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Explicit numerical approximations for SDDEs in finite and infinite horizons using the adaptive EM method: Strong convergence and almost sure exponential stability

Ulises Botija-Munoz, Chenggui Yuan *

Department of Mathematics, Swansea University, Bay Campus, SA1 8EN, UK

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ABSTRACT

In this paper we investigate explicit numerical approximations for stochastic differential delay equations (SDDEs) under a local Lipschitz condition by employing the adaptive Euler-Maruyama (EM) method. Working in both finite and infinite horizons, we achieve strong convergence results of the adaptive EM solution. We also obtain the order of convergence in finite horizon. In addition, we show almost sure exponential stability of the adaptive approximate solution for both SDEs and SDDEs.

1. Introduction

Consider the following SDDEs

$$dY_t = (-2Y_t - Y_t^3 + \frac{1}{2}Y_t \sin(Y_{t-1}))dt + \sqrt{2}Y_t \cos(Y_{t-1})dW_t \tag{1.1}$$

with initial data $\xi \in C([-1,0];\mathbb{R}), \xi(0) = c \in \mathbb{R}/\{0\}$. Using [16, Theorem 1], we can show that the exact solution of the SDDE (1.1) is almost sure exponentially stable, i.e.

$$\limsup_{t\to\infty} \frac{1}{t} \log |Y_t| \le -\lambda \text{ a.s.}, \quad \lambda > 0.$$

However, the discrete (standard) EM approximate solution

$$\begin{cases} X_k &= \xi(k\Delta) \quad k = -m, -m+1, ..., 0, \\ X_{k+1} &= X_k - X_k[(2 + X_k^2 - \frac{1}{2}X_k \sin(X_{k-1}))\Delta + \sqrt{2}\cos(X_{k-1})\Delta W_k], \quad k = 0, 1, \dots \end{cases} \tag{1.2}$$

* Corresponding author.
E-mail address: c.yuan@swansea.ac.uk (C. Yuan).

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where $\Delta = 1/m, m \in \mathbb{N}$, is not almost sure exponentially stable. This means that it does not exist a constant $\eta > 0$ and a $\Delta^* \in (0,1)$ such that for all $\Delta \in (0, \Delta^*)$

$$\limsup_{k\to\infty} \frac{1}{k\Delta} \log |X_k| \le -\eta \ \text{ a.s. .}$$

On the contrary, as we will see in Section 6, the adaptive EM approximate solution to equation (1.1) is almost sure exponentially stable.

The classical existence-and-uniqueness theorem for SDDEs requires the drift and diffusion functions to satisfy a local Lipschitz condition and a linear growth condition (see [11]). However, in applications there are many SDDEs which do not satisfy the linear growth condition. The Khasminskii-type theorem in [12] enables to prove existence-and-uniqueness for a class of SDDEs using a weaker condition than the linear growth one. Thus it is desirable, under these weaker conditions, to find numerical approximate solutions that converge strongly to the exact solution. In 2003, Mao [14] proved strong convergence using the EM scheme and assuming the boundedness of the pth moments for both the exact and the numerical solution. It is well-known that the linear growth condition implies the boundedness of the pth moments for the EM approximate solution. But when the drift function grows faster than linear, the standard EM scheme fails, see the example with polynomial growth in Hunter [7]. Therefore, modifications of the EM scheme for SDDEs that provide explicit approximate solutions have appeared in the last few years to account for this issue. Examples of these are the tamed [8] and the truncated [3] methods.

In 2020, Wei and Giles [2] obtained strong convergence for the numerical solution of a SDEs in a finite horizon under local Lipschitz and one-sided linear growth conditions. They use an adaptive EM scheme in which the time step is not a constant, but a function of the solution at that point in time. They also, under more restrictive conditions, showed strong convergence in infinite horizon. Here, in the first part of this paper we extend their work to SDDEs in both, finite and infinite horizons. Following [2], we will show the boundedness of the pth moments but in our case, this is not enough to prove strong convergence. The main difficulty is that the delay times might not match the times where the numerical solution is computed. We therefore defined an auxiliary piecewise constant process on the delay times. This varies from the standard EM method for SDDEs and requires a new proof of convergence.

Additionally, to study the stability of numerical solutions is an important topic. Moment stability for SDDEs has been studied extensively, see for example [1,13]. Almost sure (a.s.) exponential stability is usually derived from moment stability by means of the Borel-Cantelli lemma and Markov's inequality (see [6]). In Wu et al. [16], using the EM and the Backward EM (BEM) methods, a.s. exponential stability was studied for SDDEs without using moment stability. Their approach was based on the martingale convergence theorem. They required the linear growth condition when dealing with the standard EM scheme. When they weaken the linear growth condition to the one-sided linear growth condition for the diffusion function, they showed how the standard EM approximate solution loses the stability of the exact solution. Then they showed that under the one-sided linear growth condition, the a.s. exponential stability can be achieved by using the BEM method. This method is implicit and therefore more computationally expensive than explicit methods like the adaptive EM. In Song et al. [15], applying the truncated EM method, a.s. exponential stability was studied for SDDEs was also investigated. Since the adaptive EM scheme is one of the important explicit numerical method, here, we show that the adaptive EM method can also preserve a.s. exponential stability for SDDEs. We also do it for SDEs, which was not studied in [2].

The rest of the paper is structured as follows. Section 2 introduces some preliminary notation and the type of SDDE we will work with in the rest of the paper. Section 3 describes the adaptive EM method. Section 4 deals with strong convergence and order of convergence in finite horizon. In Section 5 we obtained the boundedness of the pth moments for the adaptive EM approximate solution in infinite horizon. In Section 6 we show that almost surely exponential stability of the adaptive EM solution for SDDEs can be recovered and provide illustration for counterexample (1.1). In Section 7, we present some simulations to illustrate the results in Section 6.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered complete probability space where the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfies the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $\tau > 0$ and $T > 0$ be constants and denote $C([-\tau,0];\mathbb{R}^m)$ the space of all continuous functions from $[-\tau,0]$ to $\mathbb{R}^m$ with the norm $||\phi|| = \sup_{-\tau\le\theta\le 0} |\phi(\theta)|$. Let $\{W_t\}_{0\le t\le T}$ be a standard d-dimensional Brownian motion. For a $\mathbb{R}^m$-vector v, we denote the Euclidean norm by $|v| := (|v_1|^2 + \dots + |v_m|^2)^{\frac{1}{2}}$ and the inner product of two $\mathbb{R}^m$-vectors v and w by $\langle v, w\rangle := v_1 w_1 + \dots + v_m w_m$. For a $m \times d$ matrix A, we denote the Frobenius matrix norm by $||A|| := \sqrt{\text{trace}(A^T A)}$.

Consider an m-dimensional SDDE of the form

$$\begin{cases} dY_t = f(Y_t, Y_{t-\tau})dt + g(Y_t, Y_{t-\tau})dW_t, & t > 0, \\ Y_t = \xi(t), & t \in [-\tau, 0], \end{cases} \tag{2.1}$$

where $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{m\times d}$ are Borel-measurable functions, and ξ is a $\mathcal{F}_0$-measurable $C([-\tau,0];\mathbb{R}^m)$-valued random variable such that $E\|\xi\|^p < \infty$.

3. Adaptive method

The time step is determined by a function $h^\delta : \mathbb{R}^m \to \mathbb{R}^+$ with $\delta \in (0,1)$. The family of functions $\{h^\delta\}_{0<\delta<1}$ is not specifically defined, it just has to satisfy certain conditions that we will describe later in the next assumption. To see concrete examples where the function h^δ is fully specified, see Section 4 in [2] or equation (6.9) below. We now define the adaptive method for SDDEs. Set

$$\widehat{X}_0 := \xi(0),\ h_0^\delta := h^\delta(\widehat{X}_0),\ t_1 := h_0^\delta.$$

We introduce the continuous-time step (auxiliary) process $\overline{X}$. Define

$$\overline{X}_t := \xi(t), t \in [-\tau, 0), \quad \overline{X}_t := \xi(0), t \in [0, t_1).$$

For t_1 we define the discrete-time approximate solution $\widehat{X}$ as

$$\begin{aligned}
\widehat{X}_{t_1} &:= \widehat{X}_0 + f(\overline{X}_0, \overline{X}_{-\tau})h_0^\delta + g(\overline{X}_0, \overline{X}_{-\tau})\Delta W_0,\\
h_1^\delta &:= h^\delta(\widehat{X}_{t_1}),\ t_2 = t_1 + h_1^\delta,\\
\overline{X}_t &:= \widehat{X}_{t_1}, t \in [t_1, t_2),
\end{aligned}$$

where $\Delta W_0 := W_{t_1} - W_0$. Then for a generic t_n we define

$$\begin{aligned}
\widehat{X}_{t_{n+1}} &:= \widehat{X}_{t_n} + f(\overline{X}_{t_n}, \overline{X}_{t_n-\tau})h_n^\delta + g(\overline{X}_{t_n}, \overline{X}_{t_n-\tau})\Delta W_n,\\
h_{n+1}^\delta &:= h^\delta(\widehat{X}_{t_{n+1}}),\ t_{n+2} = t_{n+1} + h_{n+1}^\delta,\\
\overline{X}_t &:= \widehat{X}_{t_{n+1}}, t \in [t_{n+1}, t_{n+2}),
\end{aligned} \tag{3.1}$$

where $\Delta W_n := W_{t_{n+1}} - W_{t_n}$. For every path $\omega \in \Omega$, we continue the recursion (3.1) until $n = N(\omega) := \inf\{n \in \mathbb{Z}^+ : t_n(\omega) \geq T\}$. Note that t_n and h_n^δ are random variables. For every ω, let $r = r(\omega)$ be such $t_r \leq \tau < t_{r+1}$. Then we define the continuous-time step (auxiliary) process $\widetilde{X}$ as

$$\begin{aligned}
&\widetilde{X}_t := \overline{X}_{-\tau}, t \in [-\tau, t_1 - \tau), \quad \widetilde{X}_t := \overline{X}_{t_1-\tau}, t \in [t_1 - \tau, t_2 - \tau), \ldots., \quad \widetilde{X}_t := \overline{X}_{t_r-\tau}, t \in [t_r - \tau, t_{r+1} - \tau),\\
&\widetilde{X}_t := \overline{X}_{t_{r+1}-\tau}, t \in [t_{r+1} - \tau, t_{r+2} - \tau), \quad \widetilde{X}_t := \overline{X}_{t_{r+n}-\tau}, t \in [t_{r+n} - \tau, t_{r+n+1} - \tau)
\end{aligned} \tag{3.2}$$

for $n = 1, \ldots, N - r$. We now define the continuous approximate solution

$$\begin{aligned}
X_t &:= \xi(t), \quad t \in [-\tau, 0];\\
X_t &:= X_0 + \int_0^t f(\overline{X}_s, \widetilde{X}_{s-\tau})ds + \int_0^t g(\overline{X}_s, \widetilde{X}_{s-\tau})dW_s, \quad t > 0.
\end{aligned} \tag{3.3}$$

Note that $\widehat{X}_{t_n} = \overline{X}_{t_n} = X_{t_n}$ for $n = 0, 1, \ldots, N$.

4. Convergence of the numerical solutions on finite time interval

In this section we will work on a finite time interval $[-\tau, T], T > 0$, and investigate the convergence of the numerical solutions to the exact solution on $[0, T]$.

Assumption 4.1. The functions f and g satisfy the local Lipschitz condition: for every $R > 0$ there exists a positive constant C_R such that

$$|f(x,y) - f(\overline{x}, \overline{y})| + ||g(x,y) - g(\overline{x}, \overline{y})|| \leq C_R(|x - \overline{x}| + |y - \overline{y}|) \tag{4.1}$$

for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}^m$ with $|x| \vee |y| \vee |\overline{x}| \vee |\overline{y}| \leq R$. Furthermore, there exist two constants $\alpha, \beta \geq 0$ such that for all $x, y \in \mathbb{R}^m$, f satisfies the one-sided linear growth condition:

$$\langle x, f(x,y)\rangle \leq \alpha(|x|^2 + |y|^2) + \beta \tag{4.2}$$

and g satisfies the linear growth condition:

$$||g(x,y)||^2 \leq \alpha(|x|^2 + |y|^2) + \beta. \tag{4.3}$$

Assumption 4.2. The time step function $h^\delta : \mathbb{R}^m \to \mathbb{R}^+$, $\delta \in (0,1)$, is continuous, strictly positive and bounded by δT, i.e.

$$0 < h^\delta(x) \leq \delta T \quad \text{for all } x \in \mathbb{R}^m. \tag{4.4}$$

Furthermore, there exist constants $\alpha, \beta > 0$ such that for all $x, y \in \mathbb{R}^m$.

$$\langle x, f(x,y)\rangle + \frac{1}{2}h^{\delta}(x)|f(x,y)|^2 \le \alpha(|x|^2 + |y|^2) + \beta. \tag{4.5}$$

Note that condition (4.5) implies condition (4.2) with the same values of α and β.

4.1. The boundedness of the pth moments of the exact solution and the numerical solutions

4.1.1. Exact solution

In this subsection we will discuss the pth moments of the exact solution to SDDE (2.1).

Lemma 4.3. *If the SDDE (2.1) satisfies Assumption (4.1), then there exists a positive constant C such that for any $p \ge 2$*

$$\mathbb{E}\left[\sup_{0\le t\le T} |Y_t|^p\right] \le C. \tag{4.6}$$

Proof. The proof is given in Lemma 3.2 in [8]. □

4.1.2. Adaptive EM numerical solutions

In this subsection, the pth moments of numerical solution will be investigated. In the standard Euler-Maruyama method the discretisation times $\{t_n\}$ are built using a constant time step Δ and a fixed number of steps $N \in \mathbb{N}$, i.e. $t_N = N\Delta = T$. However, in the adaptive method, $\{t_n\}$ is a sequence of random variables and there is no guarantee that it reaches T in a finite number of steps. Thus, we have the following definition.

Definition 4.1. We say that the time horizon T is attainable if $\{t_n\}$ reaches T in a finite number of steps N, i.e. for almost all $\omega \in \Omega$, there exists a $N(\omega)$ such that $t_{N(\omega)} = \sum_{n=0}^{N(\omega)} h^{\delta}(X_{t_n}) \ge T$.

