + (a+b)\,\mathrm{tr}\big(\delta A \wedge A\big)|_{\zeta=0} \Big] \ . \tag{1.116}$$

The absence of the second term in this expression allows for an alternative choice of boundary conditions, namely $\partial_\zeta A_\pm|_0 = 0$. Let us consider the 4dCS setup with this 1-form and the boundary conditions

$$\partial_\zeta A_\pm|_{\zeta=0} = 0 \ , \qquad A_\pm|_{\zeta=\infty} = 0 \ . \tag{1.117}$$

Relative to the PCM setup, we are choosing the same boundary conditions at $\zeta = \infty$, but swapping the boundary conditions at $\zeta = 0$. Gauge transformations will leave the action invariant if they obey similar boundary conditions given by $\partial_\zeta \hat{h}|_0 = 0$ and $\partial_\pm \hat{h}|_\infty = 0$. We will follow through the localisation to the boundary theory and see the impact that changing the boundary conditions has on the integrable sigma model.

Many of the details in this derivation remain the same from case to case, so we will be a little more brief in our presentation this time around. Nonetheless, we will still try to highlight the novel features of this setup. We start with the field redefinition

$$A = L^{\hat{g}} \equiv \hat{g}^{-1} L \hat{g} + \hat{g}^{-1}\mathrm{d}\hat{g} \ . \tag{1.118}$$

The edge mode $\hat{g}$ always enters the action against $\mathrm{d}\omega$ which is a distribution with support at the poles of ω. Therefore, the theory only cares about the boundary values of the edge mode which we denote by

$$\hat{g}|_{\zeta=0} = g \ , \qquad \hat{g}^{-1}\partial_\zeta \hat{g}|_{\zeta=0} = \phi \ , \qquad \hat{g}|_{\zeta=\infty} = \tilde{g} \ , \qquad \hat{g}^{-1}\partial_\zeta \hat{g}|_{\zeta=\infty} = \tilde{\phi} \ . \tag{1.119}$$

Under the external symmetries, the edge mode $\hat{g}$ transform as $\hat{g} \mapsto \hat{g} \cdot \hat{h}$, where the transformation parameter $\hat{h}$ must obey the boundary conditions $\partial_\zeta \hat{h}|_0 = 0$ and $\partial_\pm \hat{h}|_\infty = 0$. We will denote the relevant components of these external symmetries by

$$\hat{h}|_{\zeta=0} = h \ , \qquad \hat{h}^{-1}\partial_\zeta \hat{h}|_{\zeta=0} = \epsilon \ , \qquad \hat{h}|_{\zeta=\infty} = \tilde{h} \ , \qquad \hat{h}^{-1}\partial_\zeta \hat{h}|_{\zeta=\infty} = \tilde{\epsilon} \ . \tag{1.120}$$

The first boundary condition on $\hat{h}$ tells us that $\epsilon = 0$, whilst the second boundary condition tells us that $\tilde{h}$ is constant along spacetime. The action on the edge mode fields is given by

$$g \mapsto g \cdot h \ , \qquad \phi \mapsto \mathrm{Ad}_h^{-1}\phi \ , \qquad \tilde{g} \mapsto \tilde{g} \cdot \tilde{h} \ , \qquad \tilde{\phi} \mapsto \mathrm{Ad}_{\tilde{h}}^{-1}\tilde{\phi} + \tilde{\epsilon} \ . \tag{1.121}$$

As there are no constraints on the spacetime dependence of either h or $\tilde{\epsilon}$, these parameterise gauge symmetries which we can use to impose the gauge fixing constraints $g = \mathrm{id}$ and $\tilde{\phi} = 0$. There is still a global G symmetry preserving these constraints which is parameterised by $\tilde{h}$.

In addition to the external symmetries, we also have the internal gauge symmetries coming from a redundancy in our field redefinition. These act as

$$L \mapsto L^{\check{h}} \ , \qquad \hat{g} \mapsto \check{h}^{-1} \cdot \hat{g} \ , \tag{1.122}$$

and can be use to fix both the $\mathrm{d}\bar{\zeta}$ leg of L and the value of the edge mode at one pole. We will impose the gauge fixing constraints $L_{\bar{\zeta}} = 0$ and $\tilde{g} = \mathrm{id}$. In summary, the degrees of freedom in the edge mode are now

$$\hat{g}|_{\zeta=0} = \mathrm{id} \ , \qquad \hat{g}^{-1}\partial_\zeta \hat{g}|_{\zeta=0} = \phi \ , \qquad \hat{g}|_{\zeta=\infty} = \mathrm{id} \ , \qquad \hat{g}^{-1}\partial_\zeta \hat{g}|_{\zeta=\infty} = 0 \ . \tag{1.123}$$

The algebra-valued field ϕ will be the fundamental field of the two-dimensional integrable model. This transforms in the adjoint representation of the global G symmetry.

The next step is to solve two of the bulk equations of motion. Taking into account the pole structure of the gauge field configurations, we solve the holomorphicity conditions $\partial_{\bar{\zeta}} L_\pm = 0$ with the general solutions

$$L_+ = V_+ + \frac{U_+}{1 - \zeta/a} \ , \qquad L_- = V_- + \frac{U_-}{1 - \zeta/b} \ , \tag{1.124}$$

The fields $V_\pm$ and $U_\pm$ in this expression are independent of the spectral parameter and we can solve for them in terms of ϕ using the boundary conditions. The boundary conditions at $\zeta = \infty$ tell us that $V_\pm = 0$, whilst the boundary conditions at $\zeta = 0$ imply

$$\big(a^{-1} - \mathrm{ad}_\phi\big)\, U_+ = -\partial_+\phi \ , \qquad \big(a^{-1} + \mathrm{ad}_\phi\big)\, U_- = \partial_-\phi \ . \tag{1.125}$$

At this stage, one might worry that the operators $(a^{-1} \pm \mathrm{ad}_\phi)$ may not be invertible. In general, this is a good question to ask, but in this case we are safe because ad_ϕ is anti-symmetric $(\mathrm{ad}_\phi^T = -\mathrm{ad}_\phi)$. The sum of an anti-symmetric matrix with the identity matrix is always invertible, so there is no problem in solving these conditions to find the Lax connection

$$L_+ = -\frac{(a^{-1} - \mathrm{ad}_\phi)^{-1}\partial_+\phi}{1 - \zeta/a} \ , \qquad L_- = \frac{(a^{-1} + \mathrm{ad}_\phi)^{-1}\partial_-\phi}{1 + \zeta/a} \ . \tag{1.126}$$

The final step is to substitute this expression back into the 4dCS action and compute the integral over $\mathbb{CP}^1$. Since we have gauge fixed all of the edge modes at $\zeta = \infty$, the only contribution will come from $\zeta = 0$. This contribution is given by

$$S_{4\text{dCS}}[L, \hat{g}] = a^2 \int_\Sigma \left[\partial_\zeta \text{tr}\big(L \wedge \text{d}\hat{g}\hat{g}^{-1}\big)|_{\zeta=0} - \int_{[0,1]} \partial_\zeta \text{WZ}[\hat{g}]\big|_{\zeta=0} \right] . \tag{1.127}$$

When distributing the ∂_ζ derivative, any terms including $\text{d}\hat{g}$ will vanish as we have gauge fixed $\hat{g}|_0 = \text{id}$. The only terms that will survive occur when the ∂_ζ derivative acts on $\text{d}\hat{g}$ to produce $\text{d}\phi$. The WZ term is cubic in $\text{d}\hat{g}$, so it always vanishes after evaluation at $\zeta = 0$. The two-dimensional action is therefore given by just the first term, which evaluates to

$$S_{2\text{dIFT}}[\phi] = a^2 \int_\Sigma \text{d}^2 x \ \text{tr}\bigg(\partial_+ \phi \cdot \frac{1}{a^{-1} + \text{ad}_\phi} \cdot \partial_- \phi\bigg) . \tag{1.128}$$

This theory may be recognised as the non-Abelian T-dual (NATD) of the PCM and was first recovered from 4dCS theory in [LV21]. The residual boundary symmetry of 4dCS theory is manifested as a global G symmetry acting as $\phi \mapsto \text{Ad}_g^{-1}\phi$.

This 4dCS setup, which uses the same 1-form ω as the PCM with different boundary conditions, has produced an integrable model which is related to the PCM by NATD. In fact, this is a generic feature of 4dCS theory: changing the boundary conditions while retaining the 1-form ω leads to different integrable models which are related by dualities [Del+20; LV21]. In general, models obtained in this manner will be related by Poisson-Lie T-dualities (PLTD) which are known to have a close connection to integrability (see [Tho19] for a review).

1.3.4 Lambda deformation

The class of integrable sigma models known as integrable deformations can also be recovered from 4dCS theory [Del+20]. These theories depend on a deformation parameter, and in some limit they reduce to another known integrable model, for example the PCM or WZW model. Rather pleasingly, this deformation parameter can be realised in the geometry of 4dCS theory. Let us take the 1-form ω associated with the PCM plus WZ term and split the double pole at $\zeta = 0$ into two simple poles. Introducing a deformation parameter α, this gives the 1-form

$$\omega = \frac{(\zeta - a)(\zeta - b)}{(\zeta - \alpha)(\zeta + \alpha)} \, \text{d}\zeta . \tag{1.129}$$

In the limit $\alpha \to 0$, this reduces to the 1-form associated with the PCM plus WZ term.

We will continue to use the boundary conditions $A_\pm|_\infty = 0$ at the double pole, but we must perform a variation of the action to determine the allowed boundary conditions at the new simple poles. Neglecting the contribution from $\zeta = \infty$, the remaining boundary variation is given by

$$\int_\Sigma \left[\frac{(\alpha - a)(\alpha - b)}{2\alpha} \text{tr}\big(\delta A \wedge A\big)|_{\zeta=+\alpha} - \frac{(\alpha + a)(\alpha + b)}{2\alpha} \text{tr}\big(\delta A \wedge A\big)|_{\zeta=-\alpha} \right] . \tag{1.130}$$

One way for this boundary variation to vanish would be for each term to vanish independently. For

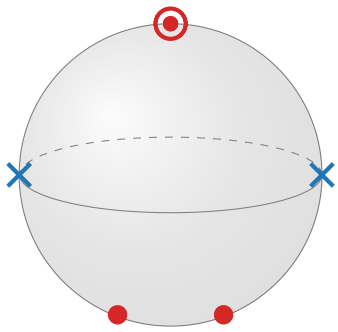


Figure 1.3: The meromorphic 1-form associated with integrable deformations has poles (red circles) at $\zeta = \pm\alpha$ and $\zeta = \infty$. It also has simple zeroes (blue crosses) at $\zeta = a$ and $\zeta = b$. This relates to the 1-form associated with the PCM plus WZ term but with the double pole at $\zeta = 0$ split into two simple poles.

example, we could impose chiral boundary conditions on the gauge field of the form $A_+|_{+\alpha} = 0$ and $A_-|_{-\alpha} = 0$. In this analysis, however, we will instead seek to relate the value of the gauge field at both simple poles such that the two terms in the boundary variation cancel against one another. With this in mind, let us consider the difference of the two coefficients,

$$\frac{(\alpha + a)(\alpha + b)}{2\alpha} - \frac{(\alpha - a)(\alpha - b)}{2\alpha} = a + b \ . \tag{1.131}$$

If we take $b = -a$, the residues of the two simple poles have the same magnitude, allowing for some special choices of boundary conditions. For this reason, we will impose this restriction on the parameters and we will also set $a = 1$ for simplicity. We will shortly see why it is important that the residues of the two simple poles have the same magnitude, and why the associated boundary conditions are special. We refer to this condition as residue matching.

Introducing an overall coefficient for convenience, we would like to consider the 1-form

$$\omega = \frac{2\alpha}{1 - \alpha^2} \frac{1 - \zeta^2}{\zeta^2 - \alpha^2} \mathrm{d}\zeta \ . \tag{1.132}$$

Computing the boundary variation (away from $\zeta = \infty$) with this 1-form gives

$$\int_\Sigma \Big[\operatorname{tr}\big(\delta A \wedge A\big)|_{\zeta=+\alpha} - \operatorname{tr}\big(\delta A \wedge A\big)|_{\zeta=-\alpha} \Big] \ . \tag{1.133}$$

One valid choice of boundary conditions, which we will refer to as the diagonal boundary conditions, is to impose $A_\pm|_{+\alpha} = A_\pm|_{-\alpha}$. It is important to highlight that the boundary variation only vanishes under these conditions because both terms come with the same coefficient. If the residues of the two simple poles had different magnitudes, we would have to introduce some parameter $q \in \mathbb{C}$ measuring the ratio between these residues. We could then choose to impose the boundary conditions $A_\pm|_{+\alpha} = q\, A_\pm|_{-\alpha}$, but we will argue that these are qualitatively different from the diagonal boundary conditions.

The diagonal boundary conditions are special because of the gauge transformations which preserve them, namely those transformations which obey $g|_{+\alpha} = g|_{-\alpha}$. In particular, there are no differential constraints on these transformations, meaning that they are true gauge symmetries of the system. In contrast, consider the boundary conditions given by $A_\pm|_{+\alpha} = q\, A_\pm|_{-\alpha}$ where $q \neq 1$ is some parameter. After a gauge transformation $A \mapsto A^g$, the boundary conditions on the transformed field are

$$\big(g^{-1} A_\pm g + g^{-1}\partial_\pm g\big)|_{\zeta=+\alpha} = q\,\big(g^{-1} A_\pm g + g^{-1}\partial_\pm g\big)|_{\zeta=-\alpha}\ . \tag{1.134}$$

Since we expect this to hold for any field configuration, the terms which are independent of A should match separately from those which depend on A. Using the fact that the original configuration obeys the boundary conditions, the first terms on each side will agree when $g|_{+\alpha} = g|_{-\alpha}$. In addition to this constraint, the second terms must also agree with one another. In the case that $q = 1$, this is implied by $g|_{+\alpha} = g|_{-\alpha}$ and the system will exhibit a gauge symmetry. However, when $q \neq 1$ there are no non-trivial solutions and only global symmetries will survive. This is why the diagonal boundary conditions, which are only possible due to the residue matching, are special[11] boundary conditions.

In summary, we choose to impose the boundary conditions

$$A_\pm|_{\zeta=+\alpha} = A_\pm|_{\zeta=-\alpha}\ , \qquad A_\pm|_{\zeta=\infty} = 0\ . \tag{1.135}$$

Let us proceed with the localisation calculation to see which two-dimensional theory this produces. We start by introducing the usual field redefinition $A = L^{\hat{g}}$ and denoting the relevant degrees of freedom in the edge mode by

$$\hat{g}|_{\zeta=+\alpha} = g_+\ , \qquad \hat{g}|_{\zeta=-\alpha} = g_-\ , \qquad \hat{g}|_{\zeta=\infty} = \tilde{g}\ , \qquad \hat{g}^{-1}\partial_\zeta \hat{g}|_{\zeta=\infty} = \tilde{\phi}\ . \tag{1.136}$$

We will now consider the external and internal symmetries of the system and impose some convenient gauge fixing constraints. The external gauge symmetries at $\zeta = \infty$ can be used to fix $\tilde{\phi} = 0$. Then, because we are working with the diagonal boundary conditions, we have an additional external gauge symmetry acting as $g_\pm \mapsto g_\pm \cdot h$. This can be used to fix either of these edge modes to the trivial map, and we will choose to impose $g_- = \mathrm{id}$. Furthermore, the field redefinition introduces an internal gauge symmetry which we will use to impose $L_{\bar{\zeta}} = 0$ and $\tilde{g} = \mathrm{id}$. For ease of notation, we will denote the remaining edge mode by $g \equiv g_+$ going forwards. The surviving degrees of freedom in the edge mode are

$$\hat{g}|_{\zeta=+\alpha} = g\ , \qquad \hat{g}|_{\zeta=-\alpha} = \mathrm{id}\ , \qquad \hat{g}|_{\zeta=\infty} = \mathrm{id}\ , \qquad \hat{g}^{-1}\partial_\zeta \hat{g}|_{\zeta=\infty} = 0\ . \tag{1.137}$$

[11] Statements about "special" boundary conditions are nicely presented in the language of the defect Lie algebra [Del+20; LV21]. An introduction to this formalism with some simple examples will be given later.

The group-valued field g will be the fundamental field of the two-dimensional sigma model. We expect this theory to have a global G symmetry corresponding to the constant external symmetry acting at $\zeta = \infty$. In order to preserve the constraint $\tilde{g} = \mathrm{id}$, we must combine this with a constant internal symmetry, and then we must also act with a constant external diagonal symmetry to preserve $g_- = \mathrm{id}$. Bringing these pieces together, this culminates in a diagonal action on the fundamental field $g \mapsto h^{-1} \cdot g \cdot h$.

The next step in the localisation is to solve the two bulk equations of motion $\partial_{\bar{\zeta}} L_\pm = 0$. Allowing for the components $L_\pm$ to have a simple pole at $\zeta = \pm 1$ respectively, the bulk equations have the general solutions

$$L_+ = V_+ + \frac{1-\alpha}{1-\zeta} U_+ \ , \qquad L_- = V_- + \frac{1+\alpha}{1+\zeta} U_- \ . \tag{1.138}$$

The fields $V_\pm$ and $U_\pm$ in this expression are constant along the spectral plane. We will solve for these constant fields in terms of g using the boundary conditions. The boundary condition at $\zeta = \infty$ implies $V_\pm = 0$, whilst the other boundary condition reads

$$\big(\lambda^{-1} - \mathrm{Ad}_g^{-1}\big) U_+ = j_+ \ , \qquad \big(\lambda - \mathrm{Ad}_g^{-1}\big) U_- = j_- \ , \qquad \lambda = \frac{1+\alpha}{1-\alpha} \ . \tag{1.139}$$

We would like to solve this equation for $U_\pm$ in terms of g, so we must ask whether or not the operator on the left-hand side is invertible. Since the adjoint action is an orthogonal matrix ($\mathrm{Ad}_g^T = \mathrm{Ad}_g^{-1}$) its eigenvalues will all have unit norm. Therefore, this operator will be invertible so long as the parameter λ does not have unit norm, in other words so long as $\alpha \neq 0$. Notably, in the limit $\alpha \to 0$ the structure of the 1-form ω changes: the two simple poles recombine into a double pole at $\zeta = 0$. Since limits of this type dramatically change the characteristics of 4dCS theory, one should be sceptical when examining any results in these regimes. We will assume that we are working away from this limit and solve the boundary conditions with

$$U_+ = \big(\lambda^{-1} - \mathrm{Ad}_g^{-1}\big)^{-1} j_+ \ , \qquad U_- = \big(\lambda - \mathrm{Ad}_g^{-1}\big)^{-1} j_- \ . \tag{1.140}$$

The final step is to substitute this solution back into the action and compute the two-dimensional integrable sigma model. The non-vanishing contributions from the integral over $\mathbb{CP}^1$ are given by

$$S_{\mathrm{4dCS}}[L, \hat{g}] = \int_\Sigma \bigg[\mathrm{tr}\big(L \wedge \mathrm{d}\hat{g}\hat{g}^{-1}\big)|_{\zeta=+\alpha} - \int_{[0,1]} \mathrm{WZ}[\hat{g}]\big|_{\zeta=+\alpha} \bigg] \ . \tag{1.141}$$

Substituting in the expression for the Lax and evaluating gives the two-dimensional action

$$S_\lambda[g] = -\int_\Sigma \mathrm{d}^2x \ \mathrm{tr}\big(j_+ \cdot \frac{1+\lambda\,\mathrm{Ad}_g}{1-\lambda\,\mathrm{Ad}_g} j_-\big) - \int_{\widetilde{\Sigma}} \mathrm{WZ}[g] \ , \qquad \lambda = \frac{1+\alpha}{1-\alpha} \ . \tag{1.142}$$

We recognise this action as the integrable λ-deformation of the WZW model, and its derivation from 4dCS theory first appeared in [Del+20]. As expected, this theory has a global diagonal G symmetry acting as $g \mapsto h^{-1}gh$. In the limit $\lambda \to 0$ this model reduces to the WZW model, whilst the limit $\lambda \to 1$ is associated with the NATD of the PCM. The treatment of this second limit is subtle as it also requires taking a limit in the parameterisation of the group element.

From the 4dCS perspective, taking the deformation parameter $\alpha \to 0$ in the 1-form corresponds to the limit $\lambda \to 1$ in the two-dimensional theory. This should give the NATD of the PCM which arises from a 4dCS setup with two double poles and the boundary conditions $A_\pm|_\infty = 0$ and $\partial_\zeta A_\pm|_0 = 0$. Let us see how to verify this expectation. In the context of the λ-model, we are imposing the diagonal boundary conditions which may be written as

$$A_\pm|_{\zeta=+\alpha} - A_\pm|_{\zeta=-\alpha} = 0 \ . \tag{1.143}$$

When we take the limit $\alpha \to 0$, the two simple poles are colliding at $\zeta = 0$ and we are left to make sense of the resulting boundary conditions. Since this produces a second order pole, the relevant degrees of freedom in the boundary variation are the value of the gauge field $A_\pm|_{\zeta=0}$ and its $\mathbb{CP}^1$ derivative $\partial_\zeta A_\pm|_{\zeta=0}$. From this perspective, our diagonal boundary conditions look like a finite difference which reduces to the $\mathbb{CP}^1$ derivative in the limit $\alpha \to 0$. We therefore deduce that the resulting boundary conditions are $\partial_\zeta A_\pm|_{\zeta=0} = 0$ as expected for the NATD of the PCM.

1.3.5 Defect Lie algebra and PLTD

In 4dCS theory, the dynamical degrees of freedom are edge modes living at the poles of ω. In the presentation we have followed so far, we start with a field $\hat{g}$ which depends on the spectral parameter, and the edge modes are the evaluation of this field (and its derivatives) at the poles. This passage from a single G-valued field to a set of edge modes is aptly presented in the language of the defect Lie algebra [Del+20; LV21]. This formalism is also useful for discussing the external symmetries which preserve a set of boundary conditions. In particular, it allows for a precise statement of the "special" property that the diagonal boundary conditions exhibit in the example of the λ-deformation. We will now provide an introduction to this formalism alongside some simple applications.

Let us consider a meromorphic 1-form ω which contains a simple pole at $\zeta = +\alpha$. As is usual in our localisation procedure, we define a field $\hat{g}$ which is a map from the 4-manifold $M_4 = \mathbb{CP}^1 \times \Sigma$ to the Lie group G. In practice, this field only appears in the action against the distribution $\mathrm{d}\omega$, so the theory only depends on its behaviour at the poles. For the present example, this edge mode can be captured by the evaluation of $\hat{g}$ at $\zeta = +\alpha$ which we will denote by $\hat{g}|_{+\alpha} = g_+$,

$$\hat{g} : \mathbb{CP}^1 \times \Sigma \to \mathsf{G} \ , \qquad g_+ : \Sigma \to \mathsf{G} \ . \tag{1.144}$$

Now, consider another simple pole at $\zeta = -\alpha$ and let us denote the corresponding edge mode by $\hat{g}|_{-\alpha} = g_-$. There are two equivalent ways of thinking about these edge modes. It is natural to think of g_+ and g_- as a pair of maps from Σ to the Lie group G. Alternatively, we can package these maps together into $\vec{g} = (g_+, g_-)$ which is a map from Σ to the product $\mathsf{G} \times \mathsf{G}$,

$$\vec{g} : \Sigma \to \mathsf{G} \times \mathsf{G} \ . \tag{1.145}$$

At the level of the Lie algebra, the product space $\mathsf{G} \times \mathsf{G}$ corresponds to a direct sum in which each factor commutes with one another,

$$\mathrm{Lie}\,(\mathsf{G} \times \mathsf{G}) = \mathfrak{g} \oplus \mathfrak{g} \ . \tag{1.146}$$

The fact that these two copies of $\mathfrak{g}$ commute is implied by the fact that the poles are at distinct locations in the spectral plane.

We can also consider second order poles in ω. In this case, we extract two fields from the edge mode, its value g at the pole and its $\mathbb{CP}^1$ derivative ϕ. The former is valued in the Lie group G, whilst the latter is valued in the Lie algebra $\mathfrak{g}$. As before, these maps can be packaged together into a single map $\vec{g} = (g, \phi)$. In this case, however, one might suspect that these factors do not commute as they live at the same point in $\mathbb{CP}^1$. It turns out that this map is valued in the tangent bundle of the Lie group,

$$\vec{g} : \Sigma \to \mathsf{TG} \cong \mathsf{G} \ltimes \mathfrak{g} \ . \tag{1.147}$$

At the level of the Lie algebra, this gives a semi-direct product between $\mathfrak{g}$ and an Abelian algebra,

$$\mathrm{Lie}\,(\mathsf{G} \ltimes \mathfrak{g}) = \mathfrak{g} \ltimes \mathbb{R}^{\dim(\mathsf{G})} \ . \tag{1.148}$$

These group multiplication and Lie bracket structures can be derived by considering consecutive gauge transformations of 4dCS theory.

These spaces are known in the literature [Del+20; LV21] as the defect Lie algebra and the defect Lie group. They are especially useful for discussing boundary conditions in 4dCS theory, as the boundary variation may be succinctly written in this language. To this end, it is necessary to introduce an inner product on the defect Lie algebra which is derived from the 1-form ω. Let us return to the example of the λ-model where the boundary variation was given by

$$\int_\Sigma \Big[\, \mathrm{tr}\big(\delta A \wedge A\big)|_{\zeta=+\alpha} \, - \, \mathrm{tr}\big(\delta A \wedge A\big)|_{\zeta=-\alpha} \Big] \ . \tag{1.149}$$

Inspired by the previous discussion, we might introduce a 1-form $\vec{A} = (A|_{+\alpha}, A|_{-\alpha})$ which is valued in the defect Lie algebra $\vec{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}$. We would like to write the boundary variation of 4dCS theory in terms of this 1-form as

$$\int_\Sigma \langle\!\langle \delta \vec{A}, \vec{A} \rangle\!\rangle \ , \qquad \langle\!\langle \cdot \, , \cdot \rangle\!\rangle : \vec{\mathfrak{g}} \times \vec{\mathfrak{g}} \to \mathbb{R} \ . \tag{1.150}$$

Here, we are denoting by $\langle\langle \cdot\,,\cdot \rangle\rangle$ an inner product on $\vec{\mathfrak{g}}$ which is derived from the boundary variation of 4dCS theory. In the present example, this inner product is defined by

$$\langle\langle (X_+, X_-), (Y_+, Y_-) \rangle\rangle = \mathrm{tr}\big(X_+ \cdot Y_+\big) - \mathrm{tr}\big(X_- \cdot Y_-\big) \ . \tag{1.151}$$

This provides a context in which to discuss boundary conditions of 4dCS theory.

Schematically speaking, the 1-form $\vec{A}$ has two types of indices: coordinate indices on the 4-manifold M_4 running over a basis of 1-forms, and gauge group indices running over a basis of the defect Lie algebra. Generically, boundary conditions could impose constraints on either of these indices. For example, we could constraint the 1-form legs of the gauge field with chiral boundary conditions such as $A_+|_0 = 0$, or we could constraint the algebra indices of $\vec{A}$ by relating its components with conditions like $A|_{+\alpha} = A|_{-\alpha}$. Restricting our attention to the second class of boundary conditions, these fit nicely into the defect Lie algebra formalism.

An isotropic subspace of $\vec{\mathfrak{g}}$ is a subspace $\vec{\mathfrak{h}} \subset \vec{\mathfrak{g}}$ such that the inner product restricts zero on $\vec{\mathfrak{h}}$,

$$\langle\langle \vec{X}, \vec{Y} \rangle\rangle = 0 \qquad \forall\, \vec{X}, \vec{Y} \in \vec{\mathfrak{h}} \ . \tag{1.152}$$

Assuming that the inner product on $\vec{\mathfrak{g}}$ is non-degenerate, an isotropic subspace $\vec{\mathfrak{h}}$ may be at most half-dimensional and these are known as maximal isotropic (or Lagrangian) subspaces. Given a maximal isotropic subspace $\vec{\mathfrak{h}}$ of the defect Lie algebra, we can impose boundary conditions which restrict $\vec{A}$ to live in this subspace. For example, the diagonal boundary conditions require $\vec{A}$ to live in the diagonal subspace $\mathfrak{g}_\Delta$ of the defect Lie algebra $\vec{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}$,

$$A|_{+\alpha} = A|_{-\alpha} \qquad \Longleftrightarrow \qquad \vec{A} \in \mathfrak{g}_\Delta = \big\{(X, X) \mid X \in \mathfrak{g}\big\} \subset \mathfrak{g} \oplus \mathfrak{g} \ . \tag{1.153}$$

The 4dCS boundary variation vanishes due to the isotropy of the subspace.

At this stage, we are ready to state the "special" property of these diagonal boundary conditions when compared to the boundary conditions $A|_{+\alpha} = q\, A|_{-\alpha}$ for some parameter $q \in \mathbb{C}$. Whilst all of these conditions will define isotropic subspaces of their associated defect Lie algebras, only the diagonal subspace is also a subalgebra, meaning it is closed under the Lie bracket. The implication of this statement is that external gauge transformations within this subalgebra will preserve the boundary conditions of 4dCS theory. In particular, there will be no differential constraints on these external symmetries, leading to an $\vec{\mathfrak{h}}$-valued gauge symmetry which we can use to fix some degrees of freedom in the edge modes.

Let us turn to the examples of the PCM and its non-Abelian T-dual (NATD). The 4dCS setups for these two integrable sigma models are related by changing the boundary conditions at a second order pole in ω. In this context, the relevant defect Lie algebra is $\vec{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^{\dim(\mathsf{G})}$ whose defect Lie group is the tangent bundle TG. To be explicit, we will work with the 1-form

$$\omega = \frac{(\zeta - a)(\zeta - b)}{\zeta^2}\, \mathrm{d}\zeta \ . \tag{1.154}$$

Focusing on the pole at $\zeta = 0$, the boundary variation of 4dCS is given by

$$\int_\Sigma \Big[\, (ab)\, \partial_\zeta \mathrm{tr}\big(\delta A \wedge A\big)|_{\zeta=0} - (a+b)\,\mathrm{tr}\big(\delta A \wedge A\big)|_{\zeta=0} \Big] \, . \tag{1.155}$$

We can introduce a 1-form $\vec{A} = (A|_0, \partial_\zeta A|_0)$ which captures the relevant degrees of freedom in this boundary variation. Then, in order to write the boundary variation as before, we are prompted to define an inner product on the defect Lie algebra by

$$\langle\!\langle (X,x),(Y,y) \rangle\!\rangle = (ab)\,\mathrm{tr}\big(X\cdot y + x\cdot Y\big) - (a+b)\,\mathrm{tr}\big(X\cdot Y\big) \, . \tag{1.156}$$

There are two natural subspaces to consider which are related to the boundary conditions $A|_0 = 0$ and $\partial_\zeta A|_0 = 0$. We will denote these subspaces (which are also subalgebras) by

$$\begin{aligned} \mathfrak{g} &= \big\{ (X,0) \mid X \in \mathfrak{g} \big\} \subset \mathfrak{g} \ltimes \mathbb{R}^{\dim(\mathsf{G})} \, , \\ \mathbb{R}^{\dim(\mathsf{G})} &= \big\{ (0,x) \mid x \in \mathbb{R}^{\dim(\mathsf{G})} \big\} \subset \mathfrak{g} \ltimes \mathbb{R}^{\dim(\mathsf{G})} \, . \end{aligned} \tag{1.157}$$

These are complementary subspaces of the defect Lie algebra, but they are not always isotropic subspaces — meaning they do not always define good boundary conditions. The second subspace $\mathbb{R}^{\dim(\mathsf{G})}$ is always isotropic, but the subspace $\mathfrak{g}$ is only isotropic when the parameters satisfy $b = -a$. This can also be phrased in terms of the boundary conditions we have encountered earlier. Restricting $\vec{A}$ to live in one of these subspaces can be translated into the aforementioned conditions,

$$\begin{aligned} \vec{A} &\in \mathfrak{g} & &\Longleftrightarrow & \partial_\zeta A|_0 &= 0 \, , \\ \vec{A} &\in \mathbb{R}^{\dim(\mathsf{G})} & &\Longleftrightarrow & A|_0 &= 0 \, . \end{aligned} \tag{1.158}$$

Considering the boundary variation, $A|_0 = 0$ is always a good boundary condition, but $\partial_\zeta A|_0 = 0$ only works when $b = -a$.

This restriction on the parameters in 4dCS theory has an impact on the associated two-dimensional sigma model. Whilst this 1-form with the boundary conditions $A|_0 = 0$ generically recovers the PCM plus WZ term, the condition $b = -a$ sets the coefficient of the WZ term to zero, leaving just the standard PCM. At this point, the boundary conditions $\partial_\zeta A|_0 = 0$ are also accessible which leads to the NATD of the PCM.

In the language presented above, these two boundary conditions define complementary subalgebras of the defect Lie algebra which are both maximal and isotropic. This structure is known as a Manin triple where the triple refers two three objects: the defect Lie algebra, and two complementary maximal isotropic subalgebras. In this context, the defect Lie algebra is also known as a Drinfeld double. These structures also appear in the context of Poisson-Lie T-duality (PLTD) of integrable sigma models (see [Tho19] for a recent review). In short, given a Manin triple $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$ where the Drinfeld double is $\mathfrak{d} = \mathfrak{g} \bowtie \tilde{\mathfrak{g}}$, one can define two integrable sigma models whose target spaces are $\mathsf{G} = \exp(\mathfrak{g})$ and $\tilde{\mathsf{G}} = \exp(\tilde{\mathfrak{g}})$ respectively. It turns out that these integrable sigma models are related by a generalisation of T-duality known as Poisson-Lie T-duality [KS95; KS96b].

In the special case that the second isotropic subalgebra is abelian, that is $\tilde{\mathfrak{g}} = \mathbb{R}^{\dim(\mathsf{G})}$, PLTD reduces to NATD. This provides an interpretation of the fact that changing the boundary conditions in our 4dCS setup had the effect of turning the PCM into its NATD. In general, 4dCS setups with the same 1-form and different boundary conditions are related by PLTD. Returning to the example of the λ-model, we might try to find different boundary conditions to impose on the two simple poles with the goal of recovering another two-dimensional sigma model which should be related to the λ-model by PLTD. This is indeed possible, and the PLTD model is the Yang-Baxter deformation. We will explore this example in the following section.

1.3.6 Yang-Baxter deformation

We can recover another integrable deformation from 4dCS theory using the same meromorphic 1-form ω but imposing different boundary conditions. Let us assume that the Lie algebra $\mathfrak{g}$ admits a solution $\mathcal{R}$ to the mcYBe,

$$[\mathcal{R}X, \mathcal{R}Y] - \mathcal{R}[X, \mathcal{R}Y] - \mathcal{R}[\mathcal{R}X, Y] + c^2[X, Y] = 0 \qquad \forall\, X, Y \in \mathfrak{g} \ . \tag{1.159}$$

Solutions of this equation fall into three classes determined by the value of $c \in \{0, 1, \mathrm{i}\}$. In the present case we will consider solutions to the inhomogeneous Yang-Baxter equation which have $c \neq 0$. For the "non-split" solutions with $c = \mathrm{i}$, one must perform a thorough analysis of the reality conditions [Del+20] to properly understand the 4dCS theory. In order to avoid these issues for the time being, we will study the "split" solutions with $c = 1$. We will also restrict to anti-symmetric solutions with respect to the trace on $\mathfrak{g}$.

Given an anti-symmetric solution to the split inhomogeneous mcYBe on $\mathfrak{g}$, we can impose the boundary conditions

$$(\mathcal{R} + 1)A_\pm|_{\zeta=+\alpha} = (\mathcal{R} - 1)A_\pm|_{\zeta=-\alpha} \ . \tag{1.160}$$

We can understand these boundary conditions in the language of the defect Lie algebra. Let us consider the 1-form $\vec{A} = (A|_{+\alpha}, A|_{-\alpha})$ which captures the boundary components of the gauge field. This 1-form is valued in the defect Lie algebra $\vec{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}$ and we might consider boundary conditions defined by isotropic subspaces of $\vec{\mathfrak{g}}$. To recover the λ-model, we imposed the diagonal boundary conditions which require the gauge field to live in the diagonal subalgebra of $\vec{\mathfrak{g}}$,

$$A|_{+\alpha} = A|_{-\alpha} \qquad \Longleftrightarrow \qquad \vec{A} \in \mathfrak{g}_\Delta = \big\{(X, X) \mid X \in \mathfrak{g}\big\} \subset \mathfrak{g} \oplus \mathfrak{g} \ . \tag{1.161}$$

The fact that this subspace is also a subalgebra explains the emergence of an additional gauge symmetry in the 4dCS theory.

Now, we would like to find a complementary subalgebra to $\mathfrak{g}_\Delta$ which is also istropic. Given a solution $\mathcal{R}$ to the split inhomogeneous mcYBe, we can define such a subalgebra by

$$\mathfrak{g}_{\mathcal{R}} = \{\big((\mathcal{R}-1)X, (\mathcal{R}+1)X\big) \mid X \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{g} \ . \tag{1.162}$$

The boundary conditions given earlier are equivalent to the requirement that the 1-form $\vec{A}$ is valued in this subspace of $\vec{\mathfrak{g}}$. Whilst the diagonal subalgebra $\mathfrak{g}_{\Delta}$ is isomorphic to $\mathfrak{g}$, the subalgebra $\mathfrak{g}_{\mathcal{R}}$ is equipped with an alternative Lie bracket defined by

$$[X,Y]_{\mathcal{R}} = [\mathcal{R}X, Y] + [X, \mathcal{R}Y] \ . \tag{1.163}$$

We will refer to this Lie bracket as the $\mathcal{R}$-bracket and one can show that it satisfies the Jacobi identity by virtue of the mcYBe. In addition, the mcYBe also implies that the maps $(\mathcal{R} \pm 1)$ are Lie algebra homomorphisms from $\mathfrak{g}_{\mathcal{R}}$ into $\mathfrak{g}$. This implies that the subspace $\mathfrak{g}_{\mathcal{R}} \subset \vec{\mathfrak{g}}$ is also a subalgebra of $\vec{\mathfrak{g}}$, meaning that the 4dCS theory will exhibit a residual gauge symmetry with this choice of boundary conditions. In fact, this subspace is complementary to the diagonal subspace, meaning that we have the vector space decomposition

$$\mathfrak{g} \oplus \mathfrak{g} \cong \mathfrak{g}_{\Delta} \oplus \mathfrak{g}_{\mathcal{R}} \ . \tag{1.164}$$

At the level of groups, this allows us to parameterise the coset $(\mathsf{G} \times \mathsf{G})/\mathsf{G}_{\mathcal{R}}$ by elements of the diagonal subgroup $\mathsf{G}_{\Delta} \subset \mathsf{G} \times \mathsf{G}$. We will shortly use this fact in our gauge fixing.

Returning to our 4dCS setup, let us derive the associated two-dimensional sigma model. We will implement the field redefinition $A = L^{\hat{g}}$ and denote the relevant degrees of freedom in the edge mode by

$$\hat{g}|_{\zeta=+\alpha} = g_+ \ , \qquad \hat{g}|_{\zeta=-\alpha} = g_- \ , \qquad \hat{g}|_{\zeta=\infty} = \tilde{g} \ , \qquad \hat{g}^{-1}\partial_\zeta \hat{g}|_{\zeta=\infty} = \tilde{\phi} \ . \tag{1.165}$$

The external symmetries at $\zeta = \infty$ allow us to fix $\tilde{\phi} = 0$. Then, we have this residual $\mathsf{G}_{\mathcal{R}}$ gauge symmetry which preserves the boundary conditions. As mentioned, we can parameterise the coset by elements of the diagonal subgroup, meaning we can fix the edge modes to obey $g_+ = g_-$. We will denote this surviving element by g for ease of notation. Then, we can use the internal gauge symmetry to fix $L_{\bar{\zeta}} = 0$ and $\tilde{g} = \mathrm{id}$. In summary, the edge mode degrees of freedom now take the form

$$\hat{g}|_{\zeta=+\alpha} = g \ , \qquad \hat{g}|_{\zeta=-\alpha} = g \ , \qquad \hat{g}|_{\zeta=\infty} = \mathrm{id} \ , \qquad \hat{g}^{-1}\partial_\zeta \hat{g}|_{\zeta=\infty} = 0 \ . \tag{1.166}$$

The group-valued field g will be the fundamental field of the two-dimensional sigma model. We expect this two-dimensional sigma model to have a global G symmetry inherited from the constant external transformations acting at $\zeta = \infty$. These act on the surviving field as a global left action $g \mapsto h^{-1} \cdot g$.

The next step is to solve the bulk equations of motion $\partial_{\bar{\zeta}} L_{\pm} = 0$. Allowing for the poles in

$L_\pm$, the general solution is given by

$$L_+ = V_+ + \frac{U_+}{1-\zeta}\ , \qquad L_- = V_- + \frac{U_-}{1+\zeta}\ . \tag{1.167}$$

We can then solve for $V_\pm$ and $U_\pm$ in terms of g using the boundary conditions. The boundary condition at $\zeta = \infty$ tells us that $V_\pm = 0$, whilst the other boundary conditions imply

$$-\frac{1}{1-\alpha^2}\big(1 \pm \alpha\mathcal{R}\big)\mathrm{Ad}_g^{-1}U_\pm = g^{-1}\partial_\pm g\ . \tag{1.168}$$

The equations can be solved for $U_\pm$ provided that the operators $(1 \mp \alpha\mathcal{R})$ are invertible. Many of the known solutions to the mcYBe satisfy the additional condition $\mathcal{R}^3 = \mathcal{R}$. In this case, its eigenvalues are $\{0, \pm 1\}$ and the operators will be invertible so long as $\alpha \neq \pm 1$. In the 4dCS setup, this means that the locations of the simple zeroes must be distinct from the locations of the simple poles. We will assume that these operators are invertible and solve for $U_\pm$ as

$$U_\pm = -\frac{1-\alpha^2}{1 \pm \alpha\mathcal{R}_g}\partial_\pm g g^{-1}\ , \qquad \mathcal{R}_g = \mathrm{Ad}_g\,\mathcal{R}\,\mathrm{Ad}_g^{-1}\ . \tag{1.169}$$

To compute the action of the two-dimensional sigma model, we substitute these expressions into the 4dCS action and compute the integral over $\mathbb{CP}^1$. The interal over $\mathbb{CP}^1$ gives

$$\begin{aligned} S_{\mathrm{4dCS}}[L,\hat{g}] = \int_\Sigma \Big[\ &\mathrm{tr}\big(L\wedge \mathrm{d}\hat{g}\hat{g}^{-1}\big)|_{+\alpha} - \int_{[0,1]} \mathrm{WZ}[\hat{g}]\big|_{+\alpha} \\ &- \mathrm{tr}\big(L\wedge \mathrm{d}\hat{g}\hat{g}^{-1}\big)|_{-\alpha} + \int_{[0,1]} \mathrm{WZ}[\hat{g}]\big|_{-\alpha}\Big]\ . \end{aligned} \tag{1.170}$$

Since the residues of the two pole have the same magnitude, the two WZ terms will cancel against each other and it remains to compute the quadratic term. This gives the two-dimensional sigma model

$$S_{\mathrm{YB}}[g] = 2\,\alpha \int_\Sigma \mathrm{d}^2x\ \mathrm{tr}\big(j_+ \cdot \frac{1}{1-\alpha\mathcal{R}} \cdot j_-\big)\ . \tag{1.171}$$

This may be recognised as the integrable Yang-Baxter deformation of the PCM which reduces to the PCM in the limit $\alpha \to 0$. From the 4dCS perspective, this limit corresponds to the two simple poles recombining at $\zeta = 0$. The operator in between the two Maurer-Cartan forms breaks the right-acting global symmetry, but the left-acting global symmetry is preserved — this matches our expectations from 4dCS theory. This model was first derived from 4dCS theory in [Del+20].

The 4dCS setup of this integrable deformation uses the same 1-form as the λ-model. The different two-dimensional theories are related by choosing different boundary conditions in 4dCS theory: they correspond to the two complementary subspaces in the decomposition of the defect Lie algebra

$$\mathfrak{g} \oplus \mathfrak{g} \cong \mathfrak{g}_\Delta \oplus \mathfrak{g}_\mathcal{R}\ . \tag{1.172}$$

This structure describes a Manin triple which underlies Poisson-Lie T-duality (PLTD) in two-dimensional sigma models. Indeed, it is known that the λ-deformation and the Yang-Baxter deformation are related by PLTD [Vic15; SST15; Kli15]. These 4dCS constructions offer an alternative perspective on this statement [Del+20; LV21].

Chapter 2

Self-dual Yang-Mills theory and twistor space

2.1 Self-dual Yang-Mills theory

Self-dual Yang-Mills (SDYM) theory is a four-dimensional gauge theory which has connections to integrability. It describes an interesting sector of the usual Yang-Mills theory which has been studied due to its relationship to instantons [Bel+75]. We will start by describing the role of the self-dual sector in usual Yang-Mills theory, before moving on to its applications in integrability.

Four-dimensional Yang-Mills theory is a gauge theory defined by the action

$$S_{\mathrm{YM}}[A] = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}\big(F \wedge \star F\big) \ . \tag{2.1}$$

This action is quadratic in the field strength $F = \mathrm{d}A + A \wedge A$ which is an algebra-valued 2-form on spacetime. The field strength satisfies the Bianchi identity $\nabla F = 0$ where $\nabla = \mathrm{d} + A$ is the covariant derivative, and the equation of motion of this theory, found by varying the action, is given by $\nabla \star F = 0$.

In four-dimensions, the Hodge star maps 2-forms to 2-forms and (in Euclidean or split signature) it squares to the identity. In these cases, we can decompose the field strength into its self-dual and anti-self-dual components,

$$F = F_+ + F_- \ , \qquad \star F_\pm = \pm F_\pm \ . \tag{2.2}$$

These two components are exchanged if we swap the choice of orientation on spacetime, so any comments about one sector equally apply to the other. In four-dimensions, it turns out that self-dual and anti-self-dual 2-forms are orthogonal with respect to the wedge product. This allows us to write the Yang-Mills action as

$$S_{\rm YM}[A] = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}\big(F_+ \wedge F_+ - F_- \wedge F_-\big) \ . \tag{2.3}$$

On the other hand, there is another gauge invariant 4-form we can consider,

$$\frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}\big(F \wedge F\big) = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}\big(F_+ \wedge F_+ + F_- \wedge F_-\big) \ . \tag{2.4}$$

This is the topological theta term which can locally be written as a total derivative of the Chern-Simons 3-form. Since this is a boundary term, one might think that it will not contribute to the action as we should ask our fields to die off at infinity. In fact, this is a little too strong — we should not require the gauge field A to vanish as $r \to \infty$, it is sufficient for the action to vanish in this limit which happens whenever A is pure gauge,

$$r \to \infty \ , \qquad A \to g^{-1}\mathrm{d}g \ . \tag{2.5}$$

Let us denote the "boundary" of spacetime at infinity by $\partial\mathbb{R}^4 = S^3$. The field g is a map from this boundary to the gauge group G,

$$g : S^3 \to \mathsf{G} \ . \tag{2.6}$$

These maps are classified by the third homotopy group $\pi_3(\mathsf{G}) \cong \mathbb{Z}$ which is isomorphic to the integers for any simple Lie group G. This is easiest to understand in the example $\mathsf{G} = \mathsf{SU}(2) \cong S^3$ where the field g is a map from S^3 to itself. Each class in the homotopy group is characterised by an integer $k \in \mathbb{Z}$ which counts the number times the map g wraps the S^3 around itself. Evaluating the topological theta term on these solutions gives

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{tr}\big(F \wedge F\big) = \frac{1}{2\pi^2}\int_{S^3} \mathrm{WZ}[g] = k \ . \tag{2.7}$$

The boundary term is the Chern-Simons 3-form which becomes the Wess-Zumino (WZ) 3-form when evaluated on gauge-trivial solutions. In turn, the WZ 3-form is known to count precisely the same winding number $k \in \mathbb{Z}$ which we just discussed. This characterisation of the gauge field at infinity decomposes the space of solutions into different topological sectors based on the value of the winding number.

There is more to say about these solutions with non-trivial winding, as can be seen by writing the Yang-Mills action as

$$S_{\rm YM}[A] = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}\big(F \wedge F\big) - \int_{\mathbb{R}^4} \mathrm{tr}\big(F_- \wedge F_-\big) \ . \tag{2.8}$$

The first term is the topological theta term which counts the winding number of the gauge field. Restricting ourselves to a given topological sector, this term is fixed and proportional to $k \in \mathbb{Z}$.

Therefore, minimising the action within this topological sector amounts to searching for solutions to $F_- = 0$. This is known as the self-dual Yang-Mills equation and can equivalently be written as

$$F = \star F \ . \tag{2.9}$$

One can check that solutions to this equation are classical solutions of the theory as the Bianchi identity implies that the equations of motion are solved. In non-trivial topological sectors, these will be finite action solutions which are known as instantons [Bel+75].

The self-dual Yang-Mills equations have been also been studied extensively in the context of integrability. To provide a Lax pair for these equations, we will first introduce some additional notation. Let $\{u^1, u^2\}$ denote complex coordinates on $\mathbb{R}^4$ such that the (Euclidean) metric and volume form are given by

$$\mathrm{d}s^2 = \mathrm{d}u^1\mathrm{d}\bar{u}^1 + \mathrm{d}u^2\mathrm{d}\bar{u}^2 \ , \qquad \mathrm{d}^4x = \mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 \wedge \mathrm{d}u^2 \wedge \mathrm{d}\bar{u}^2 \ . \tag{2.10}$$

In these conventions, a basis for the anti-self-dual 2-forms is given by

$$\mathrm{d}u^1 \wedge \mathrm{d}u^2 \ , \qquad \mathrm{d}\bar{u}^1 \wedge \mathrm{d}\bar{u}^2 \ , \qquad \omega = \mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 + \mathrm{d}u^2 \wedge \mathrm{d}\bar{u}^2 \ . \tag{2.11}$$

The third 2-form here which we denote by ω is proportional to the Kähler form on $\mathbb{R}^4$ and should not be confused with the meromorphic 1-form appearing in 4dCS theory. The self-dual Yang-Mills equations are therefore written as

$$(\partial_{u^1} \wedge \partial_{u^2}) \lrcorner F = 0 \ , \qquad (\partial_{\bar{u}^1} \wedge \partial_{\bar{u}^2}) \lrcorner F = 0 \ , \qquad (\partial_{u^1} \wedge \partial_{\bar{u}^1} + \partial_{u^2} \wedge \partial_{\bar{u}^2}) \lrcorner F = 0 \ . \tag{2.12}$$

The contractions with these bivectors pick out the anti-self-dual components of the field strength. In the complex structure where $\{u^1, u^2\}$ are holomorphic coordinates, the first two of these expressions may be written as $F^{2,0} = 0$ and $F^{0,2} = 0$.

Alternatively, consider the pair of differential operators given by

$$L = \nabla_{\bar{u}^1} - \zeta\, \nabla_{u^2} \ , \qquad M = \nabla_{\bar{u}^2} + \zeta\, \nabla_{u^1} \ . \tag{2.13}$$

Treating these as a Lax pair, their commutator is given by

$$[L, M] = F_{\bar{u}^1\bar{u}^2} - \zeta \left(F_{u^1\bar{u}^1} + F_{u^2\bar{u}^2}\right) + \zeta^2\, F_{u^1u^2} \ . \tag{2.14}$$

Asking this to vanish for all values of the spectral parameter $\zeta \in \mathbb{C}$ is equivalent to the self-dual Yang-Mills equations. These differential operators provide a Lax pair for SDYM which is central to the connection with integrability.

2.1.1 Four-dimensional WZW and LMP models

The self-dual Yang-Mills equations can be treated as a system of three equations for a gauge field A which is defined up to gauge transformations. Alternatively, we can solve some of these equations and impose some gauge fixing constraints to find a simpler yet equivalent system of equations. There are two well-known examples of this approach which we will now review.

4dWZW model

In the first case, we will solve the equations $F^{2,0} = 0$ and $F^{0,2} = 0$. These are integrability conditions which imply the existence of solutions h and $\tilde{h}$ to the equations

$$A^{1,0} = \tilde{h}^{-1}\partial\tilde{h} \ , \qquad A^{0,1} = h^{-1}\bar{\partial}h \ . \tag{2.15}$$

We can now perform a gauge transformation by $\tilde{h}$ which brings the total gauge field A to the form

$$A = g^{-1}\bar{\partial}g \ , \qquad g = h \cdot \tilde{h}^{-1} \ . \tag{2.16}$$

In this parameterisation, the field g is known as Yang's matrix [Yan77] (or alternatively as the J-matrix) and the remaining SDYM equation is known as Yang's equation [Yan77],

$$\omega \wedge \partial(g^{-1}\bar{\partial}g) = 0 \ . \tag{2.17}$$

It is worth emphasising that this represents a generic solution to the SDYM equations. The solutions h and $\tilde{h}$ were completely general, and gauge fixing is a legal move as solutions should only be considered up to gauge transformations. In other words, there is a one-to-one map between solutions to Yang's equation and solutions to the SDYM equations up to gauge.

This equation also arises as the equation of motion of the four-dimensional Wess-Zumino-Witten (4dWZW) model [Don85; Los+96]. The action of this theory is given by

$$S_{\text{4dWZW}}[g] = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}\big(g^{-1}\mathrm{d}g \wedge \star g^{-1}\mathrm{d}g\big) + \int_{\mathbb{R}^4\times[0,1]} \omega \wedge \mathrm{WZ}[g] \ . \tag{2.18}$$

This is a higher-dimensional analogue of the two-dimensional WZW model which exhibits some similar properties. The 2-form ω appearing in the second term is proportional to the Kähler form on $\mathbb{R}^4$, and this term is defined over the 5-manifold $\mathbb{R}^4 \times [0,1]$. In particular, the WZ 3-form is defined in terms of smooth homotopy from the trivial map at $\mathbb{R}^4 \times \{0\}$ to the fundamental field g at $\mathbb{R}^4 \times \{1\}$. The analogue of level quantisation in two-dimensions is that the integral over ω over any non-trivial 2-cycle must be quantised [Los+96]. This action is invariant under a semi-local $\mathsf{G} \times \mathsf{G}$ symmetry which acts as

$$g \mapsto h_\ell^{-1} \cdot g \cdot h_r \ , \qquad \bar{\partial}h_\ell = 0 \ , \qquad \partial h_r = 0 \ . \tag{2.19}$$

The presence of ω in the action breaks some of the spacetime symmetry of the 4dWZW model. Translational invariance is preserved, but only those rotations which leave the Kähler form invariant are symmetries of the theory. These form a $\mathsf{U}(2)$ subgroup of the full $\mathsf{SO}(4)$ group of rotations.

The first incarnation of this theory appeared in [Don85] where the action took the form

$$S_{\text{4dWZW}}[g] = \int_{\mathbb{R}^4} \kappa \, \mathrm{tr}\big(F \wedge F\big) \; . \tag{2.20}$$

In this expression, we have introduced a Kähler potential defined by $\omega = \partial\bar{\partial}\kappa$ and F is the field strength of the gauge field $A = g^{-1}\bar{\partial}g$. The 4dWZW model also appeared in the context of Kähler Chern-Simons theory [NS90; NS92] as a natural generalisation of the 3dCS/2dWZW correspondence, and in the context of $N = 2$ open string theory [OV90]. Important aspects of the theory were developed in [Los+96; Los+97] including finiteness of the quantum theory, a "four-dimensional Verlinde formula", and a connection to Donaldson theory.

LMP model

We now turn to the second presentation of the self-dual Yang-Mills equations. We choose to solve $F^{0,2} = 0$ in terms of a group-valued field h as before, and then we write the gauge field as

$$A = h^{-1}B^{1,0}h + h^{-1}\mathrm{d}h \; . \tag{2.21}$$

From here, we can perform a gauge transformation by h leaving the gauge field in the form $A = B^{1,0}$, such that the third SDYM equation reads

$$\omega \wedge \bar{\partial}B^{1,0} = 0 = \partial_{\bar{u}^1}B_{u^1} + \partial_{\bar{u}^2}B_{u^2} \; . \tag{2.22}$$

This implies the existence of an algebra-valued function ϕ (also known as the K-matrix) solving the equations $B_{u^1} = -\partial_{\bar{u}^2}\phi$ and $B_{u^2} = \partial_{\bar{u}^1}\phi$. Substituting this solution into the remaining SDYM equation gives

$$\partial B^{1,0} + B^{1,0} \wedge B^{1,0} = 0 = \partial_{u^1}\partial_{\bar{u}^1}\phi + \partial_{u^2}\partial_{\bar{u}^2}\phi + [\partial_{\bar{u}^1}\phi, \partial_{\bar{u}^2}\phi] \; . \tag{2.23}$$

This equation was first presented in [New78] as a technique for finding solutions to the SDYM equations. It also arises as the equation of motion of the LMP action, introduced independently by Leznov and Mukhtarov [Lez87; LM87] and later by Parkes [Par92],

$$S_{\text{LMP}}[\phi] = \int_{\mathbb{R}^4} \mathrm{tr}\left(\frac{1}{2}\,\mathrm{d}\phi \wedge \star\mathrm{d}\phi + \frac{2}{3}\,\mathrm{d}u^1 \wedge \mathrm{d}u^2 \wedge \phi \cdot \mathrm{d}\phi \wedge \mathrm{d}\phi\right) \; . \tag{2.24}$$

This theory, and also the 4dWZW model, have been used to compute amplitudes in Yang-Mills theory, see [CS96; Ber+97] for example. In more recent years, the LMP model has also appeared in the context of the double copy: a relationship between amplitudes in Yang-Mills theory and gravity. The kinematic algebra appearing in this topic was studied at the level of the action in [MO11] using the LMP model.

2.1.2 Reductions of self-dual Yang-Mills

The self-dual Yang-Mills equations are also related to lower-dimensional integrable models by reduction (see the textbook [MW91] for a useful introduction to this topic). In four dimensions, the action of the Hodge star on 2-forms determines and is determined by the conformal class of the metric. It is not so surprising that knowing the metric on spacetime up to a conformal transformation is enough to determine the action of the Hodge star on 2-forms. One might not have expected, however, that knowing the action of the Hodge star on 2-forms is enough to determine the conformal class of the metric. This observation underlies a rather miraculous reformulation of four-dimensional gravity in which the role of the fundamental field is played by a basis for the self-dual 2-forms, rather than by the metric (see the textbook [Kra20]).

In the present context, this means that the self-dual Yang-Mills equations are invariant under conformal transformations. On the other hand, a solution of the SDYM equations need not be invariant under conformal transformations: there is no reason that a solution of a given system should posses the same symmetries as the system itself, instead each solution is generically mapped to a different solution under a symmetry of the system. However, we can restrict our attention to those solutions which are invariant under some symmetries of the system. Let $\mathsf{H} \subset \mathsf{Conf}(\mathbb{R}^4)$ be a subgroup of the conformal group which acts on spacetime. We can study the system of equations given by the SDYM equations together with the constraints

$$\mathcal{L}_X A = 0 \qquad \forall\, X \in \mathsf{H}\ . \tag{2.25}$$

Here, X denotes a conformal Killing vector on spacetime which generates the action of H.

These constraints are not gauge invariant, and this has two important consequences. Firstly, gauge transformations will only preserve the space of solutions to this constraint if they also obey similar constraints. For example, if we take $X = \partial_{u^2}$ which generates translations in u^2, then the gauge transformations must be independent of this coordinate to preserve the constraint. Secondly, a connection may be gauge-equivalent to an invariant connection even though it is presented in a gauge in which it is not invariant. There is a proper geometric framework for addressing these concerns [MW91] which is especially important when the symmetry group does not act freely. Technically speaking, the action of H must be lifted from spacetime to act on the gauge group, and inequivalent lifts are available when the action of H has fixed points. For the time being, we will bypass these considerations in favour of studying some examples.

Translational reductions of 4dWZW

As well as studying reductions of the SDYM equations, we might also consider reductions of the other equivalent systems. One may partially solve the SDYM equations to recover the equations of motion of the 4dWZW model (also known as Yang's equation) or alternatively the equations of motion of the LMP model. These presentations break some of the conformal symmetry, so there will be fewer options for reduction when compared to the SDYM equations. Nonetheless, if one chooses a subgroup $\mathsf{H} \subset \mathsf{Conf}(\mathbb{R}^4)$ which leaves these equations of motion invariant, one may restrict to solutions which are also invariant under this action.

Let us start by considering reductions of Yang's equation which is the equation of motion of the 4dWZW model. In complex coordinates $\{u^1, u^2\}$ on spacetime, this equation is written as

$$\partial_{u^1}(g^{-1}\partial_{\bar{u}^1}g) + \partial_{u^2}(g^{-1}\partial_{\bar{u}^2}g) = 0 \ . \tag{2.26}$$

The simplest reductions are by translational symmetries, for example translations in u^2 and $\bar{u}^2$ which are generated by the vectors ∂_{u^2} and $\partial_{\bar{u}^2}$. We can use the coordinates $\{u^1, \bar{u}^1\}$ as coordinates on the quotient space $\mathbb{R}^4/\mathsf{H}$ as they are invariant under the action of the reduction vectors, and the reduced equations on this space are given by

$$\partial_{u^1}(g^{-1}\partial_{\bar{u}^1}g) = 0 \ . \tag{2.27}$$

These may be recognised as the equations of motion of the two-dimensional WZW model.

Alternatively, consider a different two-dimensional translational reduction generated by

$$X = \partial_{\bar{u}^1} - \partial_{u^2} \ , \qquad Y = \partial_{\bar{u}^2} - \partial_{u^1} \ . \tag{2.28}$$

We can parameterise the quotient space $\mathbb{R}^4/\mathsf{H}$ by the invariant combinations $x^+ = u^1 + \bar{u}^2$ and $x^- = \bar{u}^1 + u^2$, in terms of which the reduced equations are then written as [War85]

$$\partial_+(g^{-1}\partial_- g) + \partial_-(g^{-1}\partial_+ g) = 0 \ . \tag{2.29}$$

In this case, we find the equations of motion of the PCM. Introducing an additional parameter interpolating between these two reductions, we can recover the whole family of two-dimensional models known as the PCM plus WZ term.

As well as recovering the equations of motion for these systems, the reduction from SDYM also provides a Lax for these models. We should start with the Lax pair for SDYM and specialise to $A = g^{-1}\bar{\partial}g$ which is relevant to the 4dWZW model. The four-dimensional Lax pair for this gauge field configuration is given by

$$L = \partial_{\bar{u}^1} + g^{-1}\partial_{\bar{u}^1}g - \zeta\,\partial_{u^2} \ , \qquad M = \partial_{\bar{u}^2} + g^{-1}\partial_{\bar{u}^2}g + \zeta\,\partial_{u^1} \ . \tag{2.30}$$

If we apply the reduction which recovers the PCM and adopt coordinates $x^+ = u^1 + \bar{u}^2$ and

$x^- = \bar{u}^1 + u^2$, this Lax pair becomes

$$L = (1-\zeta)\,\partial_- + g^{-1}\partial_- g \ , \qquad M = (1+\zeta)\,\partial_+ + g^{-1}\partial_+ g \ . \tag{2.31}$$

To relate this to the usual Lax connection, we simply multiply both of these differential operators by an overall coefficient such that they take the form $(\partial_\pm + L_\pm)$. This does not change the systems of equations derived from their commutator, but brings them into the familiar form of a covariant derivative. Doing this, we find the usual components of the Lax connection,

$$L_\pm = \frac{g^{-1}\partial_\pm g}{1 \pm \zeta} \ . \tag{2.32}$$

Taking similar steps for the interpolating reductions recovers the Lax connection of the PCM plus WZ term.

Our discussion thus far has been at the level of equations of motion, and whilst this is an interesting method for deriving integrable systems of equations, a quantum treatment of these models also requires an action for the theory. Fortunately, recent developments have shown that we can perform a similar reduction at the level of the action [BS23]. Let us start with the action of the 4dWZW model,

$$S_{\text{4dWZW}}[g] = \frac{1}{2}\int_{\mathbb{R}^4} \operatorname{tr}\big(g^{-1}\mathrm{d}g \wedge \star g^{-1}\mathrm{d}g\big) + \int_{\mathbb{R}^4\times[0,1]} \omega \wedge \mathrm{WZ}[g] \ . \tag{2.33}$$

The 2-form $\omega = \mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 + \mathrm{d}u^2 \wedge \mathrm{d}\bar{u}^2$ is proportional to the Kähler form and breaks the spacetime rotational symmetry from $\mathsf{SO}(4)$ down to $\mathsf{U}(2)$. Nonetheless, the theory is still invariant under translations, and we will look to perform a reduction of this system by the vectors

$$X = \partial_{\bar{u}^1} - a\,\partial_{u^2} \ , \qquad Y = \partial_{\bar{u}^2} + b\,\partial_{u^1} \ . \tag{2.34}$$

Here, $a, b \in \mathbb{C}$ are parameters which allow us to interpolate between the PCM and WZW model reductions. First, we will demand that the group-valued field is invariant under the flow of these vector fields,

$$\mathcal{L}_X g = 0 \ , \qquad \mathcal{L}_Y g = 0 \ . \tag{2.35}$$

With this constraint imposed, the whole integrand appearing in the action is invariant under the action of H. In particular, contracting the 4-form integrand with X and Y gives a 2-form which is also invariant under H, and we will interpret this as the integrand for a theory on the quotient space $\mathbb{R}^4/\mathsf{H}$.

Starting with the kinetic term, we can write it in coordinates as

$$\int_{\mathbb{R}^4} \mathrm{d}^4x \ \operatorname{tr}\big(g^{-1}\partial_{u^1} g \cdot g^{-1}\partial_{\bar{u}^1} g + g^{-1}\partial_{u^2} g \cdot g^{-1}\partial_{\bar{u}^2} g\big) \ . \tag{2.36}$$

We will adopt the coordinates on the quotient space given by $x^+ = u^1 - b\,\bar{u}^2$ and $x^- = u^2 + a\,\bar{u}^1$ which are invariant under X and Y. After imposing the invariance condition on g, we can convert all of the derivatives into derivatives with respect to $x^\pm$. Then, contracting the volume form with X and Y gives the two-dimensional integral

$$(a-b) \int_\Sigma \mathrm{d}^2x \ \mathrm{tr}\big(g^{-1}\partial_+ g \cdot g^{-1}\partial_- g\big) \ . \tag{2.37}$$

This is a two-dimensional kinetic term for our integrable sigma model, where we are denoting the quotient space by $\Sigma \cong \mathbb{R}^4/\mathsf{H}$.

Turning to the WZ term, since the field g is invariant under X and Y, the contraction will only see the 2-form ω. The coefficient of the reduced WZ term is therefore given by

$$X \lrcorner\, Y \lrcorner\, \omega = a + b \ . \tag{2.38}$$

This can be interpreted [Cos21] as the Kähler volume of the 2-manifold generated by the flow of X and Y. Bringing these two terms together, we land on the action for the PCM plus WZ term,

$$S_{\mathrm{PCMWZ}}[g] = \frac{a-b}{2} \int_\Sigma \mathrm{tr}\big(g^{-1}\mathrm{d}g \wedge \star g^{-1}\mathrm{d}g\big) + (a+b) \int_{\widetilde{\Sigma}} \mathrm{WZ}[g] \ . \tag{2.39}$$

Here, we denote an extension of the 2-manifold by $\widetilde{\Sigma} = \Sigma \times [0,1]$. This whole family of two-dimensional sigma models can be recovered as a reduction of the 4dWZW model, and the coefficients appearing in the action are related to the choice of reduction vectors [BS23].

This procedure of demanding that the fundamental fields are invariant and then contracting the integrand with the reduction vectors is not especially familiar, so we will offer another perspective in the language of a Kaluza-Klein reduction. Let us assume that H is abelian and consider compactifying the integral curves generated by X and Y. Then, our spacetime looks like $\mathbb{R}^4 \cong \Sigma \times T^2$ where Σ is identified with the quotient space. Shrinking the radius of the torus, higher-order modes in the Fourier expansion will decouple until only the zero modes survive in the effective two-dimensional on Σ. These are precisely the fields which are invariant under the flow of X and Y and would satisfy the constraints we imposed above.

Other reductions

There are many interesting reductions of the SDYM equations and its descendants [MW91]. From the examples above, we see that one can vary the subgroup $\mathsf{H} \subset \mathsf{Conf}(\mathbb{R}^4)$ of the conformal group, the gauge group G, and the choice of gauge fixing for the SDYM connection. In addition to this, when the action of H has fixed points one can make different choices of lifts to its action on the gauge group, and there are other examples where the values of certain scalar fields can be fixed in the reduction data. We will not explore the full range of these reductions, but we will give two more examples which will be relevant to forthcoming sections.

Keeping with reductions of Yang's equation, we can consider an alternative subgroup H of the conformal group. In cylindrical coordinates on $\mathbb{R}^4$ defined by $u^1 = \rho\, e^{\mathrm{i}\phi}$ and $u^2 = y + \mathrm{i}\,\tau$, we will consider the two-dimensional subgroup generated by

$$X = \partial_\phi \ , \qquad Y = \partial_\tau \ . \tag{2.40}$$

One can check that X is part of the $\mathsf{U}(2) \subset \mathsf{SO}(4)$ subgroup which preserves the Kähler structure on $\mathbb{R}^4$ and is therefore a symmetry of Yang's equation. We impose that g is invariant under the flow of these vectors, and use $\{\rho, y\}$ as coordinates on the quotient space. The reduced equation of motion is

$$\partial_\rho\big(\rho\, g^{-1}\partial_\rho g\big) + \rho\, \partial_y\big(g^{-1}\partial_y g\big) = 0 \ . \tag{2.41}$$

This is known as the Ernst equation which also appears in the context of general relativity.

Consider Einstein's gravity in four dimensions and restrict to metrics which have two commuting isometries. If one takes such a metric and substitutes it back into Einstein's equations then they reduce to the Ernst equation given above. In this context, the field g lives in the coset $\mathsf{SL}(2,\mathbb{R})/\mathsf{SO}(2)$ and is identified with the metric components along the isometry directions. This relationship between the static, axisymmetric gravity equations and a static, axisymmetric reduction of the SDYM equations was noticed in [Wit79]. This observation inspired further work [War82] in which the machinery of twistor theory was used to generate solutions to the vacuum Einstein's equations.

We can also consider reductions of the LMP model. As a simple example, let us reduce by the two translation vector fields

$$X = \partial_{\bar{u}^1} - a\, \partial_{u^2} \ , \qquad Y = \partial_{\bar{u}^2} + b\, \partial_{u^1} \ . \tag{2.42}$$

We will demand that the fundamental field ϕ is invariant under these vectors and adopt the coordinates $x^+ = u^1 - b\,\bar{u}^2$ and $x^- = u^2 + a\,\bar{u}^1$ on the quotient space. The kinetic term reduces in a similar way to the 4dWZW model while the cubic term picks up a coefficient of (ab). This leads [BS23] to the two-dimensional action

$$S_{\text{2dIFT}}[\phi] = \int_\Sigma \mathrm{tr}\bigg(\frac{a-b}{2}\, \mathrm{d}\phi \wedge \star\mathrm{d}\phi + \frac{2ab}{3}\, \phi \cdot \mathrm{d}\phi \wedge \mathrm{d}\phi \bigg) \ . \tag{2.43}$$

This is the integrable model known as the pseudodual of the PCM [ZM78] which exhibits particle production [Nap80] in the perturbative scattering matrix. Whilst integrability in two-dimensions is often taken to be synonymous with 'no particle production', this is only true when all of the excitations are massive [HLT19b]. When the theory admits massless excitations, as is the case for the pseudodual, this argument no longer holds.

2.2 Geometry of twistor space

Twistor theory was originally introduced with the objective of unifying quantum mechanics and general relativity [Pen67; Pen68]. Heisenberg's uncertainty principle tells us that the point particle description of physical objects breaks down when considering quantum effects, and Einstein's general relativity shows that the structure of spacetime itself ultimately depends on the nature of its contents. This prompts the budding twistor theorist to abandon the classical description of spacetime in favour of a new geometric framework — twistor space. Of course, the existing description of spacetime has been incredibly successful, and such progress should not be lost in pursuit of further advancements. Much work has focused on incorporating the known physical models into the twistor space framework, but several difficulties have been encountered. On the other hand, twistor theory has offered new insight into certain areas of physics, especially in the realm of integrability.

There are two crucial differences between twistor space and the usual geometry of spacetime. The first difference is inspired by quantum mechanics, where the constituent wavefunctions are valued in the complex plane, rather taking real values. Now consider our theories of gravity, where the dynamical object is the geometry of spacetime itself. In the spirit of quantum mechanics, if one wishes to construct a theory of quantum gravity, one might expect to work with complex manifolds, rather than real ones. Such a theory should also come equipped with a mechanism for imposing reality conditions, in order to recover known results on the usual description of spacetime. Along these lines, twistor space is a complex manifold which is related to complexified spacetime, and there are a series of valid reality conditions.

The second crucial difference is the object that is taken to be fundamental. In the usual description of spacetime, points are taken to be fundamental and the toolkit of Riemannian geometry is built up from this foundation. By comparison, the primary elements of twistor space are light rays in Minkowski space (or null 2-planes in complexified spacetime). Each point in twistor space corresponds to an entire light ray in spacetime, an inherently extended and non-local object. Similarly, each point in spacetime corresponds to a complex curve embedded in twistor space. This non-local correspondence (known as the twistor correspondence) provides a very different perspective on spacetime physics. Indeed, many challenging problems on spacetime are amenable to a twistor space approach, whilst other key aspects of spacetime physics are hard to formulate on twistor space. See [ADM17] for a recent review of twistor theory, and [Ada18] for some introductory lecture notes on the subject.

In this section, we will explore the geometry of twistor space, the complex manifold underlying twistor theory. There are a few equivalent descriptions of twistor space which appear in the literature, and we will present each of these descriptions independently while showing how to move between them. When it is necessary to impose reality conditions, we will focus on Euclidean signature since this is the easiest example to understand. We will denote the twistor space of complexified spacetime by $\mathbb{PT}_{\mathbb{C}}$ and the various real slices with the subscripts $\mathbb{E}$ for Euclidean, $\mathbb{L}$ for Lorentzian, and $\mathbb{U}$ for ultrahyperbolic (or split signature).

It is important to highlight that this thesis is not really about twistor theory as a theory of quantum gravity, but rather as a tool for understanding integrability in other physical models. Our motivation stems from the observation that most known integrable systems arise from reductions of twistor space, and this is an area where the twistor description has been incredibly successful. To be more precise, given a certain structure on twistor space (a holomorphic vector bundle), there is a recipe (known as the Penrose-Ward transform) for constructing solutions to the self-dual Yang-Mills (SDYM) equations on spacetime. In turn, most known integrable systems arise as reductions of the SDYM equations, and bringing these ingredients together provides a derivation of these integrable models from twistor space. This will be the primary application of twistor space within this thesis, and the following section is presented with this in mind. In particular, we will focus on technical aspects of the geometry which are helpful for understanding the Penrose-Ward transform, whilst neglecting many of the other interesting developments in this field.

2.2.1 As the moduli space of complex structures

Euclidean twistor space is a three-dimensional complex manifold which we will denote by $\mathbb{PT}_{\mathbb{E}}$. As a real manifold (forgetting the complex structure) there is an isomorphism

$$\mathbb{PT}_{\mathbb{E}} \cong S^2 \times \mathbb{R}^4 \ . \tag{2.44}$$

The $\mathbb{R}^4$ factor can be identified with (Euclidean) spacetime, and the moduli space of complex structures on $\mathbb{R}^4$ may be identified with the S^2 factor. This six-dimensional real manifold becomes a three-dimensional complex manifold when equipped with a complex structure which varies as we move around the S^2. We will build up to this description of twistor space by starting from a discussion of complex structures on $\mathbb{R}^4$.

Firstly, we should recall some facts about equipping Riemannian manifolds with complex structures. Locally, any even-dimensional real manifold looks like $\mathbb{R}^{2n}$ and non-trivial aspects of the geometry are encoded in the transition functions between coordinate patches. Contractible manifolds such as $\mathbb{R}^4$ are particularly simple examples as they can be covered with a single coordinate patch. Complex manifolds have a very similar structure, except now each patch looks locally like $\mathbb{C}^n$ and the transition functions are required to be holomorphic. A complex structure is a recipe for turning a real manifold into a complex manifold: it tells you how to identify each copy of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and how to consistently glue them together.

In differential geometry, complex structures on a manifold M are often studied at the level of the tangent bundle TM. It turns out that a complex structure can be encoded by a linear map on each tangent space which we will denote by $J : \mathsf{TM} \to \mathsf{TM}$. The first condition on J is that it must satisfy $J^2 = -1$, in which case it is called an almost complex structure. Every even-dimensional real manifold can be equipped with an almost complex structure locally, so the important question is whether this structure extends globally. In order for J to define a complex structure (as opposed to an almost complex structure) it must satisfy an integrability condition,

$$N_J(X,Y) \equiv [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] = 0 \ , \tag{2.45}$$

for any two vector fields X and Y. The object N_J is known as the Nijenhuis tensor, and it measures a sort of curvature of the complex structure. If the Nijenhuis tensor for J vanishes, then it defines a (genuine) complex structure on M.

How does this relate to our intuitive notion of equipping a real manifold M with a set of holomorphic coordinates such that it locally looks like $\mathbb{C}^n$? Since the linear map $J : \mathsf{TM} \to \mathsf{TM}$ satisfies $J^2 = -1$, we know that its eigenvalues must be $\pm\mathrm{i}$. We can decompose each (complexified) tangent space into the eigenspaces of J and we will denote the corresponding subbundles of TM by $\mathsf{T}^{1,0}\mathsf{M}$ and $\mathsf{T}^{0,1}\mathsf{M}$,

$$\begin{aligned} J(X) &= +\mathrm{i}\,X \qquad \Longleftrightarrow \qquad X \in \mathsf{T}^{1,0}\mathsf{M} \ , \\ J(X) &= -\mathrm{i}\,X \qquad \Longleftrightarrow \qquad X \in \mathsf{T}^{0,1}\mathsf{M} \ . \end{aligned} \tag{2.46}$$

These subbundles are involutive (meaning they are closed under the Lie bracket) when the Nijenhuis tensor of J vanishes. One can then find local holomorphic coordinates $\{z^i\}$ on each patch such that J acts as

$$J\frac{\partial}{\partial z^i} = +\mathrm{i}\,\frac{\partial}{\partial z^i} \ , \qquad J\frac{\partial}{\partial \bar{z}^i} = -\mathrm{i}\,\frac{\partial}{\partial \bar{z}^i} \ . \tag{2.47}$$

The fact that these subbundles are involutive ensures that these local holomorphic coordinates can be patched together globally with holomorphic transition functions. This recovers the more intuitive definition of a complex manifold.

So far, we have equipped a smooth real manifold with a complex structure, but we have not taken into account any other structures on the manifold. Next, we will consider adding a complex structure to a Riemannian manifold, that is a smooth manifold with a Riemannian metric g. That additional ingredient here is that we require the complex structure to preserve the metric, meaning that

$$g(X,Y) = g(JX,JY) \ . \tag{2.48}$$

This ensures that the complex manifold comes equipped with a Hermitian metric h of which g is the real part. The imaginary part of h is $(1,1)$-form ω which is also preserved by the complex structure. In the case that this $(1,1)$-form is closed ($\mathrm{d}\omega = 0$) it defines a symplectic structure on the manifold. The combination of these three mutually compatible structures (a complex structure, a Riemannian metric, and a symplectic structure) defines a Kähler manifold, and in this case ω is known as the Kähler form.

Turning to the example at hand, there are many complex structures which are compatible with the Euclidean metric on $\mathbb{R}^4$. In fact, this is an example of a hyperkähler manifold: a Riemannian manifold with three complex structures I, J, K which are all Kähler with respect to the metric g and which satisfy the quaternionic relation $IJK = -1$. Every hyperkähler manifold has a two-sphere of complex structures which are Kähler with respect to g. They are defined by

$$a\,I + b\,J + c\,K\ , \qquad a^2 + b^2 + c^2 = 1\ . \tag{2.49}$$

This two-sphere of complex structures on $\mathbb{R}^4$ is precisely the two-sphere which appears in the isomorphism of real manifolds $\mathbb{PT}_{\mathbb{E}} \cong S^2 \times \mathbb{R}^4$.

We are now ready to describe the complex structure on twistor space. We identify the two-sphere S^2 with the Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ which provides it with a complex structure and an associated holomorphic coordinate ζ. One may find an explicit map[1] between this holomorphic coordinate ζ and the coordinates $\{a, b, c\}$ but it is not especially illuminating. Then, we define the complex structure on $\mathbb{R}^4$ by providing a set of holomorphic coordinates $\{v^1, v^2\}$ which we write in terms of the usual real coordinates $\{x^\mu\}$ as

$$v^1 = x^1 + \mathrm{i}\,x^2 - \zeta\,(x^3 - \mathrm{i}\,x^4)\ , \qquad v^2 = x^3 + \mathrm{i}\,x^4 + \zeta\,(x^1 - \mathrm{i}\,x^2)\ . \tag{2.50}$$

Thinking of $\mathbb{CP}^1$ as the moduli space of complex structures, we see that varying ζ changes the holomorphic coordinates on $\mathbb{R}^4$, indicating a change in the complex structure. In combination with the complex structure on S^2, this makes twistor space $\mathbb{PT}_{\mathbb{E}}$ a three-dimensional complex manifold. Notably, it is not isomorphic to $\mathbb{CP}^1 \times \mathbb{C}^2$ as a complex manifold — one can compute a certain topological class which is non-trivial for $\mathbb{PT}_{\mathbb{E}}$ but trivial for the product manifold $\mathbb{CP}^1 \times \mathbb{C}^2$ (for example, see [Ati79] for a discussion of this point).

It is often helpful to work with fixed complex coordinates on $\mathbb{R}^4$ defined by $u^1 = x^1 + \mathrm{i}\,x^2$ and $u^2 = x^3 + \mathrm{i}\,x^4$. It is important to emphasise that these are not holomorphic with respect to the complex structure on twistor space, though they do coincide with the holomorphic coordinates on the submanifold $\zeta = 0$. In terms of these fixed complex coordinates, the holomorphic coordinates on twistor space are written as

$$v^1 = u^1 - \zeta\,\bar{u}^2\ , \qquad v^2 = u^2 + \zeta\,\bar{u}^1\ . \tag{2.51}$$

This relationship between the holomorphic coordinates on twistor space and the coordinates on spacetime is known as the incidence relations.

2.2.2 As the total space of a holomorphic vector bundle

Twistor space $\mathbb{PT}_{\mathbb{C}}$ is the total space of the holomorphic vector bundle

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1\ . \tag{2.52}$$

We will break down this statement with a brief review of holomorphic line bundles over $\mathbb{CP}^1$.

The Riemann sphere $\mathbb{CP}^1$ is also known as the complex projective space as it may be identified

[1] The coordinates $\{a, b, c\}$ and ζ are related by $(a, b, c) = (1 + \zeta\bar{\zeta})^{-1}\big(\zeta + \bar{\zeta}, -\mathrm{i}(\zeta - \bar{\zeta}), -1 + \zeta\bar{\zeta}\big)$.

with the space of lines through the origin in $\mathbb{C}^2$. This may be parameterised by non-zero points in $\mathbb{C}^2$ modulo the equivalence relation of scaling,

$$\mathbb{CP}^1 = \big\{(\pi_1, \pi_2) \in \mathbb{C}^2 \mid (\pi_1, \pi_2) \neq (0,0)\big\}/\sim \,, \qquad \pi_a \sim r\,\pi_a \quad r \in \mathbb{C}^* \,. \tag{2.53}$$

These equivalence classes are often denoted by $[\pi_1 : \pi_2]$ to emphasise that they are defined up to multiplication by a non-zero constant. The coordinates π_a are known as homogeneous coordinates, in contrast to the inhomogeneous coordinates defined on each patch. If we cover $\mathbb{CP}^1$ with two patches defined by $U_1 = \{\pi_1 \neq 0\}$ and $U_2 = \{\pi_2 \neq 0\}$, then we can define inhomogeneous coordinates on these patches by

$$\zeta = \frac{\pi_2}{\pi_1} \,, \qquad \tilde{\zeta} = \frac{\pi_1}{\pi_2} \,. \tag{2.54}$$

These coordinates are related to one another on the overlap by $\zeta = \tilde{\zeta}^{-1}$.

A holomorphic line bundle over $\mathbb{CP}^1$ is a fibre bundle whose fibres are isomorphic to the complex line $\mathbb{C}$. The total space of this bundle, which we will denote by E, is therefore locally isomorphic to $\mathbb{CP}^1 \times \mathbb{C}$, though this may not hold globally. If it does hold globally, then $\mathsf{E} \to \mathbb{CP}^1$ is the trivial bundle which we will denote by $\mathcal{O}(0)$. To show that a bundle is trivial, one may provide a nowhere-vanishing global section. A section of a fibre bundle may be specified by a map from each patch to the fibre,

$$s_1 : U_1 \to \mathbb{C} \,, \qquad s_2 : U_2 \to \mathbb{C} \,. \tag{2.55}$$

These are local complex functions, and the bundle tells us how to glue them together on the overlap. In particular, a holomorphic line bundle comes with a transition function φ_{12} which relates the local sections by

$$s_1 = \varphi_{12} \cdot s_2 \,. \tag{2.56}$$

The trivial bundle is defined by the transition function $\varphi_{12} = 1$. In this case, the local sections must satisfy $s_1 = s_2$ and any non-zero constant provides an example of a nowhere-vanishing global section.

There are also non-trivial holomorphic line bundles over $\mathbb{CP}^1$. In fact, its identification with the space of lines in $\mathbb{C}^2$ provides a canonical example of a non-trivial line bundle, known as the tautological bundle. We can view the equivalence relation as a projection map from the total space $\mathsf{E} = \mathbb{C}^2 \setminus \{(0,0)\}$ to the base manifold $\mathbb{CP}^1$,

$$\sim \, : \mathsf{E} \to \mathbb{CP}^1 \,. \tag{2.57}$$

This projection maps an element of $\mathbb{C}^2$ to the corresponding equivalence class in $\mathbb{CP}^1$. We can trivialise this bundle by working with inhomogeneous coordinates on the base and introducing a coordinate for the fibre over each patch. Let us parameterise this trivialisation by

$$\begin{aligned} &U_1 \times \mathbb{C} \to \mathsf{E} \,, && U_2 \times \mathbb{C} \to \mathsf{E} \,, \\ &(\zeta, \pi_1) \mapsto (\pi_1, \zeta\, \pi_1) \,, && (\tilde{\zeta}, \pi_2) \mapsto (\tilde{\zeta}\, \pi_2, \pi_2) \,. \end{aligned} \tag{2.58}$$

Sections of this bundle are maps from $\mathbb{CP}^1$ to E, which we may define locally by two maps from the base to the fibre. On the patch U_1, a local section is specified uniquely by the value of the fibre coordinate $\pi_1 \in \mathbb{C}$ since the value of π_2 can be determined by the relation $\pi_2 = \zeta\, \pi_1$. A local section on the other patch is similarly determined by the value of $\pi_2 \in \mathbb{C}$, so a global section is defined by two maps $\pi_1 = s_1(\zeta)$ and $\pi_2 = s_2(\tilde{\zeta})$. By definition, these sections must be related on the overlap by the transition function

$$s_1 = \varphi_{12} \cdot s_2 \,, \qquad \varphi_{12} = \zeta^{-1} \,. \tag{2.59}$$

This bundle is denoted by $\mathcal{O}(-1)$ and, in general, a holomorphic line bundle over $\mathbb{CP}^1$ with transition function $\varphi_{12} = \zeta^n$ is denoted by $\mathcal{O}(n)$.

It is instructive to consider the global sections of these bundles. Locally, a (holomorphic[2]) section of these bundles is given by some polynomial in the inhomogeneous coordinates,

$$s_1 = a_0 + a_1\, \zeta + a_2\, \zeta^2 + \dots \,, \qquad s_2 = b_0 + b_1\, \tilde{\zeta} + b_2\, \tilde{\zeta}^2 + \dots \,. \tag{2.60}$$

In particular, we are restricting to positive powers so that s_1 is finite near $\zeta = 0$ and s_2 is finite near $\tilde{\zeta} = 0$. For the bundle $\mathcal{O}(n)$, these must be related on the overlap by

$$a_0 + a_1\, \zeta + a_2\, \zeta^2 + \dots = \zeta^n \left(b_0 + b_1\, \zeta^{-1} + b_2\, \zeta^{-2} + \dots\right) \,. \tag{2.61}$$

Solutions to this equation may not exist, meaning that there are no global sections of the bundle. For example, if $n < 0$ then it is not possible to match the strictly positive powers on the left-hand side with the strictly negative powers on the right-hand side. In the case of the trivial bundle with $n = 0$, there are global sections defined by $a_0 = b_0$ with the other coefficients vanishing. Choosing $a_0 \neq 0$ provides a nowhere-vanishing global section, which demonstrates that the bundle is indeed trivial. Then, for $n > 0$ all polynomials up to degree n are valid global sections, but they will all vanish somewhere on $\mathbb{CP}^1$ because the bundles are non-trivial. In summary, if we denote by $\Gamma(\mathsf{E})$ the global sections of a bundle E, we have

$$\Gamma\big(\mathcal{O}(n)\big) = 0 \quad \text{for} \quad n < 0 \,, \qquad \Gamma\big(\mathcal{O}(n)\big) \cong \mathbb{C}^{n+1} \quad \text{for} \quad n \geq 0 \,. \tag{2.62}$$

These complex numbers represent the coefficients of a general polynomial of degree n.

[2] In much the same way that the adjective "smooth" is often taken for granted in the context of real differential geometry, the adjective "holomorphic" is often implicit in discussions about complex geometry.

