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# Fracterm Calculus for Signed Common Meadows

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**Abstract**

A common meadow is an enrichment of a field with a division operator and an error value to make division total. A signed common meadow enriches a common meadow with a sign function that can be equationally axiomatised; the sign function can simulate an ordering on the underlying field but is not limited to orderings. In particular, of mathematical interest are the *weakly signed common meadows*. The prime example of a weakly signed common meadow is an expansion of a common meadow of complex numbers with a weak sign function. We show that all common meadows may be enlarged to a weakly signed common meadow. A special case are the *4-signed common meadows*, which are precisely the enlargements of ordered fields. To illustrate the equational calculus for signed common meadows, we use it as a foundation for building a probability calculus and derive some classical formulae.

## 1 Introduction

Orderings occur naturally in number systems and are essential to their practical usefulness. Mathematically, the theory of numerical orderings lifts to become the algebra and logic of ordered rings and fields. Now, rings and fields are somewhat deficient to analyse arithmetic systems that are customised to be computer arithmetics. First, the standard operations of addition, multiplication and inverse, $+, \cdot, -$ of fields need to be augmented with an explicit division operator $\div$; further, this division operator needs to be made a total function so that $x \div 0$ is defined for all $x$. A second reason is that the theory of data types makes primary the use of equations, or conditional equations, for specification and reasoning, and uses initial algebra semantics and term rewriting. The axioms for fields makes their logic more complex, beyond equational reasoning. Fields are definable in first order logic but cannot be defined by equations (by Birkhoff's Theorem [26]) or conditional equations.

## 1.1 Meadow theories

In an extensive study of the algebra of arithmetical structures, to make suitable abstract data types for specifying, reasoning and computing, we have explored several ways of addressing these deficiencies in fields. In [15], we have introduced division operators into fields to make algebras we call *meadows*, and we have examined different totalisation methods for division and found equational axiomatisations for different meadows as abstract data types. For example, we have (i) given meadows of rational numbers equational specifications under initial algebra semantics; (ii) established the basic algebra of calculating with fractions in meadows; (iii) established the completeness of the axiomatisations for some types of the meadow with respect to equational formulae.

In this paper, we turn to the role of orderings and study how orderings can be introduced into meadows so as to preserve their suitability as abstract data types for computation, and the standard methods based on equational specifications and reasoning.

Different types of meadow arise from how division is totalised on a field. From our previous work, a particular semantic option for totalising division stands out which we will study here: the *common meadow* in which

$$x \div 0 = \bot$$

for all $x$, and where $\bot$ plays the role of an error value; we give formal definitions later (Definition 2.2).

Here, we will consider augmenting common meadows with orderings whilst preserving their status as abstract data types with equational specifications. Recall that an ordered field is a field $F$ equipped with a total order $\leq$ that is compatible with its operations (Definition 3.1). In a field the relation $\leq$ gives rise to the *sign operation* $\mathsf{s}(-)$ defined for $x \in F$ by

$$\mathsf{s}(x) = 1 \text{ if } x > 0, \text{ or } \mathsf{s}(x) = 0 \text{ if } x = 0, \text{ or } \mathsf{s}(x) = -1 \text{ if } x < 0.$$

We will show how the idea of a sign function can be abstracted to create equational axiomatic theories of *signed common meadows* as abstract data types that fit closely with the established theories of common meadows. In this way, we can establish ordering in our arithmetical abstract data types.

The axioms for a sign function reveal some new insights of arithmetical interest. First, we find that a special case of signed common meadows, the *4-signed common meadows*, are precisely the enlargements of ordered fields with a four-valued sign function. But signed common meadows suggest a remarkable generalization, the *weakly signed common meadows*, which become the focus of our theory building. The prime example of a weakly signed common meadow is an expansion of a common meadow of complex numbers with a weak sign function; in contrast, the field of complex numbers cannot be ordered, nor can its common meadow. In fact, the distinction between signed and weakly signed is significant as we prove:

**Theorem**. *All common meadows of characteristic* 0 *may be enlarged to a weakly signed common meadow.*

To illustrate the equational specifications of signed common meadows, we define axioms for a probability calculus, in which we derive some classic formulae for inverse probabilities that involve orderings and division.

Finally, we compare this development with the introduction of ordering via sign functions into another type of meadow, the involutive meadow, in which division is made total by setting $x \div 0 = 0$.

### 1.2 Structure of the paper

In Section 2, we gather necessary material about fields and meadows. In Section 3, we recall the notion of an ordered field and turn to the sign function and, specifically, its axiomatisation. We give a number of theorems characterising the sign function, relating it to ordered fields. In Section 4, we develop the meadow of complex numbers and show that it can be given a weak sign function and, in Section 5, we prove the general theorem above. We derive the Bayes-Price Theorem and its expansion using the fracterm calculus of signed common meadows in Section 6. In Section 7, we compare our results with the use of sign functions on involutive meadows. Finally, we list further mathematical questions arising from this first exploration of order in Section 8.

We asume the reader is familiar with the theory of abstract data types [31] and field theory [38, 33].

## 2 Fields and meadows

### 2.1 Constructing common meadows

Recall the definitions of a commutative ring and field. A field is a commutative ring in which non-zero elements are invertible. Whilst commutative rings are defined by equations, fields are axiomatically definable by $\Pi^0_1$ first order formulae. Whilst fields are defined axiomatically, we have chosen to define meadows as constructions over fields – like matrices and polynomials.

**Definition 2.1.** *A* meadow *expands the operations of a field with a division operation. A unary inverse operation* $^{-1}$ *may also be used.*

Our study of arithmetic algebras suitable for computation led us to define this type of meadow.

**Definition 2.2.** *A* common meadow *is an enlargement of a field* $F$ *by*

*(i) extending its domain with an additional element (a peripheral number) named* $\bot$*, and*

*(ii) expanding its operations with the constant $\bot$ and a function $\div$ for division, wherein division is made total by having $\frac{x}{0} = \bot$ for all $x \in F \cup \{\bot\}$, and such that $\bot$ is absorptive w.r.t. to each operator, i.e., if $\bot$ appears as an argument to any operation then the result is $\bot$.*

*(iii) The class of all common meadows we denote* $\mathsf{CM}$.

So, common meadows have the general form

$$(F \cup \{\bot\} \mid 0, 1, \bot, x + y, -x, x \cdot y, x/y)$$

where $F$ is a field and $\bot$ is an element that behaves like an error value. The idea of a common meadow was introduced in [12].

The signatures of rings and fields are the same and are denoted $\Sigma_{\mathsf{r}}$. The signature of a meadow is denoted $\Sigma_{\mathsf{m}}$. The signature $\Sigma_{\mathsf{cm}}$ of a common meadow extends the signature $\Sigma_{\mathsf{m}}$ of meadows with the constant $\bot$. The addition of $\div$ introduces the all important fractions which in the setting of meadows can be defined unambiguously:

**Definition 2.3.** *Any term over a signature containing $\Sigma_{\mathsf{m}}$ that contains a division operator as its leading function symbol is called a* fracterm.

The terminology of fracterms is due to [14] and with more detail such terminology has been developed and motivated in [5].