Theorem 4.4. *If the SDDE (2.1) and the function h^{δ} satisfy Assumption 4.1 and 4.2 respectively, then T is attainable and for all $p > 0$ there exists a constant $C > 0$ dependent on T and p, but independent of h_n^{δ}, such that*

$$\mathbb{E}\left[\sup_{0\le t\le T} |X_t|^p\right] \le C. \tag{4.7}$$

The discrete-time approximate solution defined in (3.1) need not be bounded. In order to show that T is attainable and prove Theorem 4.4, we need to work with a bounded approximate solution. To this end we now introduce the following auxiliary scheme. Let $K > ||\xi||$. Set $\widehat{X}_0^K := \xi(0), h_0^{\delta,K} := h^{\delta}(\widehat{X}_0^K), t_1 := h_0^{\delta,K}$ and $\overline{X}_t^K := \xi(t), t \in [-\tau, 0), \overline{X}_t^K := \xi(0), t \in [0, t_1)$. Note that $t_n, n = 1, 2, \ldots$, also depend on K, but we have dropped it to ease the notation. Consider the function $\Phi_K : \mathbb{R}^m \to \mathbb{R}^m, \Phi(x) = \min(1, K/|x|)x$. Then for every $\omega \in \Omega$ and for $n = 0, 1, \ldots N^K(\omega)$ (where $N^K(\omega) := \inf\{n \in \mathbb{Z}^+ : t_n(\omega) \ge T\}$), we define

$$\begin{aligned}
\widehat{X}_{t_{n+1}}^K &:= \Phi_K(\widehat{X}_{t_n}^K + f(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)h_n^{\delta,K} + g(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)\Delta W_n) \\
h_{n+1}^{\delta,K} &:= h^{\delta}(X_{t_{n+1}}^K), t_{n+2} := t_{n+1} + h_{n+1}^{\delta,K}, \\
\overline{X}_t^K &:= \widehat{X}_{t_{n+1}}^K, t \in [t_{n+1}, t_{n+2}).
\end{aligned} \tag{4.8}$$

Define for $n = 0, \ldots, N - r$

$$\widetilde{X}_t^K := \overline{X}_{t_n-\tau}^K, t \in [t_n - \tau, t_{n+1} - \tau), \tag{4.9}$$

where $r = r(\omega)$ is such that $t_r \le \tau \le t_{r+1}$. We now define the continuous approximate solution

$$\begin{aligned}
X_t^K &:= \xi(t), \quad t \in [-\tau, 0]; \\
X_t^K &:= \Phi_K\left(\widehat{X}_{\underline{t}}^K + f(\widehat{X}_{\underline{t}}^K, \overline{X}_{\underline{t}-\tau}^K)(t - \underline{t}) + g(\widehat{X}_{\underline{t}}^K, \overline{X}_{\underline{t}-\tau}^K)(W_t - W_{\underline{t}})\right) \quad t > 0,
\end{aligned} \tag{4.10}$$

where $\underline{t} := \max\{t_n : t_n \le t\}$. Note that $X_{t_n}^K = \widehat{X}_{t_n}^K = \overline{X}_{t_n}^K$.

Lemma 4.5. *Let $p \ge 4$, the SDDE (2.1) satisfy Assumption 4.1 and the function h^{δ} satisfy Assumption 4.2. Then, for the auxiliary scheme defined by (4.10), T is attainable and for all $p \ge 4$ there exists a constant C dependent on T and p, but independent of h_n^{δ} and K such that*

$$\mathbb{E}\left[\sup_{0\le t\le T} |X_t^K|^p\right] \le C. \tag{4.11}$$

Proof. Fix $\delta \in (0,1)$. Since h^δ is continuous and strictly positive, $\inf_{|x|\le K} h^\delta(x) > 0$. This implies that for every $\omega \in \Omega$

$$\liminf_{n\to\infty} h_n^{\delta,K}(\omega) = \liminf_{n\to\infty} h^\delta(\widehat{X}_{t_n}^K(\omega)) > 0,$$

so $\lim_{n\to\infty} t_n(\omega) = \sum_{n=0}^{\infty} h_n^{\delta,K}(\omega) = \infty$ for all $\omega \in \Omega$ and T is attainable in the bounded scheme.

Now we will prove the boundedness of the pth moments and the upper bound will be independent of $h_n^{\delta,K}$ and K. To ease the notation will drop the symbols "δ" and "K" in the adaptive time-step "$h_n^{\delta,K}$". Let $t \in [0,T]$. Define $\underline{t} := \max\{t_n : t_n \le t\}$, and $n_t := \max\{n : t_n \le t\}$. Using (4.8) and since for any $x \in \mathbb{R}^m$, $|\Phi(x)|^2 \le |x|^2$, we have that for $n = 0$ to $n = n_t - 1$,

$$\begin{aligned}
|\widehat{X}_{t_{n+1}}^K|^2 &\le |\widehat{X}_{t_n}^K + f(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)h_n + g(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)\Delta W_n|^2 \\
&= \langle \widehat{X}_{t_n}^K, \widehat{X}_{t_n}^K\rangle + 2\langle \widehat{X}_{t_n}^K, f(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)h_n\rangle + \langle f(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)h_n, f(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)h_n\rangle \\
&\quad + 2\langle \widehat{X}_{t_n}^K + f(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)h_n, g(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)\Delta W_n\rangle \\
&\quad + \langle g(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)\Delta W_n, g(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)\Delta W_n\rangle \\
&= |\widehat{X}_{t_n}^K|^2 + 2h_n\left[\langle \widehat{X}_{t_n}^K, f(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)\rangle + \frac{1}{2}h_n|f(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)|^2\right] \\
&\quad + 2\langle \widehat{X}_{t_n}^K + f(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)h_n, g(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)\Delta W_n\rangle + |g(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)\Delta W_n|^2 \\
&\le |\widehat{X}_{t_n}^K|^2 + 2h_n\alpha(|\widehat{X}_{t_n}^K|^2 + |\overline{X}_{t_n-\tau}^K|^2) + 2h_n\beta \\
&\quad + 2\langle \widehat{X}_{t_n}^K + f(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)h_n, g(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)\Delta W_n\rangle + |g(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)\Delta W_n|^2,
\end{aligned}$$

where in the last step we have used condition (4.5). Solving the recurrence relation, we get

$$\begin{aligned}
|\widehat{X}_{\underline{t}}^K|^2 &\le |\widehat{X}_0^K|^2 + 2\alpha\left(\sum_{n=0}^{n_t-1} |\widehat{X}_{t_n}^K|^2 h_n + |\overline{X}_{t_n-\tau}^K|^2 h_n\right) + 2\beta\underline{t} \\
&\quad + 2\sum_{n=0}^{n_t-1}\langle \widehat{X}_{t_n}^K + f(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)h_n, g(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)\Delta W_n\rangle + \sum_{n=0}^{n_t-1}|g(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)\Delta W_n|^2.
\end{aligned} \tag{4.12}$$

Similarly, the continuous approximate solution verifies

$$\begin{aligned}
|X_t^K|^2 &\le |\widehat{X}_{\underline{t}}^K|^2 + 2(t-\underline{t})\alpha(|\widehat{X}_{\underline{t}}^K|^2 + |\overline{X}_{\underline{t}-\tau}^K|^2) + 2(t-\underline{t})\beta \\
&\quad + 2\langle \widehat{X}_{\underline{t}}^K + f(\widehat{X}_{\underline{t}}^K, \overline{X}_{\underline{t}-\tau}^K)(t-\underline{t}), g(\widehat{X}_{\underline{t}}^K, \overline{X}_{\underline{t}-\tau}^K)(W_t - W_{\underline{t}})\rangle + |g(\widehat{X}_{\underline{t}}^K, \overline{X}_{\underline{t}-\tau}^K)(W_t - W_{\underline{t}})|^2.
\end{aligned} \tag{4.13}$$

Substituting (4.12) into (4.13) yields

$$\begin{aligned}
|X_t^K|^2 &\le |\widehat{X}_0^K|^2 + 2\alpha\left(\sum_{n=0}^{n_t-1} |\widehat{X}_{t_n}^K|^2 h_n + |\overline{X}_{t_n-\tau}^K|^2 h_n + |\widehat{X}_{\underline{t}}^K|^2(t-\underline{t}) + |\overline{X}_{\underline{t}-\tau}^K|^2(t-\underline{t})\right) + 2\beta t \\
&\quad + 2\sum_{n=0}^{n_t-1}\langle \widehat{X}_{t_n}^K + f(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)h_n, g(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)\Delta W_n\rangle + \sum_{n=0}^{n_t-1}|g(\widehat{X}_{t_n}^K, \overline{X}_{t_n-\tau}^K)\Delta W_n|^2 \\
&\quad + 2\langle \widehat{X}_{\underline{t}}^K + f(\widehat{X}_{\underline{t}}^K, \overline{X}_{\underline{t}-\tau}^K)(t-\underline{t}), g(\widehat{X}_{\underline{t}}^K, \overline{X}_{\underline{t}-\tau}^K)(W_t - W_{\underline{t}})\rangle + |g(\widehat{X}_{\underline{t}}^K, \overline{X}_{\underline{t}-\tau}^K)(W_t - W_{\underline{t}})|^2.
\end{aligned}$$

Using the step processes $\overline{X}^K$ and $\widetilde{X}^K$ defined previously, the second summand on the RHS of the equation above, can be expressed as a Riemann integral. Similarly the fourth and the sixth terms can be written as an Itô integral, i.e.

$$\begin{aligned}
|X_t^K|^2 &\le |X_0^K| + 2\alpha\int_0^t (|\overline{X}_s^K|^2 + |\widetilde{X}_{s-\tau}^K|^2)ds + 2\beta t \\
&\quad + 2\int_0^t \langle \overline{X}_s^K + f(\overline{X}_s^K, \widetilde{X}_{s-\tau}^K)[h(\overline{X}_u^K)I_{[0,\underline{t})}(u) + (t-\underline{t})I_{[\underline{t},t]}(u)], g(\overline{X}_s^K, \widetilde{X}_{s-\tau}^K)dW_s\rangle \\
&\quad + \sum_{n=0}^{n_t-1}|g(\overline{X}_{t_n}^K, \widetilde{X}_{t_n-\tau}^K)\Delta W_n|^2 + |g(\overline{X}_t^K, \widetilde{X}_{t-\tau}^K)(W_t - W_{\underline{t}})|^2.
\end{aligned}$$

Hence, we have

$$|X_t^K|^p \le 6^{p/2-1}\Bigg\{ |X_0^K|^p + \left(2\alpha\int_0^t (|\overline{X}_s^K|^2 + |\widetilde{X}_{s-\tau}^K|^2)ds\right)^{p/2} + (2\beta t)^{p/2}$$
$$+ \left|2\int_0^t \langle \overline{X}_s^K + f(\overline{X}_s^K, \widetilde{X}_{s-\tau}^K)[h(\overline{X}_u^K)I_{[0,\underline{t})}(u) + (t-\underline{t})I_{[\underline{t},t]}(u)], g(\overline{X}_s^K, \widetilde{X}_{s-\tau}^K)dW_s\rangle\right|^{p/2}$$
$$+ \left(\sum_{n=0}^{n_t-1} |g(\overline{X}_{t_n}^K, \widetilde{X}_{t_n-\tau}^K)\Delta W_n|^2\right)^{p/2} + |g(\overline{X}_t^K, \widetilde{X}_{t-\tau}^K)(W_t - W_{\underline{t}})|^p \Bigg\}.$$

Taking the expectation of the supremum, one has

$$\mathbb{E}\left[\sup_{0\le s\le t} |X_s^K|^p\right] \le 6^{p/2-1}(I_1 + I_2 + I_3 + I_4),$$

where

$$I_1 := \mathbb{E}|X_0^K|^p + \mathbb{E}\left[\left(2\alpha\int_0^t (|\overline{X}_s^K|^2 + |\widetilde{X}_{s-\tau}^K|^2)ds\right)^{p/2}\right] + (2\beta t)^{p/2};$$

$$I_2 := \mathbb{E}\left[\sup_{0\le s\le t}\left|2\int_0^s \langle \overline{X}_u^K + f(\overline{X}_u^K, \widetilde{X}_{u-\tau}^K)[h(\overline{X}_u^K)I_{[0,\underline{s})}(u) + (s-\underline{s})I_{[\underline{s},s]}(u)], g(\overline{X}_u^K, \widetilde{X}_{u-\tau}^K)dW_u\rangle\right|^{p/2}\right];$$

$$I_3 := \mathbb{E}\left[\left(\sum_{n=0}^{n_t-1} |g(\overline{X}_{t_n}^K, \widetilde{X}_{t_n-\tau}^K)\Delta W_n|^2\right)^{p/2}\right];$$

$$I_4 := \mathbb{E}\left[\sup_{0\le s\le t} |g(\overline{X}_s^K, \widetilde{X}_{s-\tau}^K)(W_s - W_{\underline{s}})|^p\right].$$

Now we will establish bounds for each of the four terms above. In the remainder of the proof, C is positive constants, independent of K, that may change from line to line.

