SWANSEA UNIVERSITY

DOCTORAL THESIS

Higher rank tropicalization

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Declaration of Authorship

I, Xuan Gao, declare that this thesis titled, "Higher rank tropicalization" and the work presented in it are my own. I confirm that:

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- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
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- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
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Abstract

Faculty of Science and Engineering
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Higher rank tropicalization

by Xuan Gao

A higher rank valuation is a function that maps a field to the union of an ordered abelian group and infinity. There are studies that have shown that Kapranov's theorem still holds when the valuation is of rank $n > 1$ and the rank n tropicalization of a d-dimensional variety is a polyhedral complex of dimension nd, etc. This thesis aims to focus on higher rank valuation, we present a method about how to reduce a higher rank valuation to a sequence of classic valuations. With this method, we can describe the structure of the tropicalization over a higher rank valuation in terms of rank 1 tropicalisations, which will help us to reprove Kapranov's theorem in an alternative way.

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Contents

Chapter 1

Introduction

1.1 Introduction

This paper aims to study an alternative proof of the Fundamental theorem of tropical geometry with higher rank valuation. For illustrating it, we will introduce the background of tropical geometry and some work about higher rank valuation first.

Tropical geometry is a new field of mathematics and it's applied in many areas such as real and complex geometry, machine learning, neural networks and auctions theory, etc. Tropical geometry was proposed in the 1980s and it has experienced a rapid development since the beginning of the 21st century. Tropical geometry is a variant of algebraic geometry, in simple term, it is an intersection of algebraic geometry and combinatorial mathematics. Tropical geometry is based on the algebraic structure which is known as the tropical semiring or the min-plus algebra sum. In tropical semiring, the tropical sum $\oplus$ of two numbers is their minimum and the tropical product of two elements is their usual sum. With tropical algebra, the functions in tropical geometry are piecewise linear, and the algebraic variety also can be defined in tropical setting, which consists of convex ployhedra in $\mathbb{R}^n$. In this paper, chapter 2 will introduce the background of tropical geometry and the details of tropical geometry can be found in [1].

The tool building the tropical geometry is valuation. This is a function that mapping a field k to $\Gamma \cup \{\infty\}$ such that

$$\mathrm{val}(0) = \infty$$
$$\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$$
$$\mathrm{val}(a+b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$$

for any $a, b \in k$, and Γ here is meaning an additive ordered abelian subgroup of $\mathbb{R}$. In this thesis, a valuation is abbreviated as ν. Let ν be a valuation on field k, there exists a local ring containing those elements with non-negative valuation which is given by $R = \{a \in k \ : \ \nu(a) \geq 0\}$. Since R is a local ring, it contains a unique maximal ideal such that $\mathfrak{m}_k = \{a \in k \ : \ \nu(a) > 0\}$. Then the quotient ring $\mathbb{K} = R/\mathfrak{m}_k$ forms a field which is called the residue field of k. Now consider a Laurent polynomial $f = \sum_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} x^{\mathbf{u}} \in k[x_1^{\pm}, \ldots, x_n^{\pm}]$, the tropicalization of f via ν is taking valuation for every coefficient in f and replacing usual sum and product as tropical sum and tropical product such that

$$\operatorname{trop}(f) = \min_{\mathbf{u} \in \mathbb{Z}^n} \{\nu(c_{\mathbf{u}}) + x \cdot \mathbf{u}\} \in \mathbb{R}.$$

Obviously, $\operatorname{trop}(f)$ is a piecewise linear function mapping k to k which is a finite set of monomials and taking the minimum among this set to be the result of $\operatorname{trop}(f)$. Fix a vector $\mathbf{w} \in \mathbb{R}^n$ and substitute it into $\operatorname{trop}(f)$, we will have the minimum which is denoted as

$$\mathbf{W} = \operatorname{trop}(f)(\mathbf{w}) = \min_{\mathbf{u} \in \mathbb{Z}^n} \{\nu(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} \ : \ c_{\mathbf{u}} \neq 0\}.$$

For any valuation ν on k, there exists a splitting that $\sigma : \Gamma \longrightarrow k^*$, then the initial form of f with respect to $\mathbf{w}$ is defined as

$$\operatorname{in}_{\mathbf{w}}^{\nu}(f) = \sum_{\substack{\mathbf{u} \in \mathbb{Z}^n; \\ \mathbf{W} = \nu(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u}}} \overline{c_{\mathbf{u}} \sigma(\nu(c_{\mathbf{u}}))^{-1}} x^{\mathbf{u}} \in \mathbb{K}[x_1^{\pm}, \ldots, x_n^{\pm}].$$

The notation $\overline{c_{\mathbf{u}} \sigma(\nu(c_{\mathbf{u}}))^{-1}}$ means the image of $c_{\mathbf{u}} \sigma(\nu(c_{\mathbf{u}}))^{-1}$ in the residue field $\mathbb{K}$ of ν. In chapter 4, for clarifying the notations, we will use π to represent the mapping $R \to \mathbb{K}$. For the polynomial f, the tropical hypersurface $\mathcal{V}(\operatorname{trop}(f))$ is

$$\{\mathbf{w} \in \mathbb{R}^n \ : \ \text{the minimum in } \operatorname{trop}(f)(\mathbf{w}) \text{ is achieved at least twice}\}.$$

Then we have all the prerequisites of Kapranov's theorem. Kapranov's theorem was first stated in 1990's by Mikhail Kapranov[1]. It shows three different constructions of a tropical variety from a classic algebraic variety and those three are all the same.

Theorem 1.1.1. *(Kapranov's theorem)*
Let k be an algebraically closed field with a non-trivial valuation ν. Suppose a Laurent polynomial $f = \sum_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} x^{\mathbf{u}} \in k[x_1^{\pm}, \ldots, x_n^{\pm}]$. Then the following subset in $\mathbb{R}^n$ coincide:

1. *the tropical hypersurface* $\mathcal{V}(\mathrm{trop}(f))$*;*

2. *the set* $\{\mathbf{w} \in \mathbb{R}^n \;:\; \mathrm{in}_{\mathbf{w}}^{\nu}(f) \text{ in not a monomial}\}$*;*

3. *the closure of* $\{(\nu(y_1), \ldots, \nu(y_n)) \;:\; (y_1, \ldots, y_n) \in V(f)\}$*.*

Moreover, if f *is irreducible and* $\mathbf{w}$ *is any point in* $\Gamma_\nu^n \cap \mathrm{trop}(V(f))$*, then the set* $\{\mathbf{y} \in V(f) \;:\; \nu(\mathbf{y}) = \mathbf{w}\}$ *is Zariski dense in the hypersurface* $V(f)$*.*

With the same polynomial, the tropical hypersurface $\mathcal{V}(\mathrm{trop}(f))$ associated with tropical polynomial $\mathrm{trop}(f)$ is the set of those points in $\mathbb{R}^n$ which let $\mathrm{trop}(f)$ contains at least two minimum terms. After defining tropical hypersurface, we shall see the tropicalization of variety easily. Consider an ideal $I \subset k[x_1^{\pm}, \ldots, x_n^{\pm}]$, the variety of I is denoted as $X = V(I)$, the tropicalization of variety X: $\mathrm{trop}(X)$ is the intersection of all tropical hypersurfaces associated to the polynomials $f \in I$ such that

$$\mathrm{trop}(X) = \bigcap_{f \in I} \mathrm{trop}(V(f)) \subseteq \mathbb{R}^n.$$

The set above also called tropical variety $\mathrm{trop}(X)$ which is involved in the Fundamental theorem. Moreover, there are two other subset of $\mathbb{R}^n$ involve in the Fundamental theorem. One of them is the Zariski closure of the set of coordinate-wise valuations of points in X such that

$$\nu(X) = \{(\nu(y_1), \ldots, \nu(y_n) \;:\; (y_1, \ldots, y_n) \in X\}.$$

Then fixing the weight vector $\mathbf{w}$, for the ideal $I \subset k[x_1^{\pm}, \ldots, x_n^{\pm}]$, the initial ideal $\mathrm{in}_{\mathbf{w}}^{\nu}(I)$ is

$$\langle \mathrm{in}_{\mathbf{w}}^{\nu}(f) : f \in I \rangle \subset \mathbb{K}[x_1^{\pm}, \ldots, x_n^{\pm}].$$

And the last subset of $\mathbb{R}^n$ involve in the fundamental theorem is the set of all vectors $\mathbf{w} \in \mathbb{R}^n$ such that

$$\{\mathbf{w} \in \mathbb{R}^n \;:\; \mathrm{in}_{\mathbf{w}}^{\nu}(I) \neq \mathbb{K}[x_1^{\pm}, \ldots, x_n^{\pm}]\}.$$

Then we come to the Fundamental theorem which is the direct generalization of Kapranov's theorem from hypersurfaces to arbitrary varieties.

Theorem 1.1.2. *(Fundamental Theorem of Tropical Algebraic Geometry)[1]*
Let k *be an algebraically closed field with a non-trivial valuation* ν*, let* I *be an ideal in*

$k[x_1^{\pm}, \ldots, x_n^{\pm}]$, and let $X = V(I)$ be its variety in the algebraic torus $T^n \cong (k^*)^n$. Then the following three subsets of $\mathbb{R}^n$ coincide:

1. *the tropical variety* $\mathrm{trop}(X)$*;*
2. *the set of all vectors* $\mathbf{w} \in \mathbb{R}^n$ *with* $\mathrm{in}_{\mathbf{w}}^{\nu}(I) \neq \langle 1 \rangle$*;*
3. *the closure of the set of coordinate-wise valuations of points in X,* $\nu(X) = \{(\nu(y_1), \ldots, \nu(y_n)) \ : \ (y_1, \ldots, y_n) \in X\}$.

Furthermore, if X is irreducible and $\mathbf{w}$ is any point in $\Gamma_{\nu}^n \cap \mathrm{trop}(X)$, then the set $\{\mathbf{y} \in X \ : \ \nu(\mathbf{y}) = \mathbf{w}\}$ is Zariski dense in the classical variety X.

This paper is aim to extend the study on higher rank valuation, so we will introduce the difference between higher rank and ordinary valuation. In some paper, higher rank valuation also called Krull valuation and the definition of Krull valuation is that $\nu : k \mapsto \mathbb{G} \cup \{\infty\}$ where k is a field and $\mathbb{G}$ is an ordered group with rank $n > 1$. The rank of group $\mathbb{G}$ is the maximum length of the chain of distinct proper convex subgroups in $\mathbb{G}$. Replacing the valued group as an ordered group then ν is a higher rank valuation. Naturally, the definitions of tropical polynomial and tropical hypersurface in higher rank version is similar to the definitions in classic version.

In 2018, Fuensanta Aroca proved Kapranov's theorem holds when the valuation on the field has higher rank in [2]. And, in [3], S. Banerjee proved that rank n tropicalization of a d-dimensional variety is a polyhedral complex of dimension nd and proposed a question that $\mathrm{trop}(X)$ is connected if X is a connected variety in rank 1 case, but is that true when the rank of valuation is $n \geq 1$. In [4], Tyler Foster and Dhruv Ranganathan proved that is true.

In chapter 3, we will introduce more about higher rank valuation and show some examples of higher rank tropicalizaition and initial form with higher rank valuation. Then we will discuss about our own work on higher rank valuation, and the most important step is reducing a rank n valuation with the following proposition

Proposition 1.1.3. *Let ν be a rank n valuation on field k. Let $\nu = (\nu_1, \nu_2) : k \to \mathbb{R}^n_{\mathrm{lex}} \cup \{\infty\}$ where ν_1 is the first component and ν_2 is the remaining $n-1$ components such that $\nu_1 : k \to \mathbb{R} \cup \{\infty\}$ and $\nu_2 : k \to \mathbb{R}^{n-1}_{\mathrm{lex}} \cup \{\infty\}$. Clearly, ν_1 is a valuation and we let R be the valuation ring of ν_1, $\mathfrak{m}$ is the maximal ideal, $\mathbb{K}$ is the residue field. Now restricting the domain of ν_2 to be R such that $\nu_2 \mid_R : R \to \mathbb{R}^{n-1}_{\mathrm{lex}} \cap \{\infty\}$. Then there is a map $\overline{\nu}_2 : \mathbb{K} \longrightarrow \mathbb{R}^{n-1}_{\mathrm{lex}} \cup \infty$ as the following diagram*

$$\begin{array}{ccc} R & \xrightarrow{v_2|_R} & \mathbb{R}^{n-1}_{\text{lex}} \cup \infty \\ \downarrow & \overset{\overline{v}_2}{\nearrow} & \\ \mathbb{K} & & \end{array}$$

and the map $\overline{v}_2$ is a rank $n-1$ valuation on $\mathbb{K}$.

By induction, we shall reduce a rank n valuation ν on a field k to an n-step valuation $(v_1, \ldots, v_n)$ on k such that each v_i is a rank 1 valuation on the residue field of v_{i-1} for $i > 1$, and v_1 is a rank 1 valuation on k.

After defining n-step valuation, we will explore n-step tropicalization of a hypersurface and the initial form via an n-step valuation in chapter 4. Then we will prove that the rank n tropicalization of the hypersurface associated to a polynomial is equal to the n-step tropicalization of the hypersurface assoicated to this polynomial such that

Corollary 1.1.4. *Let ν be a rank n valuation on k and $f \in k[x_1^{\pm}, \ldots, x_m^{\pm}]$. Reducing ν to an n-step valuation $(v_1, \ldots, v_n)$ where*

$$\begin{aligned} v_1 &: k^* \to \mathbb{R} \\ v_2 &: k_1^* \to \mathbb{R} \\ &\vdots \\ v_n &: k_{n-1}^* \to \mathbb{R}. \end{aligned}$$

Fixing a weight vector $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_n) \in (\Gamma^m)^n$ and let $\operatorname{trop}_\nu(f)(\mathbf{w}) = \mathbf{W} = (\mathbf{W}_1, \ldots, \mathbf{W}_n) \in \Gamma^n$. Then we will have

$$\begin{aligned} \operatorname{trop}_{v_1}(f)(\mathbf{w}_1) &= \mathbf{W}_1 \\ \operatorname{trop}_{v_2}(\operatorname{in}^{v_1}_{\mathbf{w}_1} f)(\mathbf{w}_2) &= \mathbf{W}_2 \\ &\vdots \\ \operatorname{trop}_{v_n}(\operatorname{in}^{v_{n-1}}_{\mathbf{w}_{n-1}} \ldots \operatorname{in}^{v_1}_{\mathbf{w}_1} f)(\mathbf{w}_n) &= \mathbf{W}_n \end{aligned}$$

Moreover there is also an equivalence between the initial form of a polynomial via a rank n valuation with respect to a weight vector and the initial form with respect to the same weight vector via n-step valuation iterated such that

Proposition 1.1.5. *Let ν be a rank n valuation on field k which can be reduced to an n-step valuation on k which is supposed to be $(v_1, v_2, \ldots, v_n)$. Let $f = \sum_{\mathbf{u} \in \mathbb{Z}^m} c_{\mathbf{u}} x^{\mathbf{u}} \in k[x_1^{\pm}, \ldots, x_m^{\pm}]$*

and fix a weight vector $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n) \in \mathbb{R}^{n\times m}_{\text{lex}}$ *where* $\mathbf{w}_i \in \mathbb{R}^m$ *for all* $1 \leq i \leq n$. *Then*

$$\text{in}^{\nu}_{\mathbf{w}}(f) = \text{in}^{v_n}_{\mathbf{w}_n} \ldots \text{in}^{v_1}_{\mathbf{w}_1}(f)$$

Then rank n and n-step tropicalization vanish at the same weight vector consequently.

Corollary 1.1.6. *Let* $f \in k[x_1^{\pm}, \ldots, x_m^{\pm}]$, ν *be a rank n valuation on k which can be split as an n-step valuation* $(v_1, \ldots, v_n)$. *Suppose* $\mathbf{w}$ *be a weight vector for v such that* $\mathbf{w} \in (\mathbb{R}^m_{\text{lex}})^n$ *and* $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_n)$ *where* $\mathbf{w}_i \in \mathbb{R}^m$. *Then* $\text{trop}_{\nu}(f)$ *tropically vanishes at* $\mathbf{w}$ *if and only if* $\text{trop}_{v_1}(f)$ *tropically vanishes at* $\mathbf{w}_1$ *and* $\text{trop}_{v_i}(\text{in}^{v_{i-1}}_{\mathbf{w}_{i-1}} \ldots \text{in}^{v_1}_{\mathbf{w}_1}(f))$ *tropically vanishes at* $\mathbf{w}_i$ *for each* $1 < i \leq n$.

Finally, for proving Kapranov's theorem in higher rank version, we will prove the following proposition.

Proposition 1.1.7. *Let* ν *be a rank n valuation on an algebraically closed field k, and* f *is a polynomial in m variables over k. Fixing a weight vector* $\mathbf{w} \in \mathcal{V}(\text{trop}(f)) \cap (\Gamma^m)^n$ *and a point* $\mathbf{A} \in (\mathbb{K}^{\times})^{m\times n}$ *such that* $\mathbf{A} \in V(\text{in}^{\nu}_{\mathbf{w}}(f))$ *where* $\mathbb{K}$ *is the residue field of* ν. *Then there exists a point* $\mathbf{a} \in V(f)$ *with* $\nu(\mathbf{a}) = \mathbf{w}$ *and* $\pi(\mathbf{a}) = \mathbf{A}$.

With the proposition 1.1.7, we prove that: given a hypersurface $V(f)$ associated to a polynomial f, the coordinate-wise valuations via a rank n valuation ν of the points in $V(f)$ is equal to the tropical hypersurface associated to the rank n tropicalization of f, which means

$$\text{trop}_{\nu}(V(f)) = \mathcal{V}(\text{trop}_{\nu}(f)).$$

Chapter 2

Tropical Background

This chapter is an introduction of the tropical geometry. We will discuss the background of tropical geometry and explain the fundamental theorem of tropical geometry in the last section.

2.1 Polyhedral Geometry

Before we talking about the background of tropical geometry, polyhedral geometry is an important part which plays a significant role in the study of tropical variety. Everything in this section can be found in section 2.3 of [1] or [7]

Definition 2.1.1. Let C be a polyhedral cone in $\mathbb{R}^n$, then C is a positive hull of finite subsets of $\mathbb{R}^n$ such that

$$C = \operatorname{pos}(\mathbf{v}_1, \ldots, \mathbf{v}_n) := \left\{ \sum_{i=1}^{r} \lambda_i \mathbf{v}_i \in \mathbb{R}^n \ : \ \lambda_i \geq 0 \ \text{ for all } i \right\}.$$

In other word, a cone $C \in \mathbb{R}^n$ is a nonempty set of vectors which also contains all the linear combinations of these vectors with nonnegative coefficients. If all $\mathbf{v}_i$ are linearly independent, then the cone is simplicial.

Definition 2.1.2. A polyhedron $P \subseteq \mathbb{R}^n$ is an intersection of finitely many closed half spaces in $\mathbb{R}^n$ such that

$$P = \{\mathbf{v} \in \mathbb{R}^n \ : \ A\mathbf{v} \leq \mathbf{u}\}$$

where A is a $d \times n$ matrix and $\mathbf{u} \in \mathbb{R}^d$.

Furthermore, a polyhedral cone is a polyhedron.

Definition 2.1.3. A subset $X \subseteq \mathbb{R}^n$ is a convex set, if for all $\mathbf{u}, \mathbf{v} \in X$ we have $\lambda\mathbf{u} + (1-\lambda)\mathbf{v} \in X$ where $0 \leq \lambda \leq 1$.

With the notion of convex, it is clear that any intersection of convex sets is convex.

Definition 2.1.4. Let $X \subseteq \mathbb{R}^n$. The convex hull $\operatorname{conv}(X)$ of X is the smallest convex set which containing X. Furthermore, if $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a finite set, then $\operatorname{conv}(X) = \left\{ \sum_{i=1}^{r} \lambda_i \mathbf{x}_i \ : \ 0 \leq \lambda_i \leq 1, \sum_{i=1}^{r} \lambda_i = 1 \right\}$ is a polytope.

Polytopes are bounded polyhedra.

Example 2.1.5. Let $P \subset \mathbb{R}^2$ be a 2 dimensional polyhedron and $A = (1, -1)$, $\mathbf{u} = 0$. Then P is a polyhedron consists of the points which are below the line $x - y = 0$.

The inequality $A\mathbf{v} \leq \mathbf{u}$ represents a list of inequalities. Let $\mathbf{a}_1, \ldots, \mathbf{a}_d$ be the rows of matrix A, then $\mathbf{a}_i\mathbf{v}$ can be considered as a product of vectors which is less than u_i the corresponding component of vector $\mathbf{u}$.

Definition 2.1.6. A face of a polyhedron $P \subseteq \mathbb{R}^n$ is determined by a linear functional $\mathbf{w} \in (\mathbb{R}^n)^\vee$ such that

$$\operatorname{face}_{\mathbf{w}}(P) = \{\mathbf{x} \in P \ : \ \mathbf{w} \cdot \mathbf{x} \leq \mathbf{w} \cdot \mathbf{y} \text{ for all } \mathbf{y} \in P\}.$$

The definition of a face of a cone is the same

$$\operatorname{face}_{\mathbf{w}}(C) = \{\mathbf{x} \in C \ : \ \mathbf{w} \cdot \mathbf{x} \leq \mathbf{w} \cdot \mathbf{y} \text{ for all } \mathbf{y} \in C\}$$

Example 2.1.7. Let $P \subset \mathbb{R}^4$, and $P = \{(a, b, c, d) \in \mathbb{R}^4 \ : \ d \geq 0\}$. Let the linear functional $\mathbf{w}$ be $\mathbf{w} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, then $\operatorname{face}_{\mathbf{w}} = \{(a, b, c, d) \ : \ b = c = d = 0\}$.

A face of a polyhedron P that is not contained in any larger proper face is called a facet of this polyhedron P.

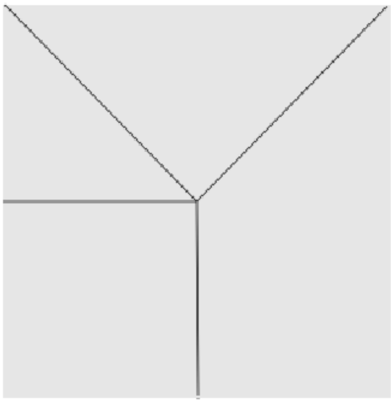
Definition 2.1.8. A polyhedral fan of $\mathbb{R}^n$ is a collection of polyhedral cones,

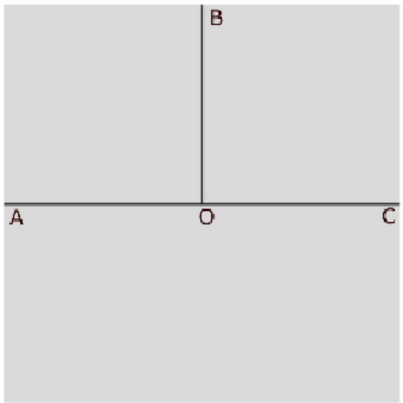
$$\mathcal{F} = \{C_1, \ldots, C_n\}$$

with the following two properties:

- Every nonempty face of a cone in the fan $\mathcal{F}$ is also a cone in $\mathcal{F}$.

- The intersection of any two cones in $\mathcal{F}$ is a face of these two cones





Example 2.1.9. The two figure above show an example of polyhedral fan and a picture which is not one. The figure on left hand side is a polyhedral fan but the right one is not. The intersection of the cone at the bottom of the figure and the cone in the upper left corner of the figure is the segment OA, but it is not a face of the cone at bottom.

Definition 2.1.10. A polyhedral complex is a collection Σ of polyhedra which satisfies the following two conditions:

- let P be a polyhedron, if P is in Σ, then any face of P is in Σ too
- let P, Q be two polyhedron, if P, Q are in Σ, then the intersection $P \bigcap Q$ is either an empty set or a face in both P and Q

The polyhedra in a polyhedral complex Σ is called the cells of polyhedral complex Σ. The cells of Σ are not faces of any larger cell are called facets of the complex, and their facets are called ridges of the complex.

Example 2.1.11. A common example is cube, a cube is a 3-dimensional polyhedral complex, the quadrangles are the facets of the cube and each edges are the ridges of the cube.

The support $|\Sigma|$ of polyhedral complex Σ is a set such that

$$\{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{x} \in P \text{ where } P \in \Sigma\}$$

Definition 2.1.12. The lineality space of a polyhedron P is the largest linear subspace $V \subset \mathbb{R}^n$ with the property that $\mathbf{x} \in P, \mathbf{v} \in V$ implies $\mathbf{x} + \mathbf{v} \in P$

The lineality space of a polyhedral complex is the intersection of all the lineality sapce of the polyhedra in the complex. The smallest affine subspace of $\mathbb{R}^n$ containing a polyhedron P is called the linear space parallel to P. The dimension of P is the dimension of the linear space parallel to P.

Definition 2.1.13. A polyhedral complex is pure of dimension d if every facet of the polyhedral complex has dimension d.

Definition 2.1.14. The relative interior of P which is denoted relint(P) is the interior of P inside its affine span. If $P = \{\mathbf{x} \in \mathbb{R}^n \ : \ A\mathbf{x} = \mathbf{b}, A'\mathbf{x} \leq \mathbf{b}'\}$, where each of the inequalities in $A'\mathbf{x} < \mathbf{b}'$ is strict for some $\mathbf{x} \in P$, then $\text{relint}(P) = \{\mathbf{x} \in \mathbb{R}^n \ : \ A\mathbf{x} = \mathbf{b}, A'\mathbf{x} < \mathbf{b}'\}$

Definition 2.1.15. Let Γ be a subgroup of $(\mathbb{R}, +)$. A Γ-rational polyhedron is

$$P = \{\mathbf{x} \in \mathbb{R}^n \ : \ A\mathbf{x} \leq \mathbf{u}\}$$

for some A is a $d \times n$ matrix with entries in $\mathbb{Q}$, and $\mathbf{u} \in \Gamma^d$.

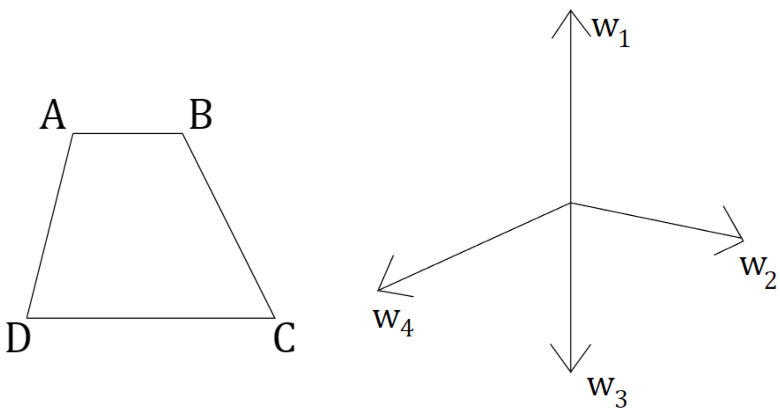
And if every polyhedron in a polyhedral complex Σ is Γ-rational, then Σ is Γ-rational.

Definition 2.1.16. Let $P \in \mathbb{R}^n$ be a polyhedron. The normal fan of P is the polyhedral fan $\mathcal{N}_P$ consisting of the cones

$$\mathcal{N}_P(\mathcal{F}) = \text{cl}(\{w \in (\mathbb{R}^n)^\vee : \text{face}_w(P) = \mathcal{F})\})$$

as $\mathcal{F}$ varies over the faces of P.

The notation cl is the closure in the Euclidean topology on $(\mathbb{R}^n)^\vee$.



Example 2.1.17. As the figure above, the normal fan of the quadrangle $ABCD$ is the graph on right side. The vector w_1 is the linear functional which determines the segment CD as a face of $ABCD$, so do vector w_2, w_3 and w_4 determine AD, AB and BC respectively. And this fan consists of nine cones, four of dimension 2 that are the four areas separated by the vectors, four of dimension 1 and one of dimension 0 which is the point intersected by the vectors.

Definition 2.1.18. Let $S = k[{x_1}^{\pm 1}, \ldots, {x_n}^{\pm 1}]$ be a Laurent polynomial ring. Let $f = \sum\limits_{u \in \mathbb{Z}^n} c_u x^u \in S$, then the Newton polytope of f is

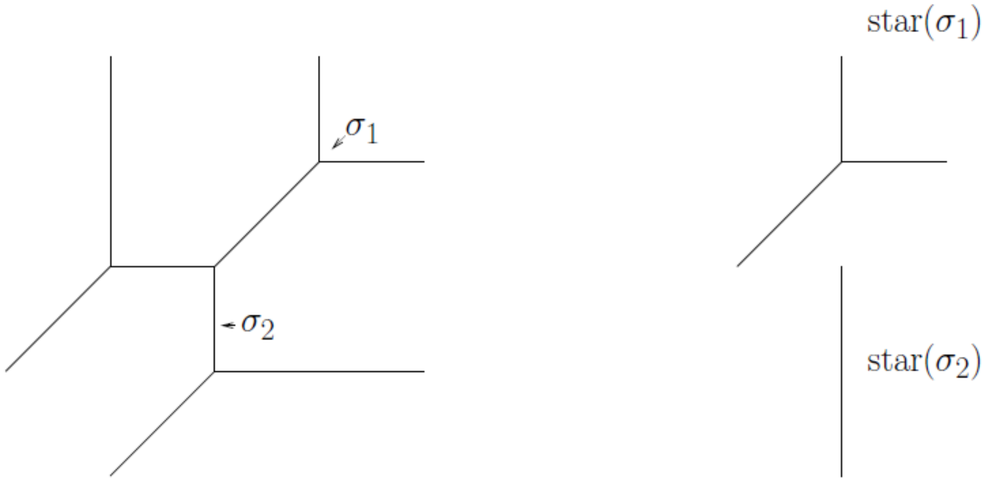
$$\mathrm{Newt}(f) = \mathrm{conv}(u : c_u \neq 0) \subset \mathbb{R}^n.$$

Example 2.1.19. Consider the polynomial $f = x^{-1} - y^{-1} + 3x - 2y + xy$, its Newton polytope is a polygon which has vertices at $(-1, 0)$, $(0, -1)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$.

Definition 2.1.20. Let Σ be a polyhedral complex in $\mathbb{R}^n$, and σ is a cell in Σ. The star of σ in Σ is a fan in $\mathbb{R}^n$, written as $\mathrm{star}_\Sigma(\sigma)$. the cones in the fan $\mathrm{star}_\Sigma(\sigma)$ are indexed by those cells τ in Σ that contains σ as a face. Then the cone of $\mathrm{star}_\Sigma(\sigma)$ that is indexed by τ is the following subset in $\mathbb{R}^n$

$$\tilde{\tau} = \{\lambda(x - y) : \lambda \geq 0, x \in \tau, y \in \sigma\}$$

As the following figure, the polyhedral complex Σ is shown on the left hand side. The affine span of the vertex σ_1 is the vertex itself and the star of σ_1 is shown on the right. So the the star of σ_2.



Definition 2.1.21. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be an ordered set of vectors in $\mathbb{R}^{n+1}$ and fix $\mathbf{w} = (w_1, \ldots, w_r) \in \mathbb{R}^r$. The regular subdivision of $\mathbf{v}_1, \ldots, \mathbf{v}_r$ induced by $\mathbf{w}$ is the polyhedral fan with the support

$$\text{pos}(\mathbf{v}_1, \ldots, \mathbf{v}_r) = \left\{ \sum_{i=1}^{r} \lambda_i \mathbf{v}_i \in \mathbb{R}^n : \lambda_i \geq 0 \text{ for all } i \right\}$$

whose cones are $\text{pos}(\mathbf{v}_i : i \in \sigma)$ for all subsets $\sigma \subseteq \{1, \ldots, r\}$ such that there exists $\mathbf{c} \in \mathbb{R}^{n+1}$ with $\mathbf{c} \cdot \mathbf{v}_i = w_i$ for $i \in \sigma$ and $\mathbf{c} \cdot \mathbf{v}_i < w_i$ for $i \notin \sigma$.