For signatures, we adopt the convention that constants stand for their interpretation by default and that we will speak of a constant symbol if the name is meant rather than its interpretation. We notice that when writing $\frac{2}{4}$ we will by default think of a number as its meaning while a fracterm will be meant only when that is made explicit or is clear from the context.

The following is immediate from Definition 2.2:

**Lemma 2.1.** *Every field $F$ can be enlarged to a common meadow $F_{\bot}$.*

Thus, important arithmetics are the common meadows of rational numbers $\mathbb{Q}_{\bot}$, real numbers $\mathbb{R}_{\bot}$, and complex numbers $\mathbb{C}_{\bot}$. This last algebra plays an important role in the theory of sign functions in Section 4.

**Example 2.1.** The field $\mathbb{C}$ of complex numbers can be made a common meadow $\mathbb{C}_{\bot}$ in the obvious way. Normally, when working with the complex numbers there are a number of important extra features we invariably need to employ. Most obviously, there is the imaginary number $\mathsf{i} = +\sqrt{-1}$ which we will need to add as a constant. Thus, we will extend the signature $\Sigma_{\mathsf{r}}$ of fields with a constant $\mathsf{i}$, so that common meadows over complex numbers have signature $\Sigma_{\mathsf{cm,i}}$. We mention some further operations later in Section 4.

## 2.2 Equational axioms for common meadows

Just as rings and fields are defined axiomatically, so we sought axiomatisations of common meadows. The totalisation of division presents opportunities to replace the logical complexity of axiomatisations from $\Pi^0_1$

formulae to equations and reasoning with equational calculi. The importance of division led us to call these equation-based formal systems *fracterm calculi* (after Definition 2.3). Fracterm calculi are quite dependent on how one deals with partiality, i.e., with division by zero.

Following the investigation of a number of formalisations, we settled upon a set, here named as $E_{\mathsf{ftc-cm}}$, of 18 equations in [20]:

**Proposition 2.1.** *Any common meadow satisfies the equational axioms of* $E_{\mathsf{ftc-cm}}$ *listed in Tables 1 and 2.*

Table 1 reveals the relationship with commutative rings: the equations 1-10 are derivable from commutative ring axioms (hence 'weak'). The equations of Table 2 add the axioms for $\div$.

In various papers, we have built a portfolio of results and applications based on the axioms in $E_{\mathsf{ftc-cm}}$ (and their equivalents). These results establish that the axioms of $E_{\mathsf{ftc-cm}}$ offer a stable foundation for working with known common meadows, and for building an abstract axiomatic theory based on equational reasoning.

Thus, at this point, we must be careful to keep track of results specific to

(a) common meadows as semantic constructions from fields – rather like rings of polynomials, power series and matrices; or

(b) all the algebras that satisfy the axioms of $E_{\mathsf{ftc-cm}}$.

**Definition 2.4.** *Let* $\mathsf{Alg}(E_{\mathsf{ftc-cm}})$ *be the class of all algebras that are models of* $E_{\mathsf{ftc-cm}}$.

One such result is this from [22]:

**Definition 2.5.** *The* equational theory of common meadows *is the set*

$$Eqn(\mathsf{CM}) = \{e \mid \forall A \in \mathsf{CM}.A \models e\}$$

*of all equations over* $\Sigma_{cm}$ *that are true in all common meadows.*

**Theorem 2.1.** *The finite equational axiomatisation* $E_{\mathsf{ftc-cm}}$*, equipped with equational logic, is sound for the class* $\mathsf{CM}$ *of all common meadows, and complete for the equational theory* $Eqn(\mathsf{CM})$ *for common meadows. Thus, for any equation* $e$ *over* $\Sigma_{cm}$*,*

$$E_{\mathsf{ftc-cm}} \vdash e \text{ if, and only if, } e \in Eqn(\mathsf{CM}).$$

A corollary of the theorem is that the equational theory for common meadows is algorithmically decidable.

## 2.3 Conditional equational logic for common meadows

Below we will discuss probability calculus in the setting of common meadows, and we find that the use of conditional equations is needed. For that reason we will introduce the Additional Value Law as introduced in [12] and listed in Table 3. In that paper additional value is proposed as another name for $\bot$.

$$(x+y)+z = x+(y+z) \tag{1}$$
$$x+y = y+x \tag{2}$$
$$x+0 = x \tag{3}$$
$$x+(-x) = 0\cdot x \tag{4}$$
$$x\cdot(y\cdot z) = (x\cdot y)\cdot z \tag{5}$$
$$x\cdot y = y\cdot x \tag{6}$$
$$1\cdot x = x \tag{7}$$
$$x\cdot(y+z) = (x\cdot y)+(x\cdot z) \tag{8}$$
$$-(-x) = x \tag{9}$$
$$0\cdot(x\cdot x) = 0\cdot x \tag{10}$$
$$x+\bot = \bot \tag{11}$$

Table 1: $E_{\mathsf{wcr},\bot}$ axioms for weak commutative rings with $\bot$

`import:` $E_{wcr,\bot}$

$$x = \frac{x}{1} \tag{12}$$
$$\frac{x}{y}\cdot\frac{u}{v} = \frac{x\cdot u}{y\cdot v} \tag{13}$$
$$\frac{x}{y}+\frac{u}{v} = \frac{(x\cdot v)+(y\cdot u)}{y\cdot v} \tag{14}$$
$$\frac{x}{y+0\cdot z} = \frac{x+0\cdot z}{y} \tag{15}$$
$$\bot = \frac{1}{0} \tag{16}$$

Table 2: $E_{\mathsf{ftc-cm}}$: equations for fracterm calculus for common meadows

$$\frac{1}{x} = \bot \to 0 \cdot x = x \tag{17}$$

Table 3: $E_{\mathsf{AVL}}$: additional value law

### 2.4 Proof rule $\mathsf{R_{cm}}$ for conditional equational fracterm calculus

Besides AVL we will also make use of an additional proof rule for conditional equational logic taken from [23].

The following proof rule is sound for common meadows, with $E$ a conjunction of equations:

$$\mathsf{R_{cm}}\ \frac{\vdash E \wedge t = \bot \to r = \bot,\ \vdash E \wedge r = \bot \to t = \bot}{\vdash E \to 0 \cdot t = 0 \cdot r}$$

In [23] it is shown that all conditional equations true in all common meadows are also derivable from the axioms in Table 2 and 3 with the help of proof rule $\mathsf{R_{cm}}$.

### 2.5 Division by zero

The objective of totalizing division is to obtain a workable and practical fracterm calculus. Common meadows provide one of several ways for totalizing division in a field of numbers. There are a number of semantical options in practical computing – such as using values error, $\infty$, $+\infty$, $NaN$ – and we have also constructed equational specifications and fracterm calculi for these other semantics:

*Involutive meadows*, in which $1/0 = 0$, [15];
*Wheels*, in which one external $\infty$ is used for totalisation $1/0 = \infty = -1/0$, together with an additional external error element $\bot$ to control the side-effects of infinity, [36, 27, 18];
*Transrationals*, in which besides the error element $\bot$, two external signed infinities $+\infty, -\infty$ are added and division is totalised by setting $1/0 = \infty$ and $-1/0 = -\infty$, [1, 30, 16].