Using Hölder's inequality, we have

$$I_1 \le \mathbb{E}|X_0^K|^p + (2\alpha)^{p/2}T^{p/2-1}2^{p/2-1}\int_0^t \mathbb{E}[|\overline{X}_s^K|^p + |\widetilde{X}_{s-\tau}^K|^p]ds + (2\beta T)^{p/2}$$
$$\le C\int_0^t \mathbb{E}\left[\sup_{0\le u\le s}|X_u^K|^p\right]ds + C.$$

By the Burkholder-Davis-Gundy (BDG) inequality (see [9]) we obtain

$$I_2 \le 2^{p/2}C\mathbb{E}\left[\left(\int_0^t |(\overline{X}_u^K + f(\overline{X}_u^K, \widetilde{X}_{u-\tau}^K)[h(\overline{X}_u^K)I_{[0,\underline{t})}(u) + (t-\underline{t})I_{[\underline{t},t]}(u)])g(\overline{X}_u^K, \widetilde{X}_{u-\tau}^K)|^2 du\right)^{p/4}\right]$$

An application of the Hölder inequality yields that

$$I_2 \le 2^{\frac{p}{2}}T^{\frac{p}{4}-1}C\mathbb{E}\left[\int_0^t \left|\overline{X}_u^K + f(\overline{X}_u^K, \widetilde{X}_{u-\tau}^K)[h(\overline{X}_u^K)I_{[0,\underline{t})}(u) + (t-\underline{t})I_{[\underline{t},t]}(u)]\right|^{\frac{p}{2}} ||g(\overline{X}_u^K, \widetilde{X}_{u-\tau}^K)||^{\frac{p}{2}}du\right] \tag{4.14}$$

Now, we bound the integrand of the integral above. Using condition (4.5) we obtain

$$|\overline{X}_u^K + f(\overline{X}_u^K, \widetilde{X}_{u-\tau}^K)[h(\overline{X}_u^K)I_{[0,\underline{t})}(u) + (t-\underline{t})I_{[\underline{t},t]}(u)]|^2 =$$
$$= |\overline{X}_u^K|^2 + 2[h(\overline{X}_u^K)I_{[0,\underline{t})}(u) + (t-\underline{t})I_{[\underline{t},t]}(u)]\Big[\langle \overline{X}_u^K, f(\overline{X}_u^K, \widetilde{X}_{u-\tau}^K)\rangle$$
$$+ \frac{1}{2}[h(\overline{X}_u^K)I_{[0,\underline{t})}(u) + (t-\underline{t})I_{[\underline{t},t]}(u)]|f(\overline{X}_u^K, \widetilde{X}_{u-\tau}^K)|^2\Big]$$
$$\le |\overline{X}_u^K|^2 + 2[h(\overline{X}_u^K)I_{[0,\underline{t})}(u) + (t-\underline{t})I_{[\underline{t},t]}(u)]\left[\alpha\left(|\overline{X}_u^K|^2 + |\widetilde{X}_{u-\tau}^K|^2\right) + \beta\right]$$

$$= (1+2\alpha T)|\overline{X}_u^K|^2 + 2\alpha T|\widetilde{X}_{u-\tau}^K|^2 + 2\beta T.$$

This implies

$$\begin{aligned}
&|\overline{X}_u^K + f(\overline{X}_u^K, \widetilde{X}_{u-\tau}^K)[h(\overline{X}_u^K)I_{[0,\underline{t})}(u) + (t-\underline{t})I_{[\underline{t},t]}(u)]|^{p/2} \\
&\quad \le 3^{p/4-1}\left[(1+2\alpha T)^{p/4}|\overline{X}_u^K|^{p/2} + (2\alpha T)^{p/4}|\widetilde{X}_{u-\tau}^K|^{p/2} + (2\beta T)^{p/4}\right] \\
&\quad \le C\left(|\overline{X}_u^K|^{p/2} + |\widetilde{X}_{u-\tau}^K|^{p/2} + 1\right).
\end{aligned}$$

Also by condition (4.3) one can see that

$$\begin{aligned}
||g(\overline{X}_u^K, \widetilde{X}_{u-\tau}^K)||^{p/2} &= \left(||g(\overline{X}_u^K, \widetilde{X}_{u-\tau}^K)||^2\right)^{p/4} \le \left[\alpha\left(|\overline{X}_u^K|^2 + |\widetilde{X}_{u-\tau}^K|^2\right) + \beta\right]^{p/4} \\
&\le C\left(|\overline{X}_u^K|^{p/2} + |\widetilde{X}_{u-\tau}^K|^{p/2} + 1\right).
\end{aligned}$$

Substituting the last two inequalities into (4.14), we obtain

$$\begin{aligned}
I_2 &\le C\mathbb{E}\left[\int_0^t \left(1 + |\overline{X}_u^K|^p + |\widetilde{X}_{u-\tau}^K|^p\right) du\right] \\
&\le C + C\left(\int_0^t \mathbb{E}\left[\sup_{0\le u\le s} |X_u^K|^p\right] ds\right).
\end{aligned}$$

Now we will bound I_3. Note that t_n is a stopping time of the filtration $\{\mathcal{F}_t^W\}$. Define

$$\mathcal{F}_{t_n} := \{A \in \mathcal{F} : A \cap \{t_n \le t\} \in \mathcal{F}_t^W\}.$$

We want to show that given p there exists a constant C dependent on d and p such that for every n we have

$$\mathbb{E}[|\Delta W_n|^p|\mathcal{F}_{t_n}] \le C h_n^{p/2}.$$

Note that $\Delta W_n = W_{t_{n+1}} - W_{t_n}$ has the same distribution as the random variable $\sqrt{h_n}Z$, where Z is a standard normal random d-dimensional vector independent of h_n and $\mathcal{F}_{t_n}$. Thus,

$$\mathbb{E}[|\Delta W_n|^p|\mathcal{F}_{t_n}] = \mathbb{E}[|\sqrt{h_n}Z|^p|\mathcal{F}_{t_n}] = |\sqrt{h_n}|^p\mathbb{E}[|Z|^p] = h_n^{p/2}C, \tag{4.15}$$

where we have used the facts that h_n is $\mathcal{F}_{t_n}$-measurable, that Z is independent of $\mathcal{F}_{t_n}$ and that $\mathbb{E}[|Z|^p]$ is a real positive number that depends only on p and d.

Combining Jensen's inequality and equation (4.15), we arrive at

$$\begin{aligned}
I_3 &\le \mathbb{E}\left[\left(\sum_{n=0}^{n_t-1} ||g(\overline{X}_{t_n}^K, \widetilde{X}_{t_n-\tau}^K)||^2|\Delta W_n|^2\right)^{p/2}\right] = \mathbb{E}\left[\left(\sum_{n=0}^{n_t-1} h_n||g(\overline{X}_{t_n}^K, \widetilde{X}_{t_n-\tau}^K)||^2\frac{|\Delta W_n|^2}{h_n}\right)^{p/2}\right] \\
&\le T^{p/2-1}\mathbb{E}\left[\sum_{n=0}^{n_t-1} h_n||g(\overline{X}_{t_n}^K, \widetilde{X}_{t_n-\tau}^K)||^p\frac{E[|\Delta W_n|^p|\mathcal{F}_{t_n}]}{h_n^{p/2}}\right] \le CT^{p/2-1}\mathbb{E}\left[\sum_{n=0}^{n_t-1} h_n||g(\overline{X}_{t_n}^K, \widetilde{X}_{t_n-\tau}^K)||^p\right] \\
&\le CT^{p/2-1}\mathbb{E}\left[\int_0^{\underline{t}} ||g(\overline{X}_s^K, \widetilde{X}_{s-\tau}^K)||^p ds\right] \le CT^{p/2-1}\mathbb{E}\left[\int_0^t ||g(\overline{X}_s^K, \widetilde{X}_{s-\tau}^K)||^p ds\right].
\end{aligned}$$

Using condition (4.3) and Hölder's inequality, we have

$$\begin{aligned}
I_3 &\le CT^{p/2-1}\mathbb{E}\left[\int_0^t \left(||g(\overline{X}_s^K, \widetilde{X}_{s-\tau}^K)||^2\right)^{p/2} ds\right] \le CT^{p/2-1}\mathbb{E}\left[\int_0^t \left(\alpha(|\overline{X}_s^K|^2 + |\widetilde{X}_{s-\tau}^K|^2) + \beta\right)^{p/2} ds\right] \\
&\le T^{p/2-1}2^{p-2}C\mathbb{E}\left[\int_0^t \left(\alpha^{p/2}(|\overline{X}_s^K|^p + |\overline{X}_{s-\tau}^K|^p) + \beta^{p/2}\right) ds\right] \\
&\le C + C\int_0^t \mathbb{E}\left[\sup_{0\le u\le s} |X_u^K|^p\right] ds.
\end{aligned}$$

For I_4, using the linear condition (4.3), we obtain

$$I_4 \le \mathbb{E}\left[\sup_{0\le s\le t}|g(\overline{X}^K_s,\widetilde{X}^K_{s-\tau})(W_s-W_{\underline{s}})|^p\right] \le \mathbb{E}\left[\sup_{0\le s\le t}\left\{[(\alpha(|\overline{X}^K_s|^p+|\widetilde{X}^K_{s-\tau})|^p)+\beta]\ |(W_s-W_{\underline{s}})|^p\right\}\right]$$

$$\le \mathbb{E}\left[\sum_{n=0}^{n_t-1}[\alpha(|\overline{X}^K_{t_n}|^p+|\widetilde{X}^K_{t_n-\tau}|^p)+\beta]\mathbb{E}\left[\sup_{t_n\le s\le t_{n+1}}|(W_s-W_{t_n})|^{p/2}|\mathcal{F}_{t_n}\right]\right.$$

$$\left.+[\alpha(|\overline{X}^K_t|^p+|\widetilde{X}^K_{t-\tau}|^p)+\beta]\mathbb{E}\left[\sup_{\underline{t}\le s\le t}|(W_s-W_{\underline{t}})|^{p/2}|\mathcal{F}_{\underline{t}}\right]\right]$$

$$\le C + C\int_0^t E\left[\sup_{0\le u\le s}|X^K_u|^p\right]ds.$$

Adding all the bounds for I_1 to I_4, we have that for all $t\in[0,T]$

$$\mathbb{E}\left[\sup_{0\le s\le t}|X^K_s|^p\right] \le C + C\int_0^t E\left[\sup_{0\le u\le s}|X^K_u|^p\right]$$

and by the Gronwall inequality we obtain

$$\mathbb{E}\left[\sup_{0\le t\le T}|X^K_t|^p\right] \le C. \quad \square$$

Remark 4.1. Note that assuming that T was attainable, we have proved the boundedness of the pth moments without using the auxiliary scheme. The only reason why we needed to work with a bounded scheme was to show that $\inf_{|x|\le K} h^\delta(x)$ is strictly positive and therefore T is attainable.

Proof of Theorem 4.4. Since h^δ is continuous and strictly positive, $\inf_{|x|\le K_\omega} h^\delta(x)>0$. This implies that for almost every $\omega\in\Omega$

$$\liminf_{n\to\infty} h^\delta_n(\omega) = \liminf_{n\to\infty} h^\delta(X_{t_n}(\omega)) \ne 0,$$

so $\lim_{n\to\infty} t_n(\omega)=\sum_{n=0}^\infty h^\delta_n(\omega)=\infty$ a.s. and T is attainable. By Lemma 4.5 and the Markov inequality

$$\mathbb{P}(\sup_{0\le t\le T}|X_t|<K) = 1-\mathbb{P}(\sup_{0\le t\le T}|X^K_t|\ge K) \ge 1-\frac{\mathbb{E}[\sup_{0\le t\le T}|X^K_t|^4}{K^4} = 1-\frac{C}{K^4}.$$

Thus

$$\lim_{K\to\infty}\mathbb{P}(\sup_{0\le t\le T}|X_t|<K)=1.$$

This means that $\sup_{0\le t\le T}|X_t|<\infty$ a.s., i.e. for almost all $\omega\in\Omega$ there exists a K_ω such that

$$\sup_{0\le t\le T}|X_t(\omega)|\le K_\omega. \tag{4.16}$$

Also, for all ω and all $0<K_1\le K_2$, we have

$$\sup_{0\le t\le T}|X^{K_1}_t(\omega)| = \min(\sup_{0\le t\le T}|X_t(\omega)|,K_1) \le \min(\sup_{0\le t\le T}|X_t(\omega)|,K_2) = \sup_{0\le t\le T}|X^{K_2}_t(\omega).| \tag{4.17}$$

Equations (4.16) and (4.17) imply that

$$\lim_{K\to\infty}\sup_{0\le t\le T}|X^K_t| = \sup_{0\le t\le T}|X_t| \quad \text{a.s.} \tag{4.18}$$

This together with Lemma 4.5, yields

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_t|^p\right] = \lim_{K\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}|X^K_t|^p\right] \le C.$$

The proof is complete for $p\ge 4$. For $0\le p<4$, the required assertion follows from the Hölder inequality. $\square$

4.1.3. Strong convergence of the numerical solutions

In order to prove the strong convergence of the approximate solution (3.3) to the exact solution of the SDDE (2.1), we need the following lemma and corollary.

Lemma 4.6. *Let the SDDE (2.1) and the function h^δ satisfy Assumption 4.1 and 4.2 respectively. Assume also that the function f satisfies the (global) linear growth condition, i.e. there exists a constant $C_1 \geq 0$ such that for all $x, y \in \mathbb{R}^m$,*

$$|f(x,y)|^2 \leq C_1(|x|^2 + |y|^2 + 1). \tag{4.19}$$

Then there exists a positive constant C such that for all $t \in [0,T]$.

$$\mathbb{E}|X_t - \overline{X}_t|^2 \leq C\delta T, \tag{4.20}$$

$$\mathbb{E}|X_t - \widetilde{X}_t|^2 \leq C\delta T. \tag{4.21}$$

Proof. Let $t \in [0,T]$. Let r be such that $t_r \leq t < t_{r+1}$. Then by definition we have $X_{t_r} = \overline{X}_{t_r} = \overline{X}_t$. Thus

$$X_t = \overline{X}_t + \int_{t_r}^{t} f(\overline{X}_s, \widetilde{X}_s)ds + \int_{t_r}^{t} g(\overline{X}_s, \widetilde{X}_s)dW_s.$$

This together with (4.19), (4.3), Assumption 4.2 and Theorem 4.4 imply that

$$\begin{aligned}
\mathbb{E}|X_t - \overline{X}_t|^2 &\leq 2\mathbb{E}\left|\int_{t_r}^{t} f(\overline{X}_s, \widetilde{X}_s)ds\right|^2 + 2\mathbb{E}\left|\int_{t_r}^{t} g(\overline{X}_s, \widetilde{X}_s)dW_s\right|^2 \\
&\leq 2\mathbb{E}[C_1(h_r^\delta)^2(1 + 2\sup_{t_r\leq s\leq t}|X_s|^2 + ||\xi||)] + 2\mathbb{E}[\alpha h_r^\delta(2\sup_{t_r\leq s\leq t}|X_s|^2 + ||\xi||) + \beta] \\
&\leq 4(\delta T)^2(1 + \mathbb{E}[\sup_{t_r\leq s\leq t}|X_s|^2] + \mathbb{E}||\xi||) + 4\alpha\delta T(\mathbb{E}[\sup_{t_r\leq s\leq t}|X_s|^2] + \mathbb{E}||\xi||) + \beta] \\
&\leq C\delta T.
\end{aligned}$$

To prove assertion (4.21), we first prove that there is a constant C such that for all $t \in [0,T]$

$$\mathbb{E}|\widetilde{X}_t - \overline{X}_t|^2 \leq C\delta T. \tag{4.22}$$

Let $t \in [0,T]$. Let k and n be such that $t_k \leq t < t_{k+1}$ and $t_n - \tau \leq t < t_{n+1} - \tau$ respectively. Let $r, 0 \leq r \leq k$ be such that $t_{k-r} \leq t_n - \tau < t_{k-r+1}$. From (3.1) and the definitions of the step processes $\overline{X}$ and $\widetilde{X}$, one can see that

$$\begin{aligned}
\widehat{X}_{t_k} &= \widehat{X}_{t_{k-r}} + \sum_{i=0}^{r-1}[f(\overline{X}_{t_{k-r+i}}, \overline{X}_{t_{k-r+i}-\tau})h_{k-r+i} + g(\overline{X}_{t_{k-r+i}}, \overline{X}_{t_{k-r+i}-\tau})\Delta W_{k-r+i}] \\
&= \widehat{X}_{t_{k-r}} + \sum_{i=0}^{r-1}\int_{t_{k-r+i}}^{t_{k-r+i+1}} f(\overline{X}_s, \widetilde{X}_{s-\tau})ds + \sum_{i=0}^{r-1}\int_{t_{k-r+i}}^{t_{k-r+i+1}} g(\overline{X}_s, \widetilde{X}_{s-\tau})dW_s \\
&= \widehat{X}_{t_{k-r}} + \int_{t_{k-r}}^{t_k} f(\overline{X}_s, \widetilde{X}_{s-\tau})ds + \int_{t_{k-r}}^{t_k} g(\overline{X}_s, \widetilde{X}_{s-\tau})dW_s.
\end{aligned}$$