We claimed that twistor space $\mathbb{PT}_{\mathbb{C}}$ was the total space of the holomorphic vector bundle

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1 \ . \tag{2.63}$$

Let us adopt the inhomogeneous coordinate ζ on a local patch of $\mathbb{CP}^1$. Then, as coordinates on the fibre, we need two local sections of $\mathcal{O}(1)$. In order for these sections to be globally well-defined, they must be polynomials of at most degree 1 in ζ. We can write these two local sections as

$$v^1 = u^1 - \zeta\, \tilde{u}^2 \ , \qquad v^2 = u^2 + \zeta\, \tilde{u}^1 \ . \tag{2.64}$$

These are determined by four complex numbers which parameterise complexified spacetime $\mathbb{C}^4$. This complexified spacetime is equipped with the quadratic form

$$\mathrm{d}s^2 = \mathrm{d}u^1 \mathrm{d}\tilde{u}^1 + \mathrm{d}u^2 \mathrm{d}\tilde{u}^2 \ . \tag{2.65}$$

We can then pick out the Euclidean real slice by imposing the reality conditions $\tilde{u}^1 = \bar{u}^1$ and $\tilde{u}^2 = \bar{u}^2$. Then, these two local sections parameterising the fibres of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ match the holomorphic coordinates on $\mathbb{R}^4$ we encountered earlier.

2.2.3 As the set of anti-self-dual null 2-planes

Twistor space $\mathbb{PT}_{\mathbb{C}}$ is the set of anti-self-dual totally null 2-planes in complexified spacetime $\mathbb{C}^4$. From the perspective of causal structures in Minkowski spacetime, null 2-planes play a distinguished role as the light cones emanating from a given point. These separate those spacetime events which may be causally connected to the original point from those which may not. One might then consider the set of all null 2-planes in Minkowski space, or more generally (and more in the spirit of twistor theory) in complexified spacetime.

We can parameterise complexified spacetime $\mathbb{C}^4$ by four complex coordinates $\{x^\mu\}$. It is also helpful to work with double null coordinates $\{u^1, \tilde{u}^1, u^2, \tilde{u}^2\}$, in terms of which the metric and volume form are written as

$$\mathrm{d}s^2 = \mathrm{d}u^1 \mathrm{d}\tilde{u}^1 + \mathrm{d}u^2 \mathrm{d}\tilde{u}^2 \ , \qquad \mathrm{d}^4x = \mathrm{d}u^1 \wedge \mathrm{d}\tilde{u}^1 \wedge \mathrm{d}u^2 \wedge \mathrm{d}\tilde{u}^2 \ . \tag{2.66}$$

We call a 2-plane (totally) null if $\mathrm{d}s^2(X, Y) = 0$ for every pair of tangent vectors X and Y. For each null 2-plane N we can construct a tangent bivector $n = X \wedge Y$, with components $n^{\mu\nu} = X^{[\mu} Y^{\nu]}$, where X and Y are any two independent tangent vectors. Varying the choice of X and Y only changes the tangent bivector n by an overall constant, and the tangent space of the null 2-plane N is completely determined by the tangent bivector. This means that there is a one-to-one correspondence between null 2-planes that pass through the origin and tangent bivectors. To specify a general null 2-plane, we also need some additional coordinates to specify its deviation from the origin.

One can show that if N is a null 2-plane, then the 2-form $n_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is either self-dual or anti-self-dual with respect to the metric. We will refer to self-dual null 2-planes as α-planes, and anti-self-dual null 2-planes as β-planes. The space of anti-self-dual bivectors is spanned by

$$\partial_{u^1} \wedge \partial_{u^2}\ , \qquad \partial_{\tilde{u}^1} \wedge \partial_{\tilde{u}^2}\ , \qquad \partial_{u^1} \wedge \partial_{\tilde{u}^1} + \partial_{u^2} \wedge \partial_{\tilde{u}^2}\ . \tag{2.67}$$

The tangent bivector of any β-plane is necessarily in the span of these bivectors, so it is possible to write the tangent bivector of any β-plane as

$$n = \ell \wedge m\ , \qquad \ell = \partial_{\tilde{u}^1} - \zeta\,\partial_{u^2}\ , \qquad m = \partial_{\tilde{u}^2} + \zeta\,\partial_{u^1}\ , \tag{2.68}$$

for some $\zeta \in \mathbb{CP}^1$ (where the point $\zeta = \infty$ should be understood as $n = \partial_{u^1} \wedge \partial_{u^2}$). This provides a one-to-one correspondence between $\zeta \in \mathbb{CP}^1$ and β-planes through the origin. A generic β-plane, which does not necessarily pass through the origin, may be specified by including two additional coordinates $v^1 = u^1 - \zeta\,\tilde{u}^2$ and $v^2 = u^2 + \zeta\,\tilde{u}^1$ which are constant over the 2-plane. These three complex coordinates $\{\zeta, v^1, v^2\}$ parameterise the set of anti-self-dual null 2-planes in complexified spacetime. This set may be identified with the complex three-dimensional manifold known as twistor space $\mathbb{PT}_{\mathbb{C}}$.

A point in twistor space (given by three coordinates $\{\zeta, v^1, v^2\}$) corresponds to an anti-self-dual null 2-plane in complexified spacetime. We can also ask this question in reverse: what does a point in complexified spacetime (given by four coordinates $\{u^1, \tilde{u}^1, u^2, \tilde{u}^2\}$) correspond to in twistor space? Varying the remaining parameter ζ, we traverse a $\mathbb{CP}^1$ submanifold of $\mathbb{PT}_{\mathbb{C}}$. In other words, the coordinates $\{u^1, \tilde{u}^1, u^2, \tilde{u}^2\}$ specify an embedding of $\mathbb{CP}^1$ into $\mathbb{PT}_{\mathbb{C}}$ which is parameterised by $\zeta \in \mathbb{CP}^1$. The precise statement is that a point in complexified spacetime corresponds to a linearly holomorphically embedded $\mathbb{CP}^1$ in twistor space.

The relationship between spacetime and twistor space is captured by the correspondence space $\mathbb{CP}^1 \times \mathbb{C}^4$. This manifold is parameterised by the coordinates $\{\zeta, x^\mu\}$ and is related to the twistor correspondence by the diagram

$$\begin{array}{ccccc} & & \mathbb{CP}^1 \times \mathbb{C}^4 & & \\ & \swarrow^{p} & & \searrow^{q} & \\ \mathbb{PT}_{\mathbb{C}} & & & & \mathbb{C}^4 \end{array}$$

The two projection maps in this diagram are defined by

$$p : (\zeta, x^\mu) \mapsto (\zeta, v^1, v^2)\ , \qquad q : (\zeta, x^\mu) \mapsto x^\mu\ . \tag{2.69}$$

Following this diagram, a point $x^\mu \in \mathbb{C}^4$ in complexified spacetime lifts to a $\mathbb{CP}^1$ in the correspondence space $q^{-1}(x^\mu) \subset \mathbb{CP}^1 \times \mathbb{C}^4$ which is then embedded into twistor space by the map $p : \mathbb{CP}^1 \times \mathbb{C}^4 \to \mathbb{PT}_{\mathbb{C}}$. In the other direction, the correspondence space is a five-dimensional complex manifold, whilst twistor space is a three-dimensional complex manifold. It is therefore reasonable to assert that a point in twistor space lifts to a 2-plane in the correspondence space which then projects down to a 2-plane in complexified spacetime.

This description also makes it easier to understand the twistor spaces associated with different real slices of complexified spacetime. The metric on $\mathbb{C}^4$ is written in terms of $\{x^\mu\}$ as

$$\mathrm{d}s^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 + (\mathrm{d}x^4)^2 \ . \tag{2.70}$$

When restricting to a real slice, different signatures of metric correspond to different reality conditions on the coordinates. For example, Euclidean signature corresponds to all of these coordinates being real, whilst Lorentzian signature may be realised by taking one coordinate to be purely imaginary with the other three coordinates real.

Each of these real slices defines a subspace of $\mathbb{C}^4$, and we will denote these by $\mathbb{E}$ for Euclidean, $\mathbb{L}$ for Lorentzian, and $\mathbb{U}$ for ultrahyperbolic (or split) signature. These subspaces can be lifted to subspaces of the correspondence space by considering their preimage under the map $q : \mathbb{CP}^1 \times \mathbb{C}^4 \to \mathbb{C}^4$. The associated twistor spaces are defined to be the image of these subspaces under the projection map $p : \mathbb{CP}^1 \times \mathbb{C}^4 \to \mathbb{PT}_{\mathbb{C}}$,

$$\mathbb{PT}_{\mathbb{E}} = p(\mathbb{CP}^1 \times \mathbb{E}) \ , \qquad \mathbb{PT}_{\mathbb{L}} = p(\mathbb{CP}^1 \times \mathbb{L}) \ , \qquad \mathbb{PT}_{\mathbb{U}} = p(\mathbb{CP}^1 \times \mathbb{U}) \ . \tag{2.71}$$

These fixed signature twistor spaces are all necessarily subspaces of the complexified twistor space $\mathbb{PT}_{\mathbb{C}}$, but differ from one another in important ways.

In the case of Euclidean signature, it turns out that the restriction of the projection map p to the subspace $\mathbb{CP}^1 \times \mathbb{E}$ is invertible. This provides an isomorphism between Euclidean twistor space $\mathbb{PT}_{\mathbb{E}}$ and $\mathbb{CP}^1 \times \mathbb{E}$ as real manifolds,

$$\mathbb{PT}_{\mathbb{E}} \cong S^2 \times \mathbb{R}^4 \ . \tag{2.72}$$

In particular, this allows one to work with either the holomorphic coordinates $\{\zeta, v^1, v^2\}$ on twistor space, or the correspondence space coordinates $\{\zeta, x^\mu\}$. Whilst the former privileges the complex structure on twistor space, the latter can be useful for making the relationship to spacetime manifest.

This isomorphism does not hold for the Lorentzian or split signature twistor spaces, which are actually lower-dimensional manifolds. Lorentzian twistor space is a real five-dimensional manifold which is often denoted by $\mathbb{PN}$ and referred to as the space of null twistors. Geometrically, this manifold may be identified with the space of real null geodesics in Minkowski space. Ultrahyperbolic twistor space is a real three-dimensional manifold which (as well as being a subspace of $\mathbb{PT}_{\mathbb{C}}$) is an open subspace of $\mathbb{RP}^3$. Points in this twistor space correspond to β-planes which lie entirely in the subspace $\mathbb{U} \subset \mathbb{C}^4$ and have real tangent bivectors.

2.2.4 Homogeneous coordinates on projective space

There are (at least) two descriptions of the non-contractible manifold $\mathbb{CP}^1$ which are useful in the context of twistor theory. Since this manifold is topologically non-trivial, it cannot be covered by one patch isomorphic to $\mathbb{C}$. On the other hand, thinking of $\mathbb{CP}^1$ as the Riemann sphere, we can define two open subsets by excluding either the north or south pole respectively. These patches cover $\mathbb{CP}^1$ and we can adopt holomorphic coordinates ζ on the southern patch and $\tilde{\zeta}$ on the northern patch. These coordinates are related on the overlap by $\zeta = \tilde{\zeta}^{-1}$ and are known as inhomogeneous coordinates.

It is often convenient to work in inhomogeneous coordinates, especially when doing local calculations. The standard tools of complex analysis are readily available, and this treatment of differential geometry is likely more familiar. The shortcomings of inhomogeneous coordinates mostly appear in the context of global properties. When asking if a given object is well-defined, if a function is holomorphic, or about the poles of a meromorphic 1-form, one must take care to consider what is happening in the other patch.

These shortcomings are certainly surmountable, and it possible to perform many calculations in inhomogeneous coordinates. Alternatively, there is another description of $\mathbb{CP}^1$ which is particularly well-adapted for questions about global properties. Whilst it is not possible to cover $\mathbb{CP}^1$ with a single patch isomorphic to $\mathbb{C}$, we can embed this manifold into $\mathbb{C}^2$ and work with coordinates on this larger space. The advantage of this approach is that we can traverse the whole $\mathbb{CP}^1$ manifold by varying the $\mathbb{C}^2$ coordinates, removing the significance of the north and south poles. However, this approach also comes with its own drawbacks. A generic object on $\mathbb{C}^2$ will not correspond to a well-defined object on $\mathbb{CP}^1$, so constraints must be imposed to ensure that things consistently descend from $\mathbb{C}^2$ to $\mathbb{CP}^1$. Atul Sharma's thesis [Sha22] was a useful resource in preparing this section and the next.

We can think of $\mathbb{CP}^1$ as the non-zero points in $\mathbb{C}^2$ modulo the equivalence relation of scaling,

$$\mathbb{CP}^1 = \big\{(\pi_1, \pi_2) \in \mathbb{C}^2 \mid (\pi_1, \pi_2) \neq (0,0)\big\}/\sim \,, \qquad \pi_a \sim r\,\pi_a \quad r \in \mathbb{C}^* \,. \tag{2.73}$$

In this context, the coordinates $\pi_a = (\pi_1, \pi_2)$ on $\mathbb{C}^2$ are known as homogeneous coordinates. Whilst there appear to be two degrees of freedom in these coordinates, they are only defined up to a rescaling, so it seems plausible that they might actually contain just one degree of freedom. For example, every class in the subspace $\pi_1 \neq 0$ has a representative of the form $(\pi_1, \pi_2) \sim (1, \zeta)$. Similarly, one can find a representative $(\pi_1, \pi_2) \sim (\tilde{\zeta}, 1)$ whenever $\pi_2 \neq 0$. On the intersection of these two subspaces, the equivalence of these representatives $(1, \zeta) \sim (\tilde{\zeta}, 1)$ implies the overlap condition $\zeta = \tilde{\zeta}^{-1}$. This recovers the description of $\mathbb{CP}^1$ in terms of inhomogeneous coordinates.

The space of functions on $\mathbb{C}^2$ is much larger than the space of functions on $\mathbb{CP}^1$. In order for a function on $\mathbb{C}^2$ to descend to a well-defined function on $\mathbb{CP}^1$, it must be single-valued on each equivalence class. This means that it must obey $f(\pi_1, \pi_2) = f(r\,\pi_1, r\,\pi_2)$ for any non-zero constant $r \in \mathbb{C}^*$. We will refer to the degree of a homogeneous polynomial in π_a as its weight, and this condition implies that only those with weight zero will descend to functions on $\mathbb{CP}^1$. If we also require such a function to be holomorphic, all that remains are constant functions (as implied by Liouville's theorem). There are also rational functions which descend to $\mathbb{CP}^1$, for example

$$\zeta = \frac{\pi_2}{\pi_1} \ , \qquad \tilde{\zeta} = \frac{\pi_1}{\pi_2} \ , \qquad \frac{\pi_2 - \pi_1}{\pi_1 + \pi_2} \ . \tag{2.74}$$

These are meromorphic functions meaning they are holomorphic everywhere except at their poles.

To phrase this constraint more geometrically, we can introduce the holomorphic and anti-holomorphic Euler vector fields on $\mathbb{C}^2$,

$$D = \pi_1 \frac{\partial}{\partial \pi_1} + \pi_2 \frac{\partial}{\partial \pi_2} \ , \qquad \bar{D} = \bar{\pi}_1 \frac{\partial}{\partial \bar{\pi}_1} + \bar{\pi}_2 \frac{\partial}{\partial \bar{\pi}_2} \ . \tag{2.75}$$

These vector fields generate the scaling action on $\mathbb{C}^2$ and therefore preserve the equivalence classes defining elements of $\mathbb{CP}^1$. If we think of the equivalence relation as defining a projection from the total space $\mathsf{E} = \mathbb{C}^2 \setminus \{(0,0)\}$ to the base manifold $\mathbb{CP}^1$, then the Euler vector fields span the tangent space to the fibres of this line bundle. In order for an arbitrary function on E to descend down to the base manifold, it must be invariant under translations along the fibres,

$$D(f) = 0 \ , \qquad \bar{D}(f) = 0 \ . \tag{2.76}$$

This is equivalent to the requirement that $f(\pi_1, \pi_2)$ is a polynomial of weight zero.

In addition to functions on $\mathbb{CP}^1$, we are also interested in sections of the holomorphic line bundles $\mathcal{O}(n) \to \mathbb{CP}^1$. Let us see what these look like in homogeneous coordinates. Recall that a section of $\mathcal{O}(n)$ may be specified by a polynomial in ζ of at most degree n when $n \geq 0$. The local section on the other patch is then uniquely determined by the overlap condition,

$$s_1 = \sum_{i=0}^{n} a_i \, \zeta^i \ , \qquad s_1 = \zeta^n \, s_2 \ . \tag{2.77}$$

In homogeneous coordinates, the overlap condition reads $\pi_1^n \, s_1 = \pi_2^n \, s_2$. Since the rational function $\zeta = \pi_2/\pi_1$ is weight zero, the expression $\pi_1^n \, s_1$ is a weight n homogeneous polynomial in π_a. Explicitly, it is written in homogeneous coordinates as

$$\pi_1^n \, s_1 = \sum_{i=0}^{n} a_i \, \pi_1^{n-i} \, \pi_2^i \ . \tag{2.78}$$

This is actually a generic homogeneous polynomial in π_a of degree n, meaning that we have a one-to-one map between weight n functions of π_a and sections of the line bundle $\mathcal{O}(n)$. Notably, weight n functions of π_a are not invariant under the action of the Euler vector fields, but satisfy the condition

$$D(f) = n \cdot f \ , \qquad \bar{D}(f) = 0 \ . \tag{2.79}$$

A function on $\mathbb{C}^2$ satisfying this condition descends to a section of the line bundle $\mathcal{O}(n)$ over $\mathbb{CP}^1$.

Moving on from functions, we can consider differential forms on $\mathbb{CP}^1$ in homogeneous coordinates. A natural basis for the 1-forms on $\mathbb{C}^2$ is given by the exterior derivatives of the coordinates $\mathrm{d}\pi_a$ and $\mathrm{d}\bar{\pi}_a$. Unfortunately, none of these 1-forms can descend to $\mathbb{CP}^1$ because they all have legs along the fibre directions. This can be seen by computing their contractions with the Euler vector fields which are all non-vanishing. Instead, let us introduce the basis of 1-forms given by

$$\begin{aligned} \langle \pi \mathrm{d}\pi \rangle &= \pi_1 \mathrm{d}\pi_2 - \pi_2 \mathrm{d}\pi_1 \ , & \langle \hat{\pi} \mathrm{d}\pi \rangle &= -\bar{\pi}_1 \mathrm{d}\pi_1 - \bar{\pi}_2 \mathrm{d}\pi_2 \ , \\ \langle \pi \mathrm{d}\hat{\pi} \rangle &= \pi_1 \mathrm{d}\bar{\pi}_1 + \pi_2 \mathrm{d}\bar{\pi}_2 \ , & \langle \hat{\pi} \mathrm{d}\hat{\pi} \rangle &= \bar{\pi}_1 \mathrm{d}\bar{\pi}_2 - \bar{\pi}_2 \mathrm{d}\bar{\pi}_1 \ . \end{aligned} \tag{2.80}$$

The left-hand sides are written in a spinor notation we will introduce later, but for the time being we can simply understand them as notation representing the expressions on the right-hand side.

Explicitly contracting these 1-forms with the Euler vector fields, one finds that $\langle \hat{\pi} \mathrm{d}\pi \rangle$ and $\langle \pi \mathrm{d}\hat{\pi} \rangle$ point along the fibres, whilst the other two 1-forms satisfy

$$D \lrcorner \, \omega = 0 \ , \qquad \bar{D} \lrcorner \, \omega = 0 \ . \tag{2.81}$$

Differential forms satisfying this condition are referred to as horizontal to indicate that they have no legs along the vertical (fibre) directions. If such a differential form is also invariant under the Euler vector fields, then it is referred to as basic. The invariance condition may be written as

$$\mathcal{L}_D(\omega) = 0 \ , \qquad \mathcal{L}_{\bar{D}}(\omega) = 0 \ . \tag{2.82}$$

Basic 1-forms are in one-to-one correspondence with 1-forms on the base manifold, so these are the 1-forms on $\mathbb{C}^2$ which descend to 1-forms on $\mathbb{CP}^1$.

It is possible to construct 1-forms on $\mathbb{C}^2$ which are both horizontal and invariant, meaning they are basic. For example, consider the meromorphic 1-form $\langle \pi \mathrm{d}\pi \rangle / \pi_1^2$ which has a second order pole at $\pi_1 = 0$. This 1-form is invariant under the action of the Euler vector fields, in essence because it is weight zero in π_a. On the other hand, we have been forced to single out a point in $\mathbb{CP}^1$ where the 1-form has a pole. Another option, which has the virtue of respecting the symmetries of $\mathbb{CP}^1$, is to work with differential forms which are valued in the line bundles $\mathcal{O}(n)$. Rather than being invariant under the Euler vector fields, $\mathcal{O}(n)$-valued differential forms satisfy

$$\mathcal{L}_D(\omega) = n \cdot \omega \ , \qquad \mathcal{L}_{\bar{D}}(\omega) = 0 \ . \tag{2.83}$$

This coincides with the condition on sections of $\mathcal{O}(n)$ when ω is an $\mathcal{O}(n)$-valued 0-form.

We will adopt the basis of $\mathcal{O}(n)$-valued 1-forms on $\mathbb{CP}^1$ given by

$$e^0 = \langle \pi \mathrm{d}\pi \rangle \in \Omega^{1,0}(\mathbb{CP}^1) \otimes \mathcal{O}(2) \ , \qquad \bar{e}^0 = \frac{\langle \hat{\pi} \mathrm{d}\hat{\pi} \rangle}{\langle \pi \hat{\pi} \rangle^2} \in \Omega^{0,1}(\mathbb{CP}^1) \otimes \mathcal{O}(-2) \ . \tag{2.84}$$

In the second expression, we have introduced some notation for the norm of π_a which is defined by $\langle\pi\hat{\pi}\rangle = \pi_1\bar{\pi}_1 + \pi_2\bar{\pi}_2$. The inclusion of this factor ensures that $\bar{e}^0$ is weight zero in $\bar{\pi}_a$ so that we only have to worry about the action of the holomorphic vector field D and not the anti-holomorphic $\bar{D}$. We will generically take the approach of including judicious factors of $\langle\pi\hat{\pi}\rangle$ so that all of our expressions are weight zero in $\bar{\pi}_a$.

A similar discussion holds for vector fields on $\mathbb{C}^2$ which descend to vector fields on $\mathbb{CP}^1$. In this case, the analog of the horizontal condition is that we consider vector fields on $\mathbb{C}^2$ modulo D and $\bar{D}$. This is justified so long as we are only acting on weight zero objects, as these are required to be invariant under D and $\bar{D}$. In addition, we impose the invariance conditions

$$\mathcal{L}_D(V) \equiv [D, V] = 0 \ , \qquad \mathcal{L}_{\bar{D}}(V) \equiv [\bar{D}, V] = 0 \ . \tag{2.85}$$

These conditions are only required to hold modulo D and $\bar{D}$, and they ensure that the vector field descends to a vector field on $\mathbb{CP}^1$. In practice, we will work with $\mathcal{O}(n)$-valued vector fields which satisfy

$$\mathcal{L}_D(V) \equiv [D, V] = n \cdot V \ , \qquad \mathcal{L}_{\bar{D}}(V) \equiv [\bar{D}, V] = 0 \ . \tag{2.86}$$

Explicitly, the $\mathcal{O}(n)$-valued vector fields dual to the 1-forms introduced above are given by

$$\begin{aligned} \partial_0 &= \frac{1}{\langle\pi\hat{\pi}\rangle}\left(\bar{\pi}_1\frac{\partial}{\partial\pi_2} - \bar{\pi}_2\frac{\partial}{\partial\pi_1}\right) \in \mathfrak{X}^{1,0}(\mathbb{CP}^1)\otimes\mathcal{O}(-2) \ , \qquad && \partial_0 \lrcorner\, e^0 = 1 \\ \bar{\partial}_0 &= \langle\pi\hat{\pi}\rangle\left(\pi_1\frac{\partial}{\partial\bar{\pi}_2} - \pi_2\frac{\partial}{\partial\bar{\pi}_1}\right) \in \mathfrak{X}^{0,1}(\mathbb{CP}^1)\otimes\mathcal{O}(2) \ , \qquad && \bar{\partial}_0 \lrcorner\, \bar{e}^0 = 1 \ . \end{aligned} \tag{2.87}$$

In particular, the dual vector field to an $\mathcal{O}(n)$-valued 1-form must be valued in the dual bundle $\mathcal{O}(-n)$ so that their contraction may be valued in the trivial bundle.

Following this discussion, we can construct differential forms and vector fields on $\mathbb{C}^2$ which descend to $\mathcal{O}(n)$-valued differential forms and vector fields on $\mathbb{CP}^1$. When performing calculations in homogeneous coordinates, we often want to apply the basic operations of calculus, including taking derivatives and computing integrals. These operations have certain subtleties in homogeneous coordinates which we will now address.

Let us say we want to integrate some object over $\mathbb{CP}^1$ which is written in terms of the homogeneous coordinates on $\mathbb{C}^2$. In order for such an integral to make sense, the argument must be single-valued on each equivalence class, meaning that it has zero weight. Notably, the integrand may be a composite object whose constituent parts are not themselves weightless. For example, it may be the product of an $\mathcal{O}(2)$-valued 2-form and a section of $\mathcal{O}(-2)$ such that the total weight of the integrand is zero. One should check that this condition is satisfied whenever writing down an integral in homogeneous coordinates.

On the other hand, let us say that we construct some $\mathcal{O}(n)$-valued differential form ω in homogeneous coordinates and we would like to compute its exterior derivative $\mathrm{d}\omega$. If we are not careful, then the new differential form $\mathrm{d}\omega$ will not satisfy the necessary conditions to be interpreted as a differential form on $\mathbb{CP}^1$. Problems arise for those ω which are valued in the non-trivial line bundles $\mathcal{O}(n)$ with $n \neq 0$. In this case, the exterior derivative on $\mathbb{C}^2$ introduces legs in $\mathrm{d}\omega$ which point along the fibre directions, spoiling the horizontal property.

We can account for this by defining a covariant derivative on the bundle $\mathcal{O}(n)$ which acts as

$$\nabla(\omega) = \mathrm{d}\omega + n\,\frac{\langle\hat{\pi}\mathrm{d}\pi\rangle}{\langle\pi\hat{\pi}\rangle} \wedge \omega \ . \tag{2.88}$$

The first term is the usual exterior derivative on $\mathbb{C}^2$, and the second term cancels any legs along the fibre directions. Explicitly, the exterior derivative of a section of $\mathcal{O}(n)$ may be written as

$$\mathrm{d}(s) = e^0\,\partial_0(s) + \bar{e}^0\,\bar{\partial}_0(s) - \frac{\langle\hat{\pi}\mathrm{d}\pi\rangle}{\langle\pi\hat{\pi}\rangle}\,D(s) \ . \tag{2.89}$$

The second term in the covariant derivative cancels the term proportional to $D(s)$, ensuring that $\nabla(s)$ is a basic differential form. The Lie derivative of $\mathcal{O}(n)$-valued objects should then be defined using the Cartan homotopy formula and this covariant derivative. In practice, when acting on sections of $\mathcal{O}(n)$, we can write this covariant derivative as

$$\nabla(s) = e^0\,\partial_0(s) + \bar{e}^0\,\bar{\partial}_0(s) \ . \tag{2.90}$$

This expression holds for any value of $n \in \mathbb{Z}$ making calculations a little simpler.

2.2.5 Spinor notation on twistor space

The geometry of twistor space is also nicely presented in spinor notation. In this context, we write the 4-vector x^μ parameterising spacetime in spinor notation as $x^{a\dot{a}}$. The homogeneous coordinate π_a on the Riemann sphere can then be thought of as a spinor with an undotted index which consequently transforms under some spacetime rotations. This allows for the incidence relations and various other expressions to be written in a succinct and compact manner. We will now present this perspective in more detail.

When considering rotations of complexified spacetime, there is an isomorphism

$$\mathsf{SO}(4,\mathbb{C}) \cong \big(\mathsf{SL}(2,\mathbb{C}) \times \mathsf{SL}(2,\mathbb{C})\big)/\mathbb{Z}_2 \ . \tag{2.91}$$

This allows us to replace an $\mathsf{SO}(4,\mathbb{C})$ vector index with a pair of $\mathsf{SL}(2,\mathbb{C})$ spinor indices. We will use undotted indices to denote fields transforming under the first copy of $\mathsf{SL}(2,\mathbb{C})$, and dotted indices for those transforming under the second. For example, the spacetime coordinate vector x^μ will be replaced by

$$x^{a\dot{a}} = \begin{pmatrix} x^1 + \mathrm{i}\,x^2 & x^3 + \mathrm{i}\,x^4 \\ -x^3 + \mathrm{i}\,x^4 & x^1 - \mathrm{i}\,x^2 \end{pmatrix} \ . \tag{2.92}$$

Spinor indices are raised and lowered using the $\mathsf{SL}(2,\mathbb{C})$-invariant Levi-Civita symbol. Because this tensor is anti-symmetric, one must be careful about raising and lower conventions. We will adopt the mnemonic 'raise from the right, lower from the left' corresponding to the conventions

$$\alpha^a = \alpha_b \, \varepsilon^{ba} \ , \qquad \alpha_a = \varepsilon_{ab} \, \alpha^b \ . \tag{2.93}$$

This is consistent with the sign choices $\varepsilon_{12} = +1$ and $\varepsilon^{12} = +1$. We make the same choices of conventions for the dotted spinor indices. The Levi-Civita symbol also defines an $\mathsf{SL}(2, \mathbb{C})$-invariant inner product on spinor indices,

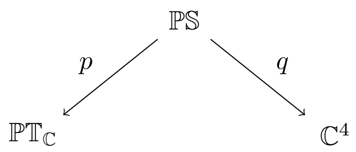
$$\langle \alpha\beta \rangle = \alpha^a \beta_a = \alpha_a \varepsilon^{ab} \beta_b \ . \tag{2.94}$$

This pairing is anti-symmetric meaning the contraction of any spinor with itself vanishes. We can use this to write the flat metric on complexified spacetime as

$$\mathrm{d}s^2 = \frac{1}{2} \, \varepsilon_{ab} \, \varepsilon_{\dot{a}\dot{b}} \, \mathrm{d}x^{a\dot{a}} \mathrm{d}x^{b\dot{b}} \ . \tag{2.95}$$

It is worth noting that norm of x^μ is proportional to the determinant of $x^{a\dot{a}}$.

We can also introduce a spinor bundle $\mathbb{S}$ over spacetime which is a trivial rank 2 vector bundle $\mathbb{S} = \mathbb{C}^2 \times \mathbb{C}^4$. The fibre coordinate $\pi_a \in \mathbb{C}^2$ lives in the fundamental representation of one copy of $\mathsf{SL}(2, \mathbb{C})$, and we will continue to use the spacetime coordinates $x^{a\dot{a}}$ on the base manifold $\mathbb{C}^4$. If we then quotient by a rescaling action on the fibres, we land on the projective spin bundle $\mathbb{PS} = \mathbb{CP}^1 \times \mathbb{C}^4$. This is precisely the correspondence space appearing in the twistor correspondence diagram,



The two projection maps in this diagram may now be written as

$$p : (\pi_a, x^{a\dot{a}}) \mapsto (\pi_a, \pi_a x^{a\dot{a}}) \ , \qquad q : (\pi_a, x^{a\dot{a}}) \mapsto x^{a\dot{a}} \ . \tag{2.96}$$

In the first expression, the four coordinates $Z^A = (\pi_a, \pi_a x^{a\dot{a}})$ play the role of homogeneous coordinates on $\mathbb{CP}^3$. Taking into account the constraint $\pi_a \neq (0, 0)$, this shows that twistor space can also be viewed as the manifold $\mathbb{PT}_{\mathbb{C}} = \mathbb{CP}^3 \setminus \mathbb{CP}^1$ where the removed $\mathbb{CP}^1$ corresponds to the submanifold $\{Z^A = (0, 0, Z^3, Z^4) \mid Z^3, Z^4 \in \mathbb{C}\} \subset \mathbb{CP}^3$.

As well as working with complexified spacetime, we are often interested in considering real slices with a fixed signature metric. These real slices may be specified by imposing reality conditions on the coordinates x^μ, such as "they are all real coordinates" or "three real, one imaginary". Reality conditions may also be specified geometrically in terms of a conjugation action on the manifold. For example, asking the coordinates x^μ to be invariant under the usual conjugation action $x^\mu \mapsto \overline{x^\mu}$ is equivalent to all the coordinates being real numbers. By comparison, if we wanted one of the coordinates, say x^2, to be pure imaginary, we could demand that it is invariant under the modified conjugation $x^2 \mapsto -\overline{x^2}$. The inclusion of a minus sign in this conjugation action ensures that the invariant subspace will be pure imaginary numbers, rather than real numbers.

These conjugation actions can then be interpreted as actions on the matrix $x^{a\dot{a}}$, and sometimes they coincide with familiar matrix operations. For example, Lorentz signature is achieved by asking the matrix $x^{a\dot{a}}$ to be invariant under Hermitian conjugation $x^{a\dot{a}} \mapsto (x^{a\dot{a}})^\dagger$. Ultrahyperbolic (split) signature corresponds to asking the entries of the matrix to be real in the usual sense, that is invariant under the conjugation $x^{a\dot{a}} \mapsto \overline{x^{a\dot{a}}}$. Euclidean conjugation is a little more complicated. We denote the matrix conjugation by $x^{a\dot{a}} \mapsto \hat{x}^{a\dot{a}}$ which is explicitly defined by

$$x^{a\dot{a}} = \begin{pmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{pmatrix} \quad \mapsto \quad \hat{x}^{a\dot{a}} = \begin{pmatrix} \overline{x^{22}} & -\overline{x^{21}} \\ -\overline{x^{12}} & \overline{x^{11}} \end{pmatrix} = \begin{pmatrix} \overline{x^1} + \mathrm{i}\,\overline{x^2} & \overline{x^3} + \mathrm{i}\,\overline{x^4} \\ -\overline{x^3} + \mathrm{i}\,\overline{x^4} & \overline{x^1} - \mathrm{i}\,\overline{x^2} \end{pmatrix} . \tag{2.97}$$

The final expression can be compared with the explicit expression for $x^{a\dot{a}}$ to show that the condition $x^{a\dot{a}} = \hat{x}^{a\dot{a}}$ implies that all of the coordinates are real.

The matrix conjugations also induce conjugation actions on the single index spinors. One can understand this by demanding that conjugation should act on scalar quantities as the usual complex conjugation. If we denote this complex conjugation by C, we can consider the equality $C(\alpha_a x^{a\dot{a}} \kappa_{\dot{a}}) = \overline{\alpha_a x^{a\dot{a}} \kappa_{\dot{a}}}$. Distributing the conjugation action on the left-hand side, we can deduce conjugation actions on the spinors $\alpha_a \mapsto C(\alpha_a)$ and $\kappa_{\dot{a}} \mapsto C(\kappa_{\dot{a}})$. For our purposes, the relevant case is Euclidean conjugation which acts on the single index spinors as

$$\alpha_a = (\alpha_1, \alpha_2) \mapsto \hat{\alpha}_a = (-\overline{\alpha_2}, \overline{\alpha_1}) . \tag{2.98}$$

The same expression applies to dotted spinors. Notably, applying this conjugation twice does not return the original spinor α_a, but rather gives $-\alpha_a$. Four consecutive applications are required to get back to the input spinor.

Turning our attention to the spinor π_a which parameterises the Riemann sphere $\mathbb{CP}^1$, we might look for a geometric interpretation of this map. Inspecting the explicit formula, we see that the map $\pi_a \mapsto \hat{\pi}_a$ has no fixed points on $\mathbb{CP}^1$ as the point $\pi_a = (0, 0)$ is excluded. Let us work in the patch $\pi_1 \neq 0$ and define the inhomogeneous coordinate $\zeta = \pi_2/\pi_1$. Euclidean conjugation acts on this inhomogeneous coordinate as

$$\zeta = \frac{\pi_2}{\pi_1} \mapsto \frac{\hat{\pi}_2}{\hat{\pi}_1} = -1/\bar{\zeta} . \tag{2.99}$$

Since π_a and $-\pi_a$ correspond to the same point in $\mathbb{CP}^1$, only two applications of this map are required to return to the same inhomogeneous coordinate. We can test this map on a few points to build some intuition. One can check that zero and infinity are exchanged $0 \leftrightarrow \infty$, as are $1 \leftrightarrow -1$ and $\mathrm{i} \leftrightarrow -\mathrm{i}$.

To get the full picture, we can use the stereographic projection to trade the complex coordinate ζ for three real coordinates $\{a, b, c\}$. These coordinate parameterise a copy of $\mathbb{R}^3$ in which $\mathbb{CP}^1 \cong S^2$ is embedded as the unit sphere. Explicitly, the $\mathbb{R}^3$ coordinates are given in terms of ζ as

$$(a, b, c) = \left(\frac{\zeta + \bar{\zeta}}{1 + \zeta\bar{\zeta}}, \frac{-\mathrm{i}(\zeta - \bar{\zeta})}{1 + \zeta\bar{\zeta}}, \frac{-1 + \zeta\bar{\zeta}}{1 + \zeta\bar{\zeta}} \right) . \tag{2.100}$$

One may check that the Euclidean conjugation acts on these coordinates as

$$(a, b, c) \mapsto (-a, -b, -c) . \tag{2.101}$$

This map sends a given point on the sphere to its antipodal point, and we will refer to it as the antipodal map.

In Euclidean signature, the map $p : \mathbb{PS}_{\mathbb{E}} \to \mathbb{PT}_{\mathbb{E}}$ is invertible, allowing us to use the coordinates $(\pi_a, x^{a\dot{a}}) \in \mathbb{CP}^1 \times \mathbb{R}^4$ on twistor space. However, the holomorphic coordinates on twistor space are given by $Z^A = (\pi_a, \pi_a x^{a\dot{a}})$ which include a non-trivial combination of the $\mathbb{CP}^1$ coordinate π_a and the spacetime coordinates $x^{a\dot{a}}$. This means that, whilst these coordinates are useful for highlighting the relationship to spacetime, they somewhat obscure the complex structure. For example, when considering differential forms on twistor space, we might be tempted to use a basis of 1-forms given by e^0, $\bar{e}^0$, and $\mathrm{d}x^{a\dot{a}}$. This would be perfectly valid, but it is often convenient to work in an alternative basis of 1-forms adapted to the complex structure.

Let us define some $\mathcal{O}(n)$-valued 1-forms, originally introduced in [BMS07], by

$$\begin{aligned} e^0 &= \langle \pi \mathrm{d}\pi \rangle \in \Omega^{1,0}(\mathbb{PT}_{\mathbb{E}}) \otimes \mathcal{O}(2) , \qquad & \bar{e}^0 &= \frac{\langle \hat{\pi} \mathrm{d}\hat{\pi} \rangle}{\langle \pi \hat{\pi} \rangle^2} \in \Omega^{0,1}(\mathbb{PT}_{\mathbb{E}}) \otimes \mathcal{O}(-2) , \\ e^{\dot{a}} &= \pi_a \mathrm{d}x^{a\dot{a}} \in \Omega^{1,0}(\mathbb{PT}_{\mathbb{E}}) \otimes \mathcal{O}(1) , \qquad & \bar{e}^{\dot{a}} &= \frac{\hat{\pi}_a \mathrm{d}x^{a\dot{a}}}{\langle \pi \hat{\pi} \rangle} \in \Omega^{0,1}(\mathbb{PT}_{\mathbb{E}}) \otimes \mathcal{O}(-1) . \end{aligned} \tag{2.102}$$

This provides a basis for 1-forms on twistor space which is adapted to the complex structure. They are globally well-defined and nowhere vanishing, but it is important to highlight that they are valued in various holomorphic line bundles $\mathcal{O}(n)$, meaning they have weight under the rescaling $\pi_a \sim r\,\pi_a$ for $r \in \mathbb{C}^*$. In particular, this means that the components of a generic 1-form in this basis will be valued in the various dual bundles. For example, consider a 1-form $\mathcal{A} = \mathcal{A}_{\dot{a}}\,\bar{e}^{\dot{a}}$. Since the 1-forms $\bar{e}^{\dot{a}}$ are valued in $\mathcal{O}(-1)$, the associated components $\mathcal{A}_{\dot{a}}$ must be sections of $\mathcal{O}(1)$ such that the overall 1-form has weight zero.

In addition, since these 1-forms are valued in $\mathcal{O}(n)$ we should take the derivatives with the covariant derivative defined earlier, rather than just the exterior derivative on $\mathbb{PS}$. This will ensure that their derivatives are still well-behaved under the equivalence relation of rescaling, and that they can be interpreted as $\mathcal{O}(n)$-valued differential forms on twistor space.

The dual basis of $\mathcal{O}(n)$-valued vector fields to this basis of 1-forms is given by

$$\begin{aligned} \partial_0 &= \frac{\hat{\pi}_a}{\langle \pi\hat{\pi}\rangle}\frac{\partial}{\partial \pi_a} \in \mathfrak{X}^{1,0}(\mathbb{PT}_{\mathbb{E}}) \otimes \mathcal{O}(-2) \ , \qquad & \bar{\partial}_0 &= -\langle \pi\hat{\pi}\rangle\, \pi_a \frac{\partial}{\partial \hat{\pi}_a} \in \mathfrak{X}^{0,1}(\mathbb{PT}_{\mathbb{E}}) \otimes \mathcal{O}(2) \ , \\ \partial_{\dot{a}} &= -\frac{\hat{\pi}^a}{\langle \pi\hat{\pi}\rangle}\partial_{a\dot{a}} \in \mathfrak{X}^{1,0}(\mathbb{PT}_{\mathbb{E}}) \otimes \mathcal{O}(-1) \ , \qquad & \bar{\partial}_{\dot{a}} &= \pi^a \partial_{a\dot{a}} \in \mathfrak{X}^{0,1}(\mathbb{PT}_{\mathbb{E}}) \otimes \mathcal{O}(1) \ . \end{aligned} \tag{2.103}$$

Notably, the dual vector field to a $\mathcal{O}(n)$-valued 1-form is valued in the dual line bundle $\mathcal{O}(-n)$. All of these 1-forms and vector fields are constructed to be weight zero in $\hat{\pi}_a$ so that we only have to keep track of the holomorphic rescaling $\pi_a \sim r\,\pi_a$ for $r \in \mathbb{C}^*$.

This basis of 1-forms is not closed, and the dual vector fields do not all commute. The non-trivial structure equations are given by

$$\begin{aligned} \nabla(e^{\dot{a}}) &= e^0 \wedge \bar{e}^{\dot{a}} \ , \qquad & \nabla(\bar{e}^{\dot{a}}) &= -\bar{e}^0 \wedge e^{\dot{a}} \\ [\partial_0, \bar{\partial}_{\dot{a}}] &= -\partial_{\dot{a}} \ , \qquad & [\bar{\partial}_0, \partial_{\dot{a}}] &= \bar{\partial}_{\dot{a}} \ . \end{aligned} \tag{2.104}$$

In this expression, ∇ is the covariant derivative acting on $\mathcal{O}(n)$-valued objects which we defined in the previous section.

2.3 Penrose-Ward transform

The Penrose-Ward transform relates solutions of the self-dual Yang-Mills (SDYM) equations on spacetime to certain holomorphic vector bundles over twistor space. In the language of gauge theory, it relates spacetime gauge fields A which solve $F = \star F$ to twistor space gauge fields $\mathcal{A}$ which solve $\mathcal{F}^{0,2} = 0$. In the original work by Ward [War77], the bundle over twistor space was described in terms of a "patching matrix" which plays the role of the transition function on the holomorphic vector bundle. In this section, we will follow the approach presented in [Ati+78] where the bundle over twistor space is described in terms of an anti-holomorphic covariant derivative operator $\nabla^{0,1} = \bar{\partial} + \mathcal{A}$. In order for this to define a complex structure on the vector bundle (turning it into a holomorphic vector bundle), the covariant derivative must square to zero, $(\nabla^{0,1})^2 = 0$. This requires the gauge field to satisfy $\mathcal{F}^{0,2} = 0$, and one can show that this is equivalent to the SDYM equation for a spacetime gauge field A which is built from $\mathcal{A}$. We will be more explicit about this construction and its relationship to self-dual Yang-Mills in the coming section.

Recall the privileged role the self-dual sector plays in Yang-Mills theory: solutions to the SDYM equation correspond to instantons in the full theory. An important application of the Penrose-Ward transform is the construction of instanton solutions on spacetime via their twistor space description. In fact, in a landmark paper [Ati+78] by Atiyah, Drinfeld, Hitchin, and Manin (ADHM), all instantons in Euclidean Yang-Mills theory were constructed following this methodology. This is possible because, whilst the SDYM equation on spacetime is a challenging non-linear differential equation, one can reduce the construction of holomorphic vector bundles over twistor space to a purely algebraic problem. In the literature, this is referred to as the ADHM construction, and it is nicely reviewed in some lecture notes by Atiyah [Ati79].

2.3.1 Inhomogeneous coordinates

We are interested in solutions to the self-dual Yang-Mills equations $F = \star F$ on Euclidean spacetime. In the double null coordinates $\{u^1, u^2\}$, the metric and volume form are written as

$$\mathrm{d}s^2 = \mathrm{d}u^1\mathrm{d}\bar{u}^1 + \mathrm{d}u^2\mathrm{d}\bar{u}^2 \ , \qquad \mathrm{d}^4x = \mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 \wedge \mathrm{d}u^2 \wedge \mathrm{d}\bar{u}^2 \ . \tag{2.105}$$

In these conventions, a basis for the anti-self-dual 2-forms is given by

$$\mathrm{d}u^1 \wedge \mathrm{d}u^2 \ , \qquad \mathrm{d}\bar{u}^1 \wedge \mathrm{d}\bar{u}^2 \ , \qquad \mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 + \mathrm{d}u^2 \wedge \mathrm{d}\bar{u}^2 \ . \tag{2.106}$$

Any anti-self-dual 2-form can be expanded in this basis, and the remaining 2-forms span the self-dual sector. The self-dual Yang-Mills equations can then be phrased as the components of F vanishing along all of the anti-self-dual 2-forms. Using a basis for the anti-self-dual bivectors, we can write the SDYM equations as

$$(\partial_{u^1} \wedge \partial_{u^2}) \lrcorner F = 0 \ , \qquad (\partial_{\bar{u}^1} \wedge \partial_{\bar{u}^2}) \lrcorner F = 0 \ , \qquad (\partial_{u^1} \wedge \partial_{\bar{u}^1} + \partial_{u^2} \wedge \partial_{\bar{u}^2}) \lrcorner F = 0 \ . \tag{2.107}$$

The contractions pick out the anti-self-dual components of F which are then set to zero. We can write these conditions concisely in terms of an auxiliary spectral parameter $\zeta \in \mathbb{C}$ as

$$(\ell \wedge m) \lrcorner F = 0 \quad \forall \zeta \in \mathbb{C} \ , \qquad \ell = \partial_{\bar{u}^1} - \zeta\, \partial_{u^2} \ , \qquad m = \partial_{\bar{u}^2} + \zeta\, \partial_{u^1} \ . \tag{2.108}$$

At a given point in spacetime, the bivector $(\ell \wedge m)$ varies over all anti-self-dual null 2-planes (known as β-planes) as we vary the spectral parameter. The self-duality condition on F is therefore equivalent to the condition that F vanishes upon restriction to every β-plane in spacetime.

Bringing twistor space into the picture, the complex structure on $\mathbb{PT}_{\mathbb{E}}$ is defined such that $\{\ell, m, \partial_{\bar{\zeta}}\}$ span the anti-holomorphic tangent space. In particular, holomorphic functions on twistor space are those satisfying

$$\ell(f) = 0 \ , \qquad m(f) = 0 \ , \qquad \partial_{\bar{\zeta}}(f) = 0 \ . \tag{2.109}$$

This is equivalent to demanding that the function only depends on the holomorphic coordinates $\{\zeta, v^1, v^2\}$. Now, let us introduce a vector bundle $\mathsf{E} \to \mathbb{PT}$ over twistor space whose fibres transform in a representation of some gauge group G. If we want to define some notion of a holomorphic section of this bundle, we need to construct an anti-holomorphic covariant derivative operator $\nabla^{0,1} = \bar{\partial} + \mathcal{A}$ such that holomorphic sections of E are those satisfying $\nabla^{0,1}(s) = 0$.

The operator $\nabla^{0,1}$ defines an almost complex structure on E which is integrable (and therefore defines a complex structure) if and only if this anti-holomorphic derivative squares to zero, $(\nabla^{0,1})^2 = 0$. In terms of the gauge field $\mathcal{A}$, this is equivalent to $\mathcal{F}^{0,2} = 0$ where $\mathcal{F} = \bar{\partial}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is the field strength of $\mathcal{A}$. Using the basis of anti-holomorphic vectors on twistor space, we can write the condition $\mathcal{F}^{0,2} = 0$ as

$$[\nabla_\ell, \nabla_m] = 0 \ , \qquad [\nabla_{\bar{\zeta}}, \nabla_\ell] = 0 \ , \qquad [\nabla_{\bar{\zeta}}, \nabla_m] = 0 \ . \tag{2.110}$$

Let us try to construct a solution to these equations from a SDYM gauge field on spacetime.

Working with the coordinates $\{\zeta, x^\mu\}$ on Euclidean twistor space, we can think of this as a fibre bundle over spacetime where the projection map simply forgets the value of ζ. We can use this projection map to pullback a gauge field A on spacetime to a gauge field on twistor space. This effectively entails thinking of $A = A_\mu \mathrm{d}x^\mu$ as a gauge field on twistor space which is independent of the spectral parameter. If we identify the twistor space gauge field $\mathcal{A}$ with (the $(0,1)$-component of) this gauge field A, then the covariant derivative operators on twistor space become

$$\nabla_\ell = \nabla_{\bar{u}^1} - \zeta\, \nabla_{u^2} \ , \qquad \nabla_m = \nabla_{\bar{u}^2} + \zeta\, \nabla_{u^1} \ , \qquad \nabla_{\bar{\zeta}} = \partial_{\bar{\zeta}} \ . \tag{2.111}$$

Since the gauge field has no legs along $\mathrm{d}\bar{\zeta}$ and is independent of the spectral parameter, two of the integrability equations are immediately satisfied. The remaining equation is equivalent to the self-dual Yang-Mills equation,

$$[\nabla_\ell, \nabla_m] = 0 \qquad \Longleftrightarrow \qquad (\ell \wedge m) \lrcorner F = 0 \ . \tag{2.112}$$

In fact, these anti-holomorphic covariant derivative operators provide a Lax pair for the SDYM equations which we previously denoted by $L \equiv \nabla_\ell$ and $M \equiv \nabla_m$.

We can also go the other way, starting with a gauge field $\mathcal{A}$ on twistor space which satisfies $\mathcal{F}^{0,2} = 0$ and recovering a SDYM gauge field on spacetime. For this, we require that $\mathcal{A}$ is gauge trivial upon restriction to every $\mathbb{CP}^1$ in twistor space. This allows us to impose the gauge fixing condition $\partial_{\bar{\zeta}} \lrcorner \mathcal{A} = 0$ which might otherwise be obstructed. Having done this, two of the integrability conditions on $\nabla^{0,1}$ become

$$\partial_{\bar{\zeta}} \mathcal{A}_\ell = 0 \ , \qquad \partial_{\bar{\zeta}} \mathcal{A}_m = 0 \ . \tag{2.113}$$

Inspired by our discussion above, let us consider the $(0,1)$-component of a spacetime gauge field,

$$A^{0,1} = \big(A_{\bar{u}^1} - \zeta\, A_{u^2}\big)\, \frac{\mathrm{d}\bar{u}^1 - \bar{\zeta}\, \mathrm{d}u^2}{1 + \zeta\bar{\zeta}} + \big(A_{\bar{u}^2} + \zeta\, A_{u^1}\big)\, \frac{\mathrm{d}\bar{u}^2 + \bar{\zeta}\, \mathrm{d}u^1}{1 + \zeta\bar{\zeta}} \ . \tag{2.114}$$

This is the pullback of a spacetime gauge field A, written in a basis of $(0,1)$-forms on twistor space, so the components of A in this expression are independent of the spectral parameter. The fractions are the dual $(0,1)$-forms to the vectors ℓ and m respectively.

It turns out that this expression for a $(0,1)$-form gauge field on twistor space is the general solution to the equations $\partial_{\bar{\zeta}}\mathcal{A}_\ell = 0$ and $\partial_{\bar{\zeta}}\mathcal{A}_m = 0$ where the "components of A" are undetermined functions of spacetime. The remaining components of $\mathcal{F}^{0,2} = 0$ are precisely the SDYM equations for the spacetime gauge field A, written in the Lax formalism. This demonstrates the Penrose-Ward transform: a one-to-one correspondence between solutions of the SDYM equation and holomorphic vector bundles over twistor space.

2.3.2 Spinor notation

The Penrose-Ward transform can also be understood in terms of spinor notation. Starting with the self-dual Yang-Mills equations on spacetime, a basis for the anti-self-dual and self-dual 2-forms is given by

$$\Sigma^{ab} = \varepsilon_{\dot{a}\dot{b}}\,\mathrm{d}x^{a\dot{a}} \wedge \mathrm{d}x^{b\dot{b}} \ , \qquad \widetilde{\Sigma}^{\dot{a}\dot{b}} = \varepsilon_{ab}\,\mathrm{d}x^{a\dot{a}} \wedge \mathrm{d}x^{b\dot{b}} \ . \tag{2.115}$$

Expanding the field strength $F = \mathrm{d}A + A \wedge A$ in this basis,

$$F_{a\dot{a}b\dot{b}}\,\mathrm{d}x^{a\dot{a}} \wedge \mathrm{d}x^{b\dot{b}} = \left(\Phi_{ab}\,\varepsilon_{\dot{a}\dot{b}} + \tilde{\Phi}_{\dot{a}\dot{b}}\,\varepsilon_{ab}\right)\mathrm{d}x^{a\dot{a}} \wedge \mathrm{d}x^{b\dot{b}} \ , \tag{2.116}$$

we can parameterise the anti-self-dual and self dual components by symmetric 2-spinors Φ_{ab} and $\tilde{\Phi}_{\dot{a}\dot{b}}$ respectively. In particular, the self-dual Yang-Mills equations are equivalent to the vanishing of the anti-self-dual component $\Phi_{ab} = 0$, which can be written succinctly as

$$\pi^a\pi^b F_{a\dot{a}b\dot{b}} = 0 \quad \forall\, \pi_a \in \mathbb{CP}^1 \ . \tag{2.117}$$

We have introduced a homogeneous coordinate for $\mathbb{CP}^1$ and given it a spinor index $\pi_a = (\pi_1, \pi_2)$.

Turning to twistor space, a holomorphic function is defined by the conditions

$$\bar{\partial}_0(f) = 0 \ , \qquad \bar{\partial}_{\dot{a}}(f) = 0 \ . \tag{2.118}$$

This is written in terms of the basis of $\mathcal{O}(n)$-valued $(0,1)$-vector fields defined in section 2.2.5, in particular we recall $\bar{\partial}_{\dot{a}} = \pi^a\partial_{a\dot{a}}$. As before, we would like to construct a vector bundle $\mathsf{E} \to \mathbb{PT}$ over twistor space whose fibres transform in a representation of some gauge group G. We will equip this vector bundle with a covariant derivative $\nabla^{0,1} = \bar{\partial} + \mathcal{A}$ and we would like this to define a complex structure on E. We can write the associated condition on the gauge field $\mathcal{F}^{0,2} = 0$ in terms of the covariant derivative operators as

$$[\nabla_{\dot{a}}, \nabla_{\dot{b}}] = 0 \ , \qquad [\nabla_0, \nabla_{\dot{a}}] = 0 \ . \tag{2.119}$$

Since we would like this twistor space gauge field $\mathcal{A}$ to descend to a spacetime gauge field A, we will require that it is trivial upon restriction to $\mathbb{CP}^1$ allowing us to impose the gauge fixing condition $\mathcal{A}_0 = 0$. Let us consider the conditions $\mathcal{F}_{0\dot{a}} = 0$ in this gauge,

$$\mathcal{F}_{0\dot{a}} = \bar{\partial}_0\mathcal{A}_{\dot{a}} = 0 \ . \tag{2.120}$$

This implies that the components $\mathcal{A}_{\dot{a}}$ are holomorphic in the $\mathbb{CP}^1$ coordinate π_a. In addition to this, recall that the basis 1-forms $\bar{e}^{\dot{a}}$ are valued in the line bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^1$, meaning that the components $\mathcal{A}_{\dot{a}}$ must be sections of $\mathcal{O}(1)$ so that the gauge field $\mathcal{A}$ has zero weight. Holomorphic sections of $\mathcal{O}(1)$ are homogeneous polynomials in π_a of degree 1, so we can write these components of the gauge field as

$$\mathcal{A}_{\dot{a}} = \pi^a A_{a\dot{a}} \ . \tag{2.121}$$

This solves the condition $\mathcal{F}_{0\dot{a}} = 0$ whenever the components $A_{a\dot{a}}$ are $\mathbb{CP}^1$-independent functions.

Having imposed the gauge fixing condition $\mathcal{A}_0 = 0$ and solved two components of the equation $\mathcal{F}^{0,2} = 0$, the remaining condition on twistor space is given by

$$[\nabla_{\dot{a}}, \nabla_{\dot{b}}] = \pi^a \pi^b [\partial_{a\dot{a}} + A_{a\dot{a}}, \partial_{b\dot{b}} + A_{b\dot{b}}] = 0 \ . \tag{2.122}$$

This is precisely the self-dual Yang-Mills equation for the spacetime gauge field $A_{a\dot{a}} \mathrm{d}x^{a\dot{a}}$. Similarly, given a spacetime gauge field A satisfying the SDYM equation, this provides a recipe for constructing a twistor space gauge field $\mathcal{A}$ satisfying $\mathcal{F}^{0,2} = 0$. This concludes our review of the Penrose-Ward transform in spinor notation.

2.4 Six-dimensional Chern-Simons theory

The action for six-dimensional Chern-Simons (6dCS) theory first appeared in the work of Witten [Wit95], and later in the context of twistor theory [Wit04] where it was used to compute perturbative scattering amplitudes in Yang-Mills theory. The action was defined over a complex three-dimensional Calabi-Yau manifold X in term of a $(0,1)$-form gauge field $\mathcal{A}$,

$$S_{\text{6dCS}}[A] = \int_X \Omega \wedge \mathrm{tr}\Big(\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Big) \ . \tag{2.123}$$

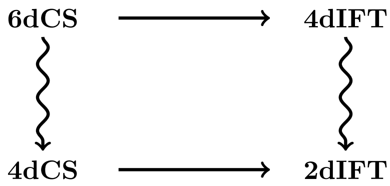
In this expression, Ω is a holomorphic $(3,0)$-form which exists because X is a Calabi-Yau manifold. Unfortunately, the twistor space of compactified spacetime[3] is $\mathbb{CP}^3$, which is not a Calabi-Yau manifold and does not have a globally well-defined Ω. Witten resolved this issue by working with the supersymmetric extension $\mathbb{CP}^{3|4}$ where the fermionic components allow for a consistent definition of Ω.

Recently, Costello proposed an alternative resolution to this problem [Cos20; Cos21], which is to allow the $(3,0)$-form Ω to be meromorphic rather than holomorphic. One must then supplement the theory with boundary conditions at the poles of Ω, in a manner that is familiar from four-dimensional Chern-Simons (4dCS) theory. It is then possible to reduce the action to an effective four-dimensional theory on spacetime, whose equations of motion turn out to be

[3] The twistor space of $\mathbb{R}^4$ may be identified with $\mathbb{PT} = \mathbb{CP}^3 \setminus \mathbb{CP}^1$. Compactifying spacetime by adding the point at infinity turns $\mathbb{R}^4$ into S^4 and the associated twistor space becomes $\mathbb{CP}^3$.

equivalent to the self-dual Yang-Mills (SDYM) equations.

Due to the close similarity between 6dCS theory and 4dCS theory, one might wonder whether there is a relationship between these two formalisms. Indeed, Bittleston and Skinner [BS23] showed that applying an alternative reduction to 6dCS theory allows one to recover 4dCS theory, which can then be further reduced to a two-dimensional integrable model. This leads to a correspondence of integrable field theories (IFTs) represented in the following diagram.



The straight arrows in this diagram represent a reduction over $\mathbb{CP}^1$ which provides an equivalence of the two theories (at least classically). The wavy arrows represent a harsher Kaluza-Klein-like reduction in which infinitely many massive modes are discarded from the theory.

In this section, we will provide an introduction to 6dCS theory, centred around some key examples. We will work in inhomogeneous coordinates on twistor space, which has the advantage of making notation and calculus simpler, at the expense of the clarity of global issues. An alternative approach is to adopt homogeneous coordinates and spinor notation, which can be found in the paper [BS23].

2.4.1 Fundamentals of 6dCS theory

Six-dimensional Chern-Simons (6dCS) theory provides an action for the Penrose-Ward transform. Its equations of motion on twistor space are $\mathcal{F}^{0,2} = 0$, and it localises to field theories on $\mathbb{R}^4$ whose equations of motion are equivalent to the SDYM equation. In preparation for defining this theory, let us consider the space of $(3,0)$-forms on twistor space.

Firstly, we recall that twistor space is isomorphic to $\mathbb{PT} \cong \mathbb{CP}^1 \times \mathbb{R}^4$ as a real manifold. We will work with the inhomogeneous coordinate ζ on the southern patch of $\mathbb{CP}^1$ and real coordinates x^μ on spacetime. As a complex manifold, twistor space $\mathbb{PT}$ has three holomorphic coordinates,

$$\zeta \ , \qquad v^1 = (x^1 + \mathrm{i}x^2) - \zeta\,(x^3 - \mathrm{i}x^4) \ , \qquad v^2 = (x^3 + \mathrm{i}x^4) + \zeta\,(x^1 - \mathrm{i}x^2) \ . \tag{2.124}$$

These are related to their counterparts on the northern patch by

$$\zeta = 1/\tilde{\zeta} \ , \qquad v^1 = \tilde{v}^1/\tilde{\zeta} \ , \qquad v^2 = \tilde{v}^2/\tilde{\zeta} \ . \tag{2.125}$$

Working in the southern patch, a natural $(3,0)$-form to consider is $\mathrm{d}\zeta \wedge \mathrm{d}v^1 \wedge \mathrm{d}v^2$. This $(3,0)$-form appears to be holomorphic, but we must also consider its behaviour on the northern patch.

The presentation of this $(3,0)$-form on the northern patch is given by

$$\mathrm{d}\zeta \wedge \mathrm{d}v^1 \wedge \mathrm{d}v^2 = -\frac{\mathrm{d}\tilde{\zeta} \wedge \mathrm{d}\tilde{v}^1 \wedge \mathrm{d}\tilde{v}^2}{\tilde{\zeta}^4} \ . \tag{2.126}$$

Despite its appearance on the southern patch, it has a singularity in the northern patch at $\tilde{\zeta} = 0$. The location and order of these poles have physical significance in the theory, and different setups will lead to different theories on $\mathbb{R}^4$. When defining 6dCS theory, one must specify a choice of $(3,0)$-form on twistor space, which amounts to choosing the location and order of any poles and zeros. The geometry of twistor space dictates that there will always be four more poles than zeros when counted with multiplicity.

In the example above, the $(3,0)$-form is nowhere vanishing and has a single fourth order pole at $\tilde{\zeta} = 0$. This is an interesting model, but let us consider an alternative $(3,0)$-form given by

$$\Omega = \frac{\mathrm{d}\zeta \wedge \mathrm{d}v^1 \wedge \mathrm{d}v^2}{\zeta^2} = -\frac{\mathrm{d}\tilde{\zeta} \wedge \mathrm{d}\tilde{v}^1 \wedge \mathrm{d}\tilde{v}^2}{\tilde{\zeta}^2} \ . \tag{2.127}$$

This $(3,0)$-form is nowhere vanishing and has two second order poles at $\zeta = 0$ and $\tilde{\zeta} = 0$. Having made a choice of $(3,0)$-form Ω, the 6dCS action is given by

$$S_{\mathrm{6dCS}}[\mathcal{A}] = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{PT}} \Omega \wedge \mathrm{tr}\left(\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) \ . \tag{2.128}$$

This does not completely define the theory, as we can see by varying the action,

$$\delta S_{\mathrm{6dCS}} = \frac{2}{2\pi\mathrm{i}} \int_{\mathbb{PT}} \Omega \wedge \mathrm{tr}\big(\delta\mathcal{A} \wedge \mathcal{F}\big) + \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{tr}\big(\delta\mathcal{A} \wedge \mathcal{A}\big) \ . \tag{2.129}$$

The first term provides the bulk equation of motion $\mathcal{F}^{0,2} = 0$, and the second term is a boundary contribution at the poles of Ω. This follows from the complex analysis identity

$$\partial_{\bar{\zeta}}\left(\frac{1}{\zeta}\right) = -2\pi\mathrm{i}\,\delta(\zeta) \ , \qquad \int_{\mathbb{CP}^1} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}\,\delta(\zeta)\, f(\zeta) = f(0) \ . \tag{2.130}$$

Since our choice of $(3,0)$-form contains second order poles, the boundary term includes derivatives of delta-functions, and higher order poles would produce higher order derivatives. The contribution from the southern patch may be explicitly evaluated to give

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{tr}\big(\delta\mathcal{A} \wedge \mathcal{A}\big) &= \int_{\mathbb{PT}} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}\,\delta(\zeta)\,\partial_\zeta \Big[\mathrm{d}v^1 \wedge \mathrm{d}v^2 \wedge \mathrm{tr}\big(\delta\mathcal{A} \wedge \mathcal{A}\big)\Big] \\ &= \int_{\mathbb{R}^4} \Big[\omega \wedge \mathrm{tr}\big(\delta\mathcal{A} \wedge \mathcal{A}\big)\big|_{\zeta=0} + \mathrm{d}u^1 \wedge \mathrm{d}u^2 \wedge \partial_\zeta \mathrm{tr}\big(\delta\mathcal{A} \wedge \mathcal{A}\big)\big|_{\zeta=0}\Big] \ . \end{aligned} \tag{2.131}$$

The 2-form $\omega = \mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 + \mathrm{d}u^2 \wedge \mathrm{d}\bar{u}^2$ is proportional to the Kähler form on $\mathbb{R}^4$. Boundary conditions must be imposed on the gauge field such that the boundary terms in the variation vanish, and these conditions may generically constrain not only the value of the gauge field at the poles, but also its $\mathbb{CP}^1$-derivatives. In this case, we will choose to impose the boundary conditions

$$\mathcal{A}\big|_{\zeta=0} = 0 \ , \qquad \mathcal{A}\big|_{\tilde{\zeta}=0} = 0 \ . \tag{2.132}$$

The second of these conditions follows from a similar analysis on the northern patch.

The Chern-Simons 3-form is invariant under infinitesimal gauge transformations up to boundary terms, which must be considered carefully in the present context. Infinitesimal gauge transformations act on the gauge field as

$$\delta\mathcal{A} = \bar{\partial}\epsilon + [\mathcal{A}, \epsilon] \ . \tag{2.133}$$

Generic transformations do not preserve the action, whose variation is given by

$$\delta S_{\text{6dCS}} = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \operatorname{tr}\big(\mathcal{A} \wedge \bar{\partial}\epsilon\big) \ . \tag{2.134}$$

This boundary term may be computed explicitly, as before, though some simplifications occur due to the boundary conditions on $\mathcal{A}$. The contribution from the southern patch is

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \operatorname{tr}\big(\mathcal{A} \wedge \bar{\partial}\epsilon\big) = \int_{\mathbb{R}^4} \mathrm{d}u^1 \wedge \mathrm{d}u^2 \wedge \operatorname{tr}\big(\partial_\zeta \mathcal{A} \wedge \bar{\partial}\epsilon\big)\big|_{\zeta=0} \ . \tag{2.135}$$

This term, along with its counterpart on the northern patch, will vanish if the gauge transformation parameter satisfies

$$\bar{\partial}\epsilon\big|_{\zeta=0} = 0 \ , \qquad \bar{\partial}\epsilon\big|_{\tilde{\zeta}=0} = 0 \ . \tag{2.136}$$

At this point, it is important to linger on the physical status of these symmetries. In general, gauge symmetries describe redundancies of a theory that should be removed by imposing a gauge fixing condition. Crucially, physical observables must not depend on this choice of gauge fixing. On the other hand, a given theory may also admit physical symmetries, which are accompanied by a conserved charge according to Noether's theorem. States in such a theory organise themselves into representations of the physical symmetries, distinguished by differing values of the conserved charge. One way to determine whether a symmetry is physical or gauge is via its conserved charge: the charge associated to a gauge symmetry always vanishes.

With this in mind, let us consider the infinitesimal transformations satisfying the boundary conditions (2.136). Those transformations which act trivially at the poles, but non-trivially in the bulk, are the genuine local gauge transformations of Chern-Simons theory. To understand the transformations which act non-trivially at the poles, it is important to accurately identify the boundary degrees of freedom in this theory. As we have seen, the boundary terms typically depend on both the value of the gauge field and its $\mathbb{CP}^1$-derivative due to the second order poles in Ω. Similarly, the infinitesimal transformations can act non-trivially at the poles in two different ways. Either the value of ϵ is non-vanishing at the poles, or it has a non-vanishing $\mathbb{CP}^1$-derivative.

A careful analysis of the Noether charges shows that the latter are gauge symmetries, whilst the former are physical symmetries with associated conservation laws given by

$$\mathrm{d}u^1 \wedge \mathrm{d}u^2 \wedge \bar{\partial}(\partial_\zeta \mathcal{A})\big|_{\zeta=0} = 0 \ , \qquad \mathrm{d}\bar{u}^1 \wedge \mathrm{d}\bar{u}^2 \wedge \bar{\partial}(\partial_{\tilde{\zeta}} \mathcal{A})\big|_{\tilde{\zeta}=0} = 0 \ . \tag{2.137}$$

In these expressions, the Noether currents are $\partial_\zeta \mathcal{A}\big|_{\zeta=0}$ and $\partial_{\tilde{\zeta}} \mathcal{A}\big|_{\tilde{\zeta}=0}$ which are generically non-vanishing. These currents are not conserved in the usual sense, but rather satisfy a holomorphicity condition. This follows from the fact that general solutions to the boundary conditions (2.136) are not global transformations, but semi-local transformations with partial dependence on $\mathbb{R}^4$.

The exponentiation of these infinitesimal transformations are the finite gauge transformations which act on the gauge field as

$$\mathcal{A} \mapsto \mathcal{A}^g = g^{-1}\mathcal{A}g + g^{-1}\bar{\partial}g \ . \tag{2.138}$$

These preservere the space of solutions to the bulk equations of motion $\mathcal{F}^{0,2} = 0$, as the field strength transforms as $\mathcal{F} \mapsto g^{-1}\mathcal{F}g$. To understand the need for boundary conditions, we should consider the transformation of the action,

$$S_{\text{6dCS}}[\mathcal{A}] \mapsto S_{\text{6dCS}}[\mathcal{A}] + \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{tr}\big(\mathcal{A} \wedge \bar{\partial}g g^{-1}\big) - \frac{1}{6\pi \mathrm{i}} \int_{\mathbb{PT}} \Omega \wedge \mathrm{tr}\big(g^{-1}\bar{\partial}g\big)^3 \ . \tag{2.139}$$

Using the boundary conditions on $\mathcal{A}$, one may see that the second term vanishes if we impose boundary conditions of the gauge transformations given by

$$g^{-1}\bar{\partial}g\big|_{\zeta=0} = 0 \ , \qquad g^{-1}\bar{\partial}g\big|_{\tilde{\zeta}=0} = 0 \ . \tag{2.140}$$

These are very similar to the boundary conditions we imposed on the infinitesimal gauge transformations (2.136). The third term is most easily understood by introducing an extension of g over the 7-manifold $\mathbb{PT} \times [0,1]$ which agrees with g on $\mathbb{PT} \times \{1\}$ and with the trivial map on $\mathbb{PT} \times \{0\}$. Denoting this extension by $\tilde{g}$, we define the Wess-Zumino (WZ) term by

$$\mathrm{WZ}[g] = \frac{1}{3}\mathrm{tr}\big(\tilde{g}^{-1}\mathrm{d}\tilde{g} \wedge \tilde{g}^{-1}\mathrm{d}\tilde{g} \wedge \tilde{g}^{-1}\mathrm{d}\tilde{g}\big) \ . \tag{2.141}$$

To relate this to the third term in the gauge transformation of the action, we first note that we may replace the Dobeault operators $\bar{\partial}$ by standard exterior derivatives whenever they are wedged against Ω. Then, we leverage Stokes's theorem and the extension described above to write the integral over the 6-manifold $\mathbb{PT}$ as a surface integral over the 7-manifold $\mathbb{PT} \times [0,1]$. Since the $\tilde{g}$-dependent 3-form is closed, the overall exterior derivative can only act on Ω, giving us the identity

$$\frac{1}{3} \int_{\mathbb{PT}} \Omega \wedge \mathrm{tr}\big(g^{-1}\bar{\partial}g \wedge g^{-1}\bar{\partial}g \wedge g^{-1}\bar{\partial}g\big) = \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \int_{[0,1]} \mathrm{WZ}[g] \ . \tag{2.142}$$

This means that we can write the gauge transformation of the action succinctly as

$$S_{6\mathrm{dCS}}[\mathcal{A}^g] = S_{6\mathrm{dCS}}[\mathcal{A}] + \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \left[\mathrm{tr}\big(\mathcal{A} \wedge \bar{\partial} g g^{-1}\big) - \int_{[0,1]} \mathrm{WZ}[g] \right] . \tag{2.143}$$

Explicitly computing the localisation of the WZ-term, one finds that the boundary conditions we have already imposed on the finite transformations are sufficient to make it vanish. As with the infinitesimal transformations, the finite transformations include both a gauge and physical component, distinguished by their Noether charges.

In addition to these transformations of the fields, we should also consider diffeomorphisms of twistor space. The action is not invariant under general diffeomorphisms, only under those which preserve Ω. This includes all translations along $\mathbb{R}^4$, but does not include all $\mathrm{SO}(4)$ rotations of spacetime. There is an $\mathrm{SU}(2)$ subgroup of rotations preserving Ω which is generated by

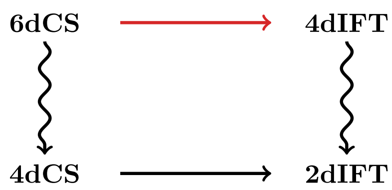
$$\begin{aligned} R_1'' &= \frac{\mathrm{i}}{2}\big(u^1\partial_{u^1} - \bar{u}^1\partial_{\bar{u}^1} - u^2\partial_{u^2} + \bar{u}^2\partial_{\bar{u}^2}\big) \ , \\ R_2'' &= \frac{\mathrm{i}}{2}\big(u^2\partial_{u^1} - \bar{u}^2\partial_{\bar{u}^1} + u^1\partial_{u^2} - \bar{u}^1\partial_{\bar{u}^2}\big) \ , \\ R_3'' &= \frac{1}{2}\big(u^2\partial_{u^1} + \bar{u}^2\partial_{\bar{u}^1} - u^1\partial_{u^2} - \bar{u}^1\partial_{\bar{u}^2}\big) \ . \end{aligned} \tag{2.144}$$

These act trivially on the $\mathbb{CP}^1$-coordinate ζ, but non-trivially on the holomorphic coordinates $\{v^1, v^2\}$ of twistor space. There is also a $\mathrm{U}(1)$ diffeomorphism acting on $\mathbb{R}^4$ and $\mathbb{CP}^1$ generated by

$$R_0'' = \frac{\mathrm{i}}{2}\big(u^1\partial_{u^1} - \bar{u}^1\partial_{\bar{u}^1} + u^2\partial_{u^2} - \bar{u}^2\partial_{\bar{u}^2}\big) + \mathrm{i}\zeta\partial_\zeta \ . \tag{2.145}$$

This commutes with the $\mathrm{SU}(2)$ subgroup of rotations, and together they form the $\mathrm{U}(2) \subset \mathrm{SO}(4)$ which preserves the Kähler form on $\mathbb{R}^4$.

2.4.2 Localisation to spacetime



Next, we would like to show that 6dCS theory localises to a field theory on spacetime whose equations of motion are equivalent to the SDYM equation. In particular, unlike a Kaluza-Klein reduction in which infinitely many modes are discarded from the theory, a finite number of fields on $\mathbb{R}^4$ capture all of the physical degrees of freedom in 6dCS theory. This is analogous to the localisation of 4dCS theory which is presented in section 1.3. In that context, we saw that the bulk degrees of freedom in the theory were gauge trivial, and the only physical degrees of freedom arose from boundary conditions imposed at poles in $\mathbb{CP}^1$. The same argument applies to 6dCS theory, where the role of the boundary is played by the poles in Ω. We will refer to the degrees of freedom living at the poles as *edge modes*, and they will become the fundamental fields of the 4d theory.

Now that we have the idea in mind, let us explicitly localise 6dCS theory to derive the theory on spacetime. It is helpful to introduce a field redefinition which separates the bulk gauge field from the edge modes. To this end, we introduce two new fundamental fields $\mathcal{A}'$ and $\hat{g}$ which are related to $\mathcal{A}$ by

$$\mathcal{A} = \mathcal{A}'^{\hat{g}} = \hat{g}^{-1}\mathcal{A}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g} \ . \tag{2.146}$$

This new parameterisation is partially redundant and introduces an *internal* gauge symmetry which acts as

$$\mathcal{A}' \mapsto \mathcal{A}'^{\check{h}} \ , \qquad \hat{g} \mapsto \check{h}^{-1} \cdot \hat{g} \ . \tag{2.147}$$

Notice that the original field $\mathcal{A}$ is invariant under this transformation meaning that it trivially preserves the action. Assuming that $\mathcal{A}'$ is trivial upon restriction to $\mathbb{CP}^1$, we can leverage this internal symmetry to impose the gauge fixing constraint

$$\partial_{\bar{\zeta}} \lrcorner \mathcal{A}' = 0 \ . \tag{2.148}$$

This constraint is familiar from our discussion of the Penrose-Ward transform in section 2.3, and it will always be applied when localising 6dCS theory. By comparison, we will shortly impose some additional gauge fixing conditions on the edge mode $\hat{g}$, but these will vary from model to model depending on the choice of Ω and boundary conditions.

Before imposing these additional gauge fixing conditions, let us revisit the action of 6dCS theory. In the new variables, it is given by

$$S_{\text{6dCS}}[\mathcal{A}', \hat{g}] = S_{\text{6dCS}}[\mathcal{A}'] + \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \left[\mathrm{tr}\big(\mathcal{A}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1}\big) - \int_{[0,1]} \mathrm{WZ}[\hat{g}] \right] . \tag{2.149}$$

The field $\hat{g}$ only appears in the action against $\bar{\partial}\Omega$, justifying its title of edge mode. In particular, while one might initially think of $\hat{g}$ as defined over the whole manifold $\mathbb{PT}$, we see that it only enters into the action through the fields

$$\hat{g}\big|_{\zeta=0} = g \ , \qquad \hat{g}^{-1}\partial_\zeta \hat{g}\big|_{\zeta=0} = \phi \ , \qquad \hat{g}\big|_{\tilde{\zeta}=0} = \tilde{g} \ , \qquad \hat{g}^{-1}\partial_{\tilde{\zeta}} \hat{g}\big|_{\tilde{\zeta}=0} = \tilde{\phi} \ . \tag{2.150}$$

These 4d fields capture all of the degrees of freedom in the edge mode $\hat{g}$, where the $\mathbb{CP}^1$-derivatives appear because of the second order poles in Ω.

However, some of these degrees of freedom are non-physical due to the presence of gauge symmetries. For example, compatible with the gauge fixing condition $\partial_{\bar{\zeta}} \lrcorner \mathcal{A}' = 0$, we may apply internal gauge transformations satisfying $\partial_{\bar{\zeta}}\check{h} = 0$. This provides sufficient freedom to gauge fix the value of $\hat{g}$ at one point on $\mathbb{CP}^1$ to the identity, for example $\tilde{g} = \mathrm{id}$. In addition, we also have access to the original gauge transformations which we will refer to as *external* symmetries. These act on the new variables as

$$\mathcal{A}' \mapsto \mathcal{A}' \ , \qquad \hat{g} \mapsto \hat{g} \cdot \hat{h} \ . \tag{2.151}$$

Notice that our field redefinition has conveniently decoupled the unconstrained bulk gauge transformations from the residual symmetries acting on the boundary degrees of freedom. As we emphasised in the previous section, the parameter $\hat{h}$ contains both gauge symmetries and semi-local physical symmetries. The gauge symmetries must be trivial at the poles, but may have non-trivial $\mathbb{CP}^1$-derivatives. These allow us to impose the constraints $\phi = 0$ and $\tilde{\phi} = 0$. In summary, the gauge fixing conditions imposed on the edge mode are

$$\hat{g}\big|_{\zeta=0} = g \ , \qquad \hat{g}^{-1}\partial_\zeta \hat{g}\big|_{\zeta=0} = 0 \ , \qquad \hat{g}\big|_{\tilde{\zeta}=0} = \mathrm{id} \ , \qquad \hat{g}^{-1}\partial_{\tilde{\zeta}} \hat{g}\big|_{\tilde{\zeta}=0} = 0 \ . \tag{2.152}$$

This exhausts the gauge symmetries of the theory, and the semi-local symmetries act on the surviving degrees of freedom as

$$\mathcal{A}' \mapsto \mathcal{A}'^{h_\ell} \ , \qquad g \mapsto h_\ell^{-1} \cdot g \cdot h_r \ , \qquad \partial h_\ell = 0 \ , \qquad \bar{\partial} h_r = 0 \ . \tag{2.153}$$

In this expressions, the transformation parameters h_ℓ and h_r only depend on $\mathbb{R}^4$, and we have written the semi-local constraints in terms of the complex coordinates $u^1 = x^1 + \mathrm{i}x^2$ and $u^2 = x^3 + \mathrm{i}x^4$. In general, when we write Dobeault operators acting on fields on $\mathbb{R}^4$, they are defined with respect to the fixed complex structure in which $\{u^1, u^2\}$ are holomorphic coordinates. This coincides with the complex structure on twistor space at $\zeta = 0$, and defines the opposite complex structure (with holomorphic and anti-holomorphic exchanged) when compared to $\tilde{\zeta} = 0$. For example, we can write the boundary conditions on the infinitesimal transformations as

$$\bar{\partial}\epsilon\big|_{\zeta=0} = \bar{\partial}\big(\epsilon\big|_{\zeta=0}\big) = 0 \ , \qquad \bar{\partial}\epsilon\big|_{\tilde{\zeta}=0} = \partial\big(\epsilon\big|_{\tilde{\zeta}=0}\big) = 0 \ . \tag{2.154}$$

We expect these residual symmetries to descend to semi-local symmetries of the theory on spacetime.

The next step in the localisation procedure is to solve the $\delta\mathcal{A}'$ bulk equations of motion. Taking into account the constraint $\partial_{\bar{\zeta}} \lrcorner \mathcal{A}' = 0$, this equation of motion is given by

$$\Omega \wedge \mathcal{L}_{\bar{\zeta}} \mathcal{A}' = 0 \ . \tag{2.155}$$

We have already seen the general solution in our review of the Penrose-Ward transform (§2.3), but here we will present a more detailed derivation. When writing $\mathcal{A}'$ in components, it is natural to work in the basis of $(0,1)$-forms given by $\{\mathrm{d}\bar{\zeta}, \mathrm{d}\bar{v}^1, \mathrm{d}\bar{v}^2\}$. Unfortunately, this basis has some drawbacks and is not particularly well adapted for localising to spacetime. Firstly, the $(0,1)$-forms along $\mathbb{R}^4$ also have legs along $\mathbb{CP}^1$, meaning that the constraint $\partial_{\bar{\zeta}} \lrcorner \mathcal{A}' = 0$ relates the various components rather than simply setting one to zero. Secondly, the $(0,1)$-forms along $\mathbb{R}^4$ are not invariant along $\mathbb{CP}^1$, meaning that the Lie derivative acts on both the components and the basis forms. Of course, one may overcome these obstacles and complete the calculation in this basis, but here we will present an alternative approach.

The localisation is made easier by working in a basis of forms which does not suffer from these downsides. It turns out that asking for invariance along $\mathbb{CP}^1$ is too restrictive, but we can find some $(0,1)$-forms $\{\bar{\theta}^1, \bar{\theta}^2\}$ satisfying

$$\partial_{\bar{\zeta}} \lrcorner\, \bar{\theta}^1 = 0 \ , \qquad \Omega \wedge \mathcal{L}_{\bar{\zeta}} \bar{\theta}^1 = 0 \ , \qquad \partial_{\bar{\zeta}} \lrcorner\, \bar{\theta}^2 = 0 \ , \qquad \Omega \wedge \mathcal{L}_{\bar{\zeta}} \bar{\theta}^2 = 0 \ . \tag{2.156}$$

These $(0,1)$-forms are explicitly given by

$$\bar{\theta}^1 = \frac{\mathrm{d}\bar{u}^1 - \bar{\zeta}\,\mathrm{d}u^2}{1+\zeta\bar{\zeta}} \ , \qquad \bar{\theta}^2 = \frac{\mathrm{d}\bar{u}^2 + \bar{\zeta}\,\mathrm{d}u^1}{1+\zeta\bar{\zeta}} \ . \tag{2.157}$$

The first basis form $\bar{\theta}^1$ is in the span of $\{\mathrm{d}\bar{\zeta}, \mathrm{d}\bar{v}^1\}$, whilst $\bar{\theta}^2$ is in the span of $\{\mathrm{d}\bar{\zeta}, \mathrm{d}\bar{v}^2\}$. These are the inhomogeneous equivalents of the basis introduced in section 2.2.5.

Working in the basis of $(0,1)$-forms given by $\{\mathrm{d}\bar{\zeta}, \bar{\theta}^1, \bar{\theta}^2\}$, the constraint $\partial_{\bar{\zeta}} \lrcorner\, \mathcal{A}' = 0$ simply implies that the coefficient of $\mathrm{d}\bar{\zeta}$ vanishes. Furthermore, the bulk equation of motion implies that the coefficients of $\bar{\theta}^1$ and $\bar{\theta}^2$ are holomorphic functions of $\mathbb{CP}^1$. Expanding these holomorphic coefficients as polynomials in ζ, the requirement that $\mathcal{A}'$ is finite everywhere on $\mathbb{CP}^1$ tells us that they may be at most degree 1. The general solution to the bulk equation of motion is therefore given by

$$\mathcal{A}' = (A_{\bar{u}^1} - \zeta\, A_{u^2})\,\bar{\theta}^1 + (A_{\bar{u}^2} + \zeta\, A_{u^1})\,\bar{\theta}^2 \ . \tag{2.158}$$

This completely specifies the $\mathbb{CP}^1$-dependence of $\mathcal{A}'$, and agrees with the result presented in section 2.3. In this expression, the components A_μ are generic $\mathbb{CP}^1$-independent functions.

Having found the $\mathbb{CP}^1$-dependence of $\mathcal{A}'$, we could now compute the integral over $\mathbb{CP}^1$ in the action, arriving at a theory on $\mathbb{R}^4$. As it stands, that theory would depend on the surviving edge mode g and the 4d gauge field A. However, we have yet to take into account the boundary conditions of 6dCS theory. These allow us to solve for A in terms of g, landing on a theory with a single fundamental field. We expect the equations of motion of this theory to be equivalent to the SDYM equation for A.

The boundary conditions on $\mathcal{A}$ may be converted into conditions on $\mathcal{A}'$ and $\hat{g}$ as

$$\mathcal{A}'\big|_{\zeta=0} = -\bar{\partial}\hat{g}\hat{g}^{-1}\big|_{\zeta=0} \ , \qquad \mathcal{A}'\big|_{\tilde{\zeta}=0} = -\bar{\partial}\hat{g}\hat{g}^{-1}\big|_{\tilde{\zeta}=0} \ . \tag{2.159}$$

Solving these for the components of $\mathcal{A}'$ gives

$$A_{u^1} = 0 \ , \qquad A_{\bar{u}^1} = -\partial_{\bar{u}^1} g g^{-1} \ , \qquad A_{u^2} = 0 \ , \qquad A_{\bar{u}^2} = -\partial_{\bar{u}^2} g g^{-1} \ . \tag{2.160}$$

Consider the corresponding 4d gauge field A, this expression may be summarised as $A^{1,0} = 0$ and

$A^{0,1} = -\bar{\partial} g g^{-1}$. This connection identically solves two of the SDYM equations,

$$F^{2,0} = 0 \ , \qquad F^{0,2} = 0 \ . \tag{2.161}$$

The first of these is solved trivially, and the second is solved due to the Maurer-Cartan identity. We expect the final SDYM equation to coincide with the equation of motion of the 4d theory,

$$\omega \wedge F = 0 \quad \Longleftrightarrow \quad \omega \wedge \partial(\bar{\partial} g g^{-1}) = 0 \ . \tag{2.162}$$

This parameterisation of a self-dual connection is well-known, and the field g is often referred to as *Yang's matrix* [Yan77]. In this context, final SDYM equation is known as Yang's equation. At this stage, one might also guess that the corresponding four-dimensional theory will be the 4dWZW model [Don85; Los+96].

To find the action of the 4d theory, we should substitute our solution for $\mathcal{A}'$ into the 6dCS action and compute the integral over $\mathbb{CP}^1$. Recall that the 6dCS action is given by

$$S_{6\text{dCS}}[\mathcal{A}', \hat{g}] = S_{6\text{dCS}}[\mathcal{A}'] + \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \left[\operatorname{tr}\big(\mathcal{A}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1}\big) - \int_{[0,1]} \mathrm{WZ}[\hat{g}] \right] . \tag{2.163}$$

Since the solution to the bulk equation of motion obeys $\partial_{\bar{\zeta}} \lrcorner \mathcal{A}' = 0$ and $\Omega \wedge \mathcal{L}_{\bar{\zeta}} \mathcal{A}' = 0$, the integrand in the first term cannot saturate the $\mathrm{d}\bar{\zeta}$ leg and therefore vanishes. It remains to compute the integral over $\mathbb{CP}^1$ in the second term, which we will do piece by piece. Since we have gauge fixed all of the degrees of freedom in the edge mode at the north pole, this integral will only receive contributions from the south pole. The first of these is given by

$$\int_{\mathbb{R}^4} \left[\omega \wedge \operatorname{tr}\big(\mathcal{A}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1}\big)\big|_{\zeta=0} + \mathrm{d}u^1 \wedge \mathrm{d}u^2 \wedge \partial_\zeta \operatorname{tr}\big(\mathcal{A}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1}\big)\big|_{\zeta=0} \right] . \tag{2.164}$$

This expression may be simplified with a couple of observations. Firstly, the value of $\mathcal{A}'$ at the south pole is $\mathcal{A}'|_{\zeta=0} = A^{0,1} = -\bar{\partial} g g^{-1}$. Similarly, since the complex structure on twistor space agrees with that on spacetime at $\zeta = 0$, we have $\bar{\partial}\hat{g}\hat{g}^{-1}|_{\zeta=0} = \bar{\partial} g g^{-1}$. Inspecting the first term, we see that the integrand would be a $(1,3)$-form on $\mathbb{R}^4$. This does not exist, so this term must vanish.