The construction is usually the following processes:

First, let the vectors $\mathbf{v}_i = (\mathbf{u}_i, 1)$ which represents a point configuration $\mathbf{u}_1, \ldots, \mathbf{u}_r$ in $\mathbb{R}^n$. So the polyhedral fan in definition 2.1.21 is a subdivision of the polytope $P = \text{conv}\{\mathbf{u}_i \ : \ 1 \leq i \leq r\}$ in $\mathbb{R}^n$. Then we can write the regular subdivision of P induced by $\mathbf{w} = (w_1, \ldots, w_r) \in \mathbb{R}^r$ as the following equation

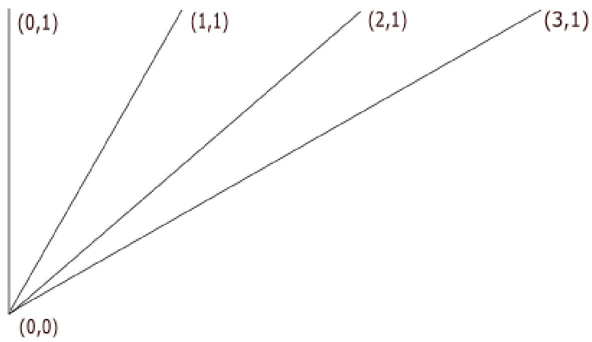
$$P_{\mathbf{w}} = \text{conv}\{(\mathbf{u}_i, w_i) \ : \ 1 \leq i \leq r\} \subset \mathbb{R}^{n+1}.$$

Let $\mathbf{c} \in (\mathbb{R}^{n+1})^{\vee}$ be an inner normal vector. For finding the regular subdivision of P, we need to find the lower faces of $P_{\mathbf{w}}$ which is the vectors in $P_{\mathbf{w}}$ with an inner normal vector $\mathbf{c} \in (\mathbb{R}^{n+1})^{\vee}$ and the last coordinate of $\mathbf{c}$ is positive. Then these lower faces project to $P \subset \mathbb{R}^n$ and form a polyhedral complex which is defined in definition 2.1.21.

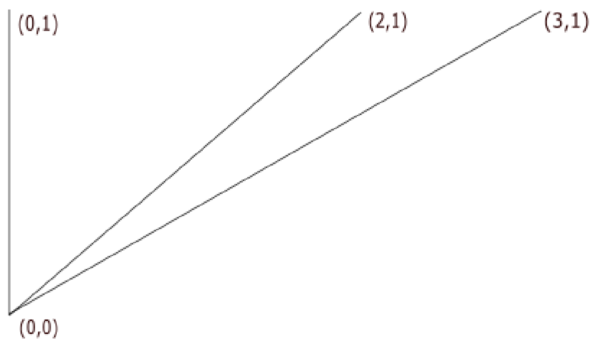
In addition, as the construction above, $\mathbf{c}$ represents the inner normal vectors of the lower faces of $P_{\mathbf{w}}$. We can rewrite them to define the vectors $\mathbf{c} \in \mathbb{R}^{n+1}$ in definition 2.1.21. Let $(\hat{\mathbf{c}}, 1)$ be an inner normal vector for a face $\text{conv}((\mathbf{u}_i, w_i) \ : \ i \in \sigma)$ of

$P_{\mathbf{w}}$. By definition and , $(\hat{\mathbf{c}}, 1) \cdot \mathbf{x} \leq (\hat{\mathbf{c}}, 1) \cdot \mathbf{y}$ where $\mathbf{x} \in \mathrm{conv}((\mathbf{u}_i, w_i) \ : \ i \in \sigma)$ and $\mathbf{y} \in P_{\mathbf{w}}$. Then let $(\hat{\mathbf{c}}, 1) \cdot (\mathbf{u}_i, w_i) \geq c_0$ for all i and the equality holds when $i \in \sigma$. Then we rewrite this inequality that is $(-\hat{\mathbf{c}}, c_0) \cdot (\mathbf{u}_i, 1) \leq w_i$, with equality when $i \in \sigma$. Let $(-\hat{\mathbf{c}}, c_0)$ be the vector $\mathbf{c} \in \mathbb{R}^{n+1}$, then we have the vector defined in definition 2.1.21.

Example 2.1.22. Let $n = 1$, $r = 4$ and a cone $\mathrm{pos}((0,1),(3,1)) \in \mathbb{R}^2$ which is spanned by the vectors $(0,1),(1,1),(2,1),(3,1)$. Suppose $\mathbf{w} = (4,2,1,2) \in \mathbb{R}^4$ then the regular subdivision are three cones: $\mathrm{pos}((0,1),(1,1))$, $\mathrm{pos}((1,1),(2,1))$ and $\mathrm{pos}((2,1),(3,1))$, which is shown in the following figure.



When we let $\mathbf{w} = (3,2,1,2) \in \mathbb{R}^4$, the regular subdivision are two cones: $\mathrm{pos}((0,1),(2,1))$ and $\mathrm{pos}((2,1),(3,1))$, as the following figure.



2.2 Valuations

From this section to the end of this chapter, we will introduce the background of Fundamental theorem in tropical geometry. And any details of this part can be found in chapter 2 and 3 in [1].

Definition 2.2.1. Let k be a field and $k^{\times}$ be the set of nonzero elements in k. A valuation ν on k is a function such that $\nu : k \longrightarrow \Gamma \cup \{\infty\}$, and ν has the following three axioms:

$$\nu(0) = \infty$$
$$\nu(ab) = \nu(a) + \nu(b)$$
$$\nu(a+b) \geq \min\{\nu(a),\ \nu(b)\},$$

for all $a, b \in k$. Γ is an additive ordered abelian subgroup of $\mathbb{R}$, which is called the value group of (k, ν).

From definition 2.2.1, we shall gain $\nu(1) = 0$. Since for any $a \in k$ we have $\nu(a) = \nu(a \cdot 1) = \nu(1) + \nu(a)$ which implies $\nu(1) = 0$.

Usually $\Gamma \cup \{\infty\}$ is abbreviated as Γ_+, and Γ_+ is an idempotent semifield with the operation $\oplus$ and $\odot$. These two operations are tropical addition and multiplication respectively, the tropical addition is taking the minimum between two elements and the tropical multiplication is the usual addition. Then we shall see that $(\Gamma, \odot)$ is an abelian group with identity element 0 and ∞ is the identity for $(\Gamma_+, \oplus)$. As ordinary arithmetic, tropical multiplication $\odot$ has higher priority than tropical addition $\oplus$, so suppose $a, b, c \in \Gamma_+$ we have

$$c \odot (a \oplus b) = c \odot a \oplus c \odot b$$

and since $(\Gamma_+, \odot)$ is commutative, $\odot$ is distributive on both sides. Also division is always possible in $(\Gamma_+, \odot)$ since $\odot$ is the usual addition. For every element $\nu(a) \in \Gamma_+$ we have $\nu(a) \oplus \nu(a) = \nu(a)$, which shows that it is idempotent.

Lemma 2.2.2. *Let k be a field and ν be a valuation on k. Let $a, b \in k$ if $\nu(a) \neq \nu(b)$ then $\nu(a+b) = \min\{\nu(a), \nu(b)\}$.*

Proof. Without loss of generality, we shall assume that $\nu(b) > \nu(a)$. We just show that $\nu(1) = 0$, then $\nu(-1) = 0$, since $(-1)^2 = 1$. Therefore we have $\nu(-b) = \nu(b)$ for all $b \in k$. By the third axiom of definition 2.2.1 there exist

$$\nu(a) = \nu((a+b) + (-b)) \geq \min\{\nu(a+b),\ \nu(-b)\} = \min\{\nu(a+b),\ \nu(b)\}.$$

So we have $\nu(a) \geq \nu(a+b)$, since we have assumed $\nu(b) > \nu(a)$. On the other hand, we have

$$\nu(a+b) \geq \min\{\nu(a), \nu(b)\} = \nu(a).$$

Hence, $\nu(a+b) = \nu(a)$. $\square$

In addition, there are several subsets in the field k need to be mentioned. First, let the valuation on k be ν and consider a subset R of field k such that

$$R = \{c \in k \ : \ \nu(c) \geq 0\}.$$

Then the set R is a local ring, and R is also called a valuation ring associated to valuation ν. Since R is a local ring, there is a unique maximal ideal which we can denote it to be $\mathfrak{m}_\nu$ such that

$$\mathfrak{m}_\nu = \{c \in k \ : \ \nu(c) > 0\}.$$

It is easy to see that $R/\mathfrak{m}_\nu$ is a subfield in k and usually we denote it to be $\mathbb{K}$, called the residue field of (k, ν).

Example 2.2.3. One of the most common example of valuation is the p-adic valuation on the field of rational numbers $\mathbb{Q}$. Let ν be a valuation such that $\nu \ : \ \mathbb{Q} \longrightarrow \mathbb{R}$ and it is defined as $\nu_p(q) = t$ where $q = p^t \frac{a}{b}$, $a, b \in \mathbb{Z}$ and p does not divide a or b. For instance,

$$\nu_2(15) = 0 \quad \text{or} \quad \nu_3(\frac{9}{14}) = 2.$$

The valuation ring R of p-adic valuation ν at prime p is the set consists of the rational numbers $\frac{m}{n}$ where p does not divide n. The maximal ideal $\mathfrak{m}_\nu$ consists of the rational numbers $\frac{m}{n}$ where p divides m but not n. Then the residue field $\mathbb{K}$ of ν is a finite field $\mathbb{Z}/p\mathbb{Z}$.

Another common field in tropical geometry is the field of Puiseux series. Puiseux series is a generalization of power series that the exponents of indeterminate in Puiseux series are allowed to be negative or fraction. Usually, we let a Puiseux series with coefficients in complex number $\mathbb{C}$, then there is an expression of the form

$$c(t) = \sum_{k=k_0}^{\infty} c_k t^{\frac{k}{n}},$$

where n is a positive integer and k_0 is an integer. For instance,

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + c_3 t^{a_3} + \dots$$

is a Puiseux series where c_i are non-zero complex numbers for all i, and a_i are rational numbers with the same denominator with $a_1 < a_2 < a_3 < \dots$. Usually, we use notation $\mathbb{C}\{\{t\}\}$ for the field of Peisuex series over $\mathbb{C}$. There is a natural valuation on the field of Peisuex series, that is $\nu : \mathbb{C}\{\{t\}\} \longrightarrow \mathbb{R}$ which is defined by taking a nonzero Peisuex series $c(t)$ to the lowest exponent a_1 that appears in $c(t)$ and we will call this valuation as t-adic valuation in this thesis. In addition, the field $k\{\{t\}\}$ is algebraically closed when k is an algebraically closed field of characteristic zero by [1, Theorem 2.1.5].

Example 2.2.4. Suppose a field of Puiseux series $\mathbb{C}\{\{t\}\}$, let $c_1(t), c_2(t) \in \mathbb{C}\{\{t\}\}$ such that

$$\begin{aligned} c_1(t) &= \frac{4t^2 - 7t^3 + 9t^5}{6 + 11t^4} = \frac{2}{3}t^2 - \frac{7}{6}t^3 + \frac{3}{2}t^5 + \dots \\ c_2(t) &= \frac{14t + 3t^2}{7t^4 + 3t^7 + 8t^8} = 2t^{-3} + \frac{3}{7}t^{-2} + \dots . \end{aligned}$$

Then the valuations are $\nu(c_1(t)) = 2$ and $\nu(c_2(t)) = -3$

2.3 Tropical polynomials

A tropical polynomial is a finite tropical linear combination of tropical monomials. For instance, let $f \in \mathbb{R}_+[x_1, \dots, x_n]$ such that

$$f = a_1 {x_1}^{i_1} {x_2}^{i_2} \dots {x_n}^{i_n} \oplus a_2 {x_1}^{j_1} {x_2}^{j_2} \dots {x_n}^{j_n} \oplus \dots$$

where $a_1, a_2, \dots \in \mathbb{R}$ are coefficients. The symbol $\oplus$ represents tropical sum which is taking minimum and multiplication in tropical polynomial means addition in classic arithmetic. So evaluating f in classic arithmetic we will have

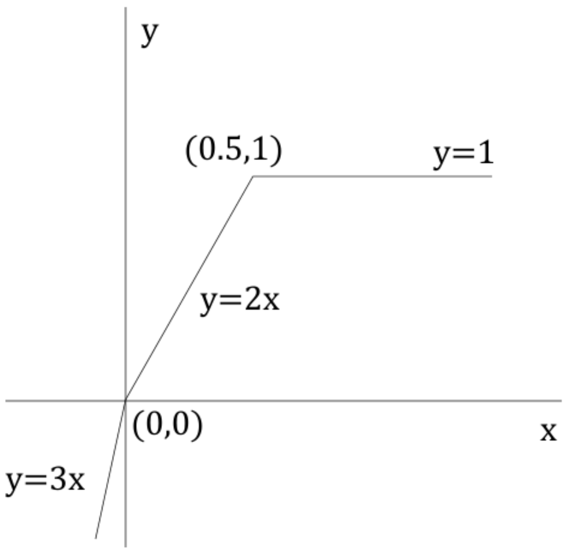
$$f = \min\{a_1 + i_1 x_1 + i_2 x_2 + \dots i_n x_n, a_2 + j_1 x_1 + j_2 x_2 + \dots j_n x_n, \dots\}.$$

Let $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$ be a point and polynomial f evaluated at $\mathbf{w}$ is $f(\mathbf{w}) = \min\{a_1 + i_1 w_1 + i_2 w_2 + \dots i_n w_n, a_2 + j_1 w_1 + j_2 w_2 + \dots j_n w_n, \dots\}$, then $f(\mathbf{w})$ is the linear function which is taking the minimum at $\mathbf{w}$. Hence tropical polynomial is a combination of linear functions.

Hence a tropical polynomial also represents a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ and satisfies the following three properties

1. f is continuous
2. f is piecewise linear and the number of pieces is finite
3. f is concave

Example 2.3.1. Let $f(x) = x^3 \oplus x^2 \oplus 1$ be a tropical polynomial, then evaluating f in classic arithmetic we have $f = \min\{3x, 2x, 1\}$. The graph of f is the following figure.



As shown in the (x, y) plane above, when $x < 0$ then minimum of f is equal to the linear function $y = 3x$, when $0 < x < 0.5$ the minimum is $y = 2x$, and when $x > 0.5$ it is $y = 1$. And if the minimum in f is more than one term such that $3x = 2x$ and $2x = 1$, it is those two points which labelled in the figure.

2.4 Tropicalization of polynomials

Definition 2.4.1. Let k be a field, f is a polynomial such that $f = \sum\limits_{\mathbf{u}\in\mathbb{N}^n} c_{\mathbf{u}} x^{\mathbf{u}} \in k[x_1, \ldots, x_n]$ and ν is a valuation on k. Then the tropicalization of f via ν is a tropical polynomial in $x_1, \ldots, x_n$ such as

$$\operatorname{trop}_\nu(f) = \bigoplus_{\mathbf{u}\in\mathbb{N}^n} \nu(c_{\mathbf{u}}) + x \cdot \mathbf{u}.$$

And if we fix a weight vector $\mathbf{w} \in \Gamma^n$, the tropicalization of f at weight vector $\mathbf{w}$ is

$$\text{trop}_\nu(f)(\mathbf{w}) = \min\Big(\nu(c_\mathbf{u}) + \mathbf{w} \cdot \mathbf{u}\Big) \text{ where } \mathbf{u} \in \mathbb{N}^n.$$

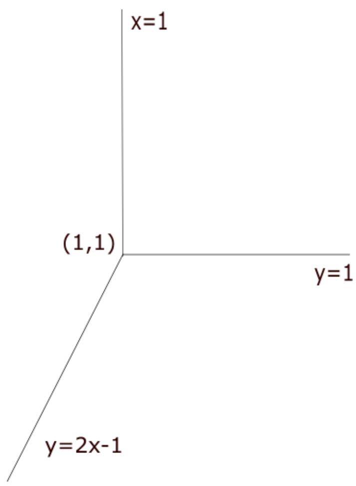
Example 2.4.2. Suppose a polynomial $f = 3x^4 + 4y^2 + 16 \in \mathbb{Q}[x^\pm, y^\pm]$. The valuation ν on $\mathbb{Q}$ is 2-adic valuation. Then the tropicalization of f is

$$\text{trop}_\nu(f) = \min\{4x, 2 + 2y, 4\}$$

For instance, let the weight vector $\mathbf{w}$ be $(2, 2)$, then the tropicalization of f at $\mathbf{w}$ is

$$\text{trop}_\nu(f)(\mathbf{w}) = \min\{8, 2 + 4, 4\} = 4$$

The following graph is a projection of the tropical line of $\text{trop}(f)$ in 2-dimensional plane. The segment $x = 1$ represents a shadow of a part of the whole tropical line. It consists of those weight vector $\mathbf{w}$ which make $\text{trop}(f)_\nu(\mathbf{w})$ have two terms $4x$ and 4, then $4x = 4$. Similarly, the lines $y = 1$ and $y = 2x - 1$ represent the value of weight vectors $\mathbf{w}$ that make $2 + 2y = 4$ and $4x = 2 + 2y$ respectively. And the point $(1, 1)$ is the solution of $4x = 2 + 2y = 4$.



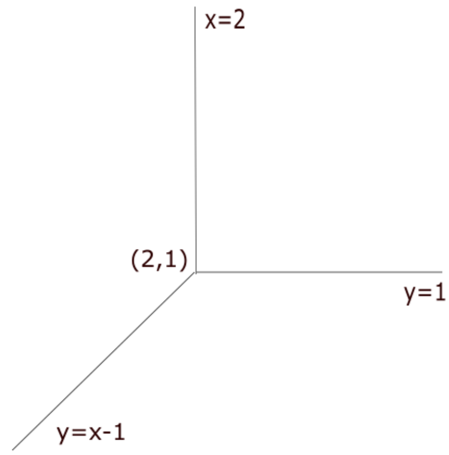
Example 2.4.3. Suppose that $k = \mathbb{C}\{\{t\}\}$, $f = (t + t^2)x + 2t^2y + t^3 \in k[x^\pm, y^\pm]$ and the valuation ν on $\mathbb{C}\{\{t\}\}$ is given by taking the lowest nonzero exponent which appears in Puiseux series. Then the tropicalization of f is

$$\text{trop}_\nu(f) = \min\{1 + x, 2 + y, 3\}$$

Let the weight vector $\mathbf{w}$ be $(1,2)$. Then the tropicalization of f at $\mathbf{w}$ is

$$\text{trop}_\nu(f)(\mathbf{w}) = \min\{1+1, 2+2, 3\} = 2$$

The same as Example 2.4.2, the following graph represents those solutions that make the minimum in trop(f) has at least two terms.



2.5 Initial forms of polynomials

Recalling Definition 2.4.1, we know that the tropicalization with a valuation ν of a polynomial $f = \sum_{\mathbf{u}\in\mathbb{N}^n} c_\mathbf{u} x^\mathbf{u} \in k[x_1,\ldots,x_n]$ is a piecewise linear function trop(f) : $\mathbb{R}^n \longrightarrow \mathbb{R}$. For the valuation $\nu : k \to \Gamma_+$, there is a mapping $\sigma : \Gamma \to k^*$ where k^* is the set of nonzero elements of k. Obviously, it is a homomorphism such that $\sigma(a+b) = \sigma(a)\sigma(b)$, for any $a, b \in \Gamma$. And this mapping σ is named as a splitting of ν. Now we can define the initial form of f.

Definition 2.5.1. Let $f = \sum_{\mathbf{u}\in\mathbb{N}^n} c_\mathbf{u} x^\mathbf{u} \in k[x_1,\ldots,x_n]$, ν is a valuation on k and $\mathbf{w} = (w_1,\ldots,w_n) \in \mathbb{R}^n$ be weight vector. Suppose

$$\mathbf{W} = \text{trop}_\nu(f)(\mathbf{w}) = \min\{\nu(c_\mathbf{u}) + \mathbf{w}\cdot\mathbf{u} \;:\; c_\mathbf{u} \neq 0\}.$$

Then the initial form of f with respect to $\mathbf{w}$ is

$$\begin{aligned}\text{in}_{\mathbf{w}}^{\nu}(f) &= \sum_{\substack{\mathbf{u}\in\mathbb{Z}^n \\ \min\{\nu(c_{\mathbf{u}})+\mathbf{w}\cdot\mathbf{u}\}=\mathbf{W}}} \overline{\sigma(\mathbf{w}\cdot\mathbf{u}-\mathbf{W})\cdot c_{\mathbf{u}}}x^{\mathbf{u}} \\ &= \sum_{\substack{\mathbf{u}\in\mathbb{Z}^n \\ \min\{\nu(c_{\mathbf{u}})+\mathbf{w}\cdot\mathbf{u}\}=\mathbf{W}}} \overline{\sigma(-\nu(c_{\mathbf{u}}))\cdot c_{\mathbf{u}}}x^{\mathbf{u}}\end{aligned}$$

The notation $\overline{\sigma(-\nu(c_{\mathbf{u}}))\cdot c_{\mathbf{u}}}$ means the image of $\sigma(-\nu(c_{\mathbf{u}}))\cdot c_{\mathbf{u}} \in k$ in the residue field $\mathbb{K}$ of ν.

Example 2.5.2. Let $f \in k[x_1, x_2, x_3]$ where $k = \mathbb{C}\{\{t\}\}$. Suppose $f = (t-t^3){x_1}^2 + 3t^2x_2 + 2t^3{x_3}^3$ then the tropicalization of f is $\text{trop}_{\nu}(f) = \min\{1+2x_1, 2+x_2, 3+3x_3\}$. Fixing a weight vector $\mathbf{w} = (1,1,1)$ then $\mathbf{W} = 3$ and the initial form of f is

$$\begin{aligned}\text{in}_{\mathbf{w}}^{\nu}(f) &= \overline{t^{-1}(t-t^3)}{x_1}^2 + \overline{t^{-2}(3t^2)}x_2 \\ &= \overline{(1-t^2)}{x_1}^2 + \overline{3}x_2 \\ &= {x_1}^2 + 3x_2.\end{aligned}$$

If $\mathbf{w} = (2,1,0)$ then $\mathbf{W} = 3$ and the initial form of f is

$$\begin{aligned}\text{in}_{\mathbf{w}}^{\nu}(f) &= \overline{t^{-2}(3t^2)}x_2 + \overline{t^{-3}2t^3}{x_3}^3 \\ &= \overline{3}x_2 + \overline{2}{x_3}^3 \\ &= 3x_2 + 2{x_3}^3.\end{aligned}$$

Definition 2.5.3. The degree of a polynomial $f = \sum_{\mathbf{u}\in\mathbb{N}^n} c_{\mathbf{u}}x^{\mathbf{u}}$ in $k[x_1, \ldots, x_n]$ is $U = \max\{|\mathbf{u}| : c_{\mathbf{u}} \neq 0\}$, where $|\mathbf{u}| = \sum_{i=1}^{n} u_i$. The homogenization $\tilde{f}$ of f is the homogeneous polynomial $\tilde{f} = \sum c_{\mathbf{u}} x_0^{U-|\mathbf{u}|}x^{\mathbf{u}} \in k[x_0, x_1, \ldots, x_n]$. The homogenization of an ideal I in $k[x_1, \ldots, x_n]$ is the ideal $I_{\text{proj}} = \langle \tilde{f} : f \in I\rangle$.

Similarly, the definition of I_{proj} for a given Laurent ideal $I \subset k[x_1^{\pm}, \ldots, x_n^{\pm}]$ is the same.

Definition 2.5.4. Let ν be a valuation on k, and I be any ideal in $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and fix a weight vector $\mathbf{w} \in \mathbb{R}^n$, then its initial ideal is

$$\text{in}_{\mathbf{w}}^{\nu}(I) = \langle \text{in}_{\mathbf{w}}^{\nu}(f) \ : \ f \in I\rangle \subset \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

where $\mathbb{K}$ is the residue field of ν.

Notice that the polynomial ring $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ consists of Laurent polynomials, then some choices of weight vector $\mathbf{w} \in \mathbb{R}^n$ may let the initial form $\text{in}_{\mathbf{w}}^{\nu}(f)$ be a unit in $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ where $\mathbb{K}$ is the residue field of ν, and $\text{in}_{\mathbf{w}}^{\nu}(I)$ will be the whole ring. So we only pay attention on those wight vectors which make $\text{in}_{\mathbf{w}}^{\nu}(I) \subset \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Lemma 2.5.5. *[1, Lemma 2.4.6] Fix a polynomial $f \in k[x_1, \ldots, x_n]$ and $\mathbf{w}, \mathbf{v} \in \mathbb{R}^n$. There exists an $\epsilon > 0$ such that, for all $\epsilon > \epsilon' > 0$, we have*

$$\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(f)) = \text{in}_{\mathbf{w}+\epsilon'\mathbf{v}}(f).$$

Proof. Let $f = \sum_{\mathbf{u} \in \mathbb{N}^n} c_{\mathbf{u}} x^{\mathbf{u}}$. Then the initial form of f with respect to $\mathbf{w}$ is

$$\text{in}_{\mathbf{w}}(f) = \sum_{\substack{\mathbf{u} \in \mathbb{Z}^n \\ \min\{\nu(c_{\mathbf{u}}) + \mathbf{w}\cdot\mathbf{u}\} = W}} \overline{c_{\mathbf{u}} t^{\mathbf{w}\cdot\mathbf{u} - W}} x^{\mathbf{u}}, \quad \text{where } \; W = \text{trop}_{\nu}(f)(\mathbf{w}).$$

Let $W' = \min\{\mathbf{v} \cdot \mathbf{u} : \nu(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} = W\}$ which means we choose the minimum among the terms in $\text{in}_{\mathbf{w}}^{\nu}(f)$ when plugging the vector weight $\mathbf{v}$ into $\text{in}_{\mathbf{w}}^{\nu}(f)$. Then

$$\text{in}_{\mathbf{v}}^{\nu}(\text{in}_{\mathbf{w}}^{\nu})(f) = \sum_{\mathbf{v}\cdot\mathbf{w} = W'} \overline{c_{\mathbf{u}} t^{\mathbf{w}\cdot\mathbf{u} - W}} x^{\mathbf{u}}.$$

and the power of x in $\text{in}_{\mathbf{v}}^{\nu}(\text{in}_{\mathbf{w}}^{\nu})$ is the subset $\{\mathbf{u} : \nu(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} = W, \; \mathbf{v} \cdot \mathbf{u} = W'\}$ of $\mathbb{N}^n$. Now consider the initial form $\text{in}_{\mathbf{w}+\mathbf{v}}^{\nu}(f)$, the power of monomials in $\text{in}_{\mathbf{w}+\mathbf{v}}^{\nu}(f)$ is $\{\mathbf{u} : \min(\nu(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u})\}$. Clearly, there is a possible that $\{\mathbf{u} : \nu(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} = W, \; \mathbf{v} \cdot \mathbf{u} = W'\} \neq \{\mathbf{u} : \min(\nu(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u})\}$, then we may choose a sufficiently small positive real number ϵ to make the changing of value by $\epsilon\mathbf{v} \cdot \mathbf{u}$ cannot influence the choosing of $\mathbf{u}$. Let ϵ be a sufficiently small positive real number such that

$$\text{trop}(f)(\mathbf{w} + \epsilon\mathbf{v}) = \min\{\nu(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} + \epsilon\mathbf{v} \cdot \mathbf{u}\} = \mathbf{W} + \epsilon\mathbf{W}'.$$

Then the exponent appearing in $\text{trop}(f)(\mathbf{w} + \epsilon\mathbf{v})$ is $\{\mathbf{u} : \nu(c_{\mathbf{u}}) + (\mathbf{w} + \epsilon\mathbf{v}) \cdot \mathbf{u} = \mathbf{W} + \epsilon\mathbf{W}'\}$. Since $\mathbf{W}' = \min\{\mathbf{v} \cdot \mathbf{u} : \nu(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} = \mathbf{W}\}$, then

$$\{\mathbf{u} : \nu(c_{\mathbf{u}}) + (\mathbf{w} + \epsilon\mathbf{v}) \cdot \mathbf{u} = \mathbf{W} + \epsilon\mathbf{W}'\} = \{\mathbf{u} : \nu(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} = \mathbf{W}, \mathbf{v} \cdot \mathbf{u} = \mathbf{W}'\}.$$

These exponents are the same as those in initial form $\text{in}_{\mathbf{v}}^{\nu}(\text{in}_{\mathbf{w}}^{\nu})(f)$. Therefore $\text{in}_{\mathbf{w}+\epsilon'\mathbf{v}}(f) = \text{in}_{\mathbf{v}}^{\nu}(\text{in}_{\mathbf{w}}^{\nu})(f)$ for all $0 < \epsilon' < \epsilon$. $\square$

Lemma 2.5.6. *[1, Lemma 2.4.2] Let I be a homogeneous ideal in $k[x_1, \ldots, x_n]$. Fix $\mathbf{w} \in \mathbb{R}^n$. Then $\text{in}_{\mathbf{w}}(I)$ is homogeneous, and we may choose a homogeneous Gröbner basis for I. Furthermore, if $g \in \text{in}_{\mathbf{w}}(I)$, then $g = \text{in}_{\mathbf{w}}(f)$ for some $f \in I$.*

Proof. Let f be an arbitrary polynomial in I. Since I is a homogeneous ideal, every polynomial in I is generated by some homogeneous polynomials. Suppose $I = \langle h_1, \ldots, h_j \rangle$ where $h_1, \ldots h_j$ are all homogeneous and $f = l_1 h_1 + \ldots l_j h_j$ and let $h_1 = c_{\mathbf{a}_1} x^{\mathbf{a}_1} + c_{\mathbf{a}_2} x^{\mathbf{a}_2} + \ldots$, $l_1 = d_{\mathbf{b}_1} x^{\mathbf{b}_1} + d_{\mathbf{b}_2} x^{\mathbf{b}_2} + \ldots$, then

$$f = c_{\mathbf{a}_1} d_{\mathbf{b}_1} x^{\mathbf{a}_1 + \mathbf{b}_1} + c_{\mathbf{a}_1} d_{\mathbf{b}_2} x^{\mathbf{a}_1 + \mathbf{b}_2} + \ldots + c_{\mathbf{a}_2} d_{\mathbf{b}_1} x^{\mathbf{a}_2 + \mathbf{b}_1} + c_{\mathbf{a}_2} d_{\mathbf{b}_2} x^{\mathbf{a}_2 + \mathbf{b}_2} + \ldots.$$

Rearrange the polynomial we get

$$f = c_{\mathbf{a}_1} d_{\mathbf{b}_1} x^{\mathbf{a}_1 + \mathbf{b}_1} + c_{\mathbf{a}_2} d_{\mathbf{b}_1} x^{\mathbf{a}_2 + \mathbf{b}_1} + \ldots + c_{\mathbf{a}_1} d_{\mathbf{b}_2} x^{\mathbf{a}_1 + \mathbf{b}_2} + c_{\mathbf{a}_2} d_{\mathbf{b}_2} x^{\mathbf{a}_2 + \mathbf{b}_2} + \ldots.$$

Since $h_1, \ldots h_j$ are homogeneous, then $|\mathbf{a}_1 + \mathbf{b}_i| = |\mathbf{a}_2 + \mathbf{b}_i| = \ldots$ for all i. Hence for any polynomial in I, it is a sum of a set of homogeneous polynomials, so we can write $f = \sum_{i \geq 0} f_i \in k[x_1, \ldots, x_n]$ with each f_i homogeneous of degree i. The initial form $\text{in}_{\mathbf{w}}(f)$ is the sum of initial forms $\text{in}_{\mathbf{w}}(f_i)$ of those f_i with $\text{trop}(f)(\mathbf{w}) = \text{trop}(f_i)(\mathbf{w})$. From the progress above, we shall see that each $f_i \in I$, and for any initial form $\text{in}_{\mathbf{w}}(f)$, it can be generated by some initial forms $\text{in}_{\mathbf{w}}(f_i) \in \text{in}_{\mathbf{w}}(I)$. The initial form of a homogeneous polynomial is homogeneous, then $\text{in}_{\mathbf{w}}(I)$ is a homogeneous ideal. Since polynomial ring is Noetherian, then $\text{in}_{\mathbf{w}}(I)$ is generated by a finite number of $\text{in}_{\mathbf{w}}(f)$ where $f \in I$, and the corresponding f form a homogeneous Gröbner basis for I. For the last claim, we let $g = \sum_{\mathbf{u} \in \mathbb{N}^n} a_{\mathbf{u}} x^{\mathbf{u}} \text{in}_{\mathbf{w}}(f_{\mathbf{u}}) \in \text{in}_{\mathbf{w}}(I)$, with $a_{\mathbf{u}} \in \mathbb{K}^*$ and $f_{\mathbf{u}} \in I$ for all $\mathbf{u}$. Then $g = \sum a_{\mathbf{u}} \text{in}_{\mathbf{w}}(x^{\mathbf{u}} f_{\mathbf{u}})$. Now choosing a lift $c_{\mathbf{u}}$ in the valuation ring R for each $a_{\mathbf{u}}$ with $\nu(c_{\mathbf{u}}) = 0$ and $\overline{c_{\mathbf{u}}} = a_{\mathbf{u}}$, and let $W_{\mathbf{u}} = \text{trop}(f_{\mathbf{u}})(\mathbf{w}) + \mathbf{w} \cdot \mathbf{u}$. Then let $f = \sum_{\mathbf{u} \in \mathbb{N}^n} c_{\mathbf{u}} t^{-W_{\mathbf{u}}} x^{\mathbf{u}} f_{\mathbf{u}}$, consider the tropicalization of f we have

$$\text{trop}(f)(\mathbf{w}) = \min_{\mathbf{u} \in \mathbb{N}^n} \{\nu(c_{\mathbf{u}}) - W_{\mathbf{u}} + W_{\mathbf{u}}\}.$$