Note that $\overline{X}_t = \widehat{X}_{t_k}$ and $\widehat{X}_{t_{k-r}} = \overline{X}_{t_{k-r}} = \overline{X}_{t_n-\tau} = \widetilde{X}_{t_n-\tau} = \widetilde{X}_t$, we have that

$$\overline{X}_t = \widetilde{X}_t + \int_{t_{k-r}}^{t_k} f(\overline{X}_s, \widetilde{X}_{s-\tau})ds + \int_{t_{k-r}}^{t_k} g(\overline{X}_s, \widetilde{X}_{s-\tau})dW_s.$$

Also, we have that

$$t_k - t_{k-r} \leq (t_{n+1} - \tau) - (t_n - \tau) + h_{k-r}^\delta = h_n^\delta + h_{k-r}^\delta \leq 2\delta T.$$

Therefore, by (4.19), (4.3), Assumption 4.2 and Theorem 4.4 we have that

$$\begin{aligned}
\mathbb{E}|\overline{X}_t - \widetilde{X}_t|^2 &\leq 2\mathbb{E}\left|\int_{t_{k-r}}^{t_k} f(\overline{X}_s, \widetilde{X}_{s-\tau})ds\right|^2 + 2\mathbb{E}\left|\int_{t_{k-r}}^{t_k} g(\overline{X}_s, \widetilde{X}_{s-\tau})dW_s\right|^2 \\
&\leq 2\mathbb{E}[C_1(t_k - t_{k-r})^2(1 + 2\sup_{t_k\leq s\leq t}|X_s|^2 + ||\xi||)] + 2\mathbb{E}[\alpha(t_k - t_{k-r})(2\sup_{t_k\leq s\leq t}|X_s|^2 + ||\xi||) + \beta] \\
&\leq 4(\delta T)^2(1 + \mathbb{E}[\sup_{t_k\leq s\leq t}|X_s|^2] + \mathbb{E}||\xi||) + 4\alpha\delta T(\mathbb{E}[\sup_{t_k\leq s\leq t}|X_s|^2] + \mathbb{E}||\xi||) + \beta]
\end{aligned}$$

$$\le C\delta T.$$

This together with (4.20) implies that

$$\mathbb{E}|X_t - \widetilde{X}_t|^2 = \mathbb{E}|X_t - \overline{X}_t|^2 + \mathbb{E}|\overline{X}_t - \widetilde{X}_t|^2 \le C\delta T. \quad \square$$

In our attempt to prove the strong convergence using the local Lipschitz condition instead of the global one, we introduce the stopping times

$$\tau_m := \inf\{t \ge 0 : |Y_t| \ge m\}, \quad \sigma_m := \inf\{t \ge 0 : |X_t| \ge m\}$$

and $\upsilon_m := \tau_m \wedge \sigma_m$. As usual we set $\inf \emptyset = \infty$. In the next corollary, we relax the global linear condition imposed to f in the previous lemma and use instead the local Lipschitz condition.

Corollary 4.7. *Let the SDDE (2.1) and the function h^δ satisfy Assumption 4.1 and 4.2 respectively. Then there exists a positive constant C_m such that for all $t \in [0,T]$.*

$$\mathbb{E}|X_{t\wedge\upsilon_m} - \overline{X}_{t\wedge\upsilon_m}|^2 \le C_m\delta T, \tag{4.23}$$

$$\mathbb{E}|X_{t\wedge\upsilon_m-\tau} - \widetilde{X}_{t\wedge\upsilon_m-\tau}|^2 \le C_m\delta T. \tag{4.24}$$

Proof. The processes $X_{t\wedge\upsilon_m}, \overline{X}_{t\wedge\upsilon_m}$ and $\widetilde{X}_{t\wedge\upsilon_m}$ are bounded by m. Thus, the local Lipschitz condition (4.1) implies condition (4.19). Therefore the corollary follows directly from Lemma 4.6. $\square$

Theorem 4.8. *If the SDDE (2.1) and the function h^δ satisfy Assumption 4.1 and 4.2 respectively, then for all $p > 0$*

$$\lim_{\delta\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |X_t - Y_t|^p\right] = 0.$$

Proof. One can see that

$$\begin{aligned}\mathbb{E}[\sup_{0\le t\le T} |Y_t - X_t|^2] &= \mathbb{E}[\sup_{0\le t\le T} |Y_t - X_t|^2 I_{\{\tau_m>T \text{ and } \sigma_m>T\}}] + \mathbb{E}[\sup_{0\le t\le T} |Y_t - X_t|^2 I_{\{\tau_m\le T \text{ or } \sigma_m\le T\}}],\\ &=: R_1 + R_2,\end{aligned} \tag{4.25}$$

where I_A es the indicator function of the set A. In order to bound R_1, we combine the definitions of the continuous-time approximation (3.3) and the exact solution (2.1) to obtain

$$\begin{aligned}&|Y_{t\wedge\upsilon_m} - X_{t\wedge\upsilon_m}|^2\\ &= \left|\int_0^{t\wedge\upsilon_m} [f(Y_s, Y_{s-\tau}) - f(\widehat{X}_s, \widetilde{X}_{s-\tau})]ds + \int_0^{t\wedge\upsilon_m} [g(Y_s, Y_{s-\tau}) - g(\widehat{X}_s, \widetilde{X}_{s-\tau})]dW_s\right|^2\\ &\le 2T\int_0^{t\wedge\upsilon_m} |f(Y_s, Y_{s-\tau}) - f(\widehat{X}_s, \widetilde{X}_{s-\tau})|^2 ds + 2\left|\int_0^{t\wedge\upsilon_m} [g(Y_s, Y_{s-\tau}) - g(\widehat{X}_s, \widetilde{X}_{s-\tau})]dW_s\right|^2\end{aligned}$$

Thus, for any $t_1 \le T$,

$$\begin{aligned}&\mathbb{E}[\sup_{0\le t\le t_1} |Y_{t\wedge\upsilon_m} - X_{t\wedge\upsilon_m}|^2]\\ &\le 2T\mathbb{E}\left[\int_0^{t\wedge\upsilon_m} |f(Y_s, Y_{s-\tau}) - f(\widehat{X}_s, \widetilde{X}_{s-\tau})|^2 ds\right] + 8\mathbb{E}\left[\int_0^{t\wedge\upsilon_m} |g(Y_s, Y_{s-\tau}) - g(\widehat{X}_s, \widetilde{X}_{s-\tau})|^2 ds\right],\end{aligned}$$

where we have used the Doob martingale inequality in the second summand. Using the local Lipschitz condition (4.1) in the RHS of the previous equation and then, adding and subtracting X_t twice yields

$$\begin{aligned}&\mathbb{E}[\sup_{0\le t\le t_1} |Y_{t\wedge\upsilon_m} - X_{t\wedge\upsilon_m}|^2]\\ &\le C_m\left(\int_0^{t_1} \mathbb{E}|Y_{s\wedge\upsilon_m} - X_{s\wedge\upsilon_m}|^2 ds + \int_0^{t_1} \mathbb{E}|Y_{s\wedge\upsilon_m-\tau} - X_{s\wedge\upsilon_m-\tau}|^2 ds\right)\end{aligned}$$

$$+ C_m \left(\int_0^{t_1} \mathbb{E}|X_{s\wedge \upsilon_m} - \overline{X}_{s\wedge \upsilon_m}|^2 ds + \int_0^{t_1} \mathbb{E}|X_{s\wedge \upsilon_m - \tau} - \widetilde{X}_{s\wedge \upsilon_m - \tau}|^2 ds \right),$$

where C_m is a positive constant that depends on T and m. By Corollary 4.7, we obtain

$$\mathbb{E}[\sup_{0\le t\le t_1} |Y_{t\wedge \upsilon_m} - X_{t\wedge \upsilon_m}|^2]$$
$$\le C_m \left(\int_0^{t_1} \mathbb{E}|Y_{s\wedge \upsilon_m} - X_{s\wedge \upsilon_m}|^2 ds + \int_0^{t_1} \mathbb{E}|Y_{s\wedge \upsilon_m - \tau} - X_{s\wedge \upsilon_m - \tau}|^2 ds \right) + C_m \delta.$$

The Gronwall inequality yields

$$R_1 = \mathbb{E}[\sup_{0\le t\le T} |Y_{t\wedge \upsilon_m} - X_{t\wedge \upsilon_m}|^2] \le C_m \delta.$$

Proceeding in exactly the same way as in [4], one can see that for all $\alpha, \beta, \eta, \mu > 0$ we have

$$R_2 \le \frac{2^{p+1}\eta C}{p} + \frac{2(p-2)C}{p\eta^{2/(p-2)} m^p}$$

where $\overline{C}$ is a positive constant. Substituting the estimates of R_1 and R_2 into (4.25), we obtain

$$\mathbb{E}[\sup_{0\le t\le T} |Y_t - X_t|^2] \le C_m \delta + \frac{2^{p+1}\eta C}{p} + \frac{2(p-2)C}{p\eta^{2/(p-2)} m^p}.$$

Now, given any $\epsilon > 0$, we can find an η sufficiently small so

$$\frac{2^{p+1}\eta C}{p} < \frac{\epsilon}{3},$$

and then m large enough so

$$\frac{2(p-2)C}{p\eta^{2/(p-2)} m^p} < \frac{\epsilon}{3},$$

and finally δ small enough such that

$$\delta C_m < \frac{\epsilon}{3}.$$

The proof is complete. □

4.2. *Order of convergence*

Now we investigate the order of convergence of the adaptive EM numerical solutions.

Assumption 4.9. There exists a constant $L > 0$ such that for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}^m$, f satisfies the one-sided Lipschitz condition

$$2\langle x - \overline{x}, f(x, y) - f(\overline{x}, \overline{y})\rangle \le L(|x - \overline{x}|^2 + |y - \overline{y}|^2) \tag{4.26}$$

and g satisfies the (global) Lipschitz condition

$$||g(x, y) - g(\overline{x}, \overline{y})||^2 \le L(|x - \overline{x}|^2 + |y - \overline{y}|^2). \tag{4.27}$$

In addition f satisfies the polynomial growth Lipschitz condition: there exist constants $\gamma, \lambda, q > 0$ such that for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}^m$

$$|f(x, y) - f(\overline{x}, \overline{y})| \le (\gamma(|x|^q + |y|^q + |\overline{x}|^q + |\overline{y}|^q) + \lambda)(|x - \overline{x}| + |y - \overline{y}|). \tag{4.28}$$

Furthermore, for any $s, t \in [-\tau, 0]$ and $q > 0$, there exists a positive constant Λ such that

$$\mathbb{E}|\xi(t) - \xi(s)| \le \Lambda |t - s|^q. \tag{4.29}$$

Theorem 4.10. *If the SDDE* (2.1) *satisfies Assumption 4.9 and the time-step function h satisfies Assumption 4.2, then for all $p > 0$, there exists a positive constant C independent of δ such that*

$$\mathbb{E}\left[\sup_{0\le t\le T} |X_t - Y_t|^p\right] \le C\delta^{p/2}.$$

Proof. The proof is similar to that of SDEs given in [2]. We only give the proof for $p \geq 4$; the result for $0 \leq p < 4$ follows from Hölder's inequality. Define $e_t := Y_t - X_t, 0 \leq t \leq T$. Hence

$$e_t = \int_0^t [f(Y_s, Y_{s-\tau}) - f(\overline{X}_s, \widetilde{X}_{s-\tau})]ds + \int_0^t [g(Y_s, Y_{s-\tau}) - g(\overline{X}_s, \widetilde{X}_{s-\tau})]dW_s.$$

Applying Itô's formula we obtain

$$\begin{aligned} |e_t|^2 &\leq 2\int_0^t \langle e_s, f(Y_s, Y_{s-\tau}) - f(\overline{X}_s, \widetilde{X}_{s-\tau})\rangle ds + \int_0^t |g(Y_s, Y_{s-\tau}) - g(\overline{X}_s, \widetilde{X}_{s-\tau})|^2 ds \\ &+ 2\int_0^t \langle e_s, (g(Y_s, Y_{s-\tau}) - g(\overline{X}_s, \widetilde{X}_{s-\tau}))dW_s\rangle \\ &\leq 2\int_0^t \langle e_s, f(Y_s, Y_{s-\tau}) - f(X_s, X_{s-\tau})\rangle ds + 2\int_0^t \langle e_s, f(X_s, X_{s-\tau}) - f(\overline{X}_s, \widetilde{X}_{s-\tau})\rangle ds \\ &+ \int_0^t |g(Y_s, Y_{s-\tau}) - g(\overline{X}_s, \widetilde{X}_{s-\tau})|^2 ds + 2\int_0^t \langle e_s, (g(Y_s, Y_{s-\tau}) - g(\overline{X}_s, \widetilde{X}_{s-\tau}))dW_s\rangle. \end{aligned} \tag{4.30}$$

Using condition (4.26) we get

$$2\langle e_s, f(Y_s, Y_{s-\tau}) - f(X_s, X_{s-\tau})\rangle \leq L(|Y_s - X_s|^2 + |Y_{s-\tau} - X_{s-\tau}|^2) = L(|e_s|^2 + |e_{s-\tau}|^2). \tag{4.31}$$