Secondly, the $\mathbb{CP}^1$-derivative of $\mathcal{A}'$ vanishes at the south pole $\partial_\zeta \mathcal{A}'|_{\zeta=0} = 0$. The $\mathbb{CP}^1$-derivative of the edge mode also vanishes due to our choice of gauge fixing, so this derivative may only act on the $\mathbb{CP}^1$-dependence inside the twistor space Dobeault operator $\bar{\partial}$. In the basis of $(0,1)$-forms $\{\mathrm{d}\bar{\zeta}, \bar{\theta}^1, \bar{\theta}^2\}$, this operator acts on functions as

$$\bar{\partial} f = \partial_{\bar{\zeta}} f \, \mathrm{d}\bar{\zeta} + (\partial_{\bar{u}^1} f - \zeta \, \partial_{u^2} f)\, \bar{\theta}^1 + (\partial_{\bar{u}^2} f + \zeta \, \partial_{u^1} f)\, \bar{\theta}^2 \ . \tag{2.165}$$

This allows us to compute the contribution of the second term which is

$$\int_{\mathbb{R}^4} \mathrm{d}u^1 \wedge \mathrm{d}u^2 \wedge \partial_\zeta \mathrm{tr}\big(\mathcal{A}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1}\big)\big|_{\zeta=0} = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}\big(g^{-1}\mathrm{d}g \wedge \star g^{-1}\mathrm{d}g\big) \ . \tag{2.166}$$

This is the standard kinetic term of the principal chiral model (PCM). Using some identities from Kähler geometry, this may also be written as

$$\frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}\big(g^{-1}\mathrm{d}g \wedge \star g^{-1}\mathrm{d}g\big) = \int_{\mathbb{R}^4} \omega \wedge \mathrm{tr}\big(g^{-1}\partial g \wedge g^{-1}\bar{\partial} g\big) \ . \tag{2.167}$$

Turning to the WZ-term, this computation is very direct. Since the $\mathbb{CP}^1$-derivative of the edge mode has been fixed, we only pick up a term of the form $\omega \wedge \mathrm{WZ}[g]$. Bringing these pieces together, the action of the 4d theory is

$$S_{\mathrm{4dWZW}}[g] = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}\big(g^{-1}\mathrm{d}g \wedge \star g^{-1}\mathrm{d}g\big) - \int_{\mathbb{R}^4\times[0,1]} \omega \wedge \mathrm{WZ}[g] \ . \tag{2.168}$$

This theory is known as the four-dimensional Wess-Zumino-Witten (4dWZW) model, and it was first recovered from 6dCS theory in [Cos20; Cos21]. As expected, the equations of motion coincide with the final SDYM equation,

$$\delta S_{\mathrm{4dWZW}} = 0 \quad \Longleftrightarrow \quad \omega \wedge \partial(\bar{\partial} g g^{-1}) = 0 \ . \tag{2.169}$$

In addition, the action is invariant under two semi-local symmetries acting as

$$g \mapsto h_\ell^{-1} \cdot g \cdot h_r \ , \qquad \partial h_\ell = 0 \ , \qquad \bar{\partial} h_r = 0 \ . \tag{2.170}$$

These are directly inherited from the residual symmetries of 6dCS theory which satisfy the boundary conditions. The associated conservation laws take the form of holomorphicity conditions,

$$\omega \wedge \partial(\bar{\partial} g g^{-1}) = 0 \ , \qquad \omega \wedge \bar{\partial}(g^{-1}\partial g) = 0 \ . \tag{2.171}$$

One may show that these agree with the conservation laws derived from 6dCS theory (2.137) when substituting in the field redefinition and solution to the bulk equation of motion. Considering the spacetime symmetries, we see that only those diffeomorphisms which preserve the Kähler form will leave the action invariant. This includes all translations of $\mathbb{R}^4$ and a subgroup $\mathrm{U}(2) \subset \mathrm{SO}(4)$ of the rotations, matching the 6dCS analysis.

Chapter 3

Integrability in gravity from Chern-Simons theory

The results in this chapter were found in collaboration with Peter Weck [Col+24a].

3.1 Introduction

Many advances in our understanding of theories of gravity have been closely linked to the study of integrability. This is particularly true in the context of black hole solutions. The integrable structure of gravity with $D-2$ commuting Killing symmetries facilitates powerful solution generating methods, which have led to the discovery of a surprisingly rich array of exact solutions [ER08]. Many black hole uniqueness theorems also rely on this integrable structure [Maz82; HY08; HY11; LT22; LT20].

Two formal ingredients underlie many of these developments. First, with this much spacetime symmetry, various D-dimensional theories of gravity can be reduced to nonlinear σ-models in two dimensions [BM87; BMG88]. The fundamental field of these 2d models roughly corresponds to the internal part of the metric and takes values in a coset group. In the canonical example, four-dimensional General Relativity (GR) is reduced by axial ∂_ϕ and stationary ∂_t Killing vectors to a nonlinear σ-model on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. The second ingredient is that these 2d models are integrable, a fact which is exhibited by the existence of a flat 'Lax' connection expressed in terms of an auxiliary complex parameter.

The goal of this chapter is to show how recent advances relating 2d integrable models to 4d Chern-Simons theories [CY19; Del+20] can be applied to stationary and axisymmetric GR (see [Lac22] for a pedagogical introduction to 4dCS theory). This framework beautifully captures the integrable structure of gravity in the language of gauge theory and geometry. The Lax connection is realized as a dynamical gauge field, and the 'spectral' parameter as a spatial coordinate. We will demonstrate how the 2d σ-model for stationary, axisymmetric gravity emerges from the boundary dynamics of this Chern-Simons theory. We believe this framework will provide fertile ground for the study of powerful but highly technical solution generating methods, not only in GR but also in higher-dimensional supergravity theories. Below we provide two divergent but related motivations for this work: one which may appeal to researchers interested in solution generating methods in theories of gravity, and the other to those interested in applications of 4d and 6d Chern-Simons theory to lower-dimensional integrable models.

3.1.1 Motivation from exact techniques in gravity

Let us first consider the application of the Lax formalism to solution generating methods in theories of gravity. As early as 1978, the Lax formalism was used by Belinski and Zakharov to generate 4d vacuum solutions describing n black holes with a common axis of symmetry [BZ78; BS79]. The BZ method takes an initial solution G_0 for the metric on the Killing directions, defines an extension G which depends on the spectral parameter Z, and constructs a new solution via a dressing transformation,

$$G \mapsto \chi G, \qquad \chi = \mathrm{id} + \sum_{i=1}^{2n} \frac{\chi_i}{Z - Z_i}. \tag{3.1}$$

Compatibility with the Lax strongly constrains the dressing matrix χ. Its Z-dependence is shown above explicitly, and the form of χ_i and Z_i is completely fixed[1] up to free parameters. These parameters set properties of the new solution such as mass, rotation, and NUT charge. The same method can be used when all the Killing vectors are spacelike, to generate various 'multi-soliton' gravitational wave solutions.

In the application to 4d stationary and axisymmetric solutions, only the $n = 1$ case of the Kerr black hole is free from non-physical features such as conical defects and closed time-like curves. However, the generalization of the BZ method (also known as the inverse scattering method) to higher dimensional GR [Pom06] has led to the discovery of a rich array of new solutions. Perhaps most fascinating are the black ring and black Saturn solutions, which have established that black holes of non-spherical horizon topology are a generic feature of vacuum gravity for $D > 4$. See [ER08] for a review of these developments. Extensions to Einstein-Maxwell theory have also been made in [Ale80; NK83].

These techniques have seen more limited application in supergravity theories even though the integrable 2d σ-model foundation widely applies. The BZ method is usually formulated directly in terms of components of the metric on Killing directions. While this is closely related to a field configuration in the coset space of the corresponding σ-model, direct application of the dressing transformation is not guaranteed to respect the coset group structure [Fig+10]. Reconciling the BZ method and the σ-model approach pioneered by Breitenlohner and Maison can present a technical challenge — see the minimal 5d supergravity case in [Fig+10], and STU supergravity

[1] A few additional constraints are required beyond preservation of the form of the Lax. These ensure the resulting metric is symmetric, real-valued, and asymptotically well-behaved.

$D \geq 4$ vacuum gravity	$\mathrm{SL}(D-2,\mathbb{R})/\mathrm{SO}(D-2)$
4d EMd gravity	$\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$
4d $N=4$ from 10d SuGra	$\mathrm{SO}(8,8)/(\mathrm{SO}(8)\times\mathrm{SO}(8))$
4d $N=8$ from 11d SuGra	$\mathrm{E}_{8(+8)}/\mathrm{SO}(16)$

Table 3.1: Examples of σ-models obtained from dimensional reduction of different theories of gravity with $D-2$ commuting Killing vectors, from [BMG88]. Note these integrable 2d models capture only the bosonic sector.

models in [KKV14]. Additionally, the dressing matrix for vacuum gravity with $D-2$ commuting Killing vectors is allowed only simple poles with residues of rank one. Higher rank residues are possible for more general coset groups [KKV14], further complicating the BZ methodology. One motivation to find simpler and more systematic ways to implement these transformations is to understand the scope of exact solutions in higher-dimensional (super)gravity theories.

Another motivation comes from the black hole microstate problem, particularly the fuzzball proposal [Ben+22; BW13]. Some black hole microstates are coherent enough to admit a semi-classical description in terms of smooth and horizonless geometries. Most of the known examples of such 'microstate geometries' are supersymmetric or represent small deformations away from supersymmetric solutions [BWW06; BW08; Ben+15]. Methods to systematically generate more general stationary and axisymmetric microstate geometries would be quite significant. Recent progress on the static case has indirectly made use of inverse scattering and integral equation methods, employing 4d Einstein-Maxwell solutions to solve higher-dimensional equations of motion. This strategy was employed in [BHW22] and [BH24] to construct the first examples of smooth and horizonless geometries which asymptotically resemble Schwarzschild black holes.

Accessing the full power of inverse scattering methods in supergravity theories may require approaches which are more systematic and better adapted to the σ-model presentation. A promising new perspective on the Lax formalism and 2d integrable field theories (IFTs) has recently emerged, relating them to Chern-Simons (CS) theory on a 4d space with defects [CY19]. The Lax connection is treated as a dynamical gauge field in this extended space with the spectral parameter serving as a coordinate. Dressing transformations, such as in the BZ method, have the form of (singular) gauge transformations on the Lax connection,

$$\mathcal{L} \mapsto \chi\mathcal{L}\chi^{-1} - \mathrm{d}\chi\chi^{-1} \ . \tag{3.2}$$

These transformations may have a natural home in the 4dCS framework. In this work, we lay the groundwork for further investigation along these lines by showing explicitly how the 2dIFT for stationary axisymmetric gravity can be derived from a novel 4dCS theory. While we often refer to the $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ σ-model of the 4d vacuum case for the sake of concreteness, our construction is group agnostic. For example it applies equally to any of the coset group σ-models identified by

Breitenlohner, Maison, and Gibbons in [BMG88] (see e.g. table 3.1.1).

3.1.2 Motivation from 4d & 6d Chern-Simons theory

For the reader who is primarily interested in 4dCS and/or 6dCS theory, this work presents several new results in these domains. To highlight the novel features of our construction, we first recall that the action of 4dCS theory [CY19] is given by

$$S_{4\mathrm{dCS}} = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{CP}^1 \times \mathbb{R}^2} \omega \wedge \mathrm{tr}\Big(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \Big) \ , \tag{3.3}$$

where ω is a meromorphic 1-form. This theory allows for the systematic construction of a wide array of 2d integrable field theories as well as their associated Lax connections. Provided with the input data defining a 4dCS setup, namely a choice of 1-form ω and boundary conditions on the gauge field A, it is possible to localise the action to an integral over $\mathbb{R}^2$. Different choices of input data will lead to different 2dIFTs, meaning that 4dCS theory is also a mechanism for exploring the space of integrable theories. For this reason, it is important to understand which properties of the input data are essential, and to find any extraneous constraints which may be relaxed. For example, it is not yet understood what the most general 1-form ω is which is both compatible with the localisation procedure and leads to an integrable field theory.

The existing literature on 4dCS theory has treated 1-forms which can be written in terms of a twist function as $\omega = \varphi(Z)\,\mathrm{d}Z$, where Z is a holomorphic coordinate on a complex curve, often taken to be $\mathbb{CP}^1$. These 1-forms only have legs along $\mathrm{d}Z$, and the twist function $\varphi(Z)$ is a meromorphic function of Z. In this chapter, we will relax both of these assumptions, allowing ω to have legs along the spacetime directions, and the coefficients to have dependence on the spacetime coordinates, which we will often denote $\{\rho, z\}$. The fact that our 4dCS theory retains its key properties after this generalisation is a surprising result. We will demonstrate that it is still possible to localise the action to a 2dIFT, and derive the associated Lax connection for the concrete example of stationary, axisymmetric GR.

Another perspective on this unfamiliar feature of our 4dCS theory is found by considering an alternative spectral parameter defined by

$$W = z + \frac{\rho}{2}\big(Z^{-1} - Z\big) \ . \tag{3.4}$$

The original Z-plane is a double covering of the W-plane and the meromorphic 1-form of interest may be expressed as $\omega = \mathrm{d}W$. This presentation brings the 1-form into a more familiar form, but now the novelty lies in the relationship between Z and W. In particular, the W-plane contains a branch cut between the points $W = z + \mathrm{i}\rho$ and $W = z - \mathrm{i}\rho$. From this perspective, we are generalising the *branch cut defects* of [CY19] to allow for spacetime-dependent endpoints.

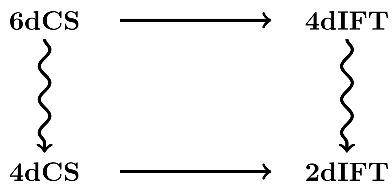
These generalisations of 4dCS theory were not the result inspired guesses, but were derived from a reduction of 6dCS theory. The 6-manifold underlying 6dCS theory [Cos20; Cos21] is twistor space $\mathbb{PT}$, which is isomorphic to $\mathbb{CP}^1 \times \mathbb{R}^4$ as a real manifold. The action is given by

$$S_{6\mathrm{dCS}} = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \Omega \wedge \mathrm{tr}\left(\mathcal{A} \wedge \bar{\partial} \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) , \tag{3.5}$$

where Ω is a meromorphic $(3,0)$-form. Much like the relationship between 4dCS theory and 2dIFTs, this six-dimensional counterpart localises to spacetime (i.e. $\mathbb{R}^4$) producing four-dimensional integrable field theories (4dIFTs). The analog of the Lax connection for 4dIFTs is a self-dual Yang-Mills (sdYM) connection which may be derived from the 6dCS description.

Integrable models in lower dimensions are known to arise as conformal reductions of the 4d sdYM equations (see the textbook [MW91]). If we restrict to solutions which are invariant under two conformal Killing vectors, the remaining sdYM equations describe 2dIFTs. For example, it was noticed in [Wit79] that the static, axisymmetric gravity equations are equivalent to a static, axisymmetric reduction of the sdYM equations. This observation inspired further work [War82] in which the machinery of twistor theory was used to generate solutions to the vacuum Einstein's equations. These developments are discussed in the textbooks [FW90] and [MW91].

When compared with this older literature on the topic, the advent of 6dCS theory gives us the opportunity to work at the level of an action, rather than just working with the equations of motion. This was recently applied in [Pen21] to derive actions for 2dIFTs coming from dimensional reductions of gravity and supergravity; including the case of 4d stationary, axisymmetric gravity which we focus on in this work. In other concurrent work on 6dCS theory [BS23], it was shown that 4dCS theory arises as a reduction of 6dCS theory, just as 2dIFTs arise as a reduction of the 4d sdYM equations. This established a commutative diagram relating Chern-Simons theories to integrable models.



Localisation is represented in the diagram by straight arrows and dimensional reduction by squiggly arrows. In particular, since the two papers [BS23] and [Pen21] first appeared within days of one another, the 4dCS theory related to stationary, axisymmetric gravity was not explored in [Pen21]. The present work seeks to fill this gap in the literature.

As mentioned earlier, our reduction of 6dCS theory results in a 4dCS theory with some novel features. These features of the 4dCS theory may be directly matched to novel features of the associated reduction. Firstly, we implement a discrete reduction which acts simultaneously on spacetime and on the gauge group. At the level of the 2d integrable model, this reduces the target space from the Lie group G to the symmetric-space G/G_0 where G_0 is the fixed subgroup of a $\mathbb{Z}_2$-automorphism. At the level of 4dCS theory, this produces the branch cut defect described above — a simple example is given in section 3.8. Secondly, our reduction features a rotational vector field. Unlike the translational reductions considered in [BS23], it is known that rotational vectors have a non-trivial lift to twistor space (see e.g. [MW91; Pen21]), mixing the spectral parameter $\mathbb{CP}^1$ with the spacetime $\mathbb{R}^4$. When applied to 6dCS theory, this forces us to define a new invariant spectral parameter which necessarily depends on the spacetime coordinates. Following this through to 4dCS theory, we find the spacetime-dependent 1-form ω discussed above.

3.1.3 Summary of contents

Let us give a short summary of the contents of this chapter. In section 3.2, we review perhaps the most famous example of integrability in gravity: stationary, axisymmetric vacuum solutions in GR. Both the Lax formalism and reduction to a 2d σ-model are presented. The remainder of the chapter will be devoted to establishing this 2dIFT in the commutative diagram of models shown above. Section 3.3 is intended to be self-contained, and describes the relationship between 4dCS theory and integrability in gravity. We show in detail how the sigma-model for axisymmetric gravity is recovered from 4dCS theory via localisation of the action to 2d defects. The reader primarily interested in this key result could focus on this section. Section 3.4 reviews how such an integrable 2d σ-model can alternatively be derived as a reduction of the 4d Wess-Zumino-Witten (WZW) model. For our purposes, the 4dWZW model serves as a stepping stone to 6dCS theory. In fact, our 4dCS model was constructed by taking the reduction vectors for the right hand side of the diagram and lifting them to twistor space, so that we could apply the equivalent reduction to 6dCS theory. This lift is explained in section 3.5. Finally, in section 3.6, we present the reduction from 6dCS to 4dCS represented on the left hand side of the diagram. We conclude with an outlook on future work made possible by this new approach to integrability in gravity.

3.2 Background on integrability in gravity

The sector of 4d vacuum General Relativity (GR) we are interested in consists of solutions with two commuting Killing vectors. These metrics can be written in the form

$$\mathrm{d}s_4^2 = e^{2\nu}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho\, G_{mn}\mathrm{d}x^m\mathrm{d}x^n, \qquad m,n \in \{3,4\} \tag{3.6}$$

where the function ν and the matrix G depend only on the Weyl canonical coordinates (ρ, z), and

$$\det G = \epsilon, \qquad \epsilon = \pm 1 \ . \tag{3.7}$$

For the sake of concreteness, let us consider *stationary* and *axisymmetric* spacetimes ($\epsilon = -1$), for which we identify the pair of commuting killing vectors with ∂_t and a vector for the azimuthal symmetry, ∂_ϕ. This includes the Schwarzschild and Kerr black holes as well as the Kerr-NUT solution. It is a particularly well-studied sector of the theory, thanks in part to the solution generating techniques developed by Belinsky and Zakharov in the 1970s [BZ78; BS79]. Based on inverse scattering methods, their technique allows for an infinite number of solutions to be constructed from a given 'seed' solution to the Einstein equations.

The reason these solution generating techniques are possible is that Einstein's equations are integrable when specialised to stationary and axisymmetric spacetimes. Substituting the metric ansatz (3.6) into Einstein's equations, one finds that they decompose into a set of equations for G and another for ν (given G). Defining a pair of 2×2 matrices U_ρ, U_z by

$$U_\rho \equiv \rho\, \partial_\rho G\, G^{-1}\ , \qquad U_z \equiv \rho\, \partial_z G\, G^{-1}\ , \tag{3.8}$$

the vacuum Einstein equations can be written

$$\partial_\rho U_\rho + \partial_z U_z = 0\ , \tag{3.9}$$

$$\partial_\rho \nu = \frac{1}{8\rho}\mathrm{tr}(U_\rho^2 - U_z^2) - \frac{1}{2\rho}\ , \quad \partial_z \nu = \frac{1}{4\rho}\mathrm{tr}(U_\rho\, U_z)\ . \tag{3.10}$$

These equations define a completely integrable system — a rare thing, especially in theories of gravity. The algebraic structure behind this integrability was first explored by Geroch, Breitenlohner, Maison, and others [Ger71; Ger72; BM87].

The integrability of this system is exhibited by the existence of a flat connection, called the Lax connection. The Lax for stationary and axisymmetric GR can be succinctly written in complex coordinates $\xi = z + i\rho$, $\bar{\xi} = z - i\rho$ as the one-form

$$\mathcal{L} = \frac{-\partial_\xi G G^{-1}}{1 - \mathrm{i}Z}\mathrm{d}\xi + \frac{-\partial_{\bar{\xi}} G G^{-1}}{1 + \mathrm{i}Z}\mathrm{d}\bar{\xi}, \qquad Z = \frac{2\mathrm{i}}{\xi - \bar{\xi}}\left(\frac{\xi + \bar{\xi}}{2} - W \pm \sqrt{(W - \xi)(W - \bar{\xi})}\right) \tag{3.11}$$

where Z and W are complex parameters known in the literature as the variable and constant spectral parameters, respectively. Readers familiar with the Principal Chiral Model (PCM) will note that its Lax connection would be identical were we to neglect the spacetime dependence of the variable spectral parameter Z. The flatness of this connection for any value of W is equivalent to the Einstein equations for G,

$$\partial_\xi \mathcal{L}_{\bar{\xi}} - \partial_{\bar{\xi}} \mathcal{L}_\xi + [\mathcal{L}_\xi, \mathcal{L}_{\bar{\xi}}] = 0 \quad \forall\, W \quad \iff \quad \text{Eq. (3.9)}. \tag{3.12}$$

Since the flatness of a connection implies the existence of solutions $\Psi(\rho, z, W)$ to the equations

$$\nabla_\xi \Psi = 0, \qquad \nabla_{\bar{\xi}} \Psi = 0\ , \qquad \nabla \equiv \mathrm{d} + \mathcal{L}\ , \tag{3.13}$$

the Einstein equations for G are sometimes presented as the compatibility conditions for this pair of linear 'Lax equations'. Some authors work in a basis where Z (rather than W) is held fixed, in which case the differential operators appearing above are mapped to

$$\partial_\xi \;\mapsto\; \partial_\xi + (\xi - \bar{\xi})^{-1}\frac{1+\mathrm{i}Z}{1-\mathrm{i}Z}\, Z\partial_Z \;, \qquad \partial_{\bar{\xi}} \;\mapsto\; \partial_{\bar{\xi}} - (\xi - \bar{\xi})^{-1}\frac{1-\mathrm{i}Z}{1+\mathrm{i}Z}\, Z\partial_Z \;. \tag{3.14}$$

For example, in the original reference [BS79] the spectral parameter λ held fixed is related those appearing here as $\lambda = -\rho Z$, in terms of which the Lax equations are

$$\left(\partial_z - \frac{2\lambda^2}{\lambda^2+\rho^2}\partial_\lambda\right)\Psi = \frac{\rho U_z - \lambda U_\rho}{\lambda^2+\rho^2}\,\Psi, \qquad \left(\partial_\rho + \frac{2\lambda\rho}{\lambda^2+\rho^2}\partial_\lambda\right)\Psi = \frac{\rho U_\rho + \lambda U_z}{\lambda^2+\rho^2}\,\Psi \;. \tag{3.15}$$

The effectively two-dimensional character of stationary, axisymmetric GR can be made explicit at the level of the action. In particular, reduction by the two Killing vectors yields a 2d non-linear σ-model [BM87]. We can see this schematically by evaluating the four-dimensional Einstein-Hilbert term for the metric (3.6),

$$\sqrt{\det g^{(4)}}\; R^{(4)} = -\,\frac{\rho}{4}\,\mathrm{tr}[(G^{-1}\partial_\rho G)^2 + (G^{-1}\partial_z G)^2] + \mathcal{B}\;, \tag{3.16}$$

where $\mathcal{B}$ collects various total derivatives. Given our determinant constraint on G, and the fact that it should be a symmetric matrix, the target space for the 2d model should be $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. In the next subsection, we carefully go through the reduction procedure of 4d GR by a pair of Killing vectors to come to the same basic conclusion. The reader uninterested in these details can skip ahead to section 3.3, where we present a new framework for understanding this 2d model for stationary and axisymmetric GR. The $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ σ-model field is found to capture the boundary dynamics of a Chern-Simons gauge field (equivalent to the Lax).

3.2.1 Reduction of 4dGR to 2dIFT

In this subsection, we consider the dimensional reduction of pure Einstein gravity to 2d by a pair of commuting Killing vectors, ∂_ϕ and ∂_t. For convenience, we work with Euclidean metric signature ($\epsilon = +1$) although results can easily be extended to Lorentzian signature. The resulting nonlinear, coset space σ-model is sometimes called the BM model in the literature, after [BM87]. We roughly follow the presentation of the reduction in that early work.

Let us distinguish indices on the two isometry directions with lowercase Latin letters, and indices on the pair of associated orbit space directions with lowercase Greek letters,

$$x^M = (x^\mu, x^m), \qquad \mu = 1,2, \qquad m = 3,4, \tag{3.17}$$

having identified ∂_ϕ, ∂_t with ∂_3, ∂_4. We will use the monikers 'external' for x^1, x^2 and 'internal' for $x^3 = \phi$, $x^4 = t$. The first step is to use local Lorentz transformations to select a lower-triangular

4d vielbein E^A_M,

$$E^A_M = \begin{pmatrix} \sqrt{\lambda}\ e^\alpha_\mu & 0 \\ \sqrt{\rho}\ \hat{e}^a_n\ B^n_\mu & \sqrt{\rho}\ \hat{e}^a_m \end{pmatrix}, \qquad g^{(4)}_{MN} = \eta_{AB} E^A_M E^B_N \tag{3.18}$$

The explicit conformal factors $\lambda(x^1, x^2)$, $\rho(x^1, x^2)$ have been included so that we can normalize the determinants of the 'purely external' vielbein e and 'purely internal' vielbein $\hat{e}$ as desired. Let us select

$$\rho = \det\ E^a_m \quad \Longleftrightarrow \quad \det\ \hat{e}^a_m = 1. \tag{3.19}$$

Given some basis of coordinate one-forms, we may want to contract (3.18) by $\mathrm{d}x^M$ and write

$$E^\alpha = \sqrt{\lambda}\ e^\alpha, \quad E^a = \sqrt{\rho}\ (\hat{e}^a + \hat{e}^a_n\ B^n), \quad \mathrm{d}s^2_4 = \eta_{\alpha\beta} E^\alpha E^\beta + \eta_{ab} E^a E^b\ , \tag{3.20}$$

where the one-forms e^α, B^n have legs only on $\mathrm{d}x^1$, $\mathrm{d}x^2$ while $\hat{e}^a$ has legs only on $\mathrm{d}\phi$, $\mathrm{d}t$. This lower-triangular ansatz for the vielbein is preserved under 2d Lorentz transformations acting on only e^α_μ or only $\hat{e}^a_m$, as well as any 2d diffeomorphisms on $\hat{e}^a_m$. The only other diffeomorphisms which preserve the lower-triangular form are $\mathrm{GL}(2, \mathbb{R})$ transformations on the internal coordinates $x^m = (\phi, t)$ and those of the form

$$x^m \mapsto x^m + \Gamma^m, \qquad B^n \mapsto B^n + \mathrm{d}\Gamma^n. \tag{3.21}$$

From the Kaluza-Klein perspective B^n is a pair of gauge fields in 2d, with field strengths

$$F^n_{\mu\nu} = \partial_\mu B^n_\nu - \partial_\nu B^n_\mu. \tag{3.22}$$

However, they are non-dynamical in 2d. To see this, define the 2d metric g and associated Ricci scalar with respect to e^α_μ, and construct an analogous matrix G from $\hat{e}^a_m$,

$$g^{(2)}_{\mu\nu} = \eta_{\alpha\beta} e^\alpha_\mu e^\beta_\nu, \qquad G_{mn} = \eta_{ab} \hat{e}^a_m \hat{e}^b_n. \tag{3.23}$$

Suppressing internal indices m, n,[2] the 4d Einstein-Hilbert term can then be written

$$\begin{aligned}\sqrt{\det\ g^{(4)}}\ R^{(4)} = \rho \sqrt{\det\ g^{(2)}}\ \Big[& R^{(2)} - \frac{1}{4}\mathrm{tr}(G^{-1}\partial_\mu G\ G^{-1}\partial^\mu G) \\ & + \frac{1}{4\lambda}\rho\ F^T_{\mu\nu}\ G\ F^{\mu\nu} + \lambda^{-1}\partial_\mu\lambda\ \rho^{-1}\partial^\mu\rho \Big]\ ,\end{aligned} \tag{3.24}$$

with all μ, ν indices contracted using the 2d metric $g^{(2)}_{\mu\nu}$. The corresponding field equations for the B^n_μ and its scalar dual F_0 read

$$\nabla_\mu(\rho^2 \lambda^{-1} G\ F^{\mu\nu}) = 0, \qquad \partial_\mu F_0 = 0. \tag{3.25}$$

[2] So e.g. $GF_{\mu\nu} = G_{mn} F^n_{\mu\nu}$

We see that the equations of motion fix $F_0 = \text{const.}$ which means F_0 and $F_{\mu\nu}$ must both be set to zero for asymptotically flat solutions. As we are primarily interested in this class of solutions, the $F^T_{\mu\nu}\, G\, F^{\mu\nu}$ term will be dropped from the Lagrangian moving forward.

The remaining equations of motion are

$$\begin{aligned} R^{(2)}_{\mu\nu} - \frac{1}{2} g^{(2)}_{\mu\nu} R^{(2)} = \frac{1}{4}\text{tr}(G^{-1}\partial_\mu G\, G^{-1}\partial_\nu G) - \lambda^{-1}\partial_{(\mu}\lambda\, \rho^{-1}\partial_{\nu)}\rho \\ - \frac{1}{2} g^{(2)}_{\mu\nu}\left[\frac{1}{4}\text{tr}(G^{-1}\partial_\sigma G\, G^{-1}\partial^\sigma G) - \lambda^{-1}\partial_\sigma\lambda\, \rho^{-1}\partial^\sigma\rho\right] , \end{aligned} \tag{3.26}$$

$$\nabla_\mu(\rho\, G^{-1}\partial^\mu G) = 0 \ , \tag{3.27}$$

$$\nabla_\mu \partial^\mu \rho = 0 \ . \tag{3.28}$$

With a suitable choice of coordinates and redefinition of λ, we can bring $g^{(2)}_{\mu\nu}$ to $\delta_{\mu\nu}$, setting the Ricci tensor and Ricci scalar to zero and replacing all covariant derivatives with partials in the equations above. The scalar equation $\partial_\mu\partial^\mu\rho = 0$ says that ρ is a harmonic function on $\mathbb{R}^2$. Together with it's conjugate harmonic function z, defined by $\mathrm{d}z = -\star_2 \mathrm{d}\rho$, it supplies us with canonical coordinates (ρ, z) in 2d. The flat-space equation for G,

$$\partial_\mu(\rho\, G^{-1}\partial^\mu G) = 0 \tag{3.29}$$

is none other than the equation of motion for the nonlinear σ-model with action

$$S = -\frac{1}{4}\int \mathrm{d}\rho\, \mathrm{d}z\, \rho\, \text{tr}[(G^{-1}\partial_\rho G)^2 + (G^{-1}\partial_z G)^2]. \tag{3.30}$$

Indeed, bringing the 2d metric to $\delta_{\mu\nu}$, this is the action for the second term in (3.24) (the first and third term having been turned off). Crucially, the σ-model field G is built from the vielbein on the isometry directions. We can use this to determine the degrees of freedom contained in G. The vielbein is a 2×2 matrix, and we have a chosen a normalisation such that it has unit determinant. This means that $\hat{e}$ lives in $\text{SL}(2,\mathbb{R})$, but the metric G only depends on $\hat{e}$ through the combination (3.23). In particular, the metric is invariant under $\text{SO}(2)$ transformations of the vielbein, meaning that the action (3.30) describes[3] an $\text{SL}(2,\mathbb{R})/\text{SO}(2)$ coset σ-model.

The final field equation (3.26) in canonical coordinates reads

$$\partial_z \log\lambda = \frac{\rho}{2}\text{tr}(G^{-1}\partial_\rho G\, G^{-1}\partial_z G), \qquad \partial_\rho \log\lambda = \frac{\rho}{4}\text{tr}[(G^{-1}\partial_\rho G)^2 - (G^{-1}\partial_z G)^2] \ . \tag{3.31}$$

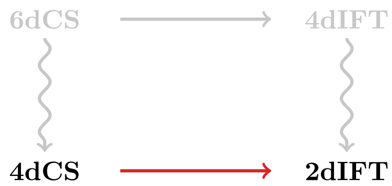
These can be interpreted as fixing λ up to a single integration constant. The integrability conditions for these equations are ensured by (3.29).

Looking back at the 4d metric ansatz in (3.6), we see how it is adapted to this integrable structure. Identifying $\lambda = \rho e^{2\nu}$ completes the match between the equations of motion in (3.31)

[3] For Lorentzian signature, the metric is invariant under $\text{SO}(1,1)$ transformations of the vielbein so the coset σ-model is $\text{SL}(2,\mathbb{R})/\text{SO}(1,1)$.

and (3.29), on the one hand, and (3.9) and (3.10) on the other. The key takeaway from this section for the remainder of this work is that the 2d σ-model given in (3.30) emerges from the reduction of pure 4d GR on a pair of isometry directions. As alluded to earlier, solutions to the full 4d Einstein equations can be constructed given a solution for the σ-model field G. Integrating (3.31) to obtain λ is obviously an important final step in these solution generating techniques. But for our present purposes, the σ-model term (3.30) alone will serve as the bridge between the actions for 4d GR and the Chern-Simons models we are interested in.

3.3 4dCS to 2dIFT



Recent developments in the field of two-dimensional integrable models have brought to light the existence of four-dimensional Chern-Simons (4dCS) theory [CY19]. This higher dimensional gauge theory provides a geometric origin for the spectral parameter appearing in the Lax formalism, making manifest the integrable structure of the lower dimensional model. The 4-manifold M_4 over which it is defined is the product of the spectral plane $\mathbb{CP}^1$ and the 2d spacetime of the integrable model $\mathbb{R}^2$. The action is built from the usual Chern-Simons 3-form for an algebra-valued gauge field A, and a meromorphic 1-form ω with poles at certain values of $Z \in \mathbb{CP}^1$. This action is given by

$$S_{4\text{dCS}} = \frac{1}{2\pi \mathrm{i}} \int_{M_4} \omega \wedge \mathrm{tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right) . \tag{3.32}$$

The poles as well as the zeros of ω constitute the essential data of the theory. Before introducing the specific ω relevant for stationary and axisymmetric gravity, let us take a moment to outline how 4dCS theories localize to 2d field theories in general. We refer the reader to [Lac22] for a pedagogical introduction to the topic.

In three dimensions, Chern-Simons theory is a topological field theory. This statement may be understood by considering the local degrees of freedom in the fundamental gauge field A. The equation of motion $F = 0$ implies that there exists a local solution g to the equation $A = g^{-1}\mathrm{d}g$. Any gauge field of this form may be fixed to $A = 0$ by a gauge transformation, so all solutions to the equations of motion are locally gauge trivial. The fact that 3dCS theory has no local degrees of freedom is a manifestation of its topological nature.

However, let us consider 3dCS theory on a manifold with boundary. We must impose boundary conditions on both the fundamental field A and on the gauge transformations. In effect, the presence of a boundary has broken some of the gauge symmetry of the theory. It may no longer be possible to transform a gauge field $A = g^{-1}\mathrm{d}g$ into $A = 0$ since the required gauge transformation may not satisfy the boundary conditions. In the bulk of the manifold (away from the boundary, that is) this has no effect and there are still no local dynamics. On the other hand, the broken gauge symmetry gives rise to local degrees of freedom which live on the boundary.

The same argument applies to 4dCS theory with the role of the boundary played by the poles in ω. It is necessary to impose boundary conditions on the gauge transformations at these points, and this leads to the emergence of physical degrees of freedom living on $\mathbb{R}^2$ alone. We will refer to these degrees of freedom as *edge modes*, and they will become the fundamental field G of the 2d theory – an $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ σ-model, for our present purposes.

3.3.1 Constructing our 4dCS theory

We will continue to work with coordinates $\{\rho, z\}$ on spacetime and consider the 4dCS theory (3.32) with meromorphic 1-form

$$\omega = -\frac{\rho}{2}\left(\frac{Z^2+1}{Z^2}\,\mathrm{d}Z\right) + \frac{Z^{-1}-Z}{2}\,\mathrm{d}\rho + \mathrm{d}z\ . \tag{3.33}$$

This may seem like an ad hoc choice. For the time being, we will adopt the philosophy that "the proof is in the pudding", and justify this by showing that it leads to the 2d integrable model known to describe stationary axisymmetric GR. However, in section 3.6 we will show that (3.33) is a consequence of the ∂_ϕ, ∂_τ reduction isometries, emerging from a corresponding reduction of 6dCS theory to our 4dCS model.

To the best of our knowledge, all previous 4dCS constructions have considered 1-forms ω which may be written in terms of a twist function $\varphi(Z)$ as $\omega = \varphi(Z)\,\mathrm{d}Z$. In particular, they only have legs on $\mathbb{CP}^1$ and only depend on the $\mathbb{CP}^1$ directions. By comparison, our meromorphic 1-form ω mixes the spacetime and $\mathbb{CP}^1$ directions. We will now describe the construction of this 1-form by starting with a more conventional 1-form and introducing a spacetime-dependent branch cut. Consider another spectral plane parameterised by $W \in \mathbb{CP}^1$ and equipped with the meromorphic 1-form $\mathrm{d}W$. This 1-form has a second order pole at $W = \infty$, which can be seen by moving to the other patch covering $\mathbb{CP}^1$. Now, let us insert a branch cut in this spectral plane between the points $W = z + \mathrm{i}\rho$ and $W = z - \mathrm{i}\rho$. This introduces a two-sheeted covering of the W-plane which we can parameterise by the coordinate

$$Z = \frac{1}{\rho}\left(z - W \pm \sqrt{(W-z)^2 + \rho^2}\right)\ . \tag{3.34}$$

Since this is a double covering, there are two values of Z for each value of W except for the points at the end of the branch cut where the radicand vanishes. We can invert this relation to find the two-to-one map from Z to W,

$$W = z + \frac{\rho}{2}\left(Z^{-1} - Z\right)\ . \tag{3.35}$$

We can move between the two sheets of the double covering with the map $Z \mapsto -Z^{-1}$, which preserves W. The meromorphic 1-form ω given in (3.33) is simply $\mathrm{d}W$ after moving to the two-sheeted cover parameterised by Z. This ω has two second order poles, at $Z = 0$ and $Z = \infty$, which are the preimages of $W = \infty$.

Despite our unconventional choice of 1-form, the essential properties of the 4dCS theory are retained. Since this is a surprising result, we will take some time to demonstrate this statement and review the fundamentals of 4dCS theory. It is convenient to work in complex coordinates on spacetime given by $\xi = z + \mathrm{i}\rho$ and $\bar{\xi} = z - \mathrm{i}\rho$. In these coordinates, the meromorphic 1-form is given by

$$\omega = \frac{\mathrm{i}(\xi - \bar{\xi})}{4}\left(\frac{Z^2+1}{Z^2}\,\mathrm{d}Z\right) + \frac{\mathrm{i}(Z-\mathrm{i})^2}{4Z}\,\mathrm{d}\xi - \frac{\mathrm{i}(Z+\mathrm{i})^2}{4Z}\,\mathrm{d}\bar{\xi}\ . \tag{3.36}$$

The singularities in ω play the role of boundaries in our theory and we must impose boundary conditions on the gauge field A at these points. To see this, consider the variation of the action,

$$\delta S_{4\mathrm{dCS}} = \frac{2}{2\pi\mathrm{i}}\int_{M_4} \omega \wedge \mathrm{tr}\big(\delta A \wedge F\big) + \frac{1}{2\pi\mathrm{i}}\int_{M_4} \mathrm{d}\omega \wedge \mathrm{tr}\big(\delta A \wedge A\big)\ . \tag{3.37}$$

The first term gives the bulk equations of motion $\omega \wedge F = 0$, while the second term is a boundary term which must be killed by imposing constraints on the gauge field. One might expect that $\mathrm{d}\omega$ is identically zero since the 1-form may be expressed as $\omega = \mathrm{d}W$. This is almost correct, but the argument fails where ω is singular. At these points, we must make use of the identities from complex analysis

$$\partial_{\bar{Z}}\left(\frac{1}{Z}\right) = -2\pi\mathrm{i}\,\delta(Z)\ , \qquad \int_{\mathbb{CP}^1} \mathrm{d}Z \wedge \mathrm{d}\bar{Z}\,\delta(Z)\,f(Z) = f(0)\ . \tag{3.38}$$

This means that $\mathrm{d}\omega$ is a distribution with support on $\mathbb{CP}^1$ at the poles of ω. It is for this reason that we refer to the second term in the variation as a boundary term. More explicitly, we can write the contribution from $Z = 0$ as

$$\frac{\mathrm{d}\omega}{2\pi\mathrm{i}} = \partial_Z\delta(Z)\ \frac{\mathrm{i}(\xi-\bar{\xi})}{4}\mathrm{d}\bar{Z} \wedge \mathrm{d}Z + \delta(Z)\frac{\mathrm{i}}{4}\,\mathrm{d}\bar{Z} \wedge \mathrm{d}\xi - \delta(Z)\frac{\mathrm{i}}{4}\,\mathrm{d}\bar{Z} \wedge \mathrm{d}\bar{\xi}. \tag{3.39}$$

Sufficient boundary conditions on the gauge field which cause the boundary term in $\delta S_{4\mathrm{dCS}}$ to vanish are given by[4]

$$A\big|_{Z=0} = 0\ , \qquad A\big|_{Z=\infty} = 0\ . \tag{3.40}$$

By a similar argument, the gauge transformations $\delta_\epsilon A = \mathrm{d}\epsilon + [A, \epsilon]$ must obey boundary conditions given by

$$\mathrm{d}\epsilon\big|_{Z=0} = 0\ , \qquad \mathrm{d}\epsilon\big|_{Z=\infty} = 0\ . \tag{3.41}$$

[4]When we write the boundary conditions on the gauge field, we would like to highlight that the $\mathrm{d}Z$ legs do not contribute due to the restriction map, that is $\mathrm{d}Z|_{Z=0} = 0$. This is in contrast to the boundary variation where the Z-component of A will appear explicitly. Despite this observation, it is sufficient to constrain only the ξ and $\bar{\xi}$-components in this basis.

The zeroes of ω also play an important role in 4dCS theory. Due to the zeros at $Z = \pm\mathrm{i}$, we should allow the gauge field A to have simple poles at these points. If A_ξ has a simple pole at $Z = \mathrm{i}$ and $A_{\bar{\xi}}$ has a simple pole at $Z = -\mathrm{i}$, the action S_{4dCS} remains finite. The presence of singularities in the gauge field is far from a problem in 4dCS theory – it is a crucial feature to capture the usual meromorphic dependence of a Lax connection. We will therefore allow these singularities in our field configurations.

Turning to the symmetries of the action, notice that we have a trivial shift symmetry acting as

$$A \mapsto A + C_\omega \, \omega \ . \tag{3.42}$$

In the 1-form basis $\{\mathrm{d}W, \mathrm{d}\bar{W}, \mathrm{d}\xi, \mathrm{d}\bar{\xi}\}$ adapted to this symmetry, $\omega = \mathrm{d}W$ and thus the component A_W does not contribute to the action – it decouples. For this reason we will often neglect to mention A_W as it can always be fixed to zero. Transforming to the basis $\{\mathrm{d}Z, \mathrm{d}\bar{Z}, \mathrm{d}\xi, \mathrm{d}\bar{\xi}\}$, this choice also sets $A_Z = 0$.

This almost completes our definition of the theory, but there is one final ingredient which we would like to introduce. The interpretation of the Z-plane as a double covering of the W-plane calls for an additional restriction on the gauge field A. Rather than having generic dependence on Z, we would like to think of the gauge field A as living on the spectral plane parameterised by W. One might think to impose the constraint $A(Z) = A(-1/Z)$ so that A is single-valued on the W-plane. We instead allow a non-trivial transformation on the Lie algebra indices of A, introducing the $\mathbb{Z}_2$-automorphism of the algebra $\mathfrak{g}$. In the context of stationary axisymmetric gravity, the appropriate Lie algebra is $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ and the automorphism $\eta : \mathfrak{g} \to \mathfrak{g}$ is given by $x \mapsto -x^T$. We impose the equivariance condition

$$A(Z) = \eta\big(A(-1/Z)\big) \ . \tag{3.43}$$

This means that the values of the gauge field on each sheet of the covering are not independent, and knowing one is sufficient to know them both. In the resulting 2dIFT, this restricts the target space from $\mathrm{SL}(2,\mathbb{R})$ to the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. It is also important that this is compatible with the boundary conditions we have imposed on the gauge field. In the literature on 4dCS theory, this equivariance condition is also known as a *branch cut defect* [CY19]. This completes the definition of our 4dCS theory.

3.3.2 Localisation to 2dIFT

Next, we would like to show that 4dCS theory localises to an integrable field theory on spacetime. In particular, unlike a Kaluza-Klein reduction in which infinitely many modes are discarded from the theory, a finite number of fields on $\mathbb{R}^2$ capture all of the physical degrees of freedom in 4dCS theory.

As a first step in this localisation, it is helpful to introduce a field redefinition which separates the bulk gauge field (i.e. away from $Z = 0$ or $Z = \infty$) from the edge modes. We introduce two new fields $\mathcal{L}$ and $\hat{g}$ defined by

$$A = \mathcal{L}^{\hat{g}}, \qquad \mathcal{L}^{\hat{g}} \equiv \hat{g}^{-1}\mathcal{L}\hat{g} + \hat{g}^{-1}\mathrm{d}\hat{g} \ . \tag{3.44}$$

This new parameterisation is partially redundant and introduces an *internal* gauge symmetry which acts as

$$\mathcal{L} \mapsto \mathcal{L}^{\check{h}} \ , \qquad \hat{g} \mapsto \check{h}^{-1}\hat{g} \ . \tag{3.45}$$

Notice that the original field A is invariant under this transformation, meaning that it preserves the action. We can leverage this internal symmetry to significantly simplify the equations of motion for the theory. Denoting the contraction of a vector field X with a differential form A by $X \lrcorner A$, we will impose the gauge fixing constraint

$$\partial_{\bar{Z}} \lrcorner \mathcal{L} = 0 \quad \Longleftrightarrow \quad \mathcal{L}_{\bar{Z}} = 0 \ . \tag{3.46}$$

There are some residual symmetries after this gauge fixing, including the portion of the original gauge symmetry satisfying the boundary conditions. In a moment, we will use the remaining gauge symmetries to impose constraints on the edge mode $\hat{g}$, but first let us return to the action. In the new variables, it is given by

$$S_{\text{4dCS}} = \frac{1}{2\pi\mathrm{i}} \int_{M_4} \omega \wedge \mathrm{tr}\big(\mathcal{L} \wedge \mathrm{d}\mathcal{L}\big) + \frac{1}{2\pi\mathrm{i}} \int_{M_4} \mathrm{d}\omega \wedge \mathrm{tr}\big(\mathcal{L} \wedge \mathrm{d}\hat{g}\hat{g}^{-1}\big) - \frac{1}{6\pi\mathrm{i}} \int_{M_4} \omega \wedge \mathrm{tr}\big(\hat{g}^{-1}\mathrm{d}\hat{g}\big)^3 \ . \tag{3.47}$$

In the second term, the edge mode $\hat{g}$ appears against the 2-form $\mathrm{d}\omega$. While it might appear that this term depends on the value of $\hat{g}$ over all of M_4, the presence of the distribution $\mathrm{d}\omega$ means that it only depends on the fields

$$\hat{g}\big|_{Z=0} = g \ , \qquad \hat{g}^{-1}\partial_Z\hat{g}\big|_{Z=0} = \phi \ , \qquad \hat{g}\big|_{Z=\infty} = \tilde{g} \ , \qquad \hat{g}^{-1}\partial_Z\hat{g}\big|_{Z=\infty} = \tilde{\phi} \ . \tag{3.48}$$

The $\mathrm{d}\omega$ term cares about both the edge mode and its $\mathbb{CP}^1$-derivatives because ω contains second order poles. Higher order poles would lead to higher order derivatives contributing to the action. Readers familiar with the 2d Wess-Zumino-Witten (WZW) model might recognise the 3-form in the third term as a Wess-Zumino (WZ) term. In that context, despite being a 3-form integrated over a 3-manifold, the WZ term only produces 2d dynamics on the boundary. Based on the similarity, one might expect that this remains true in the present context, and this is indeed the case.

First, consider an extension of $\hat{g}$ over the 5-manifold $M_5 = M_4 \times [0, 1]$ which reproduces $\hat{g}$ on $M_4 \times \{1\}$ and is the trivial map on $M_4 \times \{0\}$. Denoting this extension by the same symbol $\hat{g}$ in an abuse of notation, we may write the WZ term as a surface integral over M_5,

$$\int_{M_4} \omega \wedge \mathrm{tr}\big(\hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g}\big) = \int_{M_5} \mathrm{d}\Big[\omega \wedge \mathrm{tr}\big(\hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g}\big)\Big] \ . \tag{3.49}$$

Since the $\hat{g}$-dependent 3-form in this expression is closed, the exterior derivative on the right hand side can only act on ω, producing the desired distribution on $\mathbb{CP}^1$. This allows us to write the action as

$$S_{4\mathrm{dCS}} = \frac{1}{2\pi \mathrm{i}} \int_{M_4} \omega \wedge \mathrm{tr}\big(\mathcal{L} \wedge \mathrm{d}\mathcal{L}\big) + \frac{1}{2\pi \mathrm{i}} \int_{M_4} \mathrm{d}\omega \wedge \Big[\mathrm{tr}\big(\mathcal{L} \wedge \mathrm{d}\hat{g}\hat{g}^{-1}\big) - \mathrm{WZ}[\hat{g}]\Big] \ , \tag{3.50}$$

where we define the WZ 2-form by

$$\mathrm{WZ}[\hat{g}] = \frac{1}{3} \int_{[0,1]} \mathrm{tr}\big(\hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g}\big) \ . \tag{3.51}$$

From this expression for the action, we see that the edge mode indeed appears in the action only through the 'boundary' fields (3.48), justifying its name.

Let us consider the degrees of freedom in the edge mode more carefully. Earlier, we saw that the original gauge symmetries of 4dCS theory are constrained to obey the boundary conditions (3.41). While these restrict the spacetime derivatives of allowed gauge transformations, their $\mathbb{CP}^1$-derivatives are unconstrained, meaning that we can choose to fix $\phi = 0$ and $\tilde{\phi} = 0$. We should also consider the compatibility of our field configurations with the equivariance condition (3.43). This condition exchanges the two poles of ω, meaning that the values of the edge mode at these points must be related by $\tilde{g} = \eta(g)$. In summary, the physical degrees of freedom in the edge mode are captured by

$$\hat{g}\big|_{Z=0} = g \ , \qquad \hat{g}^{-1}\partial_Z \hat{g}\big|_{Z=0} = 0 \ , \qquad \hat{g}\big|_{Z=\infty} = \eta(g) \ , \qquad \hat{g}^{-1}\partial_Z \hat{g}\big|_{Z=\infty} = 0 \ . \tag{3.52}$$

Now that we have the degrees of freedom of our theory in hand, and have applied some helpful gauge fixing conditions, we will continue with the localisation to spacetime. The next step is to solve a subset of the equations of motion to explicitly fix the $\mathbb{CP}^1$-dependence of $\mathcal{L}$. Since the equations of motion read

$$\omega \wedge F = 0 \ , \qquad F \equiv \mathrm{d}\mathcal{L} + \frac{1}{2}[\mathcal{L}, \mathcal{L}] \ , \tag{3.53}$$

it is convenient to work in the 1-form basis $\{\mathrm{d}W, \mathrm{d}\bar{W}, \mathrm{d}\xi, \mathrm{d}\bar{\xi}\}$ and its duals vector basis[5] $\{\mathring{\partial}_W, \mathring{\partial}_{\bar{W}}, \mathring{\partial}_\xi, \mathring{\partial}_{\bar{\xi}}\}$, so that we have

$$\mathrm{d}W \wedge F = 0 \quad \Leftrightarrow \quad \mathring{F}_{\xi\bar{\xi}} = 0 \ , \quad \mathring{F}_{\bar{W}\xi} = \mathring{F}_{\bar{W}\bar{\xi}} = 0 \ . \tag{3.54}$$

The $\mathring{F}_{\xi\bar{\xi}}$ equation will become the equation of motion for the 2dIFT. Using the (internal) gauge-

[5] We use $^\circ$ to distinguish vector and form components expressed in this basis from those in the usual $Z, \bar{Z}, \xi, \bar{\xi}$ coordinate basis.

fixing (3.46), the latter two equalities tell us

$$\mathring{\partial}_{\bar{W}}\mathring{\mathcal{L}}_{\xi} = 0 \ , \qquad \mathring{\partial}_{\bar{W}}\mathring{\mathcal{L}}_{\bar{\xi}} = 0 \ , \tag{3.55}$$

meaning that these components of $\mathcal{L}$ are holomorphic functions of W. The relationship between the ∂_a and $\mathring{\partial}_a$ bases ensure $\mathring{\mathcal{L}}_{\xi}$ and $\mathring{\mathcal{L}}_{\bar{\xi}}$ are holomorphic functions of Z as well. If they were bounded, this would imply they were constant by Liouville's theorem. However, recall that we have allowed these components of our gauge field to have singularities: $\mathcal{L}_{\xi}$ is allowed a simple pole at $Z = -\mathrm{i}$ and $\mathcal{L}_{\bar{\xi}}$ is allowed a simple pole at $Z = +\mathrm{i}$. Moving to the other basis, the same can be said about the components $\mathring{\mathcal{L}}_{\xi}$ and $\mathring{\mathcal{L}}_{\bar{\xi}}$. The general solution with these properties is given by

$$\mathring{\mathcal{L}}_{\xi} = \frac{1}{1-\mathrm{i}Z}U_{\xi} - \frac{\mathrm{i}Z}{1-\mathrm{i}Z}V_{\xi} \ , \qquad \mathring{\mathcal{L}}_{\bar{\xi}} = \frac{1}{1+\mathrm{i}Z}U_{\bar{\xi}} + \frac{\mathrm{i}Z}{1+\mathrm{i}Z}V_{\bar{\xi}} \ , \tag{3.56}$$

where the Us and Vs are functions of the spacetime coordinates ξ, $\bar{\xi}$ alone. Note, we could have expressed each component of $\mathcal{L}$ as the sum of a term which is constant on $\mathbb{CP}^1$ and a term which is singular. Instead we have chosen to collect sums and differences of those two terms such that each term vanishes at either $Z = 0$ or $Z = \infty$. This parameterisation makes it particularly easy to solve the boundary conditions (3.40) which, together with the field redefinition (3.44), yields

$$U_{\xi} = -\partial_{\xi}gg^{-1} \ , \qquad U_{\bar{\xi}} = -\partial_{\bar{\xi}}gg^{-1} \ , \qquad V_{\xi} = -\partial_{\xi}\tilde{g}\tilde{g}^{-1} \ , \qquad V_{\bar{\xi}} = -\partial_{\bar{\xi}}\tilde{g}\tilde{g}^{-1} \ . \tag{3.57}$$

Substituting these solutions back into the expression for the Lax connection (3.56), it does not immediately agree with the expected Lax given in (3.11). However, a gauge transformation[6] by $\tilde{g}$ brings the Lax derived from 4dCS theory into the desired form,

$$\mathring{\mathcal{L}}_{\xi} = \frac{-\partial_{\xi}GG^{-1}}{1-\mathrm{i}Z} \ , \qquad \mathring{\mathcal{L}}_{\bar{\xi}} = \frac{-\partial_{\bar{\xi}}GG^{-1}}{1+\mathrm{i}Z} \ , \qquad G \equiv \tilde{g}^{-1}g \ . \tag{3.58}$$

This derivation of the Lax connection for the 2d integrable model is a standard feature of 4dCS theories. In the application to stationary axisymmetric GR, this G is precisely the $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ field of section 3.2, encoding the metric components along the isometry directions. Indeed, the equivariance condition (3.43) tells us that $\tilde{g} = g^T$ implying $G = g^T g$ and we see that the edge mode g is identified with the vielbein $\hat{e}$.

Having fixed the $\mathbb{CP}^1$-dependence of $\mathcal{L}$, we can return to the action (3.50) and localise it to two-dimensional spacetime. The first integrand is proportional to $\omega \wedge \mathrm{tr}(\mathcal{L} \wedge \mathring{\partial}_{\bar{W}}\mathcal{L})$ and thus vanishes on-shell. This follows from $\omega = \mathrm{d}W$ and the internal gauge fixing $\mathcal{L}_{\bar{Z}} = 0$ (which also sets $\mathring{\mathcal{L}}_{\bar{W}} = 0$), requiring $\mathrm{d}\mathcal{L}$ to saturate the $\mathrm{d}\bar{W}$ leg. This leaves the boundary contributions,

[6] One might object that the gauge transformation by $\tilde{g}$ is not a symmetry of the theory. It would fix the value of the edge mode at infinity which would break the equivariance condition $\tilde{g} = \eta(g)$. One resolution is to understand the gauge equivalence of the Lax connections as a formal statement at the level of equations of motion: the flatness conditions are equivalent. Alternatively, one might weaken the equivariance condition, demanding that it is respected only up to a gauge transformation. Both approaches seem valid at this level.

namely those terms proportional to $\mathrm{d}\omega$.

For our choice of ω and the gauge fixing conditions (3.52) the boundary WZ-term vanishes. Since $\hat{g}^{-1}\partial_Z\hat{g}$ has been fixed to zero at $Z=0$ and $Z=\infty$, $\mathrm{d}\omega$ must contribute a $\mathrm{d}Z$ leg in this term. Performing an integration by parts, the contribution from $Z=0$ is

$$\frac{1}{2\pi\mathrm{i}}\int_{M_4}\mathrm{d}\omega\wedge\mathrm{WZ}[\hat{g}]=-\frac{\mathrm{i}}{4}\int_{M_4}\mathrm{d}\bar{Z}\wedge\mathrm{d}Z\wedge\partial_Z(\mathrm{WZ}[\hat{g}])\;\delta(Z)\;(\xi-\bar{\xi})\;, \tag{3.59}$$

which vanishes, again since $\hat{g}^{-1}\partial_Z\hat{g}=0$ at the boundaries. The contribution from this term at $Z=\infty$ also vanishes. It remains to compute the second term in the action, whose contribution at $Z=0$ is given by

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_{M_4}\mathrm{d}\omega\wedge\mathrm{tr}\big(\mathcal{L}\wedge\mathrm{d}\hat{g}\hat{g}^{-1}\big)=\frac{\mathrm{i}}{4}\int_{M_4}&\mathrm{d}Z\wedge\mathrm{d}\bar{Z}\wedge\mathrm{d}\xi\wedge\mathrm{d}\bar{\xi}\;\delta(Z)(\xi-\bar{\xi})\\ \times\,\mathrm{tr}\,\big[&\big(\partial_Z\mathcal{L}_\xi+(\xi-\bar{\xi})^{-1}\mathcal{L}_Z\big)\,\partial_{\bar{\xi}}\hat{g}\hat{g}^{-1}-\big(\partial_Z\mathcal{L}_{\bar{\xi}}-(\xi-\bar{\xi})^{-1}\mathcal{L}_Z\big)\,\partial_\xi\hat{g}\hat{g}^{-1}\big]\,.\end{aligned} \tag{3.60}$$

The appearance of $\mathcal{L}_Z$ in this expression may be surprising as it is related to the W-component which we saw decouples from the theory. While is it valid to fix this component to zero from the beginning, this is unnecessary as it drops out on its own. This can be seen as follows. In order to use our solutions for the $\mathbb{CP}^1$-dependence of $\mathcal{L}$, we need to change basis using

$$\begin{aligned}(\partial_Z\mathcal{L}_\xi+(\xi-\bar{\xi})^{-1}\mathcal{L}_Z)\big|_{Z=0}&=\partial_Z\mathring{\mathcal{L}}_\xi\big|_{Z=0}\\ (\partial_Z\mathcal{L}_{\bar{\xi}}-(\xi-\bar{\xi})^{-1}\mathcal{L}_Z)\big|_{Z=0}&=\partial_Z\mathring{\mathcal{L}}_{\bar{\xi}}\big|_{Z=0}.\end{aligned}$$

Substituting (3.56) and (3.57) and integrating over $\mathbb{CP}^1$ with the help of the delta functions, we arrive at

$$\frac{1}{2}\int_{\mathbb{R}^2}\mathrm{d}\xi\wedge\mathrm{d}\bar{\xi}\;(\xi-\bar{\xi})\;\mathrm{tr}(\partial_\xi gg^{-1}-\partial_\xi\tilde{g}\tilde{g}^{-1})(\partial_{\bar{\xi}}gg^{-1}-\partial_{\bar{\xi}}\tilde{g}\tilde{g}^{-1}). \tag{3.61}$$

This is the action of the 2dIFT and can be rewritten as

$$S_{\mathrm{2dIFT}}=-\frac{1}{2}\int_{\mathbb{R}^2}\mathrm{d}\rho\wedge\mathrm{d}z\;\rho\;\mathrm{tr}[(G^{-1}\partial_\rho G)^2+(G^{-1}\partial_z G)^2]\;. \tag{3.62}$$

In particular, when the original 4dCS gauge field is valued in the Lie Algebra $\mathfrak{sl}(2,\mathbb{R})$, this is the σ-model derived from 4d stationary and axisymmetric GR in (3.30).

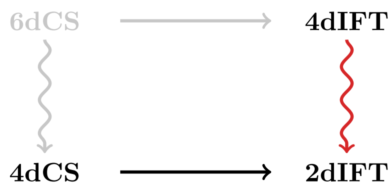
Having derived the 2dIFT describing stationary, axisymmetric gravity from 4dCS theory, one might ask what this buys you. First and foremost, many technical aspects of this integrable system are now encoded in terms the geometry of the underlying 4-manifold, and the Lax connection has a natural home as the fundamental gauge field of 4dCS theory. In terms of applications, the inverse scattering method is a natural place to start. It relies heavily on the Lax formalism and after nearly half a century remains one of the most powerful tools to find and classify exact solutions, particularly axially symmetric black hole geometries in $D = 4$ and $D = 5$. Solitonic solutions like these are a fairly generic feature of integrable models. A new solution is generated from a "seed" solution by applying a $\mathbb{CP}^1$-dependent gauge transformation to the Lax. In particular, the gauge transformation parameter does not have generic dependence on $\mathbb{CP}^1$, but is a rational function with simple poles and constraints on the residues of these poles.

Given that the 4dCS description is adapted to the Lax formalism, one might hope that the inverse scattering method has a natural home in this theory. Indeed, there is a reasonable candidate for the origin of these transformations in the residual symmetries of our theory. Recall that our derivation of the 2dIFT involved imposing the gauge fixing condition $\mathcal{L}_{\bar{Z}} = 0$ in equation (3.46). This imposes a substantial constraint on the internal gauge symmetries, namely

$$\check{h}^{-1}\partial_{\bar{Z}}\check{h} = 0 \ . \tag{3.63}$$

The solution to this constraint that is most commonly encountered in the literature on 4dCS theory is to take $\check{h}$ to be independent of $\mathbb{CP}^1$. However, if we allow $\check{h}$ to be a rational function of $\mathbb{CP}^1$ with singularities, there may be a wider class of residual symmetries solving this constraint. Generically such a transformation would alter the $\mathbb{CP}^1$-dependence of the Lax connection, producing poles at unwanted locations, but appropriate conditions on the residues of $\check{h}$ should ensure that no problems arise [HSS84]. In fact, this discussion should apply more generally to all 4dCS theories and their associated 2dIFTs. We leave this interesting application of 4dCS theory as an avenue for future work.

3.4 Origin of 2dIFT from 4dWZW



In the previous section, we constructed and studied a 4dCS theory which describes stationary, axisymmetric gravity. The main strength of this formalism is that it makes the integrable structure of the 2d model manifest, allowing these aspects of the theory to be studied directly. Key details of this construction appear mysterious, including a spacetime-dependent branch cut in the spectral plane and an equivariance condition on the Lax connection. The goal of the following sections is to provide a geometric origin for each feature of the 4dCS theory. Our approach will be to realise the 2dIFT (3.62) as a reduction of an integrable 4d theory known to arise[7] from a 6d Chern-Simons (6dCS) theory [Cos20; BS23]. Having understood this spacetime reduction (represented in red in the diagram above), we can find the corresponding reduction on twistor

[7] For a review of the relationship between the 4dWZW model and 6dCS theory, see section 2.4.

space, the underlying manifold of 6dCS theory. Applying the twistor space reduction to 6dCS theory we recover the 4dCS setup described in the previous section.

Beginning this journey, the integrable 4d theory in question is known as the 4d Wess-Zumino-Witten (4dWZW) model [Don85; Los+96] and is defined by the action

$$S_{4\mathrm{dWZW}} = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}\big(G^{-1}\mathrm{d}G \wedge \star G^{-1}\mathrm{d}G\big) - \int_{\mathbb{R}^4} \mu \wedge \mathrm{WZ}[G] \ , \tag{3.64}$$

where, for our purposes, G is an $\mathrm{SL}(2,\mathbb{R})$-valued field. Its equation of motion is $\mu \wedge \partial(\bar{\partial}GG^{-1}) = 0$. In these expressions, we have introduced a 2-form $\mu = \mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 + \mathrm{d}u^2 \wedge \mathrm{d}\bar{u}^2$ which is proportional to the Kähler form on $\mathbb{R}^4$ equipped with the Euclidean metric. The presence of this 2-form breaks some of the spacetime symmetry of the theory: the action is only invariant under diffeomorphisms which preserve the Kähler form. This includes all translations on $\mathbb{R}^4$, but only a subgroup $\mathrm{U}(2) \subset \mathrm{SO}(4)$ of rotations. This subgroup is generated by

$$\begin{aligned} R_0 =& \frac{\mathrm{i}}{2}\big(u^1\partial_{u^1} - \bar{u}^1\partial_{\bar{u}^1} + u^2\partial_{u^2} - \bar{u}^2\partial_{\bar{u}^2}\big) \ , \\ R_1 =& \frac{\mathrm{i}}{2}\big(u^1\partial_{u^1} - \bar{u}^1\partial_{\bar{u}^1} - u^2\partial_{u^2} + \bar{u}^2\partial_{\bar{u}^2}\big) \ , \\ R_2 =& \frac{\mathrm{i}}{2}\big(u^2\partial_{u^1} - \bar{u}^2\partial_{\bar{u}^1} + u^1\partial_{u^2} - \bar{u}^1\partial_{\bar{u}^2}\big) \ , \\ R_3 =& \frac{1}{2}\big(u^2\partial_{u^1} + \bar{u}^2\partial_{\bar{u}^1} - u^1\partial_{u^2} - \bar{u}^1\partial_{\bar{u}^2}\big) \ . \end{aligned} \tag{3.65}$$

The central $\mathrm{U}(1)$ is generated by R_0 while the other three generators form an $\mathrm{SU}(2)$ subalgebra, $[R_i, R_j] = \varepsilon_{ijk}R_k$.

We would like to perform a dimensional reduction to two-dimensions, by the spacetime vector fields

$$X_\phi = R_0 + R_1 = \mathrm{i}\big(u^1\partial_{u^1} - \bar{u}^1\partial_{\bar{u}^1}\big) \ , \qquad X_\tau = \mathrm{i}\big(\partial_{u^2} - \partial_{\bar{u}^2}\big) \ . \tag{3.66}$$

These vector fields are simpler in cylindrical coordinates, defined by $u^1 = \rho e^{\mathrm{i}\phi}$ and $u^2 = z + \mathrm{i}\tau$, in which they read

$$X_\phi = \partial_\phi \ , \qquad X_\tau = \partial_\tau \ . \tag{3.67}$$

Whilst it is not surprising that the requisite reduction vectors are identical to those used to obtain the 2dIFT from 4d GR, it does not seem to the authors that this had to be the case. These theories are defined on different manifolds: the spacetime of the WZW model is $\mathbb{R}^4$ equipped with a flat metric, rather than a dynamical spacetime metric in the case of 4d GR. Nevertheless, in both cases the reduction requires a restriction to stationary and axisymmetric solutions, which for the 4dWZW model imposes

$$\partial_\phi G = 0 \ , \qquad \partial_\tau G = 0 \ . \tag{3.68}$$

This constraint on the group-valued field G implies that the 2-form $\mathrm{WZ}[G]$ does not have support on either $\mathrm{d}\phi$ or $\mathrm{d}\tau$. In cylindrical coordinates, the Kähler form reads

$$\mu = 2\mathrm{i}(\rho \, \mathrm{d}\phi \wedge \mathrm{d}\rho + \mathrm{d}\tau \wedge \mathrm{d}z), \tag{3.69}$$

so it cannot compensate by saturating both $\mathrm{d}\phi$ and $\mathrm{d}\tau$. Another way of saying this [Cos21] is that the 2-torus parameterised by a compactification of ϕ and τ has zero Kähler volume, meaning that $\int_\phi \int_\tau \mu = 0$. As a result the WZ term does not contribute to the reduced 2d model. Imposing (3.68) on the surviving term in $S_{4\mathrm{dWZW}}$ and performing the reduction (contracting by X_ϕ and X_τ), we obtain the now familiar two-dimensional action

$$S_{2\mathrm{dIFT}} = -\frac{1}{2} \int_{\mathbb{R}^2} \mathrm{d}\rho \wedge \mathrm{d}z \; \rho \; \mathrm{tr}[(G^{-1}\partial_\rho G)^2 + (G^{-1}\partial_z G)^2] \; . \tag{3.70}$$

Whilst this is identical to the action we are looking for, there is a key difference between this theory and the theory which describes gravity. In the context of stationary axisymmetric GR, the matrix G parameterises the metric components along the isometry directions. In particular, this means that G should be symmetric, taking values in the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ rather than the whole of $\mathrm{SL}(2,\mathbb{R})$. We can pick out the subgroup $\mathrm{SO}(2) \subset \mathrm{SL}(2,\mathbb{R})$ by defining a $\mathbb{Z}_2$-automorphism of the algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ acting as

$$\eta : \mathfrak{g} \to \mathfrak{g} \; , \qquad \eta : x \mapsto -x^T \; . \tag{3.71}$$

Since this map squares to the identity, the eigenvalues of η are ± 1. We can decompose the algebra into the corresponding subspaces as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \; , \qquad \mathfrak{g}_0 \cong \mathfrak{so}(2) \; . \tag{3.72}$$

Here $\mathfrak{g}_0$ denotes the subspace preserved by η, which coincides with $\mathfrak{so}(2)$. We would like to restrict our group element to live in the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. This would mean that the algebra element generating G lives in $\mathfrak{g}_1$, the -1-eigenspace of η, which exponentiates to[8] the constraint $\eta(G) = G^{-1}$. For the automorphism specified above, this translates to $(G^{-1})^T = G^{-1}$ implying that G is symmetric as desired.

One might try to impose this constraint by demanding that the action is invariant under the transformation $G \mapsto \eta(G) = G^{-1}$. However, we see that the cubic term proportional to μ is not invariant under this transformation, it picks up a minus sign. To compensate for this, we will simultaneously apply a discrete spacetime transformation which preserves μ up to a sign. Such a transformation which is compatible with our reduction vectors is

$$\sigma : (\rho, \phi, z, \tau) \mapsto (\rho, -\phi, z, -\tau) \; . \tag{3.73}$$

[8] With a slight abuse of notation, we are denoting the group automorphism with the same symbol as the algebra automorphism.

This reflection generates a $\mathbb{Z}_2$-action on spacetime. The action $S_{4\text{dWZW}}$ is invariant under the combination of σ and η provided that $\eta(G) = G^{-1}$. We demand that the group element obeys this constraint as part of our reduction, which restricts the resulting 2d field to the coset $\text{SL}(2,\mathbb{R})/\text{SO}(2)$.

The 4dWZW model is integrable, and can be described by a 6dCS theory on twistor space (for an introduction to this model and the details of this relationship see section 2.4). Having established a reduction of the 4dWZW model which lands on the 2dIFT for stationary and axisymmetric GR, we will use this as a bridge to connect to 6dCS theory. In fact, this was the method by which we derived the 4dCS model presented earlier.

3.5 Lift of reduction vectors to twistor space

In this section, we lift the spacetime reduction by X_ϕ, X_τ to a reduction of twistor space so that it can be applied to 6dCS theory in section 3.6. In preparation for this endeavour, let us review some relevant aspects of the geometry of (Euclidean) twistor space, which we will denote by $\mathbb{PT}$. This section closely follows the exposition in [MW91].

As a real manifold, there is an isomorphism

$$\mathbb{PT} \cong \mathbb{CP}^1 \times \mathbb{R}^4 \ . \tag{3.74}$$

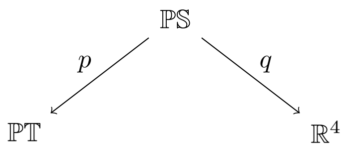
On the Riemann sphere factor, we will use a complex coordinate $\zeta \in \mathbb{CP}^1$, and we will use standard Cartesian coordinates $\{x^1, x^2, x^3, x^4\}$ on $\mathbb{R}^4$. It is also helpful to employ the complex combinations $u^1 = x^1 + \mathrm{i}x^2$ and $u^2 = x^3 + \mathrm{i}x^4$ which appeared in the previous section. While this is an isomorphism of real manifolds, meaning that these serve as good coordinates on twistor space, the complex structure on twistor space mixes the $\mathbb{CP}^1$ and $\mathbb{R}^4$ factors. Holomorphic coordinates on twistor space are given by

$$\zeta \ , \qquad v^1 = u^1 - \zeta\,\bar{u}^2 \ , \qquad v^2 = u^2 + \zeta\,\bar{u}^1 \ . \tag{3.75}$$

It is important to highlight that $\{u^1, u^2\}$ are *not* holomorphic coordinates on twistor space, though they coincide with $\{v^1, v^2\}$ on the $\mathbb{R}^4$ defined by $\zeta = 0$. Holomorphic coordinates on the northern patch are related to those on the southern patch by

$$\tilde{\zeta} = 1/\zeta \ , \qquad \tilde{v}^1 = v^1/\zeta \ , \qquad \tilde{v}^2 = v^2/\zeta \ . \tag{3.76}$$

The relationship between twistor space and spacetime, known as the twistor correspondence, is most easily seen by considering the product manifold $\mathbb{PS} = \mathbb{CP}^1 \times \mathbb{R}^4$ known as the projective spin bundle. It is captured by the double fibration



where the maps p and q are given by

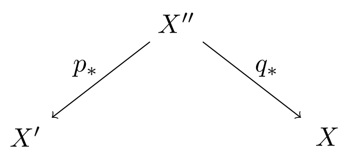
$$p : (\zeta, x^i) \mapsto (\zeta, v^1, v^2) \ , \qquad q : (\zeta, x^i) \mapsto x^i \ . \tag{3.77}$$

We should highlight that this correspondence space and double fibration exist for all signatures of metric on $\mathbb{R}^4$. In fact, it is best understood by working with a complex metric on $\mathbb{C}^4$ which becomes $\mathbb{R}^4$ with various signatures when restricted to certain real slices. The isomorphism between twistor space $\mathbb{PT}$ and the projective spin bundle $\mathbb{PS}$ does not hold for general signatures, however. This isomorphism only exists in Euclidean signature when $p : \mathbb{PS} \to \mathbb{PT}$ is invertible. It is often convenient to work in Euclidean signature, which is the approach we take here, and then analytically continue the result to other signatures of interest.

Since we are interested in performing a reduction of twistor space, we need to know how various conformal transformations on $\mathbb{R}^4$ lift to transformations of twistor space. In fact, the strategy which we will adopt is to first lift the conformal Killing vectors to the correspondence space $\mathbb{PS}$, and then to project these vectors down to twistor space using the map $p : \mathbb{PS} \to \mathbb{PT}$. A general spacetime vector field may be expressed as

$$X = X^{u^1} \partial_{u^1} + X^{\bar{u}^1} \partial_{\bar{u}^1} + X^{u^2} \partial_{u^2} + X^{\bar{u}^2} \partial_{\bar{u}^2} \ . \tag{3.78}$$

We will assume that this is a conformal Killing vector for the Euclidean spacetime metric $\mathrm{d}s^2 = \mathrm{d}u^1 \, \mathrm{d}\bar{u}^1 + \mathrm{d}u^2 \, \mathrm{d}\bar{u}^2$. If this is to be a real vector field on $\mathbb{R}^4$, then these components must also obey the relations $X^{\bar{u}^a} = \overline{X^{u^a}}$. We would like to lift this to a vector field X'' on $\mathbb{PS}$ which satisfies $q_*(X'') = X$. We will choose this lift such that the projection $X' = p_*(X'')$ is a well-defined holomorphic vector field on twistor space.



Vector fields on $\mathbb{PS}$ which are tangent to the projection map $p : \mathbb{PS} \to \mathbb{PT}$ are given by

$$V_1 = \partial_{\bar{u}^1} - \zeta \, \partial_{u^2} \ , \qquad V_2 = \partial_{\bar{u}^2} + \zeta \, \partial_{u^1} \ . \tag{3.79}$$

For each fixed $\zeta \in \mathbb{CP}^1$, one can show (using the conformal Killing equation) that

$$[X, V_1] = Q\,\partial_{u^2}\ , \qquad [X, V_2] = -Q\,\partial_{u^1}\ , \qquad \text{mod}\ \{V_1, V_2\} \tag{3.80}$$

where Q is given by

$$Q = \partial_{\bar{u}^2} X^{u^1} + \zeta\big(\partial_{u^1} X^{u^1} - \partial_{\bar{u}^2} X^{\bar{u}^2}\big) - \zeta^2 \partial_{u^1} X^{\bar{u}^2}\ . \tag{3.81}$$

The conformal Killing equation also shows that Q is constant along V_1 and V_2, meaning that the lifted vector field

$$X'' = X + Q\,\partial_\zeta + \bar{Q}\,\partial_{\bar{\zeta}}\ , \tag{3.82}$$

satisfies $[X'', V_1] = 0$ and $[X'', V_2] = 0$ modulo $\{V_1, V_2, \partial_{\bar{\zeta}}\}$. The component involving $\bar{Q}$ ensures that this is a real vector field. The projection of this vector field to twistor space is explicitly given by

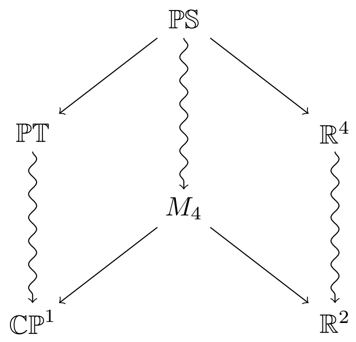
$$X' = \big(X^{u^1} - \zeta\, X^{\bar{u}^2} - \bar{u}^2\, Q\big)\partial_{v^1} + \big(X^{u^2} + \zeta\, X^{\bar{u}^1} + \bar{u}^1\, Q\big)\partial_{v^2} + Q\,\partial_\zeta\ , \tag{3.83}$$

where the components are constant along V_1 and V_2 and hence holomorphic functions of $\{\zeta, v^1, v^2\}$. The lifts and projections of some conformal Killing vectors which generate the symmetries of the 4dWZW model are presented in table 3.2.

X	Q	X'
∂_{u^1}	$-$	∂_{v^1}
$\partial_{\bar{u}^1}$	$-$	$\zeta\partial_{v^2}$
∂_{u^2}	$-$	∂_{v^2}
$\partial_{\bar{u}^2}$	$-$	$-\zeta\partial_{v^1}$
R_0	$\mathrm{i}\zeta$	$(\mathrm{i}/2)(v^1\partial_{v^1} + v^2\partial_{v^2}) + \mathrm{i}\zeta\partial_\zeta$
R_1	$-$	$(\mathrm{i}/2)(v^1\partial_{v^1} - v^2\partial_{v^2})$
R_2	$-$	$(\mathrm{i}/2)(v^2\partial_{v^1} + v^1\partial_{v^2})$
R_3	$-$	$(1/2)(v^2\partial_{v^1} - v^1\partial_{v^2})$

Table 3.2: Lifts and projections of some conformal Killing vectors.

Having understood how to move vector fields between these spaces, we would like to study the twistor correspondence under a pair of reductions. As a warm up, let us consider a reduction by two translational isometries, given by the spacetime vector fields $X = \partial_{u^2}$ and $Y = \partial_{\bar{u}^2}$. On the $\mathbb{R}^4$, the quotient space is identified with $\mathbb{R}^2$ and parameterised by coordinates $\{u^1, \bar{u}^1\}$. The lift of these vector fields to the correspondence space $\mathbb{PS} = \mathbb{CP}^1 \times \mathbb{R}^4$ is trivial, so the reduced correspondence space is $M_4 = \mathbb{CP}^1 \times \mathbb{R}^2$. Finally, the projection of these vector fields to twistor space gives $X' = \partial_{v^2}$ and $Y' = -\zeta\partial_{v^1}$. Functions on the quotient space must be independent of $\{v^1, v^2\}$, and so the remaining coordinate ζ is a holomorphic coordinate on the reduced twistor space, which is therefore identified with $\mathbb{CP}^1$. In summary, we have a reduced twistor correspondence captured by the following diagram.



Now, let us move to the reduction of interest, which is generated by the spacetime vector fields X_ϕ, X_τ introduced in (3.66). It is once more convenient to work in cylindrical coordinates (defined by $u^1 = \rho e^{\mathrm{i}\phi}$ and $u^2 = z + \mathrm{i}\tau$) in which these vector fields are $X_\phi = \partial_\phi$ and $X_\tau = \partial_\tau$. Coordinates on the quotient space of $\mathbb{R}^4$ are provided by $\{\rho, z\}$, and it is identified with $\mathbb{R}_{\geq 0} \times \mathbb{R}$. The lift of the vector fields to the correspondence space is given by

$$X''_\phi = \partial_\phi + \mathrm{i}\zeta\partial_\zeta - \mathrm{i}\bar{\zeta}\partial_{\bar{\zeta}} \ , \qquad X''_\tau = \partial_\tau \ . \tag{3.84}$$

Unlike for the translational isometries just considered, for the rotational isometry we get a non-trivial mixing of the $\mathbb{CP}^1$ and $\mathbb{R}^4$ coordinates, i.e. $Q = \mathrm{i}\zeta$. We would like to find coordinates on the quotient space M_4 which are invariant under the flow of these vector fields. The obvious candidate for the $\mathbb{CP}^1$ factor, ζ, is not invariant, but by mixing the $\mathbb{CP}^1$ and $\mathbb{R}^4$ coordinates as $Z = e^{-\mathrm{i}\phi}\zeta$ we obtain an invariant spectral parameter. We can continue to use coordinates $\{\rho, z\}$ on the spacetime factor in M_4.

The projection of these vector fields to twistor space is then

$$X'_\phi = \mathrm{i}(v^1\partial_{v^1} + \zeta\partial_\zeta) \ , \qquad X'_\tau = \mathrm{i}(\partial_{v^2} - \zeta\partial_{v^1}) \ . \tag{3.85}$$

The reduced twistor space should be a one-dimensional complex manifold with a single holomorphic coordinate. Indeed the quotient of $\mathbb{PT}$ by X'_ϕ, X'_τ may be parameterised by

$$W = \frac{1}{2}\left(v^2 + \frac{v^1}{\zeta}\right) = z + \frac{\rho}{2}\big(Z^{-1} - Z\big) \ , \tag{3.86}$$

where the first expression shows that this coordinate is holomorphic, and its invariance can be easily verified by acting with the reduction vectors. Meanwhile, the second expression exhibits the relationship of this spectral parameter to the invariant spectral parameter Z. We recognise this relationship as the two-to-one covering map from our 4dCS construction. It reveals that the $\mathbb{CP}^1$ coordinatized by Z is that of the reduced projective spin bundle, while the $\mathbb{CP}^1$ coordinatized by W is that of the reduced twistor space. The explicit relationship between these variables was derived by imposing invariance under the reduction vectors, and follows from our particular choice of reduction as well as the twistor correspondence. Alternative reductions would lead to different relationships between these parameters.

It remains to identify the lift of the discrete transformation

$$\sigma : (\rho, \phi, z, \tau) \mapsto (\rho, -\phi, z, -\tau) \ . \tag{3.87}$$

We will follow the same rational as above, lifting this to a transformation of $\mathbb{PS}$ subject to the constraint that it projects down to a holomorphic map on $\mathbb{PT}$. In the complex coordinates on $\mathbb{R}^4$, this discrete transformation acts as $\sigma : (u^1, u^2) \mapsto (\bar{u}^1, \bar{u}^2)$. Inspecting the expressions $v^1 = u^1 - \zeta\, \bar{u}^2$ and $v^2 = u^2 + \zeta\, \bar{u}^1$, we see that σ exchanges these combinations (up to an overall rescaling) if we include the map $\zeta \mapsto -\zeta^{-1}$. We therefore define the lift of this discrete transformation as

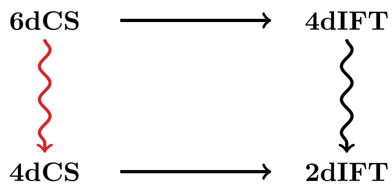
$$\sigma'' : (\zeta, \rho, \phi, z, \tau) \mapsto (-\zeta^{-1}, \rho, -\phi, z, -\tau) \ . \tag{3.88}$$

The action on the invariant spectral parameter $\sigma'' : Z \mapsto -Z^{-1}$ moves between the two sheets of the double covering over the W-plane. The projection down to twistor space acts as

$$\sigma' : (\zeta, v^1, v^2) \mapsto \left(-\frac{1}{\zeta}, -\frac{v^2}{\zeta}, \frac{v^1}{\zeta} \right) \ . \tag{3.89}$$

This is a holomorphic map, as required, and it acts trivially on the spectral parameter W.

3.6 Reduction of 6dCS theory



In the previous section, we studied geometric aspects of the reduction. We lifted the spacetime reduction vectors to the correspondence space $\mathbb{PS}$ and then projected them down to holomorphic vectors on twistor space $\mathbb{PT}$. Applying this reduction to each of these manifolds, we exhibited a reduced twistor correspondence with correspondence space $M_4 = \mathbb{CP}^1 \times \mathbb{R}^2$. This is the 4-manifold over which our 4dCS theory is defined, and in this section we will recover that theory as a reduction of 6dCS theory.

Six-dimensional Chern-Simons (6dCS) theory [Cos20; BS23; Pen21; Cos21] is to 4d integrable theories as 4dCS theory is to 2d integrable theories. The underlying 6-manifold is isomorphic to $\mathbb{CP}^1 \times \mathbb{R}^4$ and there is a localisation procedure by which one may integrate out the $\mathbb{CP}^1$. In particular, the 4dWZW model presented in section 3.4 is known to admit a 6dCS description

[Cos20; BS23]. A pedagogical introduction to this theory, and the localisation to the 4dWZW model are presented in the section 2.4. Since those results are not novel to this work, we will skip the majority of the details here and jump directly to the reduction to 4dCS theory. The action of 6dCS theory is given by

$$S_{\text{6dCS}} = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \Omega \wedge \mathrm{tr}\left(\mathcal{A} \wedge \bar{\partial} \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) , \tag{3.90}$$

where the meromorphic 3-form Ω and boundary conditions are given by

$$\Omega = \frac{\mathrm{d}\zeta \wedge \mathrm{d}v^1 \wedge \mathrm{d}v^2}{\zeta^2} , \qquad \mathcal{A}\big|_{\zeta=0} = 0 , \qquad \mathcal{A}\big|_{\tilde{\zeta}=0} = 0 . \tag{3.91}$$

Much like in 4dCS theory, the choice of meromorphic form and boundary conditions are input data, and different choices lead to different integrable field theories — now four-dimensional theories (4dIFTs) rather than two-dimensional. The fundamental gauge field is an algebra-valued $(0,1)$-form, where we are making use of the complex structure on twistor space[9] to introduce the Dobeault complex. In this notation, a (p,q)-form is a $(p+q)$-form with p legs along the holomorphic directions and q legs along the anti-holomorphic directions. Similarly, the Dobeault operator in the action is defined by

$$\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1} , \qquad \bar{\partial} = \pi_{p,q+1} \circ \mathrm{d} , \tag{3.92}$$

where $\pi_{p,q}$ is the projection from the space of $(p+q)$-forms to the subspace of (p,q)-forms. The 4dIFT associated to this theory can be derived from a similar localisation procedure to that of section 3.3, performing the integral over $\mathbb{CP}^1$ explicitly after solving the bulk equations of motion. As already noted, a more generous introduction to 6dCS theory and the details of this localisation are given in section 2.4.