In practice, these semantic conventions can be found in theorem provers, common calculators, exact numerical computation and, of course, floating point computation. Indeed, we have developed a new model, the symmetric transrationals [21]. However, their fracterm calculi are not without complications. For example, a key property for any fracterm calculus is *fracterm flattening*, a property first obtained in [12] for the fracterm calculus for common meadows. In [19] it was shown that the presence of fracterm flattening imposes requirements on a meadow which come very close to it being a common meadow.

A fracterm calculus for the case of a *partial* division function has been outlined in [24], the complexity of that approach manifesting itself in the preferred logic for reasoning (a 'short-circuit' logic, after [11]).

# 3 The sign function

## 3.1 Ordered fields

**Definition 3.1.** *An* ordered field $F_{\leq}$ *is a field $F$ with an ordering relation that is*

*(i) reflexive, antisymmetric, and transitive;*

*(ii) total: for all $x, y \in F_{\leq}$, either $x \leq y$ or $y \leq x$;*

*(iii) compatible with the operations: for all $x, y, a \in F_{\leq}$, if $x \leq y$ then $x + a \leq y + a$ and $0 \leq x$ and $0 \leq y$ then $0 \leq x \cdot y$.*

The basic theory of ordered fields can be found in many algebra textbooks, starting with standard works such as [38, 33].

## 3.2 Axioms for signed common meadows

The focus of attention is now axiomatisation. The sign function is most helpful for introducing ordering into the framework of an equational logic. We will build on the characterisation $E_{\mathsf{ftc-cm}}$ of common meadows and provide *conditional* equational axioms for common meadows equipped with sign functions.

**Definition 3.2.** *A* weakly signed common meadow *is a common meadow equipped with a sign function $\mathsf{s}(-)$ which satisfies the axioms of Table 4.*

Now, whilst $\mathsf{s}(\frac{1}{x}) = \frac{1}{\mathsf{s}(x)}$, it is not in general the case that

$$\mathsf{s}(\frac{1}{x}) = \mathsf{s}(x) \text{ because } \mathsf{s}(\frac{1}{0}) = \mathsf{s}(\bot) = \bot \neq 0 = \mathsf{s}(0)$$

This deficiency is addressed by the axiom 31 in Table 5. We prefer the axioms for signed meadows to have an equational form with the exception of $\mathsf{AVL}$ in order to be able to use the model theory for common meadows as it has been developed in [23]. The conditional equations 26, 27 and 28 can be easily inferred from the somewhat less intuitive axioms of Table 5. It is not an option to replace the conditional axioms for weakly signed common meadows by the equations for signed meadows because equations 30 and 31 are both invalid in the common meadow of complex numbers enriched with a sign function given by $\mathsf{s}(0) = 0$, $\mathsf{s}(\bot) = \bot$ and $\mathsf{s}(x) = \frac{x}{|x|}$ for non-zero, non-$\bot$ $x$, which is the main example of a weakly signed common meadow we will discuss in Section 4 below.

To see these facts we work in the axiom system of Table 5 as follows: first notice that by substituting $\mathsf{s}(x)$ for $x$ in equation 31 and applying equation 24 on derives $\mathsf{s}^2(x) \cdot \mathsf{s}(x) = \mathsf{s}(x)$.

In order to prove $\mathsf{s}(x) = 0 \cdot y \to x = 0 \cdot y$ assume $\mathsf{s}(x) = 0 \cdot y$. Now $x = \mathsf{s}^2(x) \cdot x = (0 \cdot y)^2 \cdot x = (0 \cdot y) \cdot x$. Moreover from $\mathsf{s}(x) = 0 \cdot y$ we find $0 \cdot \mathsf{s}(x) = 0 \cdot (0 \cdot y)$ and thus $\mathsf{s}(0) \cdot \mathsf{s}(x) = 0 \cdot y$ and $\mathsf{s}(0 \cdot x) = 0 \cdot y$ and with equation 25 one obtains $0 \cdot x = 0 \cdot y$, which in combination with $x = (0 \cdot y) \cdot x$ yields $x = (0 \cdot y) \cdot y = 0 \cdot y$.

For conditional equation 28 it suffices to apply equation 31: $x = \mathsf{s}^2(x) \cdot x = \bot^2 \cdot x = \bot$.

In order to derive conditional equation 26 we assume $s(x) = s(y)$ multiply both sides of equation 30 with $s(x)$ for the LHS we find $\mathsf{s}(x) \cdot$

$$\texttt{import: } E_{\mathsf{ftc-cm}}$$
$$\texttt{import: } E_{\mathsf{AVL}}$$
$$\mathsf{s}(0) = 0 \tag{18}$$
$$\mathsf{s}(1) = 1 \tag{19}$$
$$\mathsf{s}(-1) = -1 \tag{20}$$
$$\mathsf{s}(\bot) = \bot \tag{21}$$
$$\mathsf{s}(x \cdot y) = \mathsf{s}(x) \cdot \mathsf{s}(y) \tag{22}$$
$$\mathsf{s}(\frac{x}{y}) = \frac{\mathsf{s}(x)}{\mathsf{s}(y)} \tag{23}$$
$$\mathsf{s}(\mathsf{s}(x)) = \mathsf{s}(x) \tag{24}$$
$$\mathsf{s}(x + 0 \cdot y) = \mathsf{s}(x) + 0 \cdot y \tag{25}$$
$$\mathsf{s}(x) = \mathsf{s}(y) \to \mathsf{s}(x + y) = \mathsf{s}(y) \tag{26}$$
$$\mathsf{s}(x) = 0 \cdot y \to x = 0 \cdot y \tag{27}$$
$$\mathsf{s}(x) = \bot \to x = \bot \tag{28}$$

Table 4: $E_{\mathsf{ftc-wscm}}$: axioms for weakly signed common meadows

$\mathsf{s}(\mathsf{s}(x)\cdot x+\mathsf{s}(y)\cdot y) = \mathsf{s}(\mathsf{s}(x))\cdot\mathsf{s}(\mathsf{s}(x)\cdot x+\mathsf{s}(y)\cdot y) = \mathsf{s}(\mathsf{s}^2(x)\cdot x+\mathsf{s}(x)\cdot(\mathsf{s}(y)\cdot y)) = \mathsf{s}(x + \mathsf{s}(y) \cdot (\mathsf{s}(y) \cdot y)) = \mathsf{s}(x + \mathsf{s}^2(y) \cdot y) = \mathsf{s}(x + y)$. For the RHS we find: $\mathsf{s}(x)\cdot((\mathsf{s}^2(x)+\mathsf{s}^2(y))+(-(\mathsf{s}^2(x)\cdot\mathsf{s}^2(y)))) = \mathsf{s}(y)\cdot((\mathsf{s}^2(y)+\mathsf{s}^2(y))+(-(\mathsf{s}^2(y)\cdot \mathsf{s}^2(y)))) = \mathsf{s}(y) \cdot ((\mathsf{s}^2(y) + \mathsf{s}^2(y)) + (-\mathsf{s}^2(y))) = \mathsf{s}(y) \cdot \mathsf{s}^2(y) = \mathsf{s}(y)$.