Condition (4.28) implies that

$$\begin{aligned} |\langle e_s, f(X_s, X_{s-\tau}) - f(\overline{X}_s, \widetilde{X}_{s-\tau})\rangle| &\leq |e_s|\ |f(X_s, X_{s-\tau}) - f(\overline{X}_s, \widetilde{X}_{s-\tau})| \\ &\leq |e_s| Q(X_s, X_{s-\tau}, \overline{X}_s, \widetilde{X}_{s-\tau})(|X_s - \overline{X}_s| + |X_{s-\tau} - \widetilde{X}_{s-\tau}|) \\ &\leq \frac{1}{2}|e_s|^2 + \frac{1}{2}Q(X_s, X_{s-\tau}, \overline{X}_s, \widetilde{X}_{s-\tau})^2\ 2(|X_s - \overline{X}_s|^2 + |X_{s-\tau} - \widetilde{X}_{s-\tau}|^2), \end{aligned} \tag{4.32}$$

where $Q(x, y, \overline{x}, \overline{y}) := \gamma(|x|^q + |y|^q + |\overline{x}|^q + |\overline{y}|^q) + \lambda$. In addition, condition (4.27) implies that

$$\begin{aligned} ||g(Y_s, Y_{s-\tau}) - g(\overline{X}_s, \widetilde{X}_{s-\tau})||^2 &\leq L(|Y_s - \overline{X}_s|^2 + |Y_{s-\tau} - \widetilde{X}_{s-\tau}|^2) \\ &= L(|Y_s - X_s + X_s - \overline{X}_s|^2 + |Y_{s-\tau} - X_{s-\tau} + X_{s-\tau} - \widetilde{X}_{s-\tau}|^2) \\ &\leq 2L(|e_s|^2 + |e_{s-\tau}|^2 + |X_s - \overline{X}_s|^2 + |X_{s-\tau} - \widetilde{X}_{s-\tau}|^2). \end{aligned} \tag{4.33}$$

Substituting (4.31), (4.32) and (4.33) in (4.30), we have

$$\begin{aligned} |e_t|^2 &\leq \int_0^t \left[(3L+1)|e_s|^2 + 3L|e_{s-\tau}|^2\right] ds \\ &+ 2\int_0^t [Q(X_s, X_{s-\tau}, \overline{X}_s, \widetilde{X}_{s-\tau})^2 + L](|X_s - \overline{X}_s|^2 + |X_{s-\tau} - \widetilde{X}_{s-\tau}|^2) ds \\ &+ 2\int_0^t \langle e_s, (g(Y_s, Y_{s-\tau}) - g(\overline{X}_s, \widetilde{X}_{s-\tau}))dW_s\rangle. \end{aligned}$$

Using Hölder's inequality yields

$$\begin{aligned} |e_t|^p &\leq (6T)^{p/2-1}\int_0^t ((3L+1)^{p/2}|e_s|^p + (2L)^{p/2}|e_{s-\tau}|^p) ds \\ &+ (3T)^{p/2-1}2^{p/2}\int_0^t [Q(X_s, X_{s-\tau}, \overline{X}_s, \widetilde{X}_{s-\tau}) + L]^{p/2}(|X_s - \overline{X}_s|^p + |X_{s-\tau} - \widetilde{X}_{s-\tau}|^p) ds \end{aligned}$$

$$+3^{p/2-1}2^{p/2}\left|\int_0^t \langle e_s,(g(Y_s,Y_{s-\tau})-g(\overline{X}_s,\widetilde{X}_{s-\tau}))dW_s\rangle\right|^{p/2}.$$

In the remainder of the proof, C is positive constant, independent of δ, that may change from line to line.

Taking the supremum on each side of the previous inequality and then the expectation yields

$$\mathbb{E}\left[\sup_{0\le s\le t}|e_s|^p\right]\le J_1+J_2+J_3,$$

where

$$J_1:=C\int_0^t \mathbb{E}\left[\sup_{0\le u\le s}|e_u|^p\right]ds;$$

$$J_2:=C\int_0^t \mathbb{E}\left[[Q(X_s,X_{s-\tau},\overline{X}_s,\widetilde{X}_{s-\tau})+L]^{p/2}(|X_s-\overline{X}_s|^p+|X_{s-\tau}-\widetilde{X}_{s-\tau}|^p)\right]ds;$$

$$J_3:=C\mathbb{E}\left[\sup_{0\le s\le t}\left|\int_0^s \langle e_s,(g(Y_u,Y_{u-\tau})-g(\overline{X}_u,\widetilde{X}_{u-\tau}))dW_u\rangle\right|^{p/2}\right].$$

For J_2, by Hölder's inequality one has

$$J_2\le C\int_0^t \left(\mathbb{E}\left[[Q(X_s,X_{s-\tau},\overline{X}_s,\widetilde{X}_{s-\tau})+L]^p\right]\mathbb{E}\left[(|X_s-\overline{X}_s|^{2p}+|X_{s-\tau}-\widetilde{X}_{s-\tau}|^{2p})\right]\right)^{1/2}ds. \tag{4.34}$$

By Theorem 4.4 there exists a constant C such that

$$\mathbb{E}\left[[Q(X_s,X_{s-\tau},\overline{X}_s,\widetilde{X}_{s-\tau})+L]^p\right]\le C. \tag{4.35}$$

Let $\underline{s}:=\max\{t_n : t_n\le s\}$. From (3.3), we can write

$$X_s-\overline{X}_s=f(\overline{X}_{\underline{s}},\widetilde{X}_{\underline{s}-\tau})(s-\underline{s})+g(\overline{X}_{\underline{s}},\widetilde{X}_{\underline{s}-\tau})(W_s-W_{\underline{s}}).$$

Thus, by Hölder inequality

$$\begin{aligned}\mathbb{E}|X_s-\overline{X}_s|^{2p}&=\mathbb{E}|f(\overline{X}_{\underline{s}},\widetilde{X}_{\underline{s}-\tau})(s-\underline{s})+g(\overline{X}_{\underline{s}},\widetilde{X}_{\underline{s}-\tau})(W_s-W_{\underline{s}})|^{2p}\\&\le 2^{2p-1}\mathbb{E}|f(\overline{X}_{\underline{s}},\widetilde{X}_{\underline{s}-\tau})(s-\underline{s})|^{2p}+2^{2p-1}\mathbb{E}|g(\overline{X}_{\underline{s}},\widetilde{X}_{\underline{s}-\tau})(W_s-W_{\underline{s}})|^{2p}\\&\le 2^{2p-1}(\mathbb{E}[f(\overline{X}_{\underline{s}},\widetilde{X}_{\underline{s}-\tau})]^{4p}\mathbb{E}[(s-\underline{s})^{4p})^{1/2}+2^{2p-1}(\mathbb{E}[g(\overline{X}_{\underline{s}},\widetilde{X}_{\underline{s}-\tau})]^{4p}\mathbb{E}[(W_s-W_{\underline{s}})^{4p}])^{1/2}.\end{aligned} \tag{4.36}$$

By Assumption 4.2 we have

$$E[(s-\underline{s})^{4p}]\le\mathbb{E}[(h^{\delta}_{\underline{s}})^{4p}]\le(\delta T)^{4p}\le\delta^{2p}T^{4p} \tag{4.37}$$

and by condition (4.15), we get

$$\mathbb{E}[(W_s-W_{\underline{s}})^{4p}]\le C(\delta T)^{2p}. \tag{4.38}$$

Also it follows from the global Lipschitz condition (4.27) that

$$\begin{aligned}||g(\overline{X}_{\underline{s}},\widetilde{X}_{\underline{s}-\tau})||^{4p}&\le\frac{1}{2^{2p}}K^{2p}(|\overline{X}_{\underline{s}}|^2+|\widetilde{X}_{\underline{s}-\tau}|^2)^{2p}+C\\&\le C(|\overline{X}_{\underline{s}}|^{4p}+|\widetilde{X}_{\underline{s}-\tau}|^{4p}+1)\end{aligned} \tag{4.39}$$

and from the polynomial growth condition that

$$\begin{aligned}|f(\overline{X}_{\underline{s}},\widetilde{X}_{\underline{s}-\tau})|^{4p}&\le\left[(\gamma(|\overline{X}_{\underline{s}}|^q+|\widetilde{X}_{\underline{s}-\tau})|^q)+\mu)(|\overline{X}_{\underline{s}}|+|\widetilde{X}_{\underline{s}-\tau})|)+f(0,0)\right]^{4p}\\&\le C(|\overline{X}_{\underline{s}}|^{4p(q+1)}+|\widetilde{X}_{\underline{s}-\tau})|^{4p(q+1)}+1),\end{aligned} \tag{4.40}$$

so by Theorem 4.4, there exists a constant C such that

$$\mathbb{E}[|f(\overline{X}_{\underline{s}},\widetilde{X}_{\underline{s}-\tau})|^{4p}]\le C \text{ and } \mathbb{E}[|g(\overline{X}_{\underline{s}},\widetilde{X}_{\underline{s}-\tau})|^{4p}]\le C.$$

Substituting these last two expressions together with (4.37) and (4.38) into (4.36), we obtain

$$\mathbb{E}|X_s - \overline{X}_s|^{2p} \le C\delta^p. \tag{4.41}$$

Using (4.39) and (4.40), and proceeding in exactly the same way as in Lemma 4.6, yields $\mathbb{E}|X_{s-\tau} - \widetilde{X}_{s-\tau}|^{2p} \le C\delta^p$. Using this fact together with (4.41) and (4.35) in (4.34), we obtain that $J_2 \le C\delta^{p/2}$.

Now we estimate J_3. By the BDG and Hölder's inequalities one can see that

$$\begin{aligned}
J_3 &\le C\mathbb{E}\left[\left(\int_0^t |e_s|^2\ |(g(Y_s, Y_{s-\tau}) - g(\overline{X}_s, \widetilde{X}_{s-\tau}))|^2 ds\right)^{p/4}\right] \\
&\le C\mathbb{E}\left[\int_0^t |e_s|^{p/2}(|\overline{X}_s - Y_s|^{p/2} + |\widetilde{X}_{s-\tau} - Y_{s-\tau}|^{p/2}) ds\right] \\
&\le C\mathbb{E}\left[\int_0^t \frac{1}{2}|e_s|^p + |\overline{X}_s - Y_s|^p + |\widetilde{X}_{s-\tau} - Y_{s-\tau}|^p ds\right] \\
&\le C\mathbb{E}\left[\int_0^t |e_s|^p + (|\overline{X}_s - X_s|^p + |X_s - Y_s|^p + |\widetilde{X}_{s-\tau} - X_{s-\tau}|^p + |X_{s-\tau} - Y_{s-\tau}|^p) ds\right] \\
&\le C\mathbb{E}\left[\int_0^t |e_s|^p + |e_{s-\tau}|^p + (|\overline{X}_s - X_s|^p + |\widetilde{X}_{s-\tau} - X_{s-\tau}|^p) ds\right].
\end{aligned}$$

By the same argument we used with J_2 we know that

$$\mathbb{E}\left[(|\overline{X}_s - X_s|^p + |\widetilde{X}_{s-\tau} - X_{s-\tau}|^p)\right] \le C\delta^{p/2}.$$

Thus

$$J_3 \le C\int_0^t \mathbb{E}\left[\sup_{0\le u\le s} |e_u|^p\right] ds + C\delta^{p/2}.$$

Collecting the bounds for J_1, J_2 and J_3, we conclude that there exists a constant C such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |e_t|^p\right] \le C\int_0^t \mathbb{E}\left[\sup_{0\le u\le s} |e_u|^p\right] ds + C\delta^{p/2}.$$

The required assertion follows from the Gronwall inequality. □

5. Convergence of the numerical solutions on infinite time interval

In this section we will study the convergence of the numerical solutions on the time interval $[0, \infty)$. The assumptions will be stronger than the ones on the finite time interval.

Assumption 5.1. The functions f and g satisfy the local Lipschitz condition: for every $R > 0$ there exists a positive constant C_R such that

$$|f(x, y) - f(\overline{x}, \overline{y})| + ||g(x, y) - g(\overline{x}, \overline{y})|| \le C_R(|x - \overline{x}| + |y - \overline{y}|) \tag{5.1}$$

for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}^m$ with $|x|, |y|, |\overline{x}|, |\overline{y}| \le R$. Furthermore, there exists constants $\alpha_1 > \alpha_2 \ge 0$ and $\beta > 0$, such that for all $x, y \in \mathbb{R}^m$, f satisfies the dissipative one-sided linear growth condition:

$$\langle x, f(x, y)\rangle \le -\alpha_1|x|^2 + \alpha_2|y|^2 + \beta, \tag{5.2}$$

and g is globally bounded:

$$||g(x, y)||^2 \le \beta. \tag{5.3}$$

Assumption 5.2. For every δ, the time step function $h^\delta : \mathbb{R}^m \to \mathbb{R}^+$, is continuous and uniformly bounded by h^δ_{max}, where $h^\delta_{max} \in (0, \infty)$.

Furthermore, there exist constants $\alpha_1 > \alpha_2 \geq 0$ and $\beta > 0$, such that for all $x, y \in \mathbb{R}^m$.

$$\langle x, f(x,y)\rangle + \frac{1}{2}h^{\delta}(x)|f(x,y)|^2 \leq -\alpha_1|x|^2 + \alpha_2|y|^2 + \beta. \tag{5.4}$$

5.1. The boundedness of the pth moments of the exact and the numerical solutions

5.1.1. Exact solution

Lemma 5.3. *If the SDDE (2.1) satisfies Assumption 5.1, then there exists a positive constant C such that for all $t \geq 0$*

$$\mathbb{E}\left[|Y_t|^p\right] \leq C. \tag{5.5}$$

Proof. The proof is standard, we omit it here. □

5.1.2. Adaptive EM numerical solutions

The proof about attainability given for the finite time interval, is valid for the infinite time interval $[-\tau, \infty)$.