Since we set $\nu(c_{\mathbf{u}}) = 0$, then $\mathrm{trop}(f)(\mathbf{w}) = 0$ which means in the initial form $\mathrm{in}_{\mathbf{w}}(f)$ every term of f will be remained. Hence the initial form of f is

$$\begin{aligned}\mathrm{in}_{\mathbf{w}}(f) &= \sum_{\mathbf{u}\in\mathbb{N}^n} \overline{c_{\mathbf{u}} t^{-W_{\mathbf{u}}}} x^{\mathbf{u}} \mathrm{in}_{\mathbf{w}}(f_{\mathbf{u}}) \\ &= \sum_{\mathbf{u}\in\mathbb{N}^n} a_{\mathbf{u}} x^{\mathbf{u}} \mathrm{in}_{\mathbf{w}}(f_{\mathbf{u}}) \\ &= g\end{aligned}$$

□

Proposition 2.5.7. *[1, Proposition 2.6.1] Let I be an ideal in $k[x_1^{\pm}, \ldots, x_n^{\pm}]$ with a valuation ν on k, and fix $\mathbf{w} \in \mathbb{R}^n$. Then $\mathrm{in}_{\mathbf{w}}(I)$ is the image of $\mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}})$ in $\mathbb{K}[x_1^{\pm}, \ldots, x_n^{\pm}]$ where $x_0 = 1$ and $\mathbb{K}$ is the residue field of the valuation ν. Every element of $\mathrm{in}_{\mathbf{w}}(I)$ has the form $x^{\mathbf{u}}g$, where $x^{\mathbf{u}}$ is a Laurent monomial and $g = f(1, x_1, \ldots, x_n)$ for some $f \in \mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}})$.*

Proof. Suppose $f = \sum_{\mathbf{u}\in\mathbb{Z}^n} c_{\mathbf{u}} x^{\mathbf{u}} \in I \cap k[x_1, \ldots, x_n]$ and let $j_{\mathbf{u}} = (\max_{c_{\mathbf{v}}} |\mathbf{v}|) - |\mathbf{u}|$ such that $\tilde{f} = \sum_{\mathbf{u}\in\mathbb{N}^n} c_{\mathbf{u}} x^{\mathbf{u}} x_0^{j_{\mathbf{u}}}$ is the homogenization of f. Then we shall have the following equation directly.

$$\begin{aligned}W := \mathrm{trop}(f)(\mathbf{w}) =& \min\{\nu(c_{\mathbf{u}}) + \mathbf{w}\cdot\mathbf{u}\} \\ =& \min\{\nu(c_{\mathbf{u}}) + (0,\mathbf{w})\cdot(j_{\mathbf{u}},\mathbf{u})\} = \mathrm{trop}(\tilde{f})((0,\mathbf{w})).\end{aligned}$$

Then consider the initial form $\mathrm{in}_{(0,\mathbf{w})}(\tilde{f})$,

$$\mathrm{in}_{(0,\mathbf{w})}(\tilde{f}) = \sum_{\substack{\mathbf{u}\in\mathbb{Z}^n \\ \min\{\nu(c_{\mathbf{u}})+\mathbf{w}\cdot\mathbf{u}\}=W}} \overline{c_{\mathbf{u}} t^{-\nu(c_{\mathbf{u}})}} x^{\mathbf{u}} x_0^{j_{\mathbf{u}}}.$$

If we restrict x_0 at 1, then this initial form is equal to $\mathrm{in}_{\mathbf{w}}(f)$ such that

$$\mathrm{in}_{(0,\mathbf{w})}(\tilde{f})|_{x_0=1} = \sum_{\substack{\mathbf{u}\in\mathbb{Z}^n \\ \min\{\nu(c_{\mathbf{u}})+\mathbf{w}\cdot\mathbf{u}\}=W}} \overline{c_{\mathbf{u}} t^{-\nu(c_{\mathbf{u}})}} x^{\mathbf{u}} = \mathrm{in}_{\mathbf{w}}(f).$$

By multiplying some monomials, we can choose some polynomials $f_1, \ldots, f_s \in k[x_1, \ldots, x_n] \cap I$ such that $\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f_1), \ldots, \mathrm{in}_{\mathbf{w}}(f_s)\rangle$. Since we have shown that $\mathrm{in}_{(0,\mathbf{w})}(\tilde{f})|_{x_0=1} = \mathrm{in}_{\mathbf{w}}(f)$ for any $f \in k[x_1, \ldots, x_n] \cap I$, so $\mathrm{in}_{\mathbf{w}}(I) \subseteq \mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}})|_{x_0=1}$. For proving the reverse inclusion, let g be a homogeneous polynomial in I_{proj}, then we have $g = x_0^j \cdot \tilde{f}$ for some $j \in \mathbb{Z}$ and since $\tilde{f} = \sum c_{\mathbf{u}} x^{\mathbf{u}} x_0^{j_{\mathbf{u}}}$ where $j_{\mathbf{u}} = (\max_{c_{\mathbf{v}}\neq 0} |\mathbf{v}|) - |\mathbf{u}|$. Then

$f(x) = g(1, x)$, and by lemma 2.5.6 we can find a homogeneous Gröbner basis for I_{proj}. Since $\text{in}_{(0,\mathbf{w})}(\tilde{f})|_{x_0=1} = \text{in}_{\mathbf{w}}(f)$, then we have the reverse inclusion. □

Let $f \in k[x_1^{\pm}, \ldots, x_n^{\pm}]$ be a polynomial and $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{Z}^n$. If the total degree of each monomial in f is $|\mathbf{u}|$, we say that f is homogeneous with respect to the grading given by $\deg(x_i) = u_i$.

Lemma 2.5.8. *[1, Lemma 2.6.2] Let ν be a valuation on field k and I be an ideal in $k[x_1, \ldots, x_n]$. Fix $\mathbf{w} \in \mathbb{R}^n$. Then*

1. *If $\text{in}_{\mathbf{u}}^{\nu}(\text{in}_{\mathbf{w}}^{\nu}(I)) = \text{in}_{\mathbf{w}}^{\nu}(I)$ for some $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$, then $\text{in}_{\mathbf{w}}^{\nu}(I)$ is homogeneous with respect to the grading given by $\deg(x_i) = u_i$.*
2. *If $f, g \in k[x_1, \ldots, x_n]$, then $\text{in}_{\mathbf{w}}^{\nu}(fg) = \text{in}_{\mathbf{w}}^{\nu}(f)\text{in}_{\mathbf{w}}^{\nu}(g)$.*

Proof. For part 1, Suppose $\text{in}_{\mathbf{u}}^{\nu}(\text{in}_{\mathbf{w}}^{\nu}(I)) = \text{in}_{\mathbf{w}}^{\nu}(I)$. Then we shall suppose there exists $g_i \in \text{in}_{\mathbf{w}}^{\nu}(I)$ such that $\text{in}_{\mathbf{u}}^{\nu}(g_i)$ generate $\text{in}_{\mathbf{w}}^{\nu}(I)$. For any g_i, let it be

$$g_i = \sum_{\mathbf{s} \in \mathbb{N}^n} a_{\mathbf{s}} x^{\mathbf{s}} \in \mathbb{K}[x_1, \ldots, x_n],$$

then the initial form of g_i with respect to $\mathbf{u}$ is $\text{in}_{\mathbf{u}}^{\nu}(g_i) = \sum_{\mathbf{u}\cdot\mathbf{s}=\mathbf{W}} a_{\mathbf{s}} x^{\mathbf{s}}$, and $a_{\mathbf{s}} \in \mathbb{K}$. Since the monomials in $\text{in}_{\mathbf{u}}^{\nu}(g_i) = \sum_{\mathbf{u}\cdot\mathbf{s}=\mathbf{W}} a_{\mathbf{s}} x^{\mathbf{s}}$ are chosen from $\mathbf{W} = \min_{a_{\mathbf{s}} \neq 0}\{\mathbf{u} \cdot \mathbf{s}\}$ and $\mathbf{u}$ is a fixed weight vector, then the multi-degree of each monomial is the same. Hence $\text{in}_{\mathbf{w}}^{\nu}(I)$ is homogeneous.

For part 2, first we suppose $f = \sum_{\mathbf{u} \in \mathbb{N}^n} a_{\mathbf{u}} x^{\mathbf{u}}$ and $g = \sum_{\mathbf{u}' \in \mathbb{N}^n} b_{\mathbf{u}'} x^{\mathbf{u}'}$. Then let $fg = \sum_{\mathbf{s} \in \mathbb{N}^n} c_{\mathbf{s}} x^{\mathbf{s}}$ for $c_{\mathbf{s}} = \sum_{\mathbf{u}+\mathbf{u}'=\mathbf{s}} a_{\mathbf{u}} b_{\mathbf{u}'}$. Now let $\mathbf{W}_1 = \text{trop}(f)(\mathbf{w})$ and $\mathbf{W}_2 = \text{trop}(g)(\mathbf{w})$. Recall definition 2.5.1, it implies $\text{trop}(fg)(\mathbf{w}) = \mathbf{W}_1 + \mathbf{W}_2$. Then consider the initial form $\text{in}_{\mathbf{w}}^{\nu}(fg)$ we have

$$\begin{aligned} \text{in}_{\mathbf{w}}^{\nu}(fg) &= \sum_{\mathbf{W}_1+\mathbf{W}_2=\nu(c_{\mathbf{s}})+\mathbf{w}\cdot\mathbf{s}} \overline{c_{\mathbf{s}} t^{-\nu(c_{\mathbf{s}})}} x^{\mathbf{s}} \\ &= \sum_{\mathbf{W}_1+\mathbf{W}_2=\nu(c_{\mathbf{s}})+\mathbf{w}\cdot\mathbf{s}} \sum_{\mathbf{u}+\mathbf{u}'=\mathbf{s}} \overline{a_{\mathbf{u}} b_{\mathbf{u}'} t^{-\mathbf{W}_1-\mathbf{W}_2+\mathbf{w}\cdot(\mathbf{u}+\mathbf{u}')}} x^{\mathbf{s}}. \end{aligned}$$

This is just the product of $\text{in}_{\mathbf{w}}^{\nu}(f)$ and $\text{in}_{\mathbf{w}}^{\nu}(g)$ such that

$$\begin{aligned} &\left(\sum_{\nu(a_{\mathbf{u}})+\mathbf{w}\cdot\mathbf{u}=\mathbf{W}_1} \overline{a_{\mathbf{u}} t^{-\nu(a_{\mathbf{u}})}} x^{\mathbf{u}} \right) \left(\sum_{\nu(b_{\mathbf{u}'})+\mathbf{w}\cdot\mathbf{u}'=\mathbf{W}_2} \overline{b_{\mathbf{u}'} t^{-\nu(b_{\mathbf{u}'})}} x^{\mathbf{u}'} \right) \\ = &\sum_{\nu(a_{\mathbf{u}})+\mathbf{w}\cdot\mathbf{u}=\mathbf{W}_1} \sum_{\nu(b_{\mathbf{u}'})+\mathbf{w}\cdot\mathbf{u}'=\mathbf{W}_2} \overline{a_{\mathbf{u}} b_{\mathbf{u}'} t^{-\nu(a_{\mathbf{u}})-\nu(b_{\mathbf{u}'})}} x^{\mathbf{u}+\mathbf{u}'}, \end{aligned}$$

where $-\mathbf{W}_1 - \mathbf{W}_2 + \mathbf{w} \cdot (\mathbf{u} + \mathbf{u}') = -\nu(a_{\mathbf{u}}) - \nu(b_{\mathbf{u}'})$ and $\mathbf{u} + \mathbf{u}' = \mathbf{s}$. Then we have the equation $\mathrm{in}^{\nu}_{\mathbf{w}}(fg) = \mathrm{in}^{\nu}_{\mathbf{w}}(f)\mathrm{in}^{\nu}_{\mathbf{w}}(g)$. □

2.6 Tropical hypersurfaces

In algebraic geometry, a hypersurface of a polynomial $f \in k[x_1^{\pm}, \dots, x_n^{\pm}]$ is a set

$$V(f) = \{\mathbf{y} \in k^n \ : \ f(\mathbf{y}) = 0\}.$$

And the n-dimensional algebraic torus T_k^n over a field k is

$$T_k^n = \{(a_1, a_2, \dots, a_n) \ : \ a_i \in k^*\}$$

In this section, we will discuss an important lemma which will be involved in the proof of Kapranov's theorem in the next section. This lemma shows a specific subset in an algebraic torus T_k^n is Zariski dense. So we need to know what Zariski topology and Zariski dense set is. In algebraic geometry, Zariski topology is a topology which is defined by its closed sets. For instance, let $S \subset k[x_1, \dots, x_n]$, the closed set $V(S)$ in k^n is

$$V(S) = \{x \in k^n \ : \ f(x) = 0, \ \forall f \in S\}.$$

Then we shall introduce the Zariski closure.

Definition 2.6.1. The Zariski closure of a subset of the affine space k^n is the smallest affine algebraic variety containing the subset.

And then we have the definition of a Zariski dense set

Definition 2.6.2. let $V(S)$ be a subset of an affine space k^n. $V(S)$ is said to be Zariski dense if the smallest variety containing $V(S)$ is k^n.

Lemma 2.6.3. *[1, Lemma 2.2.12] Let k be a valued field with a splitting $\Gamma_v \longrightarrow k^*$ given by $w \longrightarrow t^w$, so that $v(t^w) = w$. Let $\alpha_1, \dots, \alpha_n \in \mathbb{K}^*$ where $\mathbb{K}$ is the residue field of v and $w_1, \dots, w_n \in \Gamma_v$. Consider the set of all $\mathbf{y} = (y_1, \dots, y_n)$ in T^n that satisfy $v(y_i) = w_i$ and $\overline{t^{-w_i} y_i} = \alpha_i$ for $i = 1, \dots, n$. Then this set is Zariski dense in T^n.*

Proof. We shall start from the case that $n = 1$. First of all, fixing an element z in the valuation ring R of ν and the image of z in $\mathbb{K}^*$ is α. Then $y = t^w z$, then $\nu(y) = w$, since $\alpha \in \mathbb{K}^*$ which implies that $\nu(z) = 0$. In fact, we have infinite number of elememts in k^* that satisfy the desired form. For instance, $y + t^{w+j}$ for all $j > 0$, then

$$\nu(y + t^{w+j}) = \min\{\nu(y), w + j\}.$$

Since $\nu(y) = w$ then $\nu(y + t^{w+j})$, and

$$\overline{t^{-w}(y + t^{w+j})} = \overline{t^{-w}y + t^j}.$$

Since $j > 0$ then $\nu(t^j) > 0$ and $t^j \notin \mathbb{K}^*$. So $\overline{t^{-w}(y + t^{w+j})} = \alpha$. Then for any non-zero polynomial $h \in k[x_1^{\pm 1}]$ we can choose a y_1 from the infinite number of elements which we just defined with $\nu(y_1) = w_1$ and $\overline{t^{-w_1}y_1} = \alpha_1$ to make $h(y_1) \neq 0$ when $n = 1$. When $n = 2$, let $h = h_1 \cdot x_2^j$ where $h_1 \in k[x_1^{\pm 1}]$. Repeating the process above we can choose a $\mathbf{y} = (y_1, y_2)$ which we defined with $\nu(y_i) = w_i$ and $\overline{t^{-w_i}y_i} = \alpha_i$ for $i = 1, 2$. Then for proving this induction, we can hypothesis this holds when $n = k$. Now suppose $n = k + 1$, let $h = \sum h_j \cdot x_{k+1}^j$ where $h_j \in k[x_1^{\pm}, \ldots, x_n^{\pm}]$. Obviously, we can choose $\mathbf{y}' = (y_1, \ldots, y_n) \in (\mathbb{K}^*)^n$ coordinate by coordinate with $\nu(y_i) = w_i$ and $\overline{t^{-w_i}y_i} = \alpha_i$ with $h_j(\mathbf{y}') \neq 0$ for all j. We then shall choose y_n with $\nu(y_n) = w_n$ and $\overline{t^{-w_n}y_n} = \alpha_n$ to make $h(y_1, \ldots, y_{n-1}, y_n) \neq 0$. Then there is not any non-zero polynomial in $k[x_1^{\pm}, \ldots, x_n^{\pm}]$ vanishes at those points $\mathbf{y}$. Therefore the only variety containing the set consists of $\mathbf{y}$ is T^n itself and it is the smallest variety containing those points $\mathbf{y}$, which implies the set of $\mathbf{y}$ is Zariski dense in T^n. □

Now we define the tropical hypersurface associated with f.

Definition 2.6.4. Let $f \in k[x_1^{\pm}, \ldots, x_n^{\pm}]$. Then the tropical hypersurface $\mathcal{V}(\mathrm{trop}(f))$ is the set

$$\{\mathbf{w} \in \mathbb{R}^n \ : \ \text{the minimum in } \mathrm{trop}(f)(\mathbf{w}) \text{ is achieved at least twice}\}$$

By definition 2.5.1 and 2.6.4, we shall conclude that the tropical hypersurface of $f \in k[x_1^{\pm}, \ldots, x_n^{\pm}]$ is a set of weight vectors $\mathbf{w} \in \mathbb{R}^n$ which let the initial form $\mathrm{in}_{\mathbf{w}}^{v}(f)$ have more than one monomial.

Definition 2.6.5. Given a morphism $\phi : T^n \longrightarrow T^m$ with associated ring homomorphism $\phi^* : k[x_1^{\pm}, \ldots, x_m^{\pm}] \longrightarrow k[z_1^{\pm}, \ldots, z_n^{\pm}]$, we can denote by ϕ^* the map $\mathbb{Z}^m \to \mathbb{Z}^n$ given by $\phi^*(\mathbf{e}_i) = \mathbf{u}$ where $\mathbf{e}_1, \ldots, \mathbf{e}_m$ is the standard basis of $\mathbb{Z}^m$, and $\phi^*(x_i) = z^{\mathbf{u}}$.

This gives an induced map which is called the tropicalization of ϕ

$$\operatorname{trop}(\phi) : \operatorname{Hom}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n \longrightarrow \operatorname{Hom}(\mathbb{Z}^m, \mathbb{Z}) \cong \mathbb{Z}^m.$$

For an instance, let $\phi^*(x_i) = x^{\mathbf{a}_i}$ for $\mathbf{a}_i \in \mathbb{Z}^n$ and A be the $n \times m$ matrix which each i-th column is $\mathbf{a}_i$. Then the tropicalization of ϕ is A^T. For any $\mathbf{y} = (y_1, \ldots, y_n) \in T^n$

$$\begin{aligned}
\operatorname{trop}(\phi(\mathbf{y})) = \nu(\phi(\mathbf{y})) =& (\nu(\mathbf{y}^{\mathbf{a}_1}), \ldots, \nu(\mathbf{y}^{\mathbf{a}_m})) \\
=& (\mathbf{a}_1(\nu(y_1), \ldots, \nu(y_n)), \ldots \mathbf{a}_m(\nu(y_1), \ldots, \nu(y_n))) \\
=& \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} \begin{pmatrix} \nu(y_1) \\ \vdots \\ \nu(y_n) \end{pmatrix} \\
=& A^T \begin{pmatrix} \nu(y_1) \\ \vdots \\ \nu(y_n) \end{pmatrix}
\end{aligned}$$

Lemma 2.6.6. *[1, Lemma 2.6.10] Let $\phi^* : k[x_1^{\pm}, \ldots, x_m^{\pm}] \longrightarrow k[z_1^{\pm}, \ldots, z_n^{\pm}]$ be a monomial map. Let $I \subseteq k[z_1^{\pm}, \ldots, z_n^{\pm}]$ be an ideal, and let $I' = {\phi^*}^{-1}(I)$. Then*

$$\phi^*(\operatorname{in}_{\operatorname{trop}(\phi)(\mathbf{w})}(I')) \subseteq \operatorname{in}_{\mathbf{w}}(I) \quad \text{for all } \mathbf{w} \in \mathbb{R}^n.$$

Thus, in particular, if $\operatorname{in}_{\mathbf{w}}(I) \neq \langle 1 \rangle$*, then we also have* $\operatorname{in}_{\operatorname{trop}(\phi)(\mathbf{w})}(I') \neq \langle 1 \rangle$*.*

Proof. Let $\phi^*(x_i) = z^{\mathbf{a}_i}$, where $\mathbf{a}_i \in \mathbb{Z}^n$. Then $\phi^*(x^{\mathbf{u}}) = \phi^*(x_1^{u_1}) \ldots \phi^*(x_m^{u_m}) = z^{\mathbf{a}_1 u_1} \ldots z^{\mathbf{a}_m u_m}$. Then the power of z is the product of matrices $(\mathbf{a}_1, \ldots, \mathbf{a}_m) \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$. Let $A = (\mathbf{a}_1, \ldots, \mathbf{a}_m)$ and $f = \sum c_{\mathbf{u}} x^{\mathbf{u}} \in I'$, then we have $\phi^*(f) = \sum c_{\mathbf{u}} z^{A\mathbf{u}} \in I$. Therefore fixing a weight vector $\mathbf{w} = (w_1, \ldots, w_n) \in \Gamma_\nu^n$, we have $\operatorname{trop}(\phi^*(f))(\mathbf{w}) = \min_{c_{\mathbf{u}} \neq 0}\{\nu(c_{\mathbf{u}}) + \mathbf{w} \cdot A\mathbf{u}\}$ and $\mathbf{w} \cdot A = (w_1, \ldots, w_n) \cdot (\mathbf{a}_1, \ldots, \mathbf{a}_m)$ which gets the same vector as $A^T \cdot \mathbf{w} = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$. So it is equal to $\operatorname{trop}(f)(A^T\mathbf{w})$ then we let $W = \text{trop}(f)(A^T\mathbf{w})$ and

$$\begin{aligned}\text{in}_{\mathbf{w}}(\phi^*(f)) =&\phi^*\left(\sum_{\nu(c_{\mathbf{u}})+\mathbf{w}\cdot A\mathbf{u}=W} \overline{c_{\mathbf{u}}\sigma(\nu(c_{\mathbf{u}}))^{-1}}\cdot x^{\mathbf{u}}\right)\\ =&\sum_{\nu(c_{\mathbf{u}})+\mathbf{w}\cdot A\mathbf{u}=W} \overline{c_{\mathbf{u}}\sigma(\nu(c_{\mathbf{u}}))^{-1}}\cdot z^{A\mathbf{u}}.\end{aligned}$$

In the equation above, the polynomial $\sum_{\nu(c_{\mathbf{u}})+\mathbf{w}\cdot A\mathbf{u}=W} \overline{c_{\mathbf{u}}\sigma(\nu(c_{\mathbf{u}}))^{-1}}\cdot x^{\mathbf{u}}$ can be seen as an initial form of f with respect to weight vector $\mathbf{w}\cdot A \in \Gamma_\nu^m$. Since $\text{trop}(\phi) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ then $\sum_{\nu(c_{\mathbf{u}})+\mathbf{w}\cdot A\mathbf{u}=W} \overline{c_{\mathbf{u}}\sigma(\nu(c_{\mathbf{u}}))^{-1}}\cdot x^{\mathbf{u}} = \text{in}_{\text{trop}(\phi)(\mathbf{w})}(f)$. Hence

$$\phi^*(\text{in}_{\text{trop}(\phi)(\mathbf{w})}(f)) = \text{in}_{\mathbf{w}}(\phi^*(f)) \in \text{in}_{\mathbf{w}}(I)$$

which implies that $\phi^*(\text{in}_{\text{trop}(\phi)(\mathbf{w})}(I')) \subseteq \text{in}_{\mathbf{w}}(I)$. And it is obvious that if $1 \in \text{in}_{\text{trop}(\phi)(\mathbf{w})}(I')$ then $1 \in \phi^*(\text{in}_{\text{trop}(\phi)(\mathbf{w})}(I'))$ and $1 \in \text{in}_{\mathbf{w}}(I)$ which proves the contrapositive at the end of the lemma. □

The following corollary is the direct result from lemma 2.6.6 which will be involved in the next section.

Corollary 2.6.7. *[1, Corollary 2.6.12] Let ϕ^* be a monomial automorphism of $k[x_1^{\pm}, \dots, x_n^{\pm}]$, let I be any ideal in this Laurent polynomial ring, and let $I' = \phi^{*-1}(I)$. Then*

$$\text{in}_{\mathbf{w}}(I) = \langle 1\rangle \quad \text{if and only if} \quad \text{in}_{\text{trop}(\phi)(\mathbf{w})}(I') = \langle 1\rangle.$$

2.7 The Fundamental Theorem

Before Fundamental theorem, we shall introduce Kapranov's Theorem first. Kapranov's theorem was first stated in an unpublished manuscript by Russian mathematician Mikhail Kapranov in the early 1990's. It builds a link between the classical hypersurfaces over a field k and the tropical hypersurfaces in $\mathbb{R}^n$.

Theorem 2.7.1. *(Kapranov's theorem)[1, Theorem 3.1.3]*
Let k be an algebraically closed field with a non-trivial valuation v. Fix a Laurent polynomial $f = \sum_{\mathbf{u}\in\mathbb{N}^n} c_{\mathbf{u}} x^{\mathbf{u}} \in k[x_1, \dots, x_n]$. The following three sets coincide:

1. *the tropical hypersurface $\mathcal{V}(\text{trop}(f))$ in $\mathbb{R}^n$;*

2. *the set $\{\mathbf{w} \in \mathbb{R}^n : \text{in}_{\mathbf{w}}^{\nu}(f)$ is not a monomial$\}$;*

3. *the closure in* $\mathbb{R}^n$ *of* $\{(\nu(y_1),\ldots,\nu(y_n)) \ : \ (y_1,\ldots,y_n) \in V(f)\}$

Moreover, if f *is irreducible and* $\mathbf{w}$ *is ant point in* $\Gamma_\nu^n \cap \mathcal{V}(\mathrm{trop}(f))$*, then the set* $\{\mathbf{y} \in V(f) \ : \ \nu(\mathbf{y}) = \mathbf{w}\}$ *is Zariski dense in the hypersurface* $V(f)$*.*

Proof. First suppose $\mathbf{w} = (w_1,\ldots,w_n) \in \mathcal{V}(\mathrm{trop}(f))$. By definition 2.6.4, $\mathrm{trop}(f)(\mathbf{w})$ has at least two monomials and recall definition 2.5.1 $\mathbf{w}$ is contained by set 2 clearly. Then set 2 contains set 1. Similarly, recalling definition 2.5.1 and 2.6.4, we will prove the converse direction easily. Hence this is the equality of set 1 and set 2.

Now we need to show the equality of set 1 and set 3. Since $(y_1,\ldots,y_n) \in V(f)$, we have $f(\mathbf{y}) = \sum\limits_{\mathbf{u}\in\mathbb{N}^n} c_\mathbf{u}\mathbf{y}^\mathbf{u} = 0$. Then taking the valuation we will have

$$v\Big(\sum_{\mathbf{u}\in\mathbb{N}^n} c_\mathbf{u}\mathbf{y}^\mathbf{u}\Big) = v(0) = \infty > v(c_{\mathbf{u}'}\mathbf{y}^{\mathbf{u}'})$$

for all $\mathbf{u}'$ with $c_{\mathbf{u}'} \neq 0$. By lemma 2.2.2, $v\big(\sum\limits_{\mathbf{u}\in\mathbb{N}^n} c_\mathbf{u}\mathbf{y}^\mathbf{u}\big) > \min_{\mathbf{u}\in\mathbb{N}^n}\{v(c_\mathbf{u}) + \mathbf{y}\cdot\mathbf{u}\}$ implies there are more than one minimum. Therefore $v(\mathbf{y}) \in \mathrm{trop}(V(f))$. Since $\mathrm{trop}(f)(\mathbf{w})$ is a tropical hypersurface associated to f, then it is closed. Hence set 1 contains set 3. □

The proof above isn't complete, we haven't proven the converse direction of the last part. For proving that set 3 contains set 1, we need to show the subset $\{\mathbf{y} \in V(f) \ : \ v(\mathbf{y}) = \mathbf{w}\}$ is Zariski dense in the hypersurface $V(f)$. The following proposition will show that every zero of an initial form can be lifted to a zero of the given polynomial.

Proposition 2.7.2. *[1, Proposition 3.1.5] Fix* $f \in k[x_1^{\pm},\ldots,x_n^{\pm}]$*, and let* $\mathbf{w} = (w_1,\ldots,w_n) \in \Gamma_v^n$ *where* v *is the valuation on* k *and the residue field* v *is* $\mathbb{K}$*. Suppose* $\mathrm{in}_\mathbf{w}^v(f)$ *is not a monomial and* $\alpha = (\alpha_1,\ldots,\alpha_n) \in (\mathbb{K}^*)^n$ *satisfies* $\mathrm{in}_\mathbf{w}^v(f)(\alpha) = 0$*. There exists* $\mathbf{y} = (y_1,\ldots,y_n) \in (k^*)^n$ *satisfying* $f(\mathbf{y}) = 0$*,* $v(\mathbf{y}) = \mathbf{w}$*, and* $\overline{t^{-w_i}y_i} = \alpha_i$ *for* $1 \leq i \leq n$*. If* f *is irreducible, then the set of such* $\mathbf{y}$ *is Zariski dense in the hypersurface* $V(f)$*.*

Proof. First, we shall let $n = 1$. Let $f = \sum\limits_{i=0}^{s} c_i x^i$ where $c_0, c_s \neq 0$, then we may assume that $f = \prod\limits_{j=1}^{s}(a_j x - b_j)$. By the part 3 of lemma 2.5.8, we have $\mathrm{in}_\mathbf{w}^v(f) = \prod\limits_{j=1}^{s} \mathrm{in}_\mathbf{w}^v(a_j x - b_j)$. Let $\mathrm{in}_\mathbf{w}^v(f)$ has more than one monomial and $\alpha \in \mathbb{K}^*$ where $\mathrm{in}_\mathbf{w}^v(f)(\alpha) = 0$, then for some j, we have $\mathrm{in}_\mathbf{w}^v(a_j x - b_j)(\alpha) = 0$. Since $\alpha \neq 0$ then $\mathrm{in}_\mathbf{w}^v(a_j x - b_j)$ is not a monomial either. Then $v(a_j) + \mathbf{w} = v(b_j)$ which implies that $\alpha = \overline{t^{-\mathbf{w}} b_j / a_j}$. Therefore let $\mathbf{y} = b_j/a_j \in k^*$, we have $f(\mathbf{y}) = 0$ and we shall find $\mathbf{w} = v(\mathbf{y}) = v(b_j) - v(a_j) \in \Gamma_v$. Then $\alpha = \overline{t^{-\mathbf{w}}\mathbf{y}}$.

Now assume that the proposition holds for all dimensions less than n. We first reduce this case such that no two monomials in f have the same power of x_n. Then we shall consider f as a polynomial in x_n with coefficients in $k[x_1, \dots, x_{n-1}]$. Now consider the automorphism ϕ_l^* on $k[x_1, \dots, x_{n-1}]$ given by

$$\phi_l^*(x_j) = x_j x_n^{l^j} \text{ for } 1 \leq j \leq n-1$$

and $\phi_l^*(x_n) = x_n$, where $l \in \mathbb{N}$. Then for any monomial $x^{\mathbf{u}} x_n^i$, we have

$$\phi_l^*(x^{\mathbf{u}} x_n^i) = x^{\mathbf{u}} x_n^{i + \sum_{j=1}^{n-1} u_j l^j}$$

where $\mathbf{u} = (u_1, \dots, u_{n-1}) \in \mathbb{Z}^{n-1}$. Then follow this construction, each monomial in $\phi_l^*(f)$ has a different power of x_n, for $l \gg 0$. Now suppose $\mathbf{y} = (y_1, \dots, y_n) \in T^n$ which satisfies $\phi_l^*(f)(\mathbf{y}) = 0$, $v(y_i) = w_i - l^i w_n$ and $\overline{t^{-w_i + l^i w_n} y_i} = \alpha_i \alpha_n^{-l^i}$ for $1 \leq i \leq n-1$. Also it satisfies $v(y_n) = w_n$ and $\overline{t^{-w_n} y_n} = \alpha_n$. Base on the assumption above, define $\mathbf{y}' \in T^n$ by $y_i{}' = y_i y_n^{l^i}$ for $1 \leq i \leq n-1$ and $y_n{}' = y_n$. Then

$$0 = \phi_l^*(f)(\mathbf{y}) = \sum_{\substack{\mathbf{u} \in \mathbb{Z}^{n-1} \\ i \in \mathbb{Z}}} c_{\mathbf{u}} \tilde{\mathbf{y}}^{\mathbf{u}} y_n^{i + \sum_{j=1}^{n-1} u_j l^j}$$

where $\tilde{\mathbf{y}} = (y_1, \dots, y_{n-1})$. For each monomial, $\tilde{\mathbf{y}}^{\mathbf{u}} = (y_1^{u_1}, \dots, y_{n-1}^{u_{n-1}})$ multiplying with $y_n^{\sum_{j=1}^{n-1} u_j l^j}$ we have $\tilde{\mathbf{y}}' = \tilde{\mathbf{y}}^{\mathbf{u}} \cdot y_n^{\sum_{j=1}^{n-1} u_j l^j}$ since $y_i{}' = y_i y_n^{l^i}$ for $1 \leq i \leq n-1$, and $y_n{}' = y_n$ implies that $\mathbf{y}' = \tilde{\mathbf{y}}^{\mathbf{u}} y_n^{i + \sum_{j=1}^{n-1} u_j l^j}$. Therefore $\phi_l^*(f)(\mathbf{y}) = f(\mathbf{y}') = 0$. Secondly, $v(y_i{}') = v(y_i) + v(y_n^{l^j}) = w_i - l^j w_n + l^j w_n = w_i$ for $1 \leq i \leq n-1$, so $v(\mathbf{y}') = \mathbf{w}$. Finally, $\overline{t^{-w_i + l^i w_n} y_i} = \alpha_i \alpha_n^{-l^i} = \alpha_i \cdot \overline{t^{-w_n} y_n}^{-l^j} \implies \overline{t^{-w_i + l^i w_n} y_i} \cdot \overline{t^{-w_n} y_n}^{l^j} = \alpha_i$, then we have $\overline{t^{-w_i} y_i{}'} = \alpha_i$.