Another consequence of the equations in Table 5 is $\mathsf{s}(x + \mathsf{s}(x)) = \mathsf{s}(x)$. A proof is found by substituting $s(x)$ for $y$ in equation 30 and then multiplying both sides with $\mathsf{s}(x)$.

**Definition 3.3.** *A* signed common meadow *is a common meadow equipped with a sign function* $\mathsf{s}(-)$ *which satisfies the axioms of Table 5.*

So, signed common meadows have the general form

$$F_{\bot,\mathsf{s}} = (F \cup \{\bot\} \mid 0, 1, \bot, x + y, -x, x \cdot y, x/y, \mathsf{s})$$

On a weakly signed common meadow one may define an absolute value or modulus function $|-|$ (see Table 6).

## 3.3 4-signed common meadows

An ordering of a meadow is enforced once it is required that the sign function can only take one of four values.

**Definition 3.4.** *A weakly signed common meadow is* 4-signed *if it satisfies the following disjunction:*

$$\mathsf{s}(x) = 0 \vee \mathsf{s}(x) = 1 \vee \mathsf{s}(x) = -1 \vee \mathsf{s}(x) = \bot.$$

The notion captures the signed case completely:

$$
\begin{aligned}
\texttt{import: } & E_{\mathsf{ftc-cm}} \\
\texttt{import: } & E_{\mathsf{AVL}} \\
\mathsf{s}(0) &= 0 \\
\mathsf{s}(1) &= 1 \\
\mathsf{s}(-1) &= -1 \\
\mathsf{s}(\bot) &= \bot \\
\mathsf{s}(x \cdot y) &= \mathsf{s}(x) \cdot \mathsf{s}(y) \\
\mathsf{s}(\frac{x}{y}) &= \frac{\mathsf{s}(x)}{\mathsf{s}(y)} \\
\mathsf{s}(\mathsf{s}(x)) &= \mathsf{s}(x) \\
\mathsf{s}(x + 0 \cdot y) &= \mathsf{s}(x) + 0 \cdot y \\
\mathsf{s}^2(x) &= \mathsf{s}(x) \cdot \mathsf{s}(x) && (29) \\
\mathsf{s}(\mathsf{s}(x) \cdot x + \mathsf{s}(y) \cdot y) &= (\mathsf{s}^2(x) + \mathsf{s}^2(y)) + (-(\mathsf{s}^2(x) \cdot \mathsf{s}^2(y)) && (30) \\
x &= \mathsf{s}^2(x) \cdot x && (31)
\end{aligned}
$$

Table 5: $E_{\mathsf{ftc-scm}}$: axioms for signed common meadows

**Proposition 3.1.** *A weakly signed common meadow is 4-signed if, and only if, it is signed.*

*Proof.* Let $F_{\bot,\mathsf{s}}$ be a weakly signed common meadow. First suppose that $F_{\bot,\mathsf{s}}$ is 4-signed. In order to show that $F_{\bot,\mathsf{s}}$ is signed it must be checked that $F_{\bot,\mathsf{s}} \models \mathsf{s}^2(x) \cdot x = x$. There are 4 cases for $\mathsf{s}(x)$ and in each case the verification is immediate.

For the other direction, we assume that $F_{\bot,\mathsf{s}}$ is signed. Consider $x \in F_{\bot,\mathsf{s}}$. It must be verified that $\mathsf{s}(x) = 0$ or $\mathsf{s}(x) = 1$ or $\mathsf{s}(x) = -1$ or $\mathsf{s}(x) = \bot$. If $x = 0$ or $x = \bot$ said verification is immediate. For non-zero and non-$\bot$ $x$ we know that $\mathsf{s}^2(x) \cdot x = x$; thus, in view of axioms 27 and 28 (both derivable from the equations of Table 5), $\mathsf{s}(x)$ differs from 0 and from $\bot$ so that (with equation 31), $\mathsf{s}(x) \cdot \mathsf{s}(x) = 1$. Now $\mathsf{s}(x)$ is an element of a field, equal to its own inverse, from which it follows that $\mathsf{s}(x) = 1$ or $\mathsf{s}(x) = -1$. □

The following fact expresses the soundness of the axioms for common meadows resulting from an ordered field:

**Theorem 3.1.** *Each ordered field $F_<$ can be enlarged to a signed common meadow $A$. Conversely, each signed common meadow $A$ is an enlargement of an ordered field $F_<$.*

*Proof.* First, enlarge $F$ to a common meadow $F_\bot$, then enrich $F_\bot$ with a sign function as follows: $\mathsf{s}(\bot) = \bot$, and for $a \in F$: if $a > 0$ then $\mathsf{s}(a) = 1$, if $a < 0$ then $\mathsf{s}(a) = -1$, and $\mathsf{s}(0) = 0$.

Conversely, the domain of the required field $F$ is taken as $F = A - \{\bot\}$. The ordering $<_{\mathsf{s}}$ on $A - \{\bot\}$ is given by

(i) $a <_{\mathsf{s}} b \iff \mathsf{s}(b-a) = 1$,

(ii) $a =_{\mathsf{s}} b \iff \mathsf{s}(a-b) = 0$ and,

(iii) $a \leq_{\mathsf{s}} b \iff \mathsf{s}(b-a)^2 = \mathsf{s}(b-a)$.

From equation 27 we find that $a =_{\mathsf{s}} b$ implies $a = b$. It follows from the definition of a 4-signed common meadow that for all non-$\bot$ $a$ and $b$ in $F$ either $a <_{\mathsf{s}} b$ or $a =_{\mathsf{s}} b$ or $b <_{\mathsf{s}} a$.

Reflexivity of $=_{\mathsf{s}}$ is immediate, and so is antisymmetry of $<_{\mathsf{s}}$. For transitivity assume $a <_{\mathsf{s}} b$ and $b <_{\mathsf{s}} c$, then $\mathsf{s}(b-a) = \mathsf{s}(c-b) = 1$ so that $\mathsf{s}(c-a) = \mathsf{s}((b-a)+(c-b)) = \mathsf{s}(b-a) = 1$ (with equation ReSign) whence $a <_{\mathsf{s}} c$. $\square$

### 3.4 Completeness for the conditional fracterm calculus of signed meadows

The following completeness result is available:

**Theorem 3.2.** *A conditional equation of the signature of signed meadows is true in all signed meadows if and only if it is provable from the axioms of signed meadows ($E_{\mathsf{ftc-scm}}$ of Table 5) in conditional equational logic extended with the additional proof rule $\mathsf{R}_{\mathsf{cm}}$.*

*Proof.* The proof follows the lines of the corresponding completeness proof in [23] without any further complication. We notice that the proof uses model theory for common meadows and that the proof is based on a characterisation of models of $E_{\mathsf{ftc-cm}}$ as discussed in [28] and [29]. $\square$

### 3.5 Equational specifications

The purpose of the equations $E_{\mathsf{ftc-wscm}}$ is to specify the essentials of weak signed common meadows. Among the confirmatory tests is this result:

**Proposition 3.2.** *The equational specification $E_{\mathsf{ftc-wscm}}$ constitutes an initial algebra specification of the data type of weakly signed common meadows of rational numbers.*

*Proof.* With induction on $n$, mainly using equation 26 one finds for all $n > 0$ and corresponding numerals $\underline{n}$ that $E_{\mathsf{ftc-wscm}} \vdash \mathsf{s}(\underline{n}) = 1$. Then it follows with equations 22 and 25 that for $n > 0$: $E_{\mathsf{ftc-wscm}} \vdash \frac{\underline{n}}{\underline{n}} = 1$, which is known to suffice from [12]. $\square$

**Proposition 3.3.** *The class of 4-signed common meadows has no equational specification.*

*Proof.* To see this let $E_{\mathsf{scm}}$ be the equational theory of the class of 4-signed meadows. One may add a new constant $c$ and consider the initial algebra $A$ of $E_{\mathsf{scm}}$ in the signature thus extended. Because each of $\mathsf{s}(\bot)$, $\mathsf{s}(c) = 0, \mathsf{s}(c) = 1$ and $\mathsf{s}(c) = -1$ is consistent with $E_{\mathsf{scm}}$ none of these is provable from $E_{\mathsf{scm}}$ so that in $A$, $\mathsf{s}(-)$ cannot take values in $\{0, 1, -1\}$. $\square$

# 4 Complex numbers

One of the most mathematically prominent models of the axioms of common meadows is an enlargement of a field of complex numbers. The field of complex numbers cannot be ordered, but it can possess a weak sign function as a common meadow.

## 4.1 Common meadows of complex numbers

Recalling Example 2.1, a common meadow of complex numbers has as a constant $\mathsf{i}$, such that $\mathsf{i} \cdot \mathsf{i} = -1$ is satisfied. We set:

$$E_{\mathsf{ftc-ccm}} = E_{\mathsf{ftc-cm}} \cup \{\mathsf{i} \cdot \mathsf{i} = -1\}.$$

The completeness result of [22] may or may not generalise in a straightforward manner. This is a question for which we have no answer yet (cf. Section 8.1).

**Problem 4.1.** *Is $E_{\mathsf{ftc-ccm}}$ complete for the equational theory of complex common meadows?*

## 4.2 Fracterm flattening for complex numbers

We find that fracterm flattening persists with the expansion of common meadows to signed common meadows for complex numbers. We notice that the notion of a fracterm is generic in the sense that it works for each signature containing (a name for) the division function $\div$.

**Definition 4.1.** *A fracterm $t$ is* flat *if it has the form $\frac{r}{s}$ where $r$ and $s$ have no occurrences of the division operator.*

**Proposition 4.1.** *(*Fracterm flattening.*) The specification $E_{\mathsf{ftc-ccm}}$ has fracterm flattening: for each term $t$ in the signature of complex common meadows, there is a flat fractcerm $r$ such that $E_{\mathsf{ftc-ccm}} \vdash t = r$.*

*Proof.* It is easy to check that the proof of fracterm flattening for fracterm calculus of [12] works just as well in the complex case. □

One might expect another form of flattening, *viz.* that each expression can be written in the form $P + \mathsf{i} \cdot Q$ with $P$ and $Q$ flat fracterms over the signature $\Sigma_m$ of meadows, i.e., not containing any occurrence of $\mathsf{i}$. That idea does not work, however:

**Proposition 4.2.** *Let $P \equiv \frac{1}{x+\mathsf{i}\cdot y}$, there are no flat fracterms $Q$ and $R$ over the signature of common meadows such that $P = Q + \mathsf{i} \cdot R$* all *complex common meadows.*

*Proof.* Suppose $Q$ and $R$ exist as required and let $Q \equiv \frac{a}{b}$ and $R \equiv \frac{c}{d}$ with $a, b, c, d$ polynomials. Suppose that for all complex common meadows and for all valuations, $x + \mathsf{i} \cdot y = 0$. Then either $b$ or $d$ or both must take value 0 so that $x + \mathsf{i} \cdot y$ divides either $b$ or $d$ or both. We assume that it divides $Q$ the other case being dealt with in the same manner.

We find that $b = (x + \mathsf{i} \cdot y) \cdot b'$ for some polynomial $b'$ (for instance $b' = x - \mathsf{i} \cdot y$ might work). Now $b'$ cannot be a real polynomial and it cannot have only factors $x + \mathsf{i} \cdot y$, so that it must have a different factor which in some cases (i.e., for some complex common meadow and for some valuation) has zeroes that are not zeroes for $x + \mathsf{i} \cdot y$ so that the RHS of $P = Q + \mathsf{i} \cdot R$ yields $\bot$ while the LHS does not, so that LHS = RHS cannot be satisfied. □

## 4.3 Sign and modulus

Here the focus is on the weakly signed common meadows.

**Lemma 4.1.** *The common meadow of complex numbers $\mathbb{C}_\bot$ cannot become a signed common meadow.*

*Proof.* Suppose $\mathbb{C}_\bot$ has a sign function satisfying the axioms $E_{\mathsf{ftc-scm}}$. Then, by Proposition 3.1, $\mathbb{C}_\bot$ is a 4-signed common meadow. By the proof of the converse clause of Theorem 3.1, $\mathbb{C}_\bot$ is an enlargement of the ordered field whose domain is $\mathbb{C} = \mathbb{C}_\bot - \{\bot\}$. It is well known that $\mathbb{C}$ is cannot an ordered field because $\mathsf{i} \cdot \mathsf{i} = -1$. This is a contradiction of our assumption. □

Turning to the weakly signed complex meadows, we obtain the axiomatisation $E_{\mathsf{sftc-scmc}}$ by extending $E_{\mathsf{ftc-wsmc}}$ with the equations

$$\mathsf{i} \cdot \mathsf{i} = -1 \text{ and } \mathsf{s}(\mathsf{i}) = \mathsf{i}.$$

A modulus function on signed complex numbers can be introduced as in Table 6. Notice $\frac{x}{x} = 1 \rightarrow |x| = \frac{x}{\mathsf{s}(x)}$. Inverting, we note $\mathsf{s}(x) = \frac{x}{|x|}$.

**Proposition 4.3.** *Let $\mathbb{C}$ be a field of complex numbers and let $\mathbb{C}_\bot$ the corresponding common meadow. Then $\mathbb{C}_\bot$ can be expanded with a function $\mathsf{s}(-)$ so as to become a weakly signed common meadow $\mathbb{C}_{\bot,\mathsf{s}}$.*

*Proof.* One may expand a common meadow $\mathbb{C}_\bot$ of complex numbers with a sign function $\mathsf{s}$ by defining

$$\mathsf{s}(0) = 0 \text{ and } x \neq 0 \rightarrow \mathsf{s}(x) = \frac{x}{|x|},$$

with $|-|$ the standard absolute value function for complex numbers (rather than the general modulus as given in Table 6). The absolute value function is defined by $|z| = +\sqrt{\mathsf{Re}(z)^2 + \mathsf{Im}(z)^2}$, but note that the real part, imaginary part and the square root functions do not need to be named in the signature of the algebra, only $\mathsf{s}$. Now, it is routine to check that all equations of $E_{\mathsf{ftc-wscm}}$ are satisfied by the algebra $\mathbb{C}_{\bot,\mathsf{s}}$. □

Signed complex numbers make use of the idea of polar coordinates. The range of the sign function $\mathsf{s}(x)$ of $\mathbb{C}_{\bot,\mathsf{s}}$ is infinite as all points on the unit circle $S^1$ serve as values of $\mathsf{s}$.