Theorem 5.4. *If the SDE (2.1) and the function h^{δ} satisfy Assumption 5.1 and 5.2 respectively, then for all $p > 0$ there exists a constant C dependent on $h_{max}, \beta, \alpha_1, \alpha_2$ and p, but independent of δ and t, such that for all $t \geq 0$,*

$$\mathbb{E}\left[|X_t|^p\right] \leq C. \tag{5.6}$$

Proof. The proof is given for $p \geq 4$. For $0 < p < 4$, the result holds from Hölder's inequality. Fix t and define $\underline{t} := \max\{t_n : t_n \leq t\}$, $\underline{\hat{t}} := \max\{t_n : t_n \leq t - \tau\}$ and $n_t := \max\{n : t_n \leq t\}$. Taking squared norms in (3.1), we have that for $n = 0$ to $n = n_t$,

$$\begin{aligned}|\hat{X}_{t_{n+1}}|^2 &= |\hat{X}_{t_n}|^2 + 2h_n(\langle \hat{X}_{t_n}, f(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})\rangle + \frac{1}{2}h_n|f(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})|^2) \\ &\quad + 2\langle \hat{X}_{t_n} + f(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})h_n, g(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})\Delta W_n\rangle + |g(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})\Delta W_n|^2.\end{aligned}$$

Note that, since it is irrelevant in this proof, we have dropped the term "δ" in the adaptive time-step "h_n^{δ}" to ease the notation. Using conditions (5.4) and (5.3), we obtain

$$\begin{aligned}|\hat{X}_{t_{n+1}}|^2 &\leq |\hat{X}_{t_n}|^2 - 2h_n\alpha_1|\hat{X}_{t_n}|^2 + 2h_n\alpha_2|\overline{X}_{t_n-\tau}|^2 + 2h_n\beta \\ &\quad + 2\langle \hat{X}_{t_n} + f(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})h_n, g(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})\Delta W_n\rangle + \beta|\Delta W_n|^2.\end{aligned}$$

Multiplying both sides by $e^{2\alpha_1 t_{n+1}}$ yields

$$\begin{aligned}e^{2\alpha_1 t_{n+1}}|\hat{X}_{t_{n+1}}|^2 &\leq e^{2\alpha_1 t_{n+1}}|\hat{X}_{t_n}|^2 - 2h_n\alpha_1 e^{2\alpha_1 t_{n+1}}|\hat{X}_{t_n}|^2 + 2h_n\alpha_2 e^{2\alpha_1 t_{n+1}}|\overline{X}_{t_n-\tau}|^2 \\ &\quad + 2h_n\beta e^{2\alpha_1 t_{n+1}} + 2e^{2\alpha_1 t_{n+1}}\langle \hat{X}_{t_n} + f(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})h_n, g(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})\Delta W_n\rangle + e^{2\alpha_1 t_{n+1}}\beta|\Delta W_n|^2.\end{aligned}$$

Now, taking into account that $t_{n+1} = t_n + h_n$ and using the fact that for all $x \in \mathbb{R}$, $1 + x \leq e^x$ with $x = -2h_n\alpha_1$, we obtain

$$\begin{aligned}e^{2\alpha_1 t_{n+1}}|\hat{X}_{t_{n+1}}|^2 &\leq e^{2\alpha_1 t_n}|\hat{X}_{t_n}|^2 + 2h_n\alpha_2 e^{2\alpha_1 t_{n+1}}|\overline{X}_{t_n-\tau}|^2 + 2h_n\beta e^{2\alpha_1 t_{n+1}} \\ &\quad + 2e^{2\alpha_1 t_{n+1}}\langle \hat{X}_{t_n} + f(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})h_n, g(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})\Delta W_n\rangle + e^{2\alpha_1 t_{n+1}}\beta|\Delta W_n|^2.\end{aligned}$$

Solving the recurrence, we have

$$\begin{aligned}e^{2\alpha_1 \underline{t}}|\hat{X}_{\underline{t}}|^2 &\leq |\hat{X}_0|^2 + 2\alpha_2\sum_{n=0}^{n_t-1} e^{2\alpha_1 t_{n+1}}|\overline{X}_{t_n-\tau}|^2 h_n + 2\beta\sum_{n=0}^{n_t-1} e^{2\alpha_1 t_{n+1}}h_n \\ &\quad + 2\sum_{n=0}^{n_t-1} e^{2\alpha_1 t_{n+1}}\langle \hat{X}_{t_n} + f(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})h_n, g(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})\Delta W_n\rangle + \beta\sum_{n=0}^{n_t-1} e^{2\alpha_1 t_{n+1}}|\Delta W_n|^2.\end{aligned} \tag{5.7}$$

Similarly for the partial time step from $\underline{t}$ to t, we get

$$\begin{aligned}e^{2\alpha_1 t}|X_t|^2 &\leq e^{2\alpha_1 \underline{t}}|\hat{X}_{\underline{t}}|^2 + 2(t-\underline{t})\alpha_2 e^{2\alpha_1 t}|\overline{X}_{\underline{t}-\tau}|^2 + 2(t-\underline{t})\beta e^{2\alpha_1 t} \\ &\quad + 2e^{2\alpha_1 t}\langle \hat{X}_{\underline{t}} + f(\hat{X}_{\underline{t}}, \overline{X}_{\underline{t}-\tau})h_n, g(\hat{X}_{\underline{t}}, \overline{X}_{\underline{t}-\tau})(W_t - W_{\underline{t}})\rangle + e^{2\alpha_1 t}\beta|(W_t - W_{\underline{t}})|^2.\end{aligned} \tag{5.8}$$

Substituting the penultimate inequality into the last one, we obtain

$$
\begin{aligned}
e^{2\alpha_1 t}|X_t|^2 \le |X_0|^2 &+ 2\alpha_2 \sum_{n=0}^{n_t-1} e^{2\alpha_1 t_{n+1}} |\overline{X}_{t_n-\tau}|^2 |h_n + 2\alpha_2 e^{2\alpha_1 t} |\overline{X}_{t_n-\tau}|^2 (t-\underline{t}) \\
&+ 2\beta \sum_{n=0}^{n_t-1} e^{2\alpha_1 t_{n+1}} h_n + 2\beta e^{2\alpha_1 t}(t-\underline{t}) \\
&+ 2\sum_{n=0}^{n_t-1} e^{2\alpha_1 t_{n+1}} \langle \widehat{X}_{t_n} + f(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau}) h_n, g(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau})\Delta W_n\rangle \\
&+\beta \sum_{n=0}^{n_t-1} e^{2\alpha_1 t_{n+1}} |\Delta W_n|^2 + e^{2\alpha_1 t}\beta|(W_t - W_{\underline{t}})|^2 \\
&+ 2e^{2\alpha_1 t}\langle \widehat{X}_{\underline{t}} + f(\widehat{X}_{\underline{t}}, \overline{X}_{\underline{t}-\tau})(t-\underline{t}), g(\widehat{X}_{\underline{t}}, \overline{X}_{\underline{t}-\tau})(W_t - W_{\underline{t}})\rangle.
\end{aligned}
$$

Since $t_{n+1} \le t_n + h_{max}$ and $t \le \underline{t} + h_{max}$, we can take the common factor $e^{2\alpha_1 h_{max}}$ out in the equation above. The processes $\overline{X}$ and $\widetilde{X}$, defined in (3.1) and (3.2) respectively, are a simple processes, so we express the second and the third terms in the RHS of the previous equation as a Riemann integral. The same for the fourth and fifth terms. Similarly, the sixth and ninth terms can be written together as a (pathwise) Itô integral,

$$
\begin{aligned}
e^{2\alpha_1 t}|X_t|^2 \le |X_0|^2 + e^{2\alpha_1 h_{max}} \Bigg\{ &\int_0^t e^{2\alpha_1 s} |\widetilde{X}_{s-\tau}|^2 ds + 2\beta \int_0^t e^{2\alpha_1 s} ds \\
&+ 2\int_0^t e^{2\alpha_1 s} \langle \overline{X}_s + f(\overline{X}_s, \widetilde{X}_{s-\tau})[h(\overline{X}_s) I_{[0,\underline{t})}(s) + (t-\underline{t}) I_{[\underline{t},t]}(s)], g(\overline{X}_s, \widetilde{X}_{s-\tau}) dW_s\rangle \\
&+ \beta \sum_{n=0}^{n_t-1} e^{2\alpha_1 t_n} |\Delta W_n|^2 + e^{2\alpha_1 \underline{t}} \beta |(W_t - W_{\underline{t}})|^2 \Bigg\}.
\end{aligned}
$$

Now, raising to the power $p/2$, using Hölder's inequality and taking the expectation of the supremum, we obtain

$$
e^{p\alpha_1 t}\mathbb{E}\left[\sup_{0\le s\le t} |X_t|^p\right] \le 6^{p/2-1} e^{p\alpha_1 h_{max}} (H_1 + H_2 + H_3 + H_4), \tag{5.9}
$$

where

$$
\begin{aligned}
H_1 &:= \mathbb{E}|X_0|^p + \mathbb{E}\left[\left(2\alpha_2 \int_0^t e^{2\alpha_1 s} |\widetilde{X}_{s-\tau}|^2 ds\right)^{p/2}\right] + \left(2\beta \int_0^t e^{2\alpha_1 s} ds\right)^{p/2}; \\
H_2 &:= \mathbb{E}\Bigg[\sup_{0\le s\le t} \Bigg| 2\int_0^s e^{2\alpha_1 u} \langle \overline{X}_u + f(\overline{X}_u, \widetilde{X}_{u-\tau})[h(\overline{X}_u) I_{[0,\underline{s})}(u) \\
&\qquad\qquad + (s-\underline{s}) I_{[\underline{s},s]}(u)], g(\overline{X}_s, \widetilde{X}_{u-\tau}) dW_u\rangle \Bigg|^{p/2}\Bigg]; \\
H_3 &:= \mathbb{E}\left[\left(\beta \sum_{n=0}^{n_t-1} e^{2\alpha_1 t_n} |\Delta W_n|^2\right)^{p/2}\right]; \\
H_4 &:= \beta^{p/2} e^{p\alpha_1 t} \mathbb{E}[\sup_{0\le s\le t} |(W_s - W_{\underline{s}})|^p].
\end{aligned}
$$

Now we will establish bounds for each of the four terms above. In the remainder of the proof, C is a positive constant that may depend on $\beta, \alpha_1, \alpha_2, h_{max}$ and p, but independent of t, that may change from line to line. We start by bounding H_1.

$$
\begin{aligned}
H_1 &\le \mathbb{E}|X_0|^p + \mathbb{E}\left[\left(2\alpha_2 \sup_{-\tau\le s\le t} |X_s|^2 \int_0^t e^{2\alpha_1 s} ds\right)^{p/2}\right] + \left(2\beta \int_0^t e^{2\alpha_1 s} ds\right)^{p/2} \\
&\le \mathbb{E}|X_0|^p + \left(\frac{\alpha_2}{\alpha_1}\right)^{p/2} \mathbb{E}\left[\sup_{-\tau\le s\le t} |X_s|^p\right] e^{\alpha_1 pt} + \left(\frac{2\beta}{2\alpha_1}\right)^{p/2} e^{\alpha_1 pt}
\end{aligned}
$$

$$\le e^{\alpha_1 pt}\left(C+\left(\frac{\alpha_2}{\alpha_1}\right)^{p/2}\mathbb{E}\left[\sup_{0\le s\le t}|X_s|^p\right]\right).$$

For H_2, the BDG inequality and condition (5.3) yields

$$H_2 \le 2^{p/2}\beta^{p/4}C\mathbb{E}\left[\left(\int_0^t e^{4(\alpha_1-\alpha_2)s}|(\overline{X}_s+f(\overline{X}_s,\widetilde{X}_{s-\tau})[h(\overline{X}_s)I_{[0,\underline{t})}(s)+(t-\underline{t})I_{[\underline{t},t]}(s)])|^2ds\right)^{p/4}\right].$$

Since $e^{4(\alpha_1-\alpha_2)s}=e^{2(\alpha_1-\alpha_2)\frac{p-4}{p}s}e^{2(\alpha_1-\alpha_2)(1+\frac{4}{p})s}$, by Hölder's inequality, we get

$$\begin{aligned}&\left(\int_0^t e^{4(\alpha_1-\alpha_2)s}|(\overline{X}_s+f(\overline{X}_s,\widetilde{X}_{s-\tau})[h(\overline{X}_s)I_{[0,\underline{t})}(s)+(t-\underline{t})I_{[\underline{t},t]}(s)])|^2ds\right)^{p/4}\\&\le\left(\int_0^t e^{2(\alpha_1-\alpha_2)s}ds\right)^{\frac{p-4}{4}}\\&\times\int_0^t e^{(\alpha_1-\alpha_2)\frac{p+4}{2}s}|(\overline{X}_s+f(\overline{X}_s,\widetilde{X}_{s-\tau})[h(\overline{X}_s)I_{[0,\underline{t})}(s)+(t-\underline{t})I_{[\underline{t},t]}(s)])|^{p/2}ds.\end{aligned}$$

Using Assumption (5.2), we obtain

$$\begin{aligned}&|\overline{X}_s+f(\overline{X}_s,\widetilde{X}_{s-\tau})[h(\overline{X}_s)I_{[0,\underline{t})}(s)+(t-\underline{t})I_{[\underline{t},t]}(s)]|^2\\&\le|\overline{X}_s|^2+2[h(\overline{X}_s)I_{[0,\underline{t})}(s)+(t-\underline{t})I_{[\underline{t},t]}(s)]\left(-\alpha_1|\overline{X}_s|^2+\alpha_2|\widetilde{X}_{s-\tau}|^2+\beta\right)\\&\le|\overline{X}_s|^2+2h_{max}\left(\alpha_2|\widetilde{X}_{s-\tau}|^2+\beta\right).\end{aligned}$$

Therefore,

$$\begin{aligned}H_2\le\mathbb{E}\Bigg[C&\left(\int_0^t e^{2\alpha_1 s}ds\right)^{\frac{p-4}{4}}\\&\times\int_0^t e^{\alpha_1\frac{p+4}{2}s}\left\{|\overline{X}_s|^{p/2}+(2h_{max}\alpha_2)^{p/4}|\widetilde{X}_{s-\tau}|^{p/2}+(2\beta h_{max})^{p/4}\right\}ds\Bigg].\end{aligned}$$

We can write the previous inequality as $H_2\le H_{21}+H_{22}+H_{23}$, where

$$\begin{aligned}H_{21}&:=C\mathbb{E}[\sup_{0\le s\le t}|X_s|^{p/2}]\left(\int_0^t e^{2\alpha_1 s}ds\right)^{\frac{p-4}{4}}\int_0^t e^{\alpha_1\frac{p+4}{2}s}ds;\\H_{22}&:=C(2h_{max}\alpha_2)^{p/4}\mathbb{E}[\sup_{-\tau\le s\le t}|X_s|^{p/2}]\left(\int_0^t e^{2\alpha_1 s}ds\right)^{\frac{p-4}{4}}\left(\int_0^t e^{\alpha_1\frac{p+4}{2}s}ds\right);\\H_{23}&:=C(2h_{max}\alpha_2)^{p/4}\left(\int_0^t e^{2\alpha_1 s}ds\right)^{\frac{p-4}{4}}\left(\int_0^t e^{\alpha_1\frac{p+4}{2}s}ds\right).\end{aligned}$$

Since,

$$\begin{aligned}\left(\int_0^t e^{2\alpha_1 s}ds\right)^{\frac{p-4}{4}}\int_0^t e^{\alpha_1\frac{p+4}{2}s}&=\frac{e^{\alpha_1(p-4)t}-1}{(2\alpha_1)^{\frac{p-4}{4}}}\cdot\frac{e^{\alpha_1\frac{p+4}{2}t-1}}{\alpha_1\frac{p+4}{2}}\\&\le\frac{e^{\alpha_1 pt}}{\alpha_1\frac{p+4}{2}(2\alpha_1)^{\frac{p-4}{4}}}\le Ce^{\alpha_1 pt},\end{aligned}$$

we arrive at

$$H_2 \le C\mathbb{E}[\sup_{0\le s\le t}|X_s|^{p/2}]e^{\alpha_1 pt} + C\mathbb{E}[\sup_{-\tau\le s\le t}|X_s|^{p/2}]e^{\alpha_1 pt} + Ce^{\alpha_1 pt}$$
$$= e^{\alpha_1 pt}(C\mathbb{E}[\sup_{0\le s\le t}|X_s|^{p/2}] + C).$$

Using the elementary inequality $ab \le \frac{1}{2\gamma}a^2 + \frac{\gamma}{2}b^2$ for all $\gamma \in \mathbb{R}^+$ and all $a, b \in \mathbb{R}$ with $a = C$ and $b = \mathbb{E}[\sup_{0\le s\le t}|X_s|^{p/2}$, and later Jensen's inequality, we get

$$C\mathbb{E}[\sup_{0\le s\le t}|X_s|^{p/2}] \le \frac{1}{2\gamma}C^2 + \frac{\gamma}{2}(\mathbb{E}[\sup_{0\le s\le t}|X_s|^{p/2}])^2 \le \frac{1}{2\gamma}C^2 + \frac{\gamma}{2}\mathbb{E}[\sup_{0\le s\le t}|X_s|^{p}].$$

Therefore,

$$H_2 \le e^{\alpha_1 pt}(\frac{\gamma}{2}\mathbb{E}[\sup_{0\le s\le t}|X_s|^p] + C_\gamma), \tag{5.10}$$

where the "γ" in C_γ is to emphasise that this constant depends also on γ and is not fixed yet.