Now consider the set consists of all $\tilde{y} = (y_1, \dots, y_n) \in T^{n-1}$ with $v(y_i) = w_i$ and $\overline{t^{-w_i} y_i} = \alpha_i$ for $1 \leq i \leq n-1$. By lemma 2.6.3, $\tilde{y}$ is Zariski dense in T^{n-1}. Moreover, by the induction we follow $g(x_n) = f(y_1, \dots, y_{n-1}, x_n)$ is not the zero polynomial.

Let $\mathbf{u}' \in \mathbb{Z}^{n-1}$ be the projection of $\mathbf{u} \in \mathbb{Z}^n$. Let $g = \sum d_i x_n^i$ and $d_i = c_{\mathbf{u}} \mathbf{y}^{\mathbf{u}'}$ for a unique $\mathbf{u} \in \mathbb{Z}^n$ that $u_n = i$. Note that $\operatorname{trop}(g)(w_n) = v(d_i) + w_n i = v(c_{\mathbf{u}}) + v(\mathbf{y}^{\mathbf{u}'}) + w_n i$, since we have $v(y_i) = w_i$ then $v(c_{\mathbf{u}}) + v(\mathbf{y}^{\mathbf{u}'}) + w_n i = v(c_{\mathbf{u}}) + \mathbf{w}' \cdot \mathbf{u}' + w_n u_n = v(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u}$. Hence $\operatorname{trop}(g)(w_n) = \operatorname{trop}(f)(\mathbf{w})$. Then consider the initial form of g

with respect to w_n

$$\begin{aligned} \mathrm{in}_{w_n}(g) &= \sum_{i:v(d_i)+w_n i=\mathrm{trop}(g)(w_n)} \overline{t^{-v(d_i)} d_i} x_n^i \\ &= \sum_{\mathbf{u}:v(c_{\mathbf{u}}\mathbf{y}^{\mathbf{u}'})+w_n u_n=\mathrm{trop}(g)(w_n)} \overline{t^{-v(c_{\mathbf{u}})} c_{\mathbf{u}} t^{-\mathbf{u}'\cdot\mathbf{w}'} \mathbf{y}^{\mathbf{u}'}} x_n^{u_n}. \end{aligned}$$

Recall that we let $\overline{t^{-w_i}y_i} = \alpha_i$ for all $1 \leq i \leq n-1$, let $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ we will gain $\overline{t^{-\mathbf{u}'\cdot\mathbf{w}'}\mathbf{y}^{\mathbf{u}'}} = \alpha^{\mathbf{u}'}$. Therefore

$$\begin{aligned} &\sum_{\mathbf{u}:v(c_{\mathbf{u}})+\mathbf{w}\cdot\mathbf{u}=\mathrm{trop}(f)(\mathbf{w})} \overline{t^{-v(c_{\mathbf{u}})} c_{\mathbf{u}}} \cdot \alpha^{\mathbf{u}'} x_n^{u_n} \\ =&\mathrm{in}_{\mathbf{w}}(f)(\alpha_1, \ldots, \alpha_{n-1}, x_n). \end{aligned}$$

By the case $n = 1$, for the polynomial $g \in k[x_1^{\pm}, \ldots, x_n^{\pm}]$, there exists an element in k^*, here we shall let it be y_n with $g(y_n) = 0$, $\nu(y_n) = w_n$ and $\mathrm{in}_{w_n}(g)(\alpha_n) = 0$ for which $\overline{t^{-w_n}y_n} = \alpha_n$. Thus $f(y_1, \ldots, y_{n-1}, y_n) = 0$ and this is the point $\mathbf{y} = (y_1, \ldots, y_{n-1}, y_n)$ in the hypersurface $V(f)$.

Finally, we need to show that if f is irreducible, then the set Y that consists of those points $\mathbf{y}$ is Zariski dense in $V(f)$ which means that the smallest variety containing Y is $V(f)$ itself. By lemma 2.6.3, for any $(y_1, \ldots, y_{n-1}) \in T^n$, with $\nu(y_i) = w_i$ and $\overline{t^{-w_i}y_i} = \alpha_i$ for all $1 \leq i \leq n-1$, so the set of such point $(y_1, \ldots, y_{n-1})$ is Zariski dense in T^{n-1}. Now constructing a set Y containing these points $\mathbf{y} = (y_1, \ldots, y_{n-1}, y_n)$, then the projection of Y onto the first $n-1$ coordinates is not in any hypersurface in T^{n-1}. Let g_i be those polynomials in $k[x_1^{\pm}, \ldots, x_n^{\pm}]$ and $g_i(\mathbf{y}) = 0$ for all $\mathbf{y} \in Y$ for all $i \in \mathbb{N}$. Since the set of $(y_1, \ldots, y_{n-1})$ is Zariski dense in T^{n-1}, then $\langle f, g_i \rangle \cap k[x_1^{\pm}, \ldots, x_{n-1}^{\pm}] = \{0\}$. And since f is irreducible, then for all $i \in \mathbb{N}$, g_i is a multiple of f. Therefore $V(f) = \overline{Y}$. □

Hence proposition 2.7.2 shows that the closure $\{(\nu(y_1), \ldots, \nu(y_n)) : (y_1, \ldots, y_n) \in V(f)\}$ contains $\mathcal{V}(\mathrm{trop}(f))$, which complete the proof of theorem 2.7.1

Here we have a simple example to illustrate Kapranov's theorem. This example is modified from example 3.1.4 in [1].

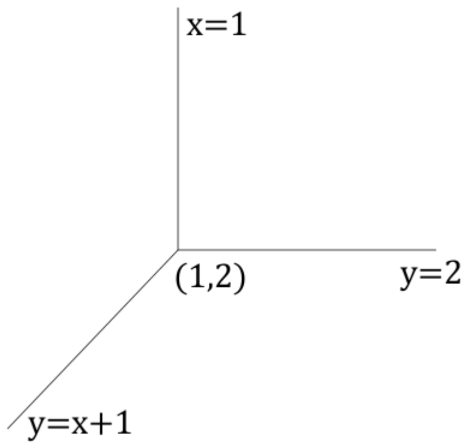
Example 2.7.3. Let $k = \mathbb{C}\{\{t\}\}$ be an algebraically closed field, and $f = 2tx - y + t^2 \in k[x^{\pm 1}, y^{\pm 1}]$. The value group Γ_{val} of k is dense in $\mathbb{R}$ where the valuation on k is t-adic valuation. Then the variety $V(f)$ is $\{(z, 2tz + t^2) : z \in k\}$. Consider the tropicalization of f, we have

$$\text{trop}(f) = \min\{1+x, y, 2\}.$$

Now we have four possibilities that trop(f) contains more than 1 term, and they are $1+x=y$, $1+x=2$, $y=2$ and $1+x=y=2$. Then the tropical hypersurfaces are

$$\{(1, 2+\lambda_1), (1+\lambda_2, 2), (\lambda_3, 1+\lambda_3), (1,2)\},$$

where $\lambda_1, \lambda_2 \in \mathbb{R}^+$ and $\lambda_3 < 1$. And the tropical line of trop$(V(f))$ is shown on the following diagram.



Now consider the weight vectors $\mathbf{w}$ which make $\text{in}_{\mathbf{w}}(f)$ be not one monomial. It is obviously that $\text{in}_{\mathbf{w}}(f)$ contains more than one monomial only when $\mathbf{w} = (1,2)$ or $(1, 2+\lambda_1)$, $(1+\lambda_2, 2)$ and $(\lambda_3, 1+\lambda_3)$ where $\lambda_1, \lambda_2 \in \mathbb{R}^+$ and $\lambda_3 < 1$. Then the set of the weight vectors which making $\text{in}_{\mathbf{w}}(f)$ contains more than one term coincides the tropical hypersurfaces of trop(f). And the $\text{in}_{\mathbf{w}}(f)$ with respect to those weight vectors are $x+y+1$, $x+1$, $y+1$ and $x+y$ respectively.

Finally, note the closure of the set of coordinatewise valuations of the points in $V(f)$. In the beginning of this example, we have $V(f) = \{(z, 2tz+t^2) : z \in k\}$, then the coordinatewise valuation of the hypersurface is $(\text{val}(z), \text{val}(2tz+t^2))$. Let $\text{val}(z) = 1$, then we have $\text{val}(2tz+t^2) > \min\{1+\text{val}(z), 2\} = 2$. Hence the points are contained in the set $\{(1, 2+\lambda_1) : \lambda_1 \in \mathbb{R}^+\}$ when $\text{val}(z) = 1$. If $\text{val}(z) > 1$, then $\text{val}(2tz+t^2) = \min\{1+\text{val}(z), 2\} = 2$. Then the points in the $\{(1+\lambda_2, 2) : \lambda_2 \in \mathbb{R}^+\}$ when $\text{val}(z) > 1$. Let $\text{val}(z) < 1$, then we have $\text{val}(2tz+t^2) = \min\{1+$

$\text{val}(z), 2\} = 1 + \text{val}(z) < 2$. Hence, the points in the set $\{(\lambda_3, 1 + \lambda_3) : \lambda_3 < 1\}$ when $\text{val}(z) < 2$. In conclusion, we have

$$(\text{val}(z), \text{val}(2tz + t^2)) = \begin{cases} (1, 2 + \lambda_1) & \text{if val}(z) = 1 \\ (1 + \lambda_2, 2) & \text{if val}(z) > 1 \\ (\lambda_3, 1 + \lambda_3) & \text{if val}(z) < 1 \\ (1, 2) & \text{otherwise} \end{cases}$$

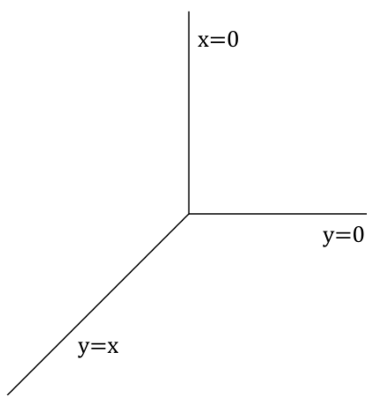
where $\lambda_1, \lambda_2 \in \mathbb{R}^+$ and $\lambda_3 < 1$. Therefore the closure of the set coordinatewise valuations of points in $V(f)$ coinsides with the tropical hypersurfaces of f and the set of weight vectors that make $\text{in}_{\mathbf{w}}(f)$ contains more than 1 term.

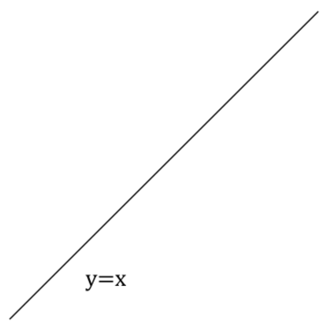
Kapranov's theorem establishes an equivalence between the hypersurface associated to the polynomial and its coordinate-wise valuation. The Fundamental Theorem of Tropical Algebraic Geometry is directly generalized from Kapranov's theorem from hypersurfaces to arbitrary varieties.

Definition 2.7.4. Let I be an ideal in $k[x_1^{\pm}, \ldots, x_n^{\pm}]$, and let $X = V(I)$ be its variety. With a valuation ν on k, the tropicalization $\text{trop}(X)$ of the variety X is the intersection of all tropical hypersurfaces defined by elements of I:

$$\text{trop}(X) = \bigcap_{f \in I} \text{trop}(V(f)) \subseteq \mathbb{R}^n.$$

Example 2.7.5. Let $I = \langle x + y + 1, x + 2y \rangle \subset \mathbb{C}\{\{t\}\}$. Then the classical variety is $X = \{(-2, 1)\}$ and the coordinate-wise valuation is $\text{trop}(X) = \{(0, 0)\}$. However the tropical hypersurfaces $\text{trop}(V(x + y + 1))$ and $\text{trop}(V(x + 2y))$ are shown on the following diagram respectively.





Then the intersection is the half-ray $\{(w_1, w_2) \in \mathbb{R}^2 : w_1 = w_2 \leq 0\}$. So $\mathrm{trop}(X) \neq \bigcap_f \mathrm{trop}(V(f))$ when f only runs over generator set of I.

Before introducing the next lemma, we need to introduce the rank of subgroup. Let G be a group and L be a subgroup of G, the smallest cardinality of a generating set for L is the rank of L.

Lemma 2.7.6. *[1, Lemma 2.2.7] Given any vector $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{Z}^n$ with the greatest common divisor of $|v_1|, \dots, |v_n|$ equal to 1, there exists a matrix $U \in GL(n, \mathbb{Z})$ with $U\mathbf{v} = \mathbf{e}_1$. Further, if L is a rank k subgroup of $\mathbb{Z}^n$ with $\mathbb{Z}^n/L$ torsion free, then there is a matrix $U \in GL(n, \mathbb{Z})$ with UL equal to the subgroup generated by $\mathbf{e}_1, \dots, \mathbf{e}_k$.*

Proof. First of all, we show that if $\mathbf{v} = (v_1, \dots, v_n)$ and the greatest common divisor of $|v_1|, \dots, |v_n|$ is equal to 1, then the group $\mathbb{Z}^n/\mathbb{Z}\mathbf{v}$ is torsion free. Suppose that $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n/\mathbb{Z}\mathbf{v}$ and there exists an integer $m \in \mathbb{Z}$ such that $m \cdot \mathbf{a} = 0 \in \mathbb{Z}^n/\mathbb{Z}\mathbf{v}$ which means that $(ma_1, \dots, ma_n) \in \mathbb{Z}\mathbf{v}$. Let $z \in \mathbb{Z}$ that $(ma_1, \dots, ma_n) = (zv_1, \dots, zv_n)$. Then we have the ratio $a_1 : \cdots : a_n = v_1 : \cdots : v_n$. Since the greatest common divisor of $|v_1|, \dots, |v_n|$ is 1, for any $1 \leq i \leq n$, a_i can only be the multiple of v_i. Then $(a_1, \dots, a_n) \in \mathbb{Z}\mathbf{v}$ and $\mathbb{Z}^n/\mathbb{Z}\mathbf{v}$ is torsion free. The condition $\mathbb{Z}^n/L$ is torsion free allows us to choose a generator set of L with cardinality k and the greatest common divisor of their absolute value is 1. Let $\mathbf{b}_1, \dots, \mathbf{b}_k$ be a generator set of L and for each $\mathbf{b}_i$ $1 \leq i \leq k$ the greatest common divisor of $|b_{i,1}|, \dots, |b_{i,n}|$ is 1. Suppose a $k \times n$ matrix A such that

$$A = \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_k \end{pmatrix}$$

By the Smith normal form of A, there exist a $k \times k$ matrix V and a $n \times n$ matrix U' such that

$$VAU' = \begin{pmatrix} c_1 & 0 & \dots & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \dots & c_k & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix}$$

where $c_i = \frac{d_i(A)}{d_{i-1}(A)}$, $d_i(A)$ is the the greatest common divisor of the determinants of all $i \times i$ minors of A. Since the greatest common divisor of $|b_{i,1}|, \dots, |b_{i,n}|$ is 1 for each $\mathbf{b}_i$ $1 \leq i \leq k$, then $c_i = 1$ for all $1 \leq i \leq k$. And according the algorithm of Smith normal form, we may have $V \in GL(k, \mathbb{Z})$ and $U' \in GL(n, \mathbb{Z})$. Since $V \in GL(k, \mathbb{Z})$, then VA is still a basis of L. Then we take $U = {U'}^T$. □

Lemma 2.7.7. *[1, Lemma 3.2.10] Let X be a d-dimensional subvariety of T^n, with ideal $I \subset k[x_1^{\pm}, \dots, x_n^{\pm}]$. Every cell in the Gröbner complex Σ whose support lies in the set*

$$\{\mathbf{w} \in \mathbb{R}^n \ : \ \text{in}_{\mathbf{w}}(I) \neq \langle 1 \rangle\}$$

has dimension at most d.

Proof. Since $|\Sigma|$ is in the set $\{\mathbf{w} \in \mathbb{R}^n \ : \ \text{in}_{\mathbf{w}}(I) \neq \langle 1 \rangle\}$, we may assume that there exists a set of $\mathbf{w} \in \Gamma_v^n$ lie in the relative interior of a maximal cell $P \in \Sigma$. Let L be a subspace of $\mathbb{R}^n$ such that $\mathbf{w} + L$ is the affine span of P. By lemma 2.7.6, we know that with a linear transformation we may assume that L is spanned by vectors $\mathbf{e}_1, \dots, \mathbf{e}_m$ for some m. And by corollary 2.6.6, we can assume that the image of $\mathbf{w}$ under the linear transformation is $\mathbf{w}'$ and $\text{in}_{\mathbf{w}'}(I) \neq \langle 1 \rangle$. Then we need to prove that $\dim(L) = m \leq d$. Since $\mathbf{w}$ is in the relative interior of P, then $\text{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I) \neq \langle 1 \rangle$ for all $\mathbf{v} \in \mathbb{R}^n \cap L$ and ϵ is sufficiently small. By lemma 2.5.5 and proposition 2.5.7, we have $\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I)) = \text{in}_{\mathbf{w}}(I)$ for all $\mathbf{v} \in \mathbb{R}^n \cap L$. Let G be a set of generators for $\text{in}_{\mathbf{w}}(I)$ such that none of the generators in G is the sum of two other polynomial in $\text{in}_{\mathbf{w}}(I)$ which have fewer monomials. Now suppose $f \in G$, since $\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I)) = \text{in}_{\mathbf{w}}(I)$ we have $\text{in}_{\mathbf{v}}(f) = f$, because if $\text{in}_{\mathbf{v}}(f) \neq f$, $\text{in}_{\mathbf{v}}(f)$ will be a polynomial with fewer monomials in $\text{in}_{\mathbf{w}}(I)$ which contradicts our assumption. Also, since we assume L is spanned by $\mathbf{e}_1, \dots, \mathbf{e}_m$, then $\text{in}_{\mathbf{e}_i}(f) = f$ for $1 \leq i \leq m$. Moreover, by proposition 2.5.7, we have $f = g \cdot \tilde{f}$ where g is a monomial and $x_1, \dots, x_m$ do not appear in $\tilde{f} \in I_{\text{proj}}$. Since monomials are units in $\mathbb{K}[x_1^{\pm}, \dots, x_n^{\pm}]$ where $\mathbb{K}$ is the residue field of the valuation on k, which implies that $\text{in}_{\mathbf{w}}(I)$ is generated by those polynomials not containing $x_1, \dots, x_m$. Therefore $m \leq \dim(\text{in}_{\mathbf{w}}(I)) \leq \dim(X) = d$. □

Proposition 2.7.8. *[1, Proposition 3.2.7]) Fix a subvariety X in T^n and $m \geq \dim(X)$. There exists a morphism $\psi : T^n \longrightarrow T^m$ whose image $\psi(X)$ is Zariski closed in T^m and satisfies $\dim(\psi(X)) = \dim(X)$. This map can be chosen so that the following hold.*

1. *The kernel of the linear map $\text{trop}(\psi) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ intersects trivially with a fixed finite arrangement of m-dimensional subspaces in $\mathbb{R}^n$.*

2. *When $n > m$, if we change coordinates so that ψ is the projection onto the first m coordinates, then the ideal I of X is generated by polynomials in $x_{m+1}, \ldots, x_n$ whose coefficients are monomials in $x_1, \ldots, x_m$.*

Proof. We shall prove this proposition by induction. The case $n = m$ is trivial. Now assume $n > m$. We first define a monomial change of variables in T^n as

$$\phi_l^*(x_i) = x_i x_n^{l^i} \text{ and } \phi_l^*(x_n) = x_n,$$

for $1 \leq i \leq n-1$ and $l \in \mathbb{N}$. Then, for any $f \in I \subset k[x_1^{\pm}, \ldots, x_n^{\pm}]$ we have

$$g = \phi_l^*(f) = f(x_1 x_n^l, x_2 x_n^{l^2}, \ldots, x_{n-1} x_n^{l^{n-1}}, x_n).$$

This is an automorphism of $k[x_1^{\pm}, \ldots, x_n^{\pm}]$. Notice that its monomials have distinct degrees in the variables x_i. Since ϕ_l^* is invertible, we can replace I by $\phi_l^*(I)$, and assume that I is generated by a set of polynomials with this property.

Let π be the map $T^n \longrightarrow T^{n-1}$ which is defined by

$$\pi(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{n-1}).$$

We need to show the image of subvariety $X \in T^n$ under π is closed. First, by [5, Theorem 3.2.2], the closure of $\pi(X)$ is the variety $\mathrm{V}(I \cap k[x_1^{\pm}, \ldots, x_{n-1}^{\pm}])$ in T^{n-1}. The difference $\overline{\pi(X)}/\pi(X)$ is contained in the variety of the leading coefficients of the polynomials in a generating set of I when viewed as polynomials in x_n. Since the leading coefficient of each generator is a monomial consists of $x_1, \ldots, x_{n-1}$, then the variety in T^{n-1} defined by those polynomials is empty. Therefore $\overline{\pi(X)} = \pi(X)$.

For proving $\dim(X) = \dim(\pi(X))$, since $I \subset k[x_1^{\pm}, \ldots, x_n^{\pm}]$ then there always exists a polynomial in I that is monic when it is regarded as a polynomial in x_n. Then $k[X]$ can be consider as a coordinate ring which is generated by x_n, and the field of fractions $k(X)$ is a finite extension of $k(\pi(X))$. Then their transcendence degrees are the same by [5, Theorem 9.5.6] (The details about this step can be found in the whole section 5 in Chapter 9 of [5]). Therefore $\dim(X) = \dim(\pi(X))$.

By induction on $n - m$, there is a morphism $\psi : T^{n-1} \longrightarrow T^m$ with $\psi(X)$ is Zariski closed in T^m and $\dim(\psi(X)) = \dim(X)$, and by induction, the second requirement on the form of the generators all follow.

For proving the first requirement, we can choose the change of the coordinates which makes the intersection of the kernel of $\mathrm{trop}(\pi) : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$ and some finite collection of subspaces empty, in the original coordinates the kernel of $\text{trop}(\pi)$ is the line spanned by $(1, l, l^2, \ldots, l^{n-1})$ in $\mathbb{R}^n$. If $l \gg 0$, the intersection of this line and any fixed finite number of hyperplanes is the origin. Then by induction on $n - m$, we can prove that ψ satisfies the first requirement. $\square$

Theorem 2.7.9. *(Fundamental Theorem of Tropical Algebraic Geometry)[1, Theorem 3.2.3] Let k be an algebraically closed field with a non-trivial valuation ν, let I be an ideal in $k[x_1^{\pm}, \ldots, x_n^{\pm}]$, and let $X = V(I)$ be its variety in the algebraic torus $T^n \cong (k^*)^n$. Then the following three subsets of $\mathbb{R}^n$ coincide:*

1. *the tropical variety* $\text{trop}(X)$*;*
2. *the set of all vectors* $\mathbf{w} \in \mathbb{R}^n$ *with* $\text{in}_{\mathbf{w}}^{\nu}(I) \neq \langle 1 \rangle$*;*
3. *the closure of the set of coordinate-wise valuations of points in X,* $\nu(X) = \{(\nu(y_1), \ldots, \nu(y_n)) \ : \ (y_1, \ldots, y_n) \in X\}$.

Furthermore, if X is irreducible and $\mathbf{w}$ is any point in $\Gamma_\nu^n \cap \text{trop}(X)$, then the set $\{\mathbf{y} \in X \ : \ \nu(\mathbf{y}) = \mathbf{w}\}$ is Zariski dense in the classical variety X.

Proof. By Kapranov's theorem 2.7.1, for any $f \in I$, if $\mathbf{y} = (y_1, \ldots, y_n) \in X$ that $f(\mathbf{y}) = 0$, then $(\nu(y_1), \ldots, \nu(y_n)) \in \text{trop}(X)$. Since $\text{trop}(X)$ is a tropical variety which is closed with Zariski topology, hence set 3 is contained by set 1.

Then we let $\mathbf{w} \in \text{trop}(X)$, hence for any $f = \sum\limits_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} x^{\mathbf{u}} \in I$ we have more than one monomial in $\{\nu(c_{\mathbf{u}}) + \mathbf{u} \cdot \mathbf{w} \ : \ c_{\mathbf{u}} \neq 0\}$. Then $\text{in}_{\mathbf{w}}^{\nu}(f)$ is not a monomial. Since $\text{in}_{\mathbf{w}}^{\nu}(f) \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, $\mathbb{K}$ is the residue field of ν, if $\text{in}_{\mathbf{w}}^{\nu}(f)$ is a monomial then there may exists a $h \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ such that $\text{in}_{\mathbf{w}}^{\nu}(f) \cdot h = 1$. Now suppose $g \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ that $\text{in}_{\mathbf{w}}^{\nu}(g) = h$, by lemma 2.5.8 we have $\text{in}_{\mathbf{w}}^{\nu}(f) \cdot \text{in}_{\mathbf{w}}^{\nu}(g) = \text{in}_{\mathbf{w}}^{\nu}(fg) = 1$. Then $\text{in}_{\mathbf{w}}^{\nu}(I) = \langle 1 \rangle$. Therefore for any $\mathbf{w}$, it lies in set 2.

Before showing that set 2 is contained by set 3, we will first prove that if a weight vector $\mathbf{w}$ is in set 2 for X, then $\mathbf{w}$ lies in the set 2 for some irreducible component f X. First of all, X is a variety in T^n then there is a decomposition that $X = X_1 \cup \ldots \cup X_s$ where X_i is irreducible for all $1 \leq i \leq s$. Then the ideals corresponding to $X_1, \ldots, X_s$ are all prime ideals, let them be $I_1, \ldots, I_s$ respectively, then $I = I_1 \cap \ldots \cap I_s$. Let $\mathbf{w} \in \mathbb{R}^n$ has $\text{in}_{\mathbf{w}}^{\nu} \neq \langle 1 \rangle$, then there is one $j \in \{1, \ldots, s\}$ such that $\text{in}_{\mathbf{w}}^{\nu}(I_j) \neq \langle 1 \rangle$. For proving this, assume that there are polynomials $g_i \in I_i$ for $1 \leq i \leq s$ with $\text{in}_{\mathbf{w}}^{\nu}(g_i) = 1$. Since $I = I_1 \cap \ldots \cap I_s$, then $\prod\limits_{i=1}^{s} g_i \in I$. By lemma 2.5.8, we have $\text{in}_{\mathbf{w}}^{\nu}\left(\prod\limits_{i=1}^{s} g_i\right) = \prod\limits_{i=1}^{s}\left(\text{in}_{\mathbf{w}}^{\nu} g_i\right)$, hence $1 \in \text{in}_{\mathbf{w}}^{\nu}(I)$, contradicting the assumption. $\square$

As the proof of Kapronov's theorem 2.7.1 we haven't shown that the set 3 containing set 2 and the rest of the Fundamental theorem. Just like the proposition 2.7.2, proposition 2.7.10 will complete the proof of theorem 2.7.9.

Proposition 2.7.10. *[1, Proposition 3.2.11] Let X be an irreducible subvariety of T^n, with prime ideal $I \subseteq k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Fix $\mathbf{w} \in \Gamma_\nu^n$ with $\text{in}_{\mathbf{w}}^\nu(I) \neq \langle 1 \rangle$, and let $\alpha \in V(\text{in}_{\mathbf{w}}^\nu(I))$. Then there exists a point $\mathbf{y} \in X$ with $\nu(\mathbf{y}) = \mathbf{w}$ and $\overline{t^{-\mathbf{w}}\mathbf{y}} = \alpha$. The set of such $\mathbf{y}$ is Zariski dense in X.*

Proof. Let $d = \dim(X)$, if $n = 1$ then it follows the beginning of the proof of proposition 2.7.2, and if $n = d + 1$, X is a hypersurface then it also follows proposition 2.7.2. So assume that $0 \leq d \leq n - 2$ and we shall use induction on n. According to lemma 2.7.7, the support of a polyhedral complex Σ lies in the set $\{\mathbf{v} \in \mathbb{R}^n : \text{in}_{\mathbf{v}}(I) \neq \langle 1 \rangle\}$, and every cell $P \in \Sigma$ the dimension of P is at most $d = \dim(X)$. Let L_P be the linear span of $P - \mathbf{w} \in \mathbb{R}^n$. Then $\dim(L_P) \leq d + 1 < n$, and $\mathbf{w} + L_P$ is the subspace spanned by P and $\mathbf{w}$.

Suppose a monomial map $\phi : T^n \longrightarrow T^{n-1}$, then by definition 2.6.5 there exists a linear map $\text{trop}(\phi) : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$. Recalling proposition 2.7.8, we can choose the linear map $\text{trop}(\phi)$ which satisfies $\ker(\text{trop}(\phi)) \cap L_P = \{\mathbf{0}\}$ for all $P \in \Sigma$. Furthermore, from proposition 2.7.8, we have that ϕ maps onto the first $n - 1$ coordinates and the image $\phi(X)$ is Zariski closed in T^{n-1}. Then we can show that $\text{trop}(\phi)$ is an injective map. Now suppose there exists another weight vector $\mathbf{w}' \in \Gamma_\nu^n$ such that $\text{in}_{\mathbf{w}'}(I) \neq \langle 1 \rangle$ and $\text{trop}(\phi)(\mathbf{w}') = \text{trop}(\phi)(\mathbf{w})$. Since $\mathbf{w}'$ is in the support of Σ then there exists a polyhedron P such that $\mathbf{w}' \in P$. And we let L_P be the affine span of $P - \mathbf{w}$ then $\mathbf{w}' \in L_P$ which implies $\mathbf{w}' \in \mathbf{w} + L_P$ and $\mathbf{w}' - \mathbf{w} \in L_P$. Since $\mathbf{w}' - \mathbf{w} \in \ker(\text{trop}(\phi))$ and we choose the linear map $\text{trop}(\phi)$ with $\ker(\text{trop}(\phi)) \cap L_P = \{\mathbf{0}\}$, then $\mathbf{w}' = \mathbf{w}$.

Recalling definition 2.6.5, the monomial map $\phi : T^n \to T^{n-1}$ is associated with the ring homomorphism $\phi^* : k[x_1^\pm, \dots, x_{n-1}^\pm] \to k[x_1^\pm, \dots, x_n^\pm]$. Therefore let $I' = \phi^{*-1}(I) = I \cap k[x_1^\pm, \dots, x_{n-1}^\pm]$ and $X' = V(I')$. By proposition 2.7.8, we have $\phi(X)$ is Zariski closed in T^{n-1}, then $X' = \phi(X)$. By lemma 2.6.6, $\text{in}_{\text{trop}(\phi)(\mathbf{w})}(I') \neq \langle 1 \rangle$, then $\text{trop}(\phi)(\mathbf{w})$ is in the tropical variety of I'. As the proof of proposition 2.7.2, we use induction start from $\phi : T^n \to T$. When $\phi : T^n \to T$, we have $\phi^* : k[x_1^\pm] \to k[x_1^\pm, \dots, x_n^\pm]$. Then as the beginning of this paragraph, we have $I_1 = \phi^{*-1}(I) = I \cap k[x_1^\pm]$, $X_1 = \phi(X)$ is Zariski closed in T and $\text{in}_{\text{trop}(\phi)(\mathbf{w})}(I_1) = \langle 1 \rangle$. Since $k[x_1^\pm]$ is a principal ideal domain, I_1 can be generated by a polynomial let it be $f_1 \in I_1$, which implies that X_1 is a hypersurface of this polynomial f_1. Then by proposition 2.7.2, there exists $y \in k^*$ satisfying $f_1(y) = 0$, $\nu(y) = w$ and $\overline{t^{-w}y} = \alpha$ where $\text{in}^{\nu}_{w}(f_1)(\alpha) = 0$. So we shall prove the Kapranov's theorem in X_1 when $\phi : T^n \to T$, then we can consider the monomial map $\phi : T^n \to T^2$. Following the description above, we have $\phi^* : k[x_1^{\pm}, x_2^{\pm}] \to k[x_1^{\pm}, \ldots, x_n^{\pm}]$ which implies that we have a new ideal $I_2 = \phi^{*-1}(I) = I \cap k[x_1^{\pm}, x_2^{\pm}]$ and $X_2 = V(I_2)$ is Zariski closed in T^2. So by induction, we will reach the step $\phi : T^n \longrightarrow T^{n-1}$. Then with the same method we can show that there is $\mathbf{y}' = (y_1, \ldots, y_{n-1}) \in X' \subset T^{n-1}$ with $\nu(y_i) = w_i$ and $\overline{t^{-w_i}y_i} = \alpha_i$ for all $1 \leq i \leq n-1$. Let $J = \langle f(y_1, \ldots, y_{n-1}, x_n) : f \in I \rangle \subseteq k[x_n^{\pm}]$. Since $k[x_n^{\pm}]$ is a ring of polynomials in one variable, $k[x_n^{\pm}]$ is a principal ideal domain which implies that there must exists a single polynomial $f \in I$ whose specialization generates the principal ideal J. By proposition 2.7.8 we may rewrite f such that $f = x_n^l + f'$ where $f' \in k[x_1^{\pm}, \ldots, x_{n-1}^{\pm}]$. The degree l is positive, hence $J \neq \langle 1 \rangle$.