In the remainder of this section, we seek to close the commutative diagram above by reducing 6dCS theory on the lifted killing vectors described in section 3.5. Recall that these can be expressed as

$$X''_\phi = \partial_\phi + \mathrm{i}\zeta\partial_\zeta - \mathrm{i}\bar{\zeta}\partial_{\bar{\zeta}} , \qquad X''_\tau = \partial_\tau . \tag{3.93}$$

Our reduction procedure follows the methodology presented in [BS23]. First, we impose invariance of the gauge field under the action of these reduction vectors, that is

$$\mathcal{L}_{X''_\phi} \mathcal{A} = 0 , \qquad \mathcal{L}_{X''_\tau} \mathcal{A} = 0 . \tag{3.94}$$

[9] Since the reduction from 6dCS to 4dCS takes place on the correspondence spaces, we think it is more precise to define 6dCS theory over $\mathbb{PS}$. In this case, the action would be defined by first pulling Ω back via $p : \mathbb{PS} \to \mathbb{PT}$ and then wedging against the Chern-Simons 3-form. The gauge field and exterior derivative would initially be considered as generic, and then one observes that the legs along Ω drop out of the action and decouple from the theory. This is corroborated by the fact that $\mathbb{PT}$ changes dimension depending on the signature of the metric on $\mathbb{R}^4$, an issue which was circumvented in [BS23] by moving to the correspondence space. We argue that this is the correct approach, even for Euclidean signature (where $\mathbb{PS}$ and $\mathbb{PT}$ are isomorphic as real manifolds).

Having done this, our Lagrangian 6-form will be invariant under the flow generated by these vector fields. We would like to compute the associated 4-form on the surviving quotient space. In practice, this is achieved by contracting the Lagrangian with the bivector $X''_\tau \wedge X''_\phi$.

There are two technical preparations we can make in order to facilitate this computation. The complex structure on twistor space mixes the $\mathbb{CP}^1$ and $\mathbb{R}^4$ directions, meaning that the natural $(1,0)$-forms $\{\mathrm{d}v^1, \mathrm{d}v^2\}$ have legs along both directions. For our purposes, it will be helpful to implement the splitting between spacetime and the spectral parameter by introducing a basis of forms which only have legs along either $\mathbb{CP}^1$ or $\mathbb{R}^4$ but not both. We would still like these to provide a basis for the $(1,0)$-forms and $(0,1)$-forms with respect to the complex structure on twistor space, and it simplifies the computation further if they are invariant under X''_ϕ and X''_τ. One finds that a suitable basis of $(1,0)$-forms is given by

$$\eta^0 = e^{-\mathrm{i}\phi}\mathrm{d}\zeta \ , \qquad \eta^1 = e^{-\mathrm{i}\phi}(\mathrm{d}u^1 - \zeta\,\mathrm{d}\bar{u}^2) \ , \qquad \eta^2 = \mathrm{d}u^2 + \zeta\,\mathrm{d}\bar{u}^1 \ . \tag{3.95}$$

Note that the functional dependence on ζ and ϕ can be repackaged into the parameter $Z = e^{-\mathrm{i}\phi}\zeta$ which is invariant under the flow of our reduction vectors. Similarly, for $(0,1)$-forms we can use

$$\bar{\eta}^0 = e^{\mathrm{i}\phi}\mathrm{d}\bar{\zeta} \ , \qquad \bar{\eta}^1 = \frac{e^{\mathrm{i}\phi}(\mathrm{d}\bar{u}^1 - \bar{\zeta}\,\mathrm{d}u^2)}{1+\zeta\bar{\zeta}} \ , \qquad \bar{\eta}^2 = \frac{\mathrm{d}\bar{u}^2 + \bar{\zeta}\,\mathrm{d}u^1}{1+\zeta\bar{\zeta}} \ . \tag{3.96}$$

In this basis, the components of the gauge field may be expressed as

$$\mathcal{A} = \mathcal{A}_0\,\bar{\eta}^0 + \mathcal{A}_1\,\bar{\eta}^1 + \mathcal{A}_2\,\bar{\eta}^2 \ . \tag{3.97}$$

The fact that the basis forms are invariant means that the constraints $\mathcal{L}_{X''_\phi}\mathcal{A} = 0$ and $\mathcal{L}_{X''_\tau}\mathcal{A} = 0$ amount to the individual components of $\mathcal{A}$ being independent of the isometry coordinates $\{\phi, \tau\}$.

In addition, it is helpful to “prepare the gauge field for reduction” by imposing the gauge fixing constraints

$$X''_\phi \lrcorner \mathcal{A} = 0 \ , \qquad X''_\tau \lrcorner \mathcal{A} = 0 \ . \tag{3.98}$$

This ensures that the bivector $X''_\tau \wedge X''_\phi$ contracting the Lagrangian acts only on Ω, dramatically simplifying the calculation. This constraint may be realised using the $(1,0)$-form shift symmetry

$$\mathcal{A} \mapsto \mathcal{A} + \left(C_0\,\bar{\eta}^0 + C_1\,\bar{\eta}^1 + C_2\,\bar{\eta}^2\right) \ . \tag{3.99}$$

This is a trivial symmetry of the action which leaves it invariant because Ω saturates the $(3,0)$-legs. Solving the constraints above for the components of the $(1,0)$-form C, one finds an expression for the shifted gauge field which is most easily expressed in terms of the variables $Z = e^{-\mathrm{i}\phi}\zeta$ and $\xi = z + \mathrm{i}\rho$. In these coordinates, the shifted gauge field may be written as

$$\mathcal{A} + C = A_{\bar{Z}}\mathrm{d}\bar{Z} + A_\xi\mathrm{d}\xi + A_{\bar{\xi}}\mathrm{d}\bar{\xi} \ . \tag{3.100}$$

The precise dependence of the new components $\{A_{\bar{Z}}, A_\xi, A_{\bar{\xi}}\}$ on the old components is not especially important as these still represent generic field configurations of the theory. What is important, however, is the analytic behaviour of these components in $Z \in \mathbb{CP}^1$. One finds that the boundary conditions on $\mathcal{A}$ imply that A_ξ and $A_{\bar{\xi}}$ vanish at both $Z = 0$ and $Z = \infty$. Furthermore one also observes that A_ξ is permitted a simple pole at $Z = -\mathrm{i}$, whilst $A_{\bar{\xi}}$ is permitted a simple pole at $Z = +\mathrm{i}$. This matches the properties of the 4dCS gauge field presented in section 3.3.

After making these preparations, computing the contraction of the bivector $X''_\tau \wedge X''_\phi$ with the Lagrangian 6-form amounts to computing its contraction with Ω. This yields the meromorphic 1-form of our 4dCS theory which is given by

$$\omega = \frac{1}{2}(X''_\tau \wedge X''_\phi) \lrcorner \Omega = \mathrm{d}\Big(z + \frac{\rho}{2}(Z^{-1} - Z)\Big) \ . \tag{3.101}$$

In summary, we have landed on the 4dCS theory described in section 3.3 with the action

$$S_{4\mathrm{dCS}} = \frac{1}{2\pi \mathrm{i}} \int_{M_4} \omega \wedge \mathrm{tr}\Big(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\Big) \ . \tag{3.102}$$

The final ingredient to consider is the discrete reduction generated by a combination of the reflection

$$\sigma'' : (\zeta, \rho, \phi, z, \tau) \mapsto (-\zeta^{-1}, \rho, -\phi, z, -\tau) \ . \tag{3.103}$$

and the $\mathbb{Z}_2$-automorphism $\eta : \mathfrak{g} \to \mathfrak{g}$. The action on the isometry coordinates ϕ and τ does not play a role in the 4dCS theory, but the reflection does act non-trivially on the invariant spectral parameter as $Z \mapsto -Z^{-1}$. Demanding that the 6dCS gauge field also be invariant under this discrete reduction implies that the 4dCS gauge field should satisfy

$$A(Z) = \eta\big(A(-Z^{-1})\big) \ . \tag{3.104}$$

This is the equivariance condition which reduced the Lie group $\mathrm{SL}(2, \mathbb{R})$ of the 2d theory to the coset $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$. In the context of 4dCS theory, these equivariance conditions are related to branch cut defects [CY19]. In this section, we have shown that branch cut defects in 4dCS theory arise from discrete reductions of 6dCS theory. Since this is interesting in its own right, we devote section 3.8 to presenting this result in the simpler case of the symmetric space PCM.

3.7 Outlook

The focus of this work has been on establishing a firm basis for the study of integrable sectors of gravity in terms of 4d Chern-Simons theory. While we have often referred explicitly to 4d vacuum GR with one spacelike and one timelike killing vector, our construction can be applied to the integrable sectors of many different gravity theories. This is due to the observation by Breitenlohner, Maison, and Gibbons [BMG88] that with $D-2$ commuting Killing vectors, the bosonic sectors of various D-dimensional supergravities yield 2d σ-models of the same form. By taking the field G to be valued in the relevant coset group, and the Chern-Simons fields in the associated Lie algebra, our analysis applies to each such case.

To describe the full field content of a given supergravity theory the corresponding 2d σ-model must be coupled to fermions (see for example [Nic87]). This is one motivation to incorporate fermionic degrees of freedom into the commutative diagram of models discussed in this work. One approach to this problem in 6dCS theory was presented in [Pen21], similar in spirit to the *order defects* of 4dCS theory described in [CY19]. This introduced sufficient degrees of freedom in the 2dIFT to describe the reduction of 4d $N = 1$ supergravity, up to quadratic order in the fermions. In general, the topic of order defects in 6dCS theory and their reduction to 4dCS theory requires further development.

Dressing transformations play a key role in generating new solutions in gravity, and have many applications in integrable models more generally where solitonic solutions are a common feature. As these transformations are adapted to the Lax formalism, it is natural to study them from the perspective of the associated 4dCS theories. As alluded to in section 3.3, we suspect that dressing transformations are related to residual gauge symmetries with simple poles in the spectral plane. One might wonder whether the constraints imposed in the BZ method and its higher-dimensional generalisations have a natural origin in the 4dCS model. For example, requiring that the analytic structure of the Lax connection is preserved implies that the residues of simple poles are rank 1 matrices [HSS84]. Based on [KKV14], extensions to more general coset models for supergravity theories will require poles with higher-rank residues. Ultimately, we hope the 4dCS perspective can shed light on solitonic solutions, both in 4d vacuum gravity and higher-dimensional supergravity theories.

Asymptotics are a topic which is extensively studied in the literature on inverse scattering methods but less understood in the 4dCS context. Hopefully large (ρ, z) conditions which are desirable from a physical standpoint can be ensured by corresponding assumptions on the CS gauge field. There is the added subtlety that the 2d spacetime of interest is often the half-plane, rather than $\mathbb{R}^2$, and thus has a boundary. This may provide an interesting arena to study integrable boundary conditions in the context of 4dCS theory.

One of the advantages of the 4dCS framework is that it facilitates a systematic classification of seemingly unrelated integrable models in terms of the properties of the one-form ω. In this work, we have introduced several new ingredients which can help expand that classification. First, we have shown how discrete reductions can be used alongside reductions by Killing vectors, not only to obtain 2dIFTs valued in symmetric spaces from 4dIFTs, but also to obtain the corresponding 4dCS theory from 6dCS. The discrete reduction has the effect of introducing a branch cut in the spectral plane. In section 3.8, we isolate this ingredient and show how to recover branch cut defects in 4dCS theory in the simpler example of the symmetric space PCM.

Reduction on an angular coordinate introduces a second new ingredient to 4dCS theory: mixed $\mathbb{CP}^1$ and spacetime dependence in the one-form ω. It is the combination of reduction by an axial Killing vector and discrete reduction which leads to spacetime-dependent branch cuts in the W-plane, or mixing of $\mathbb{CP}^1$ and spacetime derivatives with respect to its double cover Z. This is the significance (from the CS perspective) of the different formulations of the Lax for stationary axisymmetric GR in terms of a variable or constant spectral parameter, in [BS79] compared with [BM87], for example.

Both of these ingredients may be used in the future to construct new 4dCS models and their associated 2dIFTs. For example, recent work on 6dCS theory [Col+24b] showed how to recover the λ-deformation [Sfe14] by splitting the double pole at $\zeta = 0$ into two simple poles. This would break the discrete symmetry $\zeta \mapsto -\zeta^{-1}$ unless one also splits the double pole at $\zeta = \infty$ into two simple poles. We suspect that applying a discrete reduction to the 6dCS setup with four simple poles may result in the symmetric-space λ-model whose 4dCS description has been explored in [Tia20; Sch20]. Gauged integrable field theories were also derived from 6dCS theory in [Col+24a]. It would be interesting to see if symmetric-space models can be recovered using that formalism, and how it relates to the methodology presented in this work.

In addition, the admissibility of meromorphic 1-forms in 4dCS theory with spacetime dependence may be relevant to the 2dIFTs recently introduced in [HLT20]. In that work, a variety of integrable models were studied in which the couplings were allowed to depend on the 2d spacetime coordinates. It was shown that these models were classically integrable, provided that the couplings took a special form, related to the RG flow equations. Using the results of this work, it may now be possible to incorporate these models into the 4dCS formalism. This could be particularly interesting in light of the recent work relating RG flow and 4dCS theory [Del+21; Der21; Lev23; LW24]. These papers propose a mechanism for computing the RG flow of a given 2dIFT very directly, supposing that one knows the appropriate 4dCS description.

3.8 Appendix: Symmetric space PCM from 6dCS theory

In the main body of this chapter, we aimed to recover the 2dIFT related to stationary, axisymmetric gravity. This involved imposing a translational, a rotational, and a discrete reduction on 6dCS theory. Since both the rotational and discrete reductions introduce novel aspects, it may be helpful to analyse them independently. Furthermore, the symmetric space models produced by the discrete reduction are interesting in their own right, in addition to their appearance in the context of gravity. For these reasons, we will devote a section at the end to recovering the standard symmetric space principal chiral model (PCM) from 6dCS theory.

Starting from the 4dWZW model, we will apply two translational reductions, recovering the 2d PCM without a WZ term. This reduction was first performed in [BS23; Pen21] but we will introduce an additional ingredient: a discrete reduction involving a $\mathbb{Z}_2$-automorphism of the Lie algebra. The addition of this discrete reduction will reduce the target space of the PCM from the full group G to the symmetric space G/G_0. Here, $G_0 \subset G$ is the subgroup which is preserved by the $\mathbb{Z}_2$-automorphism. This reduction may then be lifted to twistor space and applied to 6dCS theory. We will show that this reproduces the branch cut and equivariance condition discussed in [CY19, §11] which is known to recover the symmetric space PCM.

Let us start with the 4dWZW model whose action is

$$S_{\mathrm{4dWZW}} = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}\big(g^{-1}\mathrm{d}g \wedge \star g^{-1}\mathrm{d}g\big) - \int_{\mathbb{R}^4} \omega \wedge \mathrm{WZ}[g] \ . \tag{3.105}$$

The second term contains a 2-form $\mu = \mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 + \mathrm{d}u^2 \wedge \mathrm{d}\bar{u}^2$ which is proportional to the Kähler form on $\mathbb{R}^4$ equipped with the Euclidean metric. We would like to apply a two-dimensional reduction by vector fields X and Y, landing on the 2dPCM whose action is

$$S_{\mathrm{2dPCM}} = \frac{1}{2}\int_{\mathbb{R}^2} \mathrm{tr}\big(g^{-1}\mathrm{d}g \wedge \star g^{-1}\mathrm{d}g\big) \ . \tag{3.106}$$

In order for the 2d theory to contain no WZ-term, we must choose the vector fields such that $(X \wedge Y) \lrcorner\, \mu = 0$. Were one to compactify the directions corresponding to these vector fields, this condition would state that the resulting 2-torus has zero Kähler volume. We can satisfy this condition with two translational vector fields

$$X = \partial_{u^1} - \partial_{\bar{u}^2} \ , \qquad Y = \partial_{u^2} - \partial_{\bar{u}^1} \ . \tag{3.107}$$

Coordinates on the quotient space must be invariant under these vector fields, and these may be provided by $\xi = u^1 + \bar{u}^2$ and $\bar{\xi} = \bar{u}^1 + u^2$. Applying this reduction to the 4dWZW model results in the 2dPCM as desired.

Next, we would like to supplement this reduction with a discrete reduction. So that we do not modify the geometry of the quotient space, we will choose a discrete action which leaves the quotient space invariant. In addition, this discrete reduction must be a symmetry of the 4dWZW model, naively meaning it must preserve the 2-form μ. In fact, it is enough for this symmetry to preserve the 2-form μ up to a sign, provided that we also transform the fundamental field as $g \mapsto g^{-1}$. Such a discrete transformation is provided by

$$\sigma : (u^1, u^2) \mapsto (\bar{u}^2, \bar{u}^1) \ . \tag{3.108}$$

In order to implement the transformation of the fundamental field, we will introduce a $\mathbb{Z}_2$-automorphism of the Lie algebra,

$$\eta : \mathfrak{g} \to \mathfrak{g} \ , \qquad \eta^2 = \mathrm{id} \ . \tag{3.109}$$

The total reduction will then be a combination of the discrete spacetime action and this Lie algebra automorphism exponentiated to a group action $g \mapsto \eta(g)$. To agree with the transformation specified above, the fundamental field must satisfy $\eta(g) = g^{-1}$ which is a constraint. This constraint restricts the target space of the 2dPCM from the full group G to the symmetric space

G/G_0 where $G_0 \subset G$ is the subgroup preserved by the $\mathbb{Z}_2$-automorphism.

Having outlined the reduction of the 4dWZW model, we may now lift this to a reduction of twistor space and then apply it to 6dCS theory. Firstly, let us study the geometry of the reduced twistor correspondence. The vector fields lift trivially to $\mathbb{PS}$ meaning that the reduced correspondence space $M_4 = \mathbb{CP}^1 \times \mathbb{R}^2$ may be provided coordinates by $\{\zeta, \xi, \bar{\xi}\}$. The projection of these vector fields to twistor space $\mathbb{PT}$ is given by

$$X' = (1+\zeta)\partial_{v^1} \ , \qquad Y' = (1-\zeta)\partial_{v^2} \ . \tag{3.110}$$

We therefore identify the reduced twistor space as $\mathbb{CP}^1$ with the invariant spectral parameter ζ.

Turning to the discrete reduction, this lifts to the correspondence space as

$$\sigma'' : (u^1, u^2, \zeta) \mapsto (\bar{u}^2, \bar{u}^1, \zeta^{-1}) \ . \tag{3.111}$$

Furthermore, the action of this discrete transformation on twistor space is

$$\sigma' : (\zeta, v^1, v^2) \mapsto \left(\frac{1}{\zeta}, -\frac{v^1}{\zeta}, \frac{v^2}{\zeta}\right) \ . \tag{3.112}$$

This is a holomorphic map which preserves the $(3,0)$-form Ω associated to the 4dWZW model. The key feature of this discrete transformation is its action on the spectral parameter ζ. To understand the implications of this discrete reduction in the 4dCS theory, it is helpful to introduce another spectral parameter which is invariant under the discrete transformation,

$$W = \frac{1}{2}\big(\zeta + \zeta^{-1}\big) \ . \tag{3.113}$$

This is known as the Joukowsky transform and the ζ-plane is a double covering of the W-plane. We can see this by considering the inverse relationship

$$\zeta = W + \sqrt{W^2 - 1} \ . \tag{3.114}$$

For each value of W, there are two values of ζ, except for at the points $W = \pm 1$. These special points are the ends of a branch cut in the W-plane whose two-sheets correspond to the two values of ζ. In particular, the discrete transformation $\zeta \mapsto \zeta^{-1}$ moves between the two sheets over the same point in the W-plane.

Combining this analysis with the $\mathbb{Z}_2$-automorphism given earlier results in an equivariance condition on the 4dCS gauge field,

$$A(\zeta) = \eta\big(A(\zeta^{-1})\big) \ . \tag{3.115}$$

In addition, the meromorphic 1-form of 4dCS theory is given by

$$\omega = \frac{1}{2}(X \wedge Y) \lrcorner \Omega = \frac{1}{2}\,\frac{1-\zeta^2}{\zeta^2}\mathrm{d}\zeta = \mathrm{d}W \ . \tag{3.116}$$

This is the appropriate 1-form to recover the 2dPCM and reproduces the setup described in [CY19, §11]. We have therefore demonstrated how to recover the symmetric space PCM and branch cut defects from 6dCS theory.

Chapter 4

Integrable deformations from twistor space

The results in this chapter were found in collaboration with Ryan A. Cullinan, Ben Hoare, Joaquin Liniado, and Daniel C. Thompson [Col+24b].

4.1 Introduction

In six-dimensional Chern-Simons theory, one has to impose boundary conditions on the gauge field $\mathcal{A}$ at the poles of Ω. From the lower-dimensional 4dCS theory, we expect different choices of Ω and boundary conditions to lead to different integrable field theories (see the background section 1.3). The first papers on 6dCS theory [Cos20; Cos21; BS23; Pen21] examined the simplest choices of boundary conditions which primarily involved setting $\mathcal{A} = 0$ at the poles.

If we hope to recover four-dimensional analogues of the integrable deformations such as the λ-deformation (see 1.1.4 for background on these models), then we have to consider a wider class of boundary conditions. Prior to this work, the generalisation to recover lines of continuous integrable deformations was not known, and obstacles to the construction of integrable deformations from 6dCS were highlighted in [HTC22]. In this work, we show how to overcome these obstacles and construct the correspondence of theories associated to the λ-deformation.

Briefly, the key results of this work are:

1. We establish the consequence of a new class of boundary conditions for hCS_6. These reduce to a wider class of boundary conditions in CS_4 than have previously been considered (relaxing the assumption of an isotropic subalgebra of the defect algebra).
2. Integrating over $\mathbb{CP}^1$ results in a novel multi-parametric IFT_4 whose equations of motion can be recast in terms of an anti-self-dual Yang-Mills connection. This new IFT_4 exhibits two semi-local symmetries, which can be understood as the residual symmetries preserving

the boundary conditions. For each of these two semi-local symmetries, the Noether currents can be used to construct two inequivalent Lax formulations of the dynamics.

3. Upon symmetry reduction, this IFT_4 descends to the 2-field λ-type IFT_2 of [GS17] providing a new multi-parametric sigma-model example of the Ward conjecture [War85]. Generically the semi-local symmetries of the IFT_4 reduce to global symmetries of the IFT_2 and the two Lax formulations of the IFT_4 give rise to two Lax connections of IFT_2.

4. When the symmetry reduction constraints are aligned to these semi-local symmetries, the IFT_2 symmetries are enhanced to either affine or fully local (gauge) symmetries. In the latter case, the IFT_2 becomes the standard (1-field) λ-model.

4.2 Holomorphic 6-Dimensional Chern-Simons Theory

Our primary interest in this work will be the hCS_6 diamond containing the λ-deformed IFT_2 originally constructed in [Sfe14]. By proposing a carefully chosen set of boundary conditions, we will be able to find a diamond of theories that arrives at a multi-parametric class of integrable λ-deformations between coupled WZW models.

To this end we restrict our study of hCS_6, defined by the action

$$S_{\mathrm{hCS}_6} = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \Omega \wedge \mathrm{Tr}\Big(\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Big) \ , \tag{4.1}$$

to the case where the $(3,0)$-form is given by, in the basis of $(1,0)$-forms defined in section 2.2.5,

$$\Omega = \frac{1}{2}\,\Phi\, e^0 \wedge e^{\dot{a}} \wedge e_{\dot{a}} \ , \qquad \Phi = \frac{K}{\langle \pi\alpha\rangle\langle\pi\tilde{\alpha}\rangle\langle\pi\beta\rangle^2} \ . \tag{4.2}$$

Here, we view $\mathbb{PT}$ as diffeomorphic to $\mathbb{R}^4 \times \mathbb{CP}^1$ and adopt the standard coordinates $x^{a\dot{a}}$ on $\mathbb{R}^4$ and homogeneous coordinates π_a on $\mathbb{CP}^1$. The constant spinors α_a, $\tilde{\alpha}_a$ and β_a should be understood as part of the definition of the model. See section 2.2 for further details of twistor notation and conventions. The gauge field is similarly written in the basis of $(0,1)$-forms as

$$\mathcal{A} = \mathcal{A}_0\, \bar{e}^0 + \mathcal{A}_{\dot{a}}\, \bar{e}^{\dot{a}} \ , \tag{4.3}$$

and the action is invariant under shifts of $\mathcal{A}$ by any $(1,0)$-form, i.e. $\mathcal{A} \mapsto \mathcal{A} + \rho$ where $\rho \in \Omega^{(1,0)}(\mathbb{PT})$.

The first step in studying the 6-dimensional theory is to impose conditions ensuring the vanishing of the ‘boundary’ term that appears in the variation of the action

$$0 = \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\mathcal{A} \wedge \delta\mathcal{A}\big) \ . \tag{4.4}$$

Since Ω is meromorphic, as opposed to holomorphic, this receives contributions from the poles at α_a, $\tilde{\alpha}_a$, and β_a. We assume that Dirichlet conditions $\mathcal{A}_{\dot{a}}|_{\pi=\beta} = 0$ are imposed at the second-order pole. At the first-order poles, we can then evaluate the integral over $\mathbb{CP}^1$ to obtain[1] the condition

$$\frac{1}{\langle\alpha\tilde{\alpha}\rangle\langle\alpha\beta\rangle^2}\int_{\mathbb{R}^4}\mathrm{vol}_4\,\varepsilon^{\dot{a}\dot{b}}\mathrm{Tr}\big(\mathcal{A}_{\dot{a}}\delta\mathcal{A}_{\dot{b}}\big)\big|_{\pi=\alpha}=\frac{1}{\langle\alpha\tilde{\alpha}\rangle\langle\tilde{\alpha}\beta\rangle^2}\int_{\mathbb{R}^4}\mathrm{vol}_4\,\varepsilon^{\dot{a}\dot{b}}\mathrm{Tr}\big(\mathcal{A}_{\dot{a}}\delta\mathcal{A}_{\dot{b}}\big)\big|_{\pi=\tilde{\alpha}}\ . \tag{4.5}$$

For reasons that will shortly become apparent, let us introduce a unit norm spinor $\mu^{\dot{a}}$ about which we can expand any spinor $X^{\dot{a}}$ as

$$X^{\dot{a}}=[X\hat{\mu}]\mu^{\dot{a}}-[X\mu]\hat{\mu}^{\dot{a}}\ . \tag{4.6}$$

Expanding the gauge field components in terms of the basis $\mu^{\dot{a}}$ and $\hat{\mu}^{\dot{a}}$, and solving locally pointwise on $\mathbb{R}^4$, this condition may be written as

$$\frac{1}{\langle\alpha\beta\rangle^2}\mathrm{Tr}\big([\mathcal{A}\mu][\delta\mathcal{A}\hat{\mu}]-[\mathcal{A}\hat{\mu}][\delta\mathcal{A}\mu]\big)\big|_{\pi=\alpha}=\frac{1}{\langle\tilde{\alpha}\beta\rangle^2}\mathrm{Tr}\big([\mathcal{A}\mu][\delta\mathcal{A}\hat{\mu}]-[\mathcal{A}\hat{\mu}][\delta\mathcal{A}\mu]\big)|_{\pi=\tilde{\alpha}}\ . \tag{4.7}$$

The boundary conditions we are led to consider are

$$[\mathcal{A}\mu]|_{\pi=\alpha}=\sigma\frac{\langle\alpha\beta\rangle}{\langle\tilde{\alpha}\beta\rangle}[\mathcal{A}\mu]|_{\pi=\tilde{\alpha}}\ ,\qquad[\mathcal{A}\hat{\mu}]|_{\pi=\alpha}=\sigma^{-1}\frac{\langle\alpha\beta\rangle}{\langle\tilde{\alpha}\beta\rangle}[\mathcal{A}\hat{\mu}]|_{\pi=\tilde{\alpha}}\ , \tag{4.8}$$

where we have introduced the free parameter σ, which will play the role of the deformation parameter in the IFT_4.

Let us note that these boundary conditions are invariant under the following discrete transformations

$$\alpha\leftrightarrow\tilde{\alpha}\ ,\qquad\sigma\mapsto\sigma^{-1}\ , \tag{4.9}$$

$$\mu\mapsto\hat{\mu}\ ,\qquad\sigma\mapsto\sigma^{-1}\ . \tag{4.10}$$

These will descend to transformations that leave the IFT_4 invariant.

4.2.1 Residual Symmetries and Edge Modes

A general feature of Chern-Simons theory with a boundary is the emergence of propagating edge modes as a consequence of the violation of gauge symmetry by boundary conditions. A similar effect underpins the emergence of the dynamical field content of the lower dimensional theories that descend from hCS_6. Generally, group-valued degrees of freedom, here denoted by h and $\tilde{h}$, would be sourced at the locations of the poles of Ω. If however, the boundary conditions (4.8) admit residual symmetries, then these will result in symmetries of the IFT_4 potentially mixing the h and $\tilde{h}$ degrees of freedom. These may be global symmetries, gauge

[1]To compute the boundary variation of the action, we have used the identities $e^{\dot{c}}\wedge e_{\dot{c}}\wedge\bar{e}^{\dot{a}}\wedge\bar{e}^{\dot{b}}=-2\,\mathrm{vol}_4\,\varepsilon^{\dot{a}\dot{b}}$ (where $\mathrm{vol}_4=\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3$) and

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{CP}^1}e^0\wedge\bar{e}^0\,\bar{\partial}_0\Big(\frac{1}{\langle\pi\alpha\rangle}\Big)f(\pi)=f(\alpha)\ .$$

symmetries, or semi-local symmetries depending on the constraints imposed by the boundary conditions. It is thus important to understand the nature of any residual symmetry preserved by the boundary conditions (4.8).

Gauge transformations act on the hCS_6 gauge field as

$$\hat{g}: \quad \mathcal{A} \mapsto \hat{g}^{-1}\mathcal{A}\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g} \ . \tag{4.11}$$

In the bulk, i.e. away from the poles of Ω, these are unconstrained, but at the poles they will only leave the action invariant if they preserve the boundary conditions. For later convenience, we will denote the values of the gauge transformation parameters at the poles by

$$\hat{g}|_{\alpha} = r \ , \quad \hat{g}|_{\tilde{\alpha}} = \tilde{r} \ , \quad \hat{g}|_{\beta} = \ell^{-1} \ . \tag{4.12}$$

Firstly, the transformation acting at β must preserve the constraint $\mathcal{A}_{\dot{a}}|_{\beta} = 0$. Initially, one might suppose that only constant ℓ would preserve this boundary condition, but in fact it is sufficient for ℓ to be holomorphic with respect to the complex structure defined by β

$$\beta^a \partial_{a\dot{a}}\ell = 0 \quad \Rightarrow \quad \frac{1}{\langle\alpha\beta\rangle}\alpha^a\partial_{a\dot{a}}\ell = \frac{1}{\langle\tilde{\alpha}\beta\rangle}\tilde{\alpha}^a\partial_{a\dot{a}}\ell \ . \tag{4.13}$$

These differential constraints arise from the fact that the anti-holomorphic vector fields $\bar{\partial}_{\dot{a}} = \pi^a\partial_{a\dot{a}}$ are valued in $\mathcal{O}(1)$. In other words, they depend explicitly on the $\mathbb{CP}^1$ coordinate (see section 2.2 for more details).

Secondly, the transformations acting at α_a and $\tilde{\alpha}_a$ must preserve the boundary conditions (4.8), implying the constraints

$$\begin{aligned} \tilde{r} &= r \ , \\ \frac{1}{\langle\alpha\beta\rangle}\mu^{\dot{a}}\alpha^a\partial_{a\dot{a}}r &= \frac{\sigma}{\langle\tilde{\alpha}\beta\rangle}\mu^{\dot{a}}\tilde{\alpha}^a\partial_{a\dot{a}}r \ , \\ \frac{1}{\langle\alpha\beta\rangle}\hat{\mu}^{\dot{a}}\alpha^a\partial_{a\dot{a}}r &= \frac{\sigma^{-1}}{\langle\tilde{\alpha}\beta\rangle}\hat{\mu}^{\dot{a}}\tilde{\alpha}^a\partial_{a\dot{a}}r \ . \end{aligned} \tag{4.14}$$

These residual symmetries are neither constant (i.e. global symmetries) nor fully local (i.e. gauge symmetries). Instead, we expect that our IFT_4 should exhibit two semi-local symmetries subject to the above differential constraints, akin to the semi-local symmetries of the 4d WZW model first identified in [NS90; NS92][2].

Symmetry reduction As we progress around the diamond, we will perform 'symmetry reduction' (see § 4.4 and § 4.5 for details). In essence, this will mean we restrict to fields and gauge parameters that are independent of two directions, i.e. they obey the further differential

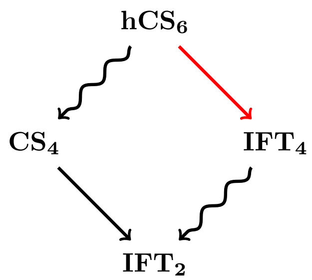
[2]Complementary to this perspective, the WZW_4 algebra can also be obtained as a global symmetry of five-dimensional Kähler Chern-Simons on a manifold with boundary [BGH96].

constraints (where γ^a is some constant spinor)

$$\mu^{\dot{a}}\gamma^a\partial_{a\dot{a}}\hat{g} = 0 \ , \quad \hat{\mu}^{\dot{a}}\hat{\gamma}^a\partial_{a\dot{a}}\hat{g} = 0 \ . \tag{4.15}$$

We can then predict some special points in the lower dimensional theories by considering how these differential constraints interact with those imposed by the boundary conditions. Generically, these four differential constraints (two from the boundary conditions and two from symmetry reduction) will span a copy of $\mathbb{R}^4$ at each pole, meaning that only constant transformations (i.e. global symmetries) will survive. However, if the symmetry reduction is carefully chosen, the two sets of constraints may partially or entirely coincide. In the case that they entirely coincide, the lower dimensional symmetry parameter will be totally unconstrained, meaning that the IFT_2 will possess a gauge symmetry. Alternatively, if the constraints partially coincide then the lower dimensional theory will have a symmetry with free dependence on half the coordinates, e.g. the chiral symmetries of the 2d WZW model.

4.3 Localisation of hCS_6 to IFT_4



Let us now proceed in navigating the top right-hand side of the diamond. By integrating over $\mathbb{CP}^1$ we will 'localise' hCS_6 on $\mathbb{PT}$ to an effective theory defined on $\mathbb{R}^4$. This resulting theory is 'integrable' in the sense that its equations of motion can be encoded in an anti-self-dual connection.

The localisation analysis is naturally presented in terms of new variables $\mathcal{A}'$ and $\hat{h}$, which are related to the fundamental field by

$$\mathcal{A} = \hat{h}^{-1}\mathcal{A}'\hat{h} + \hat{h}^{-1}\bar{\partial}\hat{h} \ . \tag{4.16}$$

However, there is some redundancy in this new parametrisation. There are internal gauge transformations (leaving $\mathcal{A}$ invariant) given by

$$\hat{g}: \quad \mathcal{A}' \mapsto \hat{g}^{-1}\mathcal{A}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g} \ , \qquad \hat{h} \mapsto \hat{g}^{-1}\hat{h} \ . \tag{4.17}$$

These allow us to impose the constraint $\mathcal{A}'_0 = 0$, i.e. it has no leg in the $\mathbb{CP}^1$-direction. This is done so that $\mathcal{A}'$ may be interpreted as an anti-self-dual Yang-Mills connection on $\mathbb{R}^4$.

There are still internal gauge transformations that are $\mathbb{CP}^1$-independent, and we can use these to fix the value of $\hat{h}$ at one pole. We will therefore impose the additional constraint $\hat{h}|_\beta = \text{id}$ so that we have resolved this internal redundancy. The values of $\hat{h}$ at the remaining poles

$$\hat{h}|_{\alpha} = h \ , \qquad \hat{h}|_{\tilde{\alpha}} = \tilde{h} \ , \tag{4.18}$$

will be dynamical edge modes as a consequence of the violation of gauge symmetry by boundary conditions. As we will now see, the entire action localises to a theory on $\mathbb{R}^4$ depending only on these edge modes.

The hCS_6 action is written in these new variables as

$$\begin{aligned} S_{\mathrm{hCS}_6} = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \Omega \wedge \mathrm{Tr}\big(\mathcal{A}' \wedge \bar{\partial}\mathcal{A}'\big) + \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\mathcal{A}' \wedge \bar{\partial}\hat{h}\hat{h}^{-1}\big) \\ - \frac{1}{6\pi \mathrm{i}} \int_{\mathbb{PT}} \Omega \wedge \mathrm{Tr}\big(\hat{h}^{-1}\bar{\partial}\hat{h} \wedge \hat{h}^{-1}\bar{\partial}\hat{h} \wedge \hat{h}^{-1}\bar{\partial}\hat{h}\big) \ . \end{aligned} \tag{4.19}$$

The cubic term in $\mathcal{A}'$ has dropped out since we have imposed $\mathcal{A}'_0 = 0$. Inspecting the terms in our action involving $\hat{h}$, we see that the second term localises to the poles due to the anti-holomorphic derivative acting on Ω. The third term similarly localises to the poles. For this, we consider a manifold whose boundary is $\mathbb{PT}$.[3] We take the 7-manifold $\mathbb{PT} \times [0,1]$ and extend our field $\hat{h}$ over this interval. We do this by choosing a smooth homotopy to a constant map, such that it's restriction to $\mathbb{PT} \times \{0\}$ coincides with $\hat{h}$. Denoting this extension with the same symbol, we see that the third term in our action may be equivalently written as

$$-\frac{1}{6\pi \mathrm{i}} \int_{\mathbb{PT} \times [0,1]} \mathrm{d}\Big[\Omega \wedge \mathrm{Tr}\big(\hat{h}^{-1}\mathrm{d}\hat{h} \wedge \hat{h}^{-1}\mathrm{d}\hat{h} \wedge \hat{h}^{-1}\mathrm{d}\hat{h}\big)\Big] \ . \tag{4.20}$$

Then, using the closure of the Wess-Zumino 3-form and the fact that all of the holomorphic legs on $\mathbb{PT}$ are saturated by Ω, this is equal to

$$S_{\mathrm{WZ}_4} = -\frac{1}{6\pi \mathrm{i}} \int_{\mathbb{PT} \times [0,1]} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\hat{h}^{-1}\mathrm{d}\hat{h} \wedge \hat{h}^{-1}\mathrm{d}\hat{h} \wedge \hat{h}^{-1}\mathrm{d}\hat{h}\big) \ . \tag{4.21}$$

Therefore, this contribution also localises, meaning that the only information contained in the

[3] More generally, a manifold whose boundary is a disjoint union of copies of $\mathbb{PT}$.

field $\hat{h} : \mathbb{PT} \to G$ are its values[4] at the poles of Ω. Explicitly, this contribution is given by[5]

$$\begin{aligned} S_{\mathrm{WZ}_4} &= -\frac{K}{\langle\alpha\tilde{\alpha}\rangle}\int_{\mathbb{R}^4\times[0,1]} \mathrm{vol}_4 \wedge \mathrm{d}\rho\, \varepsilon^{\dot{a}\dot{b}}\bigg[\frac{1}{\langle\alpha\beta\rangle^2}\,\mathrm{Tr}\big(h^{-1}\partial_\rho h\cdot \alpha^a h^{-1}\partial_{a\dot{a}}h\cdot\alpha^b h^{-1}\partial_{b\dot{b}}h\big) \\ &\qquad\qquad - \frac{1}{\langle\tilde{\alpha}\beta\rangle^2}\,\mathrm{Tr}\big(\tilde{h}^{-1}\partial_\rho \tilde{h}\cdot \tilde{\alpha}^a \tilde{h}^{-1}\partial_{a\dot{a}}\tilde{h}\cdot\tilde{\alpha}^b \tilde{h}^{-1}\partial_{b\dot{b}}\tilde{h}\big)\bigg] \\ &= \frac{K}{6\langle\alpha\tilde{\alpha}\rangle}\int_{\mathbb{R}^4\times[0,1]}\bigg[\frac{1}{\langle\alpha\beta\rangle^2}\mu_\alpha\wedge\mathrm{Tr}\big(h^{-1}\mathrm{d}h\big)^3 - \frac{1}{\langle\tilde{\alpha}\beta\rangle^2}\mu_{\tilde{\alpha}}\wedge\mathrm{Tr}\big(\tilde{h}^{-1}\mathrm{d}\tilde{h}\big)^3\bigg] , \end{aligned} \tag{4.22}$$

where

$$\mu_\alpha = \varepsilon_{\dot{a}\dot{b}}\alpha_a\alpha_b\,\mathrm{d}x^{a\dot{a}}\wedge\mathrm{d}x^{b\dot{b}} , \qquad \mu_{\tilde{\alpha}} = \varepsilon_{\dot{a}\dot{b}}\tilde{\alpha}_a\tilde{\alpha}_a\,\mathrm{d}x^{a\dot{a}}\wedge\mathrm{d}x^{b\dot{b}} , \tag{4.23}$$

are the $(2,0)$-forms with respect to the complex structure on $\mathbb{R}^4$ defined by α_a and $\tilde{\alpha}_a$ respectively.[6]

Knowing that the latter two terms in the action (4.19) localise to the poles, we are one step closer to deriving the IFT$_4$. There are two unresolved problems: the first term is still a genuine bulk term; and the second term contains $\mathcal{A}'$, rather than being written exclusively in terms of the fields h and $\tilde{h}$. Both of these problems will be resolved by invoking the bulk equations of motion for $\mathcal{A}'$. This will completely specify its $\mathbb{CP}^1$-dependence, and, combined with the boundary conditions, we will then be able to solve for $\mathcal{A}'$ in terms of the edge modes h and $\tilde{h}$.

Varying the first term in the action, which is the only bulk term, we find the equation of motion $\bar{\partial}_0\mathcal{A}'_{\dot{a}} = 0$, which implies that these components are holomorphic. Combined with the knowledge that $\mathcal{A}'_{\dot{a}}$ has homogeneous weight 1, we deduce that the $\mathbb{CP}^1$-dependence is given by $\mathcal{A}'_{\dot{a}} = \pi^a A_{a\dot{a}}$ where $A_{a\dot{a}}$ is $\mathbb{CP}^1$-independent.

Turning our attention to the boundary conditions, we first consider the double pole where we have imposed $\mathcal{A}_{\dot{a}}|_\beta = 0$. Recalling that $\hat{h}|_\beta = \mathrm{id}$, this simply translates to $\mathcal{A}'_{\dot{a}}|_\beta = 0$. This tells us that $\mathcal{A}'_{\dot{a}} = \langle\pi\beta\rangle B_{\dot{a}}$ for some $B_{\dot{a}}$, hence $A_{a\dot{a}} = \beta_a B_{\dot{a}}$. Therefore, we have that

$$\mathcal{A}_{\dot{a}} = \langle\pi\beta\rangle\mathrm{Ad}_{\hat{h}}^{-1}B_{\dot{a}} + \pi^a\hat{h}^{-1}\partial_{a\dot{a}}\hat{h} . \tag{4.24}$$

The solution for $B_{\dot{a}}$ found by solving the remaining two boundary conditions (4.8) is written more concisely if we introduce some notation. We will make extensive use of the operators

$$U_\pm = \big(1-\sigma^{\pm 1}\Lambda\big)^{-1} , \qquad \Lambda = \mathrm{Ad}_{\tilde{h}}^{-1}\mathrm{Ad}_h , \tag{4.25}$$

[4]For higher order poles in Ω, the $\mathbb{CP}^1$-derivatives of $\hat{h}$ would also contribute to the action.

[5]In principle there are also contributions from the double pole at β both in this term and the second term in the action (4.19). Since this is a double pole, these contributions may depend on $\partial_0\hat{h}|_\beta$, which is unconstrained. However, one can check that they vanish using just the boundary conditions $\mathcal{A}_{\dot{a}}|_\beta = 0$ and internal gauge-fixing $\hat{h}|_\beta = \mathrm{id}$. Alternatively, we may use part of the residual external gauge symmetry to fix $\partial_0\hat{h}|_\beta = 0$, which ensures such contributions vanish.

[6]Here we are using the fact that $\mathbb{R}^4$ is endowed with a hyper-Kähler structure such that there is a $\mathbb{CP}^1$ space of complex structures (see section 2.2).

which enjoy the useful identities

$$U_+^T + U_- = \mathrm{id} \ , \qquad U_\pm \Lambda = -\sigma^{\mp 1} U_\mp^T \ , \tag{4.26}$$

where transposition is understood to be with respect to the ad-invariant inner product on $\mathfrak{g}$. In terms of the components of $\hat{h}^{-1}\partial_{a\dot{a}}\hat{h}$, defined with useful normalisation factors,

$$\begin{aligned} j &= \langle\alpha\beta\rangle^{-1}\mu^{\dot{a}}\alpha^a h^{-1}\partial_{a\dot{a}}h \ , \qquad & \widehat{\jmath} &= \langle\alpha\beta\rangle^{-1}\hat{\mu}^{\dot{a}}\alpha^a h^{-1}\partial_{a\dot{a}}h \ , \\ \widetilde{\jmath} &= \langle\tilde{\alpha}\beta\rangle^{-1}\mu^{\dot{a}}\tilde{\alpha}^a \tilde{h}^{-1}\partial_{a\dot{a}}\tilde{h} \ , \qquad & \widehat{\widetilde{\jmath}} &= \langle\tilde{\alpha}\beta\rangle^{-1}\hat{\mu}^{\dot{a}}\tilde{\alpha}^a \tilde{h}^{-1}\partial_{a\dot{a}}\tilde{h} \ , \end{aligned} \tag{4.27}$$

we find that the solutions to the remaining boundary conditions may be written as

$$\mathrm{Ad}_h^{-1}B_{\dot{a}} = \widehat{b}\mu_{\dot{a}} - b\hat{\mu}_{\dot{a}} \ , \qquad b = U_+(j - \sigma\,\widetilde{\jmath}) \ , \qquad \widehat{b} = U_-(\widehat{\jmath} - \sigma^{-1}\widehat{\widetilde{\jmath}}) \ , \tag{4.28}$$

$$\mathrm{Ad}_{\tilde{h}}^{-1}B_{\dot{a}} = \widehat{\widetilde{b}}\mu_{\dot{a}} - \widetilde{b}\hat{\mu}_{\dot{a}} \ , \qquad \widetilde{b} = U_-^T(\widetilde{\jmath} - \sigma^{-1}j) \ , \qquad \widehat{\widetilde{b}} = U_+^T(\widehat{\widetilde{\jmath}} - \sigma\,\widehat{\jmath}) \ . \tag{4.29}$$

Note that $b = \mathrm{Ad}_h^{-1}[B\mu]$, $\widehat{b} = \mathrm{Ad}_h^{-1}[B\hat{\mu}]$, etc., and that b, $\widetilde{b}$, $\widehat{b}$ and $\widehat{\widetilde{b}}$ are related as

$$\widetilde{b} - \widetilde{\jmath} = \sigma^{-1}(b - j) \ , \qquad \widehat{\widetilde{b}} - \widehat{\widetilde{\jmath}} = \sigma(\widehat{b} - \widehat{\jmath}) \ . \tag{4.30}$$

Recovering the IFT$_4$ is then simple. The first term in the action (4.19) vanishes identically on shell, and we can substitute in our solution for $\mathcal{A}'$ in terms of h and $\tilde{h}$ to get a 4d theory only depending on these edge modes. This results in the action

$$\begin{aligned} S_{\mathrm{IFT}_4} &= \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\mathcal{A}' \wedge \bar{\partial}\hat{h}\hat{h}^{-1}\big) + S_{\mathrm{WZ}_4} \\ &= \frac{K}{\langle\alpha\tilde{\alpha}\rangle}\int_{\mathbb{R}^4} \mathrm{vol}_4\, \mathrm{Tr}\big(b(\widehat{\jmath} - \Lambda^T\widehat{\widetilde{\jmath}}) - \widehat{b}(j - \Lambda^T\widetilde{\jmath})\big) + S_{\mathrm{WZ}_4} \\ &= \frac{K}{\langle\alpha\tilde{\alpha}\rangle}\int_{\mathbb{R}^4} \mathrm{vol}_4\, \mathrm{Tr}\Big(j(U_+^T - U_-)\widehat{\jmath} + \widetilde{\jmath}(U_+^T - U_-)\widehat{\widetilde{\jmath}} - 2\sigma\,\widetilde{\jmath}\,U_+^T\,\widehat{\jmath} + 2\sigma^{-1}j\,U_-\,\widehat{\widetilde{\jmath}}\Big) + S_{\mathrm{WZ}_4} \ , \end{aligned} \tag{4.31}$$

where

$$S_{\mathrm{WZ}_4} = \frac{K}{\langle\alpha\tilde{\alpha}\rangle}\int_{\mathbb{R}^4\times[0,1]} \mathrm{vol}_4 \wedge \mathrm{d}\rho\, \mathrm{Tr}\big(h^{-1}\partial_\rho h \cdot [j, \widehat{\jmath}] - \tilde{h}^{-1}\partial_\rho\tilde{h} \cdot [\widetilde{\jmath}, \widehat{\widetilde{\jmath}}]\big) \ . \tag{4.32}$$

Observe that the 4d IFT (4.31) with (4.32) is mapped into itself under the formal transformation

$$h \leftrightarrow \tilde{h} \ , \qquad \alpha \leftrightarrow \tilde{\alpha} \ , \qquad \sigma \mapsto \sigma^{-1} \ , \tag{4.33}$$

interchanging the positions of the two poles. This directly follows from the invariance (4.9) of the hCS$_6$ boundary conditions. On the other hand, looking at how the transformation (4.10) descends to the IFT$_4$, we find[7]

$$j \mapsto \widehat{\jmath}\ , \qquad \widehat{\jmath} \mapsto -j\ , \qquad \widetilde{\jmath} \mapsto \widehat{\widetilde{\jmath}}\ , \qquad \widehat{\widetilde{\jmath}} \mapsto -\widetilde{\jmath}\ , \qquad \sigma \mapsto \sigma^{-1}\ . \tag{4.34}$$

It is then straightforward to check that the action (4.31) with (4.32) is invariant under this transformation.

Let us emphasise that, to our knowledge, the IFT$_4$ described by the action (4.31) with (4.32) has not been considered in the literature before. In the following subsections we will study some properties of this model starting with its symmetries, and moving onto its equations of motion and their relation to the 4d ASDYM equations.

4.3.1 Symmetries of the IFT$_4$

Having derived the action functional for the IFT$_4$, we will now examine those symmetries that leave this action invariant. While they may not be obvious from simply looking at the action, in § 4.2.1 we leveraged the hCS$_6$ description to predict the symmetries of the IFT$_4$. These may then be verified by explicit computation.

To this end, we recall that the hCS$_6$ gauge transformations act as

$$\hat{g}: \quad \mathcal{A} \mapsto \hat{g}^{-1}\mathcal{A}\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g}\ , \tag{4.35}$$

and we denoted the value of this transformation parameter at the poles by

$$\hat{g}|_\alpha = r\ , \quad \hat{g}|_{\tilde{\alpha}} = \tilde{r}\ , \quad \hat{g}|_\beta = \ell^{-1}\ . \tag{4.36}$$

Tracing through the derivation above, we find that these result in an induced action on the IFT$_4$ fields,

$$(\ell, r, \tilde{r}): \quad h \mapsto \ell h r\ , \quad \tilde{h} \mapsto \ell\tilde{h}\tilde{r}\ , \tag{4.37}$$

where ℓ, r and $\tilde{r}$ obey the constraints (4.13) and (4.14) respectively. One can explicitly verify that the IFT$_4$ is indeed invariant under these transformations. Key to this is exploiting a Polyakov-Wiegmann identity such the variation of S_{WZ_4} in eq. (4.32) produces a total derivative. This gives a contribution on $\mathbb{R}^4$ that cancels the variation of the remainder of eq. (4.31). Useful intermediate results to this end are

$$\begin{gathered}\mathrm{Ad}_h \mapsto \mathrm{Ad}_\ell \mathrm{Ad}_h \mathrm{Ad}_r\ , \quad \mathrm{Ad}_{\tilde{h}} \mapsto \mathrm{Ad}_\ell \mathrm{Ad}_{\tilde{h}} \mathrm{Ad}_r\ , \quad U_\pm \mapsto \mathrm{Ad}_r^{-1} U_\pm \mathrm{Ad}_r\ , \\ b \mapsto \mathrm{Ad}_r^{-1}(b + \langle\alpha\beta\rangle^{-1}\mathrm{Ad}_h^{-1}\mu^{\dot{a}}\alpha^a\ell^{-1}\partial_{a\dot{a}}\ell)\ , \quad \hat{b} \mapsto \mathrm{Ad}_r^{-1}(\hat{b} + \langle\alpha\beta\rangle^{-1}\mathrm{Ad}_h^{-1}\hat{\mu}^{\dot{a}}\alpha^a\ell^{-1}\partial_{a\dot{a}}\ell)\ ,\end{gathered} \tag{4.38}$$

in which the constraints (4.13) and (4.14) have been used.

We can also derive the Noether currents corresponding to these residual semi-local symmetries directly from hCS$_6$. The variation of the action under an infinitesimal gauge transformation

[7]Note that to derive this we use that $\hat{\mu} = -\mu$ following the "quaternionic conjugation" defined in section 2.2.

$\delta\mathcal{A} = \bar{\partial}\hat{\epsilon} + [\mathcal{A}, \hat{\epsilon}]$ is

$$\delta S_{\text{6dCS}} = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\mathcal{A} \wedge \bar{\partial}\hat{\epsilon}\big) \ . \tag{4.39}$$

First let us consider a transformation that descends to the ℓ-symmetry, i.e. one for which

$$\hat{\epsilon}|_{\alpha} = \hat{\epsilon}|_{\tilde{\alpha}} = 0 \ , \quad \hat{\epsilon}|_{\beta} = \epsilon^{(\ell)} \ .$$

The only contribution to the variation localises to β and is given by

$$\delta_{\ell} S_{\text{6dCS}} \propto \int_{\mathbb{R}^4} \mathrm{vol}_4 \, \partial_0 \bigg(\frac{1}{\langle \pi\alpha\rangle\langle\pi\tilde{\alpha}\rangle} \varepsilon^{\dot{a}\dot{b}} \mathrm{Tr}\big(\mathcal{A}_{\dot{a}} \bar{\partial}_{\dot{b}} \hat{\epsilon}\big) \bigg)\bigg|_{\beta} \ . \tag{4.40}$$

Since $\mathcal{A}_{\dot{a}}|_{\beta} \propto \langle\pi\beta\rangle$ (recall that we fix $\hat{h}|_{\beta} = \mathrm{id}$) the only way the integrand will be non-vanishing is for the ∂_0 operator to act on $\mathcal{A}_{\dot{a}}$. Noting that $\partial_0\langle\pi\beta\rangle|_{\beta} = 1$ we have that $\partial_0\mathcal{A}_{\dot{a}}|_{\beta} = B_{\dot{a}}$, and hence the variation becomes

$$\delta_{\ell} S_{\text{6dCS}} \propto \int_{\mathbb{R}^4} \mathrm{vol}_4 \, \mathrm{Tr}\Big(B^{\dot{a}} \beta^a \partial_{a\dot{a}} \epsilon^{(\ell)} \Big) \ . \tag{4.41}$$

If we think of the ℓ-symmetry as a global transformation, then this would provide the conservation law associated to the Noether current, i.e.

$$\beta^a \partial_{a\dot{a}} B^{\dot{a}} = 0 \ , \tag{4.42}$$

and indeed we will see later that this conservation law follows from the equations of motion of the IFT_4. As the parameter $\epsilon^{(\ell)}$ is allowed to be holomorphic with respect to the complex structure defined by β, the interpretation is more akin to a Kac-Moody current.

For the case corresponding to the r-symmetry we have

$$\hat{\epsilon}|_{\alpha} = \hat{\epsilon}|_{\tilde{\alpha}} = \epsilon^{(r)} \ , \quad \hat{\epsilon}|_{\beta} = 0 \ .$$

In this case the variation receives two contributions with an opposite sign

$$\delta_r S_{\text{6dCS}} \propto \int_{\mathbb{R}^4} \mathrm{vol}_4 \, \varepsilon^{\dot{a}\dot{b}} \, \mathrm{Tr}\bigg(\frac{1}{\langle\alpha\beta\rangle^2} \mathcal{A}_{\dot{a}} \bar{\partial}_{\dot{b}} \hat{\epsilon}\big|_{\alpha} - \frac{1}{\langle\tilde{\alpha}\beta\rangle^2} \mathcal{A}_{\dot{a}} \bar{\partial}_{\dot{b}} \hat{\epsilon}\big|_{\tilde{\alpha}} \bigg) \ . \tag{4.43}$$

Integrating by parts gives

$$\delta_r S_{\text{6dCS}} \propto \int_{\mathbb{R}^4} \mathrm{vol}_4 \, \mathrm{Tr}\bigg(\epsilon^{(r)} \bigg(\frac{\alpha^a}{\langle\alpha\beta\rangle^2} \partial_{a\dot{a}} \mathcal{A}^{\dot{a}}|_{\alpha} - \frac{\tilde{\alpha}^a}{\langle\tilde{\alpha}\beta\rangle^2} \partial_{a\dot{a}} \mathcal{A}^{\dot{a}}|_{\tilde{\alpha}} \bigg)\bigg) \ . \tag{4.44}$$

Introducing new currents defined by

$$\langle\alpha\beta\rangle \, C_{\dot{a}} = \mathcal{A}_{\dot{a}}|_{\alpha} \ , \quad \langle\tilde{\alpha}\beta\rangle \, \widetilde{C}_{\dot{a}} = \mathcal{A}_{\dot{a}}|_{\tilde{\alpha}} \ , \tag{4.45}$$

the conservation law associated to the r-symmetry is given by

$$\frac{1}{\langle\alpha\beta\rangle}\alpha^a\partial_{a\dot{a}}C^{\dot{a}} - \frac{1}{\langle\tilde{\alpha}\beta\rangle}\tilde{\alpha}^a\partial_{a\dot{a}}\widetilde{C}^{\dot{a}} = 0 \ . \tag{4.46}$$

Recalling from eq. (4.24) that $\mathcal{A}_{\dot{a}} = \langle\pi\beta\rangle\mathrm{Ad}_{\hat{h}}^{-1}B_{\dot{a}} + \pi^a\hat{h}^{-1}\partial_{a\dot{a}}\hat{h}$, we can relate the B current to the C and $\widetilde{C}$ currents as follows

$$C_{\dot{a}} = \mathrm{Ad}_h^{-1}B_{\dot{a}} + \frac{1}{\langle\alpha\beta\rangle}\alpha^a h^{-1}\partial_{a\dot{a}}h = (\widehat{b} - \widehat{\jmath})\mu_{\dot{a}} - (b-j)\hat{\mu}_{\dot{a}} \ , \tag{4.47}$$

$$\widetilde{C}_{\dot{a}} = \mathrm{Ad}_{\tilde{h}}^{-1}B_{\dot{a}} + \frac{1}{\langle\tilde{\alpha}\beta\rangle}\tilde{\alpha}^a\tilde{h}^{-1}\partial_{a\dot{a}}\tilde{h} = \sigma(\widehat{b} - \widehat{\jmath})\mu_{\dot{a}} - \sigma^{-1}(b-j)\hat{\mu}_{\dot{a}} \ , \tag{4.48}$$

where we have used the identities (4.30). The transformation of these currents under the (ℓ, r)-symmetries is given by

$$\begin{aligned} (\ell, r): &\quad B_{\dot{a}} \mapsto \mathrm{Ad}_\ell B_{\dot{a}} - \langle\alpha\beta\rangle^{-1}\alpha^a\partial_{a\dot{a}}\ell\ell^{-1} \ , \\ (\ell, r): &\quad C_{\dot{a}} \mapsto \mathrm{Ad}_r^{-1}C_{\dot{a}} + \langle\alpha\beta\rangle^{-1}\alpha^a r^{-1}\partial_{a\dot{a}}r \ , \\ (\ell, r): &\quad \widetilde{C}_{\dot{a}} \mapsto \mathrm{Ad}_r^{-1}\widetilde{C}_{\dot{a}} + \langle\tilde{\alpha}\beta\rangle^{-1}\tilde{\alpha}^a r^{-1}\partial_{a\dot{a}}r \ . \end{aligned} \tag{4.49}$$

As a consequence notice that the 4d ($\mathbb{CP}^1$-independent) gauge field introduced above, $A_{a\dot{a}} = \beta_a B_{\dot{a}}$, is invariant under the right action, whereas the left action acts as a conventional gauge transformation

$$(\ell, r): \quad A_{a\dot{a}} \mapsto \mathrm{Ad}_\ell A_{a\dot{a}} - \partial_{a\dot{a}}\ell\ell^{-1} \ , \tag{4.50}$$

albeit semi-local rather than fully local since ℓ is constrained as in eq. (4.13). The transformation of these currents and the 4d ASD connection also follows from the hCS$_6$ description. While the original gauge transformations act on $\mathcal{A}$, we observe that r and $\tilde{r}$ become right-actions on $\hat{h}$, leaving $\mathcal{A}'$ and $A_{a\dot{a}}$ invariant. By comparison, after fixing $\hat{h}|_\beta = \mathrm{id}$, it is only a combination of the 'internal' transformations and the original gauge transformations that preserve this constraint. In particular, ℓ has an induced action on h, $\tilde{h}$ and $\mathcal{A}'$, thus leading to the above transformations of $B_{\dot{a}}$ and $A_{a\dot{a}}$.

As we will show momentarily, the equations of motion of the theory correspond to anti-self duality of the field strength of the connection $A_{a\dot{a}}$, hence it immediately follows that the equations of motion are preserved by the symmetry transformations (4.50). To close the section we note that the action is concisely given in terms of the currents as

$$S_{\mathrm{IFT}_4} = \frac{K}{\langle\alpha\tilde{\alpha}\rangle}\int_{\mathbb{R}^4}\mathrm{vol}_4\,\epsilon^{\dot{a}\dot{b}}\mathrm{Tr}\big(\mathrm{Ad}_h^{-1}B_{\dot{a}}(C_{\dot{b}} - \Lambda^T\widetilde{C}_{\dot{b}})\big) + S_{\mathrm{WZ}_4} \ . \tag{4.51}$$

4.3.2 Equations of Motion, 4d ASDYM and Lax Formulation

The equations of motion of the IFT_4 can be obtained in a brute force fashion by performing a variation of the action (4.31). This calculation requires the variation of the operators $U_\pm$

$$\delta U_\pm(X) = U_\pm(\delta X) + U_\pm\left([\tilde{h}^{-1}\delta\tilde{h}, U_\mp^T(X)]\right) - U_\mp^T\left([h^{-1}\delta h, U_\pm(X)]\right) \ , \tag{4.52}$$

but is otherwise straightforward. The outcome is that the equations of motion can be written as

$$\begin{aligned} &-\frac{\mu^{\dot{a}}\alpha^a}{\langle\alpha\beta\rangle}\partial_{a\dot{a}}\widehat{b} + \frac{\hat{\mu}^{\dot{a}}\alpha^a}{\langle\alpha\beta\rangle}\partial_{a\dot{a}}b + [\widehat{\jmath}, b] - [j, \widehat{b}] - [\widehat{b}, b] = 0 \ , \\ &-\frac{\mu^{\dot{a}}\tilde{\alpha}^a}{\langle\tilde{\alpha}\beta\rangle}\partial_{a\dot{a}}\widehat{\widetilde{b}} + \frac{\hat{\mu}^{\dot{a}}\tilde{\alpha}^a}{\langle\tilde{\alpha}\beta\rangle}\partial_{a\dot{a}}\widetilde{b} + [\widehat{\widetilde{\jmath}}, \widetilde{b}] - [\widetilde{\jmath}, \widehat{\widetilde{b}}] - [\widehat{\widetilde{b}}, \widetilde{b}] = 0 \ , \end{aligned} \tag{4.53}$$

in which we invoke the definitions of b, $\widetilde{b}$, $\widehat{b}$ and $\widehat{\widetilde{b}}$ above in eqs. (4.28) and (4.29). These equations can be written in terms of the current $B_{\dot{a}}$ as

$$\begin{aligned} \alpha^a\partial_{a\dot{a}}B^{\dot{a}} + \frac{1}{2}\langle\alpha\beta\rangle[B_{\dot{a}}, B^{\dot{a}}] = 0 \ , \\ \tilde{\alpha}^a\partial_{a\dot{a}}B^{\dot{a}} + \frac{1}{2}\langle\tilde{\alpha}\beta\rangle[B_{\dot{a}}, B^{\dot{a}}] = 0 \ . \end{aligned} \tag{4.54}$$

Taking a weighted sum and difference equations gives

$$\begin{aligned} \big(\langle\tilde{\alpha}\beta\rangle\alpha^a - \langle\alpha\beta\rangle\tilde{\alpha}^a\big)\partial_{a\dot{a}}B^{\dot{a}} = -\langle\alpha\tilde{\alpha}\rangle\beta^a\partial_{a\dot{a}}B^{\dot{a}} = 0 \ , \\ \big(\langle\tilde{\alpha}\beta\rangle\alpha^a + \langle\alpha\beta\rangle\tilde{\alpha}^a\big)\partial_{a\dot{a}}B^{\dot{a}} + \langle\tilde{\alpha}\beta\rangle\langle\alpha\beta\rangle[B_{\dot{a}}, B^{\dot{a}}] = 0 \ . \end{aligned} \tag{4.55}$$

the first of which is the anticipated conservation equation for the ℓ-symmetry. Making use of the definitions of C and $\widetilde{C}$ in (4.47), the equations of motion are equivalently expressed as

$$\begin{aligned} \alpha^a\partial_{a\dot{a}}C^{\dot{a}} + \frac{1}{2}\langle\alpha\beta\rangle[C_{\dot{a}}, C^{\dot{a}}] = 0 \ , \\ \tilde{\alpha}^a\partial_{a\dot{a}}\widetilde{C}^{\dot{a}} + \frac{1}{2}\langle\tilde{\alpha}\beta\rangle[\widetilde{C}_{\dot{a}}, \widetilde{C}^{\dot{a}}] = 0 \ . \end{aligned} \tag{4.56}$$

Noting that $[C_{\dot{a}}, C^{\dot{a}}] = [\widetilde{C}_{\dot{a}}, \widetilde{C}^{\dot{a}}]$ we can take the difference of these equations to obtain

$$\frac{1}{\langle\alpha\beta\rangle}\alpha^a\partial_{a\dot{a}}C^{\dot{a}} - \frac{1}{\langle\tilde{\alpha}\beta\rangle}\tilde{\alpha}^a\partial_{a\dot{a}}\widetilde{C}^{\dot{a}} = 0 \ , \tag{4.57}$$

which is the anticipated conservation law for the r-symmetry.

ASDYM. We will now justify the claim that this theory is integrable by constructing explicit Lax pair formulations of the dynamics in two different fashions. First we will show the equations of motion (4.54) can be recast as the anti-self-dual equation for a Yang-Mills connection. Before demonstrating that this holds for our particular model, let us highlight that this follows from the construction of hCS$_6$ by briefly reviewing the Penrose-Ward correspondence [War77]. Recalling that we have resolved one of the hCS$_6$ equations of motion $\mathcal{F}'_{0\dot{a}} = 0$ to find $\mathcal{A}'_{\dot{a}} = \pi^a A_{a\dot{a}}$, it follows that the remaining system of equations should be equivalent to the vanishing of the other components of the field strength $\mathcal{F}'_{\dot{a}\dot{b}} = 0$. We may express this in terms of the anti-holomorphic covariant derivative $\bar{D}'_{\dot{a}} = \bar{\partial}_{\dot{a}} + \mathcal{A}'_{\dot{a}}$ as $[\bar{D}'_{\dot{a}}, \bar{D}'_{\dot{a}}] = 0$, which may also be written as

$$\pi^a \pi^b [D_{a\dot{a}}, D_{b\dot{b}}] = 0 \ . \tag{4.58}$$

This is equivalent to the vanishing of $\pi^a\pi^b F_{a\dot{a}b\dot{b}}$ where F is the field strength of the 4d connection $A_{a\dot{a}}$. To make contact with the anti-self-dual Yang-Mills equation, note that an arbitrary tensor that is anti-symmetric in Lorentz indices, e.g. $F_{\mu\nu}$, can be expanded in spinor indices as

$$F_{a\dot{a}b\dot{b}} = \varepsilon_{\dot{a}\dot{b}}\, \phi_{ab} + \varepsilon_{ab}\, \tilde{\phi}_{\dot{a}\dot{b}} \ . \tag{4.59}$$

Here, ϕ and $\tilde{\phi}$ are both symmetric and correspond to the self-dual and anti-self-dual components of the field strength respectively. Explicitly computing the contraction (4.58), we find that the remaining equation is simply $\phi = 0$ which is indeed the anti-self-dual Yang-Mills equation. In effect, this argument demonstrates that a holomorphic gauge field on twistor space (which is gauge-trivial in $\mathbb{CP}^1$) is in bijection with a solution to the 4-dimensional anti-self-dual Yang-Mills equation – this statement is the Penrose-Ward correspondence.

Now, returning to the case at hand, recall that our connection is of the form $A_{a\dot{a}} = \beta_a B_{\dot{a}}$, so the anti-self-dual Yang-Mills equation specialises to

$$\langle \pi\beta \rangle \big(\pi^a \partial_{a\dot{a}} B_{\dot{b}} - \pi^b \partial_{b\dot{b}} B_{\dot{a}} + \langle \pi\beta\rangle [B_{\dot{a}}, B_{\dot{b}}] \big) = 0 \ . \tag{4.60}$$

This should hold for any $\pi^a \in \mathbb{CP}^1$ and we can extract the key information by expanding π^a in the basis formed by α^a and $\tilde{\alpha}^a$, that is

$$\pi^a = \frac{1}{\langle \alpha\tilde{\alpha}\rangle} \Big(\langle \pi\tilde{\alpha}\rangle \alpha^a - \langle \pi\alpha\rangle \tilde{\alpha}^a \Big) \ . \tag{4.61}$$

Substituting into (4.60), we find two independent equations with $\mathbb{CP}^1$-dependent coefficients $\langle\pi\beta\rangle\langle\pi\tilde{\alpha}\rangle$ and $\langle\pi\beta\rangle\langle\pi\alpha\rangle$ respectively. These are explicitly given by the two equations in eq. (4.54). Therefore, as expected, the equations of motion for this IFT$_4$ are equivalent to the anti-self-dual Yang-Mills equation for $A_{a\dot{a}} = \beta_a B_{\dot{a}}$.

Let us comment on the relation to Ward's conjecture which postulates that many[8] integrable models arise as reductions of the ASDYM equations. It is clear that that the equations of motion for the λ-deformations explored in this paper arise as symmetry reductions of the ASDYM equations for the explicit form of the connection given above. On the other hand, a generic

[8]The original conjecture [War85] states that "many (and perhaps all?)" integrable models arise in this manner. However, a notable absentee of this proposal is the Kadomtsev-Petviashvilii (KP) equation, see [Mas17] for a discussion.

ASDYM connection can be partially gauge-fixed such that the remaining degrees of freedom are completely captured by the so-called Yang's matrix, which is the fundamental field of the WZW_4 model. In this case, the equations of motion of the WZW_4 model, known as Yang's equations, are the remaining ASDYM equations. It is natural to ask whether a generic ASDYM connection can also be partially gauge-fixed to take the explicit form found in this paper. This would provide a 1-to-1 correspondence between solutions of the ASDYM equations and solutions to our 4d IFT.

B-Lax. The anti-self-dual Yang-Mills equation is also 'integrable' in the sense that it admits a Lax formalism. Using the basis of spinors $\mu^{\dot{a}}$ and $\hat{\mu}^{\dot{a}}$, we define the Lax pair L and M by

$$L^{(B)} = \langle \pi\hat{\gamma}\rangle^{-1}\hat{\mu}^{\dot{a}}\pi^{a}D_{a\dot{a}} \ , \qquad M^{(B)} = \langle \pi\gamma\rangle^{-1}\mu^{\dot{a}}\pi^{a}D_{a\dot{a}} \ , \tag{4.62}$$

where the normalisations are for later convenience.[9] It is important to emphasise that here π^a is not just an *ad hoc* spectral parameter. It is introduced directly as a result of the hCS_6 equations of motion and is the coordinate on $\mathbb{CP}^1 \hookrightarrow \mathbb{PT}$. The vanishing of $[L^{(B)}, M^{(B)}] = 0$ for any $\pi^a \in \mathbb{CP}^1$ is equivalent to the anti-self-dual Yang-Mills equation.

C-Lax. Let us now turn to the equations of motion cast in terms of the C-currents in eq. (4.56). Evidently, looking at the definition of these currents eq. (4.47), we see that when $\sigma = 1$ we have $\widetilde{C} = C$ and the equations of motion have the same form as the B-current equations (4.54). Therefore, when $\sigma = 1$, we can package the C-equations in terms of a ASDYM connection $A^{(C)}_{a\dot{a}} = \beta_a C_{\dot{a}}$. Away from this point, when $\widetilde{C} \neq C$ it is not immediately evident if these equations follow from an ASDYM connection. Regardless, we can still give these equations a Lax pair presentation as follows.

Letting $\varrho \in \mathbb{C}$ denote a spectral parameter we define

$$\begin{aligned} L^{(C)} &= \frac{1}{n_L}\hat{\mu}^{\dot{a}}\left(\frac{\alpha^a}{\langle\alpha\beta\rangle}(1+\varrho) + \frac{\sigma^{-1}\tilde{\alpha}^a}{\langle\tilde{\alpha}\beta\rangle}(1-\varrho)\right)\partial_{a\dot{a}} + \frac{1}{n_L}\hat{\mu}^{\dot{a}}C_{\dot{a}} \ , \\ M^{(C)} &= \frac{1}{n_M}\mu^{\dot{a}}\left(\frac{\alpha^a}{\langle\alpha\beta\rangle}(1+\varrho) + \frac{\sigma\tilde{\alpha}^a}{\langle\tilde{\alpha}\beta\rangle}(1-\varrho)\right)\partial_{a\dot{a}} + \frac{1}{n_M}\mu^{\dot{a}}C_{\dot{a}} \ . \end{aligned} \tag{4.63}$$

Noting that $\mu^{\dot{a}}\widetilde{C}_{\dot{a}} = \sigma^{-1}\mu^{\dot{a}}C_{\dot{a}}$ and $\hat{\mu}^{\dot{a}}\widetilde{C}_{\dot{a}} = \sigma\hat{\mu}^{\dot{a}}C_{\dot{a}}$ one immediately sees that the terms inside $[L^{(C)}, M^{(C)}]$ linear in ϱ yield the conservation law eq. (4.57). The contributions independent of ϱ, combined with eq. (4.57), give either of eq. (4.56). It will be convenient to fix the overall normalisation of these Lax operators to be

$$n_L = \frac{\langle\alpha\hat{\gamma}\rangle}{\langle\alpha\beta\rangle}(1+\varrho) + \frac{\langle\tilde{\alpha}\hat{\gamma}\rangle}{\langle\tilde{\alpha}\beta\rangle}\sigma^{-1}(1-\varrho) \ , \qquad n_M = \frac{\langle\alpha\gamma\rangle}{\langle\alpha\beta\rangle}(1+\varrho) + \frac{\langle\tilde{\alpha}\gamma\rangle}{\langle\tilde{\alpha}\beta\rangle}\sigma(1-\varrho) \ . \tag{4.64}$$

Unlike the spectral parameter π_a entering the B-Lax, there is no clear way to associate the spectral parameter of the C-Lax, ϱ, with the $\mathbb{CP}^1$ coordinate alone. Indeed, under a natural

[9] The constant spinors γ and $\hat{\gamma}$ appear in the symmetry reduction and will be introduced in § 4.4.

assumption, we will see that ϱ has origins from both the $\mathbb{CP}^1$ geometry *and* the parameters that enter the boundary conditions.

The existence of a second Lax formulation of the theory, distinct from the ASDYM equations encoded via the B-Lax, is an unexpected feature. We will see shortly that, upon symmetry reduction, this twin Lax formulation is inherited by the IFT_2.

4.3.3 Reality Conditions and Parameters

To understand how the reality of the action of the IFT_4 (4.31) with (4.32), as well as the dependence on the parameters K, σ, α_a, $\tilde{\alpha}_a$, β_a, $\mu^{\dot{a}}$ and $\hat{\mu}^{\dot{a}}$, let us denote our coordinates

$$\begin{aligned} \mathsf{w} &= \frac{\langle\alpha\beta\rangle}{\langle\alpha\tilde{\alpha}\rangle[\mu\hat{\mu}]}\hat{\mu}_{\dot{a}}\tilde{\alpha}_a x^{a\dot{a}} , \qquad & \hat{\mathsf{w}} &= -\frac{\langle\alpha\beta\rangle}{\langle\alpha\tilde{\alpha}\rangle[\mu\hat{\mu}]}\mu_{\dot{a}}\tilde{\alpha}_a x^{a\dot{a}} , \\ \mathsf{z} &= -\frac{\langle\tilde{\alpha}\beta\rangle}{\langle\alpha\tilde{\alpha}\rangle[\mu\hat{\mu}]}\hat{\mu}_{\dot{a}}\alpha_a x^{a\dot{a}} , \qquad & \hat{\mathsf{z}} &= \frac{\langle\tilde{\alpha}\beta\rangle}{\langle\alpha\tilde{\alpha}\rangle[\mu\hat{\mu}]}\mu_{\dot{a}}\alpha_a x^{a\dot{a}} , \end{aligned} \tag{4.65}$$

such that

$$j = h^{-1}\partial_{\mathsf{w}} h , \qquad \hat{\jmath} = h^{-1}\partial_{\hat{\mathsf{w}}} h , \qquad \tilde{\jmath} = \tilde{h}^{-1}\partial_{\mathsf{z}}\tilde{h} , \qquad \hat{\tilde{\jmath}} = \tilde{h}^{-1}\partial_{\hat{\mathsf{z}}}\tilde{h} . \tag{4.66}$$

In this subsection we let $\mu^{\dot{a}}$ and $\hat{\mu}^{\dot{a}}$ be an unconstrained basis of spinors, i.e., not related by Euclidean conjugation or of fixed norm. This means the action (4.31) with (4.32) comes with an extra factor of $[\mu\hat{\mu}]^{-1}$. Writing the volume element $\text{vol}_4 = \frac{1}{12}\varepsilon_{\dot{a}\dot{b}}\varepsilon_{\dot{c}\dot{d}}\varepsilon_{ac}\varepsilon_{bd}dx^{a\dot{a}}\wedge dx^{b\dot{b}}\wedge dx^{c\dot{c}}\wedge dx^{d\dot{d}}$ in terms of the coordinates $\{\mathsf{w},\hat{\mathsf{w}},\mathsf{z},\hat{\mathsf{z}}\}$ we find

$$\text{vol}_4 = \frac{\langle\alpha\tilde{\alpha}\rangle^2[\mu\hat{\mu}]^2}{\langle\alpha\beta\rangle^2\langle\tilde{\alpha}\beta\rangle^2}\mathrm{d}\mathsf{w}\wedge\mathrm{d}\hat{\mathsf{w}}\wedge\mathrm{d}\mathsf{z}\wedge\mathrm{d}\hat{\mathsf{z}} = \frac{\langle\alpha\tilde{\alpha}\rangle^2[\mu\hat{\mu}]^2}{\langle\alpha\beta\rangle^2\langle\tilde{\alpha}\beta\rangle^2}\text{vol}_4' . \tag{4.67}$$

Substituting into the action (4.31) with (4.32), we see that the IFT_4 now only depends explicitly on two parameters

$$K' = \frac{\langle\alpha\tilde{\alpha}\rangle[\mu\hat{\mu}]}{\langle\alpha\beta\rangle^2\langle\tilde{\alpha}\beta\rangle^2}K \quad \text{and} \quad \sigma . \tag{4.68}$$

Moreover, the action is invariant under the following $GL(1;\mathbb{C})$ space-time symmetry

$$\mathsf{z}\to e^{\vartheta}\mathsf{z} , \qquad \mathsf{w}\to e^{\vartheta}\mathsf{w} , \qquad \hat{\mathsf{z}}\to e^{-\vartheta}\hat{\mathsf{z}} , \qquad \hat{\mathsf{w}}\to e^{-\vartheta}\hat{\mathsf{w}} , \tag{4.69}$$

where $\vartheta\in\mathbb{C}$.

To find a real action we should impose reality conditions on the coordinates $\{\mathsf{w},\hat{\mathsf{w}},\mathsf{z},\hat{\mathsf{z}}\}$, the fields h and $\tilde{h}$, and the parameters K' and σ. We start by observing four sets of admissible reality conditions simply found by inspection of the 4d IFT. Note that, implicitly, we will not assume Euclidean reality conditions for $x^{a\dot{a}}$. Starting from Euclidean reality conditions we complexify and take different split signature real slices. We will then turn to the hCS_6 origin of the different reality conditions.

Introducing Θ, the lift of an antilinear involutive automorphism θ of the Lie algebra $\mathfrak{g}$ to the group G, the four sets of reality conditions are as follows:

1. In the first case, the coordinates are all real, $\bar{\mathsf{w}} = \mathsf{w}$, $\bar{\hat{\mathsf{w}}} = \hat{\mathsf{w}}$, $\bar{\mathsf{z}} = \mathsf{z}$, $\bar{\hat{\mathsf{z}}} = \hat{\mathsf{z}}$; K' and σ are real; and the group-valued fields are elements of the real form, $\Theta(h) = h$ and $\Theta(\tilde{h}) = \tilde{h}$. Under conjugation we have $U_\pm \to U_\pm$.

2. In the second case, the coordinates conjugate as $\bar{\mathsf{w}} = \hat{\mathsf{w}}$, $\bar{\mathsf{z}} = \hat{\mathsf{z}}$; K' is imaginary and σ is a phase factor; and the group-valued fields are elements of the real form, $\Theta(h) = h$ and $\Theta(\tilde{h}) = \tilde{h}$. Under conjugation we have $U_\pm \to U_\mp$.

3. In the third case, the coordinates conjugate as $\bar{\mathsf{w}} = \hat{\mathsf{z}}$, $\bar{\mathsf{z}} = \hat{\mathsf{w}}$; K' and σ are real; and the group-valued fields are related by conjugation $\Theta(h) = \tilde{h}$. Under conjugation we have $U_\pm \to U_\pm^T$.

4. In the final case, the coordinates conjugate as $\bar{\mathsf{w}} = \mathsf{z}$, $\bar{\hat{\mathsf{w}}} = \hat{\mathsf{z}}$; K' is imaginary and σ is a phase factor; and the group-valued fields are related by conjugation $\Theta(h) = \tilde{h}$. Under conjugation we have $U_\pm \to U_\mp^T$.

The action (4.31) with (4.32) is real for each of these sets of reality conditions. To determine the signature for each set of reality conditions, we note that[10]

$$\varepsilon_{\dot{a}\dot{b}}\varepsilon_{ab}\mathrm{d}x^{a\dot{a}}\mathrm{d}x^{b\dot{b}} = \frac{2\langle\alpha\tilde{\alpha}\rangle[\mu\hat{\mu}]}{\langle\alpha\beta\rangle\langle\tilde{\alpha}\beta\rangle}(\mathrm{d}\mathsf{w}\mathrm{d}\hat{\mathsf{z}} - \mathrm{d}\mathsf{z}\mathrm{d}\hat{\mathsf{w}}) \ , \tag{4.70}$$

It is then straightforward to see that the four sets of reality conditions above all correspond to split signature. Note that for the metric to be real, we require the prefactor to be real in cases 1 and 4 and imaginary in cases 2 and 3. We will see that this is indeed the case when we comment on the hCS_6 origin.

In cases 1 and 4 the reality conditions are preserved by an $SO(1,1)$ space-time symmetry (4.69) with $\vartheta \in \mathbb{R}$. On the other hand, in cases 2 and 3, the reality conditions are preserved by an $SO(2)$ space-time symmetry with $|\vartheta| = 1$. In § 4.5, we will be interested in symmetry reducing while preserving the space-time symmetry, recovering an action on $\mathbb{R}^2$ or $\mathbb{R}^{1,1}$ that is invariant under the Euclidean or Poincaré groups respectively. We have freedom in how we do this since the action is not invariant under $SO(2)$ rotations acting on (z,w) and $(\hat{\mathsf{z}},\hat{\mathsf{w}})$. Therefore, we can choose symmetry reduce along different directions in each of these planes, in principle introducing

[10]Conjugating in Euclidean signature we find the reality conditions

$$\bar{\mathsf{w}} = \frac{\langle\hat{\alpha}\hat{\beta}\rangle}{\langle\hat{\alpha}\hat{\tilde{\alpha}}\rangle}\left(\frac{\langle\alpha\hat{\tilde{\alpha}}\rangle}{\langle\alpha\beta\rangle}\hat{\mathsf{w}} + \frac{\langle\tilde{\alpha}\hat{\tilde{\alpha}}\rangle}{\langle\tilde{\alpha}\beta\rangle}\hat{\mathsf{z}}\right) \ , \qquad \bar{\mathsf{z}} = \frac{\langle\hat{\tilde{\alpha}}\hat{\beta}\rangle}{\langle\hat{\alpha}\hat{\tilde{\alpha}}\rangle}\left(\frac{\langle\hat{\alpha}\tilde{\alpha}\rangle}{\langle\tilde{\alpha}\beta\rangle}\hat{\mathsf{z}} - \frac{\langle\alpha\hat{\alpha}\rangle}{\langle\alpha\beta\rangle}\hat{\mathsf{w}}\right) \ .$$

As an example, let us take $\hat{\alpha} = \tilde{\alpha}$, in which case the reality conditions simplify to $\bar{\mathsf{w}} = \frac{\langle\hat{\alpha}\hat{\beta}\rangle}{\langle\tilde{\alpha}\beta\rangle}\hat{\mathsf{z}}$ and $\bar{\mathsf{z}} = \frac{\langle\alpha\hat{\beta}\rangle}{\langle\alpha\beta\rangle}\hat{\mathsf{w}}$. Substituting into the metric we find $\frac{2\langle\alpha\hat{\alpha}\rangle[\mu\hat{\mu}]}{\langle\alpha\beta\rangle\langle\hat{\alpha}\hat{\beta}\rangle}\left(\mathrm{d}\mathsf{w}\mathrm{d}\bar{\mathsf{w}} + \psi\bar{\psi}\mathrm{d}\mathsf{z}\mathrm{d}\bar{\mathsf{z}}\right)$ where $\psi = \frac{\langle\alpha\beta\rangle}{\langle\alpha\hat{\beta}\rangle}$. Since the prefactor is real and positive, this indeed has Euclidean signature. Note that these reality conditions are distinct from case 3 above, and they do not imply reality of the 4d IFT.

an additional two parameters. We should note that in the Euclidean case, since the two planes are related by conjugation, we will break the reality properties of the action unless the two symmetry reduction directions are also related by conjugation, reducing the number of parameters by one for a real action. This is not an issue in the Lorentzian case since the coordinates are real, hence we expect to find a four-parameter real Lorentz-invariant IFT_2. We will indeed see that this is the case in § 4.5.

Origin of reality conditions from hCS_6. Let us now briefly describe the origin of the different sets of reality conditions from 6 dimensions. It is shown in [BS23] that for the hCS_6 action to be real we require that

$$\overline{C(\Phi)} = C(\Phi) \ , \tag{4.71}$$

where Φ is defined in eq. (4.2) and C is a conjugation that acts on the coordinates (x, π), not on the fixed spinors α, $\tilde{\alpha}$ and β.[11] In Euclidean signature this constraint becomes

$$\frac{\bar{K}}{\langle \pi\hat{\alpha}\rangle\langle \pi\hat{\tilde{\alpha}}\rangle\langle \pi\hat{\beta}\rangle^2} = \frac{K}{\langle \pi\alpha\rangle\langle \pi\tilde{\alpha}\rangle\langle \pi\beta\rangle^2} \ . \tag{4.72}$$

We immediately see that this has no solutions since the double pole at β is mapped to $\hat{\beta}$ and $\hat{\beta} = \beta$ has no solutions. On the other hand, in split signature we have

$$\frac{\bar{K}}{\langle \pi\bar{\alpha}\rangle\langle \pi\bar{\tilde{\alpha}}\rangle\langle \pi\bar{\beta}\rangle^2} = \frac{K}{\langle \pi\alpha\rangle\langle \pi\tilde{\alpha}\rangle\langle \pi\beta\rangle^2} \ . \tag{4.73}$$

This can be solved by taking K and β to be real, and α and $\tilde{\alpha}$ to either both be real or to form a complex conjugate pair.

We also need to ask that the boundary conditions (4.8) are compatible with the reality conditions. Imposing $C^*(\mathcal{A}_{\dot{a}}) = \theta(\mathcal{A}_{\dot{a}})$, we can either take μ and $\hat{\mu}$ to either both be real or to form a complex conjugate pair. The two choices of reality conditions for $(\alpha, \tilde{\alpha})$ and the two for $(\mu, \hat{\mu})$ give a total of four sets of reality conditions, which we anticipate will recover those in the list presented above. With the same ordering, we have the following:

1. In the first case, we take real $(\alpha, \tilde{\alpha})$ and real $(\mu, \hat{\mu})$. Analysing the boundary conditions we find that $\mathcal{A}_{\dot{a}}$ is valued in the real form at the poles, implying that h and $\tilde{h}$ are as well, and that σ is real. Since both $\langle \alpha\tilde{\alpha}\rangle$ and $[\mu\hat{\mu}]$ are real, real K implies that K' is real using eq. (4.68).

2. In the second case, we take real $(\alpha, \tilde{\alpha})$ and complex conjugate $(\mu, \hat{\mu})$. Analysing the boundary conditions we find that $\mathcal{A}_{\dot{a}}$ is valued in the real form at the poles, implying that h and $\tilde{h}$

[11] Conjugation in Euclidean signature can be defined as $C(\mu_{\dot{a}}) = \hat{\mu}_{\dot{a}} = \epsilon_{\dot{a}}{}^{\dot{b}}\bar{\mu}_{\dot{b}}$, $C(\gamma_a) = \hat{\gamma}_a = \varepsilon_a{}^b\bar{\gamma}_b$ and $C(x^{a\dot{a}}) = (\epsilon^T)^a{}_b\bar{x}^{b\dot{b}}\varepsilon_{\dot{b}}{}^{\dot{a}}$ with ${\varepsilon_1}^2 = -1$, while in split signature, we take $C(\mu_{\dot{a}}) = \bar{\mu}_{\dot{a}}$, $C(\gamma_a) = \bar{\gamma}_a$ and $C(x^{a\dot{a}}) = \bar{x}^{a\dot{a}}$. We will restrict our attention to Euclidean and split signatures since there are no ASD connections in Lorentzian signature [BS23].

are as well, and that σ is a phase factor. Since $\langle\alpha\tilde{\alpha}\rangle$ is real and $[\mu\hat{\mu}]$ is imaginary, real K implies that K' is imaginary using eq. (4.68).