Now we will estimate H_3. By the discrete Hölder's inequality we obtain

$$\left|\sum_{n=0}^{n_t-1} e^{2\alpha_1 t_n}|\Delta W_n|^2\right| = \left|\sum_{n=0}^{n_t-1}\left(h_n^{\frac{p-2}{p}} e^{2\alpha_1 t_n \frac{p-2}{p}}\right)\left(h_n^{\frac{2}{p}} e^{\frac{4\alpha_1 t_n}{p}} \frac{|\Delta W_n|^2}{h_n}\right)\right|$$
$$\le \left(\sum_{n=0}^{n_t-1} h_n e^{2\alpha_1 t_n}\right)^{\frac{p-2}{p}} \left(\sum_{n=0}^{n_t-1} h_n e^{\frac{2\alpha_1 t_n}{p}} \frac{|\Delta W_n|^p}{h_n^{p/2}}\right)^{\frac{2}{p}}.$$

By (4.15) we can derive that

$$H_3 \le \mathbb{E}\left[\beta^{p/2}\left(\sum_{n=0}^{n_t-1} h_n e^{2\alpha_1 t_n}\right)^{\frac{p-2}{2}} \sum_{n=0}^{n_t-1} h_n e^{2\alpha_1 t_n}\frac{|\Delta W_n|^p}{h_n^{p/2}}\right]$$
$$\le \beta^{p/2}\left(\int_0^t e^{2\alpha_1 s}ds\right)^{\frac{p-2}{2}} C\int_0^t e^{2\alpha_1 s}ds \le Ce^{2\alpha_1 t}.$$

Using (4.15) again, we have that

$$H_4 \le \beta^{p/2}e^{\alpha_1 pt}Ch_{max}^{p/2} \le Ce^{\alpha_1 pt}.$$

Collecting together the bounds for H_1, H_2 H_3 and H_4, we obtain

$$e^{p\alpha_1 t}\mathbb{E}[\sup_{0\le s\le t}|X_s|^p] \le e^{p\alpha_1 t}(C_\gamma + \frac{\gamma}{2}\mathbb{E}[\sup_{0\le s\le t}|X_s|^p]) + \left(\frac{\alpha_2}{\alpha_1}\right)^{p/2}\mathbb{E}[\sup_{0\le s\le t}|X_s|^p]).$$

Noting that the constant C is independent of t, $0 \le (\alpha_2/\alpha_1)^{p/2} < 1$ and taking γ small enough such that $\frac{\gamma}{2} < 1 - (\alpha_2/\alpha_1)^{p/2}$, the required assertion follows. □

6. Almost sure exponential stability for SDDEs

It was shown in [16] that among other conditions, when the drift function satisfies the linear growth condition, the Euler-Maruyama approximate solution is a.s. exponentially stable. However, when the drift function satisfies the less restrictive one-sided linear growth condition, the EM solution needs not longer to be stable. It was proved in the same paper that the BEM solution maintains the stability. But it's well known that the BEM method is much more computationally expensive than explicit methods such as the adaptive EM method. Therefore, it is desirable to find explicit methods that provide numerical solutions that maintain the stability of the exact solution. Our goal in this section is to show that the adaptive solution can be a.s. exponentially stable for some SDDEs where the EM breaks down.

Assumption 6.1. The functions f and g satisfy the local Lipschitz condition: for every $R > 0$ there exists a positive constant C_R such that

$$|f(x,y) - f(\overline{x},\overline{y})| + ||g(x,y) - g(\overline{x},\overline{y})|| \le C_R(|x-\overline{x}| + |y-\overline{y}|) \tag{6.1}$$

for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}^m$ with $|x|, |y|, |\overline{x}|, |\overline{y}| \le R$. Furthermore, there exist constants α_1, α_2 and β satisfying

$$\alpha_1 > 2\alpha_2 \ge 0 \text{ and } \beta > 0, \tag{6.2}$$

such that for all $x, y \in \mathbb{R}^m$, f satisfies

$$\langle x, f(x,y)\rangle + \frac{1}{2}||g(x,y)||^2 \le -\alpha_1|x|^2 + \alpha_2|y|^2. \tag{6.3}$$

Under this assumption, the SDDE (2.1) has a unique solution.

6.1. Counterexample (SDDE)

We now return to the counterexample (1.1).

Let X_k be defined by (1.2) The following lemma proves a much stronger result that X_k is not almost sure exponential stable. It shows that the set in which the EM solution grows at a geometric rate has positive probability.

Lemma 6.2. *Consider the EM approximate solution* (1.2) *to the SDE* (1.1). *Then*

$$\mathbb{P}\left(|X_k| \ge \frac{2^{k+3}}{\sqrt{\Delta}},\ \forall k \ge 1\right) > 0. \tag{6.4}$$

The following proof is based on the counterexample's proof given in [5].

Proof. First we show that if $|X_1| \ge 2^4/\sqrt{\Delta}$, then

$$\mathbb{P}\left(|X_k| \ge \frac{2^{k+3}}{\sqrt{\Delta}},\ \forall k \ge 1\right) \ge \exp\left(-4e^{-2/\sqrt{\Delta}}\right). \tag{6.5}$$

We start by proving the following fact:

$$|X_k| \ge \frac{2^{k+3}}{\sqrt{\Delta}} \quad \text{and} \quad |\Delta W_k| \le 2^k \quad \text{imply} \quad |X_{k+1}| \ge \frac{2^{k+4}}{\sqrt{\Delta}}. \tag{6.6}$$

To prove (6.6), assume that $|X_k| \ge \frac{2^{k+3}}{\sqrt{\Delta}}$. Then

$$\begin{aligned} |X_{k+1}| &\ge |X_k|\left||X_k|^2\Delta - |1 + 2\Delta + 1/2\sin(X_{k-1})\Delta + \sqrt{2}\cos(X_{k-1})\Delta W_k|\right| \\ &\ge |X_k|\left||X_k|^2\Delta - (|1| + |2\Delta| + |1/2\Delta| + |\sqrt{2}\Delta W_k|)\right| \\ &\ge \frac{2^{k+3}}{\sqrt{\Delta}}(2^{2k+6} - 6 - \sqrt{2}2^k) \ge \frac{2^{k+4}}{\sqrt{\Delta}}(2^{2k+5} - 3 - \sqrt{2}2^{k-1}) \\ &\ge \frac{2^{k+4}}{\sqrt{\Delta}}. \end{aligned}$$

Now, from (6.6), given that $|X_1| \ge 2^4/\sqrt{\Delta}$, for any integer $K \ge 0$, the event that $\{|X_k| \ge 2^{k+3}/\sqrt{\Delta}, \forall 1 \le k \le K\}$ contains the event that $\{|W_k| \le 2^k, \forall 1 \le k \le K\}$. Since $\{\Delta W_k\}$ are independent, we have

$$\mathbb{P}\left(|X_k| \ge \frac{2^{k+3}}{\sqrt{\Delta}},\ \forall 1 \le k \le K\right) \ge \prod_{k=1}^{K} \mathbb{P}(|\Delta W_k| \le 2^k).$$

In order to prove (6.5), the rest of the proof is identical to the one in Lemma 3.1 in [5]. To obtain the final result, Equation (6.5), we need to prove that $\mathbb{P}(|X_1| \ge 2^4/\sqrt{\Delta}) > 0$. But this is true since X_1 is a normal random variable and for a normal random variable X with density function f, we have that for all $a \in \mathbb{R}$, $\mathbb{P}(X \ge a) = \int_a^\infty f(x)dx > 0$. □

In contrast to the standard EM solution, now we will see that the adaptive EM solution, maintains the stability of the exact solution of SDDE (1.1). But previous to that, we need to impose more assumptions.

Assumption 6.3. For every δ, the time step function $h^\delta : \mathbb{R} \to \mathbb{R}^+$, is continuous and there exist constants $\alpha_1 > \alpha_2 \ge 0$ and $\beta > 0$, such that for all $x, y \in \mathbb{R}^m$,

$$\langle x, f(x,y)\rangle + \frac{1}{2}h^\delta(x)|f(x,y)|^2 + \frac{d}{2}||g(x,y)||^2 \le -\alpha_1|x|^2 + \alpha_2\frac{\min(h^\delta(y), h^\delta(x))}{h^\delta(x)}|y|^2, \tag{6.7}$$

where d is the dimension of the Brownian motion in the SDDE (2.1). Furthermore, the function h^δ is uniformly bounded by the real numbers $0 < h^\delta_{\min} < h^\delta_{\max} < 1$, where $h^\delta_{\max}$ is small enough such that

$$2\alpha_2 e^{2\alpha_1 h_{\max}} < \alpha_1. \tag{6.8}$$

Note that condition (6.7) implies condition (6.3) with the same values of α_1 and α_2. An example of function h^δ that satisfies condition (6.7) for the SDDE (1.1) is

$$h^\delta(x) := \left(\frac{1}{25} I_{\{|x|<1\}} + 0.25 I_{\{|x|\geq 1\}} \frac{|x|^2}{\max(1, |f(x,y)|^2)}\right)\delta. \tag{6.9}$$

The following is the main result of this section.

Theorem 6.4. *Consider the SDDE (2.1) with a d-dimensional Brownian motion. If f and g satisfy Assumption 6.1 and h^δ satisfies Assumption 6.3, then the adaptive approximate solution (3.1) is almost sure exponentially stable, i.e. there exists a $\lambda > 0$ such that*

$$\limsup_{n\to\infty} \frac{\log |\hat{X}_{t_n}|}{t_n} \leq -\lambda \ a.s.$$

Before proving Theorem 6.4, we show that the SDDE (1.1) satisfies Assumption 6.1

$$\langle x, f(x,y)\rangle + \frac{1}{2}|g(x,y)|^2 = -2x^2 - x^4 + \frac{1}{2}\sin(y)x^2 + x^2\cos^2(y) \leq -\frac{1}{2}x^2.$$

In order to show that h^δ satisfies (6.7) for the SDDE (1.1), we substitute (6.9) into (6.7) and differentiate between the cases $|x| < 1$ and $|x| \geq 1$. For $|x| < 1$ we have

$$\begin{aligned}
&\langle x, f(x,y)\rangle + \frac{1}{2}h^\delta(x)|f(x,y)|^2 + \frac{d}{2}||g(x,y)||^2 = -2x^2 - x^4 + \frac{1}{2}x^2\sin(y) \\
&+ \frac{1}{2}\frac{1}{25}\delta(4x^2 + 4x^4 - 2x^2\sin(y) + x^6 - x^4\sin(y) + \frac{1}{4}x^2\sin(y)) + \frac{1}{2}2x^2\cos^2(y) \\
&\leq \frac{-3x^2}{10}
\end{aligned}$$

and for $|x| \geq 1$ we have

$$\begin{aligned}
&\langle x, f(x,y)\rangle + \frac{1}{2}h^\delta(x)|f(x,y)|^2 + \frac{d}{2}||g(x,y)||^2 \\
&= -2x^2 - x^4 + \frac{1}{2}x^2\sin(y) + \frac{1}{2}\frac{1}{4}\delta|x|^2 + \frac{1}{2}2x^2\cos^2(y) \leq \frac{-3x^2}{8}.
\end{aligned}$$

Thus the adaptive approximate solution of the SDDE (1.1) implemented with h^δ defined as (6.9) is almost sure exponentially stable.

We will prove the theorem, but first we need the following lemma.

Lemma 6.5. *Consider the SDDE (2.1) with a d-dimensional Brownian motion. Suppose f and g satisfy Assumption 6.1 and h^δ satisfies Assumption 6.3. Let l be a positive integer. Then there exists $\lambda \in (0, \alpha_1)$ such that*

$$\begin{aligned}
\sum_{n=1}^{l} e^{\lambda t_n}|\hat{X}_{t_n}|^2 h_n &\leq C + C\sum_{n=1}^{l} e^{\lambda t_n}|g(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})|^2(|\Delta W_n|^2 - h_n d) \\
&\quad + C\sum_{n=1}^{l} e^{\lambda t_n}\langle \hat{X}_{t_n} + f(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})h_n, g(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})\Delta W_n\rangle \ a.s.,
\end{aligned} \tag{6.10}$$

where C is a positive constant dependent on $\omega \in \Omega$, the constants $\alpha_1, \alpha_2, h_{max}$ and λ, but independent of l or t_n.