With the monomial map in T^n in the proof of proposition 2.7.8, for any $f = \sum_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} x^{\mathbf{u}} \in k[x_1^{\pm}, \ldots, x_n^{\pm}]$ we have

$$\phi_l^*(f) = f(x_1 x_n^l, x_2 x_n^{l^2}, \ldots, x_{n-1} x_n^{l^{n-1}}, x_n)$$

where $l \in \mathbb{N}$. Then we can assume that f is a polynomial in only one variable x_n and the coefficients in f are monomials in $k[x_1^{\pm}, \ldots, x_{n-1}^{\pm}]$. So we can write $f = \sum_{i \in \mathbb{Z}} c_i x^{\mathbf{u}_i} x_n^i$ where $\mathbf{u}_i \in \mathbb{Z}^{n-1}$. Now plugging the point $\mathbf{y}'$ into $\phi_l^*(f)$, then let $g = f(y_1, \ldots, y_{n-1}, x_n) = \sum_{i \in \mathbb{Z}} c_i \mathbf{y}'^{\mathbf{u}_i} x_n^i$. By the induction, we let $\mathbf{w}' \in \mathbb{R}^{n-1}$ and assume that

$$\nu(\mathbf{y}') = \nu(y_1, \ldots, y_{n-1}) = (w_1, \ldots, w_{n-1}) = \mathbf{w}'.$$

Let w_n be a value such that $\text{trop}(f)(\mathbf{w}) = \text{trop}(g)(w_n)$ and then we consider the initial form. Suppose $\text{trop}(g)(w_n) = W$ then

$$\text{in}_{w_n}(g)(x_n) = \sum_{\substack{i \in \mathbb{Z} \\ \text{trop}(g)(w_n) = W}} \overline{c_i \mathbf{y}'^{\mathbf{u}_i} t^{-\nu(c_i \mathbf{y}'^{\mathbf{u}_i})}} x_n^i$$

Notice that $\nu(c_i \mathbf{y}'^{\mathbf{u}_i}) = \nu(c_i) + \mathbf{u}_i \cdot \nu(\mathbf{y}')$, by induction we have $\nu(\mathbf{y}') = \nu(y_1, \ldots, y_{n-1}) = (w_1 \ldots, w_{n-1}) = \mathbf{w}'$. Then for each i we have

$$\overline{c_i \mathbf{y}'^{\mathbf{u}_i} t^{-\nu(c_i \mathbf{y}'^{\mathbf{u}_i})}} = \overline{c_i t^{-\nu(c_i)} (t^{-\mathbf{w}'} \mathbf{y}')^{\mathbf{u}_i}} = \overline{c_i t^{-\nu(c_i)}} \alpha'^{\mathbf{u}_i}.$$

So the initial form $\text{in}_{w_n}(g)(x_n)$ will be

$$\text{in}_{w_n}(g)(x_n) = \sum_{\substack{i \in \mathbb{Z} \\ \text{trop}(g)(w_n)=W}} \overline{c_i t^{-\nu(c_i)}} \alpha'^{\mathbf{u}_i} \cdot x_n^i$$

which is the $n = 1$ case in proposition 2.7.2, there exists a $y_n \in k^\times$ with $g(y_n) = 0$, $\nu(y_n) = w_n$ and $\overline{t^{-w_n} y_n} = \alpha_n$. Then we have the point $\mathbf{y} = (y_1, \ldots, y_n)$ with $\nu(\mathbf{y}) = \mathbf{w} = (w_1, \ldots, w_n)$ and $\overline{t^{-w_i} y_i} = \alpha_i$ for all $1 \leq i \leq n$. And in the second paragraph of this proof, we have shown the uniqueness.

Now for the last step of this proof, we need the show that the set Y of all the points $\mathbf{y}$ is Zariski dense in X. Let X' be a proper subvariety in X and contains Y. Then $\phi(Y) \subset \phi(X')$, and for any $\mathbf{y} \in Y$ let $\phi(\mathbf{y}) = \mathbf{y}'$. By the induction, for all $\mathbf{y}' \in \phi(X)$ we have $\nu(y'_i) = w_i$ and $\overline{t^{-w_i} y_i} = \alpha_i$ for $1 \leq i \leq n-1$. By proposition 2.7.8, we have $\dim(\phi(X)) = \dim(X) > 0$. Then from $\phi : T^n \to T$, by induction we used in this proof with proposition 2.7.2, $\phi(Y)$ is Zariski dense in $\phi(X)$ then we have $\phi(X') = \phi(X)$, which contradicts $X' \subsetneq X \implies \dim(X') < \dim(X)$. Therefore subvariety X' doesn't exist, so Y is Zariski dense in X. □

Chapter 3

Higher rank background

In chapter 3, we will introduce higher rank valuation and some research related to it. Unlike classic valuation, higher rank valuation is mapping a field into an arbitrary ordered abelian group and the rank of the ordered abelian group is the rank of valuation. In the last section, we will reduce a rank n valuation to a sequence of rank 1 valuations and that is the n-step valuation we are studying.

3.1 The rank of an ordered abelian group

Definition 3.1.1. (Ordered abelian group)
An ordered abelian group $(\Gamma, +, \leq)$ is an abelian group $(\Gamma, +)$ which is totally ordered and $a < b, c < d$ implies $a + c < b + d$ for any $a, b, c, d \in \Gamma$.

A convex subgroup $\Delta \subset \Gamma$ is a subgroup such that for any $\gamma \in \Gamma$ and $0 \leq \gamma \leq \delta$ where $\delta \in \Delta$, then $\gamma \in \Delta$. And the set of all convex subgroups is linearly ordered inclusion.

Definition 3.1.2. The rank of an ordered abelian group is the maximal length of a chain of distinct proper convex subgroups.

For instance, $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$ are both rank 1 ordered abelian group with addition, because convex subgroups contained by $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$ are either $\{0\}$ or the whole group. And there exists an isomorphism between a non-trivial subgroup of $(\mathbb{R}, +)$ and an ordered abelian group.

Definition 3.1.3. An ordered abelian group $(\Gamma, +, \leq)$ is called archimedean if for all $\gamma, \varepsilon \in \Gamma$ such that $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\gamma \leq n\varepsilon$.

Then let Δ be a convex subgroup of Γ and $0 < a \in \Delta$. Suppose $0 < \gamma \in \Gamma$, then $\exists n \in \mathbb{N}$ such that $r \leq na$ where $a \in \Delta$. So we will have $\gamma \in \Delta$ which implies that $\Delta = \Gamma$. Therefore an archimedean ordered abelian group has no non-trivial convex subgroup.

Proposition 3.1.4. *[6, Proposition 2.1.1] An ordered abelian group Γ is of rank 1 if and only if it is order isomorphic to a non-trivial subgroup of $(\mathbb{R}, +)$ with the canonical order induced from $\mathbb{R}$.*

Proof. This is proof is from the proof of proposition 2.1.1 on page 26 of [6].

First, we need to show Γ is archimedean. Let $\varepsilon \in \Gamma$ with $\varepsilon > 0$ and set

$$\Delta = \{\gamma \in \Gamma : \gamma, -\gamma \leq n\varepsilon \text{ for some } n \in \mathbb{N}\}.$$

Obviously, $0 \in \Delta$ and $-\gamma \in \Delta$ for all $\gamma \in \Delta$. Let $\gamma_1, \gamma_2 \in \Delta$ then there exist $n_1, n_2 \in \mathbb{N}$ such that $\gamma_1, -\gamma_1 \in n_1\varepsilon$ and $\gamma_2, -\gamma_2 \in n_2\varepsilon$. Then we have $(\gamma_1 + \gamma_2), -(\gamma_1 + \gamma_2) \leq (n_1 + n_2)\varepsilon$. So Δ is a subgroup of Γ. Let $0 \leq \delta \leq \gamma \in \Delta$ and $\delta \in \Gamma$, by the definition of δ we give, $\delta \in \Delta$, then Δ is convex. However, Γ is of rank 1 and Δ is not a trivial subgroup, then $\Delta = \Gamma$ and Γ is archimedean.

For showing the isomorphism between Γ and a non-trivial subgroup of $(\mathbb{R}, +)$, we need to define a map which is mapping Γ to the set of Dedekind cuts of $\mathbb{Q}$. Since Dedekind cuts are real numbers, then we shall prove the map is injective. Suppose an arbitrary element $\alpha \in \Gamma$, set the following subsets of rational number

$$L(\alpha) = \{\frac{m}{n} \in \mathbb{Q} : n > 0 \text{ and } m\varepsilon \leq n\alpha\}$$
$$U(\alpha) = \{\frac{m}{n} \in \mathbb{Q} : n > 0 \text{ and } m\varepsilon \geq n\alpha\}$$

Since Γ is ordered, then either $m\varepsilon \leq n\alpha$ or $m\varepsilon \geq n\alpha$, then every rational number in $\mathbb{Q}$ is in $L(\alpha) \cup U(\alpha)$, thus $L(\alpha) \cup U(\alpha) = \mathbb{Q}$. Assume $U(\alpha) = \varnothing$, we have $L(\alpha) = \mathbb{Q}$, then $m\varepsilon \leq \alpha$ which contradicts that Γ is archimedean. Then $U(\alpha) \neq \varnothing$ and $L(\alpha) \neq \varnothing$ similarly. Now suppose $\frac{m}{n}, \frac{m'}{n'} \in \mathbb{Q}$ such that $m\varepsilon \leq n\alpha$ and $m'\varepsilon \geq n'\alpha$, then we have $mn'\varepsilon \leq n'n\alpha = nn'\alpha \leq nm'\varepsilon$. Then $mn' \leq nm'$, since $n, n' > 0$ we have $\frac{m}{n} \leq \frac{m'}{n'}$, which implies that $L(\alpha) \leq U(\alpha)$. So with these two subsets $L(\alpha), U(\alpha) \in \mathbb{Q}$, we shall define a Dedekind cut of $\mathbb{Q}$. Now consider the mapping $r : \Gamma \to (\mathbb{R}, +)$ where $L(\alpha) \leq r(\alpha) \leq U(\alpha)$ for any $\alpha \in \Gamma$, and this mapping preserves the ordering. We need to show r is a group homomorphism. Let $\alpha, \beta \in \Gamma$ and let $\frac{m}{n}$, $\frac{m'}{n'}$ be two arbitrary elements in $L(\alpha)$, $L(\beta)$ respectively. Then we have $m\varepsilon \leq n\alpha$ and $m'\varepsilon \leq n'\beta$. Clearly, we will have these inequalities $mn'\varepsilon \leq nn'\alpha$ and $m'n\varepsilon \leq nn'\beta$. Since $n, n' > 0$ and $\varepsilon > 0$ we have $\frac{mn'}{nn'} \leq \frac{\alpha}{\varepsilon}$ and $\frac{m'n}{nn'} \leq \frac{\beta}{\varepsilon}$. Then

$$\frac{mn' + m'n}{nn'} \leq \frac{\alpha + \beta}{\varepsilon}$$
$$(mn' + m'n)\varepsilon \leq nn'(\alpha + \beta).$$

So $\frac{m}{n} + \frac{m'}{n'} \in L(\alpha + \beta)$, then $L(\alpha) + L(\beta) \subseteq L(\alpha + \beta)$. Since we defined $r(\alpha) \geq L(\alpha)$ for any $\alpha \in \Gamma$, thus $r(\alpha + \beta) \geq r(\alpha) + r(\beta)$. Conversely, with the same method, we can show that $U(\alpha) + U(\beta) \subseteq U(\alpha + \beta)$ and this implies $r(\alpha + \beta) \leq r(\alpha) + r(\beta)$. Therefore $r(\alpha + \beta) = r(\alpha) + r(\beta)$. So r is a group homomorphism.

The last step of this proof is showing r is an isomorphism. By the fundamental thoerem of isomorphism, we need to prove that the kernel of r is trivial. So let $r(\alpha) = 0$, then $0 \geq \frac{\alpha}{\varepsilon} \geq \frac{m}{n}$ which means the maximum value of m is -1. Similarly, from $U(\alpha)$, we get $\frac{1}{n} \geq \frac{\alpha}{\varepsilon}$. Then $-\varepsilon \leq n\alpha \leq \varepsilon$ for all $n > 0$. However, Γ is archimedean, so the only possibility is $\alpha = 0$. □

Proposition 3.1.5. *Let Γ be an ordered abelian group of rank n then Γ is ordered isomorphic to a non-trivial subgroup of $(\mathbb{R}^n, +)$ with the canonical order induced from $\mathbb{R}$.*

Proof. For proving this proposition, we can use induction from $n = 2$. Suppose Γ is of rank 2, then there exists a chain of convex subgroups of Γ such that $\varnothing \subset \Gamma' \subset \Gamma$ where Γ' is a convex subgroup of Γ. Since Γ' is abelian, Γ/Γ' is a factor group. Since Γ is of rank 2, if there exists a convex subgroups in Γ/Γ' then then rank of Γ is bigger than 2, which contradicts. Therefore Γ/Γ' is of rank 1 and by proposition 3.1.4, Γ/Γ' is ordered isomorphic to a non-trivial subgroup of $(\mathbb{R}, +)$.

Now consider a mapping $r : \Gamma \to (\mathbb{R}^2, +)$. For any $a \in \Gamma$, let $a = p + q$ where $p \in \Gamma'$ and $q \in \Gamma$ but not in Γ'. Then define r such that $r(a) = (r_1(p), r_2(q))$ where r_1 is the isomorphism between Γ' and the subgroup of $(\mathbb{R}, +)$, and r_2 is the isomorphism between Γ/Γ' and the subgroup of $(\mathbb{R}, +)$. Let $a = p + q \in \Gamma$ and $b = s + t \in \Gamma$, where $p, s \in \Gamma'$ and $q, t \in \Gamma$ but not in Γ'. Then

$$\begin{aligned} r(a+b) &= (r_1(p+s), r_2(q+t)) \\ &= (r_1(p) + r_1(s), r_2(q) + r_2(t)) \\ &= (r_1(p), r_2(q)) + (r_1(s), r_2(t)) \\ &= r(a) + r(b). \end{aligned}$$

So r is a homomorphism. Now let $r(a) = 0$, we have $r_1(p) = 0$ and $r_2(q) = 0$. Since r_1 and r_2 are both isomorphism, then $p, q = 0$ and $a = 0$. So the kernel of r is trivial and r is an isomorphism.

Now suppose this proposition holds when the rank of Γ is $n > 2$. Let Γ_n be a rank n ordered abelian group which is isomorphic to $(\mathbb{R}^n, +)$ and we have a chain of convex subgroups of Γ such as $\varnothing \subset \Gamma_1 \subset \ldots \subset \Gamma_n$. Let $\Gamma_n \subset \Gamma_{n+1}$, we need to prove that Γ_{n+1} is isomorphic to $(\mathbb{R}^{n+1}, +)$. Consider a mapping $f : \Gamma_{n+1} \to (\mathbb{R}^{n+1}, +)$. For any $a \in \Gamma_n$, $f(a) = (\overline{f}(p), f_n(q))$ where $a = p + q$, $p \in \Gamma_n$ and $q \in \Gamma_{n+1}$ but $q \notin \Gamma_n$. Since Γ_n is isomorphic to $(\mathbb{R}^n, +)$, $\overline{f}$ can be defined as the isomorphism between Γ_n and $(\mathbb{R}^n, +)$. Meanwhile, f_n can be defined as an isomorphism between Γ_{n+1}/Γ_n and $(\mathbb{R}, +)$ because Γ_{n+1}/Γ_n is of rank 1. So let b be another element in Γ_{n+1} and $b = s + t$, then

$$\begin{aligned} f(a+b) &= (\overline{f}(p+s), f_n(q+t)) \\ &= (\overline{f}(p) + \overline{f}(s), f_n(q) + f_n(t)) \\ &= (\overline{f}(p), f_n(q)) + (\overline{f}(s), f_n(t)) \\ &= f(a) + f(b). \end{aligned}$$

And since $\overline{f}$ and f_n are both isomorphism, the kernel of f is trivial which implies that f is an isomorphism. So Γ_{n+1} is isomorphic to $(\mathbb{R}^{n+1}, +)$.

Then by induction, Γ is of rank n if and only if Γ is order isomorphic to a non-trivial subgroup of $(\mathbb{R}^n, +)$.

□

Now define lexicographic ordering of direct product of ordered abelian groups such that let $(\Gamma_1, \preceq_1)$ and $(\Gamma_2, \preceq_2)$ be two ordered abelian groups and $\Gamma = \Gamma_1 \oplus \Gamma_2$, the ordering is defined as

$$(\gamma_1, \gamma_2) \preceq (\gamma_1', \gamma_2') \text{ if either } \gamma_1 \preceq_1 \gamma_1' \text{ or } \gamma_1 = \gamma_1' \text{ and } \gamma_2 \preceq_2 \gamma_2'.$$

The rank of Γ is the sum of the rank of Γ_1 and Γ_2 which we will prove in the following proposition.

Proposition 3.1.6. *Suppose two ordered abelian groups* $(\Gamma_1, \preceq_1)$ *and* $(\Gamma_2, \preceq_2)$ *and define the order of the group* $\Gamma = \Gamma_1 \oplus \Gamma_2$ *to be*

$$(\gamma_1, \gamma_2) \preceq (\gamma_1', \gamma_2') \text{ if either } \gamma_1 \preceq_1 \gamma_1' \text{ or } \gamma_1 = \gamma_1' \text{ and } \gamma_2 \preceq_2 \gamma_2'.$$

Then the rank of Γ *is the sum of the rank of* Γ_1 *and* Γ_2.

Proof. Let the rank of Γ_1 and Γ_2 be n_1 and n_2 respectively. Let the chains of convex subgroups be

$$\varnothing \subset \Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_{n_1-1} \subset \Gamma_1 \text{ and } \varnothing \subset \Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_{n_2-1} \subset \Gamma_2.$$

First, let A, B be convex subgroups of Γ_1, Γ_2 respectively then we need to prove that if $\varnothing \subsetneq A \subsetneq \Gamma_1$ and $\varnothing \subsetneq B \subsetneq \Gamma_2$, then $A \oplus B$ may not be a convex subgroups of Γ. Let $(p, q) \in \Gamma$ and $p \preceq_1 s \in A$, then $(p, q) \preceq (s, t) \in A \oplus B$. But there is a possibility that $q \notin B$ which implies that $(p, q) \notin A \oplus B$. Hence $A \oplus B$ may not be a convex subgroup of Γ.

Now let $K_i = e_1 \oplus \Lambda_i$ and $L_i = \Delta_i \oplus \Gamma_2$, where e_1 is the identity of Γ_1. Then we have a chain of convex subgroups of Γ such that

$$\varnothing \subset K_1 \subset K_2 \subset \ldots \subset e_1 \oplus \Gamma_2 \subset L_1 \subset L_2 \subset \ldots \subset \Gamma.$$

Clearly, this chain of convex subgroups has length $n_1 + n_2$, so the rank of Γ is bigger than or equal to the sum of the rank of Γ_1 and Γ_2.

By proposition 3.1.5, Γ_1 is isomorphic to a non-trivial subgroup of $(\mathbb{R}^{n_1}, +)$ and Γ_2 is isomorphic to a non-trivial subgroup of $(\mathbb{R}^{n_2}, +)$, which we can write as $\Gamma_1 \cong R_1$ and $\Gamma_2 \cong R_2$ where R_1 and R_2 are the subgroups of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively. Then $\Gamma_1 \oplus \Gamma_2 \cong R_1 \oplus R_2$. Since $R_1 \subset \mathbb{R}^{n_1}$ and $R_2 \subset \mathbb{R}^{n_2}$, then the rank of $R_1 \oplus R_2$ is at most as big as $\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$. So the rank of $R_1 \oplus R_2$ is less than or equal to $n_1 + n_2$, so is the rank of $\Gamma_1 \oplus \Gamma_2$. Therefore the rank of Γ is the sum of the rank of Γ_1 and Γ_2. □

3.2 Higher rank valuation

In this section, we will discuss higher rank valuation. The type of valuation which maps a field into the real numbers is called classical valuation. In 1932, Wolfgang Krull extended the definition that let valuations with the values contained by an arbitrary ordered abelian group.

Definition 3.2.1. [2] Let $(\Gamma, +, \leq)$ be a total ordered group and $(\mathbb{G}, \oplus, \odot)$ be the min-plus algebra of Γ. A valuation of of a field k with values in $(\Gamma, +, \leq)$ is a map $\nu \ : \ k \longrightarrow \mathbb{G}$ such that

$$\nu(0) = \infty$$
$$\nu(ab) = \nu(a) + \nu(b)$$
$$\nu(a+b) \geq \min\{\nu(a),\ \nu(b)\},$$

for all $a, b \in k$. We say k has values in Γ. A field together with a valuation is called a valued field and $(\Gamma, +, \leq)$ is called the group of values.

Definition 3.2.1 is from Aroca's paper [2] that defines Krull valuation and we shall consider its rank. In [3], Banerjee gives the definition of the rank of a valuation ν on k which is the rank of the ordered abelian group $\nu(k^{\times})$. In general, the rank of a Krull valuation is the rank of the group of values.

Example 3.2.2. Let the field k be a Puiseux series such that $k = k_{n-1}\{\{t_{n-1}\}\}, k_{n-1} = k_{n-2}\{\{t_{n-2}\}\}, \ldots, k_1 = \mathbb{C}\{\{t_0\}\}$. Let ν be a rank n valuation on k, which is defined as

$$\nu \ : \ k \longrightarrow \mathbb{R}^n \cup \{\infty\}$$

The valued group of ν is a subgroup of $\mathbb{R}^n$, clearly there exists a chain of distinct proper convex subgroups such that $\mathbb{R}^{n-1} \supset \mathbb{R}^{n-2} \supset \ldots \supset \mathbb{R} \supset \{0\}$.

3.3 Tropicalization of polynomials in the rank n case

Definition 3.3.1. Let k be a field and ν is a rank n valuation of k with values in a total ordered group Γ of rank n. Suppose f is a non-zero polynomial in m variables with coefficients in k such that $f \in k[x_1, \ldots, x_m]$ and

$$f = \sum_{\mathbf{u} \in \mathbb{Z}^m} c_{\mathbf{u}} x^{\mathbf{u}}.$$

The rank n tropicalization of f via ν induces an element of $\Gamma[x_1, \ldots, x_m]$

$$\begin{aligned} \operatorname{trop}(f) := & \bigoplus_{\mathbf{u} \in \mathbb{Z}^m} \nu(c_{\mathbf{u}}) + x \cdot \mathbf{u} \\ & = \min_{\mathbf{u} \in \mathbb{Z}^m} \{\nu(c_{\mathbf{u}}) + x \cdot \mathbf{u}\} \end{aligned}$$

Notice that the notation $\bigoplus$ has the same meaning in the ordinary case, which we take the minimum among each term of the polynomial, but ν is rank n so we need to follow the lexicographic ordering when we taking the minimum.

Example 3.3.2. Let ν be a rank 2 valuation on field k such that $\nu : k^{\times} \longrightarrow \Gamma^2_{\text{lex}} \subset \mathbb{R}^2_{\text{lex}}$, and let k be a field of Puiseux series such that $k = k_1\{\{t\}\}$ and $k_1 = \mathbb{C}\{\{s\}\}$. Then suppose a polynomial $f \in k[x, y]$ such that

$$f = (s + t^2 + s^2t^2 + \ldots)x^2y + (s^2 + s^2t + s^2t^3 + \ldots)y^2 + (s + st + s^3t + \ldots)xy.$$

Then we have

$$\begin{aligned} \nu(s + t^2 + s^2t^2 + \ldots) &= (0, 1) \\ \nu(s^2 + s^2t + s^2t^3 + \ldots) &= (0, 2) \\ \nu(s + st + s^3t + \ldots) &= (0, 1) \end{aligned}$$

and the tropicalization of f via ν is

$$\operatorname{trop}_{\nu}(f) = (0, 1) + 2x + y \oplus (0, 2) + 2y \oplus (0, 1) + x + y$$

Now fix a weight vector $\mathbf{w} \in (\Gamma^2)^2$ such that $\mathbf{w} = (x, y) = ((1,3),(0,2))$, then we have

$$\begin{aligned}\operatorname{trop}(f)(\mathbf{w}) &= \min\{(0,1) + 2(1,3) + (0,2), (0,2) + 2(0,2), (0,1) + (1,3) + (0,2)\} \\ &= \min\{(2,9),(0,6),(1,6)\} \\ &= (0,6)\end{aligned}$$

3.4 Initial forms with higher rank valuations

Let ν be a rank n valuation on k with a splitting σ, the tropicalization of polynomial $f = \sum_{\mathbf{u} \in \mathbb{Z}^m} c_{\mathbf{u}} x^{\mathbf{u}} \in k[x_1, \dots, x_m]$ is $\operatorname{trop}(f) = \min_{\mathbf{u} \in \mathbb{Z}^m}\{\nu(c_{\mathbf{u}}) + x \cdot \mathbf{u}\}$. Fixing a weight vector $\mathbf{w} \in (\Gamma^m)^n$ then we have $W = \operatorname{trop}(f)(\mathbf{w}) = \min_{\mathbf{u} \in \mathbb{Z}^m}\{\nu(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u}\}$. Similar to definition 2.5.1, the initial form of f with respect to $\mathbf{w}$ via rank n valuation ν is

$$\operatorname{in}^{\nu}_{\mathbf{w}}(f) = \sum_{\mathbf{u} \in \mathbf{N}^m} \overline{\sigma(\mathbf{w} \cdot \mathbf{u} - W) \cdot c_{\mathbf{u}}} x^{\mathbf{u}}$$

where σ is the splitting of ν.

Example 3.4.1. Let ν be a rank 2 valuation on a field of Puiseux series $k = k_1\{\{t\}\}$ where $k_1 = \mathbb{C}\{\{s\}\}$. Then $\nu : k^\times \to \Gamma^2_{\text{lex}} \subset \mathbb{R}^2_{\text{lex}}$, which we define that the first coordinate of ν is the image of t-adic valuation on k and the second coordinate of ν is the image of s-adic valuation on k_1. Suppose a polynomial $f = (ts^2 + t^2 s)x_1 + (t^3 s^2 + t^4 s)x_2 + t^2 s x_3^2 \in k[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}]$, and let the weight vector be $\mathbf{w} = ((1,1),(1,1),(0,0)) \in (\Gamma^3)^2$. Then we have

$$\begin{aligned}\mathbf{W} = \operatorname{trop}(f)(\mathbf{w}) &= \min\{\nu(ts^2 + t^2 s) + (1,1), \nu(t^3 s^2 + t^4 s) + (1,1), \nu(t^2 s)\} \\ &= \min\{(1,2) + (1,1), (3,2) + (1,1), (2,1)\} \\ &= \min\{(2,3),(4,3),(2,1)\}.\end{aligned}$$

By lexicographical order, $\mathbf{W} = (2,1)$. Hence the initial form of f with respect to $\mathbf{w}$ is

$$\operatorname{in}^{\nu}_{\mathbf{u}}(f) = \overline{\sigma(-(2,1)) t^2 s} \cdot x_3^2$$

where σ is the splitting of ν. Then

$$\text{in}^{\nu}_{\mathbf{u}}(f) = x_3^2.$$

3.5 Higher rank tropical varieties

Definition 3.5.1. Let ν be a rank n valuation on field k such that $\nu : k \to \Gamma \cup \{\infty\}$, Γ is an ordered abelian group of rank n. Let $f \in k[x_1^{\pm}, \ldots, x_m^{\pm}]$, then the rank n tropical hypersurface $\mathcal{V}(\text{trop}_{\nu}(f))$ associated to tropical polynomial $\text{trop}_{\nu}(f)$ is the following set

$$\{\mathbf{w} \in (\Gamma^m)^n \ : \ \text{the minimum in } F(\mathbf{w}) \text{ is achieved at least twice}\}.$$

Definition 3.5.2. Let $f \in k[x_1^{\pm}, \ldots, x_m^{\pm}]$ and ν be a rank n valuation on k. Then the rank n tropicalization of the hypersurface of f is

$$\text{trop}_{\nu}(V(f)) = \{\nu(\mathbf{y}) : \mathbf{y} \in V(f)\},$$

which is the set of coordinate-wise valuations of points in $V(f)$.

In chapter 2, we introduce Kapranov's theorem when the valuation is classical. In [2], Fuensanta Aroca proves that in an algebraically closed field k with a surjective rank n valuation ν, the rank n tropical hypersurface associatied to a polynomial f is equal to the hypersurface associated to the rank n tropicalization of f such that

$$\text{trop}_{\nu}(V(f)) = \mathcal{V}(\text{trop}_{\nu}(f)) \tag{3.1}$$

The left hand side of the equation above is the rank n tropical hypersurface associated to f, which is called the rank n tropicalization of the hypersurface assoicated to f in [2]. More specific, it is the set of coordinatewise valuations of $V(f)$ such that

$$\{(\nu(y_1), \ldots, \nu(y_m)) : (y_1, \ldots, y_m) \in V(f)\},$$

The right hand side is the hypersurface associated to the rank n tropicalization of f which is the tropical hypersurface assoicaited to the rank n tropicalization of f in this paper. So it is equivalent to

$$\{\mathbf{w} \in (\Gamma^m)^n : \text{trop}_\nu(f)(\mathbf{w}) \text{ is not a monomial}\}.$$

Based on that, in [3], Soumya Banerjee showed that a rank n tropicalization of a d-dimensional variety is a polyhedral complex of dimension nd. In addition, a variety X is connected implies that the tropicalization of X is connected too when the valuation is of rank 1. Soumya left a question that if it is also true in rank n case in [3]. In [4], Tyler Foster and Dhruv Ranganathan proved that trop(X) is connected if X is connected holds when the valuation is of rank $n \geq 1$ as a corollary of the theory of analytic spaces over a higher rank valued field.

3.6 Going from rank n valuation to n-step valuation

In [3], Soumya Banerjee introduced a definition of higher dimensional local field over a field k which is an ordered sequence of fields $(\mathbb{K}_{(0)}, \mathbb{K}_{(1)}, \ldots, \mathbb{K}_{(n-1)}, \mathbb{K}_{(n)})$ such that

1. Each $\mathbb{K}_{(i)}$ is a local field with respect to a discrete rank 1 valuation $\nu_{(i)} : \mathbb{K}_{(i)}^{\times} \to \mathbb{Z}$, for all $1 \leq i \leq n$.
2. $\mathbb{K}_{(i+1)}$ is the residue field of $(\mathbb{K}_{(i)}, \nu_{(i)})$ for all $i \geq 0$.
3. $\mathbb{K}_{(0)} = k$.

This definition lists a sequence of local fields with a sequence of discrete rank 1 valuation correspondingly. Our interest is that can we reduce a rank n valuation to the sequence of valuations corresponding to n-dimensional local field? So in this section, we will focus on this question and define n-step valuation.

For defining an n-step valuation, we need to split a rank n valuation step by step. The following proposition is the important tool which will show us how to reduce a rank n valuation.