3. In the third case, we take complex conjugate $(\alpha, \tilde{\alpha})$ and complex conjugate $(\mu, \hat{\mu})$. Analysing the boundary conditions we find that $\mathcal{A}_{\dot{a}}$ at α is the conjugate of $\mathcal{A}_{\dot{a}}$ at $\tilde{\alpha}$, implying that h and $\tilde{h}$ are also conjugate, and that σ is real. Since both $\langle\alpha\tilde{\alpha}\rangle$ and $[\mu\hat{\mu}]$ are imaginary, real K implies that K' is real using eq. (4.68).

4. In the final case, we take complex conjugate $(\alpha, \tilde{\alpha})$ and real $(\mu, \hat{\mu})$. Analysing the boundary conditions we find that $\mathcal{A}_{\dot{a}}$ at α is the conjugate of $\mathcal{A}_{\dot{a}}$ at $\tilde{\alpha}$, implying that h and $\tilde{h}$ are also conjugate, and that σ is a phase factor. Since $\langle\alpha\tilde{\alpha}\rangle$ is imaginary and $[\mu\hat{\mu}]$ is real, real K implies that K' is imaginary using eq. (4.68).

Finally, one can also check that in split signature, the different reality conditions for $(\alpha, \tilde{\alpha})$ and $(\mu, \hat{\mu})$ imply the different reality conditions for the coordinates $\{\mathsf{w}, \hat{\mathsf{w}}, \mathsf{z}, \hat{\mathsf{z}}\}$ given above.

As implied above, see also [BS23], a real action in split signature in 4 dimensions is useful for symmetry reducing and constructing real 2d IFTs since both Euclidean and Lorentzian signature in 2 dimensions can be accessed. However, the lack of a real action in Euclidean signature raises questions about the quantisation of the IFT_4 itself. We will briefly return to the issue of quantisation in § 4.7.

4.3.4 Equivalent Forms of the Action and its Limits

In this section we describe alternative, but equivalent ways of writing the action of the 4d IFT (4.31) with (4.32), and consider two interesting limits of the theory. These constructions are motivated by analogous ones that are important in the context of the 2d λ-deformed WZW model.

First, let us note that the IFT_4 (4.31) with (4.32) can be written in the following two equivalent forms

$$\begin{aligned} S_{\mathrm{IFT}_4} &= K' \int_{\mathbb{R}^4} \mathrm{vol}'_4 \,\mathrm{Tr}\big((\jmath - \sigma\,\tilde{\jmath})(U_+^T - U_-)(\hat{\jmath} + \sigma^{-1}\hat{\tilde{\jmath}}) - \sigma\,\tilde{\jmath}\,\hat{\jmath} + \sigma^{-1}\jmath\,\hat{\tilde{\jmath}}\big) + S_{\mathrm{WZ}_4} \\ &= K' \int_{\mathbb{R}^4} \mathrm{vol}'_4 \,\mathrm{Tr}\big((\mathrm{Ad}_h \jmath - \mathrm{Ad}_{\tilde{h}}\,\tilde{\jmath})(\widetilde{U}_+^T - \widetilde{U}_-)(\mathrm{Ad}_h\,\hat{\jmath} - \mathrm{Ad}_{\tilde{h}}\hat{\tilde{\jmath}}) + \mathrm{Ad}_{\tilde{h}}\,\tilde{\jmath}\,\mathrm{Ad}_h\,\hat{\jmath} - \mathrm{Ad}_h \jmath\,\mathrm{Ad}_{\tilde{h}}\hat{\tilde{\jmath}}\big) + S_{\mathrm{WZ}_4} \,, \end{aligned} \tag{4.74}$$

where

$$\begin{aligned} U_\pm &= \big(1 - \sigma^{\pm 1}\Lambda\big)^{-1} \,, \qquad \Lambda = \mathrm{Ad}_{\tilde{h}}^{-1}\mathrm{Ad}_h \,, \\ \widetilde{U}_\pm &= \big(1 - \sigma^{\pm 1}\widetilde{\Lambda}\big)^{-1} \,, \qquad \widetilde{\Lambda} = \mathrm{Ad}_h \mathrm{Ad}_{\tilde{h}}^{-1} \,. \end{aligned} \tag{4.75}$$

Written in this way, it is straightforward to see that the symmetries of the 4d IFT are given by transformations of the form (4.37) with

$$(\partial_{\mathsf{w}} - \sigma\partial_{\mathsf{z}})r = (\partial_{\hat{\mathsf{w}}} - \sigma^{-1}\partial_{\hat{\mathsf{z}}})r = 0 \,, \qquad (\partial_{\mathsf{w}} - \partial_{\mathsf{z}})\ell = (\partial_{\hat{\mathsf{w}}} - \partial_{\hat{\mathsf{z}}})\ell = 0 \,, \tag{4.76}$$

which, as expected, coincide with (4.13) and (4.14) upon using the definitions (4.65).

We can also introduce auxiliary fields $B^{\dot{a}}$, $C^{\dot{a}}$ and $\widetilde{C}^{\dot{a}}$ to rewrite the action as

$$\begin{aligned}S_{\mathrm{IFT}_4} = K' \int_{\mathrm{E}_4} \mathrm{vol}'_4 \,\mathrm{Tr}\big(j\,\widehat{\jmath} - 2j\mathrm{Ad}_h^{-1}[B\hat{\mu}] + 2\,\widehat{\jmath}\,[C\mu] - 2[C\mu]\mathrm{Ad}_h^{-1}[B\hat{\mu}] \\ + \widetilde{\jmath}\widehat{\widetilde{\jmath}} - 2\widehat{\widetilde{\jmath}}\mathrm{Ad}_{\tilde{h}}^{-1}[B\mu] + 2\,\widetilde{\jmath}\,[\widetilde{C}\hat{\mu}] - 2[\widetilde{C}\hat{\mu}]\mathrm{Ad}_{\tilde{h}}^{-1}[B\mu] \\ + 2[B\mu][B\hat{\mu}] + 2\sigma^{-1}[C\mu][\widetilde{C}\hat{\mu}]\big) + S_{\mathrm{WZ}_4} \ .\end{aligned} \tag{4.77}$$

Here we take the auxiliary fields $B^{\dot{a}}$, $C^{\dot{a}}$ and $\widetilde{C}^{\dot{a}}$ to be undetermined. Varying the action and solving their equations of motion, we find that on-shell, they are given by the expressions introduced above in eqs. (4.28) and (4.48). Moreover, substituting their on-shell values back into (4.77) we recover the 4d IFT. Using the symmetry (4.9), we note that the action can also be written in a similar equivalent form, in which tilded and untilded quantities are swapped, $\sigma \to \sigma^{-1}$, $K' \to -K'$ and $\widetilde{B} = B$. This can also be seen by making the off-shell replacements $[B\mu] \to [B\mu]$, $[B\hat{\mu}] \to \mathrm{Ad}_h([C\hat{\mu}] + \widehat{\jmath})$, $[C\mu] \to \sigma[\widetilde{C}\mu]$ and $[\widetilde{C}\hat{\mu}] \to \mathrm{Ad}_{\tilde{h}}^{-1}[B\hat{\mu}] - \widehat{\widetilde{\jmath}}$, all of which are compatible with the on-shell values of the auxiliary fields.

The first limit we consider is $\sigma \to 0$, in which the action becomes

$$S_{\mathrm{IFT}_4}|_{\sigma\to 0} = \mathring{S}_{\mathrm{IFT}_4} = K' \int_{\mathbb{R}^4} \mathrm{vol}'_4 \,\mathrm{Tr}\big(j\,\widehat{\jmath} + \widetilde{\jmath}\widehat{\widetilde{\jmath}} - 2\mathrm{Ad}_h j\,\mathrm{Ad}_{\tilde{h}}\widehat{\widetilde{\jmath}}\big) + S_{\mathrm{WZ}_4} \ . \tag{4.78}$$

This has the form of a current-current coupling between two building blocks that could be described as 'holomorphic WZW$_4$' of the form

$$S_{\mathrm{hWZW4}}[h,\alpha] = \int_{\mathbb{R}^4} \mathrm{vol}'_4 \,\mathrm{Tr}\big(j\,\widehat{\jmath}\big) - \int_{\mathbb{R}^4\times[0,1]} \mathrm{vol}'_4 \wedge d\rho\,\mathrm{Tr}\big(h^{-1}\partial_\rho h[j,\widehat{\jmath}]\big) \ . \tag{4.79}$$

This somewhat unusual theory has derivatives only in the holomorphic two-plane singled out by the complex structure on $\mathbb{R}^4$ defined by α (i.e. only ∂_{w} and $\partial_{\hat{\mathsf{w}}}$ enter), although the field depends on all coordinates of $\mathbb{R}^4$. This structure is quite different (both in the kinetic term and Wess-Zumino term) from the conventional WZW$_4$ [Don85] for which the action[12] is

$$\begin{aligned}S_{\mathrm{WZW}_4}[h,\alpha,\beta] &= \int_{\mathbb{R}^4} \mathrm{Tr}\big(h^{-1}dh \wedge \star h^{-1}dh\big) + \frac{1}{6}\int_{\mathbb{R}^4\times[0,1]} \varpi_{\alpha,\beta} \wedge \mathrm{Tr}\big((h^{-1}dh)^3\big) \ , \\ \varpi_{\alpha,\beta} &= \epsilon_{\dot{a}\dot{b}}\alpha_a\beta_b \mathrm{d}x^{a\dot{a}} \wedge \mathrm{d}x^{b\dot{b}} \ .\end{aligned} \tag{4.80}$$

The Kähler point of the theory is achieved when $\beta = \hat{\alpha}$ such that $\varpi_{\alpha,\beta}$ is the Kähler form associated to the complex structure defined by α. In fact, the WZ term of our holomorphic WZW$_4$ is of this general form with $\beta = \alpha$ such that $\varpi_{\alpha,\beta}$ defines a $(2,0)$-form. However even in the $\beta = \alpha$ case, the kinetic term does not match that of the holomorphic WZW$_4$ action.

[12]This is also the 4d IFT that was found in [BS23; Pen21] from hCS$_6$ with two double poles at $\pi = \alpha$ and $\pi = \beta$, with Dirichlet boundary conditions.

Returning to the holomorphic WZW_4, we can establish that the theory is invariant under a rather large set of symmetries. Since only w and $\hat{\mathsf{w}}$ derivatives enter, it is immediate that the transformation $h \mapsto l(\hat{\mathsf{z}}, \mathsf{z}) h r(\hat{\mathsf{z}}, \mathsf{z})$ leaves the action eq. (4.79) invariant. These are further enhanced, as in a WZW_2, to

$$(\ell, r): \quad h \mapsto \ell(\mathsf{z}, \hat{\mathsf{z}}, \mathsf{w})\, h\, r(\mathsf{z}, \hat{\mathsf{z}}, \hat{\mathsf{w}}) \ . \tag{4.81}$$

From this perspective holomorphic WZW_4 can be considered the embedding of a WZW_2 in 4 dimensions. Similarly, we have a symmetry for the holomorphic WZW_4 for $\tilde{h}$

$$(\tilde{\ell}, \tilde{r}): \quad \tilde{h} \mapsto \tilde{\ell}(\hat{\mathsf{z}}, \hat{\mathsf{w}}, \mathsf{w})\, \tilde{h}\, \tilde{r}(\mathsf{z}, \hat{\mathsf{w}}, \mathsf{w}) \ . \tag{4.82}$$

The interaction term, $\mathrm{Ad}_h j\, \mathrm{Ad}_{\tilde{h}} \widehat{\tilde{\jmath}}$, in the action (4.78) preserves the right actions, but breaks the enhanced independent $\ell, \tilde{\ell}$ left actions. Instead a new 'diagonal' left action emerges

$$(\ell, r, \tilde{r}): \quad h \mapsto \ell(\mathsf{z} + \mathsf{w}, \hat{\mathsf{z}} + \hat{\mathsf{w}})\, h\, r(\mathsf{z}, \hat{\mathsf{z}}, \hat{\mathsf{w}}) \ , \quad \tilde{h} \mapsto \ell(\mathsf{z} + \mathsf{w}, \hat{\mathsf{z}} + \hat{\mathsf{w}})\, \tilde{h}\, \tilde{r}(\mathsf{z}, \mathsf{w}, \hat{\mathsf{w}}) \ . \tag{4.83}$$

It is important to emphasise that here the right actions on h and $\tilde{h}$ are independent (r and $\tilde{r}$ are not the same). This stems from the enlargement of the residual symmetries of the 6-dimensional boundary conditions. The constraints of eq. (4.14) are relaxed such that gauge parameters at different poles are unrelated but are chiral.

In this limit the currents associated to the left and right action become

$$\begin{aligned} B_{\dot{a}}|_{\sigma \to 0} &= \mathring{B}_{\dot{a}} = \mathrm{Ad}_{\tilde{h}} \widehat{\tilde{\jmath}} \mu_{\dot{a}} - \mathrm{Ad}_h j \hat{\mu}_{\dot{a}} \ , \\ C_{\dot{a}}|_{\sigma \to 0} &= \mathring{C}_{\dot{a}} = -\big(\hat{\jmath} - \Lambda^{-1} \widehat{\tilde{\jmath}}\big) \mu_{\dot{a}} \ , \\ \widetilde{C}_{\dot{a}}|_{\sigma \to 0} &= \mathring{\widetilde{C}}_{\dot{a}} = \big(\tilde{\jmath} - \Lambda j\big) \hat{\mu}_{\dot{a}} \ . \end{aligned} \tag{4.84}$$

The conservation laws become

$$\begin{gathered} \partial_{\mathsf{w}}\big(\hat{\jmath} - \Lambda^{-1} \widehat{\tilde{\jmath}}\big) = 0 \ , \qquad \partial_{\hat{\mathsf{z}}}\big(\tilde{\jmath} - \Lambda j\big) = 0 \ , \\ \partial_{\hat{\mathsf{w}}}(\mathrm{Ad}_h j) - \partial_{\mathsf{w}}(\mathrm{Ad}_{\tilde{h}} \widehat{\tilde{\jmath}}) + \partial_{\mathsf{z}}(\mathrm{Ad}_{\tilde{h}} \widehat{\tilde{\jmath}}) - \partial_{\hat{\mathsf{z}}}(\mathrm{Ad}_h j) = 0 \ . \end{gathered} \tag{4.85}$$

To compute the $\mathcal{O}(\sigma)$ correction to the action (4.78) we first note that

$$B_{\dot{a}} = \mathring{B}_{\dot{a}} + \sigma \left(\mathrm{Ad}_h \mathring{\widetilde{C}}_{\dot{a}} + \mathrm{Ad}_{\tilde{h}} \mathring{C}_{\dot{a}} \right) + \mathcal{O}(\sigma^2) \ , \tag{4.86}$$

and that the combination $C_{\dot{a}} - \Lambda^T \widetilde{C}_{\dot{a}} = \mathring{C}_{\dot{a}} - \Lambda^T \mathring{\widetilde{C}}_{\dot{a}}$ is independent of σ. Then from the expression of the IFT_4 action in terms of currents (4.51), we see that the leading correction to $\mathring{S}_{\mathrm{IFT}_4}$ is given by

$$2\sigma K' \int_{\mathbb{R}^4} \mathrm{vol}_4' \, \epsilon^{\dot a \dot b} \mathrm{Tr}\big(\mathring{\widetilde{C}}_{\dot a} \mathring{C}_{\dot a}\big) = -2\sigma K' \int_{\mathbb{R}^4} \mathrm{vol}_4 \, \mathrm{Tr}\big((\widetilde{\jmath} - \Lambda j)(\widehat{\jmath} - \Lambda^{-1}\widehat{\widetilde{\jmath}})\big) \ , \tag{4.87}$$

i.e. the perturbing operator is the product of two currents associated to the right-acting symmetries.

The second limit we consider is $\sigma \to 1$. Recall that in this limit, we have that $\widetilde{C} = C$ from eqs. (4.47) and (4.48), and a symmetry emerges interchanging B and C, as well as h and $\tilde{h}^{-1}$. This is also evident if we set $\sigma = 1$ in (4.77). An alternative way to take $\sigma \to 1$ is to first set $h = \exp(\frac{\nu}{K'})$ and $\tilde{h} = \exp(\frac{\tilde{\nu}}{K'})$, along with $\sigma = e^{\frac{1}{K'}}$ and take $K' \to \infty$. In this limit the 4d IFT becomes

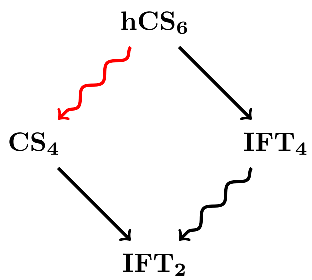
$$S_{\mathrm{IFT}_4}|_{K' \to \infty} = -\int_{\mathbb{R}^4} \mathrm{vol}_4' \, \mathrm{Tr}\big((\partial_{\mathsf{w}}\nu - \partial_{\mathsf{z}}\tilde{\nu}) \frac{1}{1 - \mathrm{ad}_\nu + \mathrm{ad}_{\tilde{\nu}}} (\partial_{\hat{\mathsf{w}}}\nu - \partial_{\hat{\mathsf{z}}}\tilde{\nu})\big) \ , \tag{4.88}$$

which is reminiscent of a 4d version of the non-abelian T-dual of the principal chiral model, albeit with two fields instead of one. If instead we take the limit in the action with auxiliary fields (4.77), also setting $[C\mu] = [B\mu] + \mathcal{O}(K'^{-1})$ and $[\widetilde{C}\hat{\mu}] = [B\hat{\mu}] + \mathcal{O}(K'^{-1})$, we find

$$\begin{aligned} S_{\mathrm{IFT}_4}|_{K' \to \infty} = \int_{\mathrm{E}_4} \mathrm{vol}_4' \, \mathrm{Tr}\Big(& 2\nu\big(\partial_{\mathsf{w}}[B\hat{\mu}] - \partial_{\hat{\mathsf{w}}}[B\mu] + [[B\hat{\mu}],[B\mu]]\big) \\ & + 2\tilde{\nu}\big(\partial_{\hat{\mathsf{z}}}[B\mu] - \partial_{\mathsf{z}}[B\hat{\mu}] + [[B\mu],[B\hat{\mu}]]\big) - 2[B\mu][B\hat{\mu}]\Big) \ , \end{aligned} \tag{4.89}$$

after integrating by parts. Integrating out the auxiliary field B^A, we recover the action (4.88). It would be interesting to instead integrate out the fields ν and $\tilde{\nu}$ to give a 4d analogue of 2d non-abelian T-duality. However, note that, unlike in 2 dimensions, ν and $\tilde{\nu}$ do not enforce the flatness of a 4d connection, hence there is no straightforward way to parametrise the general solution to their equations.

4.4 Symmetry Reduction of hCS_6 to CS_4



Returning to hCS_6, we now descend via the top left-hand side of the diamond by performing a symmetry reduction of the action. Doing so, we find the resulting theory is CS_4.

The idea of symmetry reduction is to take a truncation of a d-dimensional theory specified by a d-form Lagrangian $\mathcal{L}^d$ depending on a set of fields $\{\Phi\}$ to obtain a lower dimensional theory. We assume here that we are reducing along two directions singled out by vector fields V_i, $i = 1, 2$. The reduction procedure imposes that all fields are invariant, $L_{V_i}\Phi = 0$, with dynamics now specified by the $(d-2)$-form Lagrangian $\mathcal{L}^{d-2} = V_1 \lrcorner V_2 \lrcorner \mathcal{L}^d$. While similar in spirit to a dimensional reduction, there is no requirement that V_i span a compact space, hence there is no scale separation in this truncation.

In order to perform the symmetry reduction, we will introduce a unit norm spinor γ_a about which we can expand any spinor X_a as

$$X_a = \langle X\hat{\gamma}\rangle\gamma_a - \langle X\gamma\rangle\hat{\gamma}_a \ . \tag{4.90}$$

The spinor γ_a defines another complex structure on $\mathbb{R}^4$ which coincides with the complex structure on $\mathbb{R}^4 \subset \mathbb{PT}$ at the point $\pi_a = \gamma_a$. It coincides with the opposite complex structure – swapping holomorphic and anti-holomorphic – at the antipodal point $\pi_a = \hat{\gamma}_a$. Using the spinor $\mu^{\dot{a}}$, we can define a basis of one-forms adapted to this complex structure,

$$\begin{aligned} \mathrm{d}z &= \mu_{\dot{a}}\gamma_a \mathrm{d}x^{a\dot{a}} \ , \qquad & \mathrm{d}\bar{z} &= \hat{\mu}_{\dot{a}}\hat{\gamma}_a \mathrm{d}x^{a\dot{a}} \ , \\ \mathrm{d}w &= \hat{\mu}_{\dot{a}}\gamma_a \mathrm{d}x^{a\dot{a}} \ , \qquad & \mathrm{d}\bar{w} &= -\mu_{\dot{a}}\hat{\gamma}_a \mathrm{d}x^{a\dot{a}} \ . \end{aligned} \tag{4.91}$$

We will perform symmetry reduction along the vector fields dual to $\mathrm{d}z$ and $\mathrm{d}\bar{z}$,

$$\partial_z = \hat{\mu}^{\dot{a}}\hat{\gamma}^a\partial_{a\dot{a}} \ , \quad \partial_{\bar{z}} = \mu^{\dot{a}}\gamma^a\partial_{a\dot{a}} \ . \tag{4.92}$$

The symmetry reduction along the ∂_z and $\partial_{\bar{z}}$ directions takes us from a theory on $\mathbb{PT}$ to a theory on $\Sigma \times \mathbb{CP}^1$ in which $w, \bar{w}$ are coordinates on the worldsheet Σ.

To perform this reduction, it is expedient [Bit22] to make use of the invariance of the action (4.1) under the addition of any $(1,0)$-form to $\mathcal{A} \mapsto \hat{\mathcal{A}} = \mathcal{A} + \rho_0 e^0 + \rho_{\dot{a}} e^{\dot{a}}$. By choosing $\rho_{\dot{a}}$ appropriately, we can ensure that $\hat{\mathcal{A}}$ has no $\mathrm{d}z$ or $\mathrm{d}\bar{z}$ legs and is given by

$$\hat{\mathcal{A}} = \hat{\mathcal{A}}_w \mathrm{d}w + \hat{\mathcal{A}}_{\bar{w}} \mathrm{d}\bar{w} + \mathcal{A}_0 \bar{e}^0 \ , \tag{4.93}$$

where these components are related to those of $\mathcal{A}$ by

$$\hat{\mathcal{A}}_w = -\frac{[\mathcal{A}\mu]}{\langle\pi\gamma\rangle} \ , \qquad \hat{\mathcal{A}}_{\bar{w}} = -\frac{[\mathcal{A}\hat{\mu}]}{\langle\pi\hat{\gamma}\rangle} \ . \tag{4.94}$$

An important feature to note is that $\hat{\mathcal{A}}$ necessarily has singularities at γ and $\hat{\gamma}$. While at the 6-dimensional level this is a mere gauge-choice artefact, it plays a crucial role in the construction of the CS_4 theory.

In these variables, the boundary variation and boundary condition of hCS_6 are restated as

$$r_+ \mathrm{Tr}\big(\hat{\mathcal{A}}_w \delta\hat{\mathcal{A}}_{\bar{w}} - \hat{\mathcal{A}}_{\bar{w}}\delta\hat{\mathcal{A}}_w\big)\big|_{\pi=\alpha} = -r_- \mathrm{Tr}\big(\hat{\mathcal{A}}_w \delta\hat{\mathcal{A}}_{\bar{w}} - \hat{\mathcal{A}}_{\bar{w}}\delta\hat{\mathcal{A}}_w\big)\big|_{\pi=\tilde{\alpha}} \ , \tag{4.95}$$

$$\hat{\mathcal{A}}_w|_{\pi=\alpha} = ts\hat{\mathcal{A}}_w|_{\pi=\tilde{\alpha}} \ , \qquad \hat{\mathcal{A}}_{\bar{w}}|_{\pi=\alpha} = t^{-1}s\hat{\mathcal{A}}_{\bar{w}}|_{\pi=\tilde{\alpha}} \ , \tag{4.96}$$

where we have introduced the combinations

$$
\begin{aligned}
r_+ &= K\frac{\langle\alpha\gamma\rangle\langle\alpha\hat{\gamma}\rangle}{\langle\alpha\tilde{\alpha}\rangle\langle\alpha\beta\rangle^2} \ , & r_- &= -K\frac{\langle\tilde{\alpha}\gamma\rangle\langle\tilde{\alpha}\hat{\gamma}\rangle}{\langle\alpha\tilde{\alpha}\rangle\langle\tilde{\alpha}\beta\rangle^2} \ , \\
s &= \sqrt{-\frac{r_-}{r_+}} = \frac{\langle\alpha\beta\rangle}{\langle\tilde{\alpha}\beta\rangle}\sqrt{\frac{\langle\tilde{\alpha}\gamma\rangle\langle\tilde{\alpha}\hat{\gamma}\rangle}{\langle\alpha\gamma\rangle\langle\alpha\hat{\gamma}\rangle}} \ , & t &= \sigma s\frac{\langle\tilde{\alpha}\beta\rangle\langle\alpha\hat{\gamma}\rangle}{\langle\alpha\beta\rangle\langle\tilde{\alpha}\hat{\gamma}\rangle} \ .
\end{aligned}
\tag{4.97}
$$

Upon symmetry reduction to CS_4, $r_\pm$ will correspond to the residues of the 1-form ω.

Since the shifted gauge field $\hat{\mathcal{A}}$ manifestly has no $\mathrm{d}z$ or $\mathrm{d}\bar{z}$ legs, and we impose $L_z\hat{\mathcal{A}} = L_{\bar{z}}\hat{\mathcal{A}} = 0$, the contraction by ∂_z and $\partial_{\bar{z}}$ only hits Ω. It then follows that the symmetry reduction yields

$$
S_{\mathrm{CS}_4} = \frac{1}{2\pi \mathrm{i}}\int_{\Sigma\times\mathbb{CP}^1} \omega\wedge \mathrm{Tr}\Big(\hat{\mathcal{A}}\wedge \mathrm{d}\hat{\mathcal{A}} + \frac{2}{3}\hat{\mathcal{A}}\wedge\hat{\mathcal{A}}\wedge\hat{\mathcal{A}}\Big) \ , \tag{4.98}
$$

in which the meromorphic 1-form on $\mathbb{CP}^1$ is given by

$$
\omega = \partial_{\bar{z}} \lrcorner\, \partial_z \lrcorner\, \Omega = \Phi\, \varepsilon_{\dot{a}\dot{b}}\big(\partial_{\bar{z}} \lrcorner\, e^{\dot{a}}\big)\big(\partial_z \lrcorner\, e^{\dot{b}}\big)\, e^0 = -K\, \frac{\langle\pi\gamma\rangle\langle\pi\hat{\gamma}\rangle}{\langle\pi\alpha\rangle\langle\pi\tilde{\alpha}\rangle\langle\pi\beta\rangle^2}\, e^0 \ . \tag{4.99}
$$

To compare with the literature, we will also translate to inhomogeneous coordinates on $\mathbb{CP}^1$. The $\mathbb{CP}^1$ coordinate itself will be given by $\zeta = \pi_2/\pi_1$ on the patch $\pi_1 \neq 0$. We also specify representatives for the various other spinors in our theory. Without loss of generality we can choose

$$
\alpha_a = (1, \alpha_+) \ , \qquad \tilde{\alpha}_a = (1, \alpha_-) \ , \qquad \beta_a = (0, 1) \ , \tag{4.100}
$$

such that

$$
\langle\tilde{\alpha}\beta\rangle = \langle\alpha\beta\rangle = 1 \ , \qquad \langle\tilde{\alpha}\alpha\rangle = \alpha_+ - \alpha_- = \Delta\alpha \ . \tag{4.101}
$$

We also denote the inhomogeneous coordinates for γ_a and $\hat{\gamma}_a$ by

$$
\gamma_+ = \frac{\gamma_2}{\gamma_1} \ , \qquad \gamma_- = \frac{\hat{\gamma}_2}{\hat{\gamma}_1} = -\frac{\overline{\gamma_1}}{\overline{\gamma_2}} \ , \qquad \gamma_1\overline{\gamma_2} = \frac{1}{\gamma_+ - \gamma_-} = \frac{1}{\Delta\gamma} \ . \tag{4.102}
$$

Then, the meromorphic 1-form ω is written in inhomogeneous coordinates as

$$
\omega = \frac{K}{\Delta\gamma}\frac{(\zeta-\gamma_+)(\zeta-\gamma_-)}{(\zeta-\alpha_+)(\zeta-\alpha_-)}\mathrm{d}\zeta = \varphi(\zeta)\,\mathrm{d}\zeta \ . \tag{4.103}
$$

To complete the specification of the theory we simply note that the 6d boundary conditions immediately descend to

$$
\hat{\mathcal{A}}_w\, |_{\pi=\alpha} = ts\hat{\mathcal{A}}_w\, |_{\pi=\tilde{\alpha}} \ , \qquad \hat{\mathcal{A}}_{\bar{w}}\, |_{\pi=\alpha} = t^{-1}s\hat{\mathcal{A}}_{\bar{w}}\, |_{\pi=\tilde{\alpha}} \ . \tag{4.104}
$$

Before we discuss the residual symmetries of the CS_4 models, let us make two related observations. First, fixing the shift symmetry to ensure $\hat{\mathcal{A}}$ is horizontal with respect to the symmetry reduction introduces poles into our gauge field $\hat{\mathcal{A}}$ at γ and $\hat{\gamma}$. Thus, despite starting with potentially smooth field configurations in 6 dimensions we are forced to consider singular ones in 4 dimensions. We can understand the origin of these singularities by recalling the holomorphic coordinates on $\mathbb{R}^4$ with respect to the complex structure on $\mathbb{PT} = \mathbb{CP}^3 \backslash \mathbb{CP}^1$. Described in detail in section 2.2, $\mathbb{PT}$ is only diffeomorphic to $\mathbb{R}^4 \times \mathbb{CP}^1$, and the complex structure is more involved in these coordinates. The holomorphic coordinates on $\mathbb{R}^4$ with respect to this complex structure are given by $v^{\dot{a}} = \pi_a x^{a\dot{a}}$, which align with our coordinates $\{z, w\}$ at $\pi \sim \gamma$ and $\{\bar{z}, \bar{w}\}$ at $\pi \sim \hat{\gamma}$. It is precisely at these points that we are forced to introduce poles by the symmetry reduction procedure.

Second, in line with the singular behaviour in the gauge field, we have also introduced zeroes in ω at $\pi \sim \gamma$ and $\pi \sim \hat{\gamma}$ whereas Ω in 6 dimensions was nowhere vanishing. Of course, given the pole structure of Ω, the introduction of two zeroes was inevitable given the Riemann-Roch theorem.

4.4.1 Residual Symmetries and the Defect Algebra

Let us take a moment to consider the residual symmetries of these CS_4 models. Here the residual symmetry preserving the boundary condition (4.104) is generically constrained to only include constant modes,

$$r = \tilde{r} \ , \qquad (1 - ts)\partial_w r = 0 \ , \qquad (1 - ts^{-1})\partial_{\bar{w}} r = 0 \ . \tag{4.105}$$

At the special 'diagonal' point in parameter space where $t = s = 1$, notice these differential equations are identically solved and we find a local gauge symmetry. This enhancement of residual gauge freedom matches with previous considerations in the context of CS_4, where diagonal boundary conditions of the form $A|_\alpha = A|_{\tilde{\alpha}}$ are known to give rise to the λ-deformed WZW as an IFT_2, a theory that depends on a single field h. The residual gauge symmetries are those satisfying $\hat{g}|_\alpha = \hat{g}|_{\tilde{\alpha}}$ and can be used to reduce the number of fields appearing in the resulting IFT_2 to one (see § 5.4 in [Del+20]).

Another interesting point occurs when we take $t = s$ or $t = s^{-1}$ in which case we retain a chiral residual symmetry. When $t = 0$ the boundary conditions admit an enlarged residual symmetry as there is no requirement that $r = \hat{g}|_\alpha$ and $\tilde{r} = \hat{g}|_{\tilde{\alpha}}$ match. Instead they must be chiral and of opposite chiralities i.e. $\partial_w r = \partial_{\bar{w}} \tilde{r} = 0$. As mentioned earlier, for more generic values of t and s the residual symmetries will be constrained, preventing them from being used to eliminate any degrees of freedom. While these boundary conditions have not been yet considered for $t, s \neq 1$ and $t \neq 0$, we will see that they give rise to the multi-parametric class of λ-deformations between coupled WZW models introduced in [GS17].

To make further contact with the literature, it is helpful to rephrase the boundary conditions (4.104) in terms of a defect algebra, which in the case at hand is simply the Lie algebra $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}$

equipped with an ad-invariant pairing

$$\langle\!\langle \mathbb{X}, \mathbb{Y} \rangle\!\rangle = r_+ \mathrm{Tr}(x_1\, y_1) + r_- \mathrm{Tr}(x_2\, y_2) \ , \qquad \mathbb{X} = (x_1, x_2) \ , \mathbb{Y} = (y_1, y_2) \ . \tag{4.106}$$

We map our boundary conditions into this algebra by defining $\mathbb{A}_w = (\hat{\mathcal{A}}_w|_{\pi=\alpha}, \hat{\mathcal{A}}_w|_{\pi=\tilde{\alpha}})$ and $\mathbb{A}_{\bar{w}} = (\hat{\mathcal{A}}_{\bar{w}}|_{\pi=\alpha}, \hat{\mathcal{A}}_{\bar{w}}|_{\pi=\tilde{\alpha}})$ such that the requirement that the boundary variation vanishes locally can be recast as

$$0 = \langle\!\langle \mathbb{A}_w, \delta\mathbb{A}_{\bar{w}} \rangle\!\rangle - \langle\!\langle \mathbb{A}_{\bar{w}}, \delta\mathbb{A}_w \rangle\!\rangle \ . \tag{4.107}$$

The boundary conditions (4.104) read

$$\begin{aligned} \mathbb{A}_w &\in \mathfrak{l}_t = \mathrm{span}\{(tsx, x) \mid x \in \mathfrak{g}\} \ , \\ \mathbb{A}_{\bar{w}} &\in \mathfrak{l}_{t^{-1}} = \mathrm{span}\{\big(t^{-1}sx, x\big) \mid x \in \mathfrak{g}\} \ . \end{aligned} \tag{4.108}$$

Since $\langle\!\langle \mathfrak{l}_t, \mathfrak{l}_{t^{-1}} \rangle\!\rangle = 0$ the boundary conditions are suitable, however it should be noted that $\mathfrak{l}_t$ is itself neither a subalgebra nor an isotropic subspace of $\mathfrak{d}$. This is more general than boundary conditions previously considered[13] in the context of 4-dimensional Chern Simons theory. In particular, we might expect that generalising [BSV22; LV21; LV23] to boundary conditions defined by subspaces that are neither a subalgebra nor an isotropic subspace of $\mathfrak{d}$ will lead to novel families of 2-dimensional integrable field theories.

It is worth highlighting that these boundary conditions still define maximal isotropic subspaces, but now inside the space of algebra-valued 1-forms, rather than just the defect algebra. Consider the space of $\mathfrak{g}$-valued 1-forms on $\Sigma \times \mathbb{CP}^1$, equipped with the symplectic structure[14]

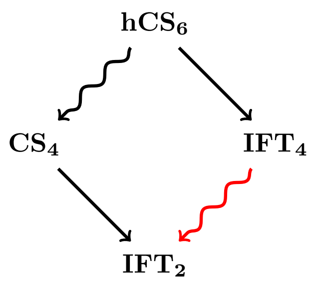
$$\mathcal{W}(X, Y) = \int_{\Sigma\times\mathbb{CP}^1} \mathrm{d}\omega \wedge \mathrm{Tr}\big(X \wedge Y\big) \ , \quad X, Y \in \Omega^1(\Sigma \times \mathbb{CP}^1) \otimes \mathfrak{g} \ . \tag{4.109}$$

The boundary conditions above define maximal isotropic subspaces with respect to this symplectic structure, that is half-dimensional subspaces $\mathcal{Y} \subset \Omega^1(\Sigma \times \mathbb{CP}^1) \otimes \mathfrak{g}$ such that $\mathcal{W}(X, Y) = 0$ for all $X, Y \in \mathcal{Y}$. Indeed, this is required for them to be 'good' boundary conditions. The isotropic subspaces of the defect algebra described earlier are then special cases of these subspaces.

[13] Of course in the limit $t, s \to 1$ $\mathfrak{l}_t$ revert to defining the diagonal isotropic subalgebra. In the special case where $t \to 0, \infty$ we recover chiral Dirichlet boundary conditions considered in [CY19; ASY23].

[14] As defined, this is not quite a symplectic structure since it is degenerate – for example, it vanishes on the subspace of 1-forms which only have legs along $\mathbb{CP}^1$. A more careful treatment would involve restricting the symplectic form to a subspace where it is non-degenerate, but we will neglect this for the purpose of our brief discussion.

4.5 Symmetry Reduction of IFT_4 to IFT_2



In this section we will apply the same symmetry reduction previously applied to hCS_6 to the IFT_4. In doing so we derive the IFT_2 corresponding to the CS_4 model described above.

Recalling that the reduction requires that the fields h and $\tilde{h}$ depend only on $w, \bar{w}$ and not on $z, \bar{z}$, we can simply set $\partial_z = \partial_{\bar{z}} = 0$ in the action eq. (4.31). To compare with the literature, when discussing 2-dimensional theories we will define $\partial_+ \equiv \partial_w$ and $\partial_- \equiv \partial_{\bar{w}}$ (implicitly rotating to 2d Minkowski space where the action is rendered real for real parameters) and denote

$$J_\pm = h^{-1}\partial_\pm h \ , \qquad \widetilde{J}_\pm = \tilde{h}^{-1}\partial_\pm \tilde{h} \ . \tag{4.110}$$

To evaluate the symmetry reduction, denoted by $\rightsquigarrow$, of the IFT_4 action we first note that

$$j \rightsquigarrow \frac{\langle\alpha\gamma\rangle}{\langle\alpha\beta\rangle} J_+ \ , \quad \widehat{\jmath} \rightsquigarrow \frac{\langle\alpha\hat{\gamma}\rangle}{\langle\alpha\beta\rangle} J_- \ , \quad \widetilde{\jmath} \rightsquigarrow \frac{\langle\tilde{\alpha}\gamma\rangle}{\langle\tilde{\alpha}\beta\rangle} \widetilde{J}_+ \ , \quad \widehat{\widetilde{\jmath}} \rightsquigarrow \frac{\langle\tilde{\alpha}\hat{\gamma}\rangle}{\langle\tilde{\alpha}\beta\rangle} \widetilde{J}_- \ . \tag{4.111}$$

The resulting 2-dimensional action is given by

$$\begin{aligned} S_{\mathrm{IFT}_4} \rightsquigarrow S_{\mathrm{IFT}_2} \ = \int_\Sigma \mathrm{vol}_2 \, \mathrm{Tr}\big(& r_+ \, J_+ (U_+^T - U_-) J_- - r_- \, \widetilde{J}_+ (U_+^T - U_-) \widetilde{J}_- + r_+ \, \mathcal{L}_{\mathrm{WZ}}(h) + r_- \, \mathcal{L}_{\mathrm{WZ}}(\tilde{h}) \\ & - 2\, t \sqrt{-r_+ \cdot r_-} \; \widetilde{J}_+ U_+^T J_- + 2\, t^{-1} \sqrt{-r_+ \cdot r_-} \; J_+ U_- \widetilde{J}_- \big) \ , \end{aligned} \tag{4.112}$$

where $\mathrm{vol}_2 = \mathrm{d}\bar{w} \wedge \mathrm{d}w = \mathrm{d}\sigma^- \wedge \mathrm{d}\sigma^+$. This theory, depending on two G-valued fields, h and $\tilde{h}$, and four independent parameters, $r_\pm$, t and σ, exactly matches a theory introduced in [GS17] as a multi-field generalisation of the λ-deformed WZW model [Sfe14]. To make a precise match with [GS17] we relate their fields (g_1, g_2) to our fields $(h, \tilde{h}^{-1})$. The model in [GS17] is defined by two WZW levels $k_{1,2}$ and by two deformation matrices which we take to be proportional to the identity with constants of proportionality $\lambda_{1,2}$. The mapping of parameters is then

$$\lambda_1 = \sigma t^{-1} \ , \qquad \lambda_2 = t \ , \qquad k_1 = r_+ \ , \qquad k_2 = -r_- \ , \qquad \lambda_0 = \sqrt{k_1/k_2} = \sqrt{-r_-/r_+} = s^{-1} \ . \tag{4.113}$$

In 2d Minkowski space, the Lagrangian (4.112) is real if the parameters $r_\pm$, s t and σ are all real. Assuming K and σ are real, this is the case if r_+ and r_- have the opposite sign and the parameters $\alpha_\pm$ and $\gamma_\pm$ lie on the same line in $\mathbb{C}$, which we can take to be the real line without loss of generality. This follows since $r_\pm$, s and t are all expressed as ratios of differences of $\alpha_\pm$ and $\gamma_\pm$, hence are invariant under translations and scalings.[15]

[15] Note that this is the subgroup of $SL(2,\mathbb{C})$ transformations that preserves the choice $\beta_a = (0,1)$.

4.5.1 Limits

The four-parameter model has a number of interesting limits, many of which are discussed in [GS17]. Here, we briefly summarise some key ones. First, let us take $t \to 0$. In order to have a well-defined limit we keep $\sigma t^{-1} = \lambda$ finite, implying $\sigma \to 0$ as well.[16] The resulting 2d Lagrangian is given by

$$S_{\mathrm{IFT}_2}|_{t,\sigma\to 0} = \int_\Sigma \mathrm{vol}_2 \, \mathrm{Tr}\big(r_+ \, J_+ J_- - r_- \, \widetilde{J}_+ \widetilde{J}_- - 2\,\lambda s r_+ J_+ \Lambda^{-1} \widetilde{J}_- \\ + r_+ \, \mathcal{L}_{\mathrm{WZ}}(h) + r_- \, \mathcal{L}_{\mathrm{WZ}}(\tilde{h}) \big) \ . \tag{4.114}$$

This current-current deformation preserves half the chiral symmetry of the $G_{r_+} \times G_{-r_-}$ WZW model, which corresponds to the UV fixed point $\lambda = 0$. Indeed, this model can be found by taking chiral Dirichlet boundary conditions in 4d CS [CY19; ASY23], corresponding to the special case $t = 0$ in the boundary conditions we find from symmetry reduction (4.108). Assuming $-r_- > r_+$, in the IR we have that $\lambda = s^{-1}$. At this point the Lagrangian can be written as

$$S_{\mathrm{IFT}_2}|_{t,\sigma\to 0, t\sigma^{-1}=s} = \int_\Sigma \mathrm{vol}_2 \, \mathrm{Tr}\big(r_+(\mathrm{Ad}_h J_+ - \mathrm{Ad}_{\tilde{h}} \widetilde{J}_+)(\mathrm{Ad}_h J_- - \mathrm{Ad}_{\tilde{h}} \widetilde{J}_-) \\ - (r_- + r_+) \, \widetilde{J}_+ \widetilde{J}_- + r_+ \, \mathcal{L}_{\mathrm{WZ}}(\tilde{h}^{-1} h) + (r_- + r_+) \, \mathcal{L}_{\mathrm{WZ}}(\tilde{h}) \big) \ . \tag{4.115}$$

Redefining $h \to \tilde{h} h$, we find the $G_{r_+} \times G_{-r_- - r_+}$ WZW model. In the case of equal levels $r_- = -r_+$ this reduces to the G_{r_+} WZW model.

The equal-level, $r_- = -r_+$, version of (4.114), whose classical integrability was first shown in [BBS97], is canonically equivalent [GSS17] and related by a path integral transformation [HLT19a] to the λ-deformed WZW model. Indeed, from the point of view of 4d CS, these two models have the same twist function. To recover (4.114) with equal levels, we take chiral Dirichlet boundary conditions, $t = 0$, $s = 1$ in (4.108), while to recover the λ-deformed WZW model we take diagonal boundary conditions $t = s = 1$.

It follows that if we take $t = s = 1$ in eq. (4.112), we expect to recover the λ-deformed WZW model. Indeed, setting $r_- = -r_+$ and $t = 1$, the Lagrangian (4.112) becomes

$$S_{\mathrm{IFT}_2}|_{t=s=1} = \int_\Sigma \mathrm{vol}_2 \, \mathrm{Tr}\big(r_+ \, (J_+ - \widetilde{J}_+)(U_+^T - U_-)(J_- - \widetilde{J}_-) + r_+ \, \mathcal{L}_{\mathrm{WZ}}(h \tilde{h}^{-1})\big) \ . \tag{4.116}$$

As explained in subsection 4.2.1, at this point in parameter space the symmetry reduction directions are aligned such that the constrained symmetry transformations (4.14) become a gauge symmetry of the IFT_2. This allows us to fix $\tilde{h} = 1$, recovering the standard form of the λ-deformed WZW model [Sfe14] with σ playing the role of λ. Further taking $\sigma \to 0$, we recover the G_{r_+} WZW model.

Another point in parameter space where we expect a gauge symmetry to emerge is when the symmetry reduction preserves the left-acting symmetry. This corresponds to setting $t = \sigma$ and

[16] An analogous limit is to take $\sigma \to 0$ and keep t finite.

$s = 1$, i.e. $r_- = -r_+$. Doing so we find

$$S_{\mathrm{IFT}_2}|_{t=\sigma,s=1} = \int_\Sigma \mathrm{vol}_2\, \mathrm{Tr}\big(r_+(\mathrm{Ad}_h J_+ - \mathrm{Ad}_{\tilde h}\widetilde{J}_+)(\widetilde{U}_+^T - \widetilde{U}_-)(\mathrm{Ad}_h J_- - \mathrm{Ad}_{\tilde h}\widetilde{J}_-) \\ + r_+ \mathcal{L}_{\mathrm{WZ}}(\tilde h^{-1}h)\big) \ , \tag{4.117}$$

where we recall that $\widetilde{U}_\pm$ are defined in eq. (4.75). This Lagrangian is invariant under a left-acting gauge symmetry as expected, which can be used to fix $\tilde h = 1$. We again recover the standard form of the λ-deformed WZW model with σ playing the role of λ. The CS_4 description of this limit is analysed in appendix 4.8.

Before we move onto the integrability of the 2d IFT and its origin from the 4d IFT, let us briefly note the symmetry reduction implications of the formal transformations (4.33) and (4.34), which in turn descended from the discrete invariances of the hCS_6 boundary conditions (4.9) and (4.10). The first (4.33) implies that the 2d IFT is invariant under

$$r_+ \leftrightarrow r_- \ , \qquad \sigma \to \sigma^{-1} \ , \qquad t \to t^{-1} \ , \qquad h \leftrightarrow \tilde h \ . \tag{4.118}$$

recovering the 'duality' transformation of [GS17]. Since the second involves interchanging w and z, it tells us the parameters are transformed if we symmetry reduce requiring that the fields h and $\tilde h$ only depend on $z, \bar z$, instead of $w, \bar w$. We find that $\sigma \to \sigma^{-1}$ and $t \to t\sigma^{-2}$.

4.5.2 Integrability and Lax Formulation

The analysis of [GS17] shows that the equations of motion of (4.112) are best cast in terms of auxiliary fields[17] $B_\pm, C_\pm$ which are related to the fundamental fields by

$$\begin{aligned} J_- &= \mathrm{Ad}_h^{-1} B_- - \lambda_0^{-1}\lambda_2^{-1} C_- \ , & \tilde J_- &= \lambda_0 \lambda_1^{-1} \mathrm{Ad}_{\tilde h}^{-1} B_- - C_- \ , \\ J_+ &= \lambda_0^{-1}\lambda_1^{-1} \mathrm{Ad}_h^{-1} B_+ - C_+ \ , & \tilde J_+ &= \mathrm{Ad}_{\tilde h}^{-1} B_+ - \lambda_0 \lambda_2^{-1} C_+ \ . \end{aligned} \tag{4.119}$$

The equations of motion for h and $\tilde h$, together with the Bianchi identities obeyed by their associated Maurer-Cartan forms, can be repackaged into the flatness of two Lax connections with components

$$\mathcal{L}^1_\pm = \frac{2\zeta_{\mathrm{GS}}}{\zeta_{\mathrm{GS}} \mp 1} \frac{1 - \lambda_0^{\mp 1}\lambda_1}{1 - \lambda_1^2} B_\pm \ , \qquad \mathcal{L}^2_\pm = \frac{2\zeta_{\mathrm{GS}}}{\zeta_{\mathrm{GS}} \mp 1} \frac{1 - \lambda_0^{\pm 1}\lambda_2}{1 - \lambda_2^2} C_\pm \ . \tag{4.120}$$

Here ζ_{GS} is the spectral parameter used in [GS17]. Taken together, the flatness of this pair of Lax connections implies both the Bianchi identities and the equations of motions. However, if one is prepared to enforce the definition (4.119) of auxiliary fields in terms of fundamental fields (such that the Bianchi equations are automatically satisfied) then either Lax will generically (i.e. away

[17] To avoid conflict of notation $B_\pm, C_\pm$ here correspond to $A_\pm, B_\pm$ of [GS17].

from special points in parameter space such as $\lambda_i = 1$) imply the equations of motion[18] of the theory.

We can relate this discussion to the construction above by symmetry reducing the 4d Lax operators (4.62) and (4.63). First we note that the currents corresponding to the (ℓ, r)-symmetries reduce to simple combinations of the auxiliary fields introduced in eq. (4.119)

$$\begin{aligned} B_{\dot{a}} &\rightsquigarrow \frac{\langle\alpha\hat{\gamma}\rangle}{\langle\alpha\beta\rangle} B_- \mu_{\dot{a}} - \frac{\langle\tilde{\alpha}\gamma\rangle}{\langle\tilde{\alpha}\beta\rangle} B_+ \hat{\mu}_{\dot{a}} \ , \\ C_{\dot{a}} &\rightsquigarrow \frac{\langle\tilde{\alpha}\hat{\gamma}\rangle}{\langle\tilde{\alpha}\beta\rangle} \sigma^{-1} C_- \mu_{\dot{a}} - \frac{\langle\alpha\gamma\rangle}{\langle\alpha\beta\rangle} C_+ \hat{\mu}_{\dot{a}} \ . \end{aligned} \tag{4.121}$$

Notice that all explicit appearances of the operators $U_\pm$ have dropped out such that these currents reduce exactly to the 2-dimensional auxiliary gauge fields.

Using the complex coordinates adapted for symmetry reduction defined in eq. (4.91), and introducing a specialised inhomogeneous coordinate on $\mathbb{CP}^1$ given by $\varsigma = \langle\pi\hat{\gamma}\rangle/\langle\pi\gamma\rangle$, the 4d B-Lax pair (4.62) may be written as

$$L^{(B)} = D_{\bar{w}} - \varsigma^{-1} D_z \ , \quad M^{(B)} = D_w + \varsigma D_{\bar{z}} \ . \tag{4.122}$$

We can symmetry reduce the 4d Lax pairs, $L^{(B/C)}, M^{(B/C)}$ of eqs. (4.62) and (4.63) to obtain

$$\begin{aligned} L^{(B)} &\rightsquigarrow \partial_- + (\langle\beta\gamma\rangle - \varsigma^{-1}\langle\beta\hat{\gamma}\rangle)\frac{\langle\alpha\hat{\gamma}\rangle}{\langle\alpha\beta\rangle} B_- \ , \quad M^{(B)} \rightsquigarrow \partial_+ + (\varsigma\langle\beta\gamma\rangle - \langle\beta\hat{\gamma}\rangle)\frac{\langle\tilde{\alpha}\gamma\rangle}{\langle\tilde{\alpha}\beta\rangle} B_+ \ , \\ L^{(C)} &\rightsquigarrow \partial_- - \frac{1}{(1-\varrho) + \lambda_0\lambda_2(1+\varrho)} C_- \ , \quad M^{(C)} \rightsquigarrow \partial_+ - \frac{1}{(1+\varrho) + \lambda_0^{-1}\lambda_2(1-\varrho)} C_+ \ . \end{aligned} \tag{4.123}$$

Now using the inhomogeneous coordinates introduced in eqs. (4.100) and (4.102), and the relations between parameters (4.113), the 4d Lax operators immediately descend upon symmetry reduction to the 2d Lax connections (4.120), provided the 4d and 2d spectral parameters are related as

$$\begin{aligned} L^{(B)} &\rightsquigarrow \partial_- + \mathcal{L}^1_- \ , \quad M^{(B)} \rightsquigarrow \partial_+ + \mathcal{L}^1_+ \ , \quad \zeta_{\text{GS}} = \frac{\bar{\gamma}_2 + \gamma_1\varsigma}{-\bar{\gamma}_2 + \gamma_1\varsigma} \ , \\ L^{(C)} &\rightsquigarrow \partial_- + \mathcal{L}^2_- \ , \quad M^{(C)} \rightsquigarrow \partial_+ + \mathcal{L}^2_+ \ , \quad \zeta_{\text{GS}} = \frac{1-\lambda_2^2}{(\lambda_0 - \lambda_0^{-1})\lambda_2 - (1-\lambda_0\lambda_2)(1-\lambda_0^{-1}\lambda_2)\varrho} \ . \end{aligned} \tag{4.124}$$

The relation between ζ_{GS} and ς can be recast in the standard $\mathbb{CP}^1$ homogeneous coordinate $\pi \sim (1, \zeta)$ as

$$\zeta_{\text{GS}} = \frac{\gamma_+ - \gamma_-}{2\zeta - (\gamma_+ + \gamma_-)} \ , \tag{4.125}$$

such that if we choose to fix $\gamma_\pm = \pm 1$ then $\zeta_{\text{GS}} = \zeta^{-1}$. If we make the assumption that the ζ_{GS} entering in the two different Lax formulations have the same origin then we can map between

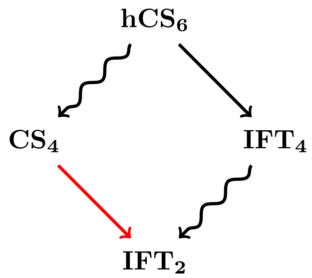
[18]It is less evident in contrast if *all* the non-local conserved charges of the theory can be obtained from a single Lax.

between ϱ and the $\mathbb{CP}^1$ homogeneous coordinate

$$\varrho = -\frac{1+\zeta}{2}\frac{1+ts}{1-ts} + \frac{1-\zeta}{2}\frac{1+ts^{-1}}{1-ts^{-1}} \ . \tag{4.126}$$

Therefore, under this assumption, we see that ϱ depends on the parameter t, which is part of the specification of boundary conditions and not just geometric data of $\mathbb{CP}^1$. Indeed ϱ becomes constant when $t \to 1$, hence there is no spectral parameter dependence left. In contrast, when $t \to 0$, we have $\varrho \to -\zeta$.

4.6 Localisation of CS_4 to IFT_2



Finally, we localise the CS_4 theory obtained by symmetry reduction of hCS_6 in § 4.4. This will result in a 2-dimensional theory on Σ, which matches the IFT_2 derived from symmetry reduction of the IFT_4 in § 4.5.

In the following discussion we will make use of the $\mathcal{E}$-model formulation of CS_4 [BSV22; LV21; LV23]. In this approach we accomplish localisation via algebraic means, constructing from the data of our CS_4 a *defect algebra* and *projectors*. The choice of boundary conditions in CS_4 corresponds to a choice of two mutually orthogonal subspaces of our defect algebra, from which we can then write down the action and Lax connection for the corresponding 2d IFT. To obtain the IFT_2 we could also follow an analogous route to that taken § 4.3, namely integrating out the $\mathbb{CP}^1$ directions directly. Details of this approach are presented in appendix of [Col+24b].

The gauge field $\hat{\mathcal{A}}$ is related to the 2-dimensional Lax connection by a change of variables that takes the form of a gauge transformation, which importantly does not preserve the boundary conditions,

$$\hat{\mathcal{A}} = \hat{h}^{-1}\mathrm{d}\hat{h} + \hat{h}^{-1}\mathcal{L}\hat{h} \ . \tag{4.127}$$

As before, we fix the redundancy in this parametrisation by demanding that $\mathcal{L}$ has no legs in the $\mathbb{CP}^1$ direction, though may of course depend on it functionally, and that $\hat{h}|_\beta = \mathrm{id}$. The bulk equations of motion, $\omega \wedge F[\hat{\mathcal{A}}] = 0$, ensures that $\mathcal{L}$ is flat and meromorphic in ζ with analytic structure mirroring ω. The key idea is that the field $\hat{h}$ evaluated at the poles serve as edge modes that become the degrees of freedom of the IFT_2, and the boundary conditions will determine the form of the Lax connection $\mathcal{L}$ in terms of these fields. However, the complete construction requires a more careful treatment, especially when Ω has higher order poles [BSV22].

Let us start by recalling the boundary conditions eq. (4.108) phrased in terms of $\mathbb{A}_w$ and $\mathbb{A}_{\bar{w}}$ valued in the defect algebra $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}$. These are that $\mathbb{A}_w \in \mathfrak{l}_t$ and $\mathbb{A}_{\bar{w}} \in \mathfrak{l}_{t^{-1}}$, where these subspaces are mutually orthogonal $\langle\!\langle \mathfrak{l}_t, \mathfrak{l}_{t^{-1}} \rangle\!\rangle = 0$. Given these boundary conditions for the gauge field at the simple poles, we introduce a group valued field $\vec{h} = (h|_\alpha, h|_{\tilde{\alpha}}) \in \mathbb{D} = \exp \mathfrak{d}$ and an algebra element $\mathbb{L} = (\mathcal{L}|_\alpha, \mathcal{L}|_{\tilde{\alpha}}) \in \mathfrak{d}$ such that

$$\begin{aligned} \mathbb{A}_w &= \vec{h}^{-1}\partial_w \vec{h} + \vec{h}^{-1}\mathbb{L}_w \vec{h} \in \mathfrak{l}_t \\ \mathbb{A}_{\bar{w}} &= \vec{h}^{-1}\partial_{\bar{w}} \vec{h} + \vec{h}^{-1}\mathbb{L}_{\bar{w}} \vec{h} \in \mathfrak{l}_{t^{-1}} \end{aligned} \tag{4.128}$$

$\mathbb{L}$ can be understood as the 2d Lax connection lifted to the double by evaluating it at the poles of the spectral parameter. From this, and the known singularity structure of $\hat{\mathcal{A}}$, we will recover the full Lax connection.

It is important to emphasise that most previous treatments have assumed that $\mathbb{A}_w$ and $\mathbb{A}_{\bar{w}}$ lie in the same subspace, and moreover that this space is a maximal isotropic subalgebra $\mathfrak{l} \subset \mathfrak{d}$. The only exception that we know of are the chiral Dirichlet boundary conditions of [CY19; ASY23], which are a special case of our boundary conditions. Taking $\mathfrak{l}$ to be an isotropic subalgebra of $\mathfrak{d}$ ensures that the resulting IFT$_2$ has a residual gauge symmetry given by the left action of $\exp(\mathfrak{l})$ on $\vec{h}$. This can be fixed by setting $\vec{h} \in \mathbb{D}/\exp(\mathfrak{l})$. For example, if we take $r_+ = -r_-$, then $\mathfrak{l} = \mathfrak{g}_{\text{diag}}$, the diagonally embedded $\mathfrak{g}$ in $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}$, is a suitable isotropic subalgebra and denoting $\vec{h} = (h, \tilde{h})$ the residual symmetry can be fixed by setting $\tilde{h} = \text{id}$. Here, however, we do not have any such residual gauge symmetry in general and the 2-dimensional theory will depend on the entire field content in $\vec{h}$.

As above, we switch notation in 2 dimensions to $\partial_w = \partial_+$ and $\partial_{\bar{w}} = \partial_-$. The IFT$_2$ action is [LV23; Kli22][19][20]

$$\begin{aligned} S_{2\text{d}}(\vec{h}) = \int \Big(&\langle\langle \partial_- \vec{h}\vec{h}^{-1}, W^+_{\vec{h}}(\partial_+ \vec{h}\vec{h}^{-1}) \rangle\rangle \\ &- \langle\langle \partial_+ \vec{h}\vec{h}^{-1}, W^-_{\vec{h}}(\partial_- \vec{h}\vec{h}^{-1}) \rangle\rangle \Big) \mathrm{d}\sigma^- \wedge \mathrm{d}\sigma^+ + S_{\text{WZ}}[\vec{h}] \ . \end{aligned} \tag{4.129}$$

The projectors $W^\pm_{\vec{h}} : \mathfrak{d} \to \mathfrak{d}$ are defined via their kernel and image

$$\text{Ker}\, W^\pm_{\vec{h}} = \text{Ad}_{\vec{h}} \mathfrak{l}_{t^{\pm 1}} \ , \qquad \text{Im} W^\pm_{\vec{h}} = \left\{ \left(\frac{\mathsf{y}}{\alpha_+ - \gamma_\pm}, \frac{\mathsf{y}}{\alpha_- - \gamma_\pm} \right) \,\middle|\, \mathsf{y} \in \mathfrak{g} \right\} , \tag{4.130}$$

where we recall that $\zeta = \gamma_\pm$ are the zeroes of ω. Care is required to correlate the zeroes of omega with the pole structure of $\mathcal{A}$, which has been determined by our symmetry reduction data. In the case at hand for instance, the zeroes at $\pi = \gamma$ and $\pi = \hat{\gamma}$ are associated to poles in $\mathcal{A}_w$ and in $\mathcal{A}_{\bar{w}}$ respectively.

In order to unpack the IFT$_2$ action, let us outline the calculation of $W^+_{\vec{h}}(\partial_+ \vec{h}\vec{h}^{-1})$. Defining

[19]The action below is related to the action in [LV23] with the redefinition $\vec{h} \to \vec{h}^{-1}$. This is due to the convention on gauge transformations. Indeed, there they consider $A^h = hAh^{-1} - \mathrm{d}hh^{-1}$ in contrast to our choice $A^h = h^{-1}Ah + h^{-1}\mathrm{d}h$.

[20]Note that we have chosen $\frac{1}{2\pi i}$ as an overall coefficient in equation (4.98), in contrast to $\frac{i}{4\pi}$ considered in [LV23].

the useful combinations[21]

$$v_\pm = \alpha_\pm - \gamma_+ \ , \qquad u_\pm = \alpha_\pm - \gamma_- \ , \tag{4.131}$$

we can parameterise the kernel and image of $W^+_{\vec{h}}$ as

$$\mathrm{Ker}W^+_{\vec{h}} = \left\{ \left(\frac{\mathrm{Ad}_h \mathsf{x}}{v_+}, \frac{\sigma^{-1}\mathrm{Ad}_h \Lambda^{-1}\mathsf{x}}{v_-} \right) \middle| \ \mathsf{x} \in \mathfrak{g} \right\}, \qquad \mathrm{Im}W^+_{\vec{h}} = \left\{ \left(\frac{\mathsf{y}}{v_+}, \frac{\mathsf{y}}{v_-} \right) \middle| \ \mathsf{y} \in \mathfrak{g} \right\} . \tag{4.132}$$

We decompose $\partial_+ \vec{h}\vec{h}^{-1}$ into the kernel and image by solving

$$\left(\mathrm{Ad}_h J_+, \mathrm{Ad}_h \Lambda^{-1} \widetilde{J}_+ \right) = \left(\frac{\mathrm{Ad}_h \mathsf{x}}{v_+}, \frac{\sigma^{-1}\mathrm{Ad}_h \Lambda^{-1}\mathsf{x}}{v_-} \right) + \left(\frac{\mathsf{y}}{v_+}, \frac{\mathsf{y}}{v_-} \right) . \tag{4.133}$$

This yields

$$\mathsf{y} = \mathrm{Ad}_h U_+ \left(J_+ v_+ - \sigma \widetilde{J}_+ v_- \right) = v_- B_+ \ , \tag{4.134}$$

in which we see the reappearance of the auxiliary combinations encountered earlier in eq. (4.119). It follows that

$$W^+_{\vec{h}}(\partial_+ \vec{h}\vec{h}^{-1}) = \left(\frac{v_-}{v_+} B_+, B_+ \right) \ , \tag{4.135}$$

from which we can evaluate (trace implicit)

$$\begin{aligned} \langle\!\langle \partial_- \vec{h}\vec{h}^{-1}, W^+_{\vec{h}}(\partial_+ \vec{h}\vec{h}^{-1}) \rangle\!\rangle &= \frac{v_-}{v_+} r_+ J_- \mathrm{Ad}_h^{-1} B_+ + r_- \widetilde{J}_- \Lambda \mathrm{Ad}_h^{-1} B_- \\ &= r_+ J_+ U_+^T J_- + r_- \widetilde{J}_+ U_- \widetilde{J}_- - t\sqrt{-r_- r_+} \widetilde{J}_+ U_+^T J_- + t^{-1} \sqrt{-r_- r_+} J_+ U_- \widetilde{J}_- \ . \end{aligned} \tag{4.136}$$

In a similar fashion we find that

$$W^-_{\vec{h}}(\partial_- \vec{h}\vec{h}^{-1}) = \left(B_-, \frac{u_+}{u_-} B_- \right) \ , \tag{4.137}$$

and

$$\begin{aligned} &\langle\!\langle \partial_+ \vec{h}\vec{h}^{-1}, W^-_{\vec{h}}(\partial_- \vec{h}\vec{h}^{-1}) \rangle\!\rangle \\ &= r_+ J_+ U_- J_- + r_- \widetilde{J}_- U_+^T \widetilde{J}_+ + t\sqrt{-r_- r_+} \widetilde{J}_+ U_+^T J_- - t^{-1} \sqrt{-r_- r_+} J_+ U_- \widetilde{J}_- \ . \end{aligned} \tag{4.138}$$

Taking the difference of eq. (4.136) and eq. (4.138) we find that the Lagrangian of the 2-dimensional action eq. (4.129) exactly matches the IFT_2 obtained previously in eq. (4.112) by descent on the other side of the diamond. This explicitly verifies our diamond of theories.

Let us note that this IFT_2 has also been constructed from CS_4 in a two-step process in

[21] In terms of these parameters, the relations (4.97) become

$$r_+ = K \frac{u_+ v_+}{\Delta\gamma\Delta\alpha} \ , \qquad r_- = -K \frac{u_- v_-}{\Delta\gamma\Delta\alpha} \ , \qquad s = \sqrt{\frac{u_- v_-}{u_+ v_+}} \ , \qquad t = \sigma s \frac{u_+}{u_-} \ .$$

[BL20]. First, a more general 2-field model based on a twist function with additional poles and zeroes, and the familiar isotropic subalgebra boundary conditions, is constructed. Second, a special decoupling limit is taken, where a subset of these poles and zeroes collide. It remains to understand how to recover our boundary conditions (4.104) from those considered in [BL20].

To complete the circle of ideas we can also directly obtain a Lax formulation from CS_4. This is essentially achieved by undoing the map into the defect algebra as follows. Given an element $\mathbb{X} = (x, y) \in \mathfrak{d}$, we determine $a, b \in \mathfrak{g}$ such that

$$(x, y) = \left(\frac{a}{u_+} + \frac{b}{v_+}, \frac{a}{u_+} + \frac{b}{v_-}\right) . \tag{4.139}$$

We introduce a map $\wp$ into the space of $\mathfrak{g}$-valued meromorphic functions

$$\wp : \mathbb{X} \mapsto \frac{a}{\zeta - \gamma_-} + \frac{b}{\zeta - \gamma_+} , \tag{4.140}$$

in terms of which the components of the Lax connection are given by

$$\mathcal{L}_\pm = \wp W^\pm_{\vec{h}}(\partial_\pm \vec{h}\vec{h}^{-1}) = \frac{\alpha_\mp - \gamma_\pm}{\zeta - \gamma_\pm} B_\pm . \tag{4.141}$$

In this way we recover the Lax $\mathcal{L}^1$ that we obtained from symmetry reduction of the 4d ASDYM Lax pair with the identification (4.125). There does not appear an algebraic derivation in this spirit of the other Lax $\mathcal{L}^2$. This in contrast to the derivation of this model from CS_4 in [BL20] where the extra data associated to the additional poles means that both Lax in (4.120) can be directly constructed. More generally, this highlights an interesting question that we leave for the future about the integrability and the counting of conserved charges, beyond the existence of a Lax connection, when we consider boundary conditions not based on isotropic subalgebras.

4.6.1 RG Flow

Let us recall the RG equations given in [GS17]

$$\dot{\lambda}_i = -\frac{c_G}{2\sqrt{k_1 k_2}} \frac{\lambda_i^2(\lambda_i - \lambda_0)(\lambda_i - \lambda_0^{-1})}{(1 - \lambda_i^2)^2} , \quad i = 1, 2 , \tag{4.142}$$

where dot indicates the derivative with respect to RG 'time' $\frac{d}{d\log\mu}$ and c_G is the dual Coxeter number. The levels k_1 and k_2 and $\lambda_0 = \sqrt{k_1/k_2}$ are RG invariants. In this section we will interpret this flow in terms of the data that is more natural from the perspective of 4d CS, namely the poles and zeroes of the differential

$$\omega = \frac{K}{\Delta\gamma} \frac{(\zeta - \gamma_+)(\zeta - \gamma_-)}{(\zeta - \alpha_+)(\zeta - \alpha_-)} \mathrm{d}\zeta = \varphi(\zeta)\, \mathrm{d}\zeta , \tag{4.143}$$

and the boundary conditions of the theory.

Using the map between parameters given in eq. (4.113) we can infer from eq. (4.142) a flow on the parameters $\{t, \alpha_\pm, \gamma_\pm, K\}$. Let us first consider the parameter $t = \lambda_2$. As discussed in [GS17], there is a flow from $t = 0$ in the UV to $t = \lambda_0$ in the IR (assuming that $\lambda_0 < 1$). Explicitly the flow equation

$$\dot{t} = \frac{c_G}{2k_2\lambda_0}\frac{t^2}{(1-t^2)^2}(t-\lambda_0)(t-\lambda_0^{-1}) \ , \tag{4.144}$$

has the solution

$$f(\lambda_0, t) + f(\lambda_0^{-1}, t) + t + t^{-1} = \frac{c_G}{2\sqrt{k_1 k_2}}\log \mu/\mu_{t_0} \ , \quad f(x,t) = x \log\left(\frac{t^{-1}-x}{t-x}\right) \ . \tag{4.145}$$

The interesting observation is that the boundary conditions

$$\begin{aligned} \mathbb{A}_w &\in \mathfrak{l}_t = \mathrm{span}\{\big(t\lambda_0^{-1}x, x\big) \mid x \in \mathfrak{g}\} \ , \\ \mathbb{A}_{\bar{w}} &\in \mathfrak{l}_{t^{-1}} = \mathrm{span}\{\big(\lambda_0^{-1}x, tx\big) \mid x \in \mathfrak{g}\} \ , \end{aligned} \tag{4.146}$$

display algebraic enhancements at the fixed points. In the UV, $t = 0$ limit, these boundary conditions become chiral, $\mathbb{A}_w \in \mathfrak{g}_R \subset \mathfrak{d}$ and $\mathbb{A}_{\bar{w}} \in \mathfrak{g}_L \subset \mathfrak{d}$. While $\mathfrak{g}_{L,R}$ are now subalgebras, neither are isotropic with respect to the inner product (4.106). In non-doubled notation the UV limit becomes

$$\hat{\mathcal{A}}_w|_\alpha = 0 \ , \qquad \hat{\mathcal{A}}_{\bar{w}}|_{\tilde{\alpha}} = 0 \ . \tag{4.147}$$

On the other hand in the IR limit, $t = \lambda_0$, we see that $\mathbb{A}_w \in \mathfrak{g}_{\mathrm{diag}} \subset \mathfrak{d}$, again a subalgebra, but only an isotropic one for $k_1 = k_2$, i.e. $r_+ = -r_-$. In non-doubled notation the IR limit becomes[22]

$$\hat{\mathcal{A}}_w|_\alpha = \hat{\mathcal{A}}_w|_{\tilde{\alpha}} \ , \qquad k_1\hat{\mathcal{A}}_{\bar{w}}|_\alpha = k_2\hat{\mathcal{A}}_{\bar{w}}|_{\tilde{\alpha}} \ . \tag{4.148}$$

While in general, there are no residual gauge transformations preserving the boundary conditions, in the UV and IR limits we notice chiral boundary symmetries emerging. For example, in the IR these are those satisfying $g^{-1}\partial_{\bar{w}} g = 0$, which corresponds to $t = s^{-1}$ in eq. (4.105).

Let us now turn to the action of RG on the differential ω. An immediate observation is that the RG invariant WZW levels are given by monodromies about simple poles[23]

$$\pm k_{1,2} = r_\pm = \frac{1}{2\pi i}\oint_{\alpha_\pm} \omega = \mathrm{res}_{\zeta=\alpha_\pm}\varphi(\zeta) \ , \tag{4.149}$$

exactly in line with the conjecture of Costello (reported and supported by Derryberry [Der21]). While there are more parameters in ω than there are RG equations, we can form the ratios of

[22] The seemingly more democratic boundary condition of $t = 1$,

$$\sqrt{k_1}\hat{\mathcal{A}}_w|_\alpha = \sqrt{k_2}\hat{\mathcal{A}}_w|_{\tilde{\alpha}} \ , \quad \sqrt{k_1}\hat{\mathcal{A}}_{\bar{w}}|_\alpha = \sqrt{k_2}\hat{\mathcal{A}}_{\bar{w}}|_{\tilde{\alpha}}$$

which does define an isotropic space of $\mathfrak{d}$ (not a subalgebra however) is *not* attained along this flow.

[23] The monodromy about the double pole at infinity is trivially RG invariant since the sum of all the residues vanishes.

poles and zeroes

$$q_\pm = \frac{\alpha_\pm - \gamma_+}{\alpha_\pm - \gamma_-} = \frac{v_\pm}{u_\pm} \ , \tag{4.150}$$

in terms of which the RG system of [GS17] translates to

$$\dot{q}_\pm = -\frac{c_G}{2K}\frac{(1+q_\mp)}{(-1+q_\mp)}q_\pm \ , \qquad \dot{K} = -\frac{c_G}{2}\frac{q_- + q_+}{(1-q_-)(1-q_+)} \ . \tag{4.151}$$

The RG invariants are given by

$$k_1 k_2 = \frac{K^2 q_- q_+}{(q_+ - q_-)^2} \ , \qquad \frac{k_1}{k_2} = \lambda_0^2 = \frac{q_+}{(1-q_+)^2}\frac{(1-q_-)^2}{q_-} \ , \tag{4.152}$$

which allows us to retain either of $q_\pm$ as independent variables. We can directly solve these equations

$$\sqrt{k_1 k_2}\frac{q_+ - q_-}{\sqrt{q_+ q_-}} + k_1 \log q_+ - k_2 \log q_- = \frac{c_G}{2}\log \mu/\mu_{q_0} \ , \tag{4.153}$$

and a remarkable feature, also conjectured by Costello, is that this quantity is precisely the contour integral between zeroes

$$\frac{\mathrm{d}}{\mathrm{d}\log\mu}\int_{\gamma_-}^{\gamma_+}\omega = \frac{c_G}{2} \ . \tag{4.154}$$

To best understand the action of the RG flow on the locations of the poles directly, we replace K with the RG invariant k_2 (or k_1), and fix the zeroes to be located at $\gamma_\pm = \pm 1$. This yields the RG invariant relation

$$1 - \alpha_+^2 - \lambda_0^2(1-\alpha_-^2) = 0 \ , \tag{4.155}$$

and a flow equation

$$\dot{\alpha}_- = \frac{c_G}{8k_2}\frac{\alpha_+(1-\alpha_-^2)^2}{\alpha_- - \alpha_+} \ , \tag{4.156}$$

the solution of which is

$$\frac{\alpha_+ - \alpha_-}{1-\alpha_+^2} + \frac{1}{2}\log\frac{\alpha_+ + 1}{\alpha_+ - 1} - \frac{1}{2\lambda_0^2}\log\frac{\alpha_- + 1}{\alpha_- - 1} = \frac{c_G}{4k_1}\log\mu/\mu_{\alpha_0} \ . \tag{4.157}$$

As illustrated in fig. 4.1, this system displays a finite RG trajectory linking fixed points. In the UV limit the poles accumulate to different zeroes, and in the IR the poles accumulate to the same zero. Let us consider the upper red trajectory of fig. 4.1 in which we choose $\lambda_0 < 1$ and pick the positive branch of the solution $\alpha_+ = +\sqrt{1-\lambda_0^2(1-\alpha_-^2)}$. With this choice we see that there are finite fixed points[24] such that the right hand side of eq. (4.156) vanishes at

$$\mathrm{UV}: \quad (\alpha_-,\alpha_+) = (-1,1) \ , \quad \lambda_1 = 0 \ , \qquad \mathrm{IR}: \quad (\alpha_-,\alpha_+) = (1,1) \, , \quad \lambda_1 = \lambda_0 \ , \tag{4.158}$$

[24] There are also fixed points to the RG flow at $\alpha_+ = 0$ with $\alpha_-^2 = 1 - \frac{k_2}{k_1}$ however by assumption $k_2 > k_1$, and so these do not correspond to real values of α_- and consequently λ_1 is imaginary. We do not consider such complex limits here.

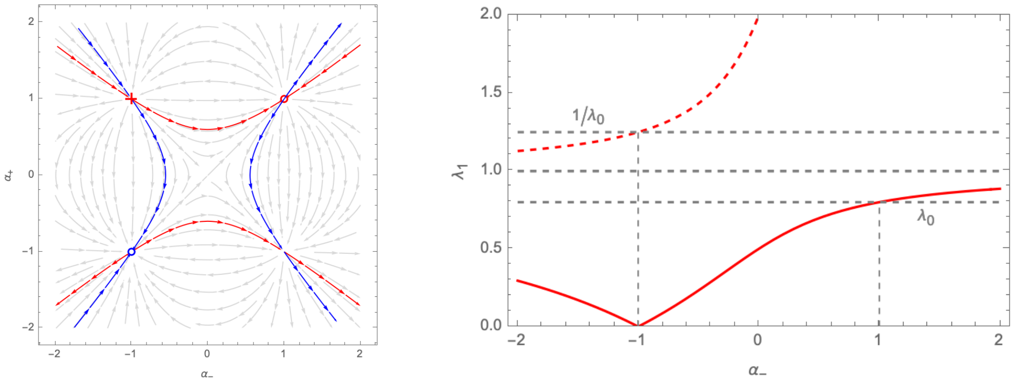


Figure 4.1: Left: RG flow across the α_+, α_- plane with arrows directed to the IR. The highlighted parabola are the solutions that lie on the locus of the RG invariant quantity $\lambda_0^2 = k_1/k_2$, plotted here for $\lambda_0 = 0.8$ (red) and $\lambda_0 = 1.2$ (blue). Right: The value of λ_1 plotted along the red loci of the left panel (upper branch solid and lower branch dotted). In both cases $\lambda_1 \to 1$ asymptotically as $\alpha_- \to \pm\infty$. Of note is the flow displayed by the upper red branch between the UV fixed point $(\alpha_-, \alpha_+) = (-1, 1)$ with $\lambda_1 = 0$ and the IR fixed point $(\alpha_-, \alpha_+) = (1, 1)$ with $\lambda_1 = \lambda_0$.

in which we recall the map

$$\lambda_1 = \left(\frac{(1+\alpha_-)(-1+\alpha_+)}{(-1+\alpha_-)(1+\alpha_+)} \right)^{\frac{1}{2}} . \tag{4.159}$$

One of the appealing features of the IFT$_2$ (4.112) is that it provides a classical Lagrangian interpolation that includes its own UV and IR limits [GS17]. That is to say these CFTs can be obtained directly from the Lagrangian eq. (4.112) by tuning the parameters of the theory to their values at the end points of the RG flow. Given the interpretation of these RG flows as describing poles colliding with zeroes it is natural to expect that a similar interpolation can be obtained directly in 4d by taking limits of the differential ω in eq. (4.143).

Here we will explore how this works for the IFT$_2$ (4.112) in the IR. The limit we will consider is to collide the poles at $\alpha_\pm$ with the zero at γ_+, following the upper red trajectory in fig. 4.1. This corresponds to taking $q_\pm \to 0$. In order to be consistent with the RG invariants (4.152), we take this limit as

$$q_+ = k_1\epsilon + \mathcal{O}(\epsilon^2) , \qquad q_- = k_2\epsilon + \mathcal{O}(\epsilon^2) , \qquad K = k_1 - k_2 + \mathcal{O}(\epsilon) , \qquad \epsilon \to 0 . \tag{4.160}$$

Taking this limit in (4.143), and redefining the spectral parameter such that the remaining pole and zero are fixed to 1 and -1 respectively, yields

$$\omega \to \frac{k_1 - k_2}{2} \frac{\zeta + 1}{\zeta - 1} d\zeta . \tag{4.161}$$

Let us consider the implication of this limit from the CS$_4$ perspective. Given that the pole structure of ω is modified in this limit, so will the double $\mathfrak{d}$, and thus we should be careful in our interpretation of the boundary conditions. If we take ω to be given by (4.161) and consider the boundary conditions (4.146) with $t = 0$, the condition $\mathbb{A}_w \in \mathfrak{l}_t$ becomes $\hat{\mathcal{A}}_w|_\alpha = 0$. From eq. (4.94) we know that $\hat{\mathcal{A}}_w$ has a pole at γ_+. In other words, we can write

$$\hat{\mathcal{A}}_w = \frac{(\zeta - \alpha_+)}{(\zeta - \gamma_+)}\Xi(\zeta) \ , \tag{4.162}$$

with $\Xi(\zeta)$ regular as $\alpha_+ \to \gamma_+$. Hence, in the IR there is no boundary condition for $\hat{\mathcal{A}}_w$ at $\zeta = 1$. On the other hand, the boundary condition $\mathbb{A}_{\bar{w}} \in \mathfrak{l}_{t^{-1}}$ for $t = 0$ is $\hat{\mathcal{A}}_{\bar{w}}|_{\tilde{\alpha}} = 0$ which in the limit $\alpha_+, \gamma_+ \to 1$ becomes a chiral boundary condition for the $\bar{w}$ component

$$\hat{\mathcal{A}}_{\bar{w}}|_{\zeta=1} = 0 \ . \tag{4.163}$$

For this choice of boundary condition one can localise the CS_4 action following the procedure described in § 4.4, and the resulting two dimensional IFT is the WZW model at level $k_2 - k_1$.

In contrast, from the 2-dimensional perspective it is known that the full result at this IR fixed point is actually a product WZW model on $G_{k_2} \times G_{k_1 - k_2}$ [GS17]. This indicates that there is some delicacy in taking the IR limit directly as a Lagrangian interpolation in 4d even when it is possible in 2d. One reason for this is there is also the freedom to perform redefinitions of the spectral parameter, which can, in general, produce non-equivalent limits of ω. Such limits are known as decoupling limits [Del+19; BL20] in the literature, and have been investigated for the UV fixed point of the bi-Yang-Baxter model in [KLT24].

4.7 Discussion and Outlook

In this work we have constructed a diamond of integrable models related by localisation and symmetry reduction. Starting from holomorphic Chern-Simons theory with the meromorphic (3,0)-form (4.2), we have found a new choice of admissible boundary conditions, which leads to a well-defined 6-dimensional theory. This generalises the analysis carried out in [BS23; Pen21] to a new class of boundary conditions.

By first viewing twistor space, $\mathbb{PT}$, as a $\mathbb{CP}^1$ bundle over $\mathbb{R}^4$, we solved the equations of motion along the $\mathbb{CP}^1$ fibres. In doing so we fully specified the dependence of the integrand on the $\mathbb{CP}^1$ fibre and thus could perform a fibrewise integration along those directions. Consequently our 6-dimensional theory then localised to the poles of Ω, leading to a new 4-dimensional theory on $\mathbb{R}^4$ given by the action (4.31). Indeed, this 4-dimensional theory is 'integrable' in the sense that its equations of motion can be encoded in an anti-self-dual connection, as expected from the Penrose-Ward correspondence. Moreover, this new IFT_4 exhibits two semi-local symmetries, which can be understood as the residual symmetries preserving the boundary conditions of hCS_6. For each of these semi-local symmetries, the Noether currents can be used to construct inequivalent Lax formulations of the dynamics.

On the other hand, symmetry reducing hCS_6 along two directions of the $\mathbb{R}^4$ in $\mathbb{PT} \cong \mathbb{CP}^1 \times \mathbb{R}^4$, leads to an effective CS_4 theory on $\mathbb{CP}^1 \times \mathbb{R}^2$. Under this procedure, the meromorphic $(3,0)$-form reduces to the meromorphic $(1,0)$-form used in [Del+20] to construct the λ-model, whereas the 6-dimensional boundary conditions reduce to a class of boundary conditions in CS_4 that have not been previously considered. Specifically, they relax the assumption of an isotropic subalgebra of the defect algebra. By performing the standard localisation procedure of CS_4 we obtain the 2-field λ-type IFT_2 introduced in [GS17].

Notably, this same multi-parametric class of integrable λ-deformations between coupled WZW models can be obtained by symmetry reduction (along the same directions) of the novel IFT_4 mentioned above. Furthermore, the semi-local symmetries of the IFT_4 reduce to global symmetries of the IFT_2 and the two Lax formulations of the IFT_4 give rise to two Lax connections for the IFT_2. When the directions of the symmetry reduction are aligned to these semi-local symmetries, the IFT_2 symmetries are enhanced to either affine or fully local (gauge) symmetries. In the latter case, the IFT_2 becomes the standard (1-field) λ-model.

This work opens up a range of interesting further directions. There are a selection of direct generalisations that can be made to incorporate the wide variety of integrable deformations known in the literature. Perhaps the most interesting outcome of this would be the construction of swathes of new four-dimensional integrable field theories. Our work focused on the case where Ω was nowhere vanishing; it would be interesting to explore the relaxation of this condition together with its possible boundary conditions, and how the ASDYM equations are modified. Moreover, one might hope that the study of boundary conditions in hCS_6 could lead to a full classification of the landscape of integrable sigma-models in 2d, and perhaps result in theories not yet encountered in the literature.

From the perspective of the IFT_2, there is a close relationship between the notions of Poisson-Lie symmetry, duality and integrability [Vic15; HT15; SST15; Kli15]. This poses an interesting question as to the implications of such dualities for both the IFT_4 and hCS_6. For the model considered here, we might seek to understand the semi-local symmetries of IFT_4 in the context of the q-deformed symmetries expected to underpin the IFT_2.

In this work, the integrable models we have studied can be viewed as descending from the open string sector of a type B topological string. An interesting direction for future work is to consider the closed string sector [Ber+94] and its possible integrable descendants. A tantalising prospect is to understand the closed string counterparts of the integrable deformations we have considered in the context of the non-linear graviton construction for self-dual space-times [Pen76; Hit79; War80].

By coupling the open and closed string sectors [CL15; CG18; CL20; Cos21] one can find an anomaly free quantization to all loop orders in perturbation theory of the coupled hCS_6-BCOV action. This mechanism has already proven a powerful tool in the context of the top-down approach to celestial holography [CPS23; BSS23]. This could provide an angle of attack to address the important questions of when the IFT_4 can be quantised, if the IFT_4 is *quantum* integrable, and if there is a higher dimensional origin of IFT_2 as quantum field theories.

4.8 Appendix: Alternative CS_4 Setup for the λ-Model

In this section, we will consider an alternative symmetry reduction of our hCS_6 setup which also recovers the λ-deformed IFT_2. In order to recover a 1-field IFT_2, we need one of the semi-local residual symmetries of the IFT_4 to become a gauge symmetry under symmetry reduction. Let us denote the symmetry reduction vector fields by V_1 and V_2. Taking the example of the residual left-action parameterised by ℓ, this must obey the constraint $\beta^a \partial_{a\dot{a}} \ell = 0$. In order for this to become a gauge symmetry of the IFT_2, the symmetry reduction constraints $L_{V_1}\ell$ and $L_{V_2}\ell = 0$ must coincide with the pre-existing constraints on ℓ. This means that we must choose to symmetry reduce along the vector fields

$$V_1 = \mu^{\dot{a}} \beta^a \partial_{a\dot{a}} \ , \qquad V_2 = \hat{\mu}^{\dot{a}} \beta^a \partial_{a\dot{a}} \ . \tag{4.164}$$

Following the recipe described elsewhere in this paper, we deduce that the CS_4 1-form is given by[25]

$$\omega = K \frac{1}{(\zeta - \alpha_+)(\zeta - \alpha_-)} \, d\zeta \ . \tag{4.165}$$

In the 4d CS description, we can already see that we have eliminated one degree of freedom relative to other symmetry reductions. The symmetry reduction zeroes have eliminated the double pole at β, effectively removing one field from the IFT_2.