Notice that the resulting structure $\mathbb{C}_{\bot,\mathsf{s}}$ is not a signed meadow as it is not a 4-signed meadow. Specifically, the condition that $\mathsf{s}(x) = 0 \vee \mathsf{s}(x) = 1 \vee \mathsf{s}(x) = -1 \vee \mathsf{s}(x) = \bot$, for all $x$, is not satisfied. To see the latter we notice that

$$|\mathsf{i}| = +\sqrt{\mathsf{Re}(\mathsf{i})^2 + \mathsf{Im}(\mathsf{i})^2} = +\sqrt{0^2 + 1^2} = +\sqrt{1^2} = 1$$

$$\texttt{import: } E_{\mathsf{ftc-wscm}} \tag{32}$$
$$|0| = 0 \tag{33}$$
$$|\bot| = \bot \tag{34}$$
$$\frac{x}{x} = 1 \to |x| = \frac{x}{\mathsf{s}(x)} \tag{35}$$

Table 6: $E_{\mathsf{ftca-wscm}}$: (conditional) equations for for weakly signed common meadows with absolute value or modulus function

and so $\mathsf{s}(\mathsf{i}) = \frac{\mathsf{i}}{1} = \mathsf{i}$.

We do not know if the expansion of $\mathbb{C}_{\bot,\mathsf{s}}$ to a weakly signed common meadow is unique.

It follows immediately from the equations in Table 5 that fracterm flattening extends to the fracterm calculus for signed common meadows of complex numbers.

**Proposition 4.4.** *(*Fracterm flattening.*) For each term $t$ in the signature of weakly signed common meadows of complex numbers there is a flat fractcerm $r$ over the signature $\Sigma_{\bot,\mathsf{s}}$ of signed common meadows such that $E_{\mathsf{ftc-ccm}} \vdash t = r$.*

The most plausible interpretation of the sign function being $\mathsf{s}(x) = \frac{x}{|x|}$ one finds that square roots come into play for expressing $|x|$. Given that square root is partial on the rationals, the rational complex numbers cannot be extended to a weakly signed meadow using the absolute value function to define a modulus function $|-|$. More generally:

**Problem 4.2.** *Prove that a common meadow of complex rational numbers cannot be expanded to a signed common meadow.*

Only in certain (possibly algebraic) extensions of common meadows of complex numbers it is plausible that a sign function can be introduced.

**Problem 4.3.** *Find an algebraic specification of some appropriate signed common submeadow of $\mathbb{C}_{\bot,\mathsf{s}}$.*

We expect that auxiliary functions will be needed, such as $\mathsf{Re}$ and $\mathsf{Im}$ for the real and imaginary part of a complex number, and a square root function. Some preliminary work regarding such specifications has already been done in [8] for the square root, though merely in a setting of a simpler case. In [17], a specification is given for an involutive meadow of rational complex numbers.

# 5 Enlargement of common meadows to weakly signed common meadows

We will now argue that any meadow can be enlarged to a weakly signed common meadow. We first consider a simple case.

**Example 5.1.** Let $\mathbb{Q}(\alpha)$ be a field of rationals expanded with a single transcendental number $\alpha$. Let $\mathbb{Q}(\alpha)_\perp$ be the corresponding common meadow? Can $\mathbb{Q}(\alpha)_\perp$ be enlarged to a signed meadow? This is indeed possible as $\mathbb{Q}(\alpha)_\perp$ is isomorphic to $\mathbb{Q}(\pi)_\perp$ for $\pi \in \mathbb{R}$ which is a submeadow of an ordered meadow of real numbers.

This observation can be generalized as follows:

**Theorem 5.1.** *Let $F$ be a field of characteristic 0 with cardinality of the continuum or less, then $F_\perp$ can be enlarged to a weakly signed common meadow.*

*Proof.* Let $A$ be a transcendence base for $F$ over its prime subfield $\mathbb{Q}$ and let $B$ be a transcendence base of $\mathbb{C}$ over $\mathbb{Q}$. Choose an injection $\phi : A \to B$, which is possible due to the cardinality constraint on $F$. Now $\phi$ extends in a natural manner to an injection $F \to \mathbb{C}$. Thus, $F_\perp$ can be isomorphically embedded in a weakly signed meadow $\mathbb{C}_{\perp,\mathsf{s}}$ from which it trivially follows that $F_\perp$ can be enlarged to a weakly signed common meadow. □

Using known results about formally real fields (e.g., see [38]), one finds that each common meadow which is the enlargement of an ordered field can be enlarged to a weakly signed common meadow. In fact:

**Theorem 5.2.** *Let $F$ be any field of characteristic 0 then $F_\perp$ can be enlarged to a weakly signed common meadow.*

*Proof.* Let $A$ be a transcendence basis of $F$ then the extension $F(A)$ is formally real, i.e., no finite sum of squares equals $-1$. Indeed, as otherwise, there would be a non-trivial (possibly multivariate) polynomial with a solution made up from elements of the transcendence base. Thus, using the core result of Artin-Schreier theory the field $F(A)$ can be ordered and enlarged to a real closed field $F'(A)$.

Subsequently, $F'(A)$ can be once more enlarged by introducing $\sqrt{-1}$, thereby obtaining $F'(A, \sqrt{-1})$. The resulting field is algebraically closed so that $F$ can be embedded into it – as all elements of $F$ are algebraic over the transcendence base $A$.

On $F'(A, \sqrt{-1})$ one may define functions $\mathsf{Re}$ and $\mathsf{Im}$ for the real part and the imaginary parts. The norm $|-|$ is given by $|a| = \sqrt{\mathsf{Re}(a)^2 + \mathsf{Im}(a)^2}$ and the sign function is found as $\mathsf{s}(a) = \frac{a}{|a|}$ thus obtaining a weakly signed common meadow $F'(A, \sqrt{-1})_{\perp,\mathsf{s}}$ which allows an embedding of the common meadow $F_\perp$ so that $F_\perp$ may be enlarged to an isomorphic copy of $F'(A, \sqrt{-1})_{\perp,\mathsf{s}}$ which is a weakly signed common meadow just as well. □

# 6 Probability calculus

The modern form of the Bayes-Price theorem on inverse probabilities can be formulated as a modest equation involving division and ordering, namely:

$$P(X|Y) = \frac{P(Y|X) \cdot P(X)}{P(Y)}$$

which is often claimed under the constraint that $P(Y) > 0$. In fact the theorem was published, refined and first applied by Richard Price; see [3].

$$(X \cup Y) \cap Y = Y \tag{36}$$
$$(X \cap Y) \cup Y = Y \tag{37}$$
$$X \cap (Y \cup Z) = (Y \cap Z) \cup (Z \cap X) \tag{38}$$
$$X \cup (Y \cap Z) = (Y \cup Z) \cap (Z \cup X) \tag{39}$$
$$X \cup \overline{X} = \mathbb{U} \tag{40}$$
$$X \cap \overline{X} = \emptyset \tag{41}$$

Table 7: $E_{\mathsf{sba}}$, symmetric equations for Boolean algebra

The formula can be derived from some simple postulates about probability. Indeed the derivations can involve divisions by $P(X)$ thus requiring the constraint that $P(X) > 0$ as well, which is often left unmentioned. Here we will illustrate the use of signed common meadows in (i) providing an axiomatisation of probability calculus, (ii) providing a formally rigorous statement of the Bayes-Price theorem, and (iii) providing a formal proof using conditional equational deduction of the theorem.