Proposition 3.6.1. *Let ν be a rank n valuation on field k. Let $\nu = (\nu_1, \nu_2) : k \to \mathbb{R}^n_{\text{lex}} \cup \{\infty\}$ where ν_1 is the first component and ν_2 is the remaining $n-1$ components such that $\nu_1 : k \to \mathbb{R} \cup \{\infty\}$ and $\nu_2 : k \to \mathbb{R}^{n-1}_{\text{lex}} \cup \{\infty\}$. Clearly, ν_1 is a valuation and we let R be the valuation ring of ν_1, $\mathfrak{m}$ is the maximal ideal, $\mathbb{K}$ is the residue field. Now restricting the domain of ν_2 to be R such that $\nu_2 \mid_R : R \to \mathbb{R}^{n-1}_{\text{lex}} \cap \{\infty\}$. Then there is a map $\overline{\nu}_2 : \mathbb{K} \longrightarrow \mathbb{R}^{n-1}_{\text{lex}} \cup \infty$ as the following diagram*

$$\begin{array}{ccc} R & \xrightarrow{\nu_2|_R} & \mathbb{R}^{n-1}_{\text{lex}} \cup \infty \\ \downarrow & \overset{\overline{\nu}_2}{\nearrow} & \\ \mathbb{K} & & \end{array}$$

and the map $\overline{\nu}_2$ is a rank $n-1$ valuation on $\mathbb{K}$.

Proof. For showing that restricting the domain of ν_2 to R, let $a \in \mathfrak{m}$ and $r \in R$ with $\nu_1(r) = 0$. Then consider $\nu_2(r+a)$, we have

$$\begin{aligned} \nu_2(r+a) &= \nu_2(r(1+\frac{a}{r})) \\ &= \nu_2(r) + \nu_2(1+\frac{a}{r}). \end{aligned}$$

Since $a \in \mathfrak{m}$, then $\nu_1(a) > 0$ and we let $\nu_1(r) = 0$. This implies that $\nu_1(\frac{a}{r}) = \nu_1(a) - \nu_1(r) > 0$, so $\frac{a}{r} \in \mathfrak{m}$. Since $\nu(1+\frac{a}{r}) = \min\{0, \nu(\frac{a}{r})\}$ and $\nu_1(\frac{a}{r}) > 0$, then $\nu(1+\frac{a}{r}) = 0$, which implies that $\nu_2(1+\frac{a}{r}) = 0$. Therefore $\nu_2(r+a) = \nu_2(r)$, and this shows that ν_2 descends to a well defined map $\overline{\nu}_2$ on the residue field $\mathbb{K}$.

Now we show that $\overline{\nu}_2$ is a valuation on $\mathbb{K}$. It is obvious that $\overline{\nu}_2(0) = \infty$ and $\overline{\nu}_2(ab) = \overline{\nu}_2(a) + \overline{\nu}_2(b)$. Now we should prove $\overline{\nu}_2$ satisfying the third property of valuation which is the inequality $\overline{\nu}_2(a+b) \geq \min(\overline{\nu}_2(a), \overline{\nu}_2(b))$. Let the corresponding elements of a and b in R be $\tilde{a}$ and $\tilde{b}$, then $\nu_1(\tilde{a}) = 0$ and $\nu_1(\tilde{b}) = 0$. Consider the inequality

$$\begin{aligned} \nu(\tilde{a}+\tilde{b}) &\geq \min\{\nu(\tilde{a}), \nu(\tilde{b})\} \\ (\nu_1(\tilde{a}+\tilde{b}), \nu_2(\tilde{a}+\tilde{b})) &\geq \min\{(\nu_1(\tilde{a}), \nu_2(\tilde{a})), (\nu_1(\tilde{b}), \nu_2(\tilde{b}))\} = (0, \min\{\nu_2(\tilde{a}), \nu_2(\tilde{b})\}), \end{aligned}$$

so we have two possibilities such that

$$\begin{gathered} \nu_1(\tilde{a}+\tilde{b}) = 0 \implies \nu_2(\tilde{a}+\tilde{b}) \geq \min\{\nu_2(\tilde{a}), \nu_2(\tilde{b})\} \\ \text{or} \\ \nu_1(\tilde{a}+\tilde{b}) > 0 \implies \exists \nu_2(\tilde{a}+\tilde{b}) < \min\{\nu_2(\tilde{a}), \nu_2(\tilde{b})\}. \end{gathered}$$

If $a+b = 0$ in $\mathbb{K}$, then $\overline{\nu}_2(a+b) = \infty$ which means it is greater than anything, so $\overline{\nu}_2(a+b) \geq \min(\overline{\nu}_2(a), \overline{\nu}_2(b))$.

If $a+b \neq 0$, we suppose the homompohism $R \longrightarrow \mathbb{K}$ to be π, then $\pi(\tilde{a}+\tilde{b}) = \pi(\tilde{a}) + \pi(\tilde{b}) = a+b$. Since $\mathbb{K} = R/\mathfrak{m}$, we can write $\tilde{a}$ and $\tilde{b}$ as $\tilde{a} = a_0 + a_1$ and $\tilde{b} = b_0 + b_1$ respectively, where $\nu_1(a_0) = \nu_1(b_0) = 0$ and $a_1, b_1 \in \mathfrak{m}$. Then $a+b \neq 0$ implies $a_0 \neq -b_0$, because $\pi(a_1) = \pi(b_1) = 0$ in $\mathbb{K}$. If $a_0 = -b_0$ then $a+b = 0$ in $\mathbb{K}$, which contradicts. Therefore

$$\nu_1(\tilde{a}+\tilde{b}) = \nu_1((a_0+b_0)+(a_1+b_1)) \geq \min\{\nu_1(a_0+b_0), \nu_1(a_1+b_1)\},$$

since $a_0, b_0 \in \mathbb{K}$ and $a_1, b_1 \in \mathfrak{m}$, $\nu_1(a_0+b_0) = 0$ and $\nu_1(a_1+b_1) > 0$. By lemma 2.2.2, we have $\nu_1(\tilde{a}+\tilde{b}) = 0$ which implies $\nu_2(\tilde{a}+\tilde{b}) \geq \min\{\nu_2(\tilde{a}), \nu_2(\tilde{b})\}$. Since $\overline{\nu}_2$ is restricted from ν_2, then we shall gain $\overline{\nu}_2(a+b) \geq \min\{\overline{\nu}_2(a), \overline{\nu}_2(b)\}$ and $\overline{\nu}_2$ is a valuation on $\mathbb{K}$. □

Base on proposition 3.6.1, we shall reduce a rank n valuation step by step and gain a set of valuations of rank 1 finally. Let ν be a rank n valuation on field k, then we have $\nu = \nu_1 \times \nu_2 \ : \ k^{\times} \longrightarrow \mathbb{R} \times \mathbb{R}^{n-1}$. As the proof of proposition 3.6.1, ν_1 is a valuation on k but ν_2 is not. Then we restrict ν_2 to the valuation ring of ν_1 and proposition 3.6.1 shows that it descends to a rank $n-1$ valuation $\overline{\nu}_2$ on the residue field k_1 of ν_1. Then repeating the process, we have $\overline{\nu}_2 = \nu_2 \times \nu_3 \ : \ {k_1}^{\times} \longrightarrow \mathbb{R} \times \mathbb{R}^{n-2}$, and ν_2 is a valuation on k_1, while ν_3 is restricted to the valuation ring of ν_2 and descends to a rank $n-2$ valuation $\overline{\nu}_3$ on the residue field k_2 of ν_2. Et cetera.

Inductively, for each valuation $\overline{\nu}_i$ in the process above for $1 < i < n$, we have

$$\overline{\nu}_i = \nu_i \times \nu_{i+1} \ : \ k_{i-1}^{\times} \longrightarrow \mathbb{R} \times \mathbb{R}^{n-i}$$

ν_i is a valuation on field k_{i-1}, let R_i be the valuation ring of ν_i, then k_i is the residue field of ν_i. By proposition 3.6.1, restrict ν_{i+1} to R_i and it descends to a rank $n-i$ valuation on k_i which can be denoted by $\overline{\nu}_{i+1}$ as the following diagram.

$$\begin{array}{ccc} R_i & \xrightarrow{\ \nu_{i+1}|_{R_i}\ } & \mathbb{R}^{n-i} \cup \{\infty\} \\ \downarrow & \overset{\overline{\nu}_{i+1}}{\nearrow} & \\ k_i & & \end{array}$$

Then by induction, there exists a sequence of rank 1 valuations ν_i for $1 \leq i \leq n$. In order not to confuse, let's call these rank 1 valuations v_i for $1 \leq i \leq n$ such that

$$\begin{aligned} v_1 \ &: k \longrightarrow \mathbb{R} \cup \{\infty\} \\ v_2 \ &: k_1 \longrightarrow \mathbb{R} \cup \{\infty\} \\ &\cdots\cdots \\ v_{n-1} \ &: k_{n-2} \longrightarrow \mathbb{R} \cup \{\infty\} \\ v_n \ &: k_{n-1} \longrightarrow \mathbb{R} \cup \{\infty\}. \end{aligned}$$

And we shall define this set of rank 1 valuation as an n-step valuation on field k.

Definition 3.6.2. Let k be a field. An n-step valuation on k is a sequence of rank 1 valuations such that $(v_1, v_2, \dots, v_n)$ where each v_i is a rank 1 valuation on the residue field k_{i-1} of v_{i-1} for all $i > 1$, and v_1 is a rank 1 valuation on k.

In this thesis, to distinguish rank n and n-step valuation, we are going to use ν and $(v_1, \dots, v_n)$ to represent the rank n and n-step valuation respectively.

The following example illustrates Proposition 3.6.1 in action.

Example 3.6.3. Let $k = k_1\{\{t\}\}$ and $k_1 = \mathbb{C}\{\{s\}\}$ and we define a map ν on k such that $\nu : k^\times \to \mathbb{R}^2$ with the lexicographical order. Suppose P be a Puiseux series in k. Then $\nu(P)$ is given by

$$\nu(P) = (\alpha, \beta)$$

where α is the lowest exponent of t in the series P and β is the exponent of s of the coefficient of the leading term.

Let P_1, P_2 be two arbitrary Puiseux series in k that can be written as

$$P_1 = c_1 t^{\alpha_1} s^{\beta_1} + c_2 t^{\alpha_2} s^{\beta_2} + c_3 t^{\alpha_3} s^{\beta_3} + \dots$$
$$P_2 = d_1 t^{\lambda_1} s^{\mu_1} + d_2 t^{\lambda_2} s^{\mu_2} + d_3 t^{\lambda_3} s^{\mu_3} + \dots,$$

$c_i, d_i \in \mathbb{C}$ and $\alpha_i, \beta_i, \lambda_i, \mu_i \in \mathbb{Q}$ for all i. Clearly, $\nu(P_1) = (\alpha_1, \beta_1)$ and $\nu(P_2) = (\lambda_1, \mu_1)$, then we shall consider the case of the product of P_1 and P_2.

$$\nu(P_1P_2) = \nu(c_1 d_1 t^{\alpha_1+\lambda_1} s^{\beta_1+\mu_1} + \dots) = (\alpha_1 + \lambda_1, \beta_1 + \mu_1) = \nu(P_1) + \nu(P_2).$$

So from the equation above we gain that $\nu(P_1P_2) = \nu(P_1) + \nu(P_2)$ which satisfies the second axiom of valuation. And now we need to check the case of $P_1 + P_2$.

$$\nu(P_1 + P_2) = \nu(c_1 t^{\alpha_1} s^{\beta_1} + d_1 t^{\lambda_1} s^{\mu_1} + \dots).$$

By the given definition at the beginning of example 3.6.3, the first coordinate of $\nu(P_1 + P_2)$ is the lowest exponent of t and the second coordinate of $\nu(P_1 + P_2)$ is the exponent of s of the leading term. Hence we have two situations

1. if $c_1 + d_1 \neq 0$, then the first coordinate is $\min\{\alpha_1, \lambda_1\}$ and the second coordinate is the power of s in the term with the power of t be $\min\{\alpha_1, \lambda_1\}$.

2. if $c_1 + d_1 = 0$ the term with lowest power of t becomes 0, so we shall find the lowest power of t among the rest terms.

In case 1, assume the first coordinate of $\nu(P_1 + P_2)$ is α_1 then

$$\nu(P_1 + P_2) = (\alpha_1, \beta_1) = \min_{\text{lex}}\{(\alpha_1, \beta_1), (\lambda_1, \mu_1)\} = \min_{\text{lex}}\{\nu(P_1), \nu(P_2)\}.$$

It is the same when $\nu(P_1 + P_2) = (\lambda_1, \mu_1)$.

In case 2, let the lowest power of t among the rest terms be ρ and let the power of s in this term be ω. It is obvious that

$$\nu(P_1 + P_2) = (\rho, \omega) > \min_{\text{lex}}\{(\alpha_1, \beta_1), (\lambda_1, \mu_1)\} = \min_{\text{lex}}\{\nu(P_1), \nu(P_2)\}.$$

We can see that for any $P_1, P_2 \in k$, we have $\nu(P_1 + P_2) \geq \min_{\text{lex}}\{\nu(P_1), \nu(P_2)\}$. Then ν satisfies the axioms of valuation, and notice that $\nu : k \to \mathbb{R}^2$ where $\varnothing \subset \mathbb{R} \subset \mathbb{R}^2$ has rank 2, so ν is a rank 2 valuation. Now by proposition 3.6.1, we can reduce ν as (v_1, v_2) and follow the process we have shown in proposition 3.6.1, we shall see that v_1 is a t-adic valuation on k and v_2 is a s-adic valuation on k_1.

Example 3.6.4. Let $k = k_1\{\{t\}\}$ and $k_1 = \mathbb{C}\{\{s\}\}$, let P_1, P_2 be two explicit puiseux series such as

$$\begin{aligned} P_1 &= -2ts + 3t^2s + 5t^3s + \ldots \\ P_2 &= 2ts + 4ts^3 + 3t^2s^3 + \ldots. \end{aligned}$$

Define ν to be a rank 2 valuation on k as example 3.6.3, so we have $\nu(P_1) = (1, 1)$ and $\nu(P_2) = (1, 1)$ then we have

$$\nu(P_1P_2) = \nu(-4t^2s^2 - 8t^2s^4 + 6t^3s^2 + \ldots) = (2, 2) = \nu(P_1) + \nu(P_2)$$

and

$$\nu(P_1 + P_2) = \nu(4ts^3 + 3t^2s + 3t^2s^3 + 5t^3s + \ldots) = (1, 3) > \min\{\nu(P_1), \nu(P_2)\}.$$

By proposition 3.6.1, ν can be reduced as (v_1, v_2), where v_1 is a t-adic valuation on k and v_2 is a s-adic valuation on the residue field k_1 of v_1.

Proposition 3.6.5. *Let ν be a rank n valuation such that $\nu : k \longrightarrow \Gamma \cup \infty$ where $\Gamma = \mathbb{R}^n$. By proposition 3.6.1, ν can be induced as (ν, ν_2) where $\nu_1 : k \longrightarrow \mathbb{R} \cup \infty$, and ν_2 will be restricted to the valuation ring of ν_1 and descends to a rank $n-1$ valuation on the residue field k_1 of ν_1 such that $\overline{\nu}_2 : k_1 \longrightarrow \mathbb{R}^{n-1} \cup \infty$. Suppose σ is a splitting of ν; then this induces splittings of ν_1 and $\overline{\nu}_2$.*

Proof. Let $(a, b) \in \Gamma$, then $\sigma(a, b) \in k$. Consider $\nu(\sigma(a, 0)) = (a, 0)$, since $\nu_1(\sigma(a, 0)) = a$ then we shall define σ_1 to be the splitting of ν_1 by $\sigma_1(a) = \sigma(a, 0)$ which induces the splitting of ν_1 such that $\nu_1(\sigma_1(a)) = a$ for any $a \in k$. Now consider $\nu(\sigma(0, b)) = (0, b)$, it is clear that $\nu_1(\sigma(0, b)) = 0$, then $\sigma(0, b) \in k_1 \subset R$ where k_1 is the residue field of ν_1 and R is the valuation ring of ν_1. Then we can define that $\sigma_2(b) = \overline{\sigma(0, b)}$ where $\overline{\nu}_2(\overline{\sigma(0, b)}) = b$ and the notation $\overline{\sigma(0, b)}$ means we take the image of $\sigma(0, b)$ in the residue field k_1 of ν_1, since $\overline{\nu}_2$ descends to a rank $n-1$ valuation on k_1. □

Chapter 4

The Main Theorem

In Chapter 3, we introduced rank n valuations and reviewed some work on it. By proposition 3.6.1, we show how to reduce a rank n valuation to a sequence of rank 1 valuations and we define this sequence of valuation to be an n-step valuation. So this Chapter will focus on higher rank tropicalization, as the strategy in Chapter 3, we will split higher rank tropicalization of a polynomial as a sequence of tropical polynomials which we will define it to be n-step tropicalization of the polynomial. In section 2 of this Chapter, we will show the equivalence between rank n and n-step tropicalization of the hypersurface associated to a polynomial, and with this equivalence there is an alternative method to prove Kapranov's Theorem in higher rank version.

4.1 n-step tropicalization of hypersurface associated to a polynomial

Recalling definition 3.6.2, an n-step valuation $(v_1, \ldots, v_n)$ on k is a sequence

$$\begin{aligned} &v_1 \text{ is a rank 1 valuation on } k \\ &v_2 \text{ is a rank 1 valuation on the residue field } k_1 \text{ of } v_1 \\ &\vdots \\ &v_n \text{ is a rank 1 valuation on the residue field } k_{n-1} \text{ of } v_{n-1}. \end{aligned}$$

Before we introducing the following tropicalization, we need to clear up some notations we will use. In Chapter 2 and 3, we used to use $\bar{a}$ to represent the image of $a \in k$ in the residue field $\mathbb{K}$ of the valuation on k. But in this Chapter, there are more one valuation ring and residue field, so in order not to confuse, we need to some new notations. First let the valuation ring of v_i be R_i, correspondingly, the residue field of ν_i is k_i for all $1 \leq i \leq n$. Similarly, the valuation ring and residue field of ν are denoted by R and $\mathbb{K}$. Now define that π and π_i are the projections from these valuation rings to these residue fields such as

$$\begin{array}{ccc} \nu : k \to \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_n & \sigma : \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_n \to k & \pi : R \to \mathbb{K} \\ v_1 : k \to \Gamma_1 & \sigma_1 : \Gamma_1 \to k & \pi_1 : R_1 \to k_1 \\ v_2 : k_1 \to \Gamma_2 & \sigma_2 : \Gamma_2 \to k_1 & \pi_2 : R_2 \to k_2 \\ \vdots & \vdots & \vdots \\ v_n : k_{n-1} \to \Gamma_n & \sigma_n : \Gamma_n \to k_{n-1} & \pi_n : R_n \to k_n \end{array}$$

Note that $\mathbb{K} = k_n$. Based on that let's consider the following sequence of tropical polynomials:

Suppose a polynomial $f = \sum_{\mathbf{u} \in \mathbb{Z}^m} c_{\mathbf{u}} x^{\mathbf{u}} \in k[x_1^{\pm}, \ldots, x_m^{\pm}]$ and fix a sequence of weight vectors $(\mathbf{w}_1, \ldots, \mathbf{w}_n)$ where $\mathbf{w}_i \in \Gamma^m$, for all $1 \leq i \leq n$. Then it is easy to calculate the tropicalization of f via v_1 with respect to $\mathbf{w}_1$ such that

$$\operatorname{trop}_{v_1}(f)(\mathbf{w}_1) = \bigoplus_{\mathbf{u} \in \mathbb{Z}^m} v_1(c_{\mathbf{u}}) + \mathbf{w}_1 \cdot \mathbf{u}.$$

By the definition of initial form we shall determine the initial form $\operatorname{in}_{\mathbf{w}_1}^{v_1}(f)$ of f with respect to $\mathbf{w}_1$ easily such that

$$\operatorname{in}_{\mathbf{w}_1}^{v_1}(f) = \sum_{v_1(c_{\mathbf{u}}) + \mathbf{w}_1 \cdot \mathbf{u} = \mathbf{W}_1} \pi_1(\sigma_1(-v_1(c_{\mathbf{u}}))c_{\mathbf{u}}) x^{\mathbf{u}}$$

where $\mathbf{W}_1 = \operatorname{trop}_{v_1}(f)(\mathbf{w}_1) = \min\{v_1(c_{\mathbf{u}}) + \mathbf{w}_1 \cdot \mathbf{u}\}$ and σ_1 is the splitting of v_1 such that $\sigma_1 : \Gamma_{v_1} \longrightarrow k^{\times}$. Notice that $\operatorname{in}_{\mathbf{w}_1}^{v_1}(f) \in k_1[x_1^{\pm}, \ldots, x_m^{\pm}]$ and v_2 is rank 1 valuation on k_1. Then we shall determine the tropicalization of $\operatorname{in}_{\mathbf{w}_1}^{v_1}(f)$ via v_2 with respect to $\mathbf{w}_2$

$$\operatorname{trop}_{v_2}(\operatorname{in}_{\mathbf{w}_1}^{v_1}(f))(\mathbf{w}_2) = \bigoplus_{\mathbf{u} \in \mathbb{Z}^m} v_2(\pi_1(\sigma_1(-v_1(c_{\mathbf{u}}))c_{\mathbf{u}})) + \mathbf{w}_2 \cdot \mathbf{u}.$$

For simplicity, let $\pi_1(\sigma_1(-v_1(c_{\mathbf{u}}))c_{\mathbf{u}}) = d_{\mathbf{u}}$, then the initial form of $\operatorname{in}_{\mathbf{w}_1}^{v_1}(f)$ with respect to $\mathbf{w}_2$ is

$$\operatorname{in}_{\mathbf{w}_2}^{v_2}(\operatorname{in}_{\mathbf{w}_1}^{v_1}(f)) = \sum_{v_2(d_{\mathbf{u}}) + \mathbf{w}_2 \cdot \mathbf{u} = \mathrm{W}_2} \pi_2(\sigma_2(-v_2(d_{\mathbf{u}}))d_{\mathbf{u}}) x^{\mathbf{u}}.$$

So if we keep repeating the process above we shall gain a sequence of tropical polynomials, and each one is the tropicalization of the previous initial form via the corresponding valuation. Then the sequence of tropical polynomials we described above is defined to be an n-step tropicalization of f via the n-step valuation $(v_1, \ldots, v_n)$ in this paper. Hence we shall give the definition of n-step tropicalization.

Definition 4.1.1. Let $(v_1, \ldots, v_n)$ be an n-step valuation on field k and $f \in k[x_1^{\pm}, \ldots, x_n^{\pm}]$. Fixing a sequence of weight vectors $(\mathbf{w}_1, \ldots, \mathbf{w}_{n-1})$ where $\mathbf{w}_i \in \mathbb{R}^m$ for all $1 \leq i \leq n-1$. Then the n-step tropicalization of f at $(\mathbf{w}_1, \ldots, \mathbf{w}_{n-1})$ is the sequence of tropical polynomials such that

$$\begin{aligned}
&\operatorname{trop}_{v_1}(f) \\
&\operatorname{trop}_{v_2}(\operatorname{in}_{\mathbf{w}_1}^{v_1}(f)) \\
&\cdots\cdots \\
&\operatorname{trop}_{v_n}(\operatorname{in}_{\mathbf{w}_{n-1}}^{v_{n-1}} \ldots \operatorname{in}_{\mathbf{w}_1}^{v_1}(f)).
\end{aligned}$$

Now we have the definition of n-step topicalization, and recall definition 2.6.4 about ordinary tropical hypersurface. It's easy to define the n-step tropical hypersurface of a polynomial directly.

Definition 4.1.2. Let $(v_1, \ldots, v_n)$ be an n-step valuation on field k and $f \in k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$. Then the **$n$-step tropicalization of the hypersurface associated to** f via $(v_1, \ldots, v_n)$ is the set of all weight vectors $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_n) \in \mathbb{R}^{m \times n}$ where $\mathbf{w}_1$ is a point in the tropical hypersurface of $\operatorname{trop}_{v_1}(f)$ and for each $1 < i \leq n$, $\mathbf{w}_i$ is a point in the tropical hypersurface of the tropical polynomial $\operatorname{trop}_{v_i}(\operatorname{in}_{\mathbf{w}_{i-1}}^{v_{i-1}} \ldots \operatorname{in}_{\mathbf{w}_1}^{v_1}(f))$.

Example 4.1.3. Let k be a field of Puiseux series such that $k = k_1\{\{t\}\}$ and $k_1 = \mathbb{C}\{\{s\}\}$, ν is a rank 2 valuation on k which can be reduced as (v_1, v_2) where v_1 is a t-adic valuation on k and v_2 is an s-adic valuation on the residue field of v_1. Suppose a polynomial $f = sx - s^2y + 4 \in k[x^{\pm 1}, y^{\pm 1}]$ then the hypersurface of f is

$$\{(sz - 4s^{-1}, z) : z \in k\}.$$

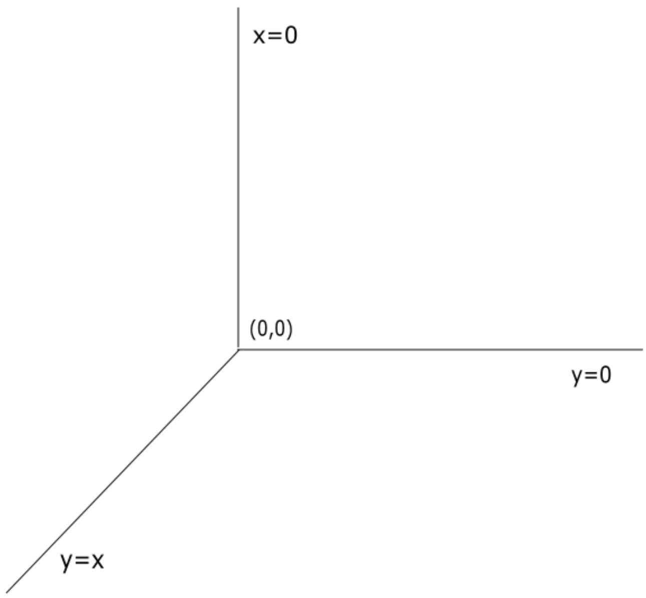
The tropicalization of f via v_1 is

$$\text{trop}_{v_1}(f) = \min\{x, y, 0\},$$

and the tropical hypersurface of $\text{trop}_{v_1}(f)$ are

$$\{(0, \lambda_1), (\lambda_2, 0), (\lambda_3, \lambda_3), (0, 0) \ : \ \lambda_1 > 0, \lambda_2 > 0, \lambda_3 < 0 \text{ and } \lambda_1, \lambda_2, \lambda_3 \in R_1\},$$

where R_1 is the valuation ring of v_1, and the tropical hypersurface of $\text{trop}_{v_1}(f)$ is the following dagram.



Clearly, the weight vectors which let $\text{trop}_{v_1}(f)$ contains at least two terms form the diagram above. Therefore we shall choose the weight vectors as $\mathbf{w}_1 = (0, 1)$, $\mathbf{w}_2 = (1, 0)$, $\mathbf{w}_3 = (-1, -1)$ and $\mathbf{w}_4 = (0, 0)$ and we have the initial forms with respect to the corresponding $\mathbf{w}_i$ for $1 \leq i \leq 4$ as the following

$$\begin{aligned}
\text{in}^{v_1}_{\mathbf{w}_1}(f) &= sx + 4 \\
\text{in}^{v_1}_{\mathbf{w}_2}(f) &= -s^2y + 4 \\
\text{in}^{v_1}_{\mathbf{w}_3}(f) &= sx - s^2y \\
\text{in}^{v_1}_{\mathbf{w}_4}(f) &= sx - s^2y + 4.
\end{aligned}$$

Then for each initial form we shall gain the tropicalization of $\text{in}_{\mathbf{w}_i}(f)$ via v_2 which are

$$\begin{aligned}
\text{trop}_{v_2}(\text{in}^{v_1}_{\mathbf{w}_1}(f)) &= \min\{1+x, 0\} \\
\text{trop}_{v_2}(\text{in}^{v_1}_{\mathbf{w}_2}(f)) &= \min\{2+y, 0\} \\
\text{trop}_{v_2}(\text{in}^{v_1}_{\mathbf{w}_3}(f)) &= \min\{1+x, 2+y\} \\
\text{trop}_{v_2}(\text{in}^{v_1}_{\mathbf{w}_4}(f)) &= \min\{1+x, 2+y, 0\}.
\end{aligned}$$

Then we need to find those weight vectors which let the tropicalization of $\text{in}^{v_1}_{\mathbf{w}_i}(f)$ via v_2 contain at least two terms. Let those weight vectors be $\hat{\mathbf{w}}_i$ for $1 \leq i \leq 4$, then we shall see that $\hat{\mathbf{w}}_1$ can be picked form the line $(-1, \mu_1)$, $\hat{\mathbf{w}}_2$ can be chosen form $(\mu_2, -2)$, $\hat{\mathbf{w}}_3$ can be chosen form $(\mu_3 + 1, \mu_3)$ and $\hat{\mathbf{w}}_4$ is the point $(-1, -2)$ where μ_1, μ_2 and μ_3 can be any complex number. Therefore the sets of the weight vectors are

$$\begin{aligned}
&\{\mathbf{w} = ((0,-1),(\lambda_1, \mu_1)) \;:\; \lambda_1 > 0,\ \lambda \in R_1,\ \mu_1 \in R_2\} \\
&\{\mathbf{w} = ((\lambda_2, \mu_2),(0,-2)) \;:\; \lambda_2 > 0,\ \lambda_2 \in R_1\ \mu_2 \in R_2\} \\
&\{\mathbf{w} = ((\lambda_3, \mu_3+1),(\lambda_3, \mu_3)) \;:\; \lambda_3 < 0,\ \lambda_3 \in R_1,\ \mu_3 \in R_2\} \\
&\{\mathbf{w} = ((0,-1),(0,2))\}.
\end{aligned}$$

and the union of these sets are the 2-step tropicalization of the hypersurface associated of f with respect to the 2-step valuation (v_1, v_2).

4.2 Equivalence of rank n and n-step tropicalization

In chapter 3, we have introduced how to construct an n-step valuation from a rank n valuation by proposition 3.6.1. In the previous section, we define the n-step tropicalization and the tropical hypersurface of polynomial with n-step valuation. It is natural to conjecture that there exists an equivalence between rank n and n-step tropicalization. So in this section, our purpose is to find the equivalence between them.