Furthermore, if we denote the surviving coordinates on Σ by $y^1 = \hat{\mu}^{\dot{a}} \hat{\beta}^a \partial_{a\dot{a}}$ and $y^2 = -\mu^{\dot{a}} \hat{\beta}^a \partial_{a\dot{a}}$, the boundary conditions reduce to

$$\hat{\mathcal{A}}_1|_\alpha = \sigma \hat{\mathcal{A}}_1|_{\tilde{\alpha}} \ , \qquad \hat{\mathcal{A}}_2|_\alpha = \sigma^{-1} \hat{\mathcal{A}}_2|_{\tilde{\alpha}} \ . \tag{4.166}$$

Since the localisation from CS_4 to the IFT_2 has been described in detail elsewhere, we will be brief in this section. In the parametrisation

$$\hat{\mathcal{A}} = \hat{h}^{-1} \mathcal{L} \hat{h} + \hat{h}^{-1} d\hat{h} \ , \tag{4.167}$$

we fix the constraints $\mathcal{L}_{\bar{\zeta}} = 0$ and denote the values of $\hat{h}$ at the poles by $\hat{h}|_\alpha = h$ and $\hat{h}|_{\tilde{\alpha}} = \mathrm{id}$. We can then use the bulk equations of motion and the boundary conditions to solve for $\mathcal{L}_1$ and $\mathcal{L}_2$ in terms of h. We find the solutions

$$\mathcal{L}_1 = (\sigma - \mathrm{Ad}_h^{-1})^{-1} h^{-1} \partial_1 h \ , \qquad \mathcal{L}_2 = (\sigma^{-1} - \mathrm{Ad}_h^{-1})^{-1} h^{-1} \partial_2 h \ . \tag{4.168}$$

[25] Since β^a appears in both of our symmetry reduction vector fields, the two zeroes from symmetry reduction have cancelled the double pole. Similarly, the boundary condition $\mathcal{A}_{\dot{a}}|_\beta = 0$ can be interpreted as a simple zero in each component of the gauge field. These simple zeroes cancel the simple poles introduced in symmetry reduction, leaving a gauge field with no singularities.

Finally, the action localises to 2d and is given, up to an overall factor of $K/(\alpha_+ - \alpha_-)$, by

$$-\int_\Sigma \mathrm{d}y^1 \wedge \mathrm{d}y^2 \,\mathrm{Tr}\bigg(h^{-1}\partial_1 h \cdot \frac{1+\sigma \mathrm{Ad}_h^{-1}}{1-\sigma \mathrm{Ad}_h^{-1}} h^{-1}\partial_2 h \bigg) - \frac{1}{6}\int_{\Sigma\times[0,1]} \mathrm{Tr}\big(h^{-1}\mathrm{d}h \wedge h^{-1}\mathrm{d}h \wedge h^{-1}\mathrm{d}h\big)\,. \tag{4.169}$$

This can be recognised as the λ-deformed IFT_2.

Chapter 5

Integrable gauged models from twistor space

The results in this chapter were found in collaboration with Ryan A. Cullinan, Ben Hoare, Joaquin Liniado, and Daniel C. Thompson [Col+24a].

5.1 Introduction

Quantum field theories (QFTs) in two dimensions have both direct applications in condensed matter systems and as the worldsheet theories of strings, and can provide a tractable sandpit for the study of quantum field theory more generally. Special examples are provided by conformal field theories (CFTs) and integrable field theories (IFTs), for which powerful infinite-dimensional symmetries enable us to exactly determine certain key properties and observables.

One longstanding goal has been to provide a constructive origin of these integrable systems from some putative parent theory, perhaps in higher dimensions. For instance, Ward suggested [War85] that all integrable equations may arise as reductions of the 4d anti-self-dual Yang-Mills (ASDYM) equation. Alternatively, motivated by the similarity between Reidemeister moves in knot theory and the Yang-Baxter equation that underpins integrability, Witten suggested [Wit89a] that integrable models might have a description in terms of Chern-Simons theory. The realisation of this idea came some years later, with Costello's understanding [Cos13; Cos14] that the gauge theory description should combine the topological nature of Chern-Simons theory with the holomorphic nature of the spectral parameter characterising IFTs. The theory proposed in [Cos13; Cos14] was extended and developed in a sequence of papers [CWY18a; CWY18b; CY19] describing a Chern-Simons theory, which we denote by CS_4, defined over a four-manifold $\Sigma \times C$.

An elegant origin of both the CS_4 and the ASDYM descriptions was provided in the work of Bittleston and Skinner [BS23] in terms of a six-dimensional holomorphic Chern-Simons theory (hCS_6), first proposed in [Cos20; Cos21]. The theory is defined over (the Euclidean slice of)

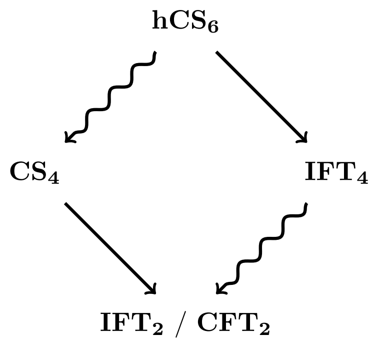


Figure 5.1: The diamond correspondence of integrable avatars, in which wavy arrows indicate a descent by reduction and straight arrows involve localisation i.e. integration over $\mathbb{CP}^1$.

Penrose's twistor space [Pen67] with the action functional

$$S_{\mathrm{hCS}_6}[\mathcal{A}] = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \Omega \wedge \mathrm{Tr}\left(\mathcal{A} \wedge \bar{\partial} \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) , \tag{5.1}$$

in which Ω is some meromorphic $(3,0)$ form. This action is supplemented by a choice of boundary conditions at the poles of Ω. The various lower-dimensional descriptions follow from exploiting the fibration structure $\mathbb{CP}^1 \hookrightarrow \mathbb{PT} \twoheadrightarrow \mathbb{R}^4$. Reducing along two directions within $\mathbb{R}^4$, hCS$_6$ descends to CS$_4$. Alternatively, one may instead first choose to localise over $\mathbb{CP}^1$, and this leads to IFT$_4$ of the ASDYM description. Indeed, the integrability properties of ASDYM are fundamentally tied to this twistorial origin and evidence suggests that at a quantum level this twistor space is the natural arena to consider [Cos21; BSS23]. Applying the reduction along $\mathbb{R}^4$ to this IFT$_4$ produces an IFT$_2$ which may also be recovered by localising the CS$_4$ description. In this way, we have a diamond correspondence of theories illustrated in Figure 5.1. Other recent work on hCS$_6$ includes [Pen21; Cos21; Col+24b].

Given an IFT$_2$ or CFT$_2$ it is sometimes possible to obtain another I/CFT$_2$ via gauging. Perhaps the most famous example is the GKO G/H coset CFTs [GKO85], which can be given a Lagrangian description by taking a WZW$_2$ CFT on G and gauging a (vectorially acting) H subgroup [GK88; Kar+89; BCR90; Wit92]. This motivates the core question of this work:

How can the diamond correspondence be gauged?

Resolving this question dramatically expands the scope of theories that can be given a higher-dimensional avatar. A significant clue is given by the rather remarkable Polyakov-Wiegmann (PW) identity, which shows that the G/H gauged WZW model is actually equivalent to the difference of a G WZW model and an H WZW model. This points towards a general resolution that integrable gauged models might be obtained as differences of ungauged models. This is less obvious than it might first seem; it was noted in [Los+96] that for a PW identity to apply for

WZW_4 one requires that the gauging is performed by connections with field strength restricted to be type $(1, 1)$. The six-dimensional origin of such a constraint is rather intriguing and will be elucidated in this work. In the context of CS_4, Stedman recently proposed [Ste21] considering the difference of CS_4 to give rise to gaugings of IFT_2. We will recover this construction as a reduction of hCS_6 theory in the present work, as well as uncovering some additional novelties in the CS_4 description.

At the top of the diamond, we will consider a theory of two-connections $\mathcal{A} \in \Omega^1(\mathbb{PT}) \otimes \mathfrak{g}$ and $\mathcal{B} \in \Omega^1(\mathbb{PT}) \otimes \mathfrak{h}$ for a subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The action of this theory is

$$S_{\mathrm{ghCS}_6} = S_{\mathrm{hCS}_6}[\mathcal{A}] - S_{\mathrm{hCS}_6}[\mathcal{B}] + S_{\mathrm{int}}[\mathcal{A}, \mathcal{B}] \,, \tag{5.2}$$

in which the term S_{int} couples the two gauge fields. We will develop this story by means of two explicit examples: choosing Ω to have two double poles, we will study the diamond relevant to the gauged WZW theory; and with Ω containing a single fourth-order pole we will study the gauged LMP model. This seemingly simple setup gives rise to a rich story whose results we will now summarise:

1. Our investigations indicate that general gaugings of the WZW_4 model break integrability in four dimensions. Integrability is preserved if the gauge field B is constrained to satisfy two of the three anti-self-dual Yang-Mills equations, namely $F^{2,0}[B] = 0$ and $F^{0,2}[B] = 0$.

2. The two gauge fields $\mathcal{A}$ and $\mathcal{B}$ of $ghCS_6$ source various degrees of freedom in the gauged WZW_4. In particular, as well as the fundamental field g and the 4d gauge field B, auxiliary degrees enter as Lagrange multipliers for $F^{2,0}[B] = 0$ and $F^{0,2}[B] = 0$.

3. Reducing by two dimensions, we recover a variety of IFT_2 including the special case of gauged WZW_2. In general, we find a coupled model between a gauged IFT_2 and a Hitchin system [Hit87] involving the gauge field B and a pair of adjoint scalar fields. These scalars may source a potential for the gauged WZW_2 in which case we recover the complex sine-Gordon model and more broadly the homogeneous sine-Gordon models [Fer+97]. At the special point associated to the 2d PCM, Lagrange multipliers ensure that the gauge field is flat and hence trivial — this is essential as the gauged PCM is not generically integrable.

4. We also use this formalism to perform an integrable gauging of the LMP model. Just as in the gauging of WZW_4, the field strength of the gauge field must be constrained to obey two of the anti-self-dual Yang-Mills equations, this time $F^{2,0}[B] = 0$ and $\varpi \wedge F^{1,1}[B] = 0$. It is noteworthy that the two equations which are enforced by Lagrange multipliers agree with the two equations that are identically solved in the ungauged case. This is true for both the WZW_4 and the LMP model. In addition, we show that the gauged LMP model obeys a PW-like identity such that it may be expressed the difference of an LMP model on $\mathfrak{g}$ and $\mathfrak{h}$.

Let us outline the structure of this chapter. We begin in section5.2 with a review of the diamond correspondence of theories for the ungauged WZW model. In section5.3, we introduce the gauging of this diamond concentrating in particular on the right hand side. We recover the gauged IFT_4 and demonstrate that its equations of motion may be rewritten as ASDYM. The wide array of IFT_2 are explored in section5.4 where we also show that they are integrable and provide the associated Lax connection. Following the gauging of the WZW_4, section5.5 fleshes out the left hand side of the diamond, connecting to four-dimensional Chern-Simons by first reducing, and then to IFT_2 by localisation. section5.6 describes the diamond in the context of the gauged LMP theory. We conclude with a brief outlook in section 5.7. Although the subject matter necessarily entails a degree of technical complexity we have endeavoured to keep the main presentation streamlined and complement this with a number of technical appendices.

5.2 The ungauged WZW diamond

In this section, we briefly describe the diamond correspondence of theories in which the two-dimensional theory is the WZW_2 CFT. This is a summary of some analysis first presented in [BS23] which will serve to fix conventions and recapitulate key steps relevant to later sections.

5.2.1 hCS_6 with double poles

We begin at the top of the diamond with 6d holomorphic Chern-Simons theory (hCS_6) whose fundamental field is an algebra-valued connection $\mathcal{A} \in \Omega^{0,1}(\mathbb{PT}) \otimes \mathfrak{g}$. The six-dimensional action is given by

$$S_{\text{hCS}_6}[\mathcal{A}] = \frac{1}{2\pi \text{i}} \int_{\mathbb{PT}} \Omega \wedge \text{Tr}\Big(\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Big) \,, \tag{5.3}$$

in which we have introduced a meromorphic $(3,0)$-form Ω. As a real manifold, there is an isomorphism $\mathbb{PT} \cong \mathbb{R}^4 \times \mathbb{CP}^1$ and we will introduce coordinates $x^{a\dot{a}} \in \mathbb{R}^4$ and $\pi_a \in \mathbb{CP}^1$. In these coordinates, the meromorphic $(3,0)$-form (which has two double poles at $\alpha_a, \beta_a \in \mathbb{CP}^1$) is given by[1]

$$\Omega = \frac{1}{2}\Phi(\pi)\, \epsilon_{\dot{a}\dot{b}}\, \pi_a \text{d}x^{a\dot{a}} \wedge \pi_b \text{d}x^{b\dot{b}} \wedge \langle \pi \text{d}\pi \rangle \,, \qquad \Phi = \frac{\langle \alpha\beta \rangle^2}{\langle \pi\alpha \rangle^2 \langle \pi\beta \rangle^2} \,. \tag{5.4}$$

The poles of Ω in $\mathbb{CP}^1$ play the role of boundaries in hCS_6 because total derivatives pick up a contribution from $\bar{\partial}\Omega$ which is a distribution with support at these poles. To ensure a well defined variational principal, we impose boundary conditions on the gauge field at these poles given by

$$\mathcal{A}|_{\pi=\alpha} = 0 \,, \qquad \mathcal{A}|_{\pi=\beta} = 0 \,. \tag{5.5}$$

Turning to the symmetries of this model, the theory is invariant under gauge transformations acting as

$$\hat{\gamma} : \quad \mathcal{A} \mapsto (\mathcal{A})^{\hat{\gamma}} = \hat{\gamma}^{-1}\mathcal{A}\hat{\gamma} + \hat{\gamma}^{-1}\bar{\partial}\hat{\gamma} \,, \tag{5.6}$$

[1] Spinor contractions are defined to be $\langle \alpha\beta \rangle = \epsilon^{ab}\alpha_a\beta_b$, see section 2.2.5 for further details of spinor conventions.

so long as they preserve the boundary conditions. This amounts to restrictions on the allowed transformations at the poles of Ω which are given by

$$\pi^a \partial_{a\dot{a}} \hat{\gamma}|_{\pi=\alpha} = 0\,, \quad \pi^a \partial_{a\dot{a}} \hat{\gamma}|_{\pi=\beta} = 0\,. \tag{5.7}$$

5.2.2 Localisation of hCS$_6$ with double poles to WZW$_4$

Surprisingly, all of the physical degrees of freedom in hCS$_6$ can be captured by a four-dimensional integrable field theory (IFT$_4$). This field theory is derived by localising the hCS$_6$ action, integrating out the $\mathbb{CP}^1$ and landing on a theory on $\mathbb{R}^4$. For the choice of meromorphic $(3,0)$-form Ω and boundary conditions given above, this 4d theory is WZW$_4$. This localisation is possible because of the substantial gauge symmetry in Chern-Simons theories. Indeed, the dynamical fields arise precisely where this gauge symmetry is broken, namely at the poles of Ω. Fields capturing these degrees of freedom are known as 'edge modes' which enter via the field redefinition

$$\mathcal{A} = (\mathcal{A}')^{\hat{g}} = \hat{g}^{-1}\mathcal{A}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g}\,. \tag{5.8}$$

Expressing the action $S_{\text{hCS}_6}[\mathcal{A}]$ in terms of the fields $\mathcal{A}'$ and $\hat{g}$ one obtains

$$\begin{aligned} S_{\text{hCS}_6}[\mathcal{A}] = S_{\text{hCS}_6}[\mathcal{A}'] + \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\mathcal{A}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1}\big) \\ - \frac{1}{6\pi \mathrm{i}} \int_{\mathbb{PT}\times[0,1]} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g}\big)\ , \end{aligned} \tag{5.9}$$

where, with a slight abuse of notation, we are also denoting by $\hat{g}$ a smooth homotopy to a constant map in the last term (such abuse will be perpetuated later without further comment). Notably, the edge mode $\hat{g}$ only appears in this action against the 4-form $\bar{\partial}\Omega$ which is a distribution with support at the poles of Ω. This means that the action only depends on $\hat{g}$ through its value (and $\mathbb{CP}^1$-derivative) at the poles of Ω which we will denote by

$$\hat{g}|_{\pi=\alpha} = g\ , \qquad \hat{g}^{-1}\partial_0\hat{g}|_{\pi=\alpha} = u\ , \qquad \hat{g}|_{\pi=\beta} = \tilde{g}\ , \qquad \hat{g}^{-1}\partial_0\hat{g}|_{\pi=\beta} = \tilde{u}\ . \tag{5.10}$$

Let us consider the symmetries of the theory in this new parameterisation. The gauge transformation (5.6) acts trivially on $\mathcal{A}'$ whilst $\hat{g}$ transforms with a right-action as

$$\hat{\gamma}: \quad \mathcal{A}' \mapsto \mathcal{A}'\,, \qquad \hat{g} \mapsto \hat{g}\hat{\gamma}\,. \tag{5.11}$$

In addition, the new parameterisation has introduced a redundancy (which we dub an internal gauge symmetry) acting as

$$\check{\gamma}: \quad \mathcal{A}' \mapsto \check{\gamma}^{-1}\mathcal{A}'\check{\gamma} + \check{\gamma}^{-1}\bar{\partial}\check{\gamma}\,, \qquad \hat{g} \mapsto \check{\gamma}^{-1}\hat{g}\,. \tag{5.12}$$

We can exploit these symmetries to impose gauge fixing conditions on the fields $\mathcal{A}'$ and $\hat{g}$. Let us fix $\mathcal{A}'$ such that it has no $\mathbb{CP}^1$-leg, and fix the value of $\hat{g}$ at $\pi = \beta$ to the identity.[2] The surviving edge mode at the other pole $g = \hat{g}|_{\pi=\alpha}$ will become the fundamental field of the WZW_4.

Returning to the action (5.9), the first term is a genuine six-dimensional bulk term which we eliminate by going on-shell. The bulk equation of motion imposes holomorphicity of $\mathcal{A}'$, which may be solved in terms of a series of $\mathbb{CP}^1$-independent components $A'_{a\dot{a}}$ as

$$\mathcal{A}' = \pi^a A'_{a\dot{a}} \bar{e}^{\dot{a}} \ , \qquad \bar{e}^{\dot{a}} = \frac{\hat{\pi}_a \mathrm{d}x^{a\dot{a}}}{\langle \pi\hat{\pi}\rangle} \, . \tag{5.13}$$

In this expression, $\bar{e}^{\dot{a}}$ is a basis $(0,1)$-form on twistor space defined in section 2.2.5. This completely specifies the $\mathbb{CP}^1$-dependence of $\mathcal{A}'$, and the boundary conditions eq. (5.5) may be solved to determine $A'_{a\dot{a}}$ in terms of g,

$$A'_{a\dot{a}} = -\frac{\beta_a \alpha^b}{\langle \alpha\beta\rangle} \partial_{b\dot{a}} g g^{-1} \, . \tag{5.14}$$

From these components, we can construct a 4d connection $A' = A'_{a\dot{a}} \mathrm{d}x^{a\dot{a}}$, and this parameterisation of A' in terms of g is known in the literature as Yang's parameterisation (g being called Yang's matrix). This solution for $\mathcal{A}'$ may now be substituted into the action and the integral over $\mathbb{CP}^1$ can be computed explicitly. The second and third term of (5.9) localise to a four-dimensional action, and the detailed manipulations are presented in appendix 5.10. We land on the WZW_4 theory defined by

$$S_{\mathrm{WZW}_4} = \frac{1}{2} \int_{\mathbb{R}^4} \mathrm{Tr}\big(g^{-1}\mathrm{d}g \wedge \star g^{-1} \mathrm{d}g\big) + \int_{\mathbb{R}^4 \times [0,1]} \omega_{\alpha,\beta} \wedge \mathcal{L}_{\mathrm{WZ}}[g] \ . \tag{5.15}$$

In the second term, we have introduced a 2-form defined by

$$\omega_{\alpha,\beta} = \frac{1}{\langle\alpha\beta\rangle} \alpha_a \beta_b \, \epsilon_{\dot{a}\dot{b}} \, \mathrm{d}x^{a\dot{a}} \wedge \mathrm{d}x^{b\dot{b}} \ , \tag{5.16}$$

and the WZ 3-form

$$\mathcal{L}_{\mathrm{WZ}}[g] = \frac{1}{3} \mathrm{Tr}\big(\tilde{g}^{-1}\mathrm{d}\tilde{g} \wedge \tilde{g}^{-1}\mathrm{d}\tilde{g} \wedge \tilde{g}^{-1}\mathrm{d}\tilde{g}\big), \tag{5.17}$$

defined, as is usual, using a suitable extension $\tilde{g}$ of g. The equation of motions of this theory are given by

$$0 = \mathrm{d}\left(\star - \omega_{\alpha,\beta}\wedge\right) \mathrm{d}g g^{-1} \quad \Leftrightarrow \quad \epsilon^{\dot{a}\dot{b}} \beta^a \partial_{a\dot{a}} \left(\alpha^b \partial_{b\dot{b}} g g^{-1}\right) = 0 \, . \tag{5.18}$$

The six-dimensional gauge transformations (constrained by boundary conditions) descend to semi-local symmetries of this action ($\gamma_L = \hat{\gamma}|_\beta$ and $\gamma_R = \hat{\gamma}|_\alpha$) which act as

$$g \to \gamma_L^{-1} \cdot g \cdot \gamma_R \, , \qquad \alpha^a \partial_{a\dot{a}} \gamma_R = 0 \, , \qquad \beta^a \partial_{a\dot{a}} \gamma_L = 0 \ . \tag{5.19}$$

[2] At this point, we may further fix the $\mathbb{CP}^1$-derivative of $\hat{g}$ at both $\pi = \alpha$ and $\pi = \beta$ to zero. However, such terms drop out of the action in this ungauged case anyway without specifying this fixing.

Of particular interest is the case where $\beta = \hat{\alpha}$ (i.e. the poles of Ω are antipodal on $\mathbb{CP}^1$) in which case $\omega_{\alpha,\hat{\alpha}} = \varpi$ is proportional to the Kähler form on $\mathbb{R}^4$. Here, we are referring to the Kähler form with respect to the complex structure $\mathcal{J}_\alpha$ which is defined[3] by the point $\alpha \in \mathbb{CP}^1$. In this case, the semi-local symmetries can be interpreted as a holomorphic left-action and anti-holomorphic right-action (akin to the two-dimensional WZW current algebra).

5.2.3 Interpretation as ADSYM

A 4d Yang-Mills connection A' with curvature $F[A'] = \mathrm{d}A' + A' \wedge A'$ is said to be anti-self dual if it obeys $F = -\star F$. After converting to bi-spinor notation, the anti-self-dual Yang-Mills (ASDYM) equations can be expressed as

$$\pi^a \pi^b F_{a\dot{a}b\dot{b}} = 0\,, \qquad \forall\, \pi_a \in \mathbb{CP}^1\,. \tag{5.20}$$

This contains three independent equations which can be extracted by introducing some basis spinors α_a and β_a satisfying $\langle\alpha\beta\rangle \neq 0$. The three independent equations are then expressed in terms of contractions with these basis spinors as

$$\alpha^a \alpha^b F_{a\dot{a}b\dot{b}} = 0\,, \tag{5.21}$$

$$\beta^a \beta^b F_{a\dot{a}b\dot{b}} = 0\,, \tag{5.22}$$

$$(\alpha^a \beta^b + \beta^a \alpha^b) F_{a\dot{a}b\dot{b}} = 0\,. \tag{5.23}$$

The six-dimensional origin of WZW_4 (and indeed all such constructed IFT_4) ensures that the connection A' introduced in the previous section satisfies the ASDYM equation when evaluated on solutions to the WZW_4 equation of motion. This follows from the six-dimensional equation $\Omega \wedge \mathcal{F}[\mathcal{A}'] = 0$ which encodes both the holomorphicity of $\mathcal{A}'$ and eq. (5.20). To see this explicitly for WZW_4 where the connection A' in given by eq. (5.14), we note that the β-contracted eq. (5.22) holds because $\langle\beta\beta\rangle = 0$, and the α-contracted eq. (5.21) holds due to the Maurer-Cartan identity. The remaining eq. (5.23) yields the equation of motion of WZW_4 (5.18).

5.2.4 Reduction of WZW_4 to WZW_2

Next, we will apply a two-dimensional reduction to WZW_4 specified by two vector fields V_i on $\mathbb{R}^4$ with $i = 1, 2$. The idea of reduction is to restrict to field configurations which are invariant under the flow of these vector fields. The two-dimensional dynamics of the reduced theory will be specified by the Lagrangian $\mathcal{L}_{\mathrm{IFT}_2} = (V_1 \wedge V_2) \lrcorner \mathcal{L}_{\mathrm{IFT}_4}$ where $\mathcal{L}_{\mathrm{IFT}_4}$ is the Lagrangian density of the parent theory and we denote the contraction of a vector field V with a differential form X by $V \lrcorner X$.

[3]Recall that $\mathbb{R}^4$ is a hyper-Kähler manifold which has a $\mathbb{CP}^1$'s worth of complex structures, see section 2.2.

Let us introduce a pair of unit norm spinors γ_a and $\kappa_{\dot a}$ and define the basis of 1-forms on $\mathbb{R}^4$

$$\mathrm{d}z = \gamma_a \kappa_{\dot a} \mathrm{d}x^{a\dot a} \ , \qquad \mathrm{d}\bar z = \hat\gamma_a \hat\kappa_{\dot a} \mathrm{d}x^{a\dot a} \ , \qquad \mathrm{d}w = \gamma_a \hat\kappa_{\dot a} \mathrm{d}x^{a\dot a} \ , \qquad \mathrm{d}\bar w = -\hat\gamma_a \kappa_{\dot a} \mathrm{d}x^{a\dot a} \ . \tag{5.24}$$

These are adapted to the complex structure $\mathcal{J}_\gamma$ defined by $\gamma_a \in \mathbb{CP}^1$. We choose to reduce along the vector fields dual to $\mathrm{d}z$ and $\mathrm{d}\bar z$ by demanding that $\partial_z g = \partial_{\bar z} g = 0$.[4] Then, contracting the WZW_4 Lagrangian with these vector fields results in the two-dimensional action of a principal chiral model (PCM) plus Wess-Zumino (WZ) term:

$$S_{\mathrm{PCM}+k\mathrm{WZ}_2}[g] = \frac{1}{2}\int_\Sigma \mathrm{Tr}\big(g^{-1}\mathrm{d}g \wedge \star g^{-1}\mathrm{d}g\big) \ + \ \frac{\mathrm{i}k}{3}\int_{\Sigma\times[0,1]} \mathrm{Tr}(\hat g^{-1}\mathrm{d}\hat g \wedge \hat g^{-1}\mathrm{d}\hat g \wedge \hat g^{-1}\mathrm{d}\hat g)\,. \tag{5.25}$$

In this action, the relative coefficient between the WZ-term and the PCM term is given by

$$k = \frac{\alpha+\beta}{\alpha-\beta}\,, \qquad \alpha = \frac{\langle\gamma\alpha\rangle}{\langle\alpha\hat\gamma\rangle}\,, \qquad \beta = \frac{\langle\gamma\beta\rangle}{\langle\beta\hat\gamma\rangle}\,. \tag{5.26}$$

Varying the basis spinor γ_a in these expressions changes the choice of reduction vector fields and interpolates between a family of two-dimensional theories. The WZW_2 CFT limit is obtained when $k \to 1$ with $\alpha\beta$ held fixed. This can be achieved by starting at the Kähler point in 4d, with $\beta_a = \hat\alpha_a$, and choosing the reduction to be aligned with the complex structure, i.e. setting $\gamma_a = \alpha_a$. An alternative reduction which turns off the WZ term is achieved by setting $\beta = -\alpha$.

For general choices of reduction, the four-dimensional semi-local symmetries descend to a global $G_L \times G_R$ symmetry; this is because, for example, the conditions $\alpha^a \partial_{a\dot a}\gamma_R = 0$ and $\partial_z \gamma_R = \partial_{\bar z}\gamma_R = 0$ generically contain four independent constraints leaving only constant solutions. However, when the reduction is taken to the CFT point, this system of four constraints is not linearly independent, and chiral symmetries emerge satisfying $\partial_w \gamma_R = 0$ (and vice versa for γ_L).

Lax connection. A virtue of this approach is that a $\mathfrak{g}^{\mathbb{C}}$-valued Lax connection for the dynamics of the resultant IFT_2 may be derived from the 4d connection A':

$$\begin{aligned} \mathscr{L}_{\bar w} &= \frac{1}{\langle\pi\hat\gamma\rangle}\hat\kappa^{\dot a}\pi^a(\partial_{a\dot a} + A'_{a\dot a}) = \partial_{\bar w} + \frac{(\beta-\zeta)}{(\alpha-\beta)}\partial_{\bar w} g g^{-1}\,, \\ \mathscr{L}_{w} &= \frac{1}{\langle\pi\gamma\rangle}\kappa^{\dot a}\pi^a(\partial_{a\dot a} + A'_{a\dot a}) = \partial_{w} + \frac{\alpha(\beta-\zeta)}{\zeta(\alpha-\beta)}\partial_{w} g g^{-1}\,, \end{aligned} \tag{5.27}$$

where the spectral parameter is given as $\zeta = \frac{\langle\gamma\pi\rangle}{\langle\pi\hat\gamma\rangle}$. Flatness of this connection for all values of ζ invokes the field equation of the PCM + WZ theory

$$\alpha\partial_{\bar w}(\partial_w g g^{-1}) - \beta\partial_w(\partial_{\bar w} g g^{-1}) = 0 \quad \Leftrightarrow \quad \mathrm{d}(\star - \mathrm{i}k)\mathrm{d}g g^{-1} = 0\,. \tag{5.28}$$

[4]In this case for reality we have $\mathcal{L}_{\mathrm{IFT}_2} = \mathrm{i}(\partial_z \wedge \partial_{\bar z}) \lrcorner \mathcal{L}_{\mathrm{IFT}_4}$.

Notice that in the CFT limit $k \to 1$ with $\beta \to \infty$, $\alpha \to 0$ the Lax connection becomes chiral and spectral parameter independent.

5.2.5 Reduction of hCS_6 to CS_4

Instead of first integrating over $\mathbb{CP}^1$ and then reducing to two dimensions, one could instead directly apply the reduction to hCS_6. This produces CS_4 with action

$$S_{\mathrm{CS}_4}[A] = \frac{1}{2\pi \mathrm{i}} \int_{\Sigma\times\mathbb{CP}^1} \omega \wedge \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right) . \tag{5.29}$$

Here Σ is the $\mathbb{R}^2 \subset \mathbb{R}^4$ with coordinates by $w, \bar{w}$, and the meromorphic 1-form ω is given by

$$\omega = \mathrm{i}(\partial_z \wedge \partial_{\bar{z}}) \lrcorner \, \Omega \, . \tag{5.30}$$

A crucial feature here is that this contraction introduces zeroes in ω to complement its poles, as required by the Riemann-Roch theorem. For the case at hand, ω is given explicitly by

$$\omega = \mathrm{i} \frac{\langle \alpha\beta\rangle^2 \langle \pi\gamma\rangle\langle\pi\hat{\gamma}\rangle}{\langle\pi\alpha\rangle^2\langle\pi\beta\rangle^2} \langle \pi \mathrm{d}\pi\rangle \, , \tag{5.31}$$

and the zeros are introduced at the points $\pi_a = \gamma_a, \hat{\gamma}_a$. The details of the reduction show that, whilst our six-dimensional gauge field was regular, the connection A entering in CS_4 develops poles at the zeros of ω. In particular, the component A_w will have a simple pole at $\pi_a = \gamma_a$ and $A_{\bar{w}}$ will have a simple pole at $\pi_a = \hat{\gamma}_a$. The four-dimensional Chern-Simons connection is subject to the same boundary conditions as its parent, namely it vanishes at the points α and β in $\mathbb{CP}^1$. Subsequent localisation of CS_4 then gives the same PCM+WZ theory derived by reducing WZW_4.

5.3 The gauged WZW diamond

We now come to the main results of this work. In this section, we will construct a diamond correspondence of theories which realises the gauged WZW_2 model, i.e. the G/H coset CFT.

5.3.1 Gauged WZW Models

First let us review the gauging of the WZW model and the crucial Polyakov-Wiegmann identity. Letting G be a Lie group and $g \in C^\infty(\Sigma, G)$ a smooth G-valued field, the WZW_2 action is[5]

$$S_{\mathrm{WZW}_2}[g] = \frac{1}{2}\int_\Sigma \mathrm{Tr}_{\mathfrak{g}}\big(g^{-1}\mathrm{d}g \wedge \star g^{-1}\mathrm{d}g\big) \ + \ \frac{1}{3}\int_{\Sigma\times[0,1]} \mathrm{Tr}_{\mathfrak{g}}(\hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g}) \, . \tag{5.32}$$

[5]To minimise factors of imaginary units we momentarily adopt Lorentzian signature. Schematically, we have $S_{\mathrm{Lorentz}} = \mathrm{i}S_{\mathrm{Euclid}}|_{\star\to\mathrm{i}\star}$.

Gauging a vectorial H-action of the principal chiral model term is straightforward. We introduce an $\mathfrak{h}$-valued connection $B \in \Omega^1(\Sigma) \otimes \mathfrak{h}$ transforming as

$$\ell \in C^\infty(\Sigma, H): \quad B \mapsto \ell^{-1} B \ell + \ell^{-1} \mathrm{d}\ell \,, \quad g \mapsto \ell^{-1} g \ell \,, \tag{5.33}$$

with field strength $F[B] = \mathrm{d}B + B \wedge B$. The principal chiral term is then gauged by replacing the exterior derivatives with covariant derivatives $\mathrm{d}g \to \mathrm{D}g = \mathrm{d}g + [B, g]$. Less trivially, the gauge completion of the WZ 3-form is [Wit92; FS94a; FS94b; FM05]

$$\mathcal{L}_{\mathrm{gWZ}}[g, B] = \mathcal{L}_{\mathrm{WZ}}[g] + \mathrm{d}\,\mathrm{Tr}_{\mathfrak{g}}(g^{-1}\mathrm{d}g \wedge B + \mathrm{d}g g^{-1} \wedge B + g^{-1} B g \wedge B) \,. \tag{5.34}$$

Adding these two pieces together gives the gauged WZW_2 action,

$$S_{\mathrm{gWZW}_2}[g, B] = S_{\mathrm{WZW}_2}[g] + \int_\Sigma \mathrm{Tr}_{\mathfrak{g}}(g^{-1}\mathrm{d}g \wedge (1-\star) B + \mathrm{d}g g^{-1} \wedge (1+\star) B + B \wedge \star B + g^{-1} B g \wedge (1-\star) B) \,. \tag{5.35}$$

Notice that chiral couplings between currents and gauge fields emerge from combinations of the PCM and WZ contributions. A remarkable feature,

$$\mathcal{L}_{\mathrm{WZ}}[g_1 g_2] = \mathcal{L}_{\mathrm{WZ}}[g_1] + \mathcal{L}_{\mathrm{WZ}}[g_2] + \mathrm{d}\,\mathrm{Tr}_{\mathfrak{g}}\left(\mathrm{d}g_2 g_2^{-1} \wedge g_1^{-1} \mathrm{d}g_1\right) \,, \tag{5.36}$$

ensures that (5.35) can instead be cast as a difference of two WZW_2 models. To see this we choose a parameterisation of the gauge field B in terms of two smooth H-valued fields

$$B = \frac{1+\star}{2} a^{-1}\mathrm{d}a + \frac{1-\star}{2} b^{-1}\mathrm{d}b \,, \quad a, b \in C^\infty(\Sigma, H) \,. \tag{5.37}$$

In two dimensions, this is not a restriction on the field content of the gauge field, but simply a way of parameterising the two independent components of B. With such a parameterisation, if we then further define $\tilde{g} = a g b^{-1} \in C^\infty(\Sigma, G)$ and $\tilde{h} = a b^{-1} \in C^\infty(\Sigma, H)$ the gauged model (5.35) can be written as the difference of two WZW_2 models:

$$S_{\mathrm{gWZW}_2}[g, B] = S_{\mathrm{WZW}_2}[\tilde{g}] - S_{\mathrm{WZW}_2}[\tilde{h}] \,. \tag{5.38}$$

This is known as the Polyakov-Wiegmann (PW) identity [PW83].

5.3.2 Gauging of the WZW_4 model

Let us now consider the four-dimensional WZW_4 model, given by eq. (5.15). The gauging procedure follows in the exact same manner, producing an analogous gauged WZW_4 action,

$$S^{(\alpha,\beta)}_{\mathrm{gWZW}_4}[g, B] = \frac{1}{2} \int_{\mathbb{R}^4} \mathrm{Tr}(g^{-1}\nabla g \wedge \star g^{-1} \nabla g) + \int_{\mathbb{R}^4 \times [0,1]} \omega_{\alpha\beta} \wedge \mathcal{L}_{\mathrm{gWZ}}[g, B] \,. \tag{5.39}$$

Here, we denote the covariant derivative by $\nabla g = \mathrm{d}g + [B, g]$. A critical difference between two and four dimensions is the applicability of the PW identity, as was pointed out by [Los+96]. In two dimensions, this mapping relies on the relation (5.37). To extend it to four dimensions, we consider the operator on 1-forms

$$\mathcal{J}_{\alpha,\beta}(\sigma) = -\mathrm{i} \star (\omega_{\alpha,\beta} \wedge \sigma) \,. \tag{5.40}$$

One may check that $\mathcal{J}^2_{\alpha,\beta} = -\mathrm{id}$, so that we can introduce useful projectors

$$P = \frac{1}{2}\left(\mathrm{id} - \mathrm{i}\mathcal{J}\right) \qquad \bar{P} = \frac{1}{2}\left(\mathrm{id} + \mathrm{i}\mathcal{J}\right) \;, \tag{5.41}$$

which furnish a range of identities detailed in appendix 5.9. With these in mind, we can write a four-dimensional analogue to (5.37),

$$B = P\left(a^{-1}\mathrm{d}a\right) + \bar{P}\left(b^{-1}\mathrm{d}b\right) \qquad a, b \in C^\infty(\mathbb{R}^4, H) \,. \tag{5.42}$$

With this parameterisation of the gauge field, it is indeed possible to use the composite fields $\tilde{g} = agb^{-1} \in C^\infty(\mathbb{R}^4, G)$ and $\tilde{h} = ab^{-1} \in C^\infty(\mathbb{R}^4, H)$ to express the gauged WZW_4 action in a fashion akin to eq. (5.38) as

$$S^{(\alpha,\beta)}_{\mathrm{gWZW}_4}[g, B] = S^{(\alpha,\beta)}_{\mathrm{WZW}_4}[\tilde{g}] - S^{(\alpha,\beta)}_{\mathrm{WZW}_4}[\tilde{h}] \,. \tag{5.43}$$

However, unlike in two dimensions, this parameterisation of the gauge field eq. (5.42) is not generic. It implies a restriction on the connection, namely that its curvature satisfies

$$\alpha^a \alpha^b F_{a\dot{a}b\dot{b}}[B] = 0 \,, \quad \beta^a \beta^b F_{a\dot{a}b\dot{b}}[B] = 0 \,. \tag{5.44}$$

This can be thought of as analogue to imposing that F be strictly a $(1,1)$-form (which indeed this becomes when $\beta = \hat{\alpha}$ and the WZW_4 is taken at the Kähler point). It is noteworthy that these constraints on the background gauge field agree with two of the three ASDYM equations; the same two equations that were identically satisfied by the Yang parameterisation of the connection A'. In the forthcoming analysis, we will see how this arises from the hCS_6 construction.

5.3.3 A six-dimensional origin

We now turn to the six-dimensional holomorphic Chern-Simons theory on twistor space that will descend to the above gauged WZW models in two and four dimensions. Given the factorisation of gWZW_2 to the difference of WZW_2 models, a natural candidate here is to consider simply the difference of hCS_6 theories to generalize the six-dimensional action introduced in [Cos20; BS23; Pen21; Cos21]. Indeed, a similar idea was proposed by [Ste21] in the construction of 2d coset models from the difference of CS_4 theories. However, how this should work in six dimensions is less clear as the factorisation of gWZW$_4$ requires the curvature of the gauge field to be constrained.

The fundamental fields of our theory are two connections $\mathcal{A} \in \Omega^{0,1}(\mathbb{PT})\otimes\mathfrak{g}$ and $\mathcal{B} \in \Omega^{0,1}(\mathbb{PT})\otimes\mathfrak{h}$, which appear in the six-dimensional action

$$S_{\mathrm{ghCS}_6}[\mathcal{A},\mathcal{B}] = S_{\mathrm{hCS}_6}[\mathcal{A}] - S_{\mathrm{hCS}_6}[\mathcal{B}] - \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\mathcal{A}\wedge\mathcal{B}\big)\,, \tag{5.45}$$

where the functional S_{hCS_6} is defined in eq. (5.3). As well as the bulk hCS$_6$ functionals, we have also included a coupling term between the two connections which contributes on the support of $\bar{\partial}\Omega$, i.e. at the poles of Ω. We will shortly provide a motivation for this boundary term related to the boundary conditions we will impose on the theory.

This definition is slightly imprecise; strictly speaking, the inner product denoted by 'Tr' should be defined separately for each algebra, i.e. $\mathrm{Tr}_{\mathfrak{g}}$ and $\mathrm{Tr}_{\mathfrak{h}}$. In the coupling term, where $\mathcal{B}$ enters inside $\mathrm{Tr}_{\mathfrak{g}}$, we should first act on $\mathcal{B}$ with some Lie algebra homomorphism from $\mathfrak{h}$ to $\mathfrak{g}$, and in principle this homomorphism could be chosen differently at each pole of Ω. We discuss more general gaugings, beyond the vectorial gauging hereby considered, in appendix 5.11.

To complete the specification of the theory, we must supply boundary conditions which ensure the vanishing of the boundary term in the variation of (5.45),

$$\delta S_{\mathrm{ghCS}_6}\big|_{\mathrm{bdry}} = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\big((\delta\mathcal{A}+\delta\mathcal{B})\wedge(\mathcal{A}-\mathcal{B})\big)\,. \tag{5.46}$$

Since $\bar{\partial}\Omega$ only has support at the poles of Ω, the integral over $\mathbb{CP}^1$ may be computed explicitly in this term. As well as contributions proportional to delta-functions on $\mathbb{CP}^1$, this will also include $\mathbb{CP}^1$-derivatives of delta-functions since the poles in Ω are second order. Using the localisation formula in the appendix 5.10, we find

$$\begin{aligned}\delta S_{\mathrm{ghCS}_6}\big|_{\mathrm{bdry}} = -\int_{\mathbb{R}^4}\Big[&\frac{\alpha_a\beta_b\Sigma^{ab}}{\langle\alpha\beta\rangle}\wedge\mathrm{Tr}\big((\delta\mathcal{A}+\delta\mathcal{B})\wedge(\mathcal{A}-\mathcal{B})\big)\\ &+\frac{1}{2}\alpha_a\alpha_b\Sigma^{ab}\wedge\partial_0\mathrm{Tr}\big((\delta\mathcal{A}+\delta\mathcal{B})\wedge(\mathcal{A}-\mathcal{B})\big)\Big]+\alpha\leftrightarrow\beta\,.\end{aligned} \tag{5.47}$$

In this expression, we introduce a basis for the self-dual 2-forms defined by $\Sigma^{ab} = \varepsilon_{\dot{a}\dot{b}}\mathrm{d}x^{a\dot{a}}\wedge\mathrm{d}x^{b\dot{b}}$. Let us also introduce an orthogonal decomposition of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{k}\,,\qquad \mathrm{Tr}\big(X\cdot Y\big) = \mathrm{Tr}\big(X^{\mathfrak{h}}\cdot Y^{\mathfrak{h}}\big) + \mathrm{Tr}\big(X^{\mathfrak{k}}\cdot Y^{\mathfrak{k}}\big)\,. \tag{5.48}$$

To attain the vanishing of the boundary variation, we consider the boundary conditions

$$\mathcal{A}^{\mathfrak{k}}\big|_{\alpha,\beta} = 0\,,\qquad \mathcal{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \mathcal{B}\big|_{\alpha,\beta}\,,\qquad \partial_0\mathcal{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \partial_0\mathcal{B}\big|_{\alpha,\beta}\,. \tag{5.49}$$

This completes our definition of the gauged hCS$_6$ theory.

One might choose to think of the boundary term in the variation as being a potential for a 'symplectic' form[6] [7]

$$\Theta = \delta S_{\text{ghCS}_6}\big|_{\text{bdry}} \ , \qquad \Omega = \delta\Theta = -\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \Big(\mathrm{Tr}_{\mathfrak{g}}\big(\delta\mathcal{A}\wedge\delta\mathcal{A}\big) - \mathrm{Tr}_{\mathfrak{h}}\big(\delta\mathcal{B}\wedge\delta\mathcal{B}\big)\Big) \ , \tag{5.50}$$

such that our boundary conditions define a Lagrangian (i.e. maximal isotropic) subspace. We should like to really interpret this as a symplectic form on an appropriate space of fields defined over $\mathbb{R}^4$. Evaluating the integral over $\mathbb{CP}^1$ and writing $\Omega = \Omega_{\mathcal{A}} - \Omega_{\mathcal{B}}$, this symplectic form is given by

$$\Omega_{\mathcal{A}} = \int_{\mathbb{R}^4}\left[\frac{\alpha_a\beta_b\Sigma^{ab}}{\langle\alpha\beta\rangle}\wedge \mathrm{Tr}_{\mathfrak{g}}\big(\delta\mathcal{A}\wedge\delta\mathcal{A}\big)\big|_{\alpha} + \frac{1}{2}\alpha_a\alpha_b\Sigma^{ab}\wedge\partial_0\mathrm{Tr}_{\mathfrak{g}}\big(\delta\mathcal{A}\wedge\delta\mathcal{A}\big)\big|_{\alpha}\right] + \alpha\leftrightarrow\beta \ , \tag{5.51}$$

with an analogous expression for $\Omega_{\mathcal{B}}$. Because our boundary conditions are identical at each pole, we concentrate now only on the contributions associated to the pole at α. The symplectic form is not sensitive to the entire field configuration $\mathcal{A}\in\Omega^1(\mathbb{PT})\otimes\mathfrak{g}$, but rather to the evaluation of $\mathcal{A}$ at the poles and its first derivatives,

$$\vec{\mathcal{A}} = \big(\mathcal{A}|_{\alpha}, \partial_0\mathcal{A}|_{\alpha}\big) \ . \tag{5.52}$$

This data may be interpreted as defining a 1-form (more precisely a $(0,1)$-form with respect to the complex structure defined by α) on $\mathbb{R}^4$ valued in the algebra[8] $\vec{\mathfrak{g}} = \mathfrak{g}\ltimes\mathbb{R}^{\dim(G)}$. With this in mind, it is more accurate to say that the contribution from the pole at $\pi_a = \alpha_a$ in Ω is a symplectic form on the space of configurations

$$\big(\vec{\mathcal{A}},\vec{\mathcal{B}}\big)\in\Omega^{0,1}(\mathbb{R}^4)\otimes\big(\vec{\mathfrak{g}}\oplus\vec{\mathfrak{h}}\big) \ . \tag{5.53}$$

This symplectic form may be succinctly written by introducing an inner product on the algebra $\vec{\mathfrak{g}}\oplus\vec{\mathfrak{h}}$, and our boundary conditions describe an isotropic subspace with respect to this inner product.[9]

To be explicit we associate $\mathbb{R}^{\dim G}$ with the dual $\mathfrak{g}^*$ and denote the natural pairing of the algebra and its dual with $\langle\bullet,\bullet\rangle$. We let $\vec{X} = (x,\tilde{x})\in\vec{\mathfrak{g}}$ such that bracket on $\vec{\mathfrak{g}}$ is defined by

$$[\vec{X},\vec{Y}]_{\vec{\mathfrak{g}}} = ([x,y], \mathrm{ad}^*_x\tilde{y} - \mathrm{ad}^*_y\tilde{x}) \ , \tag{5.54}$$

[6] Precedent in the literature dictates that we denote the symplectic form as Ω; we trust that context serves to disambiguate from the meromorphic differential Ω.

[7] This is slightly loose as the 2-form is degenerate, so properly speaking we should restrict to symplectic leaves.

[8] The dimension of $\vec{\mathfrak{g}}$ is $2\dim(G)$, so it must be isomorphic to $\mathbb{R}^{\dim(G)}\oplus\mathbb{R}^{\dim(G)}$ as a vector space. The Lie algebra structure may be derived by considering consecutive infinitesimal gauge transformations. In the CS_4 literature, these structures have been studied under the name 'defect Lie algebra' [BSV22; LV21].

[9] This need not be the case, as our boundary conditions could generically intertwine constraints on the algebra and spacetime components, meaning they could not be captured by a subspace of the algebra alone. They would always, however, define an isotropic subspace of $\Omega^{0,1}(\mathbb{R}^4)\otimes\big(\vec{\mathfrak{g}}\oplus\vec{\mathfrak{h}}\big)$ by definition. Examples of this more general type of boundary condition can be found in [Col+24b].

where the co-adjoint action is $\langle x, \mathrm{ad}^*_y \tilde{x}\rangle = \langle [x,y], \tilde{x}\rangle$. We equip $\vec{\mathfrak{g}}$ with the inner product

$$\langle \vec{X}, \vec{Y}\rangle_{\vec{\mathfrak{g}}} = \frac{\langle \beta\hat{\alpha}\rangle}{\langle \alpha\beta\rangle\langle\alpha\hat{\alpha}\rangle} \mathrm{Tr}_{\mathfrak{g}}(x\cdot y) + \frac{1}{2}\Big(\langle x, \tilde{y}\rangle + \langle y, \tilde{x}\rangle\Big)\,, \tag{5.55}$$

such that the relevant contribution to the symplectic form is given by

$$\Omega_{\mathcal{A}} = \int_{\mathbb{R}^4} \mu_\alpha \wedge \langle \delta\vec{\mathcal{A}}, \delta\vec{\mathcal{A}}\rangle_{\vec{\mathfrak{g}}}\ . \tag{5.56}$$

where $\mu^\alpha = \alpha_a\alpha_b\Sigma^{ab}$ is the $(2,0)$-form defined by the complex structure associated to $\alpha \in \mathbb{CP}^1$.

In a similar fashion we will let $\vec{U} = (u, \tilde{u})$ and $\vec{V} = (v, \tilde{v})$ be elements of $\vec{\mathfrak{h}}$ which is equipped with a bracket and pairing via the same recipe. We consider the commuting direct sum $\vec{\mathfrak{g}} \oplus \vec{\mathfrak{h}}$ equipped with pairing and bracket

$$\langle\!\langle (\vec{X}, \vec{U}), (\vec{Y}, \vec{V})\rangle\!\rangle = \langle \vec{X}, \vec{Y}\rangle_{\vec{\mathfrak{g}}} - \langle \vec{U}, \vec{V}\rangle_{\vec{\mathfrak{h}}}\,, \qquad [\![(\vec{X}, \vec{U}), (\vec{Y}, \vec{V})]\!] = \big([\vec{X}, \vec{Y}]_{\vec{\mathfrak{g}}}\,, [\vec{U}, \vec{V}]_{\vec{\mathfrak{h}}}\big)\,, \tag{5.57}$$

such that the total symplectic form coming from the pole at α is just

$$\Omega = \int_{\mathbb{R}^4} \mu_\alpha \wedge \langle\!\langle (\delta\vec{\mathcal{A}}, \delta\vec{\mathcal{B}}), (\delta\vec{\mathcal{A}}, \delta\vec{\mathcal{B}})\rangle\!\rangle\ . \tag{5.58}$$

Then, our boundary conditions can be expressed as $\big(\vec{\mathcal{A}}, \vec{\mathcal{B}}\big) \in \Omega^{0,1}(\mathbb{R}^4)\otimes L$ where we introduce a subspace

$$L = \big\{(\vec{X}, \vec{U}) \in \vec{\mathfrak{g}} \oplus \vec{\mathfrak{h}} \mid x = u\,,\ P^*_{\mathfrak{h}}\tilde{x} = \tilde{u}\big\}\ , \tag{5.59}$$

in which $P^*_{\mathfrak{h}}$ is the dual to the projector $P_{\mathfrak{h}}$ into the subgroup i.e. $\langle x, P^*_{\mathfrak{h}}\tilde{x}\rangle = \langle P_{\mathfrak{h}}x, \tilde{x}\rangle$. As L is defined by $\dim\mathfrak{g} + \dim\mathfrak{h}$ constraints, it is half-dimensional and it is also isotropic with respect to $\langle\!\langle \cdot, \cdot\rangle\!\rangle$, hence defining a Lagrangian subspace. Moreover, assuming that G/H is reductive, L is a subalgebra[10]. Pre-empting the following section, this analysis indicates that there will be a residual $\vec{\mathfrak{h}}$ gauge symmetry associated to the pole at α, and similarly at β.

We can make one further observation[11] of the role of the boundary contribution from a symplectic perspective that is best illustrated by a finite-dimensional analogy. Recall that the cotangent bundle $\mathcal{M} = T^*X$ is a symplectic manifold; if we let $\{x^i\}$ be local coordinates on X and $\{\xi_i\}$ the components of a 1-form $\xi = \xi_i dx^i \in T^*_x X$, then $p = (x^i, \xi_i)$ provide local coordinates for $\mathcal{M}$ in terms of which the canonical symplectic form is $\Omega = \mathrm{d}\xi_i \wedge \mathrm{d}x^i$. The tautological potential (which admits a coordinate free definition in terms of the projection $\pi : T^*X \to X$) for this is given by $\Theta = \xi_i \mathrm{d}x^i$. The zero section, i.e. points $p = (x^i, \xi_i = 0)$ of T^*X is a Lagrangian and notice that Θ vanishes trivially here. Now Weinstein's tubular neighbourhood theorem ensures that in the vicinity of a Lagrangian L, any symplectic manifold $\mathcal{M}$ locally looks like T^*L with L

[10] If $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$ is not assumed to be reductive then the stabiliser of L consists of elements of the form

$$\mathrm{stab}_L = \Big\{(\vec{X}, \vec{U}) \in \vec{\mathfrak{g}} \oplus \vec{\mathfrak{h}} \mid x = u\,,\ P^*_{\mathfrak{h}}\tilde{x} = \tilde{u}\,,\ [u, \mathfrak{k}] = 0\,,\ ([\mathfrak{h}, \mathfrak{k}], \tilde{u}) = 0\Big\}\,.$$

[11] We thank A. Arvanitakis for this suggestion.

given by the zero section. In the case at hand, our boundary conditions are of the schematic form $\xi = \mathcal{A} - \mathcal{B} = 0$, and the effect of including the specific boundary contribution to the Lagrangian ensures that the resultant symplectic potential is the tautological one.

To close this section, let us comment that at the special point for which $\alpha_a = \hat{\beta}_a$, one of the terms in the inner product eq. (5.55) vanishes. This allows for a larger class of admissible boundary conditions, even for the ungauged model, including the examples

$$\mathcal{A}\big|_{\hat{\alpha}} = 0 \;, \qquad \partial_0\mathcal{A}\big|_{\alpha} = 0 \qquad \text{or} \qquad \partial_0\mathcal{A}\big|_{\hat{\alpha}} = 0 \;, \qquad \partial_0\mathcal{A}\big|_{\alpha} = 0 \,. \tag{5.60}$$

We leave these for future development.

5.3.4 Localisation to gWZW$_4$

The localisation procedure follows in a similar fashion to the ungauged model. However, given that there are now two gauge fields $\mathcal{A}$ and $\mathcal{B}$, some care is required to account for degrees of freedom and residual symmetries.

We introduce a new pair of connections $\mathcal{A}' \in \Omega^{0,1}(\mathbb{PT}) \otimes \mathfrak{g}$ and $\mathcal{B}' \in \Omega^{0,1}(\mathbb{PT}) \otimes \mathfrak{h}$, along with group valued fields $\hat{g} \in C^\infty(\mathbb{PT}, G)$ and $\hat{h} \in C^\infty(\mathbb{PT}, H)$ related to the original gauge fields by

$$\begin{aligned} \mathcal{A} &= \hat{g}^{-1}\mathcal{A}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g} \equiv \mathcal{A}'^{\hat{g}} \;, \\ \mathcal{B} &= \hat{h}^{-1}\mathcal{B}'\hat{h} + \hat{h}^{-1}\bar{\partial}\hat{h} \equiv \mathcal{B}'^{\hat{h}} \;. \end{aligned} \tag{5.61}$$

The redundancy in this parameterisation is given by the action of $\check{\gamma} \in C^\infty(\mathbb{PT}, G)$ and $\check{\eta} \in C^\infty(\mathbb{PT}, H)$:

$$\mathcal{A}' \mapsto \check{\gamma}^{-1}\mathcal{A}'\check{\gamma} + \check{\gamma}^{-1}\bar{\partial}\check{\gamma} \,, \qquad \hat{g} \mapsto \check{\gamma}^{-1}\hat{g} \,, \tag{5.62}$$

$$\mathcal{B}' \mapsto \check{\eta}^{-1}\mathcal{B}'\check{\eta} + \check{\eta}^{-1}\bar{\partial}\check{\eta} \,, \qquad \hat{h} \mapsto \check{\eta}^{-1}\hat{h} \,, \tag{5.63}$$

which leave $\mathcal{A}$ and $\mathcal{B}$ invariant. As before, this is partially used to fix away the $\mathbb{CP}^1$ legs

$$\mathcal{A}'_0 = \mathcal{B}'_0 = 0 \,. \tag{5.64}$$

The localisation procedure will produce a four-dimensional boundary theory with fields given by the evaluations of $\hat{g}, \hat{h}$ and their $\mathbb{CP}^1$-derivatives at the poles α and β of Ω. Since the $\mathbb{CP}^1$-derivatives will have an important role, we give them names,

$$\hat{u} = \hat{g}^{-1}\partial_0\hat{g} \,, \quad \hat{v} = \hat{h}^{-1}\partial_0\hat{h} \,. \tag{5.65}$$

After fixing (5.64), we note that there is still some remaining symmetry given by internal gauge transformations (5.62) and (5.63) which are $\mathbb{CP}^1$-independent. We use this residual symmetry to fix the values

$$\hat{g}|_\beta = \mathrm{id}\,, \quad \hat{h}|_\beta = \mathrm{id}\,. \tag{5.66}$$

On the other hand, the action (5.45) is invariant under gauge transformations acting on $\mathcal{A}$ and $\mathcal{B}$ which preserve the boundary conditions (5.49). These are given by smooth maps $\hat{\gamma} \in C^\infty(\mathbb{PT}, G)$ and $\hat{\eta} \in C^\infty(\mathbb{PT}, H)$ satisfying[12]

$$\hat{\gamma}|_{\alpha,\beta} = \hat{\eta}|_{\alpha,\beta}\,, \quad \partial_0\hat{\gamma}|_{\alpha,\beta} = \partial_0\hat{\eta}|_{\alpha,\beta}\,. \tag{5.67}$$

The induced action of these gauge transformations on the new field content is

$$\mathcal{A}' \mapsto \mathcal{A}'\,, \qquad \hat{g} \mapsto \hat{g}\hat{\gamma}\,, \qquad \hat{u} \mapsto \hat{\gamma}^{-1}\hat{u}\hat{\gamma} + \hat{\gamma}^{-1}\partial_0\hat{\gamma} \tag{5.68}$$

$$\mathcal{B}' \mapsto \mathcal{B}'\,, \qquad \hat{h} \mapsto \hat{h}\hat{\eta}\,, \qquad \hat{v} \mapsto \hat{\eta}^{-1}\hat{v}\hat{\eta} + \hat{\eta}^{-1}\partial_0\hat{\eta}\,. \tag{5.69}$$

We want to use this symmetry to further fix degrees of freedom. Note that whereas the right action on the fields $\hat{g}$ and $\hat{h}$ at α is entirely unconstrained, we would like the action at β to preserve the gauge fixing condition (5.66). This is achieved by performing both an internal and external gauge transformation simultaneously, and requiring $\hat{\gamma}|_\beta = \check{\gamma}$ and $\hat{\eta}|_\beta = \check{\eta}$. This results in an induced left action on the fields $\hat{g}$ and $\hat{h}$ at α. In summary, introducing some notation for simplicity, we have our boundary degrees of freedom

$$\hat{g}|_\alpha \coloneqq g\,, \qquad \hat{g}|_\beta = \mathrm{id}\,, \qquad \hat{u}|_\alpha \coloneqq u\,, \qquad \hat{u}|_\beta \coloneqq \tilde{u} \tag{5.70}$$

$$\hat{h}|_\alpha \coloneqq h\,, \qquad \hat{h}|_\beta = \mathrm{id}\,, \qquad \hat{v}|_\alpha \coloneqq v\,, \qquad \hat{v}|_\beta \coloneqq \tilde{v}\,, \tag{5.71}$$

and boundary gauge transformations

$$\hat{\gamma}|_\alpha = \hat{\eta}|_\alpha \coloneqq r\,, \qquad \hat{\gamma}^{-1}\partial_0\hat{\gamma}|_\alpha = \hat{\eta}^{-1}\partial_0\hat{\eta}|_\alpha \coloneqq \epsilon\,, \tag{5.72}$$

$$\hat{\gamma}|_\beta = \hat{\eta}|_\beta \coloneqq \ell^{-1}\,, \qquad \hat{\gamma}^{-1}\partial_0\hat{\gamma}|_\beta = \hat{\eta}^{-1}\partial_0\hat{\eta}|_\beta \coloneqq \tilde{\epsilon}\,, \tag{5.73}$$

which act on the boundary fields as

$$g \mapsto \ell g r\,, \quad u \mapsto r^{-1}ur + \epsilon\,, \quad \tilde{u} \mapsto \ell\tilde{u}\ell^{-1} + \tilde{\epsilon} \tag{5.74}$$

$$h \mapsto \ell h r\,, \quad v \mapsto r^{-1}vr + \epsilon\,, \quad \tilde{v} \mapsto \ell\tilde{v}\ell^{-1} + \tilde{\epsilon}\,. \tag{5.75}$$

with $\ell, r \in C^\infty(\mathbb{R}^4, H)$ and $\epsilon, \tilde{\epsilon} \in C^\infty(\mathbb{R}^4, \mathfrak{h})$. Based on our expectation of a gauge theory containing a G-valued field and a vectorial H-gauge symmetry, we use the above symmetries to fix

$$h = \mathrm{id}\,, \qquad v = \tilde{v} = 0\,. \tag{5.76}$$

[12] This requires that G/H is reductive meaning $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}$.

We are thus left with a residual symmetry $r = \ell^{-1}$ acting as

$$g \mapsto \ell g \ell^{-1}\,, \quad u \mapsto \ell u \ell^{-1}\,, \quad \tilde{u} \mapsto \ell \tilde{u} \ell^{-1}\,, \quad B \mapsto \ell B \ell^{-1} - \mathrm{d}\ell \ell^{-1}\,, \tag{5.77}$$

which will become the gauge symmetry of our 4d theory.

We now proceed with the localisation of the six-dimensional action. As with the ungauged model, the first step is to write the action in terms of $\mathcal{A}', \mathcal{B}'$ and $\hat{g}, \hat{h}$. Given that the localisation formula (5.231) introduces at most one ∂_0 derivative, all dependence on $\hat{h}$ will drop due to our gauge fixing choices (5.71) and (5.76). Hence, there will be no contribution from $S_{\mathrm{hCS}_6}[\mathcal{B}]$ to the four-dimensional action. As per eq. (5.9) we find that the bulk equations (i.e. contributions to the variation of the action that are not localised to the poles of Ω) enforce $\bar{\partial}_0 \mathcal{A}'_{\dot{a}} = \bar{\partial}_0 \mathcal{B}'_{\dot{a}} = 0$. This implies that the components $\mathcal{A}'_{\dot{a}}, \mathcal{B}'_{\dot{a}}$ are holomorphic, which (combined with the fact that they have homogeneous weight 1) allows us to deduce that

$$\mathcal{A}'_{\dot{a}} = \pi^a A'_{a\dot{a}}\,, \quad \mathcal{B}'_{\dot{a}} = \pi^a B'_{a\dot{a}}\,, \tag{5.78}$$

in which $A'_{a\dot{a}}, B'_{a\dot{a}}$ are $\mathbb{CP}^1$-independent. Imposing this bulk equation, and the gauge fixings described above, the remaining contributions in (5.45) are given by

$$\begin{aligned} S_{\mathrm{ghCS}_6}[\mathcal{A},\mathcal{B}] = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} & \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\mathcal{A}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1} - (\hat{g}^{-1}\mathcal{A}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g}) \wedge \mathcal{B}'\big) \\ & - \frac{1}{6\pi \mathrm{i}} \int_{\mathbb{PT}\times[0,1]} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g}\big)\,. \end{aligned} \tag{5.79}$$

In the ungauged model, the next step was to solve the boundary conditions for $\mathcal{A}'$ in terms of $\hat{g}$. Here, the boundary conditions do not fully determine $A'_{a\dot{a}}, B'_{a\dot{a}}$ and instead relate them as[13]

$$A'_{a\dot{a}} = B'_{a\dot{a}} + \Theta_{a\dot{a}} := B'_{a\dot{a}} - \frac{1}{\langle \alpha\beta\rangle} \beta_a \alpha^b \nabla_{b\dot{a}} g g^{-1}\,, \tag{5.81}$$

where the covariant derivative is given by $\nabla_{a\dot{a}} g g^{-1} = \partial_{a\dot{a}} g g^{-1} + B'_{a\dot{a}} - \mathrm{Ad}_g B'_{a\dot{a}}$. Equation (5.81) allows us to express (5.79) entirely in terms of $\mathcal{B}'$, $\Theta = \pi^a \Theta_{a\dot{a}} \bar{e}^{\dot{a}}$ and $\hat{g}$. Many of the terms combine to produce a gauged Wess-Zumino Lagrangian contribution (eq. (5.34)) with the result

$$S_{\mathrm{ghCS}_6}[\mathcal{A},\mathcal{B}] = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\Theta \wedge \big(\nabla\hat{g}\hat{g}^{-1} - \mathcal{B}'\big)\big) - \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}\times[0,1]} \bar{\partial}\Omega \wedge \mathcal{L}_{\mathrm{gWZ}}[\hat{g}, \mathcal{B}']\,. \tag{5.82}$$

[13] The boundary conditions on the $\mathbb{CP}^1$ derivatives of the gauge fields impose

$$\frac{\alpha^a}{\langle \alpha\beta\rangle} \big(\nabla_{a\dot{a}} g g^{-1}\big)^{\mathfrak{h}} = -\beta^a \nabla_{a\dot{a}} \tilde{u}^{\mathfrak{h}}\,, \qquad \frac{\beta^a}{\langle \alpha\beta\rangle} \big(g^{-1}\nabla_{a\dot{a}} g\big)^{\mathfrak{h}} = -\alpha^a \nabla_{a\dot{a}} u^{\mathfrak{h}}\,, \tag{5.80}$$

however we will not invoke these since they will follow as equations of motion of the 4d theory due to the addition of the boundary term in the gauged hCS_6 action (5.45). For more details see appendix 5.11.

Given that both $B_{a\dot{a}}$ and $\Theta_{a\dot{a}}$ are $\mathbb{CP}^1$-independent, we have that

$$\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}(\Theta \wedge \mathcal{B}') = 0\,, \tag{5.83}$$

with cancelling contributions from the two end points of the integral. Hence we are left with a manifestly covariant result

$$S_{\mathrm{ghCS}_6}[\mathcal{A},\mathcal{B}] = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\Theta \wedge \big(\nabla \hat{g}\hat{g}^{-1}\big) - \mathcal{L}_{\mathrm{gWZ}}[\hat{g},\mathcal{B}']\big)\,. \tag{5.84}$$

Application of the localisation formula in the appendix (5.235) yields the four-dimensional action

$$\begin{aligned} S_{\mathrm{IFT}_4} =& \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{Tr}\big(\nabla g g^{-1} \wedge \star \nabla g g^{-1}\big) + \int_{\mathbb{R}^4\times[0,1]} \omega_{\alpha,\beta} \wedge \mathcal{L}_{\mathrm{gWZ}}[g,B'] \\ & - \int_{\mathbb{R}^4} \mu_\alpha \wedge \mathrm{Tr}(u \cdot F[B']) + \mu_\beta \wedge \mathrm{Tr}(\tilde{u}\cdot F[B'])\ . \end{aligned} \tag{5.85}$$

At this point only the $\mathfrak{h}$-components of u and $\tilde{u}$ contribute to the action, and so henceforth, to ease notation and without loss of generality, we set their $\mathfrak{k}$-components to zero.

Something rather elegant has occurred; we have found the localisation of the six-dimensional theory returns not only the gauging of the WZW_4 model, but also residual edge modes serving as Lagrange multipliers that constrain the field strength to obey exactly those conditions of eq. (5.44) which ensure the theory can be written as the difference of WZW_4 models. The constraints $F^{2,0} = 0$ and $F^{0,2} = 0$ have also been imposed by Lagrange multipliers in the context of 5d Kähler Chern-Simons theory [NS90; NS92]. This theory bears a similar relationship to WZW_4 as 3d Chern-Simons theory bears to WZW_2. This poses a natural question: what is the direct relationship between this 5d Kähler Chern-Simons theory and 6d holomorphic Chern-Simons theory? We suspect the mechanism here is rather similar to that which relates CS_4 and CS_3 [Yam19]; we comment further on this in the outlook.

5.3.5 Equations of motion and ASDYM

Making use of the projectors previously introduced in eq. (5.41), the equations of motion then read

$$\begin{aligned} \delta B' :&\quad 0 = \bar{P}\nabla g g^{-1}|_{\mathfrak{h}} - P g^{-1}\nabla g|_{\mathfrak{h}} + \star(\mu_\alpha \wedge \nabla u + \mu_\beta \wedge \nabla \tilde{u})\ , \\ \delta g :&\quad 0 = \nabla \star \nabla g g^{-1} - \omega_{\alpha,\beta} \wedge \nabla(\nabla g g^{-1}) + 2\omega_{\alpha,\beta} \wedge F[B']\ , \\ \delta u :&\quad 0 = \mu_\alpha \wedge F[B']\ , \\ \delta \tilde{u} :&\quad 0 = \mu_\beta \wedge F[B']\ . \end{aligned} \tag{5.86}$$

We can exploit the projectors to extract from the B' equation of motion the two independent contributions:

$$\delta B' : \quad \begin{array}{l} 0 = \bar{P}\left(\nabla g g^{-1}|_{\mathfrak{h}} + \star(\mu_\beta \wedge \nabla \tilde{u})\right) \\ 0 = P\left(g^{-1}\nabla g|_{\mathfrak{h}} - \star(\mu_\alpha \wedge \nabla u)\right) \end{array} . \tag{5.87}$$

In fact, these are exactly the conditions that arise from the $\mathbb{CP}^1$ derivative components of the boundary condition $\partial_0 \mathcal{A}^{\mathfrak{h}}|_{\alpha,\beta} = \partial_0 \mathcal{B}|_{\alpha,\beta}$.

Making use of the identity

$$\nabla(\omega_{\alpha\beta} \wedge \star(\mu_\beta \wedge \nabla \tilde{u})) = \nabla(\mu_\beta \wedge \nabla \tilde{u}) = \mu_\beta \wedge F[B'] \cdot \tilde{u} \ , \tag{5.88}$$

we obtain an on-shell integrability condition for the first of eq. (5.87), namely that

$$\nabla(\omega_{\alpha\beta} \wedge \bar{P}(\nabla g g^{-1}|_{\mathfrak{h}})) = 0 \, . \tag{5.89}$$

Hence, using the projection of the δg equation of motion into $\mathfrak{h}$, we have that $\omega_{\alpha,\beta} \wedge F[B'] = 0$ follows on-shell.

Let us turn back to the ASDYM equations which we can recast as

$$\mu_\alpha \wedge F = 0 \, , \quad \mu_\beta \wedge F = 0 \, , \quad \omega_{\alpha,\beta} \wedge F = 0 \, . \tag{5.90}$$

In differential form notation, the solution of the boundary condition eq. (5.81) can be written as

$$A' = B' - \bar{P}(\nabla g g^{-1}) \, . \tag{5.91}$$

By virtue of the identities obeyed by the projectors, eq. (5.211), and the covariant Maurer-Cartan identity obeyed by $R^\nabla = \nabla g g^{-1}$,

$$\nabla R^\nabla - R^\nabla \wedge R^\nabla = (1 - \mathrm{Ad}_g) F[B'] \, , \tag{5.92}$$

one can readily establish

$$\mu_\beta \wedge F[A'] \;=\; \mu_\beta \wedge F[B'] \, , \tag{5.93}$$

$$\mu_\alpha \wedge F[A'] \;=\; \mu_\alpha \wedge \mathrm{Ad}_g F[B'] \, , \tag{5.94}$$

$$2\omega_{\alpha\beta} \wedge F[A'] \;=\; 2\omega_{\alpha\beta} \wedge F[B'] + 2\omega_{\alpha\beta} \wedge \nabla \bar{P}(R^\nabla) \tag{5.95}$$

$$\;=\; 2\omega_{\alpha\beta} \wedge F[B'] - \nabla(\star \nabla g g^{-1}) + \omega_{\alpha\beta} \wedge \nabla(\nabla g g^{-1}) \, . \tag{5.96}$$

Hence we conclude that the $\delta g, \delta u, \delta \tilde{u}$ equations of motion are equivalent to the ASDYM equations for the connection A'. Demanding that the B' connection is also ASD requires in addition that $\omega_{\alpha,\beta} \wedge F[B'] = 0$, and as shown above this is indeed a consequence of the B' equations of motion.

5.3.6 Constraining then reducing

We now proceed to the bottom of the diamond by reduction of the IFT_4. In this section, we shall first implement the constraints imposed by the Lagrange multipliers $u, \tilde{u}$ in the 4d theory and then reduce. While not the most general reduction, this will allow us to directly recover the gauged WZW coset CFT. In section 5.4, we will investigate more general reductions, in particular what happens if we reduce without first imposing constraints.

Imposing the reduction ansatz that $\partial_z = \partial_{\bar{z}} = 0$ in the complex coordinates of eq. (5.24), we have that the solution to the constraints $B = P(a^{-1}\mathrm{d}a) + \bar{P}(b^{-1}\mathrm{d}b)$ becomes

$$\begin{aligned} B' = B'_{a\dot{a}}\mathrm{d}x^{a\dot{a}} &= \frac{1}{\alpha-\beta}\left(\alpha b^{-1}\partial_w b - \beta a^{-1}\partial_w a\right)\mathrm{d}w - \frac{1}{\alpha-\beta}\left(\beta b^{-1}\partial_{\bar{w}} b - \alpha a^{-1}\partial_{\bar{w}} a\right)\mathrm{d}\bar{w} \\ &\quad + \frac{1}{\alpha-\beta}\left(b^{-1}\partial_{\bar{w}} b - a^{-1}\partial_{\bar{w}} a\right)\mathrm{d}z + \frac{\alpha\beta}{\alpha-\beta}\left(b^{-1}\partial_w b - a^{-1}\partial_w a\right)\mathrm{d}\bar{z}\ . \end{aligned} \tag{5.97}$$

For simplicity, let us first consider the Kähler point and align the reduction to the complex structure (implemented simply by taking $\alpha \to 0$ and $\beta \to \infty$). In this scenario, the reduction ansatz enforces that $B'_z = 0$ and $B'_{\bar{z}} = 0$ with the remaining components of B' parameterising a generic two-dimensional gauge field. Effectively, we can simply ignore the constraints altogether but impose $B'_z = 0$ and $B'_{\bar{z}} = 0$ as part of the specification of a reduction ansatz. This could be interpreted as demanding $D_z = D_{\bar{z}} = 0$ acting on fields. In this case it is immediate that the 4d gauged WZW reduces to a 2d gauged WZW.

Away from the Kähler point and aligned reduction, i.e. not fixing α and β, one must keep account of contributions coming from B'_z and $B'_{\bar{z}}$. We can still view B'_w and $B'_{\bar{w}}$ components of eq. (5.97) as a parametrisation of a generic 2d gauge field, but there is no way in which we can view the B'_z and $B'_{\bar{z}}$ as a local combination of the B'_w and $B'_{\bar{w}}$. We forced to work with the variables a and b rather than a 2d gauge field. Fortunately, however, the reduction can still be performed immediately if we use the composite fields $\tilde{g} = agb^{-1}$ and $\tilde{h} = ab^{-1}$. These composite variables are invariant under the $\mathfrak{h}$-gauge symmetry, but a new would-be-affine symmetry emerges under $a \to \ell a$, $b \to br^{-1}$ with $\alpha^b\partial_{b\dot{b}} r = \beta^b\partial_{b\dot{b}}\ell = 0$. These leave B', g, h invariant but act as $\tilde{g} \to \ell\tilde{g}r$ and $\tilde{h} \to \ell\tilde{h}r$. At the Kähler point and aligned reduction, these symmetries descend to affine symmetries, but in general descend only to global transformations. Recall that the 4d gWZW becomes

$$S^{(\alpha,\beta)}_{\text{gWZW}_4}[g, B'] = S^{(\alpha,\beta)}_{\text{WZW}_4}[\tilde{g}] - S^{(\alpha,\beta)}_{\text{WZW}_4}[\tilde{h}]\,. \tag{5.98}$$

It is then immediate that this reduces to the difference of PCM+WZ theories of eq. (5.25) with WZ coefficient k:

$$S_{\text{IFT}_2}[\tilde{g}, \tilde{h}] = S_{\text{PCM}+k\text{WZ}_2}[\tilde{g}] - S_{\text{PCM}+k\text{WZ}_2}[\tilde{h}]\,. \tag{5.99}$$

Away from the CFT point, $k = 1$, this cannot be recast in terms of a deformation of the gauged WZW expressed as a local function of B', g.

Lax formulation. To obtain the Lax of the resultant IFT_2 we first note that the four-dimensional gauge fields, upon solving the constraints on B', are gauge equivalent to

$$A'_{a\dot{a}} = -\frac{1}{\langle\alpha\beta\rangle}\beta_a\alpha^b\partial_{b\dot{a}}\tilde{g}\tilde{g}^{-1}\,, \quad B'_{a\dot{a}} = -\frac{1}{\langle\alpha\beta\rangle}\beta_a\alpha^b\partial_{b\dot{a}}\tilde{h}\tilde{h}^{-1}\,.$$

Thus, we may simply follow the construction of the Lax from the ungauged model of eq. (5.27), with the connection A' producing a Lax for the $S_{\text{PCM}+k\text{WZ}_2}[\tilde{g}]$ and the B' producing one for $S_{\text{PCM}+k\text{WZ}_2}[\tilde{h}]$.

5.4 More General IFT_2 from IFT_4: Reducing then Constraining

In the previous section, we reduced from the gauged WZW_4 model to an IFT_2, but prior to reduction we enforced the constraints imposed by the Lagrange multiplier fields. These constraints determine implicit relations between the components of the gauge field as per eq. (5.97). In the simplest case, where we work at the Kähler point and align the reduction directions with the complex structure, the constraints enforce $B'_z = B'_{\bar{z}} = 0$. However, if we do not impose the constraints in 4d, the standard reduction ansatz would only require that B'_z and $B'_{\bar{z}}$ are functionally independent of z and $\bar{z}$, a weaker condition.

In this section, we shall explore the consequences of reducing without first constraining. Denoting the reduction with $\rightsquigarrow$ we anticipate that the lower-dimensional description will include additional fields as[14]

$$\begin{aligned} &B'_w(w,\bar{w},z,\bar{z}) \rightsquigarrow B_w(w,\bar{w})\,, \quad B'_{\bar{w}}(w,\bar{w},z,\bar{z}) \rightsquigarrow B_{\bar{w}}(w,\bar{w})\,, \\ &B'_z(w,\bar{w},z,\bar{z}) \rightsquigarrow \bar{\Phi}(w,\bar{w})\,, \quad B'_{\bar{z}}(w,\bar{w},z,\bar{z}) \rightsquigarrow \Phi(w,\bar{w})\,, \end{aligned} \tag{5.100}$$

where Φ and $\bar{\Phi}$ will be adjoint scalars in the lower-dimensional theory (sometimes called Higgs fields in the literature). These will enter explicitly in the lower-dimensional theory through the reduction of covariant derivatives

$$\nabla_z g g^{-1} \rightsquigarrow \bar{\Phi} - g\bar{\Phi}g^{-1}\,, \qquad \nabla_{\bar{z}} g g^{-1} \rightsquigarrow \Phi - g\Phi g^{-1}\,. \tag{5.101}$$

On-shell the 4d gauge field B' is ASD and couples to matter in the gWZW_4 model. It is well-known that the reduction of an ASDYM connection leads to the Hitchin system, and we shall see this feature in the lower-dimensional dynamics below.

The two-dimensional Lagrangian that arises from reducing eq. (5.85) without first constraining

[14]Note, we are dropping the prime on the 2d gauge field B.

is[15]

$$
\begin{aligned}
L_{\mathrm{IFT}_2} = {}& \frac{1}{2}\mathrm{Tr}\big(g^{-1}D_w g g^{-1}D_{\bar w} g\big) + \frac{1}{2}\frac{\alpha+\beta}{\alpha-\beta}L_{\mathrm{gWZ}} + \mathrm{Tr}\Big(\Phi\bar\Phi + \frac{\alpha}{\alpha-\beta}\Phi\mathrm{Ad}_g\bar\Phi - \frac{\beta}{\alpha-\beta}\Phi\mathrm{Ad}_g^{-1}\bar\Phi\Big) \\
& + \frac{1}{\alpha-\beta}\mathrm{Tr}\big(\Phi\,(g^{-1}D_{\bar w}g + D_{\bar w}gg^{-1}) + \alpha\beta\,\bar\Phi\,(g^{-1}D_w g + D_w g g^{-1})\big) \\
& + \mathrm{Tr}\big(\tilde u(F_{\bar w w} - \beta^{-1}D_{\bar w}\Phi - \beta D_w\bar\Phi - [\bar\Phi,\Phi])\big) + \mathrm{Tr}\big(u(F_{\bar w w} - \alpha^{-1}D_{\bar w}\Phi - \alpha D_w\bar\Phi - [\bar\Phi,\Phi])\big)\,,
\end{aligned}
\tag{5.102}
$$

where we denote the 2d covariant derivative as $D = \mathrm{d} + \mathrm{ad}_B$ and note that we have rescaled $\tilde u \to \frac{\tilde u}{\langle\beta\gamma\rangle\langle\beta\hat\gamma\rangle}$ and $u \to \frac{u}{\langle\alpha\gamma\rangle\langle\alpha\hat\gamma\rangle}$. The fields of the IFT_2 are $g \in G$ and $B_{w,\bar w}, \Phi, \bar\Phi, u, \tilde u \in \mathfrak{h}$.

In addition to the overall coupling, the IFT_2 eq. (5.102) only depends on a single parameter. This can be seen by introducing[16]

$$
\mathsf{k} = \frac{\alpha+\beta}{\alpha-\beta}\,, \qquad \mathsf{k}' = -\frac{2\sqrt{\alpha\beta}}{\alpha-\beta}\,, \qquad \mathsf{k}^2 - \mathsf{k}'^2 = 1\,,
\tag{5.103}
$$

rescaling $\Phi \to \sqrt{\alpha\beta}\Phi$ and $\bar\Phi \to \frac{1}{\sqrt{\alpha\beta}}\bar\Phi$, and defining $X^- = \mathsf{k}'^{-1}(u+\tilde u)$ and $\tilde X^+ = \mathsf{k}'^{-1}(u-\tilde u)$. The Lagrangian eq. (5.102) can be rewritten as

$$
\begin{aligned}
L_{\mathrm{IFT}_2} = {}& \frac{1}{2}\mathrm{Tr}\big(g^{-1}D_w g g^{-1}D_{\bar w}g\big) + \frac{\mathsf{k}}{2}L_{\mathrm{gWZ}} + \mathrm{Tr}\big(\Phi\mathcal{O}\bar\Phi + \Phi\,V_{\bar w} + \bar\Phi\,V_w\big) \\
& + \mathrm{Tr}\big(X^-(\mathsf{k}'(F_{\bar w w} - [\bar\Phi,\Phi]) + \mathsf{k}(D_w\bar\Phi + D_{\bar w}\Phi))\big) + \mathrm{Tr}\big(\tilde X^+(D_w\bar\Phi - D_{\bar w}\Phi)\big)\,,
\end{aligned}
\tag{5.104}
$$

where

$$
\mathcal{O} = 1 - \frac{\mathsf{k}+1}{2}\mathrm{Ad}_g + \frac{\mathsf{k}-1}{2}\mathrm{Ad}_g^{-1}\,, \qquad V_{w,\bar w} = -\frac{\mathsf{k}'}{2}(g^{-1}D_{w,\bar w}g + D_{w,\bar w}gg^{-1})\,.
\tag{5.105}
$$

Note that the CFT points $\mathsf{k}=1$ or $\mathsf{k}=-1$ correspond to taking $\gamma_a \to \hat\alpha_a$ or $\gamma_a \to \alpha_a$, i.e. when the zeroes of the twist function coincide with the poles.

By construction, as the reduction of gWZW_4, the equations of motion of this theory are equivalent to the zero curvature of Lax connections, whose components are given by the $\mathrm{d}w$ and

[15] 2d Lagrangians are defined as $S_{\mathrm{IFT}_2} = 2\mathrm{i}\int_{\mathbb{R}^2}\mathrm{d}w\wedge\mathrm{d}\bar w\,L_{\mathrm{IFT}_2}$. We denote

$$
L_{\mathrm{gWZ}} = L_{\mathrm{WZ}}(g) + \mathrm{Tr}\big((g^{-1}\partial_w g + \partial_w gg^{-1})B_{\bar w} - (g^{-1}\partial_{\bar w}g + \partial_{\bar w}gg^{-1})B_w + B_w\mathrm{Ad}_g B_{\bar w} - B_w\mathrm{Ad}_g^{-1}B_{\bar w}\big),
$$

where $\int \mathrm{d}w\wedge\mathrm{d}\bar w\,L_{\mathrm{WZ}}(g) = \int_{\mathbb{R}^2\times[0,1]}\mathcal{L}_{\mathrm{WZ}}(\hat g) = \frac{1}{3}\int_{\mathbb{R}^2\times[0,1]}\mathrm{Tr}\big(\hat g^1\mathrm{d}\hat g\wedge\hat g^1\mathrm{d}\hat g\wedge\hat g^1\mathrm{d}\hat g\big)$.

[16] Here, we have implicitly assumed that $\alpha\beta \geq 0$, which implies that $|\mathsf{k}| \geq 1$. The other regime of interest, $\alpha\beta \leq 0$ and $|\mathsf{k}| \leq 1$ is related by an analytic continuation $\mathsf{k}' \to -\mathrm{i}\mathsf{k}'$.

$\mathrm{d}\bar{w}$ legs of the 4d gauge fields. Explicitly, these Lax connections are given by

$$\begin{aligned}\mathcal{L}_w^{(A)} &= \partial_w + B_w - \frac{\mathsf{k}+1}{2}K_w - \frac{1}{\zeta}\big(\Phi + \frac{\mathsf{k}'}{2}K_w\big)\,,\\ \mathcal{L}_{\bar{w}}^{(A)} &= \partial_{\bar{w}} + B_{\bar{w}} + \frac{\mathsf{k}-1}{2}K_{\bar{w}} + \zeta\big(\bar{\Phi} + \frac{\mathsf{k}'}{2}K_{\bar{w}}\big)\,,\end{aligned} \tag{5.106}$$

$$\mathcal{L}_w^{(B)} = \partial_w + B_w - \frac{1}{\zeta}\Phi\,, \qquad \mathcal{L}_{\bar{w}}^{(B)} = \partial_{\bar{w}} + B_{\bar{w}} + \zeta\bar{\Phi}\,, \tag{5.107}$$

where we have also redefined the spectral parameter $\zeta \to \sqrt{\alpha\beta}\zeta$ compared to section 5.2.4 and we have introduced the currents

$$K_w = D_w g g^{-1} + \frac{\mathsf{k}-1}{\mathsf{k}'}(1-\mathrm{Ad}_g)\Phi\,, \qquad K_{\bar{w}} = D_{\bar{w}} g g^{-1} - \frac{\mathsf{k}+1}{\mathsf{k}'}(1-\mathrm{Ad}_g)\bar{\Phi}\,. \tag{5.108}$$

5.4.1 Lax formulation

Before analysing the Lagrangian eq. (5.104) in more detail, let us show explicitly that its equations of motion are indeed equivalent to the zero-curvature condition for the Lax connections eq. (5.106) and eq. (5.107). The equations of motion that follow from the Lagrangian eq. (5.104) varying $\tilde{X}^+$, X^- and g are

$$\begin{aligned}\delta\tilde{X}^+ :&\quad \mathcal{E}_+ \equiv D_w\bar{\Phi} - D_{\bar{w}}\Phi = 0\,,\\ \delta X^- :&\quad \mathcal{E}_- \equiv \mathsf{k}'\big(F_{\bar{w}w} - [\bar{\Phi},\Phi]\big) + \mathsf{k}\big(D_w\bar{\Phi} + D_{\bar{w}}\Phi\big) = 0\,,\\ \delta g g^{-1} :&\quad \mathcal{E}_g \equiv \frac{\mathsf{k}-1}{2}\Big(D_w K_{\bar{w}} + \frac{\mathsf{k}+1}{\mathsf{k}'}[\bar{\Phi},K_w]\Big) - \frac{\mathsf{k}+1}{2}\Big(D_{\bar{w}}K_w - \frac{\mathsf{k}-1}{\mathsf{k}'}[\Phi,K_{\bar{w}}]\Big)\\ &\qquad\qquad + \frac{\mathsf{k}}{\mathsf{k}'}\mathcal{E}_- - \frac{1}{\mathsf{k}'}\big(D_w\bar{\Phi} + D_{\bar{w}}\Phi\big) = 0\,.\end{aligned} \tag{5.109}$$

We also have the Bianchi identity following from the zero-curvature of the Maurer-Cartan form $\mathrm{d}gg^{-1}$

$$\mathcal{Z} \equiv D_w K_{\bar{w}} + \frac{\mathsf{k}+1}{\mathsf{k}'}[\bar{\Phi},K_w] - D_{\bar{w}}K_w + \frac{\mathsf{k}-1}{\mathsf{k}'}[\Phi,K_{\bar{w}}] + [K_{\bar{w}},K_w] + \frac{1}{\mathsf{k}'}(1-\mathrm{Ad}_g)(\mathcal{E}_- + \mathcal{E}_+) = 0\,. \tag{5.110}$$

The zero curvature of the A-Lax eq. (5.106) gives rise to three equations that are linear combinations of the equations of motion eq. (5.109) and the Bianchi identity eq. (5.110):

$$\begin{aligned}0 &= \frac{\mathsf{k}-1}{2}\mathcal{Z}' - \mathcal{E}_g + \frac{\mathsf{k}}{\mathsf{k}'}\mathcal{E}_- - \frac{1}{\mathsf{k}'}\mathcal{E}_+\,,\\ 0 &= \mathsf{k}'^2\mathcal{Z}' - 2\mathsf{k}\mathcal{E}_g + 2\mathsf{k}'\mathcal{E}_-\,,\\ 0 &= \frac{\mathsf{k}+1}{2}\mathcal{Z}' - \mathcal{E}_g + \frac{\mathsf{k}}{\mathsf{k}'}\mathcal{E}_- + \frac{1}{\mathsf{k}'}\mathcal{E}_+\,,\end{aligned} \tag{5.111}$$

where we have defined $\mathcal{Z}' \equiv \mathcal{Z} - \frac{1}{\mathsf{k}'}(1-\mathrm{Ad}_g)(\mathcal{E}_- + \mathcal{E}_+)$. On the other hand, the zero curvature of the B-Lax (5.107) defines the Hitchin system:

$$0 = D_{\bar{w}}\Phi\,, \qquad 0 = F_{\bar{w}w} - [\bar{\Phi}, \Phi]\,, \qquad 0 = D_w \bar{\Phi}\,, \tag{5.112}$$

which can be rewritten as the three equations $\mathcal{E}_\pm = 0$ and $\mathcal{E}_0 \equiv D_w\bar{\Phi} + D_{\bar{w}}\Phi = 0$. Therefore, the two Lax connections give rise to five independent equations, which are linear combinations of the equations of motion (5.109), the Bianchi identity (5.110), and the additional equation $\mathcal{E}_0 = 0$.