The totalisation of division allows us to accommodate conditions on equational formulae, which is needed.

## 6.1 Axioms for a probability function

First, we determine plausible equations for the postulates that define probability. Such axioms are given in Table 8. To do this we define a probability function on a Boolean algebra, requiring it to take its values $P(X)$ in a 4-signed meadow.

The axioms for a Boolean algebra in Table 7 have been taken (modulo a change of notation) from [35], where the completeness of these axioms is shown.

We notice that axiom 50 encodes by way of an equation, $0 \cdot P(X) = 0$, the condition $P(X) \neq \bot$. Similarly, since $\mathsf{s}(x) = 1$ encodes $x > 0$, for probabilities, in view of Axiom 48, $\mathsf{s}(P(X)) = 1$ encodes $P(X) \neq 0$.

## 6.2 Reformulating the Bayes-Price' theorem

The Bayes-Price theorem, which appears in nearly all introductions to probability theory (and usually without mention of Price), and which is often taken for the mathematical core of Bayesian statistics and probability takes the form of a conditional equation over signed meadows. The particular property of this form, which seems to be somehow novel, is that the non-zeroness condition is used fo $P(X)$ rather than for $P(Y)$ (in the formulation of Proposition 6.2).

**Proposition 6.1.** *The following form of Bayes-Price's theorem is provable from the axioms of* $E^{P}_{\mathsf{ftca-scm}}$*:*

$$\mathsf{s}(P(X)) = 1 \to P(X|Y) = \frac{P(Y|X) \cdot P(X)}{P(Y)}.$$

$$\texttt{import: } E_{\mathsf{ftca-wscm}} \tag{42}$$
$$\texttt{import: } E_{\mathsf{ftc-scm}} \tag{43}$$
$$\texttt{import: } E_{\mathsf{AVL}} \tag{44}$$
$$\texttt{import: } E_{\mathsf{sba}} \tag{45}$$
$$P(\emptyset) = 0 \tag{46}$$
$$P(\mathbb{U}) = 1 \tag{47}$$
$$P(X) = |P(X)| \tag{48}$$
$$P(X \cup Y) = (P(X) + P(Y)) - P(X \cap Y) \tag{49}$$
$$0 \cdot P(X) = 0 \tag{50}$$

Table 8: $E^P_{\mathsf{ftca-scm}}$: equations for a probability function $P$ into a signed common meadow

$$P(X|Y) = \frac{P(X \cap Y)}{P(Y)} \tag{51}$$

Table 9: Conditional probability definition

*Proof.* First, notice that given $\mathsf{s}(P(X)) = 1$,

$$\frac{P(X)}{P(X)} = \frac{\mathsf{s}(P(X))}{\mathsf{s}(P(X))} = \frac{1}{1} = 1.$$

Now, starting with conditional probability in Table 9,

$$P(X|Y) = \frac{P(X \cap Y) \cdot 1}{P(Y)} \tag{52}$$
$$= \frac{P(X \cap Y) \cdot \frac{P(X)}{P(X)}}{P(Y)} \tag{53}$$
$$= \frac{\frac{P(X \cap Y)}{P(X)} \cdot P(X)}{P(Y)} \tag{54}$$
$$= \frac{P(Y|X) \cdot P(X)}{P(Y)} \tag{55}$$

□

A companion to the theorem is the following expansion:

**Proposition 6.2.** *Decomposition of probability into conditional probabilities:*

$$\mathsf{s}(P(X)) = 1 \wedge \mathsf{s}(P(\overline{X})) = 1 \rightarrow P(Y) = P(Y|X) \cdot P(X) + P(Y|\overline{X}) \cdot P(\overline{X})$$

*Proof.* As in the previous proof, we may use

$$\frac{P(X)}{P(X)} = \frac{P(\overline{X})}{P(\overline{X})} = 1.$$

Now, using the Boolean algebra axioms we begin:

$$\begin{aligned}
P(Y) &= P((Y \cap X) \cup (Y \cap \overline{X})) && (56)\\
&= (P(Y \cap X) + P(Y \cap \overline{X})) + P((Y \cap X) \cap (Y \cap \overline{X})) && (57)\\
&= P(Y \cap X) + P(Y \cap \overline{X}) && (58)\\
&= P(Y \cap X) \cdot \frac{P(X)}{P(X)} + P(Y \cap \overline{X}) \cdot \frac{P(\overline{X})}{P(\overline{X})} && (59)\\
&= \frac{P(Y \cap X)}{P(X)} \cdot P(X) + \frac{P(Y \cap \overline{X})}{P(\overline{X})} \cdot P(\overline{X}) && (60)\\
&= P(Y|X) \cdot P(X) + P(Y|\overline{X}) \cdot P(\overline{X}). && (61)
\end{aligned}$$

□

Formulating the Bayes-Price theorem in a manner which takes care of division by zero and which is sound in say a structure where division is a partial function is not entirely trivial. For instance in the Stanford Encyclopedia entry on Bayes' theorem [32] the main statement of Bayes' theorem is given in 1.2 where the left-hand side may be defined while the right-hand side has no value.

The axiomatisation for probability given above calculus is complete in the following sense.

**Theorem 6.1.** *A conditional equation $E \to t = r$ of the signature of signed meadows, expanded with a Boolean algebra and a probability function $P$ is true for all probability functions into any signed meadow if and only if $E \to t = r$ is provable from the axioms of signed meadows (including* AVL *and the axioms of Table 5) in conditional equational logic extended with proof rule* $\mathsf{R_{cm}}$*, together with the equations for a Boolean algebra and the equations for a probability function $P$.*

*Proof.* The proof uses the completeness result for the conditional equational fracterm caclulus 3.2 above, and the works just as the corresponding proof in the case of Suppes-Ono fracterm calculus, i.e. the case of involutive meadows in [13]. □

# 7 Involutive meadows

We consider a simple way of totalising division without the introduction of a 'peripheral number', such as $\perp, \infty, +\infty$, as in other methods.

**Definition 7.1.** *An* involutive meadow *is an expansion of a field $F$ by adding a function $\div$ for division, wherein division is made total by having $\frac{x}{0} = 0$ for all $x \in F$. The class of all involutive meadows we denote $IM$. The name derives from the fact that inverse is an involution: $(x^{-1})^{-1} = x$ for all $x$.*

The assumption that $\frac{x}{0} = 0$ for all $x$ was proposed by Suppes in 1957 in [37] (see also the discussion in [2]) and was studied in detail in by Ono in 1983 in [34]. For these reasons, we have coined the eponym *Suppes-Ono fracterm calculus* for our equational specifications of this semantic option.