Proposition 4.2.1. *Let ν be a rank n valuation on field k and $f \in k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$. By proposition 3.6.1, we shall induce ν as $(\nu_1, \overline{\nu}_2)$ where*

$$\nu_1 \ : \ k^* \longrightarrow \mathbb{R}$$
$$\overline{\nu}_2 \ : \ k_1^* \longrightarrow \mathbb{R}^{n-1},$$

k_1 is the residue field of ν_1. Let σ be a splitting of ν, according to proposition 3.6.5, σ can be induced splittings of ν_1 and $\overline{\nu}_2$ which are supposed to be σ_1 and σ_2 such that

$$\sigma_1 \ : \ \mathbb{R} \longrightarrow k^*$$
$$\sigma_2 \ : \ \mathbb{R}^{n-1} \longrightarrow k_1^*.$$

Fixing a set of weight vectors $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2) \in \Gamma_1^m \times \Gamma_2^m \times \ldots \times \Gamma_n^m$ such that $\mathbf{w}_1 \in \Gamma_1^m$ and $\mathbf{w}_2 \in \Gamma_2^m \times \ldots \times \Gamma_n^m$. Let $\mathrm{trop}_\nu(f)(\mathbf{w}) = \mathbf{W}$ then $\mathbf{W} = (\mathbf{W}_1, \mathbf{W}_2)$ and $\mathbf{W}_1 \in \Gamma_1$, $\mathbf{W}_2 \in \Gamma_2 \times \ldots \times \Gamma_n$. Then this implies that

$$\mathrm{trop}_{\nu_1}(f)(\mathbf{w}_1) = \mathbf{W}_1$$
$$\mathrm{trop}_{\overline{\nu}_2}(\mathrm{in}_{\mathbf{w}_1}^{\nu_1} f)(\mathbf{w}_2) = \mathbf{W}_2.$$

Proof. First, let $\nu = (\nu_1, \nu_2)$ where ν_1 is the function which maps the first coordinate of ν and ν_2 maps the rest coordinates. Suppose $f = \sum_{\mathbf{u} \in \mathbb{Z}^m} c_{\mathbf{u}} x^{\mathbf{u}}$, then the tropicalization via ν of f is

$$\begin{aligned}
\mathbf{W} = \mathrm{trop}_\nu(f)(\mathbf{w}) &= \bigoplus_{\mathbf{u} \in \mathbb{Z}^m} \nu(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} \\
&= \bigoplus_{\mathbf{u} \in \mathbb{Z}^m} (\nu_1(c_{\mathbf{u}}), \nu_2(c_{\mathbf{u}})) + (\mathbf{w}_1 \cdot \mathbf{u}, \mathbf{w}_2 \cdot \mathbf{u}) \\
&= \bigoplus_{\mathbf{u} \in \mathbb{Z}^m} (\nu_1(c_{\mathbf{u}}) + \mathbf{w}_1 \cdot \mathbf{u}, \nu_2(c_{\mathbf{u}}) + \mathbf{w}_2 \cdot \mathbf{u}) \\
&= \min_{\mathrm{lex}}(\nu_1(c_{\mathbf{u}}) + \mathbf{w}_1 \cdot \mathbf{u}, \nu_2(c_{\mathbf{u}}) + \mathbf{w}_2 \cdot \mathbf{u}),
\end{aligned}$$

then following the condition in proposition 4.2.1, we shall let $\mathbf{W}_1$ be the minimum among the terms $\nu_1(c_{\mathbf{u}}) + \mathbf{w}_1 \cdot \mathbf{u}$. As for $\mathbf{W}_2$, since the lexicographical ordering, when we taking the minimum $\mathbf{W}_2$ among $\nu_2(c_{\mathbf{u}}) + \mathbf{w}_2 \cdot \mathbf{u}$ we only focus on those terms with exponents $\mathbf{u}$ such that $\nu_1(c_{\mathbf{u}}) + \mathbf{w}_1 \cdot \mathbf{u} = \mathbf{W}_1$. Then the exponents $\mathbf{u}$ appearing in the first coordinate of $\mathrm{trop}_\nu(f)(\mathbf{w})$ make up the following set

$$\arg\min_{\mathbf{u}\in\mathbb{Z}^m}(\nu_1(c_{\mathbf{u}})+\mathbf{w}_1\cdot\mathbf{u}) = \{\mathbf{u}\in\mathbb{Z}^m \ : \ \nu_1(c_{\mathbf{u}})+\mathbf{w}_1\cdot\mathbf{u} \le \nu_1(c_{\mathbf{s}})+\mathbf{w}_1\cdot\mathbf{s} \ \ \forall \mathbf{s}\in\mathbb{Z}^m\}$$

and the exponents $\mathbf{u}$ in the rest coordinates are

$$\begin{aligned}&\arg\min_{\mathbf{u}\in\arg\min_{\mathbf{u}\in\mathbb{Z}^m}(\nu_1(c_{\mathbf{u}})+\mathbf{w}_1\cdot\mathbf{u})}(\nu_2(c_{\mathbf{u}})+\mathbf{w}_2\cdot\mathbf{u})\\ =&\{\mathbf{u}\in\mathbf{u}\in\arg\min_{\mathbf{u}\in\mathbb{Z}^m}(\nu_1(c_{\mathbf{u}})+\mathbf{w}_1\cdot\mathbf{u}) \ : \ \nu_2(c_{\mathbf{u}})+\mathbf{w}_2\cdot\mathbf{u}\le\nu_2(c_{\mathbf{s}})+\mathbf{w}_2\cdot\mathbf{s} \ \ \forall\mathbf{s}\in\mathbb{Z}^m\}.\end{aligned}$$

This is obvious that $\mathbf{W}_1 = \bigoplus_{\mathbf{u}\in\mathbb{Z}^m}\nu_1(c_{\mathbf{u}})+\mathbf{w}_1\cdot\mathbf{u}$ is equal to $\mathrm{trop}_{\nu_1}(f)(\mathbf{w}_1)$ which is the tropicalization of f via the first valuation ν_1 in $(\nu_1,\overline{\nu}_2)$.

For showing $\mathrm{trop}_{\overline{\nu}_2}(\mathrm{in}^{\nu_1}_{\mathbf{w}_1}f)(\mathbf{w}_2) = \mathbf{W}_2$, notice that $\bigoplus$ means taking the minimum with lexicographical ordering, then

$$\mathbf{W}_2 = \min_{\nu_1(c_{\mathbf{u}})+\mathbf{w}_1\cdot\mathbf{u}=\mathbf{W}_1}\{\nu_2(c_{\mathbf{u}})+\mathbf{w}_2\cdot\mathbf{u}\}$$

and those terms $\nu_1(c_{\mathbf{u}})+\mathbf{w}_1\cdot\mathbf{u}$ being minimum implies that $\mathbf{u}$ are those exponents appearing in $\mathrm{in}^{\nu_1}_{\mathbf{w}_1}f$. Recalling that σ_1 is a splitting of ν_1 induced from σ, then we have the initial form of f with respect to $\mathbf{w}_1$

$$\mathrm{in}^{\nu_1}_{\mathbf{w}_1}(f) = \sum_{\mathbf{W}_1=\mathrm{trop}_{\nu_1}(f)(\mathbf{w}_1)}\overline{\sigma_1(\mathbf{w}_1\cdot\mathbf{u}-\mathbf{W}_1)c_{\mathbf{u}}}x^{\mathbf{u}}.$$

Then the tropicalization with $\overline{\nu}_2$ of $\mathrm{in}^{\nu_1}_{\mathbf{w}_1}(f)$ is

$$\mathrm{trop}_{\overline{\nu}_2}(\mathrm{in}^{\nu_1}_{\mathbf{w}_1}(f)) = \bigoplus_{\substack{\mathbf{u}\in\mathbb{Z}^m\\ \mathbf{W}_1=\mathrm{trop}_{\nu_1}(f)(\mathbf{w}_1)}}\overline{\nu}_2(\overline{\sigma_1(\mathbf{w}_1\cdot\mathbf{u}-\mathbf{W}_1)c_{\mathbf{u}}})+x\cdot\mathbf{u}$$

According to proposition 3.6.1, we have the diagram $\begin{array}{ccc} R & \xrightarrow{\nu_2|_R} & \mathbb{R}^{n-1}_{\mathrm{lex}}\cup\infty \\ \downarrow & \overset{\overline{\nu}_2}{\nearrow} & \\ K & & \end{array}$. Then from the diagram above we have the equation

$$\begin{aligned}\overline{\nu}_2(\overline{\sigma_1(\mathbf{w}_1\cdot\mathbf{u}-\mathbf{W}_1)c_{\mathbf{u}}}) &= \nu_2(\sigma_1(\mathbf{w}_1\cdot\mathbf{u}-\mathbf{W}_1)c_{\mathbf{u}})\\ &= \nu_2(\sigma_1(\mathbf{w}_1\cdot\mathbf{u}-\mathbf{W}_1))+\nu_2(c_{\mathbf{u}}).\end{aligned}$$

Now recalling the proof of proposition 3.6.5, let $z\in k$, $\nu(z) = (\zeta_1,\zeta_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$

where $\nu_1(z) = \zeta_1$ and $\nu_2(z) = \zeta_2$ then we have $\nu(\sigma(\zeta_1, \zeta_2)) = (\zeta_1, \zeta_2)$ and σ induces two splitting of ν_1 and $\overline{\nu}_2$ such that $\nu(\sigma_1(\zeta_1, \zeta_2)) = \zeta_1 \in \mathbb{R}$ and $\nu(\sigma_2(\zeta_1, \zeta_2)) = \zeta_2 \in \mathbb{R}^{n-1}$. Then $\sigma_1(\zeta_1)$ and $\sigma_2(\zeta_2)$ are defined to be $\sigma(\zeta_1, 0)$ and $\overline{\sigma(0, \zeta_2)}$ correspondingly. Clearly, $\nu_2(\sigma_1(\mathbf{w}_1 \cdot \mathbf{u} - \mathbf{W}_1)) = 0$, since $\sigma_1(\mathbf{w}_1 \cdot \mathbf{u} - \mathbf{W}_1) = \sigma(\mathbf{w}_1 \cdot \mathbf{u} - \mathbf{W}_1, 0)$. Then we have $\overline{\nu}_2(\overline{\sigma_1(\mathbf{w}_1 \cdot \mathbf{u} - \mathbf{W}_1) c_{\mathbf{u}}}) = \nu_2(c_{\mathbf{u}})$. So we have

$$\mathrm{trop}_{\overline{\nu}_2}(\mathrm{in}^{\nu_1}_{\mathbf{w}_1}(f))(x) = \bigoplus_{\substack{\mathbf{u} \in \mathbb{Z}^m \\ \nu(c_{\mathbf{u}}) + \mathbf{w}_1 \cdot \mathbf{u} = \mathbf{W}_1}} \nu_2(c_{\mathbf{u}}) + x \cdot \mathbf{u}.$$

If we plug in the $\mathbf{w}_2$ we will get the following equation immediately

$$\mathrm{trop}_{\overline{\nu}_2}(\mathrm{in}^{\nu_1}_{\mathbf{w}_1}(f))(\mathbf{w}_2) = \bigoplus_{\substack{\mathbf{u} \in \mathbb{Z}^m \\ \nu(c_{\mathbf{u}}) + \mathbf{w}_1 \cdot \mathbf{u} = \mathbf{W}_1}} \nu_2(c_{\mathbf{u}}) + \mathbf{w}_2 \cdot \mathbf{u}$$

which implies that $\mathrm{trop}_{\overline{\nu}_2}(\mathrm{in}^{\nu_1}_{\mathbf{w}_1}(f))(\mathbf{w}_2) = \mathbf{W}_2$. □

Example 4.2.2. Let $k = k_1\{\{t\}\}$ and $k_1 = \mathbb{C}\{\{s\}\}$ with the lexicographic ordering $t > s$. Suppose ν be a rank 2 valuation $\nu : k^\times \mapsto \Gamma_1 \times \Gamma_2$ such that

$$\nu(a) = (a_1, a_2)$$

where $a \in k$. let a_1 be the lowest exponent of t in a and a_2 be the exponent of s of the coefficient of the leading term in a. By proposition 3.6.1, ν can be reduced as a 2-step valuation (v_1, v_2), where v_1 is the t-adic valuation on k and v_2 be the s-adic valuation on the residue field k_1 of v_1, then

$$v_1 : k^\times \mapsto \mathbb{R}$$
$$v_2 : {k_1}^\times \mapsto \mathbb{R}.$$

Let $f \in k[x^\pm, y^\pm, z^\pm]$ such that

$$f = (s^5 + 3t^5 s^2 + \ldots)xyz + (s^3 + 6ts + \ldots)x^2 y^3 z + (ts^2 + t^3 s^4 + \ldots)x^3 y^3 z.$$

Now fixing a weight vector $\mathbf{w} \in (\Gamma^3)^2$ and let it be $\mathbf{w} = \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{pmatrix}$ where $\mathbf{w}_1, \mathbf{w}_2 \in \Gamma^3$. Suppose that

$$\mathbf{w} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

then $\mathbf{w}_1 = (1,0,1)$ and $\mathbf{w}_2 = (1,0,1)$. Now we calculate $\mathbf{W} = \mathrm{trop}_\nu(f)(\mathbf{w})$ first. Clearly, $\nu(s^5 + 3t^5s^2 + \ldots) = (0,5)$, $\nu(s^3 + 6ts + \ldots) = (0,3)$ and $\nu(ts^2 + t^3s^4 + \ldots) = (1,2)$ then

$$\begin{aligned}\mathbf{W} = \mathrm{trop}_\nu(f)(\mathbf{w}) = \min\nolimits_{\mathrm{lex}}\{&(0,5) + (1,1) + (0,0) + (1,1),\\ &(0,3) + (2,2) + (0,0) + (1,1),\\ &(1,2) + (3,3) + (0,0) + (1,1)\}.\end{aligned}$$

By lexicographic ordering, $\mathbf{W} = (2,7)$. By proposition, 4.2.1, we shall set $\mathbf{W}_1 = 2$ and $\mathbf{W}_2 = 7$ where $\mathrm{trop}_{v_1}(f)(\mathbf{w}_1) = \mathbf{W}_1$ and $\mathrm{trop}_{v_2}(\mathrm{in}^{v_1}_{\mathbf{w}_1}(f))(\mathbf{w}_2) = \mathbf{W}_2$, which we shall check now.

$$\begin{aligned}\mathrm{trop}_{v_1}(f)(\mathbf{w}_1) =& \min\{v_1(s^5 + 3t^5s^2 + \ldots) + 1 + 0 + 1,\\ & v_1(s^3 + 6ts + \ldots) + 2 + 0 + 1, v_1(ts^2 + t^3s^4 + \ldots) + 3 + 0 + 1\}\\ =& \min\{0 + 1 + 0 + 1, 0 + 2 + 0 + 1, 1 + 3 + 0 + 1\}\\ =& 2.\end{aligned}$$

Hence $\mathrm{trop}_{v_1}(f)(\mathbf{w}_1) = \mathbf{W}_1$. Then we shall find the initial form $\mathrm{in}^{v_1}_{\mathbf{w}_1}(f)$ easily that $\mathrm{in}^{v_1}_{\mathbf{w}_1}(f) = (s^5 + 3t^5s^2 + \ldots)xyz$. Finally, we need to check whether $\mathrm{trop}_{v_2}(\mathrm{in}^{v_1}_{\mathbf{w}_1}(f))(\mathbf{w}_2) = \mathbf{W}_2$.

$$\begin{aligned}\mathrm{trop}_{v_2}(\mathrm{in}^{v_1}_{\mathbf{w}_1}(f))(\mathbf{w}_2) &= v_2(s^5 + 3t^5s^2 + \ldots) + 1 + 0 + 1\\ &= 7.\end{aligned}$$

Therefore $\mathrm{trop}_{v_2}(\mathrm{in}^{v_1}_{\mathbf{w}_1}(f))(\mathbf{w}_2) = \mathbf{W}_2$.

From proposition 4.2.1 and recalling proposition 3.6.1, we shall prove the following corollary directly.

Corollary 4.2.3. *Let ν be a rank n valuation on k and $f \in k[x_1^{\pm}, \ldots, x_m^{\pm}]$. By propsoition 3.6.1, we can induce ν as an n-step valuation $(v_1, \ldots, v_n)$ where*

$$\begin{aligned} v_1 &: k^* \to \mathbb{R} \\ v_2 &: k_1^* \to \mathbb{R} \\ &\vdots \\ v_n &: k_{n-1}^* \to \mathbb{R}. \end{aligned}$$

As the description in proposition 4.2.1, fixing a weight vector $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_n) \in \Gamma_1^m \times \ldots \times \Gamma_n^m$ and let $\operatorname{trop}_\nu(f)(\mathbf{w}) = \mathbf{W} = (\mathbf{W}_1, \ldots, \mathbf{W}_n) \in \Gamma_1 \times \ldots \times \Gamma_n$. Then we will have

$$\begin{aligned} \operatorname{trop}_{v_1}(f)(\mathbf{w}_1) &= \mathbf{W}_1 \\ \operatorname{trop}_{v_2}(\operatorname{in}_{\mathbf{w}_1}^{v_1} f)(\mathbf{w}_2) &= \mathbf{W}_2 \\ &\vdots \\ \operatorname{trop}_{v_n}(\operatorname{in}_{\mathbf{w}_{n-1}}^{v_{n-1}} \ldots \operatorname{in}_{\mathbf{w}_1}^{v_1} f)(\mathbf{w}_n) &= \mathbf{W}_n \end{aligned}$$

The proof of corollary 4.2.3 is the same as the proof of proposition 4.2.1. In proposition 4.2.1, we show that

$$\operatorname{trop}_\nu(f)(\mathbf{w}) = \left(\operatorname{trop}_{\nu_1}(f)(\mathbf{w}_1),\ \operatorname{trop}_{\overline{\nu}_2}(\operatorname{in}_{\mathbf{w}_1}^{\nu_1} f)(\mathbf{w}_2)\right)$$

where $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$ and $\mathbf{w}_1 \in \Gamma_1^m$, $\mathbf{w}_2 \in \Gamma_2^m \times \ldots \times \Gamma_n^m$. Then we shall start the induction from $\operatorname{trop}_{\overline{\nu}_2}(\operatorname{in}_{\mathbf{w}_1}^{\nu_1} f)(\mathbf{w}_2)$, by proposition 3.6.1, we shall induce $\overline{\nu}_2$ as $(\nu_2, \overline{\nu}_3)$ where ν_2 is a rank 1 valuation such that $\nu_2 : k_1^* \to \mathbb{R}$ and $\overline{\nu}_3$ is a rank $n-2$ valuation such that $\overline{\nu}_3 : k_2^* \to \mathbb{R}^{n-2}$. Then with the same proof of propsoition 4.2.1, we shall easily see that

$$\operatorname{trop}_\nu(f)(\mathbf{w}) = \left(\operatorname{trop}_{\nu_1}(f)(\mathbf{w}_1),\ \operatorname{trop}_{\nu_2}(\operatorname{in}_{\mathbf{w}_1}^{\nu_1} f)(\mathbf{w}_2), \operatorname{trop}_{\overline{\nu}_3}(\operatorname{in}_{\mathbf{w}_2}^{\nu_2}\operatorname{in}_{\mathbf{w}_1}^{\nu_1} f)(\mathbf{w}_3)\right).$$

Therefore by induction we will finally prove that if $\operatorname{trop}_\nu(f)(\mathbf{w}) = \mathbf{W} \in \Gamma_1 \times \ldots \times \Gamma_n$ then

$$\mathbf{W} = (\operatorname{trop}_{v_1}(f)(\mathbf{w}_1), \operatorname{trop}_{v_2}(\operatorname{in}_{\mathbf{w}_1}^{v_1} f)(\mathbf{w}_2), \ldots, \operatorname{trop}_{v_n}(\operatorname{in}_{\mathbf{w}_{n-1}}^{v_{n-1}} \ldots \operatorname{in}_{\mathbf{w}_1}^{v_1} f)(\mathbf{w}_n)).$$

Now we prove the equivalence between rank n and n-step tropicalization, and we can go further on this. Recalling definition 2.5.1, we can hypothesise the equivalence also holds in initial form.

Proposition 4.2.4. *Let ν be a rank n valuation on k. By proposition 3.6.1 and 3.6.5, let ν_1 and $\overline{\nu}_2$ be the valuations induced from ν with the splitting σ_1 and σ_2 induced from σ. Let $f = \sum_{\mathbf{u}\in\mathbb{Z}^m} c_{\mathbf{u}} x^{\mathbf{u}} \in k[x_1^{\pm}, \ldots, x_m^{\pm}]$ and fix a weight vector $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2) \in \Gamma_1^m \times \ldots \times \Gamma_n^m$ with $\mathbf{w}_1 \in \Gamma_1^m$, $\mathbf{w}_2 \in \Gamma_2^m \times \ldots \times \Gamma_n^m$. Then*

$$\text{in}_{\mathbf{w}_2}^{\overline{\nu}_2}(\text{in}_{\mathbf{w}_1}^{\nu_1}(f)) = \text{in}_{\mathbf{w}}^{\nu}(f).$$

Proof. By proposition 4.2.1, fixing a weight vector $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$, and let $\mathbf{W} = (\mathbf{W}_1, \mathbf{W}_2)$, we have

$$\text{trop}_{\nu}(f)(\mathbf{w}) = \mathbf{W}$$
$$\text{trop}_{\nu_1}(f)(\mathbf{w}_1) = \mathbf{W}_1$$
$$\text{trop}_{\overline{\nu}_2}(\text{in}_{\mathbf{w}_1}^{\nu_1} f)(\mathbf{w}_2) = \mathbf{W}_2$$

In $\text{trop}_{\nu_1}(f)(\mathbf{w}_1) = \mathbf{W}_1$, the exponents $\mathbf{u}$ satisfies $\nu_1(c_{\mathbf{u}}) + \mathbf{w}_1 \cdot \mathbf{u} = \mathbf{W}_1$. So the initial form $\text{in}_{\mathbf{w}_1}^{\nu_1}(f)$ is

$$\text{in}_{\mathbf{w}_1}^{\nu_1}(f) = \sum_{\substack{\mathbf{u}\in\mathbb{Z}^n \\ \mathbf{W}_1 = \nu_1(c_{\mathbf{u}}) + \mathbf{w}_1\cdot\mathbf{u}}} \pi_1(\sigma_1(\mathbf{w}_1 \cdot \mathbf{u} - \mathbf{W}_1) c_{\mathbf{u}}) x^{\mathbf{u}}.$$

Let $\pi_1(\sigma_1(\mathbf{w}_1 \cdot \mathbf{u} - \mathbf{W}_1)c_{\mathbf{u}}) = d_{\mathbf{u}}$, then in the equation $\text{trop}_{\overline{\nu}_2}(\text{in}_{\mathbf{w}_1}^{\nu_1} f)(\mathbf{w}_2) = \mathbf{W}_2$, the exponents $\mathbf{u}$ which we taking from the minimum of $\nu_1(c_{\mathbf{u}}) + \mathbf{w}_1 \cdot \mathbf{u}$ must satisfy that $\overline{\nu}_2(d_{\mathbf{u}}) + \mathbf{w}_2 \cdot \mathbf{u} = \mathbf{W}_2$. By proposition 4.2.1, we have $\overline{\nu}_2(d_{\mathbf{u}}) + \mathbf{w}_2 \cdot \mathbf{u} = \mathbf{W}_2 = \nu_2(c_{\mathbf{u}}) + \mathbf{w}_2 \cdot \mathbf{u}$. Now notice that the exponents appearing in $\text{in}_{\mathbf{w}_1}^{\nu_1}(f)$ consists of the following set

$$\arg\min_{\mathbf{u}\in\mathbb{Z}^m}(\nu_1(c_{\mathbf{u}}) + \mathbf{w}_1 \cdot \mathbf{u}) = \{\mathbf{u} \in \mathbb{Z}^m \ : \ \nu_1(c_{\mathbf{u}}) + \mathbf{w}_1 \cdot \mathbf{u} \le \nu_1(c_{\mathbf{s}}) + \mathbf{w}_1 \cdot \mathbf{s} \ \ \forall \mathbf{s} \in \mathbb{Z}^m\}.$$

Then the exponents in $\text{in}_{\mathbf{w}_2}^{\overline{\nu}_2}\text{in}_{\mathbf{w}_1}^{\nu_1}(f)$ are the following set

$$\begin{aligned}&\underset{\mathbf{u}\in\arg\min_{\mathbf{u}\in\mathbb{Z}^m}(\nu_1(c_\mathbf{u})+\mathbf{w}_1\cdot\mathbf{u})}{\arg\min}(\overline{\nu}_2(d_\mathbf{u})+\mathbf{w}_2\cdot\mathbf{u})\\ =&\{\mathbf{u}\in\underset{\mathbf{u}\in\mathbb{Z}^m}{\arg\min}(\nu_1(c_\mathbf{u})+\mathbf{w}_1\cdot\mathbf{u})\ :\ \overline{\nu}_2(d_\mathbf{u})+\mathbf{w}_2\cdot\mathbf{u}\leq\overline{\nu}_2(d_\mathbf{s})+\mathbf{w}_2\cdot\mathbf{s}\ \ \forall\mathbf{s}\in\mathbb{Z}^m\}.\end{aligned}$$

Now consider the exponents $\mathbf{u}$ in $\text{in}_{\mathbf{w}}^{\nu}(f)$

$$\underset{\mathbf{u}\in\mathbb{Z}^m}{\arg\min}(\nu(c_\mathbf{u})+\mathbf{w}\cdot\mathbf{u})=\{\mathbf{u}\in\mathbb{Z}^m\ :\ \nu(c_\mathbf{u})+\mathbf{w}\cdot\mathbf{u}\leq_{\text{lex}}\nu(c_\mathbf{s})+\mathbf{w}\cdot\mathbf{s}\ \ \forall\mathbf{s}\in\mathbb{Z}^m\}.$$

Notice that the inequality $\nu(c_\mathbf{u})+\mathbf{w}\cdot\mathbf{u}\leq_{\text{lex}}\nu(c_\mathbf{s})+\mathbf{w}\cdot\mathbf{s}$ is

$$(\nu_1(c_\mathbf{u})+\mathbf{w}_1\cdot\mathbf{u},\nu_2(c_\mathbf{u})+\mathbf{w}_2\cdot\mathbf{u})\leq_{\text{lex}}(\nu_1(c_\mathbf{s})+\mathbf{w}_1\cdot\mathbf{s},\nu_2(c_\mathbf{s})+\mathbf{w}_2\cdot\mathbf{s}).$$

Then the exponents appearing in $\text{in}_{\mathbf{w}}^{\nu}(f)$ and $\text{in}_{\mathbf{w}_2}^{\overline{\nu}_2}(\text{in}_{\mathbf{w}_1}^{\nu_1}(f))$ are the same.

In the rest of proof, we need to show that the coefficients of $\text{in}_{\mathbf{w}}^{\nu}(f)$ are the same as the coefficients of $\text{in}_{\mathbf{w}_2}^{\overline{\nu}_2}\text{in}_{\mathbf{w}_1}^{\nu_1}(f)$. With the definition of the initial form, the initial form of f with respect to $\mathbf{w}$ is

$$\text{in}_{\mathbf{w}}^{\nu}(f)=\sum_{\substack{\mathbf{u}\in\mathbb{Z}^m\\ \nu(c_\mathbf{u})+\mathbf{w}\cdot\mathbf{u}=\mathbf{W}}}\pi(\sigma(\mathbf{w}\cdot\mathbf{u}-\mathbf{W})c_\mathbf{u})x^\mathbf{u}.$$

With the initial form $\text{in}_{\mathbf{w}_1}^{\nu_1}(f)$ at the beginning of this proof we shall have the initial form $\text{in}_{\mathbf{w}_2}^{\overline{\nu}_2}\text{in}_{\mathbf{w}_1}^{\nu_1}(f)$ such that

$$\text{in}_{\mathbf{w}_2}^{\overline{\nu}_2}\text{in}_{\mathbf{w}_1}^{\nu_1}(f)=\sum_{\substack{\mathbf{u}\in\mathbb{Z}^m\\ \nu_1(c_\mathbf{u})+\mathbf{w}_1\cdot\mathbf{u}=\mathbf{W}_1\\ \overline{\nu}_2(c_\mathbf{u})+\mathbf{w}_2\cdot\mathbf{u}=\mathbf{W}_2}}\pi_2(\sigma_2(\mathbf{w}_2\cdot\mathbf{u}-\mathbf{W}_2)\pi_1(\sigma_1(\mathbf{w}_1\cdot\mathbf{u}-\mathbf{W}_1)c_\mathbf{u}))x^\mathbf{u}$$

By proposition 3.6.5, a splitting σ of ν can be induced as splittings σ_1 and σ_2 of ν_1 and ν_2. Also we have these equations $\sigma_1(a)=\sigma(a,0)$ and $\sigma_2(b)=\pi_1(\sigma(0,b))$ where $a\in\Gamma_1$ and $b\in\Gamma_2\times\ldots\times\Gamma_n$. Then we shall rewrite the coefficients in $\text{in}_{\mathbf{w}_2}^{\overline{\nu}_2}\text{in}_{\mathbf{w}_1}^{\nu_1}(f)$ as

$$\pi_2(\pi_1(\sigma(0,\mathbf{w}_2\cdot\mathbf{u}-\mathbf{W}_2))\pi_1(\sigma(\mathbf{w}_1\cdot\mathbf{u}-\mathbf{W}_1,0)c_\mathbf{u})).$$

Notice that $\nu_1(\sigma_1(0,\mathbf{w}_2\cdot\mathbf{u}-\mathbf{W}_2))=0$ and

$$\begin{aligned}\nu_1(\sigma(\mathbf{w}_1\cdot\mathbf{u}-\mathbf{W}_1,0)c_\mathbf{u})&=\nu_1(\sigma(-\nu_1(c_\mathbf{u}),0))+\nu_1(c_\mathbf{u})\\&=-\nu_1(c_\mathbf{u})+\nu_1(c_\mathbf{u})=0\end{aligned}$$

then both $\sigma(0, \mathbf{w}_2 \cdot \mathbf{u} - \mathbf{W}_2)$ and $\sigma(\mathbf{w}_1 \cdot \mathbf{u} - \mathbf{W}_1, 0)c_{\mathbf{u}}$ are in the valuation ring R_1 of ν_1. Since $\pi_1 : R_1 \to k_1$ is a homomorphism then

$$\begin{aligned}
&\pi_2(\pi_1(\sigma(0, \mathbf{w}_2 \cdot \mathbf{u} - \mathbf{W}_2))\pi_1(\sigma(\mathbf{w}_1 \cdot \mathbf{u} - \mathbf{W}_1, 0)c_{\mathbf{u}})) \\
=&\pi_2(\pi_1(\sigma(0, \mathbf{w}_2 \cdot \mathbf{u} - \mathbf{W}_2)\sigma(\mathbf{w}_1 \cdot \mathbf{u} - \mathbf{W}_1, 0)c_{\mathbf{u}})) \\
=&\pi_2(\pi_1(\sigma(\mathbf{w}_1 \cdot \mathbf{u} - \mathbf{W}_1, \mathbf{w}_2 \cdot \mathbf{u} - \mathbf{W}_2)c_{\mathbf{u}})) \\
=&\pi_2(\pi_1(\sigma(\mathbf{w} \cdot \mathbf{u} - \mathbf{W})c_{\mathbf{u}})).
\end{aligned}$$

Since $\nu(\sigma(\mathbf{w} \cdot \mathbf{u} - \mathbf{W})c_{\mathbf{u}}) = (0, 0)$, $\sigma(\mathbf{w} \cdot \mathbf{u} - \mathbf{W})c_{\mathbf{u}}$ is in the valuation ring R of ν.

The final step of this proof is showing that $\pi_1(\sigma(\mathbf{w} \cdot \mathbf{u} - \mathbf{W})c_{\mathbf{u}})$ is in the valuation ring R_2 of $\overline{\nu}_2$. For any $x \in R$, we have $\nu(x) = (\nu_1(x), \nu_2(x)) \geq_{\text{lex}} (0, 0)$ which implies that $x \in R_1$. If $\nu_1(x) > 0$ then $\pi_1(x) = 0 \implies \pi_1(x) \in R_2$ and if $\nu_1(x) = 0$ then $\nu_2(x) \geq_{\text{lex}} 0 \implies \overline{\nu}_2(\pi_1(x)) \geq_{\text{lex}} 0$ so $\pi_1(x) \in R_2$. So for any $x \in R$ we have $\pi_1(x) \in R_2$. Then for proving this mapping $\pi_1 : R \to R_2$ is surjective, let y be an arbitrary element in R_2 and $\pi_1(x) = y$ where $x \in R_1$. Since $y \in R_2$, we have $\overline{\nu}_2(y) \geq_{\text{lex}} 0 \implies \overline{\nu}_2(\pi_1(x)) \geq_{\text{lex}} 0 \implies \nu_2(x) \geq_{\text{lex}} 0$. Since $x \in R_1$, $\nu_1(x) \geq 0$ so we have two possibilities, $\nu_1(x) > 0$ or $\nu_1(x) = 0$ and $\nu_2(x) \geq_{\text{lex}} 0$, both imply that $x \in R$, then $\pi_1 : R \to R_2$ is surjective. Therefore

$$\pi_2(\pi_1(\sigma(\mathbf{w} \cdot \mathbf{u} - \mathbf{W})c_{\mathbf{u}})) = \pi(\sigma(\mathbf{w}_1 \cdot \mathbf{u} - \mathbf{W}_1, \mathbf{w}_2 \cdot \mathbf{u} - \mathbf{W}_2)c_{\mathbf{u}}).$$

So $\text{in}^{\overline{\nu}_2}_{\mathbf{w}_2}(\text{in}^{\nu_1}_{\mathbf{w}_1}(f))$ has the same coefficients as $\text{in}^{\nu}_{\mathbf{w}}(f)$ and we have already proven that the exponents in these two initial forms are the same. Therefore $\text{in}^{\overline{\nu}_2}_{\mathbf{w}_2}(\text{in}^{\nu_1}_{\mathbf{w}_1}(f)) = \text{in}^{\nu}_{\mathbf{w}}(f)$. □

With proposition 4.2.4, by induction we will have the following immediate consequence.