To recover this final equation from the equations of motion, let us consider the variational equations for B_w, $B_{\bar{w}}$, $\bar{\Phi}$ and Φ

$$\begin{aligned}
\delta B_w: \quad & \mathcal{E}_B \equiv \mathsf{k}' D_{\bar{w}} X^- - [\bar{\Phi}, \tilde{X}^+ + \mathsf{k}X^-] + \frac{\mathsf{k}-1}{2} P_{\mathfrak{h}} K_{\bar{w}} + \frac{\mathsf{k}+1}{2} P_{\mathfrak{h}} \mathrm{Ad}_g^{-1} K_{\bar{w}} - \frac{\mathsf{k}+1}{\mathsf{k}'} P_{\mathfrak{h}} (1 - \mathrm{Ad}_g^{-1})\bar{\Phi} = 0\,, \\
\delta B_{\bar{w}}: \quad & \mathcal{E}_{\bar{B}} \equiv \mathsf{k}' D_w X^- - [\Phi, \tilde{X}^+ - \mathsf{k}X^-] + \frac{\mathsf{k}+1}{2} P_{\mathfrak{h}} K_w + \frac{\mathsf{k}-1}{2} P_{\mathfrak{h}} \mathrm{Ad}_g^{-1} K_w - \frac{\mathsf{k}-1}{\mathsf{k}'} P_{\mathfrak{h}} (1 - \mathrm{Ad}_g^{-1})\Phi = 0\,, \\
\delta \Phi: \quad & \mathcal{E}_\Phi \equiv D_{\bar{w}}(\tilde{X}^+ - \mathsf{k}X^-) + \mathsf{k}'[\bar{\Phi}, X^-] - \frac{\mathsf{k}'}{2} P_{\mathfrak{h}}(1 + \mathrm{Ad}_g^{-1}) K_{\bar{w}} + P_{\mathfrak{h}}(1 - \mathrm{Ad}_g^{-1})\bar{\Phi} = 0\,, \\
\delta \bar{\Phi}: \quad & \mathcal{E}_{\bar{\Phi}} \equiv D_w(\tilde{X}^+ + \mathsf{k}X^-) + \mathsf{k}'[\Phi, X^-] + \frac{\mathsf{k}'}{2} P_{\mathfrak{h}}(1 + \mathrm{Ad}_g^{-1}) K_w - P_{\mathfrak{h}}(1 - \mathrm{Ad}_g^{-1})\Phi = 0\,.
\end{aligned} \tag{5.113}$$

These can be understood as a first-order system of equations for $\tilde{X}^+$ and X^-. Consistency of the system implies that they should satisfy the integrability conditions $[D_{\bar{w}}, D_w]\tilde{X}^+ = [F_{\bar{w}w}, \tilde{X}^+]$ and $[D_{\bar{w}}, D_w]X^- = [F_{\bar{w}w}, X^-]$. We find that

$$\mathsf{k}'[D_{\bar{w}}, D_w]X^- - \mathsf{k}'[F_{\bar{w}w}, X^-] = [X^+, \mathcal{E}_+] + [X^-, \mathcal{E}_-] + P_{\mathfrak{h}}(1 - \mathrm{Ad}_g^{-1})\mathcal{E}_g + \mathsf{k}P_{\mathfrak{h}}\mathrm{Ad}_g^{-1}\mathcal{Z}\,, \tag{5.114}$$

hence, using the Bianchi identity (5.110), this vanishes on the equations of motion for $\tilde{X}^+$, X^- and g (5.109). On the other hand, we have

$$\mathsf{k}'[D_{\bar{w}}, D_w]\tilde{X}^+ - \mathsf{k}'[F_{\bar{w}w}, \tilde{X}^+] = [X^+, \mathcal{E}_-] + [X^-, \mathcal{E}_+] + \frac{2\mathsf{k}}{\mathsf{k}'}\mathcal{E}_- - \frac{2}{\mathsf{k}'}\mathcal{E}_0 - P_{\mathfrak{h}}(1 + \mathrm{Ad}_g^{-1})\mathcal{E}_g + \mathsf{k}P_{\mathfrak{h}}\mathrm{Ad}_g^{-1}\mathcal{Z}\,. \tag{5.115}$$

Here we see that in addition to the Bianchi identity (5.110) and equations of motion (5.109), we also require $\mathcal{E}_0 = 0$, recovering the final equation from the Lax system.

5.4.2 Relation to known models

As we will shortly see, if we take H to be abelian, the Lagrangian (5.104) can be related to known models, including the homogeneous sine-Gordon models and the PCM plus WZ term. However, for non-abelian H (5.104) has not been considered before, and defines a new integrable field theory in two dimensions. Moreover, by integrating out Φ, $\bar{\Phi}$ and the gauge field $B_{w,\bar{w}}$, it leads to an integrable sigma model for the fields g, $\tilde{X}^+$ and X^-. We leave the study of these models for future work.

To recover a sigma model from the Lagrangian (5.104) for abelian H, in addition to integrating out B_w and $B_{\bar{w}}$, we have two options. The first is to integrate out Φ and $\bar{\Phi}$. The second is to solve the constraint imposed by the Lagrange multiplier $\tilde{X}^+$. For abelian H the Lagrangian (5.104) simplifies to

$$\begin{aligned} L^{\text{ab}}_{\text{IFT}_2} = \frac{1}{2}\text{Tr}\big(g^{-1}D_w g g^{-1} D_{\bar{w}} g\big) + \frac{\mathsf{k}}{2} L_{\text{gWZ}} + \text{Tr}\big(\Phi \mathcal{O}\bar{\Phi} + \Phi\, V_{\bar{w}} + \bar{\Phi}\, V_w\big) \\ + \text{Tr}\big((X^-(\mathsf{k}' F_{\bar{w}w} + \mathsf{k}(\partial_w \bar{\Phi} + \partial_{\bar{w}}\Phi))\big) + \text{Tr}\big(\tilde{X}^+(\partial_w\bar{\Phi} - \partial_{\bar{w}}\Phi)\big)\,. \end{aligned} \tag{5.116}$$

This takes the form of the first-order action in the Buscher procedure, and it follows that the two sigma models will be T-dual to each other with dual fields X^+ and $\tilde{X}^+$. Explicitly the Lagrangians, before integrating out B_w and $B_{\bar{w}}$, are

$$\begin{aligned} L^{\tilde{X}}_{\text{IFT}_2} = \frac{1}{2}\text{Tr}\big(g^{-1}D_w g g^{-1} D_{\bar{w}} g\big) + \frac{\mathsf{k}}{2} L_{\text{gWZ}} + \mathsf{k}'\text{Tr}\big(X^- F_{\bar{w}w}\big) \\ + \text{Tr}\big((\partial_w \tilde{X}^+ - V_w + \mathsf{k}\partial_w X^-)\mathcal{O}^{-1}(\partial_{\bar{w}}\tilde{X}^+ + V_{\bar{w}} - \mathsf{k}\partial_{\bar{w}} X^-)\big)\,, \end{aligned} \tag{5.117}$$

and

$$\begin{aligned} L^{X}_{\text{IFT}_2} = \frac{1}{2}\text{Tr}\big(g^{-1}D_w g g^{-1} D_{\bar{w}} g\big) + \frac{\mathsf{k}}{2} L_{\text{gWZ}} + \mathsf{k}'\text{Tr}\big(X^- F_{\bar{w}w}\big) \\ + \frac{1}{4}\text{Tr}\big(\partial_w X^+ \mathcal{O}\partial_{\bar{w}} X^+ + 2\partial_w X^+(V_{\bar{w}} - \mathsf{k}\partial_{\bar{w}} X^-) + 2\partial_{\bar{w}} X^+(V_w - \mathsf{k}\partial_w X^-)\big)\,, \end{aligned} \tag{5.118}$$

where in the second we have locally solved the constraint imposed by the Lagrange multiplier $\tilde{X}^+$ by setting

$$\Phi = \frac{1}{2}\partial_w X^+\,, \qquad \bar{\Phi} = \frac{1}{2}\partial_{\bar{w}} X^+\,, \qquad X^+ \in \mathfrak{h}\,. \tag{5.119}$$

As mentioned above, the first approach can also be straightforwardly applied for non-abelian H. Generalising the second approach is more subtle. The constraint imposed by the Lagrange multiplier $\tilde{X}^+$ in the Lagrangian (5.104) implies that

$$D_w\bar{\Phi} - D_{\bar{w}}\Phi = 0. \tag{5.120}$$

Typically the full solution to this equation would be expressed in terms of path-ordered exponentials of B_w and $B_{\bar{w}}$. To avoid non-local expressions, we can restrict Φ and $\bar{\Phi}$ to be valued in the centre of $\mathfrak{h}$, denoted $\mathcal{Z}(\mathfrak{h})$. Note that this is not a restriction if H is abelian. With this restriction, the Lagrangian (5.104) again simplifies to (5.116), and the constraint (5.120) then becomes $\partial_w\bar{\Phi} - \partial_{\bar{w}}\Phi = 0$, which we can again locally solve by (5.119) now with $X^+ \in \mathcal{Z}(\mathfrak{h})$, similarly leading to the Lagrangian (5.118).

Relation to PCM plus WZ term. Taking H to be abelian, we can relate the Lagrangian (5.116) to that of the PCM + WZ term for $G \times H$ through a combination of T-dualities and field redefinitions. We start by parametrising

$$g = e^{\frac{1}{2}\tau} g e^{\frac{1}{2}\tau}\,, \qquad \tau \in \mathfrak{h}\,, \tag{5.121}$$

and setting $\partial_{\bar{w},w}\tau \to 2C_{w,\bar{w}}$. We also integrate by parts and set $\partial_w X^- \to 2\Psi$ and $\partial_{\bar{w}} X^- \to 2\bar{\Psi}$. To maintain equivalence with the Lagrangian that we started with, we add $\mathrm{Tr}\big(\tilde{\tau}(\partial_w C_{\bar{w}} - \partial_{\bar{w}} C_w)\big) + \mathrm{Tr}\big(\tilde{X}^-(\partial_w \bar{\Psi} - \partial_{\bar{w}}\Psi)\big)$, i.e., the Lagrange multipliers $\tilde{\tau}$ and $\tilde{X}^-$ locally impose $C_{w,\bar{w}} = \frac{1}{2}\partial_{\bar{w},w}\tau$, $\Psi = \frac{1}{2}\partial_w X^-$ and $\bar{\Psi} = \frac{1}{2}\partial_{\bar{w}} X^-$. We can then redefine the fields as[17]

$$\begin{aligned}
&B_w \to B_w - \frac{\mathsf{k}}{\mathsf{k}'}\Phi\,, && C_w \to C_w - \frac{1}{\mathsf{k}'}\Phi\,, && \Psi \to \Psi + \frac{\mathsf{k}}{\mathsf{k}'^2}\Phi\,, \\
&B_{\bar{w}} \to B_{\bar{w}} + \frac{\mathsf{k}}{\mathsf{k}'}\bar{\Phi}\,, && C_{\bar{w}} \to C_{\bar{w}} - \frac{1}{\mathsf{k}'}\bar{\Phi}\,, && \bar{\Psi} \to \bar{\Psi} + \frac{\mathsf{k}}{\mathsf{k}'^2}\bar{\Phi}\,, \\
&\tilde{X}^+ \to \frac{1}{\mathsf{k}'}\tilde{X}^+ - \frac{\mathsf{k}}{\mathsf{k}'}\tilde{X}^- + \frac{1}{\mathsf{k}'}\tilde{\tau}\,, && \tilde{X}^- \to \mathsf{k}'\tilde{X}^-\,, && \tilde{\tau} \to \tilde{\tau}\,.
\end{aligned} \tag{5.122}$$

Doing so, we arrive at the following Lagrangian

$$\begin{aligned}
L^{\mathrm{ab}}_{\mathrm{IFT}_2} = &\frac{1}{2}\mathrm{Tr}\big(g^{-1}\partial_w g g^{-1}\partial_{\bar{w}} g\big) + \frac{\mathsf{k}}{2}L_{\mathrm{WZ}}(g) \\
&+ \frac{1-\mathsf{k}}{2}\mathrm{Tr}\big(g^{-1}\partial_w g(C_{\bar{w}} - B_{\bar{w}}) + \partial_{\bar{w}} g g^{-1}(C_w + B_w) + (C_w + B_w)\mathrm{Ad}_g(C_{\bar{w}} - B_{\bar{w}})\big) \\
&+ \frac{1+\mathsf{k}}{2}\mathrm{Tr}\big(g^{-1}\partial_{\bar{w}} g(C_w - B_w) + \partial_w g g^{-1}(C_{\bar{w}} + B_{\bar{w}}) + (C_w - B_w)\mathrm{Ad}_g^{-1}(C_{\bar{w}} + B_{\bar{w}})\big) \\
&+ \mathrm{Tr}\big(B_w B_{\bar{w}} + C_w C_{\bar{w}} + \mathsf{k} C_w B_{\bar{w}} - \mathsf{k} B_w C_{\bar{w}}\big) \\
&+ \mathrm{Tr}\big(\tilde{\tau}(\partial_w C_{\bar{w}} - \partial_{\bar{w}} C_w)\big) + \mathsf{k}'\mathrm{Tr}\big(\tilde{X}^-(\partial_w \bar{\Psi} - \partial_{\bar{w}}\Psi)\big) + 2\mathsf{k}'\mathrm{Tr}\big(\Psi B_{\bar{w}} - B_w, \bar{\Psi}\big) \\
&+ \frac{1}{\mathsf{k}'}\mathrm{Tr}\big(\tilde{X}^+(\partial_w\bar{\Phi} - \partial_{\bar{w}}\Phi)\big) - \frac{2}{\mathsf{k}'^2}\mathrm{Tr}\big(\Phi\bar{\Phi}\big)\,.
\end{aligned} \tag{5.123}$$

The final steps are to integrate out $\tilde{\tau}$, Ψ and $\bar{\Psi}$, and Φ and $\bar{\Phi}$, leading us to set

$$C_{w,\bar{w}} = \frac{1}{2}\partial_{w,\bar{w}}\tau, \qquad B_{w,\bar{w}} = -\frac{1}{2}\partial_{w,\bar{w}}\tilde{X}^-, \qquad \Phi = -\frac{\mathsf{k}'}{2}\partial_w\tilde{X}^+, \quad \bar{\Phi} = \frac{\mathsf{k}'}{2}\partial_{\bar{w}}\tilde{X}^+. \tag{5.124}$$

Redefining $g \to e^{-\frac{1}{2}(\tau+\tilde{X}^-)} g e^{-\frac{1}{2}(\tau-\tilde{X}^-)}$, we find the difference of the PCM plus WZ term Lagrangians for G and H

$$L_{\mathrm{PCM+kWZ}_2} = \frac{1}{2}\mathrm{Tr}\big(g^{-1}\partial_w g g^{-1}\partial_{\bar{w}} g\big) + \frac{\mathsf{k}}{2}L_{\mathrm{WZ}}(g) - \frac{1}{2}\mathrm{Tr}\big(\partial_w\tilde{X}^+\partial_{\bar{w}}\tilde{X}^+\big), \tag{5.125}$$

where we recall that for abelian H the WZ term vanishes.

[17] To arrive at this field redefinition, we first look for the shifts of $B_{w,\bar{w}}$, $C_{w,\bar{w}}$, Ψ and $\bar{\Psi}$ that decouple Φ and $\bar{\Phi}$ from all other fields apart from $\tilde{X}^+$. Since both C_w and $C_{\bar{w}}$ transform in the same way, as do Ψ and $\bar{\Psi}$, we can then easily compute the transformation of $\tilde{\tau}$, $\tilde{X}^-$ and $\tilde{X}^+$ by demanding that the triplet of terms $\mathrm{Tr}\big(\tilde{\tau}F_{w\bar{w}}(C) + \tilde{X}^- F_{w\bar{w}}(\Psi) + \tilde{X}^+ F_{w\bar{w}}(\Phi)\big)$ is invariant up to a simple rescaling, i.e., it becomes $\mathrm{Tr}\big(\tilde{\tau}F_{w\bar{w}}(C) + \mathsf{k}'\tilde{X}^- F_{w\bar{w}}(\Psi) + \frac{1}{\mathsf{k}'}\tilde{X}^+ F_{w\bar{w}}(\Phi)\big)$.

To summarise, starting from the sigma model (5.118) we T-dualise in τ, X^+ and X^-, we then perform a GL(3) transformation on the dual coordinates, and finally T-dualise back in τ to recover (5.125), the difference of the PCM plus WZ term Lagrangians for G and H. This relation may have been anticipated since this is the model we would expect to find starting from the ghCS$_6$ action (5.45) and instead imposing the boundary conditions $\mathcal{A}|_{\alpha,\beta} = \mathcal{B}|_{\alpha,\beta} = 0$.

$\mathsf{k} \to 1$ **limit.** As we have seen, the $\mathsf{k} \to 1$ limit is special since if we first constrain and then reduce we recover the gauged WZW coset CFT. By first reducing and then constraining, we can recover massive integrable perturbations of these theories. We consider the setup where Φ and $\bar{\Phi}$ are restricted to lie in $\mathcal{Z}(\mathfrak{h})$ and solve the constraint imposed by the Lagrange multiplier $\tilde{X}^+$ by (5.119). Taking $\mathsf{k} \to 1$ the Lagrangian (5.118) simplifies further to

$$\begin{aligned} L_{\mathrm{IFT}_2} = &\frac{1}{2}\mathrm{Tr}\big(g^{-1}D_w g g^{-1}D_{\bar{w}} g\big) + \frac{1}{2}L_{\mathrm{gWZ}} \\ &+ \frac{1}{4}\mathrm{Tr}\big(\partial_w X^+(1-\mathrm{Ad}_g)\partial_{\bar{w}}X^+ - 2\partial_w X^+\partial_{\bar{w}}X^- - 2\partial_{\bar{w}}X^+\partial_w X^-\big)\,, \end{aligned} \tag{5.126}$$

This is reminiscent of a sigma model for a pp-wave background, with the kinetic terms for the transverse fields described by the gauged WZW model for the coset G/H, except that the would-be light-cone coordinates X^+ and X^- have $\dim \mathcal{Z}(\mathfrak{h})$ components. Nevertheless, we still have the key property that the equation of motion for X^- is $\partial_w\partial_{\bar{w}}X^+ = 0$, whose general solution is $X^+ = Y(w) + \bar{Y}(\bar{w})$. Substituting into the Lagrangian (5.126) we find

$$L_{\mathrm{IFT}_2} = \frac{1}{2}\mathrm{Tr}\big(g^{-1}D_w g g^{-1}D_{\bar{w}} g\big) + \frac{1}{2}L_{\mathrm{gWZ}} + \frac{1}{4}\mathrm{Tr}\big(Y'\bar{Y}' - Y'\mathrm{Ad}_g\bar{Y}'\big)\,. \tag{5.127}$$

In the special case that $Y = w\Lambda$ and $\bar{Y} = \bar{w}\bar{\Lambda}$ this is the gauged WZW model for the coset G/H deformed by a massive potential $V = \mathrm{Tr}(\Lambda\mathrm{Ad}_g\bar{\Lambda}) - \mathrm{Tr}(\Lambda\bar{\Lambda})$ as studied in [Par94]. Taking the limit $k \to 1$ directly at the level of the Lax connection given by eq. (5.106), keeping track of the definitions of the currents $K_w, K_{\bar{w}}$ which depend on k, we find

$$\mathcal{L}_w \to \partial_w + B_w - D_w g g^{-1} + \frac{1}{2\zeta}\Lambda\,, \qquad \mathcal{L}_{\bar{w}} = \partial_{\bar{w}} + B_{\bar{w}} - \frac{\zeta}{2}\mathrm{Ad}_g\bar{\Lambda}\,, \tag{5.128}$$

recovering the Lax given in [Par94; Fer+97].

When G is compact and $H = U(1)^{\mathrm{rk}_G}$, Λ and $\bar{\Lambda}$ can be chosen such that these models have a positive-definite kinetic term and a mass gap. These are known as the homogeneous sine-Gordon models [Fer+97]. For $G = SU(2)$ and $H = U(1)$ the homogeneous sine-Gordon model becomes the complex sine-Gordon model after integrating out the gauge fields B_w and $B_{\bar{w}}$. Note that if $\mathcal{Z}(\mathfrak{h})$ is one-dimensional and $Y(w)$ and $\bar{Y}(\bar{w})$ are both non-constant then we can always use the classical conformal symmetry to reach $Y = w\Lambda$ and $Y' = \bar{w}\Lambda$, hence recovering a constant potential. For higher-dimensional $\mathcal{Z}(\mathfrak{h})$, this is not the case.

5.4.3 Example: $SL(2)/U(1)_V$

To illustrate the features of this construction, let us consider the example of $SL(2)/U(1)_V$ for which the 2d gauged WZW describes the trumpet CFT. To be explicit we use $\mathfrak{sl}(2)$ generators

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{5.129}$$

and parametrise the group element as

$$g = \begin{pmatrix} \cos(\theta)\sinh(\rho) + \cosh(\rho)\cos(\tau) & \sin(\theta)\sinh(\rho) + \cosh(\rho)\sin(\tau) \\ \sin(\theta)\sinh(\rho) - \cosh(\rho)\sin(\tau) & \cosh(\rho)\cos(\tau) - \cos(\theta)\sinh(\rho) \end{pmatrix}. \tag{5.130}$$

We choose the $U(1)$ vector action generated by T_3 such that

$$\delta g = \epsilon[g, T_3] \quad \Rightarrow \quad \delta\rho = \delta\tau = 0\,, \quad \delta\theta = \epsilon, \tag{5.131}$$

hence we gauge fix by setting $\theta = 0$. The analysis here is simplified by the observation that there is no WZ term since there are no 3-forms on the two-dimensional target space.

The CFT point. For orientation, we first work at the CFT point corresponding to $\mathsf{k} = 1$. Recall from the discussion in §5.3, that first constraining in 4d and then reducing, enforces $\bar{\Phi} = \Phi = 0$ and the Lagrange multiplier sector vanishes. This gives the conventional gauged WZW model described by a target space geometry

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \coth^2\rho\,\mathrm{d}\tau^2\,. \tag{5.132}$$

Let us now consider the IFT$_2$ that results from taking the same reduction that would lead to the CFT, but now in our reduction ansatz set $\Phi = \frac{m}{2}T_3$ and $\bar{\Phi} = -\frac{m}{2}T_3$. The Lagrangian that follows is

$$L_{\mathrm{CsG}} = \partial_w\rho\partial_{\bar{w}}\rho + \coth^2\rho\,\partial_w\tau\partial_{\bar{w}}\tau - m^2\sinh^2\rho\,. \tag{5.133}$$

This theory is well known as the complex sinh-Gordon model, a special case of the integrable massive deformations of G/H gauged WZW models known as the homogeneous sine-Gordon models [Par94; Fer+97].

Unconstrained reduction: integrating out Φ, $\bar{\Phi}$ and $B_{w,\bar{w}}$. We now turn to the more general story, away from the CFT point, by considering the reduction without first imposing constraints. Taking the IFT$_2$ (5.116) and integrating out Φ, $\bar{\Phi}$ and the gauge field $B_{w,\bar{w}}$ while retaining X^- and $\tilde{X}^+$, results in the sigma model with target space metric and B-field

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\rho^2 + \coth^2\rho\,\mathrm{d}\tau^2 + \mathrm{csch}^2\rho\left(\mathrm{d}\tilde{X}^{+2} - \mathrm{d}X^{-2}\right) \\ B_2 &= \mathcal{V}\wedge\mathrm{d}\tilde{X}^+\,, \qquad \mathcal{V} = \mathsf{k}\mathrm{csch}^2\rho\,\mathrm{d}X^- + \mathsf{k}'\coth^2\rho\,\mathrm{d}\tau\,. \end{aligned} \tag{5.134}$$

Unconstrained reduction: the dual. On the other hand, if we solve the constraint imposed by the Lagrange multiplier $\tilde{X}^+$ setting $\Phi = \frac{1}{2}\partial_w X^+$ and $\bar{\Phi} = \frac{1}{2}\partial_{\bar{w}} X^+$, we find the sigma model with target space geometry

$$\begin{aligned} ds^2 &= d\rho^2 + \coth^2\rho\, d\tau^2 - \operatorname{csch}^2\rho\, dX^{-2} + \sinh^2\rho\,(dX^+ + \mathcal{V})^2\,, \\ B_2 &= 0\,. \end{aligned} \tag{5.135}$$

This can of course be recognised as the T-dual of (5.134) along $\tilde{X}^+$. In the limit $\mathsf{k} \to 1$ (5.135) becomes the pp-wave background

$$\begin{aligned} ds^2 &= d\rho^2 + \coth^2\rho\, d\tau^2 + \sinh^2\rho\, dX^{+2} + 2dX^+ dX^-\,, \\ B_2 &= 0\,, \end{aligned} \tag{5.136}$$

and if we light-cone gauge fix, $X^+ = m(w - \bar{w})$, in the associated sigma model we recover the complex sinh-Gordon Lagrangian (5.133) as expected.

Relation to PCM plus WZ term. Finally we demonstrate a relation between the models above and the PCM plus WZ term. Let us start with the metric and B-field for the PCM plus WZ term for $G = \mathrm{GL}(2)$

$$\begin{aligned} ds^2 &= d\tilde{X}^{+2} + d\rho^2 - \cosh^2\rho d\tau^2 + \sinh^2\rho d\tilde{X}^{-2}\,, \\ B &= \mathsf{k}\cosh^2\rho\, d\tau \wedge d\tilde{X}^-\,. \end{aligned} \tag{5.137}$$

Note that $dB = \mathsf{k}\sinh 2\rho\, d\rho \wedge d\tau \wedge d\tilde{X}^-$, which is proportional to the volume for $SL(2)$. We first T-dualise $\tau \to \tilde{\tau}$, and then perform the following field redefinition

$$\tilde{X}^+ \to \mathsf{k}'\tilde{X}^+ + \frac{\mathsf{k}}{\mathsf{k}'}\tilde{X}^- - \tilde{\tau}\,, \qquad \tilde{X}^- \to \frac{1}{\mathsf{k}'}\tilde{X}^-\,. \tag{5.138}$$

It is straightforward to check that this is the inverse transformation to (5.122). Finally, T-dualising back, $\tilde{X}^+ \to X^+$, $\tilde{X}^- \to X^-$ and $\tilde{\tau} \to \tau$, we precisely recover the background (5.135), demonstrating that it can be understood as a generalised TsT transformation of the PCM plus WZ term.

5.4.4 The LMP limit

The PCM plus WZ term admits a limit in which it becomes the 2d analogue of the LMP model, otherwise known as the pseudodual of the PCM [ZM78], see, e.g. [HLT19b]. It is possible to generalise this limit to the gauged model (5.104) by setting $g = \exp(\varepsilon U)$, $\mathsf{k} = \varepsilon^{-1}\ell$, $\tilde{X}^+ \to \varepsilon^2\tilde{X}^+$, $X^- \to \varepsilon^3 X^- - \varepsilon P_{\mathfrak{h}} U$ rescaling the Lagrangian by ε^{-2}, and taking $\varepsilon \to 0$. Implementing this limit in (5.104) we find

$$
\begin{aligned}
L_{\mathrm{IFT}_2}^{\mathrm{LMP}} = &\frac{1}{2}\mathrm{Tr}\big(D_w U D_{\bar{w}} U + [\Phi, U][\bar{\Phi}, U]\big) - \frac{\ell}{6}\mathrm{Tr}\big((D_w U + [\Phi, U][U, (D_{\bar{w}} U - [\bar{\Phi}, U]]\big) \\
&+ \ell\mathrm{Tr}\big(X^-(F_{\bar{w}w} - [\bar{\Phi}, \Phi] + D_w \bar{\Phi} + D_{\bar{w}}\Phi)\big) + \mathrm{Tr}\big((\tilde{X}^+(D_w\bar{\Phi} - D_{\bar{w}}\Phi)\big) \\
&+ \frac{1}{2\ell}\mathrm{Tr}\big(U(F_{\bar{w}w} - [\bar{\Phi}, \Phi] - D_w\bar{\Phi} - D_{\bar{w}}\Phi)\big)\,.
\end{aligned} \tag{5.139}
$$

Similarly we can take the limit in the Lax connections (5.106) and (5.107). The B-Lax (5.107) is unchanged, while the A-Lax (5.106) becomes

$$
\begin{aligned}
\mathcal{L}_w^{(A)} &= \partial_w + B_w - \frac{\ell}{2}K_w^{\mathrm{LMP}} - \frac{1}{\zeta}\big(\Phi + \frac{\ell}{2}K_w^{\mathrm{LMP}}\big)\,, \\
\mathcal{L}_{\bar{w}}^{(A)} &= \partial_{\bar{w}} + B_{\bar{w}} + \frac{\ell}{2}K_{\bar{w}}^{\mathrm{LMP}} + \zeta\big(\bar{\Phi} + \frac{\ell}{2}K_{\bar{w}}^{\mathrm{LMP}}\big)\,,
\end{aligned} \tag{5.140}
$$

where

$$
K_w^{\mathrm{LMP}} = D_w U + [\Phi, U]\,, \qquad K_{\bar{w}}^{\mathrm{LMP}} = D_{\bar{w}} U - [\bar{\Phi}, U]\,. \tag{5.141}
$$

As we will see in §5.6 this model can also be found directly from 6d hCS and 4d CS by considering a twist function with a single fourth-order pole.

As in the gauged WZW case, we can again find an integrable sigma model from (5.139) by integrating out Φ, $\bar{\Phi}$ and the gauge field $B_{w,\bar{w}}$. For abelian H we can also construct the dual model by solving the constraint imposed by the Lagrange multiplier $\tilde{X}^+$ and integrating out B_w and $B_{\bar{w}}$. For $SL(2)/U(1)_V$ the resulting backgrounds can be found by taking the LMP limit

$$
\begin{aligned}
&\rho \to \varepsilon\rho - \frac{1}{6}\varepsilon^3\rho\tau^2\,, \qquad \tau \to \varepsilon\tau - \frac{1}{3}\varepsilon^3\rho^2\tau\,, \qquad (\mathrm{d}s^2, B_2) \to \varepsilon^{-2}(\mathrm{d}s^2, B_2)\,, \qquad \mathsf{k} \to \varepsilon^{-1}\ell\,, \\
&X^- \to \varepsilon^3 X^- - \varepsilon\tau\,, \qquad \tilde{X}^+ \to \varepsilon^2\tilde{X}^+\,, \qquad X^+ \to X^+\,, \qquad \varepsilon \to 0\,,
\end{aligned} \tag{5.142}
$$

in eqs. (5.134) and (5.135). This limit breaks the manifest global symmetry given by shifts of the coordinate τ. This is in agreement with the fact that the Lagrangian (5.139) is not invariant under $U \to U + H_0$ ($H_0 \in \mathfrak{h}$), while its gauged WZW counterpart (5.104) is invariant under $g \to h_0 g h_0$ ($h_0 \in H$) for abelian H.

Curiously, we can actually take a simplified LMP limit

$$
\begin{aligned}
&\rho \to \varepsilon\rho\,, \qquad \tau \to \varepsilon\tau\,, \qquad (\mathrm{d}s^2, B_2) \to \varepsilon^{-2}(\mathrm{d}s^2, B_2)\,, \qquad \mathsf{k} \to \varepsilon^{-1}\ell\,, \\
&X^- \to \varepsilon^3 X^- - \varepsilon\tau\,, \qquad \tilde{X}^+ \to \varepsilon^2\tilde{X}^+\,, \qquad X^+ \to X^+\,, \qquad \varepsilon \to 0\,,
\end{aligned} \tag{5.143}
$$

in the backgrounds (5.134) and (5.135) that preserves this global symmetry. Taking this limit in eq. (5.134) we find

$$
\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}\rho^2 + \mathrm{d}\tau^2 + \frac{1}{\rho^2}\mathrm{d}\tilde{X}^{+2} + \frac{2}{\rho^2}\mathrm{d}X^-\mathrm{d}\tau\,, \\
B_2 &= \mathcal{V}\wedge \mathrm{d}\tilde{X}^+\,, \qquad \mathcal{V} = \frac{\ell}{\rho^2}\mathrm{d}X^- + \left(\ell - \frac{1}{2\ell\rho^2}\right)\mathrm{d}\tau\,,
\end{aligned}
\tag{5.144}
$$

while the limit of eq. (5.135) is

$$
\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}\rho^2 + \mathrm{d}\tau^2 + \rho^2(\mathrm{d}\tilde{X}^+ + \mathcal{V})^2 + \frac{2}{\rho^2}\mathrm{d}X^-\mathrm{d}\tau\,, \\
B_2 &= 0\,.
\end{aligned}
\tag{5.145}
$$

As for the gauged WZW case these two backgrounds above can also be constructed as a generalised TsT transformation of the background for the LMP model on GL(2)

$$
\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}\tilde{X}^{+2} + \mathrm{d}\rho^2 - \mathrm{d}\tau^2 + \rho^2\mathrm{d}\tilde{X}^{-2}\,, \\
B_2 &= \ell\rho^2 d\tau\wedge \mathrm{d}\tilde{X}^-\,.
\end{aligned}
\tag{5.146}
$$

Explicitly, if we first T-dualise $\tau\to\tilde{\tau}$, then perform the following field redefinition

$$
\tilde{X}^+ \to \ell\tilde{X}^+ + \frac{1}{2\ell^2}\tilde{X}^- - \tilde{\tau}\,, \qquad \tilde{X}^- \to \frac{1}{\ell}\tilde{X}^-\,, \qquad \tilde{\tau}\to\tilde{\tau} - \frac{1}{2\ell^2}\tilde{X}^-\,,
\tag{5.147}
$$

and finally T-dualise back,[18] $\tilde{X}^+ \to X^+$, $\tilde{X}^- \to X^-$ and $\tilde{\tau}\to\tau$, we recover the background (5.144).

5.5 Reduction to gCS$_4$ and localisation

Having discussed the right hand side of the diamond, we briefly describe the left hand side that follows from first reducing to obtain a gauged 4d Chern-Simons theory on $\mathbb{R}^2\times\mathbb{CP}^1$ and then integrating over $\mathbb{CP}^1$ to localise to a two-dimensional field theory on $\mathbb{R}^2$. We show that the resulting IFT$_2$ matches (5.102).

We recall the six-dimensional coupled action

$$
S_{\mathrm{ghCS}_6}[\mathcal{A},\mathcal{B}] = S_{\mathrm{hCS}_6}[\mathcal{A}] - S_{\mathrm{hCS}_6}[\mathcal{B}] - \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}}\bar{\partial}\Omega\wedge\mathrm{Tr}\left(\mathcal{A}\wedge\mathcal{B}\right)\,.
\tag{5.148}
$$

We note that the three terms in the action are invariant under the transformations $\mathcal{A}\mapsto\hat{\mathcal{A}} = \mathcal{A} + \rho^{\mathcal{A}}_{\dot{a}}e^{\dot{a}} + \rho^{\mathcal{A}}_0 e^0$ and $\mathcal{B}\mapsto\hat{\mathcal{B}} = \mathcal{B} + \rho^{\mathcal{B}}_{\dot{a}}e^{\dot{a}} + \rho^{\mathcal{B}}_0 e^0$, given that both Ω and $\bar{\partial}\Omega$ are top forms in the holomorphic directions. By choosing $\rho^{\mathcal{A}}$ and $\rho^{\mathcal{B}}$ appropriately, we can ensure that neither $\hat{\mathcal{A}}$ nor

[18] Note that here the order of T-dualities matters. In particular, we cannot first T-dualise $\tilde{\tau}$ after the coordinate redefinition since it turns out to be a null coordinate.

$\hat{\mathcal{B}}$ have dz or d$\bar{z}$ legs, so

$$\hat{\mathcal{A}} = \hat{\mathcal{A}}_w \mathrm{d}w + \hat{\mathcal{A}}_{\bar{w}} \mathrm{d}\bar{w} + \mathcal{A}_0 \bar{e}^0 \quad \text{with} \quad \hat{\mathcal{A}}_w = -\frac{[\mathcal{A}\kappa]}{\langle \pi\gamma \rangle}\,, \hat{\mathcal{A}}_{\bar{w}} = -\frac{[\mathcal{A}\hat{\kappa}]}{\langle \pi\hat{\gamma} \rangle}\,, \tag{5.149}$$

$$\hat{\mathcal{B}} = \hat{\mathcal{B}}_w \mathrm{d}w + \hat{\mathcal{B}}_{\bar{w}} \mathrm{d}\bar{w} + \mathcal{B}_0 \bar{e}^0 \quad \text{with} \quad \hat{\mathcal{B}}_w = -\frac{[\mathcal{B}\kappa]}{\langle \pi\gamma \rangle}\,, \hat{\mathcal{B}}_{\bar{w}} = -\frac{[\mathcal{B}\hat{\kappa}]}{\langle \pi\hat{\gamma} \rangle}\,. \tag{5.150}$$

To perform the reduction we follow the procedure outlined in §5.2.4. Namely, we contract the six-dimensional Lagrangian of (5.148) with the vector fields ∂_z and $\partial_{\bar{z}}$, and restrict to gauge connections which are invariant under the flow of these vector fields. Thus, since the shifted gauge fields $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ manifestly have no dz or d$\bar{z}$ legs, and we are restricting to field configurations satisfying $L_{\partial_z}\hat{\mathcal{A}} = L_{\partial_z}\hat{\mathcal{B}} = L_{\partial_{\bar{z}}}\hat{\mathcal{A}} = L_{\partial_{\bar{z}}}\hat{\mathcal{B}} = 0$, the contraction by ∂_z and $\partial_{\bar{z}}$ only hits Ω in the first two terms, and $\bar{\partial}\Omega$ in the third. In particular, we find

$$(\partial_z \wedge \partial_{\bar{z}}) \lrcorner\, \Omega = \frac{\langle \alpha\beta \rangle^2}{2} \frac{\langle \pi\gamma \rangle \langle \pi\hat{\gamma} \rangle}{\langle \pi\alpha \rangle^2 \langle \pi\beta \rangle^2} e^0\,, \qquad (\partial_z \wedge \partial_{\bar{z}}) \lrcorner\, \bar{\partial}\Omega = -\frac{\langle \alpha\beta \rangle^2}{2} \bar{\partial}_0 \left(\frac{\langle \pi\gamma \rangle \langle \pi\hat{\gamma} \rangle}{\langle \pi\alpha \rangle^2 \langle \pi\beta \rangle^2} \right) e^0 \wedge \bar{e}^0\,. \tag{5.151}$$

Hence, the six-dimensional action reduces to a four-dimensional coupled Chern-Simons action

$$S_{\mathrm{gCS}_4}[\hat{A}, \hat{B}] = \int_X \omega \wedge \mathrm{CS}[\hat{A}] - \int_X \omega \wedge \mathrm{CS}[\hat{B}] - \frac{1}{2\pi i} \int_X \bar{\partial}\omega \wedge \langle \hat{A}, \hat{B} \rangle\,, \tag{5.152}$$

where $X = \mathbb{CP}^1 \times \mathbb{R}^2$,

$$\omega = \frac{\langle \alpha\beta \rangle^2}{2} \frac{\langle \pi\gamma \rangle \langle \pi\hat{\gamma} \rangle}{\langle \pi\alpha \rangle^2 \langle \pi\beta \rangle^2} e^0\,, \tag{5.153}$$

and $\hat{A}$ and $\hat{B}$ are the restrictions of $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ to X. Similarly, the boundary conditions (5.49) descend to analogous boundary conditions on $\hat{A}$ and $\hat{B}$. The action (5.152) has been considered before in [Ste21], albeit not with the choice of ω hereby discussed.

With the gauged 4d Chern-Simons action at hand, we may now localize. The procedure is entirely analogous to the one depicted in §5.3.4 so we shall omit some of the details. We begin by reparametrising our four-dimensional gauge fields $\hat{A}$ and $\hat{B}$ in terms of a new pair of connections $\hat{A}', \hat{B}'$ and smooth functions $\hat{g} \in C^\infty(X, G)$ and $\hat{h} \in C^\infty(X, H)$. We use the redundancy in the reparametrisation to fix $\hat{A}'_0 = \hat{B}'_0 = 0$. The boundary degrees of freedom of the resulting IFT_2 will be *a priori* be given by the evaluation of $\hat{g}$, $\hat{h}$, $\hat{u}$ and $\hat{v}$ at α and β. However, as in the 6d setting, we have some residual symmetry we can use to fix $\hat{g}|_\beta = \mathrm{id}$, $\hat{h}|_{\alpha,\beta} = \mathrm{id}$, and similarly, $\hat{v}|_{\alpha,\beta} = 0$. We are thus left with

$$\hat{g}|_\alpha \coloneqq g\,, \quad \hat{u}|_\alpha \coloneqq u\,, \quad \hat{u}|_\beta = \tilde{u}\,. \tag{5.154}$$

In terms of these variables, the bulk equations of motion of gCS_4 theory imply

$$\bar{\partial}_0 \hat{A}'_i = 0\,, \qquad \bar{\partial}_0 \hat{B}'_i = 0\,, \tag{5.155}$$

away from the zeroes of ω, namely γ and $\hat{\gamma}$. The on-shell gCS$_4$ action can be thus written as

$$
\begin{aligned}
S_{\text{gCS}_4}[\hat{A}', \hat{B}'] = \frac{1}{2\pi \mathrm{i}} \int_X & \bar{\partial}\omega \wedge \operatorname{Tr}\big(\hat{A}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1} - (\hat{g}^{-1}\hat{A}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g}) \wedge \hat{B}'\big) \\
& - \frac{1}{6\pi \mathrm{i}} \int_{X\times[0,1]} \bar{\partial}\omega \wedge \operatorname{Tr}\big(\hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g}\big) \,.
\end{aligned}
\tag{5.156}
$$

To obtain the IFT$_2$ we begin by looking at the bulk equations of motion (5.155). Liouville's theorem shows that the only bounded, holomorphic functions on $\mathbb{CP}^1$ are constant functions. We are after something a little more general than this, however, as we do not require the components of our gauge field to be bounded at the zeroes of ω. Indeed, we allow the w-component to have a pole at $\pi \sim \gamma$ and the $\bar{w}$-component to have a pole at $\pi \sim \hat{\gamma}$. With this analytic structure in mind, we can parameterise the solution of the bulk equation for $\hat{B}'$ by

$$
\hat{B}'_w = B_w + \frac{\langle \pi\hat{\gamma}\rangle}{\langle \pi\gamma\rangle}\,\Phi\,, \qquad \hat{B}'_{\bar{w}} = B_{\bar{w}} - \frac{\langle \pi\gamma\rangle}{\langle \pi\hat{\gamma}\rangle}\,\bar{\Phi}\,,
\tag{5.157}
$$

where we have conveniently used the field variables introduced in (5.100) to ease comparison with (5.102) after localisation to the IFT$_2$. In particular, under π-independent gauge transformations $B_w, B_{\bar{w}}$ have the transformation of 2d gauge fields, whilst Φ and $\bar{\Phi}$ transform as adjoint scalars.

Note that in the singular piece of these solutions, we have chosen to align the zero of each with the pole of the other. Notice that this choice is also completely general, since moving the zeros in the singular pieces amounts to field redefinitions between B_w and Φ, respectively, $B_{\bar{w}}$ and $\bar{\Phi}$. This is convenient since the flatness condition on $\hat{B}'$ reproduces Hitchin's equations,

$$
F_{w\bar{w}}[\hat{B}'] = F_{w\bar{w}}[B] - [\Phi, \bar{\Phi}] - \frac{\langle \pi\gamma\rangle}{\langle \pi\hat{\gamma}\rangle} D_w\bar{\Phi} - \frac{\langle \pi\hat{\gamma}\rangle}{\langle \pi\gamma\rangle} D_{\bar{w}}\Phi \,.
\tag{5.158}
$$

On the other hand, for the $\hat{A}'$ gauge field a convenient choice of parameterisation when solving the bulk equation of motion (5.155) is

$$
\hat{A}'_i = \frac{\langle \pi\alpha\rangle}{\langle \pi\gamma\rangle}\frac{\langle \beta\gamma\rangle}{\langle \beta\alpha\rangle}\,U_i + \frac{\langle \pi\beta\rangle}{\langle \pi\gamma\rangle}\frac{\langle \alpha\gamma\rangle}{\langle \alpha\beta\rangle}\,V_i\,, \quad i = w, \bar{w}\,.
\tag{5.159}
$$

This parametrisation, in which we have chosen the coefficients such that one term vanishes at $\pi \sim \alpha$ while the other vanishes at $\pi \sim \beta$, is adapted to the boundary conditions which can be solved for U_i and V_i to yield

$$
\begin{aligned}
\hat{A}'_w &= \hat{B}'_w - \frac{\langle \pi\beta\rangle}{\langle \pi\gamma\rangle}\frac{\langle \alpha\gamma\rangle}{\langle \alpha\beta\rangle}\left(D_w g g^{-1} + \frac{\langle \alpha\hat{\gamma}\rangle}{\langle \alpha\gamma\rangle}(1 - \mathrm{Ad}_g)\Phi\right) \\
\hat{A}'_{\bar{w}} &= \hat{B}'_{\bar{w}} - \frac{\langle \pi\beta\rangle}{\langle \pi\hat{\gamma}\rangle}\frac{\langle \alpha\hat{\gamma}\rangle}{\langle \alpha\beta\rangle}\left(D_{\bar{w}} g g^{-1} - \frac{\langle \alpha\gamma\rangle}{\langle \alpha\hat{\gamma}\rangle}(1 - \mathrm{Ad}_g)\bar{\Phi}\right) \,.
\end{aligned}
\tag{5.160}
$$

Replacing (5.157) and (5.160) in (5.156) and integrating[19] along $\mathbb{CP}^1$ we recover the IFT_2 given in (5.102).

5.6 Gauged LMP action

In the previous sections, we analysed a ghCS_6 setup where the meromorphic $(3,0)$-form Ω had two double poles, showing that such a theory leads to a gauged WZW_4 upon localisation to $\mathbb{R}^4$. To highlight some of the universal features of this procedure, we will now focus on another example in which the meromorphic $(3,0)$-form will have a single fourth order pole. Such a configuration of the ungauged hCS_6 was shown in [Bit22] to lead to the LMP action for ASDYM [LM87], [Par92].

5.6.1 LMP action from hCS_6

Let us start by reviewing the ungauged localisation of hCS_6 with a fourth order pole. We start with the action and $(3,0)$-form defined by

$$S_{\text{hCS}_6}[\mathcal{A}] = \frac{1}{2\pi \text{i}} \int_{\mathbb{PT}} \Omega \wedge \text{CS}(\mathcal{A}) \ , \qquad \Omega = k \, \frac{e^0 \wedge e^{\dot{a}} \wedge e_{\dot{a}}}{\langle \pi \alpha \rangle^4} \ . \tag{5.162}$$

As is usual in hCS_6, we must impose boundary conditions on the gauge field $\mathcal{A}$ to ensure the vanishing of the boundary variation

$$\delta S_{\text{hCS}_6}\big|_{\text{bdry}} = \frac{1}{2\pi \text{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \text{tr}(\delta\mathcal{A} \wedge \mathcal{A}) \ . \tag{5.163}$$

Evaluating the above integral is achieved by making use of the localisation formula (see appendix 5.10)

$$\frac{1}{2\pi \text{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge Q = \frac{k}{6} \int_{\mathbb{R}^4} \alpha_a \alpha_b \Sigma^{ab} \wedge \partial_0^3 Q\big|_\alpha \ . \tag{5.164}$$

Then, one finds that the boundary variation vanishes if we impose the boundary conditions

$$\mathcal{A}|_{\pi=\alpha} = 0 \quad \text{and} \quad \partial_0 \mathcal{A}|_{\pi=\alpha} = 0 \ . \tag{5.165}$$

Admissible gauge transformations. We now check which residual gauge symmetries survive with the preceding choice of boundary conditions. We proceed in a familiar fashion, introducing a new parameterisation of our gauge field $\mathcal{A}$ as

$$\mathcal{A} = \hat{g}^{-1}\mathcal{A}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g} \ , \qquad \mathcal{A}'_0 = 0 \ . \tag{5.166}$$

[19] To do so, we use the localisation formula in homogeneous coordinates

$$\frac{1}{2\pi \text{i}} \int_X \bar{\partial}\omega \wedge Q = -\frac{1}{2} \int_{\mathbb{R}^2} \left[\frac{\langle \alpha\gamma \rangle \langle \beta\hat{\gamma} \rangle + \langle \alpha\hat{\gamma} \rangle \langle \beta\gamma \rangle}{\langle \alpha\beta \rangle} Q|_\alpha + \langle \alpha\gamma \rangle \langle \alpha\hat{\gamma} \rangle (\partial_0 Q)|_\alpha \right] \quad + \quad \alpha \leftrightarrow \beta \ . \tag{5.161}$$

for any $Q \in \Omega^2(X)$.

This parameterisation has both external and internal gauge symmetries which act as

$$\begin{aligned} \text{External} \quad \hat{\gamma}: \quad & \mathcal{A} \mapsto \mathcal{A}^{\hat{\gamma}} , \quad \mathcal{A}' \mapsto \mathcal{A}' , \quad \hat{g} \mapsto \hat{g}\hat{\gamma} , \\ \text{Internal} \quad \check{\gamma}: \quad & \mathcal{A} \mapsto \mathcal{A} , \quad \mathcal{A}' \mapsto \mathcal{A}'^{\check{\gamma}} , \quad \hat{g} \mapsto \check{\gamma}^{-1}\hat{g} . \end{aligned} \tag{5.167}$$

The internal gauge transformations must satisfy $\bar{\partial}_0\check{\gamma} = 0$ to preserve the condition $\mathcal{A}'_0 = 0$. These transformations leave the value of $\mathcal{A}$ invariant and as such they are fully compatible with the boundary conditions. We will use the internal gauge symmetry to fix $\hat{g}|_{\pi=\alpha} = \mathrm{id}$. The story for the external gauge symmetries is slightly different, under external gauge transformations $\mathcal{A} \mapsto \mathcal{A}^{\hat{\gamma}}$ and so the value of $\mathcal{A}$ at the poles is not generically invariant. As such we must proceed with caution: we require our boundary conditions to be invariant under external gauge transformations, imposing constraints on the admissible symmetries at $\pi = \alpha$. This limits the amount of symmetry available for gauge fixing. The gauge transformation of the first boundary condition reads

$$\mathcal{A}^{\hat{\gamma}}|_{\pi=\alpha} = (\hat{\gamma}^{-1}\mathcal{A}\hat{\gamma} + \hat{\gamma}^{-1}\bar{\partial}\hat{\gamma})|_{\pi=\alpha} = 0 \quad \Longrightarrow \quad \gamma^{-1}\alpha^a\partial_{a\dot{a}}\gamma = 0 , \tag{5.168}$$

where we have defined

$$\hat{\gamma}|_{\pi=\alpha} = \gamma .$$

Here, we have derived the fact that at $\pi = \alpha$ we restrict our gauge transformations such that they are holomorphic on $\mathbb{R}^4$ with respect to the complex structure given by the point $\pi = \alpha$. Another way of stating this is that our admissible external gauge symmetries on $\mathbb{PT}$ localise to semi-local symmetries in the effective theory on $\mathbb{R}^4$. However this restriction is derived from only one half of the boundary conditions. Introducing the notation

$$\hat{\mathbf{\Gamma}} := \hat{\gamma}^{-1}\partial_0\hat{\gamma} ,$$

the gauge transformation of the second boundary condition reads

$$\begin{aligned} \partial_0\mathcal{A}^{\hat{\gamma}}|_{\pi=\alpha} &= \partial_0\big(\hat{\gamma}^{-1}\mathcal{A}\hat{\gamma} + \hat{\gamma}^{-1}\bar{\partial}\hat{\gamma}\big)|_{\pi=\alpha} = 0 , \\ &= \big([\hat{\gamma}^{-1}\mathcal{A}\,\hat{\gamma},\, \hat{\mathbf{\Gamma}}] + \hat{\gamma}^{-1}\partial_0\mathcal{A}\hat{\gamma} + \bar{\partial}\hat{\mathbf{\Gamma}} + [\hat{\gamma}^{-1}\bar{\partial}\hat{\gamma}, \hat{\mathbf{\Gamma}}] + \hat{\gamma}^{-1}\partial_{\dot{a}}\hat{\gamma}\; \bar{e}^{\dot{a}}\big)|_{\pi=\alpha} . \end{aligned} \tag{5.169}$$

Imposing the original boundary conditions we arrive at the constraint equation

$$\alpha^a\partial_{a\dot{a}}\mathbf{\Gamma} + \gamma^{-1}\hat{\alpha}^a\partial_{a\dot{a}}\gamma = 0 , \tag{5.170}$$

where we have used $\langle\alpha\hat{\alpha}\rangle = 1$ and defined

$$\hat{\mathbf{\Gamma}}|_{\pi=\alpha} = \mathbf{\Gamma} .$$

One solution is that the external gauge transformations are global symmetries of the localised effective theory $\mathrm{d}_{\mathbb{R}^4}\gamma = 0$, and $\mathbf{\Gamma}$ is holomorphic on $\mathbb{R}^4$ with respect to choice of complex structure given by the point $\alpha \in \mathbb{CP}^1$.

Tentatively, our localised theory should have 4 degrees of freedom, known as 'edge modes',

$$\underline{\mathbf{u}} := (g, \mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3) \ . \tag{5.171}$$

where

$$g = \hat{g}|_{\pi=\alpha} \ , \quad \mathbf{u}^1 := \hat{g}^{-1}\partial_0 \hat{g}|_{\pi=\alpha} \ , \quad \mathbf{u}^2 := \hat{g}^{-1}\partial_0^2 \hat{g}|_{\pi=\alpha} \ , \quad \mathbf{u}^3 := \hat{g}^{-1}\partial_0^3 \hat{g}|_{\pi=\alpha} \ . \tag{5.172}$$

However, some of these fields are spurious and can be gauged fixed away using the admissible gauge symmetries. We have already used the internal gauge symmetry to fix $g = \mathrm{id}$.. Furthermore, the second and third ∂_0-derivatives of the external gauge transformations are unconstrained by the boundary conditions, so they can be used to gauge fix $\mathbf{u}^2 = \mathbf{u}^3 = 0$. This leaves us with one dynamical degree of freedom in the localised theory on $\mathbb{R}^4$, namely $\mathbf{u}^1 : \mathbb{R}^4 \to \mathfrak{g}$ which we will now denote by $\mathbf{u}$ for brevity. In conclusion, after gauge fixing we have

$$\underline{\mathbf{u}} = (\mathrm{id}, \mathbf{u}, 0, 0) \ . \tag{5.173}$$

Solving the boundary conditions. Using the boundary conditions, we will solve for $\mathcal{A}'$ in the parametrisation (5.166) in terms of the edge modes. The first boundary condition tells us

$$\mathcal{A}'|_{\pi=\alpha} = 0 \quad \Rightarrow \quad \alpha^a A_{a\dot{a}} = 0 \quad \Rightarrow \quad A_{a\dot{a}} = \alpha_a C_{\dot{a}} \ . \tag{5.174}$$

Then, the second boundary condition equation is written as

$$\partial_0 \mathcal{A}'|_{\pi=\alpha} + \bar{\partial}\mathbf{u} = 0 \ , \tag{5.175}$$

which allows us to conclude that

$$C_{\dot{\alpha}} = \alpha^a \partial_{\alpha\dot{\alpha}} \mathbf{u} \ . \tag{5.176}$$

We now have all the ingredients to localise the hCS$_6$ action to $\mathbb{R}^4$.

Localisation to $\mathbb{R}^4$. We can write the action (5.162) in the new variables as

$$S = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}(\mathcal{A}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1}) - \frac{1}{6\pi \mathrm{i}} \int_{\mathbb{PT}\times[0,1]} \bar{\partial}\Omega \wedge \mathrm{Tr}(\hat{g}^{-1}\mathrm{d}\hat{g})^3 \ , \tag{5.177}$$

where in the second term we have extended $\mathbb{PT}$ to the 7-manifold $\mathbb{PT} \times [0, 1]$, whose boundary is a disjoint union of two copies of $\mathbb{PT}$. We have also extended our fields via a smooth homotopy $\hat{g} \to \hat{g}(t)$ so that $\hat{g}(0) = \mathrm{id}$ and $\hat{g}(1) = \hat{g}$. Applying the localisation formula (5.164) and the choice of gauge fixing (5.173), we arrive at the spacetime action

$$S_{\rm LMP}[\mathbf{u}] = \frac{k}{3}\int_{\mathbb{R}^4} \frac{1}{2}{\rm Tr}({\rm d}\mathbf{u}\ \wedge \star{\rm d}\mathbf{u}) + \frac{1}{3}\alpha_a\alpha_b\Sigma^{ab}\wedge {\rm Tr}(\,\mathbf{u}[{\rm d}\mathbf{u},{\rm d}\mathbf{u}]\,)\ . \tag{5.178}$$

We identify the action (5.178) as the LMP model for ASDYM. Upon performing reduction to $\mathbb{R}^2$ the above action becomes the pseudo-dual of the PCM [LV23].

5.6.2 Gauged LMP action from ghCS$_6$

In the previous subsection, we derived the LMP action from hCS$_6$. Next, we shall consider the same fourth order pole structure for gauged hCS$_6$. The starting point is to calculate the boundary variation and make a choice of isotropic subspace such that it vanishes.

Boundary conditions. Starting from the action

$$S_{\rm ghCS_6}[\mathcal{A},\mathcal{B}] = S_{\rm hCS_6}[\mathcal{A}] - S_{\rm hCS_6}[\mathcal{B}] - \frac{1}{2\pi {\rm i}}\int_{\mathbb{PT}} \bar\partial\Omega\wedge{\rm Tr}\big(\mathcal{A}\wedge\mathcal{B}\big)\ , \tag{5.179}$$

the boundary variation is given by

$$\delta S_{\rm ghCS_6}\big|_{\rm bdry} = \frac{1}{2\pi{\rm i}}\int_{\mathbb{PT}}\bar\partial\Omega\wedge{\rm Tr}\big(\delta\mathcal{A}\wedge(\mathcal{A}-\mathcal{B}) - \delta\mathcal{B}\wedge(\mathcal{B}-\mathcal{A})\big)\ . \tag{5.180}$$

Following in a parallel fashion to the hCS$_6$ case, one finds a suitable choice of boundary conditions is given by

$$\mathcal{A}|_{\pi=\alpha} = \mathcal{B}|_{\pi=\alpha}\ ,\quad \partial_0\mathcal{A}|_{\pi=\alpha} = \partial_0\mathcal{B}|_{\pi=\alpha}\ ,\quad \partial_0^2\mathcal{A}^\flat|_{\pi=\alpha} = \partial_0^2\mathcal{B}|_{\pi=\alpha}\ ,\quad \partial_0^3\mathcal{A}^\flat|_{\pi=\alpha} = \partial_0^3\mathcal{B}|_{\pi=\alpha}\ . \tag{5.181}$$

Gauge fixing. Gauge fixing will once again prove dividends in completing the localisation calculation, as such, we will consider the set of admissible gauge transformations respecting our boundary conditions. Performing a gauge transformation on the first boundary condition, one arrives at

$$\big(\hat\gamma^{-1}\mathcal{A}\hat\gamma + \hat\gamma^{-1}\bar\partial\hat\gamma\big)\,|_{\pi=\alpha} = \big(\hat\eta^{-1}\mathcal{B}\hat\eta + \hat\eta^{-1}\bar\partial\hat\eta\big)\,|_{\pi=\alpha}\ , \tag{5.182}$$

from which one concludes that the admissible gauge transformations must obey $\hat\gamma|_\alpha = \hat\eta|_\alpha$. Running through systematically, the second boundary condition requires

$$\begin{aligned}\Big([\hat\gamma^{-1}\mathcal{A}\,\hat\gamma + \hat\gamma^{-1}\bar\partial\hat\gamma,\,\hat{\mathbf{\Gamma}}] + \hat\gamma^{-1}\partial_0\mathcal{A}\hat\gamma + \bar\partial\hat{\mathbf{\Gamma}} + \hat\gamma^{-1}\partial_{\dot a}\hat\gamma\,\bar e^{\dot a}\Big)\,|_{\pi=\alpha}&\\ = \Big([\hat\eta^{-1}\mathcal{B}\,\hat\eta + \hat\eta^{-1}\bar\partial\hat\eta,\,\hat{\mathbf{N}}] + \hat\eta^{-1}\partial_0\mathcal{B}\hat\eta + \bar\partial\hat{\mathbf{N}} + \hat\eta^{-1}\partial_{\dot a}\hat\eta\,\bar e^{\dot a}\Big)\,|_{\pi=\alpha}\ ,&\end{aligned} \tag{5.183}$$

where we have denoted $\hat{\mathbf{\Gamma}} = \hat\gamma^{-1}\partial_0\hat\gamma$ and $\hat{\mathbf{N}} = \hat\eta^{-1}\partial_0\hat\eta$. Making use of the original boundary condition and the constraint $\hat\gamma|_\alpha = \hat\eta|_\alpha$, we conclude that admissible gauge transformations must also obey $\hat{\mathbf{\Gamma}}|_{\pi=\alpha} = \hat{\mathbf{N}}|_{\pi=\alpha}$. In a similar fashion, from the third boundary condition we conclude that $\hat{\mathbf{\Gamma}}^{(2)}_{\mathfrak{h}}|_\alpha = \hat{\mathbf{N}}^{(2)}|_\alpha$ where $\hat{\mathbf{\Gamma}}^{(2)} := \hat{\gamma}^{-1}\partial_0^2\hat{\gamma}$ and $\hat{\mathbf{N}}^{(2)} := \hat{\eta}^{-1}\partial_0^2\hat{\eta}$. Finally, from the fourth boundary condition we find $\hat{\mathbf{\Gamma}}^{(3)}_{\mathfrak{h}}|_\alpha = \hat{\mathbf{N}}^{(3)}|_\alpha$ where $\hat{\mathbf{\Gamma}}^{(3)} := \hat{\gamma}^{-1}\partial_0^3\hat{\gamma}$ and $\hat{\mathbf{N}}^{(3)} := \hat{\eta}^{-1}\partial_0^3\hat{\eta}$.

Armed with the admissible gauge symmetries of our theory, we set ourselves the task of gauge fixing our degrees of freedom. Naively, we would think there are 8 degrees of freedom in our theory,

$$\begin{aligned} \underline{\mathbf{u}} &:= (g, \mathbf{u}, \mathbf{u}^2, \mathbf{u}^3) \ , \\ \underline{\mathbf{v}} &:= (h, \mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3) \ . \end{aligned} \tag{5.184}$$

First, one considers the internal gauge symmetries of $\mathcal{A}$ and $\mathcal{B}$ which one can use to set both g and h to the identity. Next, one should note that the H-valued external gauge transformations of $\mathcal{B}$ parameterised by $\hat{\eta}$ are unconstrained at the point $\pi = \alpha$. As such, one can gauge fix $\mathbf{v}^i = 0$ for $i = 1, 2, 3$. Now, as our external gauge transformations of $\mathcal{A}$ parameterised by $\hat{\gamma}$ are constrained to coincide with $\hat{\eta}$ at $\pi = \alpha$, and we have used these symmetries in our choice of gauge fixing, we find that we are unable to gauge fix $\mathbf{u}^i$. As such, each of these degrees of freedoms will appear as fields in our effective theory on $\mathbb{R}^4$. In summary, after gauge fixing one has,

$$\begin{aligned} \underline{\mathbf{u}} &= (\mathrm{id}, \mathbf{u}, \mathbf{u}^2, \mathbf{u}^3) \ , \\ \underline{\mathbf{v}} &= (\mathrm{id}, 0, 0, 0) \ . \end{aligned} \tag{5.185}$$

Solving the boundary conditions. The first boundary condition reads

$$(\hat{g}^{-1}\mathcal{A}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g})|_{\pi=\alpha} = (\hat{h}^{-1}\mathcal{B}'\hat{h} + \hat{h}^{-1}\bar{\partial}\hat{h})|_{\pi=\alpha} \ . \tag{5.186}$$

Given our choice of gauge fixing (5.185) and the explicit solutions e.g. $\mathcal{A}'_{\dot{a}} = \pi^a A_{a\dot{a}}$, this implies

$$\mathcal{A}'|_{\pi=\alpha} = \mathcal{B}'|_{\pi=\alpha} \quad \Rightarrow \quad \alpha^a A_{a\dot{a}} = \alpha^a B_{a\dot{a}} \quad \Rightarrow \quad A_{a\dot{a}} = B_{a\dot{a}} - \alpha_a Q_{\dot{a}} \ . \tag{5.187}$$

We can then use the second boundary condition to solve for $Q_{\dot{a}}$,

$$\partial_0\mathcal{A}|_{\pi=\alpha} = \partial_0\mathcal{B}|_{\pi=\alpha} \ , \qquad \Longrightarrow \qquad Q_{\dot{a}} = -\alpha^a([B_{a\dot{a}}, \mathbf{u}] + \partial_{a\dot{a}}\mathbf{u}) = -\alpha^a\nabla_{a\dot{a}}\mathbf{u} \ . \tag{5.188}$$

These two boundary conditions are sufficient to solve for $A_{a\dot{a}}$ in terms of the other degrees of freedom,

$$A_{a\dot{a}} = B_{a\dot{a}} + \alpha_a\alpha^b\nabla_{b\dot{a}}\mathbf{u} \ . \tag{5.189}$$

Localisation to $\mathbb{R}^4$. Writing the action (5.179) in terms of the new field variables, one can see the only terms that will contribute to the effective action given our choice of gauge (5.185) will be

$$S_{\mathrm{ghCS}_6} = \frac{1}{2\pi i}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}(\mathcal{A}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1} - \hat{g}^{-1}\mathcal{A}'\hat{g} \wedge \mathcal{B}' - \hat{g}^{-1}\bar{\partial}\hat{g} \wedge \mathcal{B}') - \frac{1}{6\pi i}\int_{\mathbb{PT}\times[0,1]} \bar{\partial}\Omega \wedge \mathrm{Tr}(\hat{g}^{-1}\mathrm{d}\hat{g})^3 \ . \tag{5.190}$$

The localisation calculation of the gauged model is slightly more involved than the ungauged case due to the additional degrees of freedom appearing. However, as seen in the calculations in previous sections we expect $\mathbf{u}^2$ and $\mathbf{u}^3$ to appear only as Lagrange multipliers, in particular imposing self-duality type constraints for our gauge field B. With this tenet in mind, one can show that the 4d theory is given by

$$S_{\text{gLMP}}[\mathbf{u}, B] = k \int_{\mathbb{R}^4} \text{vol}_4 \frac{1}{2} \text{Tr}\,(\nabla^{a\dot{a}}\mathbf{u}\nabla_{a\dot{a}}\mathbf{u}) + \frac{1}{3}\epsilon^{\dot{a}\dot{b}}\,\text{Tr}\,(\mathbf{u}[\alpha^a\nabla_{a\dot{a}}\mathbf{u}, \alpha^b\nabla_{b\dot{b}}\mathbf{u}]) + \mathbf{u}\,\epsilon^{\dot{a}\dot{b}}\hat{\alpha}^a\hat{\alpha}^b F_{a\dot{a}b\dot{b}}(B)$$
$$+ \frac{1}{2}\mathbf{u}^2\epsilon^{\dot{a}\dot{b}}(\alpha^a\hat{\alpha}^b + \hat{\alpha}^a\alpha^b)F_{a\dot{a}b\dot{b}}(B) + \tilde{\mathbf{u}}^3\epsilon^{\dot{a}\dot{b}}\alpha^a\alpha^b F_{a\dot{a}b\dot{b}}(B)\ , \tag{5.191}$$

where we have performed a field redefinition $\mathbf{u}^3 \to \tilde{\mathbf{u}}^3 := \frac{1}{6}(\mathbf{u}^3 + 2[\mathbf{u}, \mathbf{u}^2])$. Upon reducing along a particular $\mathbb{R}^2$ subgroup, and appropriately performing redefinitions of our fields and parameters, one finds that the gauged LMP action matches the two-dimensional action eq. (5.139).

Implementing the Lagrange multipliers. In section 5.2.3 we reviewed how solutions to the ASDYM can be formulated in terms of a Yang's matrix after a partial gauge fixing of the ASD connection. In this section we will look to integrate out our Lagrange multiplier fields present in the action eq. (5.191) by solving the self duality constraints they impose in a similar vein. Indeed, one may understand the LMP equations of motion as the remaining ASDYM equation after these two constraints have been solved. This is analogous to the statement that the WZW_4 equation of motion is the remaining ASDYM equation for Yang's matrix.

The equation of motion found by varying $\tilde{\mathbf{u}}^3$ is an integrability condition along the 2-plane defined by α^a, and it maybe be solved by

$$\epsilon^{\dot{a}\dot{b}}\alpha^a\alpha^b F_{a\dot{a}b\dot{b}}(B) = 0 \qquad \Longrightarrow \qquad \alpha^a B_{a\dot{a}} = h^{-1}\alpha^a\partial_{a\dot{a}}h\ , \tag{5.192}$$

where $h \in C^\infty(\mathbb{R}^4) \otimes H$. It is helpful to parameterise the remaining degrees of freedom in $B_{a\dot{a}}$ in terms of a new field $C_{\dot{a}}$, defined by the relation

$$B_{a\dot{a}} = h^{-1}\partial_{a\dot{a}}h - \alpha_a\, h^{-1}C_{\dot{a}}h\ . \tag{5.193}$$

Then, the $\mathbf{u}^2$ equation of motion becomes

$$\epsilon^{\dot{a}\dot{b}}(\alpha^a\hat{\alpha}^b + \hat{\alpha}^a\alpha^b)F_{a\dot{a}b\dot{b}}(B) = 0 \qquad \Longleftrightarrow \qquad \epsilon^{\dot{a}\dot{b}}\alpha^a\partial_{a\dot{a}}C_{\dot{b}} = 0\ . \tag{5.194}$$

This may be solved explicitly by $C_{\dot{a}} = \alpha^a\partial_{a\dot{a}}f$ for $f \in C^\infty(\mathbb{R}^4) \otimes \mathfrak{h}$, such that the gauge field B is given by

$$B_{a\dot{a}} = h^{-1}\partial_{a\dot{a}}h + h^{-1}X_{a\dot{a}}h\ , \text{ where } \ X_{a\dot{a}} = -\alpha_a\alpha^b\partial_{b\dot{a}}f\ . \tag{5.195}$$

Reinserting this expression into the action eq. (5.191), the resulting theory may be written as a difference of two LMP actions. This can be done by performing a field redefinition $h\mathbf{u}h^{-1} = v - f$, for $v \in C^\infty(\mathbb{R}^4) \otimes \mathfrak{g}$, such that one arrives at the action

$$S_{\mathrm{gLMP}}[\mathbf{u}, B] = k \int_{\mathbb{R}^4} \frac{1}{2}\mathrm{Tr}(\mathrm{d}v \wedge \star \mathrm{d}v) + \frac{1}{3}\alpha_a \alpha_b \Sigma^{ab} \wedge \mathrm{Tr}(v[\mathrm{d}v, \mathrm{d}v]) \\ - k \int_{\mathbb{R}^4} \frac{1}{2}\mathrm{Tr}(\mathrm{d}f \wedge \star \mathrm{d}f) + \frac{1}{3}\alpha_a \alpha_b \Sigma^{ab} \wedge \mathrm{Tr}(f[\mathrm{d}f, \mathrm{d}f]) \ . \tag{5.196}$$

This demonstrates the conclusion

$$S_{\mathrm{gLMP}}[\mathbf{u}, B] = S_{\mathrm{LMP}}[v] - S_{\mathrm{LMP}}[f] \ . \tag{5.197}$$

5.7 Outlook

The construction presented in this work has led us to new integrable field theories in both four and two dimensions. We conclude by highlighting a number of interesting future directions prompted by these results.

Motivated by the observation that the gauged WZW model on the coset G/H in two dimensions can be written as the difference of WZW models for the groups G and H, we took the difference of two hCS_6 theories as our starting point. The boundary conditions (5.49) led us to add a boundary term resulting in the action (5.45). It is worth highlighting that the boundary variation vanishes on the boundary conditions (5.49) whether or not the boundary term is included, and the contribution of the boundary term to the IFT_4 vanishes if we invoke all the boundary conditions. However, while the algebraic boundary conditions, $\mathcal{A}^{\mathfrak{k}}|_{\alpha,\beta} = 0$ and $\mathcal{A}^{\mathfrak{h}}|_{\alpha,\beta} = \mathcal{B}_{\alpha,\beta}$ can be straightforwardly solved, this is not the case for the differential one $\partial_0\mathcal{A}^{\mathfrak{h}}|_{\alpha,\beta} = \partial_0\mathcal{B}_{\alpha,\beta}$. Therefore, we relaxed this condition meaning that the contribution of the boundary term no longer vanishes. Remarkably, for the specific boundary term added in (5.45), the constraints implied by the differential boundary condition now follow as on-shell equations of motion, leading to fully consistent IFT_4 and IFT_2.

There are compelling reasons to follow this strategy, including that the symplectic potential becomes tautological upon including the boundary term. However, it remains to understand why the differential boundary condition can be consistently dropped for this particular choice of boundary term, and a systematic interpretation of this is an open question. To address this, it would be appropriate to pursue a more formal study, complementing a homotopic analysis (along the lines done for CS_4 in [BSV22]) with a symplectic/Hamiltonian study of the 6d holomorphic Chern-Simons theory (similar to [Vic21] in the context of CS_4).

A second arena for formal development is the connection between 6d holomorphic Chern-Simons and five-dimensional Kähler Chern-Simons (KCS_5) theory [NS90; NS92]. This should mirror the relationship between CS_4 and CS_3 theories described by Yamazaki [Yam19]. To make this suggestion precise in the present context one may consider a Kaluza-Klein expansion around the $U(1)$ rotation in the $\mathbb{CP}^1$ that leaves fixed the location of the double poles, retaining the transverse coordinate as part of the bulk five-manifold of KCS_5. The details of this are left for future study.

It would also be interesting to explore the new integrable IFT_4 and IFT_2 that we have constructed. G/H coset CFTs in two dimensions have a rich spectrum of paraferminonic operators [BCR90; BCH91]. It would be very interesting to establish the lift or analogue of these objects in the context of the IFT_4. The natural framework for this is likely to involve the study of co-dimension one defects and associated higher-form symmetries.

For abelian H we find IFT_2 that, in the $\mathsf{k} \to 1$ limit, are related to massive integrable perturbations of the G/H gauged WZW models known as homogeneous sine-Gordon models [Par94; Fer+97]. These include the sine-Gordon and complex sine-Gordon models as special cases, two of the most well-understood IFT_2. There is nothing in our construction that prohibits non-abelian H and it would be instructive to study the resulting models in more detail.

An important class of IFT_2 are the symmetric space sigma models. These can be constructed either by restricting fields to parameterise G/H directly or by gauging a left action of H on the PCM. These theories have been realised in CS_4 through branch cut defects [CY19] and recently in hCS_6 [CW24]. One might explore the realisation the gauging construction of such models within the current framework, and generalise to $\mathbb{Z}_4$ graded semi-symmetric spaces (relevant for applications of CS_4 to string worldsheet theories [CS20; BP24]).

When G/H is a symmetric space, an alternative class of massive integrable perturbations of the G/H gauged WZW model are known as the symmetric space sine-Gordon models [BPS96; Fer+97]. In the landscape of IFT_2 these are related to the $\lambda \to 0$ limit [HMS14b; HT15] of the λ-deformation of the symmetric space sigma model [Sfe14]. Note that $\mathsf{k} \to 1$ and $\lambda \to 0$ both correspond to conformal limits and it would be instructive to explore the relation between the two constructions. More generally, it would be interesting to generalise the construction in this work to deformed models, in particular splitting one or both double poles in the meromorphic (3,0)-form Ω into simple poles, or dual models, for example considering the alternative boundary conditions (5.60).

Finally, recently novel approaches to constructing IFT_3 using higher Chern-Simons theory in 5d has been explored in [SV24; CL24]. Given that there is an overlap between the models that can be obtained from these constructions and from hCS_6, or more precisely its reduction to five dimensions, CS_5 on the mini-twistor correspondence space $\mathbb{PN}$ [BS23], it would be exciting to understand the link between the two, and investigate the existence of categorical generalisations of hCS_6.

5.8 Appendix: Homogeneous and Inhomogeneous Coordinates

Homogeneous coordinates on $\mathbb{CP}^1$ will be denoted by $\pi_a = (\pi_1, \pi_2)$ which are defined up to the equivalence relation $\pi_a \sim s\,\pi_a$ for any non-zero $s \in \mathbb{C}^*$. These have the advantage of being globally defined on $\mathbb{CP}^1$ but can lead to technical challenges in certain calculations. It can also be useful to work with inhomogeneous coordinates on two patches covering $\mathbb{CP}^1 \cong S^2$. Introducing an arbitrary spinor γ_a which satisfies $\langle\gamma\hat{\gamma}\rangle = 1$, the two patches covering $\mathbb{CP}^1$ will be defined by

$$U_1 = \{\pi_a \mid \langle\pi\hat{\gamma}\rangle \neq 0\} \ , \qquad U_2 = \{\pi_a \mid \langle\pi\gamma\rangle \neq 0\} \ . \tag{5.198}$$

Inhomogeneous coordinates may be defined on each patch by

$$\zeta = \frac{\langle\gamma\pi\rangle}{\langle\pi\hat{\gamma}\rangle} \ , \qquad \xi = \frac{\langle\pi\hat{\gamma}\rangle}{\langle\gamma\pi\rangle} \ , \qquad \xi = \zeta^{-1} \ . \tag{5.199}$$

In this section, we will restrict our attention to U_1 and the inhomogeneous coordinate ζ, knowing that an analogous discussion may be had on the other patch. The complex conjugate of the inhomogeneous coordinate ζ is

$$\bar{\zeta} = -\frac{\langle\hat{\pi}\hat{\gamma}\rangle}{\langle\gamma\hat{\pi}\rangle} \ . \tag{5.200}$$

Forms and vector fields on $\mathbb{CP}^1$ written in these coordinates are related to one another by

$$\begin{aligned} \mathrm{d}\zeta &= \frac{e^0}{\langle\pi\hat{\gamma}\rangle^2} \ , \qquad & \mathrm{d}\bar{\zeta} &= \frac{\langle\pi\hat{\pi}\rangle^2}{\langle\gamma\hat{\pi}\rangle^2}\bar{e}^0 \ , \\ \partial_\zeta &= \langle\pi\hat{\gamma}\rangle^2\partial_0 \ , \qquad & \partial_{\bar{\zeta}} &= \frac{\langle\gamma\hat{\pi}\rangle^2}{\langle\pi\hat{\pi}\rangle^2}\bar{\partial}_0 \ . \end{aligned} \tag{5.201}$$

It is also helpful to defined a weight zero basis of $(1,0)$-forms on $\mathbb{R}^4 \subset \mathbb{PT}$ by

$$\theta^{\dot{a}} = \frac{e^{\dot{a}}}{\langle\pi\hat{\gamma}\rangle} = \mathrm{d}x^{a\dot{a}}\,\gamma_a + \zeta\,\mathrm{d}x^{a\dot{a}}\,\hat{\gamma}_a \ . \tag{5.202}$$

Likewise the weight zero basis of $(0,1)$-forms on $\mathbb{R}^4 \subset \mathbb{PT}$ are defined by

$$\bar{\theta}^{\dot{a}} = \langle\pi\hat{\gamma}\rangle\,\bar{e}^{\dot{a}} = \frac{1}{1+\zeta\bar{\zeta}}(dx^{a\dot{a}}\,\hat{\gamma}_a - \bar{\zeta}\,dx^{a\dot{a}}\,\gamma_a) \ . \tag{5.203}$$

Given a point on $\mathbb{CP}^1$ defined by α_a in homogeneous coordinates, we denote the corresponding point in the inhomogeneous coordinate ζ by

$$\alpha = \frac{\langle\gamma\alpha\rangle}{\langle\alpha\hat{\gamma}\rangle} = \zeta|_{\pi_a\sim\alpha_a} \ . \tag{5.204}$$

We also have the relationship

$$\frac{\langle\pi\alpha\rangle}{\langle\pi\hat{\gamma}\rangle\langle\hat{\gamma}\alpha\rangle} = (\zeta - \alpha) \ . \tag{5.205}$$

5.9 Appendix: Projector Technology

Let us denote some relevant 2-forms by

$$\omega_{\alpha,\beta} = \frac{1}{\langle\alpha\beta\rangle}\Sigma_{\alpha,\beta}\,, \quad \mu_\alpha = \Sigma_{\alpha,\alpha}\,, \quad \mu_\beta = \Sigma_{\beta,\beta}\,. \tag{5.206}$$

We consider the operator on 1-forms on $\mathbb{R}^4$ given by

$$\mathcal{J}_{\alpha,\beta}(\sigma) = -\mathrm{i}\star(\omega_{\alpha,\beta}\wedge\sigma)\,, \quad \mathcal{J}^2_{\alpha,\beta} = -\mathrm{id}\,, \tag{5.207}$$

which allows us to define projectors

$$P = \frac{1}{2}\left(\mathrm{id} - \mathrm{i}\mathcal{J}\right) \qquad \bar{P} = \frac{1}{2}\left(\mathrm{id} + \mathrm{i}\mathcal{J}\right)\ . \tag{5.208}$$

This is suggestive of a complex structure and indeed if we take $\alpha = \gamma$ and $\beta = \hat{\gamma}$ then $\mathcal{J}_{\gamma,\hat{\gamma}}$ is a complex structure. We can compactly express the corresponding complex structure to a spinor γ_a as

$$\mathcal{J}_\gamma = -i(\gamma^a\partial_{a\dot{a}})\otimes(\hat{\gamma}_b \mathrm{d}x^{b\dot{a}}) - i(\hat{\gamma}^a\partial_{a\dot{a}})\otimes(\gamma_b\mathrm{d}x^{b\dot{a}})\,, \tag{5.209}$$

to which adapted complex coordinates are given by

$$\mathrm{d}z = \gamma_a\kappa_{\dot{a}}\mathrm{d}x^{a\dot{a}}\ , \qquad \mathrm{d}\bar{z} = \hat{\gamma}_a\hat{\kappa}_{\dot{a}}\mathrm{d}x^{a\dot{a}}\ , \qquad \mathrm{d}w = \gamma_a\hat{\kappa}_{\dot{a}}\mathrm{d}x^{a\dot{a}}\ , \qquad \mathrm{d}\bar{w} = -\hat{\gamma}_a\kappa_{\dot{a}}\mathrm{d}x^{a\dot{a}}\ . \tag{5.210}$$

The projectors P and $\bar{P}$ project onto $(1,0)$ and $(0,1)$ components thus realising the Dolbeault complex. These projectors enjoy a range of identities that we deploy in calculation.

$$\bar{P}(\star(\mu_\alpha\wedge\sigma)) = 0\,, \quad P(\star(\mu_\beta\wedge\sigma)) = 0\,, \quad \mu_\beta\wedge\bar{P}(\sigma) = 0\,, \quad \mu_\alpha\wedge P(\sigma) = 0\,, \tag{5.211}$$

$$\omega_{\alpha,\beta}\wedge\bar{P}(\sigma) = -\star\bar{P}(\sigma)\,, \quad \omega_{\alpha,\beta}P(\sigma) = \star P(\sigma) \tag{5.212}$$

$$\omega_{\alpha,\beta}\wedge\bar{P}(\sigma)\wedge\tau = \omega\wedge\sigma\wedge P(\tau)\,, \quad \omega_{\alpha,\beta}\wedge\bar{P}(\sigma)\wedge\bar{P}(\tau) = 0\,. \tag{5.213}$$

To move between form and component notation is useful to observe that

$$P(\sigma)_{a\dot{a}} = -\frac{1}{\langle\alpha\beta\rangle}\alpha_a\beta^b\sigma_{b\dot{a}}\,, \quad \bar{P}(\sigma)_{a\dot{a}} = \frac{1}{\langle\alpha\beta\rangle}\beta_a\alpha^b\sigma_{b\dot{a}}\,. \tag{5.214}$$

Further relations, used for processing the $\mathbb{CP}_1$ derivative boundary conditions, are

$$\alpha^a\sigma_{a\dot{a}}\bar{e}^{\dot{a}}|_\alpha = 2\star(\mu_\alpha\wedge a)\,, \quad \beta^a\tau_{a\dot{a}}\bar{e}^{\dot{a}}|_\alpha = -\langle\alpha\beta\rangle P(\tau)\,, \tag{5.215}$$

$$\beta^a\sigma_{a\dot{a}}\bar{e}^{\dot{a}}|_\beta = 2\star(\mu_\beta\wedge a)\,, \quad \alpha^a\tau_{a\dot{a}}\bar{e}^{\dot{a}}|_\beta = \langle\alpha\beta\rangle\bar{P}(\tau)\,. \tag{5.216}$$

As an application of this projector technology let us consider the (ungauged) WZW_4 model for which the equation of motion can be cast in terms of the right-invariant Maurer Cartan form

$R = \mathrm{d}gg^{-1}$ that obeys $\mathrm{d}R = R \wedge R$ as

$$\mathrm{d} \star \bar{P}(R) = \frac{1}{2}\mathrm{d}\,(\star - \omega_{\alpha,\beta}\wedge)\,\mathrm{d}gg^{-1} = 0\,. \tag{5.217}$$

We consider now a Yang-Mills connection $A = -\bar{P}(X)$. The equations for this to be anti-self dual are

$$\mu_\beta F[A] = 0\,, \quad \mu_\alpha F[A] = 0\,, \quad \omega_{\alpha,\beta} \wedge F[A] = 0\,. \tag{5.218}$$

The first of these vanishes identically by virtue of the fact that $\mu_\beta \wedge A = 0$. Since $\mu_\alpha \wedge A = -\mu_\alpha \wedge X$, the second provides a Bianchi identity

$$\mu_\alpha F[A] = -\mu_\alpha \wedge (\mathrm{d}X - X \wedge X) \tag{5.219}$$

and hence solved with $X = R$. The final equation returns the desired equation of motion as

$$\omega_{\alpha,\beta} \wedge F[A] = -\mathrm{d}(\omega_{\alpha,\beta} \wedge \bar{P}(R)) + \omega_{\alpha,\beta} \wedge \bar{P}(R) \wedge \bar{P}(R) = \mathrm{d} \star \bar{P}(R)\,. \tag{5.220}$$

At the Kähler point $\beta = \hat{\alpha} = \hat{\gamma}$, we can simply cast the ASDYM equations as

$$F^{2,0} = 0\ , \qquad F^{0,2} = 0\ , \qquad \varpi \wedge F^{1,1} = 0\ . \tag{5.221}$$

In this case, the connection is given by $A = -\bar{\partial}gg^{-1}$ is of type $(0,1)$, hence $F^{2,0} = 0$ automatically, $F^{0,2} = 0$ is zero by Bianchi identity and the equation of motion of WZW_4 is

$$\varpi \wedge \partial(\bar{\partial}gg^{-1}) = 0\,. \tag{5.222}$$

5.10 Appendix: Derivation of Localisation Formulae

In this work we are required to evaluate integrals of the form

$$I = \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge Q\ , \qquad Q \in \Omega^{0,2}(\mathbb{PT})\ . \tag{5.223}$$

In this appendix, we will derive general formulae for these integrals for the cases in which Ω has either two double poles or a single fourth order pole as used in this chapter. To compute these integrals efficiently we will move to inhomogeneous coordinates and make use of the identity

$$\partial_{\bar{\zeta}}\left(\frac{1}{\zeta - \alpha}\right) = -2\pi \mathrm{i}\,\delta^2(\zeta - \alpha)\ , \qquad \int_{\mathbb{CP}^1} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}\,\delta^2(\zeta - \alpha)\,f(\zeta) = f(\alpha)\ . \tag{5.224}$$

5.10.1 Two double poles

We consider the $(3,0)$-form given by

$$\Omega = \frac{1}{2}\frac{\langle\alpha\beta\rangle^2}{\langle\pi\alpha\rangle^2\langle\pi\beta\rangle^2}e^0\wedge e^{\dot a}\wedge e_{\dot a} = \frac{1}{2}\frac{(\alpha-\beta)^2}{(\zeta-\alpha)^2(\zeta-\beta)^2}\mathrm{d}\zeta\wedge\theta^{\dot a}\wedge\theta_{\dot a}\ . \tag{5.225}$$

Substituting this into our integral gives

$$I = -\frac{1}{2}\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}}\mathrm{d}\zeta\wedge\mathrm{d}\bar\zeta\,\partial_{\bar\zeta}\left(\frac{(\alpha-\beta)^2}{(\zeta-\alpha)^2(\zeta-\beta)^2}\right)\wedge\theta^{\dot a}\wedge\theta_{\dot a}\wedge Q\ . \tag{5.226}$$

Then, using the identity (5.224) gives

$$I = -\frac{(\alpha-\beta)^2}{2}\int_{\mathbb{PT}}\mathrm{d}\zeta\wedge\mathrm{d}\bar\zeta\left[\frac{\partial_\zeta\delta(\zeta-\alpha)}{(\zeta-\beta)^2}+\frac{\partial_\zeta\delta(\zeta-\beta)}{(\zeta-\alpha)^2}\right]\wedge\theta^{\dot a}\wedge\theta_{\dot a}\wedge Q\ . \tag{5.227}$$

Since the integral is symmetric under $\alpha\leftrightarrow\beta$, we will only compute the first term explicitly. Integration by parts and evaluating the integral over $\mathbb{CP}^1$ gives

$$I = \frac{(\alpha-\beta)^2}{2}\int_{\mathbb{R}^4}\partial_\zeta\left(\frac{\theta^{\dot a}\wedge\theta_{\dot a}\wedge Q}{(\zeta-\beta)^2}\right)\bigg|_\alpha\quad+\quad\alpha\leftrightarrow\beta\ . \tag{5.228}$$

We will first distribute the ∂_ζ derivative, leaving the 2-form Q completely general, resulting in

$$\begin{aligned}I = \frac{(\alpha-\beta)^2}{2}\int_{\mathbb{R}^4}\bigg[&\frac{-2}{(\zeta-\beta)^3}\theta^{\dot a}\wedge\theta_{\dot a}\wedge Q+\frac{2}{(\zeta-\beta)^2}\hat\gamma_a\mathrm{d}x^{a\dot a}\wedge\theta_{\dot a}\wedge Q\\ &+\frac{\theta^{\dot a}\wedge\theta_{\dot a}}{(\zeta-\beta)^2}\wedge\partial_\zeta Q\bigg]\bigg|_\alpha\quad+\quad\alpha\leftrightarrow\beta\ .\end{aligned} \tag{5.229}$$

The overall factor of $(\alpha-\beta)^2$ outside the integral cancels with the denominators in the integrand. We will also make use of (5.205) to return to spinor notation, and introduce self-dual 2-forms defined by $\Sigma^{ab}=\varepsilon_{\dot a\dot b}\mathrm{d}x^{a\dot a}\wedge\mathrm{d}x^{b\dot b}$.

$$\begin{aligned}I = \frac{1}{2}\int_{\mathbb{R}^4}\bigg[&\frac{-2\langle\hat\gamma\beta\rangle}{\langle\alpha\beta\rangle\langle\alpha\hat\gamma\rangle}\alpha_a\alpha_b\Sigma^{ab}\wedge Q|_\alpha+\frac{2}{\langle\alpha\hat\gamma\rangle}\hat\gamma_a\alpha_b\Sigma^{ab}\wedge Q|_\alpha\\ &+\alpha_a\alpha_b\Sigma^{ab}\wedge\frac{\partial_\zeta Q}{\langle\pi\hat\gamma\rangle^2}\bigg|_\alpha\bigg]\quad+\quad\alpha\leftrightarrow\beta\ .\end{aligned} \tag{5.230}$$

Expanding α_a in the basis formed by $\hat\gamma_a$ and β_a, we see that one component of the first term cancels the entire second term, and only a term proportional to $\alpha_a\beta_b\Sigma^{ab}$ survives. In the third term of the action, we recognise the combination ∂_0 acting on Q and make this replacement. In conclusion, we have the general formula

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}}\bar\partial\Omega\wedge Q = \int_{\mathbb{R}^4}\left[\frac{\alpha_a\beta_b\Sigma^{ab}}{\langle\alpha\beta\rangle}\wedge Q|_\alpha+\frac{1}{2}\alpha_a\alpha_b\Sigma^{ab}\wedge(\partial_0 Q)|_\alpha\right]\quad+\quad\alpha\leftrightarrow\beta\ , \tag{5.231}$$

or in differential form notation

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}}\bar{\partial}\Omega\wedge Q=\int_{\mathbb{R}^4}\left[\omega_{\alpha,\beta}\wedge Q|_\alpha+\frac{1}{2}\mu_\alpha\wedge(\partial_0 Q)|_\alpha\right]\quad+\quad\alpha\leftrightarrow\beta\ . \tag{5.232}$$

It is also helpful to specialise to 2-forms of the form $Q=\pi^a\pi^b Q_{a\dot{a}b\dot{b}}\bar{e}^{\dot{a}}\wedge\bar{e}^{\dot{b}}$ which we will often encounter in practice. In this case, we may make use of the identity

$$e^{\dot{c}}\wedge e_{\dot{c}}\wedge\bar{e}^{\dot{a}}\wedge\bar{e}^{\dot{b}}=-2\,\mathrm{vol}_4\,\varepsilon^{\dot{a}\dot{b}}\ . \tag{5.233}$$

and its generalisation valid for any spinors α_a and β_a

$$\alpha_a\beta_b\Sigma^{ab}\wedge\bar{e}^{\dot{a}}\wedge\bar{e}^{\dot{b}}=-2\,\mathrm{vol}_4\,\frac{\langle\alpha\hat{\pi}\rangle\langle\beta\hat{\pi}\rangle}{\langle\pi\hat{\pi}\rangle^2}\,\varepsilon^{\dot{a}\dot{b}}\ . \tag{5.234}$$

Using these identities on the above formula in the case $Q=\pi^a\pi^b Q_{a\dot{a}b\dot{b}}\bar{e}^{\dot{a}}\wedge\bar{e}^{\dot{b}}$ gives

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}}\bar{\partial}\Omega\wedge Q=-\int_{\mathbb{R}^4}\mathrm{vol}_4\left[\frac{\varepsilon^{\dot{a}\dot{b}}(\alpha^a\beta^b+\beta^a\alpha^b)}{\langle\alpha\beta\rangle}Q_{a\dot{a}b\dot{b}}|_\alpha+\varepsilon^{\dot{a}\dot{b}}\alpha^a\alpha^b(\partial_0 Q_{a\dot{a}b\dot{b}})|_\alpha\right]\quad+\quad\alpha\leftrightarrow\beta\ . \tag{5.235}$$

One final specialism is the case when $Q_{a\dot{a}b\dot{b}}=X_{a\dot{a}}Y_{b\dot{b}}$, in which case the answer can be recast again in differential form notation as

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}}\bar{\partial}\Omega\wedge Q=\int_{\mathbb{R}^4}\left[\omega_{\alpha,\beta}\wedge X\wedge Y|_\alpha+\frac{1}{2}\mu_\alpha\partial_0\wedge(X\wedge Y)|_\alpha\right]\quad+\quad\alpha\leftrightarrow\beta\ . \tag{5.236}$$

To apply these formulae we need the following $\mathbb{CP}_1$ derivatives :

$$\partial_0(\mathrm{d}\hat{g}\hat{g}^{-1})=\hat{g}\mathrm{d}\hat{u}\hat{g}^{-1}\,, \tag{5.237}$$

$$\partial_0(\hat{g}^{-1}\mathrm{d}\hat{g})=\mathrm{d}\hat{u}+[\hat{g}^{-1}\mathrm{d}\hat{g},\hat{u}]\,, \tag{5.238}$$

$$\partial_0(A)=\partial_0(B)=0\,, \tag{5.239}$$

$$\partial_0(\hat{g}^{-1}A\hat{g})=[\hat{g}^{-1}A\hat{g},\hat{u}]\,, \tag{5.240}$$

$$\partial_0\frac{1}{3}\mathrm{Tr}(\hat{g}^{-1}\mathrm{d}g)^3=\mathrm{d}\ \mathrm{Tr}(\hat{u}(\hat{g}^{-1}\mathrm{d}g)^2)\,, \tag{5.241}$$

in which used the definition $\hat{u}=\hat{g}^{-1}\partial_0\hat{g}$.