Following [34], Suppes-Ono fracterm calculus provides finite and complete axiomatizations of the equational theory as well as of the conditional equational theory of the class of so-called involutive meadows.

However, one of our main reasons to prefer working with common meadows rather than Suppes-Ono fracterm calculus, is that, when the assumption $\frac{x}{0} = 0$ is adopted, fracterm flattening *fails*; this was shown in [10]. Indeed, the best one can achieve is that all fracterm expressions can be re-written as a finite sum of flat fracterms, and no upper bound to the number of summands may be given.

## 7.1 3-signed involutive meadows

Signed involutive meadows, i.e., meadows that conform the Suppes-Ono fracterm calculus were introduced in [8] and discussed in [9] and used in the context of probability theory in [13] and [4]. For involutive meadows 3-signed meadows are the important special case in view of the absence of $\bot$. 3-signed involutive meadows are enrichments of ordered fields. In contrast to the case for common meadows a completeness theorem has been obtained for the 3-signed case (see [8]).

We notice that 3-signed common meadows satisfy the equations of Tables 4 and 5 with the exception of the equation $\mathsf{s}(\bot) = \bot$. Trivially, 3-signed common meadows are 4-signed as well. Moreover a definition of weakly signed involutive cancellation meadows can be given in a similar way as for common meadows, with corresponding results.

However, in the course of the development of involutive meadows the naming conventions are somewhat different: involutive meadows are defined as the models of a given set of axioms while involutive meadows that are expansions of a field are called *involutive cancellation meadows*. The above results are all about expansions and enlargements of fields (which in case of common meadows is taken for granted as a part of the definition) so that corresponding results in the case of involutive meadows must be stated with regard to involutive cancellation meadows. With similar reasoning as for Theorem 5.2, one finds:

**Proposition 7.1.** *Every involutive cancellation meadow of characteristic* 0 *can be enlarged to a weakly signed involutive cancellation meadow.*

Now, as evident in [8], using Suppes-Ono fracterm calculus simplifies the introduction of a sign function when compared with the case of common meadows. In [13], it was noticed that an ordering is essential for developing probability in a setting of meadows.

The precise details of the Bayes-Price theorem depend on the fracterm calculus which is used. In Suppes-Ono fracterm calculus, a plausible version of the theorem can be formulated without making use of conditions; in the case of signed common meadows, however, a condition seems to be needed. We note that the axioms of Table 9 are identical to the axioms for a probability function $P$ into an involutive meadow as presented in [13] –

$$\texttt{import}\ \ E_{\mathsf{ftc-cm}} + \mathsf{AVL}$$

$$x \lhd 0 \rhd y = y \tag{62}$$

$$x \lhd \bot \rhd y = \bot \tag{63}$$

$$x \lhd (y + (0 \cdot u)) \rhd z = (x \lhd y \rhd z) + (0 \cdot u) \tag{64}$$

$$(x + (0 \cdot u)) \lhd y \rhd (z + (0 \cdot u)) = (x \lhd y \rhd z) + (0 \cdot u) \tag{65}$$

$$\frac{y}{y} \cdot (x \lhd y \rhd z) = \frac{y}{y} \cdot x \tag{66}$$

Table 10: $E_{\mathsf{ftc-Cond\_op}}$: axioms for fracterm calculus with a conditional operator

except for axiom 50 that comes for free in the case of Suppes-Ono fracterm calculus (i.e. involutive meadows).

Opting for working in common meadows does have its price for we have no information on a number of questions, which we list in the next section

# 8 Concluding remarks

It is plausible to extend the signature of common meadows with a conditional operator. In particular we propose to work with $x \lhd y \rhd z$ understood as "if $y = 0$ then $z$ else $x$, where $\bot$ is returned if $y = \bot$. The conditional operator is not strict in all arguments, e.g. $0 \lhd 1 \rhd \bot = 0$. Nevertheless it is easily shown from the axioms in Table 10 that for $a \neq \bot$, $\phi_a(x) = x + 0 \cdot a$ is a homomorphism w.r.t. the conditional operator. It follows that the completeness result for the conditional equational logic of common meadows of [23] can be generalised to the signature expanded with the conditional operator.

## 8.1 Open questions

In addition to the problems noted earlier, we are unaware of the answers to the several algebraic and logical questions. An obvious algebraic question arising from the Theorem 5.2 for characteristic 0 is this:

**Question 8.1.** *Can every common meadow of any characteristic be expanded to a weakly signed meadow?*

Turning to axiomatisations there are several technical questions we would like to answer. First, the role of conditional equations is quite prominent in the signed case. In the case of common meadows, from [20], we know that the conditional equation

$$\Phi \equiv (\frac{1}{x} = \bot \rightarrow 0 \cdot x = x)$$

is true in all common meadows while it cannot be derived from $E_{\mathsf{ftc-cm}}$. Together with the completeness of $E_{\mathsf{ftc-cm}}$ we find that $\Phi$ is not derivable from the equational theory of common meadows. We have no information concerning the following question:

**Question 8.2.** *Does the conditional equational theory of common meadows possess a finite (conditional equational) basis?*

In the signed case similar questions are open for common meadows:

**Question 8.3.** *(i) Does the equational theory of weakly 4-signed common meadows possess a finite (equational) basis?*

*(ii) Does the conditional equational theory of weakly 4-signed common meadows possess a finite (conditional equational) basis?*

*(iii) Does the equational theory of signed common meadows possess a finite (equational) basis? (The issue is to do without an additional proof rule such as* $\mathsf{R}_{\mathsf{cm}}$*.)*

*(iv) Does the conditional equational theory of signed common meadows possess a finite (conditional equational) basis?*

Perhaps the major open question that comes about from our paper is Problem (iii) from the Questions 8.3. A positive result, if ever obtained, extends the result of [8] and [9] for common meadows to the case of signed meadows.

A consequence of a positive result is that the axioms for the probability calculus, based on signed common meadows are comparable with the axioms for probability calculus based on Suppes-Ono fracterm calculus (as expounded in [13] and [4]). A negative result on question (iv) will imply that signed meadow based probability calculus requires an infinite axiomatization, or as is done above, the use of an additional proof rule (such as $\mathsf{R}_{\mathsf{cm}}$) which is specific for classes of algebras with a 0 and a product $\_ \cdot \_$ such that $x \neq \bot \to 0 \cdot x = 0$ always holds.

In practice one needs an informal account of elementary mathematics. A perspective of informal accounts of elementary arithmetic, and initial steps regarding the systematic investigation thereof, can be found in [24], where "naive fracterm calculus" is coined as a label for the conventional daily practice of working with fractions, and in [25], where "synthetic fracterm calculus" is coined as a label for an informal account of elementary arithmetic which is closer to the well-known intuitions of first order logic. We expect that the axiomatisation of probability calculus given above may serve as the logical background for an informal exposition of the basics of probability theory in the style of synthetic fracterm calculus as introduced in [25].

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