Proposition 4.2.5. *Let ν be a rank n valuation on field k and according to proposition 3.6.1 there exists an n-step valuation on k which is supposed to be $(v_1, v_2, \ldots, v_n)$. Let $f = \sum_{\mathbf{u} \in \mathbb{Z}^m} c_{\mathbf{u}} x^{\mathbf{u}} \in k[x_1^{\pm}, \ldots, x_m^{\pm}]$ and fix a weight vector $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n) \in \mathbb{R}^{n \times m}_{\text{lex}}$ where $\mathbf{w}_i \in \mathbb{R}^m$ for all $1 \leq i \leq n$. Then*

$$\text{in}^{\nu}_{\mathbf{w}}(f) = \text{in}^{v_n}_{\mathbf{w}_n} \ldots \text{in}^{v_1}_{\mathbf{w}_1}(f)$$

Example 4.2.6. Let $k = k_1\{\{t\}\}$ and $k_1 = \mathbb{C}\{\{s\}\}$. Suppose $\nu : k^\times \longrightarrow \Gamma_1 \times \Gamma_2$ be a rank 2 valuation on k such that

$$\nu(a) = (a_1, a_2),$$

where $a \in k$, a_1 is the lowest exponent of t in a and a_2 is the exponent of s of the coefficient of the leading term. By proposition 3.6.1, ν can be reduced as a 2-step valuation (v_1, v_2), where v_1 is the t-adic valuation on k and v_2 is the s-adic valuation on the residue field k_1 of v_1. Suppose a polynomial $f \in k[x^\pm, y^\pm]$ such that

$$f = (t^4s^6 + t^6s)xy + (t^3s + t^3s^2)x^2y + t^3sxy^2 + (t^4s^2 + t^5s^2)x^2y^2.$$

Let $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$ be a weight vector, $\mathbf{w}_1 = (1, 1)$ and $\mathbf{w}_2 = (2, 2)$. Then the tropicalization of f of v_1 at $\mathbf{w}_1$ is

$$\mathbf{W}_1 = \operatorname{trop}_{v_1}(f)(1, 1) = \min\{6, 6, 6, 8\}$$

Therefore we have the initial form of f with respect to $\mathbf{w}_1$

$$\operatorname{in}^{v_1}_{\mathbf{w}_1}(f) = (s^6 + t^2s)xy + (s + s^2)x^2y + sxy^2.$$

Next, consider the tropicalization of $\operatorname{in}^{v_1}_{\mathbf{w}_1}(f)$ of v_2 at $\mathbf{w}_2$

$$\mathbf{W}_2 = \operatorname{trop}_{v_2}(\operatorname{in}^{v_1}_{\mathbf{w}_1}(f))(2, 2) = \min\{10, 7, 7\}$$

then the initial form is

$$\operatorname{in}^{v_2}_{\mathbf{w}_2}\operatorname{in}^{v_1}_{\mathbf{w}_1}(f) = (1 + s)x^2y + xy^2.$$

On the other hand, we need to check the initial form of f with respect to $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$. First, the tropicalization of f with ν at $\mathbf{w}$ is

$$\begin{aligned}\mathbf{W} = \operatorname{trop}_\nu(f)((1, 1), (2, 2)) &= \min\{(6, 10), (6, 7), (6, 7), (8, 10)\} \\ &= \{(6, 7), (6, 7)\}.\end{aligned}$$

Hence the initial form of f with respect to $\mathbf{w}$ is

$$\operatorname{in}^\nu_{\mathbf{w}}(f) = (1 + s)x^2y + xy^2.$$

Therefore $\text{in}_{\mathbf{w}}^{\nu}(f) = \text{in}_{\mathbf{w}_2}^{v_2}\text{in}_{\mathbf{w}_1}^{v_1}(f)$.

Now recalling proposition 4.2.1, let ν be a rank n valuation on k and $(v_1, \ldots, v_n)$ be an n-step valuation on k which is reduced from ν and fix a weight vector $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_n) \in (\mathbb{R}^m)^n$ where $\mathbf{w}_i \in \mathbb{R}^m$ for all $1 \le i \le n$. By induction, we can deduce the following equation easily

$$\text{trop}_{\nu}(f)(\mathbf{w}) = \Big(\text{trop}_{v_1}(f)(\mathbf{w}_1), \text{trop}_{v_2}(\text{in}_{\mathbf{w}_1}^{v_1}(f)(\mathbf{w}_2), \ldots, \text{trop}_{v_n}(\text{in}_{\mathbf{w}_{n-1}}^{v_{n-1}} \ldots \text{in}_{\mathbf{w_1}}^{v_n}(f)(\mathbf{w}_n)\Big).$$

From the equation above, we have the hypothesis that $\text{trop}_{\nu}(f)$ has more than one terms at $\mathbf{w}$ if and only if $\text{trop}_{v_i}(\text{in}_{\mathbf{w}_{i-1}}^{v_{i-1}} \ldots \text{in}_{\mathbf{w}_1}^{v_1}(f))$ also has more than one terms at $\mathbf{w}_i$ for all $1 \le i \le n$.

We can now state and prove our main theorem.

Theorem 4.2.7. *As Proposition 4.2.1, let a polynomial* $f = \sum_{\mathbf{u} \in \mathbb{Z}^m} c_{\mathbf{u}} x^{\mathbf{u}} \in k[x_1^{\pm}, \ldots, x_m^{\pm}]$ *and v be a rank n valuation on k which can be induced as $(\nu_1, \overline{\nu}_2)$ where ν_1 is on k and $\overline{\nu}_2$ is on the residue field of ν_1. Consider a weight vector $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2) \in \mathbb{R} \times (\mathbb{R}^m)^{n-1}$. Then* $\text{trop}_{\nu}(f)$ *tropically vanishes at $\mathbf{w}$ if and only if* $\text{trop}_{\nu_1}(f)$ *tropical vanishes at $\mathbf{w}_1$ and* $\text{trop}_{\overline{\nu}_2}(\text{in}_{\mathbf{w}_1}^{\nu_1}(f))$ *tropical vanishes at $\mathbf{w}_2$.*

Proof. First, Let $\text{trop}_{\nu}(f)$ tropically vanishes at $\mathbf{w}$, then $\text{trop}_{\nu}(f)(\mathbf{w})$ contains at least two terms. So we can let the index of these terms be $\mathbf{u}_i$ such that $i \in \{1, \ldots, s\}$ and $s \ge 2$, then we have

$$\mathbf{W} = \text{trop}_{\nu}(f)(\mathbf{w}) = \nu(c_{\mathbf{u}_1}) + \mathbf{w} \cdot \mathbf{u}_1 = \ldots = \nu(c_{\mathbf{u}_s}) + \mathbf{w} \cdot \mathbf{u}_s.$$

Now let $\nu = (\nu_1, \nu_2)$ where ν_1 maps the first coordinate of ν and ν_2 maps the rest. Then fixing a weight vector $\mathbf{w} \in \Gamma_1^m \times \ldots \times \Gamma_n^m$ and let it be $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$ where $\mathbf{w}_1 \in \Gamma_1^m$ and $\mathbf{w}_2 \in \Gamma_2^m \times \ldots \times \Gamma_n^m$. Then we have

$$\mathbf{W} = \nu(c_{\mathbf{u}_i}) + \mathbf{w} \cdot \mathbf{u}_i$$
$$\mathbf{W} = (\nu_1(c_{\mathbf{u}_i}), \nu_2(c_{\mathbf{u}_i})) + (\mathbf{w}_1, \mathbf{w}_2) \cdot \mathbf{u}_i$$
$$\mathbf{W} = (\nu_1(c_{\mathbf{u}_i}), \nu_2(c_{\mathbf{u}_i})) + (\mathbf{w}_1 \cdot \mathbf{u}_i, \mathbf{w}_2 \cdot \mathbf{u}_i)$$
$$\mathbf{W} = (\nu_1(c_{\mathbf{u}_i}) + \mathbf{w}_1 \cdot \mathbf{u}_i, \nu_2(c_{\mathbf{u}_i}) + \mathbf{w}_2 \cdot \mathbf{u}_i)$$

for any $1 \leq i \leq s$. By proposition 4.2.1, we show that if $\text{trop}_\nu(f)(\mathbf{w}) = \mathbf{W} = (\mathbf{W}_1, \mathbf{W}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ then

$$\text{trop}_{\nu_1}(f)(\mathbf{w}_1) = \mathbf{W}_1$$
$$\text{trop}_{\overline{\nu}_2}(\text{in}^{\nu_1}_{\mathbf{w}_1} f)(\mathbf{w}_2) = \mathbf{W}_2.$$

Therefore $\text{trop}_{\nu_1}(f)(\mathbf{w}_1) = \nu_1(c_{\mathbf{u}_i}) + \mathbf{w}_1 \cdot \mathbf{u}_i$ for all $1 \leq i \leq s$ and $\text{trop}_{\nu_2}(\text{in}^{\nu_1}_{\mathbf{w}_1} f)(\mathbf{w}_2) = \overline{\nu}_2(c_{\mathbf{u}_i}) + \mathbf{w}_2 \cdot \mathbf{u}_i$ for all $1 \leq i \leq s$, then both $\text{trop}_{\nu_1}(f)(\mathbf{w}_1)$ and $\text{trop}_{\overline{\nu}_2}(\text{in}^{\nu_1}_{\mathbf{w}_1} f)(\mathbf{w}_2)$ contain more than one term. Hence, $\text{trop}_{\nu_1}(f)$ and $\text{trop}_{\overline{\nu}_2}(\text{in}^{\nu_1}_{\mathbf{w}_1} f)$ tropical vanish at $\mathbf{w}_1$ and $\mathbf{w}_2$ respectively.

Conversely, suppose $\text{trop}_{\nu_1}(f)$ tropical vanishes at $\mathbf{w}_1$ and $\text{trop}_{\overline{\nu}_2}(\text{in}^{\nu_1}_{\mathbf{w}_1} f)$ tropical vanishes at $\mathbf{w}_2$. We will have an immediate consequence that there is more than one index i satisfies $\nu_1(c_{\mathbf{u}_i}) + \mathbf{w}_1 \cdot \mathbf{u}_i = \text{trop}_{\nu_1}(f)(\mathbf{w}_1)$ and $\overline{\nu}_2(c_{\mathbf{u}_i}) + \mathbf{w}_2 \cdot \mathbf{u}_i = \text{trop}_{\overline{\nu}_2}(\text{in}^{\nu_1}_{\mathbf{w}_1} f)(\mathbf{w}_2)$. By proposition 4.2.1, $\text{trop}_\nu(f)(\mathbf{w}) = (\text{trop}_{\nu_1}(f)(\mathbf{w}_1), \text{trop}_{\overline{\nu}_2}(\text{in}^{\nu_1}_{\mathbf{w}_1} f)(\mathbf{w}_2))$, so $\text{trop}_\nu(f)(\mathbf{w})$ has at least two terms so that $\text{trop}_\nu(f)$ tropical vanishes at $\mathbf{w}$. □

Beside the proof above, proposition 4.2.4 provides an alternative method to theorem 4.2.7.

By proposition 4.2.4, $\text{in}^{\overline{\nu}_2}_{\mathbf{w}_2}(\text{in}^{\nu_1}_{\mathbf{w}_1}(f)) = \text{in}^{\nu}_{\mathbf{w}}(f)$, then $\text{in}^{\overline{\nu}_2}_{\mathbf{w}_2}(\text{in}^{\nu_1}_{\mathbf{w}_1}(f))$ is not monomial if and only if $\text{in}^{\nu}_{\mathbf{w}}(f)$ is not. If $\text{in}^{\overline{\nu}_2}_{\mathbf{w}_2}(\text{in}^{\nu_1}_{\mathbf{w}_1}(f))$ has more than 1 monomial then it implies that the initial form $\text{in}^{\nu_1}_{\mathbf{w}_1}(f)$ has more than 1 monomial too. By proposition 4.2.4, $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$, then $\mathbf{w} \in \mathcal{V}(\text{trop}_\nu(f))$ if and only if $\mathbf{w}_1 \in \mathcal{V}(\text{trop}_{v_1}(f))$ and $\mathbf{w}_2 \in \mathcal{V}(\text{trop}_{\overline{v}_2}\text{in}^{v_1}_{\mathbf{w}_1}(f))$, which shall prove theorem 4.2.7.

Example 4.2.8. As example 4.2.6, let $k = k_1\{\{t\}\}$ and $k_1 = \mathbb{C}\{\{s\}\}$. Let ν be a rank 2 valuation on k as example 4.2.6 and $f \in k[x_1, x_2, x_3]$ such that

$$f = (t + st + st^2 + \ldots)x_1 + (s + s^2 + s^2t + \ldots)x_1x_2^2 + (st^2 + st^3 + s^2t^3 + \ldots)x_2^2x_3^3.$$

Then the tropicalization of f via ν is

$$\text{trop}_\nu(f) = \min\{(1,0) + x_1, (0,1) + x_1 + 2x_2, (2,1) + 2x_2 + 3x_3\}. \tag{4.1}$$

Fixing a weight vector $\mathbf{w} \in (\Gamma^3_{\text{lex}})^2$ which let $\text{trop}_\nu(f)$ tropical vanishes at $\mathbf{w}$. Then let $\mathbf{w} = ((0,1), (\frac{1}{2}, -\frac{1}{2}), (1,1))$, we will have the W such that

$$W = \text{trop}_\nu(f)(\mathbf{w}) = \min\{(1,1), (1,1), (6,3)\} = \{(1,1), (1,1)\}.$$

On the other hand, recalling proposition 3.6.1, we shall reduce ν as a 2-step valuation (v_1, v_2) where v_1 is a on k and v_2 is on the residue field k_1 of v_1. And we set two weight vectors $\mathbf{w}_1, \mathbf{w}_2$ such that $\mathbf{w} = \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{pmatrix}$. Then consider the tropicalization of f via v_1, we have

$$\text{trop}_{v_1}(f) = \min\{1 + x_1, 0 + x_1 + 2x_2, 2 + 2x_2 + 3x_3\}.$$

$\mathbf{w}_1 = (0, \frac{1}{2}, 1)$, then we have the $W_1 = \text{trop}_{v_1}(f)(\mathbf{w}_1) = \{1, 1\}$. So the initial form $\text{in}^{v_1}_{\mathbf{w}_1}(f)$ is

$$\text{in}^{v_1}_{\mathbf{w}_1}(f) = (1 + s + st + \ldots)x_1 + (s + s^2 + s^2t + \ldots)x_1x_2^2.$$

Then we will calculate the tropicalization of $\text{in}^{v_1}_{\mathbf{w}_1}(f)$ via v_2

$$\text{trop}_{v_2}(\text{in}^{v_1}_{\mathbf{w}_1}(f)) = \min\{0 + x_1, 1 + x_1 + 2x_2\}.$$

Plugging $\mathbf{w}_2 = (1, -\frac{1}{2}, 1)$, we have $W_2 = \{1, 1\}$ there are more than one terms remain. Therefore $\text{trop}_{v_1}(f)$ tropical vanishes at $\mathbf{w}_1$ and $\text{trop}_{v_2}(\text{in}^{v_1}_{\mathbf{w}_1}(f))$ tropical vanishes at $\mathbf{w}_2$.

It is easy to see, by induction, we can extend theorem 4.2.7 as the following corollary.

Corollary 4.2.9. *Let $f \in k[x_1^\pm, \ldots, x_m^\pm]$, ν be a rank n valuation on k which can be split as an n-step valuation $(v_1, \ldots, v_n)$ by proposition 3.6.1. Suppose $\mathbf{w}$ be a weight vector for v such that $\mathbf{w} \in (\mathbb{R}^m_{\text{lex}})^n$ and $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_n)$ where $\mathbf{w}_i \in \mathbb{R}^m$. Then $\text{trop}_\nu(f)$ tropically vanishes at $\mathbf{w}$ if and only if $\text{trop}_{v_1}(f)$ tropically vanishes at $\mathbf{w}_1$ and $\text{trop}_{v_i}(\text{in}^{v_{i-1}}_{\mathbf{w}_{i-1}} \ldots \text{in}^{v_1}_{\mathbf{w}_1}(f))$ tropically vanishes at $\mathbf{w}_i$ for each $1 < i \leq n$.*

Notice that the description about $(\mathbf{w}_1, \ldots, \mathbf{w}_n)$ is definition 4.1.2, so we can conclude that

Theorem 4.2.10. *Let ν be a rank n valuation on k and $(v_1, \ldots, v_n)$ be an n-step valuation which is induced from ν. Given a variety X, then the rank n tropicalization of X is equal to the n-step tropicalization of X.*

In [2], Fuensanta Aroca proves that in an algebraically closed field k with a surjective rank n valuation ν, the rank n tropicalization of the hypersurface associatied to a polynomial f is equal to the tropical hypersurface associated to the rank n tropicalization of f such that

$$\mathrm{trop}_\nu(V(f)) = \mathcal{V}(\mathrm{trop}_\nu(f)).$$

In this thesis, we shall prove the main theorem in [2] again in a different way.

Recalling the Kapranov's theorem in ordinary version, for any Laurent polynomial f in m variables over an algebraically closed field k with a valuation ν, that theorem shows the equivalence between the following three subsets in $\mathbb{R}^m$

1. the tropical hypersurface $\mathrm{trop}(V(f))$ in $\mathbb{R}^m$;
2. the set $\{\mathbf{w} \in \mathbb{R}^m : \mathrm{in}^\nu_{\mathbf{w}}(f) \text{ is not a monomial}\}$
3. the closure in $\mathbb{R}^m$ of $\{(\nu(y_1), \ldots, \nu(y_m)) : (y_1, \ldots, y_m) \in V(f)\}$

By definition 2.6.4, set 2 can be rewritten as $\mathcal{V}(\mathrm{trop}_\nu(f))$, then we have the following diagram

$$\begin{array}{ccc} V(f) & \subset & (k^\times)^m \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu} \\ \mathcal{V}(\mathrm{trop}_\nu(f)) & \subset & \mathbb{R}^m \end{array}$$

If $a \in k$, then $\sigma(-\nu(a)) \cdot a$ is in the valuation ring, and we can take its image in the residue field $\mathbb{K}$. Denote this element $\pi(a)$. According to Theorem 2.7.1 and proposition 2.7.2, there is the following proposition as the immediately result, and we can use this to prove the Kapranov's Theorem holds when the valuation has rank n.

Proposition 4.2.11. *Let k be an algebraically closed field with a nontrivial valuation ν, and f is a Laurent polynomial in m variables over k. Fixing a weight vector $\mathbf{w} \in \mathcal{V}(\text{trop}(f)) \bigcap \Gamma^m$ and a point $\mathbf{A} \in (\mathbb{K}^\times)^m$ such that $\mathbf{A} \in V(\text{in}^\nu_\mathbf{w}(f))$, then there exists a point $\mathbf{a} \in V(f)$ with $\nu(\mathbf{a}) = \mathbf{w}$ and $\pi(\mathbf{a}) = \mathbf{A}$.*

Kapranov proved the above result in the rank 1 case. Our purpose is to show that proposition 4.2.11 also holds when rank of ν is $n > 1$. The method is to deduce it from the rank 1 case applied n times in sequence. Then proposition we are trying to prove is

Proposition 4.2.12. *Let ν be a rank n valuation on an algebraically closed field k, and f is a polynomial in m variables over k. Fixing a weight vector $\mathbf{w} \in \mathcal{V}(\text{trop}(f)) \cap \Gamma_1^m \times \ldots \times \Gamma_n^m$ and a point $\mathbf{A} \in (k_1^\times)^m \times (k_2^\times)^m \ldots \times (k_n^\times)^m$ such that $\mathbf{A} \in V(\text{in}^\nu_\mathbf{w}(f))$ where $k_n = \mathbb{K}$ is the residue field of ν. Then there exists a point $\mathbf{a} \in V(f)$ with $\nu(\mathbf{a}) = \mathbf{w}$ and $\pi(\mathbf{a}) = \mathbf{A}$.*

Proof. Let the valuation ν be a rank n valuation, by the manipulation described in proposition 3.6.1, ν can be reduced as an n-step valuation $(v_1, \ldots, v_n)$ finally. For clarifying the notations, recalling the list at beginning of this Chapter

$$\begin{array}{ccc}
\nu : k \to \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_n & \sigma : \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_n \to k & \pi : R \to \mathbb{K} \\
v_1 : k \to \Gamma_1 & \sigma_1 : \Gamma_1 \to k & \pi_1 : R_1 \to k_1 \\
v_2 : k_1 \to \Gamma_2 & \sigma_2 : \Gamma_2 \to k_1 & \pi_2 : R_2 \to k_2 \\
\vdots & \vdots & \vdots \\
v_n : k_{n-1} \to \Gamma_n & \sigma_n : \Gamma_n \to k_{n-1} & \pi_n : R_n \to k_n
\end{array}$$

Each k_i is the residue field of v_i for $1 \le i \le n-1$ and by proposition 3.6.5, σ induces the splittings of $v_1, \ldots, v_n$ where are $\sigma_1, \ldots, \sigma_n$ correspondingly. In addition, let the image in the residue fields $k_1, k_2, \ldots, k_n$ represented by the notations $\pi_1, \pi_2, \ldots, \pi_n$.

By proposition 4.2.5, we have $\text{in}^\nu_\mathbf{w}(f) = \text{in}^{v_n}_{\mathbf{w}_n} \ldots \text{in}^{v_1}_{\mathbf{w}_1}(f)$. Let $\mathbf{A} \in V(\text{in}^\nu_\mathbf{w}(f))$, then $\mathbf{A} \in V(\text{in}^{v_n}_{\mathbf{w}_n} \ldots \text{in}^{v_1}_{\mathbf{w}_1}(f))$

Now we have $\mathbf{A}$ be a point in $V(\text{in}^{v_n}_{\mathbf{w}_n} \ldots \text{in}^{v_1}_{\mathbf{w}_1}(f))$. Let $\mathbf{A} = \mathbf{A}_n$ and note that $\mathbf{w}_n \in \mathcal{V}(\text{trop}_{v_n}(\text{in}^{v_{n-1}}_{\mathbf{w}_{n-1}} \ldots \text{in}^{v_1}_{\mathbf{w}_1}(f)))$. By proposition 4.2.11, there exists a point let's say $\mathbf{A}_{n-1} \in V(\text{in}^{v_{n-1}}_{\mathbf{w}_{n-1}} \ldots \text{in}^{v_1}_{\mathbf{w}_1}(f))$ such that $v_n(\mathbf{A}_{n-1}) = \mathbf{w}_n$ and $\pi_n(\mathbf{A}_{n-1}) = \mathbf{A}_n$.

$$\mathbf{w}_n \in \mathcal{V}(\mathrm{trop}_{v_n} \mathrm{in}^{v_{n-1}}_{\mathbf{w}_{n-1}} \ldots \mathrm{in}^{v_1}_{\mathbf{w}_1}(f))$$

$$\mathbf{A}_{n-1} \in V(\mathrm{in}^{v_{n-1}}_{\mathbf{w}_{n-1}} \ldots \mathrm{in}^{v_1}_{\mathbf{w}_i}(f)) \nearrow \searrow$$

$$\mathbf{A}_n \in V(\mathrm{in}^{v_n}_{\mathbf{w}_n} \ldots \mathrm{in}^{v_1}_{\mathbf{w}_1}(f))$$

Suppose $\mathbf{w}_{n-1} \in \mathcal{V}(\mathrm{trop}_{v_{n-1}}(\mathrm{in}^{v_{n-2}}_{\mathbf{w}_{n-2}} \ldots \mathrm{in}^{v_1}_{\mathbf{w}_1}(f)))$, then we can apply proposition 4.2.11 again. There exists a point $\mathbf{A}_{n-2} \in V(\mathrm{in}^{v_{n-2}}_{\mathbf{w}_{n-2}} \ldots \mathrm{in}^{v_1}_{\mathbf{w}_1}(f))$ with $v_{n-1}(\mathbf{A}_{n-2}) = \mathbf{w}_{n-1}$ and $\pi_{n-1}(\mathbf{A}_{n-2}) = \mathbf{A}_{n-1}$.

$$\mathbf{w}_{n-1} \in \mathcal{V}(\mathrm{trop}_{v_{n-1}} \mathrm{in}^{v_{n-2}}_{\mathbf{w}_{n-2}} \ldots \mathrm{in}^{v_1}_{\mathbf{w}_1}(f))$$

$$\mathbf{A}_{n-2} \in V(\mathrm{in}^{v_{n-2}}_{\mathbf{w}_{n-2}} \ldots \mathrm{in}^{v_1}_{\mathbf{w}_i}(f)) \nearrow \searrow$$

$$\mathbf{A}_{n-1} \in V(\mathrm{in}^{v_{n-1}}_{\mathbf{w}_{n-1}} \ldots \mathrm{in}^{v_1}_{\mathbf{w}_1}(f))$$

Therefore we can hypothesis that the process above holds until $n-k$. Then we have $\mathbf{A}_{n-k} \in V(\mathrm{in}^{v_{n-k}}_{\mathbf{w}_{n-k}} \ldots \mathrm{in}^{v_1}_{\mathbf{w}_1}(f))$ and let $\mathbf{w}_{n-k} \in \mathcal{V}(\mathrm{trop}_{n-k}(\mathrm{in}^{v_{n-k+1}}_{\mathbf{w}_{n-k+1}} \ldots \mathrm{in}^{v_1}_{\mathbf{w}_1}(f)))$. We can apply the proposition 4.2.11 again, so there exists a point $\mathbf{A}_{n-k+1} \in V(\mathrm{in}^{v_{n-k+1}}_{\mathbf{w}_{n-k+1}} \ldots \mathrm{in}^{v_1}_{\mathbf{w}_1}(f))$ with $v_{n-k}(\mathbf{A}_{n-k+1}) = \mathbf{w}_{n-k}$ and $\pi_{n-k}(\mathbf{A}_{n-k+1}) = \mathbf{A}_{n-k}$.

$$\mathbf{w}_{n-k} \in \mathcal{V}(\mathrm{trop}_{n-k}(\mathrm{in}^{v_{n-k+1}}_{\mathbf{w}_{n-k+1}} \ldots \mathrm{in}^{v_1}_{\mathbf{w}_1}(f)))$$

$$\mathbf{A}_{n-k+1} \in V(\mathrm{in}^{v_{n-k+1}}_{\mathbf{w}_{n-k+1}} \ldots \mathrm{in}^{v_1}_{\mathbf{w}_1}(f)) \nearrow \searrow$$

$$\mathbf{A}_{n-k} \in V(\mathrm{in}^{v_{n-k}}_{\mathbf{w}_{n-k}} \ldots \mathrm{in}^{v_1}_{\mathbf{w}_1}(f))$$

Therefore by the induction above we can repeat the progress to the point $\mathbf{a} \in V(f)$. According to corollary 4.2.9, there exists a $\mathbf{w} = \begin{pmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n \end{pmatrix}$ which $\mathrm{trop}_\nu(f)$ tropically vanishes at $\mathbf{w}$. And by proposition 3.6.5, σ induces $\sigma_1, \sigma_2, \ldots \sigma_n$ where are the splitting of $v_1, v_2, \ldots, v_n$ correspondingly. Then we have $\nu(a) = \mathbf{w}$. Hence for the polynomial $f \in k[x_1^{\pm}, \ldots, x_m^{\pm}]$ and the rank n valuation ν on k, fixing a $\mathbf{w} \in \mathcal{V}(\mathrm{trop}_\nu(f)) \subset (\mathbb{R}^m)^n$ and let $\mathbf{A} = \mathbf{A}_n \in V(\mathrm{in}^\nu_{\mathbf{w}}(f)) = V(\mathrm{in}^{v_n}_{\mathbf{w}_1} \ldots \mathrm{in}^{v_1}_{\mathbf{w}_1}(f)) \subset (k_n^\times)^m$ there exists a point $\mathbf{a} \in V(f)$ with $\nu(\mathbf{a}) = \mathbf{w}$ and the its image in k_n is $\mathbf{A}$. $\square$

Therefore for any polynomial $f \in k[x_1^{\pm}, \ldots, x_m^{\pm}]$ with a rank n valuation ν, proposition 4.2.12 holds. Since $\mathrm{in}_{\mathbf{w}_n}^{v_n} \ldots \mathrm{in}_{\mathbf{w}_1}^{v_1}(f)$ is not a monomial, then $\mathrm{in}_{\mathbf{w}}^{\nu}(f)$ is not monomial neither. Hence $\mathbf{w} \in \mathcal{V}(\mathrm{trop}_{\nu}(f))$, meanwhile $\mathbf{a} \in V(f)$ and $\nu(\mathbf{a}) = \mathbf{w}$ then we have

$$\mathcal{V}(\mathrm{trop}_{\nu}(f)) = \mathrm{trop}_{\nu}(V(f))$$

when the valuation is rank $n > 1$. Then we prove the Kapranov's theorem holds when valuation is rank $n > 1$.

Example 4.2.13. Let $k = k_1\{\{t\}\}$ and $k_1 = \mathbb{C}\{\{s\}\}$, and $f = 2x + sy \in k[x, y]$. Set ν be a rank 2 valuation on k, as many examples in this paper, let $\nu(c) = (c_1, c_2)$ where c_1 is the lowest exponent of t in c and c_2 is the exponent of s of the coefficient of the leading term in c. Clearly, the classic hypersurface of f is

$$V(f) = \{(\lambda, -\frac{2}{s}\lambda) \ : \ \lambda \in k\}.$$

By proposition 3.6.1, reducing ν as (v_1, v_2), v_1 is on k and v_2 is on k_1. Then the tropicalization of f via v_1 is

$$\mathrm{trop}_{v_1}(f) = \min\{x, y\}.$$

Consider the coordinate-wise valuation $\Big(v_1(\lambda), v_1(-\frac{2}{s}\lambda)\Big) = \Big(v_1(\lambda), v_1(-\frac{2}{s}) + v_1(\lambda)\Big) = (v_1(\lambda), v_1(\lambda))$. It is easy to see that the image $(v_1(\lambda), v_1(\lambda)) \in \Gamma_1^2$ are those weight vectors which make $\mathrm{trop}_{v_1}(f)$ contain more than 1 term, so we can let the tropical hypersurface $\mathrm{trop}_{v_1}(V(f))$ be the set of the points which we let them be $\mathbf{w}_1$ such that

$$\mathbf{w}_1 = (v_1(\lambda), v_1(\lambda)) \ : \ \lambda \in k.$$

Then we consider the initial form of f via v_1 with respect to the weight vectors in this set is $\mathrm{in}_{\mathbf{w}_1}^{v_1}(f) = 2x + sy$. Now suppose a point $\mathbf{a} = (s, -2)$ which is in $V(f)$, then we have $v_1(\mathbf{a}) = (0, 0)$ and $\pi_1(\mathbf{a}) = (s, -2)$. It is clear that $(0, 0)$ is the weight vector which makes $\mathrm{trop}_{v_1}(f)$ contain at least two terms and the initial form $\mathrm{in}_{\mathbf{w}_1}^{v_1}(f)$ vanishes at $\pi_1(\mathbf{a}) = (s, -2)$. Then $\mathbf{w}_1 = (0, 0)$ and $\mathbf{A}_1 = (s, -2)$.

Now consider that tropicalization of $\text{in}^{v_1}_{\mathbf{w}_1}(f)$ via v_2 such that

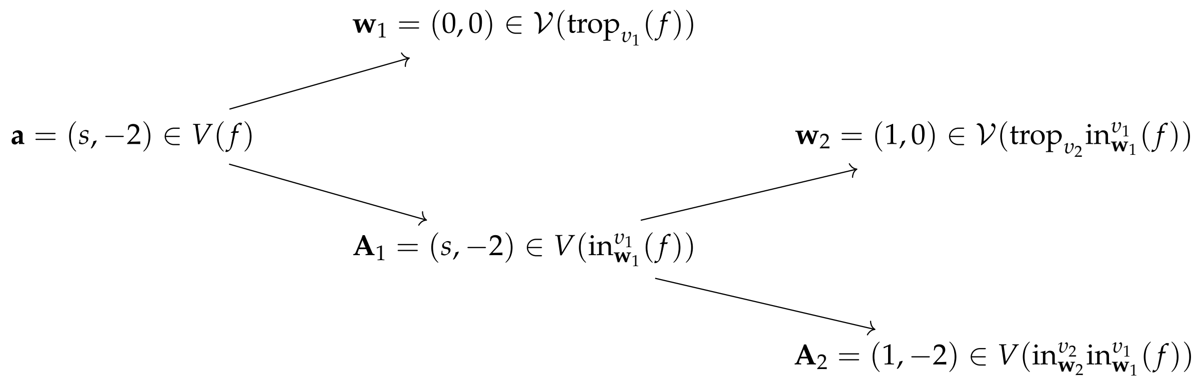
$$\text{trop}_{v_2}(\text{in}^{v_1}_{\mathbf{w}_1}(f)) = \min\{x, 1+y\},$$

then the tropical hypersurface $\text{trop}_{v_2}(V(\text{in}^{v_1}_{\mathbf{w}_1}(f)))$ is the set of the points which we let them be $\mathbf{w}_2$ such that

$$\{\mathbf{w}_2 = (\mu, \mu - 1) \in \Gamma^2 \ : \ \mu \in k_2\}.$$

So the initial form of $\text{in}^{v_1}_{\mathbf{w}_1}(f)$ via v_2 with respect to the weight vectors above is $\text{in}^{v_2}_{\mathbf{w}_2}\text{in}^{v_1}_{\mathbf{w}_1}(f) = 2x + y$. We have $\mathbf{A}_1 = (s, -2)$, then $v_2(\mathbf{A}_1) = (1, 0)$ and $\pi_2(\mathbf{A}_1) = (1, -2)$. It is easy to check that $v_2(\mathbf{A}_1)$ is the weight vector $\mathbf{w}_2$ and $\text{in}^{v_2}_{\mathbf{w}_2}\text{in}^{v_1}_{\mathbf{w}_1}(f)$ vanishes at $\pi_2(\mathbf{A}_1)$.

Therefore we have $\mathbf{a} = (s, -2)$, $\mathbf{w} = \begin{pmatrix} 0,0 \\ 1,0 \end{pmatrix}$ and $\mathbf{A} = (1, -2)$ such that $\nu(\mathbf{a}) = \mathbf{w}$ and $\pi(\mathbf{a}) = \mathbf{A}$, which is shown by the following diagram.



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