5.10.2 Fourth order Pole

In section 5.6, we considered a different $(3,0)$-form given by

$$\Omega=k\frac{e^0\wedge e^{\dot{\alpha}}\wedge e_{\dot{\alpha}}}{\langle\pi\alpha\rangle^4}=\frac{k'}{\langle\hat{\gamma}\alpha\rangle^4}\frac{d\zeta\wedge\theta^{\dot{\alpha}}\wedge\theta_{\dot{\alpha}}}{(\zeta-\alpha)^4}\ . \tag{5.242}$$

Substituting this into the general integral expression above gives

$$I = -\frac{k}{\langle\hat{\gamma}\alpha\rangle^4}\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}} \mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}\,\partial_{\bar{\zeta}}\bigg(\frac{1}{(\zeta-\alpha)^4}\bigg)\wedge\theta^{\dot{a}}\wedge\theta_{\dot{a}}\wedge Q\ . \tag{5.243}$$

Then, using the identity (5.224), we find

$$I = -\frac{k}{6\langle\hat{\gamma}\alpha\rangle^4}\int_{\mathbb{PT}} \mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}\,\Big(\partial_\zeta^3\delta(\zeta-\alpha)\Big)\wedge\theta^{\dot{a}}\wedge\theta_{\dot{a}}\wedge Q\ . \tag{5.244}$$

Applying integration by parts and completing the integral over $\mathbb{CP}^1$ gives

$$I = \frac{k}{6\langle\hat{\gamma}\alpha\rangle^4}\int_{\mathbb{R}^4} \partial_\zeta^3\Big(\theta^{\dot{a}}\wedge\theta_{\dot{a}}\wedge Q\Big)\Big|_\alpha\ . \tag{5.245}$$

In order to distribute this over the argument, it is helpful to have the identities

$$\theta^a\big|_\alpha = \frac{\alpha_a\mathrm{d}x^{a\dot{a}}}{\langle\hat{\gamma}\alpha\rangle}\ ,\qquad \partial_\zeta\theta^a\big|_\alpha = \hat{\gamma}_a\mathrm{d}x^{a\dot{a}}\ ,\qquad \partial_\zeta^2\theta^a\big|_\alpha = 0\ . \tag{5.246}$$

Then, distributing the three ∂_ζ-derivatives gives

$$I = \frac{k}{6\langle\hat{\gamma}\alpha\rangle^4}\int_{\mathbb{R}^4}\bigg[\frac{\alpha_a\alpha_b\Sigma^{ab}}{\langle\alpha\hat{\gamma}\rangle^2}\wedge\partial_\zeta^3 Q\big|_\alpha + 6\,\frac{\alpha_a\hat{\gamma}_b\Sigma^{ab}}{\langle\alpha\hat{\gamma}\rangle}\wedge\partial_\zeta^2 Q\big|_\alpha + 6\,\hat{\gamma}_a\hat{\gamma}_b\Sigma^{ab}\wedge\partial_\zeta Q\big|_\alpha\bigg]\ . \tag{5.247}$$

Converting this expression back into homogeneous coordinates (and making use of the fact that Q was a $(0,2)$-form on twistor space meaning $\hat{\alpha}_a\mathrm{d}x^{a\dot{a}}\wedge Q|_\alpha = 0$), this integral becomes

$$I = \frac{k}{6}\int_{\mathbb{R}^4}\alpha_a\alpha_b\Sigma^{ab}\wedge\partial_0^3 Q\big|_\alpha\ . \tag{5.248}$$

5.11 Appendix: Localisation Derivation with General Gaugings

In this appendix we describe in more detail the derivation of the gauged WZW_4 model from the gauged hCS_6 theory and the application of the localisation formulae above. We shall do this in a more general manner, allowing the gauging of an H subgroup that acts as

$$g\mapsto\rho_\beta(\ell)g\rho_\alpha(\ell^{-1})\,,\quad B\mapsto\ell B\ell^{-1}-\mathrm{d}\ell\ell^{-1}\,,\quad \ell\in H\subset G\,, \tag{5.249}$$

in which $\rho_i : H\to G$ are group homomorphisms (algebra homomorphisms will be denoted with the same symbol). The covariant derivative is then given by

$$\nabla gg^{-1} = \mathrm{d}gg^{-1} + B_\beta - gB_\alpha g^{-1}\mapsto\rho_\beta(\ell)(\nabla gg^{-1})\rho_\beta(\ell^{-1})\,, \tag{5.250}$$

in which we ease notation by setting $B_i = \rho_i(B)$.

The starting point is the six-dimensional theory

$$S_{\text{ghCS}_6}[\mathcal{A},\mathcal{B}] = S_{\text{hCS}_6}[\mathcal{A}] - S_{\text{hCS}_6}[\mathcal{B}] + S_{\text{bdy}}[\mathcal{A},\mathcal{B}]\,, \tag{5.251}$$

in which we specify a boundary interaction term

$$S_{\text{bdy}}[\mathcal{A},\mathcal{B}] = -\frac{q}{2\pi \text{i}}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \text{Tr}_{\mathfrak{g}}\left(\mathcal{A}\wedge\rho(\mathcal{B})\right)\,. \tag{5.252}$$

Here we have introduced a parameter q, which will ultimately be set to one, to keep track of the contributions from this boundary term. To specify this term we include an algebra homomorphism ρ which only needs to be defined piecewise on the components of the support of $\partial\Omega$. We could choose to dispense the higher-dimensional covariance and simply add different boundary terms specified only at the location of the poles but it is convenient to formally consider ρ to be a defined as a piecewise map that takes values $\rho|_{\pi=\alpha,\beta} = \rho_{\alpha,\beta}$.

To define a six-dimensional theory requires imposing conditions that ensure the vanishing of the boundary term

$$\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \left(\text{Tr}_{\mathfrak{g}}\left(\delta\mathcal{A}\wedge(\mathcal{A}-q\rho(\mathcal{B})) + q\rho(\delta\mathcal{B})\wedge\mathcal{A}\right) - \text{Tr}_{\mathfrak{h}}(\delta\mathcal{B}\wedge\mathcal{B})\right)\,. \tag{5.253}$$

We are required to cancel a term involving the inner product on the algebra $\mathfrak{h}$ with one on $\mathfrak{g}$, which can be achieved demanding

$$\text{Tr}_{\mathfrak{g}}(\rho(x)\rho(y))|_{\alpha,\beta} = \text{Tr}_{\mathfrak{h}}(xy) \quad \forall x,y\in\mathfrak{h}\,. \tag{5.254}$$

Note that as a consequence this implies

$$\text{Tr}_{\mathfrak{g}}(\rho_\alpha(x)\rho_\alpha(y)) = \text{Tr}_{\mathfrak{h}}(xy) = \text{Tr}_{\mathfrak{g}}(\rho_\beta(x)\rho_\beta(y))\,, \tag{5.255}$$

which is the anomaly-free condition allowing for the construction of a gauge-invariant extension to the WZW model for the gauge symmetry (5.249). With this condition the boundary term produced by variation is given by

$$\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \big(\text{Tr}_{\mathfrak{g}}\big(\delta\mathcal{A}\wedge(\mathcal{A}-q\rho(\mathcal{B})) + q\rho(\delta\mathcal{B})\wedge(\mathcal{A}-q^{-1}\wedge\rho(\mathcal{B}))\big)\big)\,, \tag{5.256}$$

and is set to zero by the conditions

$$\mathcal{A}^{\mathfrak{k}}\big|_{\alpha,\beta} = 0\,, \qquad \mathcal{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \rho(\mathcal{B})\big|_{\alpha,\beta}\,, \qquad \partial_0\mathcal{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \rho(\partial_0\mathcal{B})\big|_{\alpha,\beta}\,. \tag{5.257}$$

It is noteworthy that if we impose all of these conditions from the outset, the contribution from the explicit boundary term $S_{\text{bdy}}[\mathcal{A},\mathcal{B}]$ would vanish. However, from a four-dimensional perspective the boundary conditions on $\partial_0(\mathcal{A})$ lead to *differential* constraints on the fundamental fields and it is not clear that one should, or could, naively invoke them to produce a Lagrangian description. Instead what we shall do is *only* impose the conditions $\mathcal{A}^{\mathfrak{k}}\big|_{\alpha,\beta} = 0$ and $\mathcal{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \rho(\mathcal{B})\big|_{\alpha,\beta}$, which can be solved algebraically and substituted into the Lagrangian without concern. Doing this one finds that $S_{\text{bdy}}[\mathcal{A},\mathcal{B}]$ does contribute, and when $q=1$ in particular, it provides a gauge invariant completion of the action. The boundary conditions that we have not imposed have not been forgotten, instead when $q=1$ they are recovered as on-shell equations in this four-dimensional theory. This provides an alternative view of the procedure; the explicit boundary term is serving to implement the constraints arising from $\partial_0\mathcal{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \rho(\partial_0\mathcal{B})\big|_{\alpha,\beta}$ at the Lagrangian level. We can see this explicitly by observing that if we just impose $\mathcal{A}^{\mathfrak{k}}\big|_{\alpha,\beta} = 0$ and $\mathcal{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \rho(\mathcal{B})\big|_{\alpha,\beta}$ then

$$\begin{aligned}
&\Big(\delta\mathcal{A}\wedge(\mathcal{A}-q\rho(\mathcal{B})) + q\rho(\delta\mathcal{B})\wedge\mathcal{A} - \rho(\delta\mathcal{B})\wedge\rho(\mathcal{B})\Big)|_{\alpha,\beta} = 0\,,\\
&\partial_0\Big(\delta\mathcal{A}\wedge(\mathcal{A}-q\rho(\mathcal{B})) + q\rho(\delta\mathcal{B})\wedge\mathcal{A} - \rho(\delta\mathcal{B})\wedge\rho(\mathcal{B})\Big)|_{\alpha,\beta}\\
&\qquad = (1-q)\delta(\partial_0\mathcal{A}-\rho(\partial_0\mathcal{B}))\wedge\rho(\mathcal{B})|_{\alpha,\beta} + (1+q)\rho(\delta\mathcal{B})\wedge(\partial_0\mathcal{A}-\rho(\partial_0\mathcal{B}))|_{\alpha,\beta}\,.
\end{aligned} \tag{5.258}$$

Therefore, for $q=1$ we see that the boundary equation of motion for $\mathcal{B}$ is precisely $\partial_0\mathcal{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \rho(\partial_0\mathcal{B})\big|_{\alpha,\beta}$.

The localisation proceeds as follows. First, we change parametrisation $\mathcal{A} = \mathcal{A}'^{\hat{g}}$ and $\mathcal{B} = \mathcal{B}'^{\hat{h}}$ fixing some redundancy by demanding $\mathcal{A}'$ and $\mathcal{B}'$ have no $\mathbb{CP}_1$ legs. Second, we fix some of the residual symmetry preserved by the boundary conditions to set $\hat{g}|_\beta = \hat{h}|_{\alpha,\beta} = \text{id}$ and $\partial_0\hat{h}|_{\alpha,\beta} = 0$. The remaining fields are $\hat{g}|_\alpha = g$, $\hat{g}^{-1}\partial_0\hat{g}|_\alpha = u$, $\hat{g}^{-1}\partial_0\hat{g}|_\beta = \tilde{u}$ and the four-dimensional gauge fields A and B that arise from $\mathcal{A}'$ and $\mathcal{B}'$ once their holomorphicity is imposed.

We may now directly apply the localisation formulae (5.236) to show that the hCS terms localise (without imposing any boundary conditions) to give

$$\begin{aligned}
S_{\text{hCS}}[\mathcal{A}] \simeq \int_{\mathbb{R}^4} &\omega_{\alpha,\beta}\wedge\text{Tr}_{\mathfrak{g}}(A^g\wedge g^{-1}\text{d}g) - \omega_{\alpha,\beta}\wedge\mathcal{L}_{\text{WZ}}[g]\\
&+\frac{1}{2}\mu_\alpha\wedge\text{Tr}_{\mathfrak{g}}(A^g\wedge\text{d}u) + \frac{1}{2}\mu_\beta\wedge\text{Tr}_{\mathfrak{g}}(A\wedge\text{d}\tilde{u})\,,
\end{aligned} \tag{5.259}$$

while $S_{\text{hCS}}[\mathcal{B}]$ yields zero in this gauge. Let us first consider the terms involving $\omega_{\alpha,\beta}$. Since the gauge completion of the WZ term is

$$\mathcal{L}_{\text{gWZ}}[g,B] = \mathcal{L}_{\text{WZ}}[g] + \text{Tr}_{\mathfrak{g}}\left(g^{-1}\text{d}g\wedge B_\alpha + \text{d}gg^{-1}\wedge B_\beta + g^{-1}B_\beta g B_\alpha\right)\,, \tag{5.260}$$

we may express them (trace left implicit) as

$$\begin{aligned}
&\omega_{\alpha,\beta}\wedge\left(A^g\wedge g^{-1}\text{d}g - \mathcal{L}_{\text{WZ}}[g]\right)\\
=&\,\omega_{\alpha,\beta}\wedge\left(A^g\wedge g^{-1}\text{d}g - \mathcal{L}_{\text{gWZ}}[g,B] + g^{-1}\text{d}g\wedge B_\alpha + \text{d}gg^{-1}B_\beta + g^{-1}B_\beta g B_\alpha\right)\\
=&\,\omega_{\alpha,\beta}\wedge\left((A^g - B_\alpha)\wedge g^{-1}\nabla g - \mathcal{L}_{\text{gWZ}}[g,B] + A^g\wedge B_\alpha - A\wedge B_\beta\right)\,.
\end{aligned} \tag{5.261}$$

To proceed we invoke the algebraic boundary conditions of eq. (5.257), which in differential form notation become

$$A = B_\beta - \bar{P}(\nabla g g^{-1}) \Leftrightarrow A^g = P(g^{-1}\nabla g) + B_\alpha \,. \tag{5.262}$$

such that

$$\begin{aligned} &\omega_{\alpha,\beta} \wedge \left(A^g \wedge g^{-1}\mathrm{d}g - \mathcal{L}_{\mathrm{WZ}}[g]\right) \\ =&\,\omega_{\alpha,\beta} \wedge \left(P(g^{-1}\nabla g) \wedge g^{-1}\nabla g - \mathcal{L}_{\mathrm{gWZ}}[g,B] + A^g \wedge B_\alpha - A \wedge B_\beta\right) \\ =& -\frac{1}{2} g^{-1}\nabla g \wedge \star(g^{-1}\nabla g) - \omega_{\alpha,\beta} \wedge \left(\mathcal{L}_{\mathrm{gWZ}}[g,B] - A^g \wedge B_\alpha + A \wedge B_\beta\right) \,. \end{aligned} \tag{5.263}$$

Here in the last line we made use of the identity $\omega \wedge P(\sigma) \wedge \sigma = -\frac{1}{2}\sigma \wedge \star\sigma$ for a 1-form σ. To treat the terms involving μ_α and μ_β we may combine the algebraic boundary conditions with the properties $\mu_\alpha \wedge P(X) = \mu_\beta \wedge \bar{P}(X) = 0$ such that $\mu_a \wedge A^g = \mu_\alpha B_\alpha$ and $\mu_\beta \wedge A = \mu_\beta B_\beta$. In summary we find

$$\begin{aligned} S_{\mathrm{hCS}}[\mathcal{A}] \simeq \int_{\mathbb{R}^4} &-\frac{1}{2}\mathrm{Tr}_{\mathfrak{g}} \left(g^{-1}\nabla g \wedge \star g^{-1}\nabla g\right) - \omega_{\alpha,\beta} \wedge \left(\mathcal{L}_{\mathrm{gWZ}}[g,B] + \mathrm{Tr}_{\mathfrak{g}}(A \wedge B_\beta - A^g B_\alpha)\right) \\ &+ \frac{1}{2}\mu_\alpha \wedge \mathrm{Tr}(B_\alpha \wedge \mathrm{d}u) + \frac{1}{2}\mu_\beta \wedge \mathrm{Tr}(B_\beta \wedge \mathrm{d}\tilde{u}) \,. \end{aligned} \tag{5.264}$$

The localisation of the explicit boundary term yields, after using $\mu_a \wedge A^g = \mu_\alpha B_\alpha$,

$$\begin{aligned} S_{\mathrm{bdy}}[\mathcal{A},\mathcal{B}] \simeq -q\int_{\mathbb{R}^4} &\omega_{\alpha,\beta} \wedge \mathrm{Tr}_{\mathfrak{g}}(A^g B_\alpha - A B_\beta) \\ &+ \frac{1}{2}\mu_\alpha \wedge \mathrm{Tr}_{\mathfrak{g}}((\mathrm{d}u + [B_\alpha, u])B_\alpha) + \frac{1}{2}\mu_\beta \wedge \mathrm{Tr}_{\mathfrak{g}}((\mathrm{d}\tilde{u} + [B_\beta, \tilde{u}])B_\beta) \,. \end{aligned} \tag{5.265}$$

The significance of the boundary term now becomes clear, as it serves to ensure manifest gauge invariance. When $q = 1$ the terms $\omega_{\alpha,\beta} \wedge \mathrm{Tr}(A^g B_\alpha - A B_\beta)$ directly cancel. The contributions of the entire localised action that are wedged against μ_α sum to

$$\mu_\alpha \wedge \mathrm{Tr}_{\mathfrak{g}} \left((1-q)\, \mathrm{d}u \wedge B_\alpha + 2q\, uF[B]_\alpha - 2q\, \mathrm{d}(B_\alpha u)\right) \,. \tag{5.266}$$

We can see that for $q = 1$ we find a gauge-invariant field strength together with a total derivative term that we discard. The terms wedged against μ_β give a similar contribution. Hence the fully localised action becomes

$$\begin{aligned} S \simeq \int_{\mathbb{R}^4} &-\frac{1}{2}\mathrm{Tr}_{\mathfrak{g}} \left(g^{-1}\nabla g \wedge \star g^{-1}\nabla g\right) - \omega_{\alpha,\beta} \wedge \mathcal{L}_{\mathrm{gWZ}}[g,B] \\ &+ \mu_\alpha \wedge \mathrm{Tr}_{\mathfrak{g}}(uF[B]_\alpha) + \mu_\beta \wedge \mathrm{Tr}_{\mathfrak{g}}(\tilde{u}F[B]_\beta) \,. \end{aligned} \tag{5.267}$$

Noting that the components of u and $\tilde{u}$ in complement of $\mathfrak{h}$ decouple we can view u and $\tilde{u}$ as

$\mathfrak{h}$-valued and write

$$S \simeq \int_{\mathbb{R}^4} -\frac{1}{2}\mathrm{Tr}_{\mathfrak{g}}\left(g^{-1}\nabla g \wedge \star g^{-1}\nabla g\right) - \omega_{\alpha,\beta} \wedge \mathcal{L}_{\mathrm{gWZ}}[g,B] \\ + \mu_\alpha \wedge \mathrm{Tr}_{\mathfrak{h}}(uF[B]) + \mu_\beta \wedge \mathrm{Tr}_{\mathfrak{h}}(\tilde{u}F[B])\,. \tag{5.268}$$

Chapter 6

Unifying approaches to chiral bosons

The results in this chapter were found in collaboration with Alex S. Arvanitakis, Ondrej Hulik, Alexander Sevrin, and Daniel C. Thompson [Arv+23].

6.1 Introduction

Chiral bosons are an example of self-dual p-forms: p-form fields ϕ in a $(2p+2)$-dimensional space-time whose $(p+1)$-form field strength $\mathrm{d}\phi$ is self-dual, i.e. $\mathrm{d}\phi = \star\mathrm{d}\phi$. Such fields are pervasive within string theory as they are often required to complete multiplets which furnish supersymmetry. Familiar examples include the self-dual 2-form, living in the 6-dimensional $\mathcal{N} = (2, 0)$ multiplet, pertinent to both the M5 brane and the self-dual string. Another example is the self-dual Ramond-Ramond 4-form of type IIB supergravity. In two dimensions, i.e. $p = 0$, the self-duality condition amounts to the chiral boson being purely left-moving, i.e. $\partial_-\phi = 0$. These self-dual scalars form a critical component of the "doubled" approaches to string theory [Duf90; Tse90; Tse91; Hul05; Hul07] in which T-duality is promoted to a manifest symmetry of the worldsheet. Indeed, this behaviour is generic for duality-invariant formalisms and also appears in the context of gauge theory more generally [HT88; BLM19; Mkr19; BEM21; BG23].

Given these motivations, it is highly desirable to have a quantum treatment of chiral bosons. The quantisation of a given theory is made significantly easier if one can find a manifestly Lorentz-invariant Lagrangian, written in terms of Lorentz-covariant objects. However, formulating a Lorentz-invariant Lagrangian for chiral bosons is somewhat challenging, even classically, in essence because the chirality condition is a first order differential equation, whereas one anticipates second order differential equations to arise for bosonic fields. Over the years, many attempts have been made towards finding a suitable Lagrangian for chiral bosons, but most of these formalisms require making some other concession in order to accommodate manifest Lorentz-invariance.

We will provide a more detailed overview of existing approaches to chiral bosons in section 6.2, but first, let's summarise the key aspects of some well-known formalisms. Early approaches, such as those of Siegel [Sie84] and Floreanini and Jackiw [FJ87], sacrificed manifest Lorentz-invariance in order to describe the correct degrees of freedom. Manifest Lorentz-invariance was recovered by Pasti, Sorokin, and Tonin [PST97] at the expense of adding an auxiliary field to the Lagrangian in a non-polynomial manner. More recently, Mkrtchyan [Mkr19] added yet another auxiliarly field to the PST action, rendering it polynomial. Unfortunately, the non-polynomial origin of this action is hiding just beneath the surface; in order to demonstrate that the Mkrtchyan action describes a chiral boson, one must leverage a rather mysterious non-polynomial symmetry. As well as improving upon the aforementioned shortcomings, duality-invariant string theory motivates us to include non-abelian fields and twisted self-duality.

Alternative to the two-dimensional approaches, the boundary dynamics of three-dimensional Chern-Simons theory are known to describe chiral bosons [Eli+89]. This approach has a few appealing features: the quantisation of Chern-Simons theory has proved very successful, e.g. [Eli+89; Wit89b; ADW91]; the manifestly Lorentz-invariant action comes equipped with a standard, polynomial symmetry; the non-abelian generalisation is already known; and we will see that the inclusion of a twisted self-duality relation is incredibly simple. While Chern-Simons theory is known to describe chiral bosons, its relationship to the various two-dimensional approaches is less evident. Indeed, one of the key purposes of this note is to clarify these relationships.

In the forthcoming sections, we will demonstrate that, alongside the Floreanini-Jackiw action, both the PST and Mkrtchyan formalisms can be derived from Chern-Simons theory on a manifold with boundary. These derivations will centre around algebraic manipulations of the actions, and (in the abelian case) many of the calculations will also apply to the higher-form versions of the PST and Mkrtchyan actions. Using this relationship, we will also be able to provide the non-abelian and twisted self-duality generalisations of both formalisms. Having done this, we will also have a derivation of the Floreanini-Jackiw action with twisted self-duality, more commonly referred to in the literature as the $\mathcal{E}$-model where it arises in the context of Poisson-Lie T-duality [KS95; KS96a; KS96b]. While these generalisations were known for the PST action [DST16], we believe that they were previously unknown for the Mkrtchyan action. In both cases, we hope that this derivation will offer a novel perspective on these two-dimensional approaches to chiral bosons. Furthermore, we hope that this note will provide additional motivation to pursue the Chern-Simons approach to chiral bosons.

6.2 Approaches to Chiral Bosons in two-dimensions

Let's start by reviewing some of the two-dimensional approaches to chiral bosons which we mentioned in the introduction. Some other approaches to chiral bosons which, for the sake of brevity, we will not explore include those of Henneaux-Teitelboim and Beckaert-Henneaux [BH99; HT88], McClain-Wu-Yu [MYW90], Perry-Schwarz [PS97], Sen [Sen16; Sen20] and its recent extensions [BG24; And+22], and Townsend [Tow20; MT22].

We will consider a bosonic field ϕ living on a two-dimensional Lorentzian[1] manifold Σ. Starting with the action for a free non-chiral boson, we may try to incorporate a gauge symmetry $\delta_\xi \phi = \xi \partial_- \phi$, such that the only physical content obeys $\partial_- \phi = 0$. This can be done by gauging a chiral conformal symmetry. First, we introduce a gauge field h transforming as[2] $\delta_\xi h = \partial_+ \xi + \xi \partial_- h - h \partial_- \xi$, and then we write the action in terms of the would-be-covariant derivatives $\nabla_+ \phi = \partial_+ \phi - h \partial_- \phi$ and $\nabla_- \phi = \partial_- \phi$. Doing this, one arrives at Siegel's gauge invariant action [Sie84],

$$S_{\mathrm{S}}[\phi] = \int_\Sigma \mathrm{d}^2\sigma\, \nabla_+ \phi \nabla_- \phi = \int_\Sigma \mathrm{d}^2\sigma \left(\partial_+ \phi \partial_- \phi - h (\partial_- \phi)^2 \right) . \tag{6.1}$$

A challenge with this approach is that, even after gauge fixing, the h equation of motion remains as a constraint to be invoked. Whilst the constraint $(\partial_- \phi)^2 \approx 0$ evidently implies $\partial_- \phi \approx 0$, its matrix of first derivatives is degenerate on the constraint surface making treatment difficult.

Gauge fixing $h = 1$ leads us to the Floreanini-Jackiw [FJ] action [FJ87],

$$S_{\mathrm{FJ}}[\phi] = \int_\Sigma \mathrm{d}^2\sigma \left(\partial_\sigma \phi \partial_- \phi \right) , \tag{6.2}$$

whose equation of motion, although second order, has the general solution

$$\partial_- \phi = g(\tau) . \tag{6.3}$$

By virtue of another gauge symmetry[3],

$$\tilde{\delta}_{\mathrm{FJ}} \phi = h(\tau) , \tag{6.4}$$

the general solution is gauge equivalent to the chirality condition $\partial_- \phi = 0$. An evident downside of this approach, common also to [HT88; PS97], is that two-dimensional Lorentz invariance is not manifest at the Lagrangian level. Although some one-loop calculations can be done for chiral fields in such a framework, it becomes rather challenging to extend to higher loops.

Notice that this action has another symmetry,

$$\hat{\delta}_{\mathrm{FJ}} \phi = \varepsilon(\sigma^+) . \tag{6.5}$$

At first, we might worry that this kills all of the degrees of freedom of our chiral boson. Fortunately,

[1] We choose coordinates (τ, σ) and define the metric and orientation by $\mathrm{d}s^2 = \mathrm{d}\tau^2 - \mathrm{d}\sigma^2$ and $\mathrm{d}^2\sigma = \mathrm{d}\tau \wedge \mathrm{d}\sigma$ respectively. In lightcone coordinates $\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)$, these are given by $\mathrm{d}s^2 = 4\, \mathrm{d}\sigma^+ \mathrm{d}\sigma^-$ and $\mathrm{d}^2\sigma = -2\, \mathrm{d}\sigma^+ \wedge \mathrm{d}\sigma^-$. The Hodge star acts as $\star \mathrm{d}\tau = \mathrm{d}\sigma$, $\star \mathrm{d}\sigma = \mathrm{d}\tau$, and $\star \mathrm{d}\sigma^\pm = \pm \mathrm{d}\sigma^\pm$. Then, a self-dual field $\mathrm{d}\phi = \star \mathrm{d}\phi$ is one for which $\partial_- \phi = 0$. Given a 1-form ω, we define $\|\omega\|^2 = g^{\alpha\beta} \omega_\alpha \omega_\beta = \omega_+ \omega_-$.

[2] Weights are naturally $h \equiv h_{++}$ and $\xi \equiv \xi^-$. A complementary perspective is to consider the metric on the worldsheet as $\mathrm{d}s^2 = 4 \mathrm{d}\sigma^+ \mathrm{d}\sigma^- + 4h (\mathrm{d}\sigma^+)^2$ with the gauge transformation being the conformal transformations that preserve this metric form.

[3] The Noether charge corresponding to $\tilde{\delta}_{\mathrm{FJ}}$ is zero.

we find that its Noether charge is non-vanishing[4], and this means that we should not interpret it as a gauge symmetry but rather as a chiral affine symmetry. This distinction between gauge and affine symmetries will be discussed in greater detail in section 6.6.1.

Instead of working with the FJ action, one might introduce auxiliary fields so as to restore Lorentz invariance. This is done in the Pasti-Sorokin-Tonin [PST] formalism [PST97] which adopts the action

$$S_{\rm PST}[\phi,f]=\int_\Sigma {\rm d}^2\sigma\left(\partial_+\phi\partial_-\phi-\frac{\partial_+f}{\partial_-f}(\partial_-\phi)^2\right)\ . \tag{6.6}$$

Although we have displayed the result with indices explicit, this action can be cast in a manifestly Lorentz invariant fashion, and may be extended to higher form fields. The addition of the field f, which is best thought of as the local parameterisation of a closed 1-form $\omega={\rm d}f$, is complemented with additional symmetries,

$$\delta_{\rm PST}\phi=\epsilon\frac{\partial_-\phi}{\partial_-f}\ ,\quad \delta_{\rm PST}f=\epsilon\ , \tag{6.7}$$

$$\tilde\delta_{\rm PST}\phi=\Lambda(f)\ ,\quad \tilde\delta_{\rm PST}f=0\ . \tag{6.8}$$

Commensurate with this symmetry, the equation of motion for f is automatically satisfied[5] given that of ϕ which reads

$$\partial_-\left(\partial_+\phi-\frac{\partial_+f}{\partial_-f}\partial_-\phi\right)=0\ . \tag{6.9}$$

To see that this encodes a chiral field, it is convenient to introduce a 1-form v whose components read $v_\pm=\sqrt{\frac{\partial_\pm f}{\partial_\mp f}}$ and a scalar $\chi=v_+\partial_-\phi$. In terms of these, the equation of motion becomes

$$ {\rm d}(v\chi)=0\ , \tag{6.10}$$

and the desired self-duality condition now follows as the homogeneous solution $\chi=0$. Analogous to the general solution of the FJ equation of motion, the inhomogeneous solution $v\chi={\rm d}\Gamma(f)$ is irrelevant as it can be eliminated by a gauge transformation $\tilde\delta_{\rm PST}(v\chi)={\rm d}\Lambda(f)$.

Upon gauge fixing the $\delta_{\rm PST}$-symmetry by setting $f(\tau,\sigma)=\tau$, the PST action reduces exactly to the FJ action, with the residual $\tilde\delta_{\rm PST}$-symmetry becoming $\tilde\delta_{\rm FJ}$. There are some evident downsides, however, to the PST approach: first, the non-polynomial form of the action requires some careful consideration; second, at the functional level, one should restrict to configurations where $\omega={\rm d}f$ is nowhere vanishing, the existence of which is not a given when taking this approach beyond Minkowski space; third, the PST gauge symmetry appears rather exotic.

Resolving the first of these downsides, Mkrtchyan and collaborators have recently developed an approach which addresses the non-polynomial nature of the PST action [Mkr19; BEM21; EM23]. In the spirit of Hubbard-Stratonovich, the PST action can be rendered polynomial by

[4] The Noether charge corresponding to $\hat\delta_{\rm FJ}$ is $Q=2\int{\rm d}\sigma\,\varepsilon\,\partial_\sigma\phi$.

[5] Precisely, the f equation of motion is $\frac{\partial_-\phi}{\partial_-f}\partial_-\left(\partial_+\phi-\frac{\partial_+f}{\partial_-f}\partial_-\phi\right)=0$.

the introduction of an additional scalar field α to give the Mkrtchyan action,

$$S_{\mathrm{M}}[\phi, f, \alpha] = \int_{\Sigma} \mathrm{d}^2\sigma \left(\partial_+\phi\partial_-\phi - 2\alpha\partial_+ f\partial_-\phi + \alpha^2\partial_+ f\partial_- f\right) . \tag{6.11}$$

Provided that $\partial_\pm f \neq 0$, one can eliminate α by its equation of motion, $\alpha = \frac{\partial_-\phi}{\partial_- f}$, to recover the PST action. Furthermore, the δ_{PST}-symmetry is uplifted to

$$\delta_{\mathrm{M}}\phi = \epsilon\alpha \ , \quad \delta_{\mathrm{M}} f = \epsilon \ , \quad \delta_{\mathrm{M}}\alpha = \epsilon\frac{\partial_+\alpha}{\partial_+ f} \ . \tag{6.12}$$

In [Mkr19; BEM21; EM23], the gauge parameter of this symmetry is redefined according to $\varphi = \epsilon\frac{\partial_+\alpha}{\partial_+ f}$, such that it is viewed as a shift symmetry on α rather than on f. As we can see, while the action is now polynomial, dealing with this symmetry may still prove challenging since it remains non-polynomial. The second symmetry of the PST action also lifts easily to the Mkrtchyan action,

$$\tilde{\delta}_{\mathrm{M}}\phi = \Lambda(f) \ , \quad \tilde{\delta}_{\mathrm{M}} f = 0 \ , \quad \tilde{\delta}_{\mathrm{M}}\alpha = \Lambda'(f) \ . \tag{6.13}$$

It is useful to define the 1-form $\mu = \mathrm{d}\phi - \alpha\,\mathrm{d}f$, whose self-dual component is gauge invariant,

$$\delta_{\mathrm{M}}\mu_+ = 0 \ , \quad \delta_{\mathrm{M}}\mu_- = \frac{\epsilon}{\partial_+ f}(\partial_+\mu_- - \partial_-\mu_+) \ , \tag{6.14}$$

and, in terms of which, the action may be written as

$$S_{\mathrm{M}}[\phi, f, \alpha] = \int_{\Sigma} \left(\mu \wedge \star\mu + 2\alpha\,\mathrm{d}f \wedge \mathrm{d}\phi\right) . \tag{6.15}$$

The existence of this gauge invariant combination $\mu + \star\mu$ is not immediately obvious from the 2d perspective, and indeed it has no analog in the PST formalism. By comparison, we will see later that it appears naturally in the Chern-Simons approach. In [AEM22], this gauge invariant combination was essential for extending the free Mkrtchyan formalism presented above to include self-interactions. Indeed, if one wishes to preserve the gauge symmetries of the action, one must be careful to add polynomials of the fields which are independently gauge invariant, and polynomials of $\mu + \star\mu$ are a prime candidate.

Turning to the equations of motion for the Mkrtchyan action (denoting on-shell equivalence by $\simeq$), we have

$$\frac{\delta S_{\mathrm{M}}}{\delta\alpha} \simeq 0 \quad \Rightarrow \quad \mu_-\partial_+ f \simeq 0 \ , \tag{6.16}$$

$$\frac{\delta S_{\mathrm{M}}}{\delta\phi} \simeq 0 \quad \Rightarrow \quad \partial_-\mu_+ \simeq 0 \ , \tag{6.17}$$

$$\frac{\delta S_{\mathrm{M}}}{\delta f} \simeq 0 \quad \Rightarrow \quad \mu_-\partial_+\alpha + \alpha\partial_-\mu_- \simeq 0 \ . \tag{6.18}$$

Assuming that $\partial_\pm f$ are nowhere zero, the f equation of motion is redundant, reflecting the

δ_{M}-symmetry, and the ϕ and α equations of motion invoke a flatness,

$$\mathrm{d}\mu = \mathrm{d}(\alpha\,\mathrm{d}f) \simeq 0 \ . \tag{6.19}$$

Solving with $\alpha\,\mathrm{d}f = \mathrm{d}\Gamma$, and performing a field redefinition $\phi \to \phi' = \phi - \Gamma$, we then have that $\mu_- = 0$ invokes the desired chirality condition,

$$\partial_- \phi' = 0 \ . \tag{6.20}$$

Shortly, we will encounter the structure we see here, a flatness condition combined with a covariant chirality condition, from another perspective.

6.3 The Chern-Simons Approach to Chiral Bosons

Chiral bosons famously also emerge as the boundary dynamics of Chern-Simons [CS] theory [Eli+89; Wit97; BM06]. Consider CS theory on a 3-manifold $M = \mathbb{R} \times D$ with the topology of a solid cylinder (the length of the cylinder viewed as the time with coordinate τ, and the disk D parameterised by radial and angular coordinates ρ and σ). To properly define the action for Abelian CS theory,

$$S_{\mathrm{CS}}[A] = \kappa \int_M A \wedge \mathrm{d}A \ , \tag{6.21}$$

when the manifold M has a boundary, one typically imposes boundary conditions on the connection. These are chosen such that the boundary term in the variation of the action vanishes. In [Eli+89], they chose to impose the boundary condition $A_\tau|_{\partial M} = 0$, whereas we will deviate and instead impose[6] $A_\tau|_{\partial M} = A_\sigma|_{\partial M}$. Splitting the connection as $A = A_\tau \mathrm{d}\tau + A^D$, where A^D is a (time-dependent) 1-form on the disk D, the action may be rewritten (after integrating by parts and invoking the boundary condition) as

$$S_{\mathrm{CS}}[A] = \kappa \int_M \left(2A_\tau \mathrm{d}\tau \wedge \mathrm{d}^D A^D + A^D \wedge \mathrm{d}\tau \wedge \partial_\tau A^D\right) + \kappa \int_{\partial M} \mathrm{d}^2\sigma\, A_\sigma A_\sigma \ . \tag{6.22}$$

The component A_τ serves as a Lagrange multiplier enforcing the flatness of A^D which we solve with $A^D = \mathrm{d}^D\phi = \partial_\rho\phi\,\mathrm{d}\rho + \partial_\sigma\phi\,\mathrm{d}\sigma$. Subject to this, the action becomes localised on the boundary after integration by parts, and reads

$$S_{\mathrm{CS}} = -\kappa \int_{\partial M} \mathrm{d}^2\sigma\, \partial_\sigma\phi \partial_-\phi \ , \tag{6.23}$$

which we recognise as the FJ action. Having demonstrated that Chern-Simons theory indeed describes chiral bosons, we will now consider some augmentations of the action. These will not alter the physical content of the theory, but simply make it more amenable to our future analysis.

[6] The boundary condition $A_\tau|_{\partial M} = A_\sigma|_{\partial M}$ is a self-duality condition $A|_{\partial M} = \star(A|_{\partial M})$.

Returning to CS theory, we can obtain the self-duality relation as a Neumann type boundary condition by adding a boundary term to the action so that the combined boundary variation takes the form $\int \delta A \wedge (A - \star A)$. (This approach can also be seen in [MMS01, Appendix A].) To this end, we define the new functional $S'_{\rm CS}$ by

$$S'_{\rm CS}[A] = \kappa \int_M A \wedge {\rm d}A - \frac{\kappa}{2} \int_{\partial M} A \wedge \star A \ , \tag{6.24}$$

the variation of which is

$$\delta S'_{\rm CS}[A] = \kappa \int_M 2\delta A \wedge {\rm d}A + \kappa \int_{\partial M} \delta A \wedge (1 - \star) A \ , \tag{6.25}$$

and we set the boundary term to zero with $A|_{\partial M} = \star(A|_{\partial M})$. Of course, in general, CS theory with a boundary is not fully gauge invariant, and instead it is only invariant under those gauge transformations which preserve the boundary condition. Here, including our boundary term, a gauge transformation $\delta_\lambda A = {\rm d}\lambda$ transforms the action as

$$\delta_\lambda S'_{\rm CS}[A] = \kappa \int_{\partial M} A \wedge (1 - \star) {\rm d}\lambda \ , \tag{6.26}$$

and thus we only have invariance under self-dual gauge transformations. This may also be understood by considering the gauge transformation of the boundary condition directly.

Morally, this breaking of the gauge symmetry means that would-be pure-gauge modes become propagating edge modes on the boundary. For an alternative perspective, we may restore the gauge invariance by coupling the bulk theory to new boundary degrees of freedom i.e. a classical version of anomaly inflow, which has been recently considered in the context of 4d CS theory [LV21; BSV22]. Let us define $\chi \in C^\infty(\partial M)$ as a boundary field with a gauge transformation $\delta_\lambda \chi = -\lambda$ such that $A^\chi \equiv A + {\rm d}\chi$ is gauge invariant[7].

Evidently, making the replacement $A \to A^\chi$ in our action $S'_{\rm CS}[A]$ will result in a manifestly gauge invariant action, even with a boundary. Less obvious, however, is that the theory does not depend on the extension of χ into the bulk, but indeed we find the action

$$S_{\rm inv}[A, \chi] \equiv S'_{\rm CS}[A^\chi] = S_{\rm CS}[A] + \kappa \int_{\partial M} \left(A \wedge {\rm d}\chi - \tfrac{1}{2} A^\chi \wedge \star A^\chi \right) \ . \tag{6.27}$$

The first two contributions here have also been considered in [ABO03] and [CS10] to restore the gauge invariance of CS theory with a boundary, and the final term is a stand-alone gauge invariant boundary term. In this larger theory, we have the full gauge freedom with ${\rm d}\lambda$ no longer constrained to be self-dual on the boundary. We can use part of this larger symmetry to fix $\chi = 0$ such that eq. (6.27) abbreviates to eq. (6.24) demonstrating that the physical content of these

[7] It will be helpful for subsequent generalisation to notice that the combination $A^\chi \equiv A + {\rm d}\chi$ can also be thought of as the gauge transformation of the connection where, in the spirit of Stueckelberg, the gauge parameter is promoted to a dynamical field.

theories is equivalent.

In this gauge invariant presentation, we recover the chiral boson equations of motion by varying the action,

$$\delta S_{\text{inv}}[A,\chi] = \kappa \int_M 2\delta A \wedge \mathrm{d}A + \kappa \int_{\partial M} \delta A \wedge (1-\star)A^\chi + \kappa \int_{\partial M} \delta\chi\big(2\mathrm{d}A - \mathrm{d}(1-\star)A^\chi\big) \ . \quad (6.28)$$

The bulk variation of A (i.e. the first term) tells us that A is on-shell flat (which trivially implies that A^χ is also on-shell flat). Meanwhile, the boundary variation of A gives us the desired self-duality relation on the flat field, $A^\chi|_{\partial M} = \star(A^\chi|_{\partial M})$. Solving the flatness as $A = \mathrm{d}(\phi - \chi)$, the boundary equation gives $\mathrm{d}\phi = \star\mathrm{d}\phi$. Similar to some of the two-dimensional models, the χ equation of motion is implied by the other equations of motion, a consequence of the fact that it is gauge trivial. This ensures that upon gauge fixing $\chi = 0$ one can also disregard its equation of motion and return to the pure CS theory.

At this stage, one might wonder where the self-dual boundary gauge transformations have disappeared to in this presentation. While it might, at first, seem like they have been washed out by the introduction of the edge modes, a simple degrees of freedom counting argument demonstrates that this must not be true. Indeed, the new action $S_{\text{inv}}[A,\chi]$ actually comes with an additional gauge symmetry which acts exclusively on the edge modes. Consider the gauge transformation $\delta_\theta A = 0$ and $\delta_\theta \chi = \theta$. Under this, the action transforms as

$$\delta_\theta S_{\text{inv}}[A,\chi] = \int_{\partial M} \big(A \wedge (\mathrm{d}\theta - \star\mathrm{d}\theta) - \mathrm{d}\chi \wedge \star\mathrm{d}\theta\big) \ . \quad (6.29)$$

So, while the action is not generally invariant under this transformation, if we take $\mathrm{d}\theta = \star\mathrm{d}\theta$, then the first term vanishes and the second is a total derivative which we may ignore. These self-dual boundary gauge transformations are precisely the chiral affine symmetries we saw earlier, but this identification will become clearer in the ensuing sections.

The benefit of the edge mode presentation is that this self-dual gauge parameter can be exclusively defined over the boundary, i.e. $\theta \in C^\infty(\partial M)$, and acts trivially on the gauge field A. Furthermore, the λ gauge symmetry is completely unconstrained and has exactly the right degrees of freedom to render A (on-shell) locally trivial everywhere. Conceptually, this makes the transition from a bulk theory to a boundary theory much smoother.

6.4 From Chern-Simons to PST and Mkrtchyan

Having seen how to obtain the FJ action through manipulations of CS theory, our next goal is to understand how the PST and Mkrtchyan actions can also be recovered. The idea here is *not* to single out the A_τ component as auxiliary, but instead to introduce a more general decomposition of the bulk gauge field.

The arguments we present here are, to a large extent, independent of dimension, and can equally be applied to self-dual 2-forms in 6d as to chiral scalars in 2d. Accordingly, we will leave the dimension fairly general, and work in $(2n+1)$-dimensional Chern-Simons theory for an Abelian n-form field $A \in \Omega^n(M)$ on a manifold with a boundary $\partial M \equiv \Sigma$. To restore complete gauge invariance under $\delta_\lambda A = \mathrm{d}\lambda$ for $\lambda \in \Omega^{n-1}(M)$, we again invoke a Stueckelberg compensator field $\chi \in \Omega^{n-1}(\partial M)$ which transforms with a shift symmetry $\delta_\lambda \chi = -\lambda$. We then use the action $S_{\mathrm{inv}}[A,\chi]$ (6.27), now understood in this general setting.

One further comment on the number of dimensions must be made. When Σ is four-dimensional with Euclidean signature, the bulk Chern-Simons term $A \wedge \mathrm{d}A$ for a 2-form $A \in \Omega^2(M)$ is a total derivative and eq. (6.27) immediately simplifies to a boundary action,

$$\frac{\kappa}{2}\int_{\partial M}\left(A\wedge A + 2A\wedge \mathrm{d}\chi - A^\chi \wedge \star A^\chi\right) . \tag{6.30}$$

In terms of the (anti)-self-dual projections of the field, $A_\pm = \frac{1}{2}(1\pm\star)A$ and $\mathrm{d}\chi_\pm = \frac{1}{2}(1\pm\star)\mathrm{d}\chi$, the action becomes

$$\frac{\kappa}{2}\int_{\partial M}\left(2A_-\wedge A_- + 4A_-\wedge \mathrm{d}\chi_- - \mathrm{d}\chi_+\wedge \mathrm{d}\chi_+ + \mathrm{d}\chi_-\wedge \mathrm{d}\chi_-\right) . \tag{6.31}$$

The self-dual component of A has dropped entirely, and the anti-self-dual component is algebraically eliminated as $A_- = -\mathrm{d}\chi_-$, giving the action

$$-\frac{\kappa}{2}\int_{\partial M}\mathrm{d}\chi\wedge \mathrm{d}\chi\ , \tag{6.32}$$

which evidently carries no degrees of freedom and vanishes when χ is global. Thus, we continue under the specification that A is an odd-degree form (i.e. n is odd), and $\Sigma \equiv \partial M$ is Lorentzian, such that $\star^2(A|_{\partial M}) = A|_{\partial M}$. (Note that we could also include Euclidean signature when n is odd by considering imaginary self duality, i.e. $\star A|_{\partial M} = iA|_{\partial M}$.)

In order to define a more general decomposition of the gauge field which will not break Lorentz invariance, we introduce a 1-form $\omega \in \Omega^1(\partial M)$ and its normalised dual vector $v = \omega^\sharp/\|\omega\|^2$ (such that $\iota_v\omega = 1$) and extend them to live on the whole manifold M. With this data, we can decompose the exterior derivative into

$$\mathrm{d} = \mathrm{d}^\perp + \mathrm{d}^\parallel\ , \quad \mathrm{d}^\parallel = \omega\wedge\mathcal{L}_v\ , \tag{6.33}$$

such that $(\mathrm{d}^\perp)^2 = 0$ and $(\mathrm{d}^\parallel)^2 = 0$ when $\mathrm{d}\omega = 0$, which we shall hence assume. The connection similarly decomposes as

$$A = A^\perp + \omega\wedge\iota_v A\ , \quad \iota_v A^\perp = 0\ . \tag{6.34}$$

To make contact with the derivation of the FJ action from CS in section 6.3, one could specify $\omega = \mathrm{d}\tau$. Here, however, we keep ω arbitrary which will allow us to maintain Lorentz covariance in the resultant boundary theory.

Substituting this decomposition of the gauge field into our action (6.27), the bulk Chern-Simons term becomes

$$S_{\text{CS}}[A] = \kappa \int_M \left(2\omega \wedge \iota_v A \wedge \mathrm{d}A^\perp + A^\perp \wedge \mathrm{d}A^\perp\right) + \kappa \int_{\partial M} \omega \wedge \iota_v A \wedge A^\perp \ , \tag{6.35}$$

and $\iota_v A$ acts as a Lagrange multiplier enforcing the constraint $\omega \wedge \mathrm{d}A^\perp = 0$. This has the general solution $A^\perp = \mathrm{d}\phi - \omega \wedge C$ [BEM21, appendix C] which is further fixed by the constraint $\iota_v A^\perp = 0$, implying $C = \iota_v \mathrm{d}\phi$. We can therefore write the total gauge field as

$$A = \mathrm{d}\phi - \omega \wedge \alpha \ , \quad \alpha \equiv \iota_v(\mathrm{d}\phi - A) \ , \tag{6.36}$$

and the gauge invariant combination as $A^\chi = \mathrm{d}b - \omega \wedge \alpha$ where $b = \phi + \chi$. Notice that this expression for A^χ can be identified with the combination we called μ in the introduction, and we will henceforth refer to it as such.

Substituting this back into the action (6.27) gives

$$S_{\text{inv}} = -\frac{\kappa}{2} \int_{\partial M} \left(\mu \wedge \star\mu + 2\omega \wedge \alpha \wedge \mathrm{d}b\right) \ , \tag{6.37}$$

which may be expanded out as

$$S_{\text{inv}} = -\frac{\kappa}{2} \int_{\partial M} \left(\mathrm{d}b \wedge \star\mathrm{d}b + 2\omega \wedge \alpha \wedge \mathcal{X} + \omega \wedge \alpha \wedge \star(\omega \wedge \alpha)\right) \ , \tag{6.38}$$

where $\mathcal{X} = \mathrm{d}b - \star\mathrm{d}b$. We recognise this as the sought-after Mkrtchyan action for chiral p-forms. We already know that this action comes equipped with a local symmetry, which is the uplift of the PST symmetry to the Mkrtchyan action,

$$\delta_\epsilon b = \epsilon\alpha \ , \quad \delta_\epsilon \omega = \mathrm{d}\epsilon \ , \quad \delta_\epsilon \alpha = \epsilon \iota_v (1 - \star)\mathrm{d}\alpha \ , \tag{6.39}$$

under which we have

$$\delta_\epsilon \mu = \epsilon(1 - \star)\iota_v(\omega \wedge \mathrm{d}\alpha) = \epsilon(1 - \star)\iota_v \mathrm{d}\mu \ . \tag{6.40}$$

Notice that this represents a *zilch* symmetry: after imposing the constraint from the parallel component of the CS connection, we have a new symmetry proportional to $\mathrm{d}\mu \equiv \mathrm{d}A^\chi = \mathrm{d}A$ which vanishes on-shell for both the CS and Mkrtchyan theories. Thus this symmetry can be understood as a trivial symmetry which arises once auxiliary fields are integrated out (in much the same way that SUSY closes only on-shell once auxiliaries are eliminated). To experts on these formalisms, this may not be particularly surprising as the PST symmetry is also known to be a *zilch* symmetry in the same fashion [PST12; DST16].

To go from the Mkrtchyan to the PST action we use the equation of motion for α,

$$\omega \wedge \left(\mathcal{X} + \star(\omega \wedge \alpha)\right) = 0 \ . \tag{6.41}$$

We can partially determine this solution as

$$\alpha = \iota_v \mathcal{X} + \iota_v \star(\omega \wedge \rho)\,, \tag{6.42}$$

where ρ is undetermined, but then, using the general identity $\iota_v \star \beta = \star(\beta \wedge v^\flat)$ and $v^\flat \equiv \omega/\|\omega\|^2$, we conclude that the second term does not contribute and $\alpha = \iota_v \mathcal{X}$. Making use of a further identity $\star\iota_v\beta = (-1)^{p-1} v^\flat \wedge \star\beta$ for a general p-form $\beta \in \Omega^p(\partial M)$, we can entirely eliminate α from the action (6.38) to recover the p-form version of the PST action,

$$S_{\text{inv}} = -\frac{\kappa}{2} \int_{\partial M} \left(\mathrm{d}b \wedge \star \mathrm{d}b - \|\omega\|^2 \iota_v \mathcal{X} \wedge \star \iota_v \mathcal{X} \right) \,. \tag{6.43}$$

6.5 Comments on the double world sheet

Let us return momentarily to the two-dimensional Mkrtchyan action of eq. (6.11), which we recast as

$$S_{\text{M}}[\phi, f, \alpha] = \int_\Sigma \mathrm{d}^2\sigma \left((\partial_+\phi - \alpha\partial_+ f)(\partial_-\phi \alpha \partial_- f) - \alpha \partial_+ f \partial_- \phi + \alpha \partial_- f \partial_+ \phi \right) \,. \tag{6.44}$$

There is another rather direct way to see the emergence of a Chern Simons description following the approach of [DST16]. In [DST16] a similar first order formalism was obtained (in the context of non-linear sigma models) by invoking some partial gauge fixing of a two-dimensional gauge field $A_\pm = \alpha \partial_\pm f$. By reverse engineering we are led to consider an action,

$$S'_{\text{M}}[\phi, A] = \int_\Sigma \mathrm{d}^2\sigma \left((\partial_+\phi - A_+)(\partial_-\phi - A_-) - A_+ \partial_- \phi + A_- \partial_+ \phi \right) \,, \tag{6.45}$$

such that upon fixing the aforementioned gauge, returns the Mkrtchyan Lagrangian. In this way we have arrived at an action (upto a trivial integration by parts) originally proposed by Witten [Wit97] in a similar context. This, however, is not gauge invariant under $\delta\phi = \epsilon$ and $\delta A_\pm = \partial_\pm \epsilon$ as

$$\delta S'_{\text{M}}[\phi, A] = \int_\Sigma \mathrm{d}^2\sigma \left(\epsilon (\partial_- A_+ - \partial_+ A_-) \right) \,. \tag{6.46}$$

Thus, now with opposite logic, to restore gauge invariance we add a Chern-Simons bulk action to arrive again at (6.27).

Let us now consider the case where we have $2n$ bosonic fields, denoted by $\mathbb{X}$, that enter in a generalisation of the Mkrtchyan action:

$$S \;=\; \frac{1}{2} \int d^2\sigma \Big((\partial_+ f \mathbb{A} - \partial_+ \mathbb{X}) \mathcal{H} (\partial_- f \mathbb{A} - \partial_- \mathbb{X}) + \partial_- f \mathbb{A} \eta \partial_+ \mathbb{X} - \partial_+ f \mathbb{A} \eta \partial_- \mathbb{X} \Big) \,.$$

This is relevant to the doubled world sheet description of strings on a toroidal background where $\mathbb{X} = \{x^i, \tilde{x}_i\}$ constitutes the coordinates of the target space together with their T-duals. The couplings are specified by a split signature pairing

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{6.47}$$

and a generalised metric encoding the target space metric and Kalb-Ramond data

$$\mathcal{H} = \begin{pmatrix} g - bg^{-1}g & -bg^{-1} \\ g^{-1}b & g^{-1} \end{pmatrix} . \tag{6.48}$$

Note that the generalised metric is an element of $O(n,n)$ (the transformations that preserve η) and so defines an almost product structure $\mathcal{E} = \mathcal{H}\eta^{-1}$ that obeys $\mathcal{E}^2 = 1$. Elimination of the scalar fields $\mathbb{A}$ from the action yields a PST formulation of the doubled string, as in [DST16], from which twisted self-duality $d\mathbb{X} = \star\mathcal{E}d\mathbb{X}$ follows as the equation of motion. From this twisted self-duality constraint one can eliminate half the variables, the $\tilde{x}_i$ say, to yield second order equations for the other half, the x^i which reproduce those of the standard string world sheet.

We now follow the same strategy as above and propose to undo a gauge fixing $\mathbb{A}_\pm = \mathbb{A}\partial_\pm f$ by considering

$$S[\mathbb{A}] = \frac{1}{2}\int d^2\sigma \Big((\mathbb{A}_+ - \partial_+\mathbb{X})\mathcal{H}(\mathbb{A}_- - \partial_-\mathbb{X}) + \partial_+\mathbb{X}\eta\mathbb{A}_- - \partial_-\mathbb{X}\eta\mathbb{A}_+ \Big) . \tag{6.49}$$

This action is not gauge invariant under the full $U(1)^{2n}$ symmetry $\delta\mathbb{A} = d\epsilon$, $\delta\mathbb{X} = \epsilon$. It is however invariant when we consider the gauge fields to take values in an $U(1)^n$ sub-algebra that is isotropic with respect to η, i.e. if we gauge half the symmetries with $\eta_{IJ}\mathbb{A}^I\mathbb{A}^J = \eta_{IJ}\epsilon^I\epsilon^J = \eta_{IJ}\epsilon^I\mathbb{A}^J = 0$. This idea of gauging an isotropic sub-algebra was invoked by Hull in his approach to the doubled string [Hul07] and developed in [LP14]. If instead we wish to restore a full $U(1)^{2n}$ invariance we add a Chern-Simons term to arrive at the form

$$S_{\rm inv}[\mathbb{A},\mathbb{X}] = S_{\rm CS}[\mathbb{A}] + \int_{\partial M} \eta\big(\mathbb{A}\wedge \mathrm{d}\mathbb{X} - \tfrac{1}{2}(\mathbb{A} - \mathrm{d}\mathbb{X})\wedge\star\mathcal{E}(\mathbb{A}-\mathrm{d}\mathbb{X})\big) . \tag{6.50}$$

where $U(1)^{2n}$ algebra indices are contracted with the product η. This discussion can be expanded to the case where T-duality acts in an internal space that is a fibration over some base manifold. To do so one simply allows $\mathcal{H} = \mathcal{H}(y)$ to depend on the coordinates y^a of the base manifold and couple to a background gauge field $\mathbb{B} = \mathbb{B}_a dy^a$, an $O(n,n)$ vector that contains *off-diagonal* metric and two-form data (see [DST16] eq. 2.3), by making a minimal coupling substitution $\mathrm{d}\mathbb{X} \to \nabla\mathbb{X} = \mathrm{d}\mathbb{X} - \mathbb{B}$.

6.6 Non-Abelian Chern-Simons

Having seen that our formalism reduces to others found in the literature, we will now leverage the simplicity of our approach to provide a novel generalisation. By starting with non-Abelian

Chern-Simons theory in 3-dimensions, we will derive a polynomial action for non-Abelian chiral bosons. This will be the non-Abelian generalisation of the Mkrtchyan action and integrating out an auxiliary field will yield the PST action for non-Abelian chiral bosons.

Let G be a Lie group whose algebra $\mathfrak{g}$ is equipped with an ad-invariant inner product $\langle \bullet, \bullet \rangle$, and consider the algebra-valued 1-forms $A \in \Omega^1(M, \mathfrak{g})$. The action to consider is

$$S'_{\text{CS}}[A] = \int_M \left(\langle A, \mathrm{d}A \rangle + \frac{1}{3} \langle A, [A, A] \rangle \right) - \frac{1}{2} \int_{\partial M} \langle A, \star A \rangle \ , \tag{6.51}$$

which is the non-Abelian CS action plus a boundary term which renders the boundary condition $A|_{\partial M} = \star(A|_{\partial M})$ Neumann.

As in the Abelian case, the presence of a boundary spoils the gauge invariance under $A \to A^g \equiv g^{-1} A g + g^{-1} \mathrm{d}g$, but we may restore the invariance of the action by coupling the bulk CS theory to a boundary edge mode. Let $h \in C^\infty(\partial M, G)$ be a boundary field which transforms as $h \to g^{-1} h$ such that $A^h \equiv h^{-1} A h + h^{-1} \mathrm{d}h$ is invariant. Then, as before, we simply replace $A \to A^h$ in our action to find the gauge invariant theory,

$$S_{\text{inv}}[A, h] \equiv S'_{\text{CS}}[A^h] = S_{\text{CS}}[A] + S_{\text{WZ}}[h] + \int_{\partial M} \left(\langle A, \mathrm{d}h h^{-1} \rangle - \tfrac{1}{2} \langle A^h, \star A^h \rangle \right) \ , \tag{6.52}$$

where the Wess-Zumino [WZ] term is defined by

$$S_{\text{WZ}}[h] = -\frac{1}{6} \int_M \langle h^{-1} \mathrm{d}h, [h^{-1} \mathrm{d}h, h^{-1} \mathrm{d}h] \rangle \ . \tag{6.53}$$

As the action (6.52) is now gauge invariant under the bulk transformation, one might wonder how the quantisation of the CS level arises since the conventional argument about large gauge transformations is rendered moot. The answer is simply that we are required to extend the edge mode h into the bulk to define the WZ term, and, in the standard fashion, demanding that the path integral is insensitive to the choice of such an extension invokes the desired quantisation condition.

Following the same recipe, we introduce a 1-form $\omega \in \Omega^1(\partial M)$ and its normalised dual vector $v = \omega^\sharp / \|\omega\|^2$ (such that $\iota_v \omega = 1$), which we extend to live on the whole manifold M. With this data, we can decompose the exterior derivative into

$$\mathrm{d} = \mathrm{d}^\perp + \mathrm{d}^\parallel \ , \quad \mathrm{d}^\parallel = \omega \wedge \mathcal{L}_v \ , \tag{6.54}$$

such that $(\mathrm{d}^\perp)^2 = 0$ and $(\mathrm{d}^\parallel)^2 = 0$ when $\mathrm{d}\omega = 0$, which we shall hence assume. The connection similarly decomposes as

$$A = A^\perp + \iota_v A \, \omega \ , \quad \iota_v A^\perp = 0 \ , \tag{6.55}$$

where $\iota_v A \in C^\infty(M, \mathfrak{g})$ is now valued in the algebra.

Under this decomposition, the bulk CS term becomes

$$S_{\rm CS}[A] = \int_M \big(2\omega\wedge\langle\iota_v A, F^\perp\rangle + \langle A^\perp, {\rm d}A^\perp\rangle\big) + \int_{\partial M}\omega\wedge\langle\iota_v A, A^\perp\rangle\ , \tag{6.56}$$

where $F^\perp = {\rm d}A^\perp + A^\perp\wedge A^\perp$, and we see that $\iota_v A$ is again acting as a Lagrange multiplier fixing

$$\omega\wedge F^\perp = 0\ . \tag{6.57}$$

In order to write the most general solution to this equation we will need to slightly generalise the argument presented in [BEM21, appendix C] to non-Abelian fields. Let us assume that the closed 1-form ω is sufficiently *nice*, meaning $\omega\sim{\rm d}f$ where f is a good global coordinate and we can foliate our 3-manifold by slices of constant f. On each slice of constant f, the constraint above reduces to a non-Abelian flatness condition $F^\perp = 0$, which we can solve with $A^\perp = g^{-1}{\rm d}g$ using the non-Abelian Poincaré lemma. We can now glue these solutions back together to form the solution on the whole manifold, and the only piece we might have missed is a component parallel to ω. Thus, the most general solution is

$$A^\perp = g^{-1}{\rm d}g - C\,\omega\ . \tag{6.58}$$

We can fix this solution further by imposing the constraint $\iota_v A^\perp = 0$, implying $C = \iota_v(g^{-1}{\rm d}g)$. We can therefore write the total gauge field as

$$A = g^{-1}{\rm d}g - h\alpha h^{-1}\,\omega\ ,\quad \alpha\equiv h^{-1}\iota_v(g^{-1}{\rm d}g - A)h\ , \tag{6.59}$$

and the gauge invariant combination is written as $A^h = m^{-1}{\rm d}m - \alpha\,\omega$ where $m = gh$. To match notation with the previous section, we will again relabel this combination as $\mu\equiv A^h$.

Substituting this solution back into the action (6.52), we find that it may be succinctly expressed as

$$S_{\rm inv} = S_{\rm WZ}[m] - \frac{1}{2}\int_{\partial M}\big(\langle\mu,\star\mu\rangle + 2\omega\wedge\langle\alpha, m^{-1}{\rm d}m\rangle\big)\ . \tag{6.60}$$

It is actually easiest to compute this action by substituting the expression for $\mu\equiv A^h$ directly into eq. (6.51). Defining $\mathcal{X} = m^{-1}{\rm d}m - \star(m^{-1}{\rm d}m)$, this may be expanded out as

$$S_{\rm inv} = S_{\rm WZW}[m] - \frac{1}{2}\int_{\partial M}\big(2\omega\wedge\langle\alpha,\mathcal{X}\rangle + \langle\alpha,\alpha\rangle\,\omega\wedge\star\omega\big)\ , \tag{6.61}$$

to provide the non-Abelian generalisation of the Mkrtchyan action (6.38).

This action is invariant under the local transformation

$$\delta_\epsilon m = \epsilon m\alpha\ ,\quad \delta_\epsilon\omega = {\rm d}\epsilon\ ,\quad \delta_\epsilon\alpha = \epsilon\iota_v(1-\star)\nabla\alpha\ , \tag{6.62}$$

where $\nabla\bullet = {\rm d}\bullet + [m^{-1}{\rm d}m,\bullet]$. To demonstrate this invariance, it is helpful to use the action (6.60) and note that $\delta(\alpha\,\omega) = \nabla(\epsilon\alpha) - \delta\mu$ and $\omega \wedge \delta\mu = \epsilon\,\omega \wedge \nabla\alpha$. Also, the covariant derivative ∇ is nilpotent, i.e. $\nabla^2 = 0$, and satisfies

$$\int_{\partial M} \langle A_1, \nabla A_2 \rangle = (-1)^{\deg A_1 + 1} \int_{\partial M} \langle \nabla A_1, A_2 \rangle \tag{6.63}$$

for any pair of $\mathfrak{g}$-valued forms A_1, A_2. The calculation showing that the action is invariant under these transformations is given in more detail in appendix 6.8.

As in the Abelian case, this symmetry is a *zilch* symmetry arising from the elimination of auxiliary fields. This can be most easily seen by computing

$$\delta_\epsilon \mu = \epsilon(1-\star)\iota_v(\omega \wedge \nabla\alpha) = \epsilon(1-\star)\iota_v(\mathrm{d}\mu + \mu \wedge \mu) \ , \tag{6.64}$$

and the right hand side is proportional to the field strength of $\mu \equiv A^h$, related to the regular CS field strength by $F[\mu] \equiv F[A^h] = h^{-1}F[A]h$, which vanishes on-shell.

The elimination of α by its equation of motion proceeds as in the Abelian case, *mutatis mutandis*, to yield the non-abelian PST action,

$$S_{\rm inv} = S_{\rm WZW}[m] + \frac{1}{2}\int_{\partial M} \star\|\omega\|^2 \langle \iota_v \mathcal{X}, \iota_v \mathcal{X} \rangle \ , \tag{6.65}$$

and setting $\omega = \mathrm{d}\tau$ gives the FJ-type action for non-Abelian chiral bosons [Son88],

$$S_{\rm inv} = -\int_{\partial M} \mathrm{d}^2\sigma \, \langle m^{-1}\partial_\sigma m, m^{-1}\partial_- m \rangle + S_{\rm WZ}[m] \ . \tag{6.66}$$

6.6.1 Self-dual gauge transformations as affine transformations

In the above, we saw that our coupled bulk-boundary action (6.52) was invariant under the conventional transformation for the gauge field combined with a left action on the edge mode,

$$A \to A^g \equiv g^{-1}Ag + g^{-1}\mathrm{d}g \ , \quad h \to g^{-1}h \ , \quad A^h \to A^h \ . \tag{6.67}$$

We can, however, consider a further set of local transformations which follow from the *right* action on h leaving A invariant,

$$A \to A \ , \quad h \to hg \ , \quad A^h \to A^{hg} \equiv g^{-1}A^h g + g^{-1}\mathrm{d}g \ . \tag{6.68}$$

Under this right action, the action (6.52) transforms as

$$\begin{aligned} S_{\rm inv}[A,h] \to S_{\rm inv}[A,h] + S_{\rm WZ}[g] + &\int_{\partial M} \langle A^h, (1-\star)\mathrm{d}g g^{-1} \rangle \\ &- \frac{1}{2}\int_{\partial M} \langle \mathrm{d}g g^{-1}, \star(\mathrm{d}g g^{-1}) \rangle \ . \end{aligned} \tag{6.69}$$

For self-dual "gauge" transformations, i.e. those which obey $\mathrm{d}gg^{-1} = \star(\mathrm{d}gg^{-1})$, the two boundary terms are zero. Notice that this condition may also be written as $\partial_- gg^{-1} = 0$. In order to kill the WZ term, we assume that the bulk extension of g can be suitably chosen so as to also obey this constraint. Pushed through to eq. (6.66) these transformations correspond to an affine right action,

$$m \to mg(\sigma^+) \ , \tag{6.70}$$

and gives rise to a chiral current algebra [Son88, page 18].

To see that these self-dual boundary transformations are in fact affine transformations and not gauge symmetries, we should compute their Noether charges. First, working directly at the level of the chiral WZW model, eq. (6.66), we compute the Noether charge for the infinitesimal affine right action to be

$$Q = 2 \int \mathrm{d}\sigma \, \langle \varepsilon, m^{-1}\partial_\sigma m \rangle \ . \tag{6.71}$$

The fact that this charge is non-vanishing implies that these transformations are not gauge symmetries of the theory. Furthermore, we can compute the same Noether charge in the Chern-Simons theory,

$$Q = \int \mathrm{d}\sigma \, \langle \varepsilon, (A_\tau + A_\sigma) \rangle \ , \tag{6.72}$$

which comes from the conserved current $J = A + \star A$. If we impose the boundary condition $A_\tau = A_\sigma$ and substitute in the solution to the equation of motion, we see that these Noether charges agree on-shell.

Finally, we observe that this same calculation generalises immediately to the case of an abelian chiral p-form field (with self-dual $(p+1)$-form field strength in $2(p+1)$ dimensions) and yields a global "affine" symmetry of the chiral p-form gauge theory, with (nonvanishing) $p+1$-form Noether current $J = A + \star A$ as before.

6.7 Twisted Self-Duality and PLTD

One further generalisation which we wish to consider is the possibility for a twisted self-duality relation of the form $A|_{\partial M} = \mathcal{E}\star(A|_{\partial M})$. This boundary condition has been considered recently in the context of PLTD [Šev16] and 4d CS theory [LV21]. Given our choice of boundary $\Sigma = \partial M$ such that[8] $\star^2(A|_{\partial M}) = A|_{\partial M}$, we have the self-consistency condition $\mathcal{E}^2 = 1$, and therefore we should think of $\mathcal{E}$ as an involution on the algebra. For many purposes we might take $\mathcal{E}$ to be constant (though this is not essential and indeed there are natural examples in which $\mathcal{E}$ could be a function of other fields in a larger system) and the above relation can be understood to hold pointwise.

Generalising our formalism to include twisted self-duality is actually rather straight forward. Instead of adding the boundary term $A \wedge \star A$ to the CS action, we add the term $A \wedge \mathcal{E}\star A$, and the

[8]Note that we could also include Euclidean signature by considering imaginary self duality, i.e. $\star(A|_{\partial M}) = iA|_{\partial M}$ and $\star^2(A|_{\partial M}) = -A|_{\partial M}$.

rest of the analysis follows through as before, so long as we impose the condition $\langle \bullet, \mathcal{E}\bullet\rangle = \langle \mathcal{E}\bullet, \bullet\rangle$. This can be thought of as a compatibility condition between the involution $\mathcal{E}$ and the inner product $\langle \bullet, \bullet\rangle$, or alternatively as the requirement that $(\bullet\,, \bullet) = \int_\Sigma \langle \bullet, \mathcal{E}\star\bullet\rangle$ is a symmetric inner product on algebra-valued 1-forms.

So, taking this compatibility condition as given, and starting from the action

$$S_{\rm inv}[A,h] = S_{\rm CS}[A] + S_{\rm WZ}[h] + \int_{\partial M} \left(\langle A, \mathrm{d}h h^{-1}\rangle - \tfrac{1}{2}\langle A^h, \mathcal{E}\star A^h\rangle \right) , \tag{6.73}$$

our derivation culminates in

$$\begin{aligned} S_{\rm inv} = S_{\rm WZ}[m] - \frac{1}{2}\int_{\partial M} \langle m^{-1}\mathrm{d}m, \mathcal{E}\star(m^{-1}\mathrm{d}m)\rangle - \int_{\partial M} \omega\wedge\langle \alpha, \mathcal{X}\rangle \\ - \frac{1}{2}\int_{\partial M} \langle \alpha, \mathcal{E}\alpha\rangle\, \omega\wedge\star\omega \ , \end{aligned} \tag{6.74}$$

where $\mathcal{X} = m^{-1}\mathrm{d}m - \mathcal{E}\star(m^{-1}\mathrm{d}m)$. This is the non-abelian Mkrtchyan action, eq. (6.61), but now with twisted self-duality.

The elimination of α by its equation of motion proceeds directly to give the non-abelian twisted PST action,

$$S_{\rm inv} = S_{\rm WZ}[m] - \frac{1}{2}\int_{\partial M} \langle m^{-1}\mathrm{d}m, \mathcal{E}\star(m^{-1}\mathrm{d}m)\rangle + \frac{1}{2}\int_{\partial M} \star\|\omega\|^2 \langle \iota_v\mathcal{X}, \iota_v\mathcal{X}\rangle \ . \tag{6.75}$$

Introducing projectors $\mathcal{P}_\pm = \frac{1}{2}(1\pm\mathcal{E})$ we can write $\iota_v\mathcal{X}$ in lightcone coordinates as

$$\iota_v\mathcal{X} = \frac{1}{\|\omega\|^2}\left(\omega_-\mathcal{P}_-(m^{-1}\partial_+ m) + \omega_+\mathcal{P}_+(m^{-1}\partial_- m)\right) , \tag{6.76}$$

and substituting this into eq. (6.75) gives

$$\begin{aligned} S = S_{\rm WZ}[m] &- \frac{1}{2}\int_{\partial M} \mathrm{d}^2\sigma\, \langle m^{-1}\partial_+ m, \mathcal{E}(m^{-1}\partial_- m)\rangle \\ &+ \frac{1}{2}\int_{\partial M} \mathrm{d}^2\sigma\, \frac{\omega_+}{\omega_-}\langle m^{-1}\partial_- m, \mathcal{P}_+(m^{-1}\partial_- m)\rangle \\ &- \frac{1}{2}\int_{\partial M} \mathrm{d}^2\sigma\, \frac{\omega_-}{\omega_+}\langle m^{-1}\partial_+ m, \mathcal{P}_-(m^{-1}\partial_+ m)\rangle \ . \end{aligned} \tag{6.77}$$

This is now easily compared with [DST16, eq. 3.23].

Finally, fixing $\omega = \mathrm{d}\tau$ returns the FJ form with twisted self-duality,

$$S = S_{\rm WZ}[m] - \int_{\partial M} \mathrm{d}^2\sigma \left(\langle m^{-1}\partial_\sigma m, m^{-1}\partial_\tau m\rangle - \langle m^{-1}\partial_\sigma m, \mathcal{E}(m^{-1}\partial_\sigma m)\rangle \right) . \tag{6.78}$$

In the special case where the algebra is a Drinfeld double $\mathfrak{d} = \mathfrak{g} \bowtie \tilde{\mathfrak{g}}$, and the inner product is the natural pairing $\eta = \langle \bullet, \bullet\rangle$, the compatibility condition $\langle \bullet, \mathcal{E}\bullet\rangle = \langle \mathcal{E}\bullet, \bullet\rangle$ is precisely that which appears in the context of Poisson-Lie T-duality. In that context, the FJ form (6.78) is often denoted the $\mathcal{E}$-model [KS95; KS96a; KS96b].

One can now consider the transformations which gave rise to affine symmetries in the untwisted case. In general, one expects a non-trivial choice of $\mathcal{E}$ to reduce the symmetries of the theory. Indeed, in order for the action to be invariant under $m \to mg$, the gauge parameter g is required to obey

$$\mathcal{E} = \mathrm{Ad}_g \circ \mathcal{E} \circ \mathrm{Ad}_{g^{-1}} \quad \text{and} \quad \mathrm{d}gg^{-1} = \mathcal{E}\star(\mathrm{d}gg^{-1}) \ . \tag{6.79}$$

Given these constraints, we can compute the Noether charge in the Chern-Simons model,

$$Q = \int \mathrm{d}\sigma \, \langle \varepsilon, (A_\tau + \mathcal{E}A_\sigma) \rangle \ , \tag{6.80}$$

which comes from the conserved current $J = A + \mathcal{E}\star A$. As before, the fact that this charge is non-vanishing tells us that this is not a gauge symmetry for the theory. Instead, we should interpret it as a twisted affine symmetry.

As an example of the above it is informative to consider the case where $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ for which isotropic spaces of $\langle \bullet, \bullet \rangle$ are the diagonal and anti-diagonal subsets (of which only the diagonal is a Lie sub-algebra). This case is relevant to the construction of the WZW model, and its integrable λ-deformation [Kli15]. We define t_a as generators of $\mathfrak{g}$ with $f_{ab}{}^c$ the structure constants and κ_{ab} the Cartan-Killing metric. A basis of $\mathfrak{d}$ is formed by $T_a = \{t_a, -t_a\}$ and $\tilde{T}^a = \kappa^{ab}\{t_a, t_a\}$ spanning the two isotropics and we specify $\mathcal{E}$ by setting $\mathcal{E}(T_a) = \mu\, \kappa_{ab}\tilde{T}^b$. In this case the algebraic part of condition eq. (6.79), working infinitesimally in the affine symmetry parameter $g = \exp(\xi) = \exp(\xi^a T_a + \tilde{\xi}_a \tilde{T}^a)$, yields one non-trivial constraint

$$0 = (\mu^{-1} - \mu) f_{ab}{}^c \xi^a \tag{6.81}$$

which is also trivially solved, with no further condition on ξ, when $\mu = 1$ (corresponding to the un-deformed WZW model). The differential condition in eq. (6.79) then implies $\mathrm{d}\xi^a = \star\kappa^{ab}\mathrm{d}\tilde{\xi}_b$, or equivalently that $\xi = \{\xi_L(\sigma^+), \xi_R(\sigma^-)\}$. The $\mathcal{E}$-model descends to a WZW model, and its λ-deformation parameterised by μ, upon reduction to the coset $G_{\mathrm{diag}} \diagdown \exp(\mathfrak{d})$ defined by the equivalence relation $\{g_1, g_2\} \sim \{hg_1, hg_2\}$. We choose a coset representative $m = \{1, h\}$ for a group element h which will be the field of the WZW model. We see that the affine symmetry acts as

$$m \mapsto \{h_L(\sigma^+), hh_R(\sigma^-)\} \sim \{1, h_L^{-1}(\sigma^+) hh_R(\sigma^-)\} \ . \tag{6.82}$$

Thus the twisted chiral affine transformation on the double algebra descends to the anticipated $G_L \times G_R$ affine symmetry of the WZW model.

6.8 Appendix: Invariance of the non-Abelian Mkrtchyan action

In this appendix, we will demonstrate that the action (6.60),

$$S_{\mathrm{inv}} = S_{\mathrm{WZ}}[m] - \frac{1}{2}\int_{\partial M}\big(\langle\mu,\star\mu\rangle + 2\omega\wedge\langle\alpha, m^{-1}\mathrm{d}m\rangle\big)\ , \tag{6.83}$$

is invariant under the local transformation (6.62)

$$\delta_\epsilon m = \epsilon m\alpha\ , \quad \delta_\epsilon\omega = \mathrm{d}\epsilon\ , \quad \delta_\epsilon\alpha = \epsilon\iota_v(1-\star)\nabla\alpha\ , \tag{6.84}$$

where $\nabla\bullet = \mathrm{d}\bullet + [m^{-1}\mathrm{d}m, \bullet]$ is the covariant derivative with respect to the flat connection $m^{-1}\mathrm{d}m$. The covariant derivative ∇ is nilpotent $\nabla^2 = 0$ and satisfies

$$\int_{\partial M}\langle A_1, \nabla A_2\rangle = (-1)^{\deg A_1+1}\int_{\partial M}\langle\nabla A_1, A_2\rangle \tag{6.85}$$

for any pair of $\mathfrak{g}$-valued forms A_1, A_2. In terms of this operator we have a general formula for the variation of $m^{-1}\mathrm{d}m$ which is $\delta_\epsilon(m^{-1}\mathrm{d}m) = \nabla(m^{-1}\delta m)$. From (6.62) the last formula gives $\delta_\epsilon(m^{-1}\mathrm{d}m) = \nabla(\epsilon\alpha)$. We have

$$\delta_\epsilon S_{\mathrm{WZ}}[m] = -\int_{\partial M}\langle\epsilon\alpha, \nabla(m^{-1}\mathrm{d}m)\rangle \tag{6.86}$$

and the variation of the remainder — that must cancel against this — reads

$$-\int_{\partial M}\langle\delta_\epsilon\mu, \star\mu\rangle + \langle\delta_\epsilon(\alpha\omega), m^{-1}\mathrm{d}m\rangle + \langle\alpha\omega, \nabla(\epsilon\alpha)\rangle \tag{6.87}$$

The expression $\delta_\epsilon(\alpha\omega)$ is terrible and we will cancel it. We use $\star\delta_\epsilon\mu = -\delta_\epsilon\mu$ to simplify the first term inside the integral to $-\langle\mu, \delta_\epsilon\mu\rangle$. Then the first and last terms together yield

$$\begin{aligned} -\mu\delta_\epsilon\mu + \alpha\omega\nabla(\epsilon\alpha) &= (m^{-1}\mathrm{d}m)\delta_\epsilon\mu - \alpha\omega\delta_\epsilon\mu + \alpha\omega\nabla(\epsilon\alpha)\,, &(6.88)\\ &= (m^{-1}\mathrm{d}m)\delta_\epsilon\mu - \epsilon\alpha\omega\nabla\alpha + \alpha\omega\nabla(\epsilon\alpha)\,; &(6.89)\end{aligned}$$

the last two terms vanish by integration by parts (using $\nabla\omega = \mathrm{d}\omega = 0$). Therefore

$$-\mu\delta_\epsilon\mu + \alpha\omega\nabla(\epsilon\alpha) = (m^{-1}\mathrm{d}m)\delta_\epsilon\mu = -(m^{-1}\mathrm{d}m)\nabla(\epsilon\alpha) + (m^{-1}\mathrm{d}m)\delta_\epsilon(\alpha\omega)\,. \tag{6.90}$$

This last term cancels $\delta_\epsilon(\alpha\omega)m^{-1}\mathrm{d}m$ inside the remainder; the latter reduces to

$$+\int_{\partial M}\langle m^{-1}\mathrm{d}m, \nabla(\epsilon\alpha)\rangle = -\int_{\partial M}\langle\nabla(\epsilon\alpha), m^{-1}\mathrm{d}m\rangle\,, \tag{6.91}$$

which indeed cancels the variation of $S_{\mathrm{WZ}}[m]$.

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