Proof. From (3.1) and (6.7), we have

$$\begin{aligned}
|\hat{X}_{t_{n+1}}|^2 &= |\hat{X}_{t_n}|^2 + 2h_n(\langle \hat{X}_{t_n}, f(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})\rangle + \frac{1}{2}h_n|f(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})|^2) \\
&+ 2\langle \hat{X}_{t_n} + f(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})h_n, g(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})\Delta W_n\rangle + |g(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})\Delta W_n|^2 \\
&\leq |\hat{X}_{t_n}|^2 + 2h_n(\langle \hat{X}_{t_n}, f(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})\rangle + \frac{1}{2}h_n|f(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})|^2 + \frac{d}{2}|g(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})|^2) \\
&+ 2\langle \hat{X}_{t_n} + f(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})h_n, g(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})\Delta W_n\rangle + |g(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})|^2(|\Delta W_n|^2 - h_n d) \\
&\leq |\hat{X}_{t_n}|^2 - 2\alpha_1 h_n|\hat{X}_{t_n}|^2 + 2\alpha_2 h^\delta(\overline{X}_{t_n-\tau})|\overline{X}_{t_n-\tau}|^2 \\
&+ 2\langle \hat{X}_{t_n} + f(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})h_n, g(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})\Delta W_n\rangle + |g(\hat{X}_{t_n}, \overline{X}_{t_n-\tau})|^2(|\Delta W_n|^2 - h_n d).
\end{aligned}$$

Multiplying by $e^{\alpha_1 t_{n+1}}$ and using the fact that $1 + x \leq e^x$ with $x = -h_n\alpha_1$, yields

$$
\begin{aligned}
e^{\alpha_1 t_{n+1}}|\hat{X}_{t_{n+1}}|^2 \le\ & e^{\alpha_1 t_n}|\hat{X}_{t_n}|^2 + 2\alpha_2 h^\delta(\overline{X}_{t_n-\tau})e^{\alpha_1 t_{n+1}}|\overline{X}_{t_n-\tau}|^2 \\
& + e^{\alpha_1 t_{n+1}}|g(\hat{X}_{t_n},\overline{X}_{t_n-\tau})|^2(|\Delta W_n|^2 - h_n d) \\
& + 2e^{\alpha_1 t_{n+1}}\langle \hat{X}_{t_n} + f(\hat{X}_{t_n},\overline{X}_{t_n-\tau})h_n, g(\hat{X}_{t_n},\overline{X}_{t_n-\tau})\Delta W_n\rangle.
\end{aligned}
$$

Solving the recurrence and using the bound $h_{\max}$, one can see that

$$
\begin{aligned}
e^{\alpha_1 t_n}|\hat{X}_{t_n}|^2 \le |X_0|^2 + e^{\alpha_1 h_{\max}}\Bigg\{ & \sum_{k=0}^{n-1} e^{\alpha_1 t_k}|g(\hat{X}_{t_k},\overline{X}_{t_k-\tau})|^2(|\Delta W_k|^2 - h_k d) \\
+ 2\alpha_2 \sum_{k=0}^{n-1} e^{\alpha_1 t_k}|\overline{X}_{t_k-\tau}|^2 h^\delta(\overline{X}_{t_k-\tau}) & + 2\sum_{k=0}^{n-1} e^{\alpha_1 t_k}\langle \hat{X}_{t_k} + f(\hat{X}_{t_k},\overline{X}_{t_k-\tau})h_k, g(\hat{X}_{t_k},\overline{X}_{t_k-\tau})\Delta W_k\rangle \Bigg\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|\hat{X}_{t_n}|^2 \le e^{-\alpha_1 t_n}|X_0|^2 + e^{\alpha_1 h_{\max}}\Bigg\{ & e^{-\alpha_1 t_n}\sum_{k=0}^{n-1} e^{\alpha_1 t_k}|g(\hat{X}_{t_k},\overline{X}_{t_k-\tau})|^2(|\Delta W_k|^2 - h_k d) \\
& + 2\alpha_2 e^{-\alpha_1 t_n}\sum_{k=0}^{n-1} e^{\alpha_1 t_k}|\overline{X}_{t_k-\tau}|^2 h^\delta(\overline{X}_{t_k-\tau}) \\
& + 2e^{-\alpha_1 t_n}\sum_{k=0}^{n-1} e^{\alpha_1 t_k}\langle \hat{X}_{t_k} + f(\hat{X}_{t_k},\overline{X}_{t_k-\tau})h_k, g(\hat{X}_{t_k},\overline{X}_{t_k-\tau})\Delta W_k\rangle \Bigg\}.
\end{aligned}
$$

So, for any $\lambda \in (0,\alpha_1)$ we have

$$
\begin{aligned}
\sum_{n=1}^{l} e^{\lambda t_n}|\hat{X}_{t_n}|^2 h_n \le e^{-(\alpha_1-\lambda)t_n}|X_0|^2 h_n + e^{\alpha_1 h_{\max}}\Bigg\{ & \sum_{n=0}^{l} e^{-(\alpha_1-\lambda)t_n} h_n \sum_{k=0}^{n-1} e^{\alpha_1 t_k}|g(\hat{X}_{t_k},\overline{X}_{t_k-\tau})|^2 \\
& (|\Delta W_k|^2 - h_k d) + 2\alpha_2 \sum_{n=0}^{l} e^{-(\alpha_1-\lambda)t_n} h_n \sum_{k=0}^{n-1} e^{\alpha_1 t_k}|\overline{X}_{t_k-\tau}|^2 h^\delta(\overline{X}_{t_k-\tau}) \\
& + 2\sum_{n=0}^{l} e^{-(\alpha_1-\lambda)t_n} h_n \sum_{k=0}^{n-1} e^{\alpha_1 t_k}\langle \hat{X}_{t_k} + f(\hat{X}_{t_k},\overline{X}_{t_k-\tau})h_k, g(\hat{X}_{t_k},\overline{X}_{t_k-\tau})\Delta W_k\rangle \Bigg\}. \quad (6.11)
\end{aligned}
$$

Moreover, we can see that

$$
\begin{aligned}
& 2\alpha_2 e^{\alpha_1 h_{\max}} \sum_{n=1}^{l} e^{-(\alpha_1-\lambda)t_n} h_n \sum_{k=0}^{n-1} e^{\alpha_1 t_k}|\overline{X}_{t_k-\tau}|^2 h^\delta(\overline{X}_{t_k-\tau}) \\
& \qquad = 2\alpha_2 e^{\alpha_1 h_{\max}} \sum_{n=1}^{l} e^{\alpha_1 t_n}|\overline{X}_{t_n-\tau}|^2 h^\delta(\overline{X}_{t_n-\tau}) \sum_{k=n}^{l} e^{-(\alpha_1-\lambda)t_k} h_k.
\end{aligned}
$$

Now since the function $e^{-(\alpha_1-\lambda)s}$ is decreasing on s, we see that

$$
\sum_{k=n}^{l} e^{-(\alpha_1-\lambda)t_k} h_k = \sum_{k=n}^{l} e^{(\alpha_1-\lambda)h_k} e^{-(\alpha_1-\lambda)t_{k+1}} h_k \le e^{(\alpha_1-\lambda)h_{\max}} \int_{t_n}^{t_l} e^{-(\alpha_1-\lambda)s}\,ds \le \frac{e^{\alpha_1 h_{\max}}}{\alpha_1-\lambda} e^{-(\alpha_1-\lambda)t_n}.
$$

Thus

$$
\begin{aligned}
& 2\alpha_2 e^{\alpha_1 h_{\max}} \sum_{n=1}^{l} e^{-(\alpha_1-\lambda)t_n} h_n \sum_{k=0}^{n-1} e^{\alpha_1 t_k}|\overline{X}_{t_k-\tau}|^2 h^\delta(\overline{X}_{t_k-\tau}) \\
& \qquad \le \frac{2\alpha_2 e^{2\alpha_1 h_{max}}}{\alpha_1-\lambda}\left(\sum_{n=1}^{l} e^{\lambda t_n}|\overline{X}_{t_n-\tau}|^2 h^\delta(\overline{X}_{t_n-\tau})\right). \quad (6.12)
\end{aligned}
$$

Let $M = M(\omega)$ be such that $t_M \le \tau < t_{M+1}$. Then we can write

$$
\begin{aligned}
\sum_{n=1}^{l} e^{\lambda t_n}|\overline{X}_{t_n-\tau}|^2 h^\delta(\overline{X}_{t_n-\tau}) &= \sum_{n=1}^{M} e^{\lambda t_n}|\overline{X}_{t_n-\tau}|^2 h^\delta(\overline{X}_{t_n-\tau}) + \sum_{n=M+1}^{l} e^{\lambda t_n}|\overline{X}_{t_n-\tau}|^2 h^\delta(\overline{X}_{t_n-\tau}) \\
&\le C + e^{\lambda h_{\max} M} \sum_{n=1}^{l} e^{\lambda t_n}|\hat{X}_{t_n}|^2 h_n. \quad (6.13)
\end{aligned}
$$

Substituting Equation (6.13) into (6.12), we obtain

$$2\alpha_2 e^{\alpha_1 h_{\max}} \sum_{n=1}^{l} e^{-(\alpha_1-\lambda)t_n} h_n \sum_{k=0}^{n-1} e^{\alpha_1 t_k} |\overline{X}_{t_k-\tau}|^2 h_k \\ \leq C + \frac{2\alpha_2 e^{2\alpha_1 h_{max}} e^{\lambda h_{max} M}}{\alpha_1-\lambda} \sum_{n=1}^{l} e^{\lambda t_n} |\widehat{X}_{t_n}|^2 h_n. \tag{6.14}$$

Similarly we obtain

$$e^{\alpha_1 h_{\max}} \sum_{n=0}^{l} e^{-(\alpha_1-\lambda)t_n} h_n \sum_{k=0}^{n-1} e^{\alpha_1 t_k} |g(\widehat{X}_{t_k}, \overline{X}_{t_k-\tau})|^2 (|\Delta W_k|^2 - h_k d) \\ \leq \frac{2e^{\alpha_1 h_{\max}}}{\alpha_1-\lambda} \sum_{n=1}^{l} e^{\lambda t_n} |g(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau})|^2 (|\Delta W_n|^2 - h_n d), \tag{6.15}$$

and

$$e^{\alpha_1 h_{\max}} 2 \sum_{n=0}^{l} e^{-(\alpha_1-\lambda)t_n} h_n \sum_{k=0}^{n-1} e^{\alpha_1 t_k} \langle \widehat{X}_{t_k} + f(\widehat{X}_{t_k}, \overline{X}_{t_k-\tau}) h_n, g(\widehat{X}_{t_k}, \overline{X}_{t_k-\tau}) \Delta W_k \rangle \\ \leq \frac{2e^{2\alpha_1 h_{\max}}}{\alpha_1-\lambda} \sum_{n=1}^{l} e^{\lambda t_n} \langle \widehat{X}_{t_n} + f(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau}) h_n, g(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau}) \Delta W_n \rangle. \tag{6.16}$$

We observe that by condition (6.8), h_{max} is such that $0 < 2\alpha_2 e^{2\alpha_1 h_{max}} < \alpha_1$. Then by choosing λ small enough so $0 < \frac{2\alpha_2 e^{2\alpha_1 h_{max}} e^{\lambda h_{max} M}}{\alpha_1-\lambda} < 1$ and by substituting Equations (6.14), (6.15) and (6.16) into (6.11), we obtain the final result. □

We are now in the position to give

Proof of Theorem 6.4. From (3.1) and (6.7), we have

$$\begin{aligned} |\widehat{X}_{t_{n+1}}|^2 &= |\widehat{X}_{t_n}|^2 + 2h_n(\langle \widehat{X}_{t_n}, f(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau})\rangle + \frac{1}{2} h_n |f(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau})|^2) \\ &\quad + 2\langle \widehat{X}_{t_n} + f(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau}) h_n, g(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau}) \Delta W_n \rangle + |g(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau}) \Delta W_n|^2 \\ &\leq |\widehat{X}_{t_n}|^2 + 2h_n(\langle \widehat{X}_{t_n}, f(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau})\rangle + \frac{1}{2} h_n |f(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau})|^2 + \frac{d}{2} |g(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau})|^2) \\ &\quad + 2\langle \widehat{X}_{t_n} + f(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau}) h_n, g(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau}) \Delta W_n \rangle + |g(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau})|^2 (|\Delta W_n|^2 - h_n d) \\ &\leq |\widehat{X}_{t_n}|^2 - 2\alpha_1 h_n |\widehat{X}_{t_n}|^2 + 2\alpha_2 h^\delta(\overline{X}_{t_n-\tau}) |\overline{X}_{t_n-\tau}|^2 \\ &\quad + 2\langle \widehat{X}_{t_n} + f(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau}) h_n, g(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau}) \Delta W_n \rangle + |g(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau})|^2 (|\Delta W_n|^2 - h_n d). \end{aligned}$$

Now we multiply by $e^{\lambda t_{n+1}}$, where $\lambda \in (0, \alpha_1)$ is the one from Lemma 6.5, which makes equation (6.10) to hold true. Then using the fact that $1 + x \leq e^x$ with $x = -2h_n\alpha_1$, yields

$$\begin{aligned} e^{\lambda t_{n+1}} |\widehat{X}_{t_{n+1}}|^2 &\leq e^{\lambda t_n} |\widehat{X}_{t_n}|^2 + 2\alpha_2 e^{\lambda t_{n+1}} |\overline{X}_{t_n-\tau}|^2 h_n + e^{\lambda t_{n+1}} |g(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau})|^2 (|\Delta W_n|^2 - h_n d) \\ &\quad + 2e^{\lambda t_{n+1}} \langle \widehat{X}_{t_n} + f(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau}) h_n, g(\widehat{X}_{t_n}, \overline{X}_{t_n-\tau}) \Delta W_n \rangle. \end{aligned}$$

Note that in the equation above we have used the fact that $e^{-h_n\alpha_1} \leq e^{-h_n\lambda}$. Solving the recurrence and using the bound $h_{\max}$ we have

$$e^{\lambda t_n} |\widehat{X}_{t_n}|^2 \leq |X_0|^2 + e^{\lambda h_{\max}} \Bigg\{ \sum_{k=0}^{n-1} e^{\lambda t_k} |g(\widehat{X}_{t_k}, \overline{X}_{t_k-\tau})|^2 (|\Delta W_k|^2 - h_k d) \\ + 2\alpha_2 \sum_{k=0}^{n-1} e^{\lambda t_k} |\overline{X}_{t_k-\tau}|^2 h^\delta(\overline{X}_{t_k-\tau}) + 2 \sum_{k=0}^{n-1} e^{\lambda t_k} \langle \widehat{X}_{t_k} + f(\widehat{X}_{t_k}, \overline{X}_{t_k-\tau}) h_k, g(\widehat{X}_{t_k}, \overline{X}_{t_k-\tau}) \Delta W_k \rangle \Bigg\}.$$

Using (6.13), we obtain

$$e^{\lambda t_n} |\widehat{X}_{t_n}|^2 \leq |X_0|^2 + e^{\lambda h_{\max}} \Bigg\{ \sum_{k=0}^{n-1} e^{\lambda t_k} |g(\widehat{X}_{t_k}, \overline{X}_{t_k-\tau})|^2 (|\Delta W_k|^2 - h_k d) + C \\ + e^{\lambda h_{max} M} \sum_{k=1}^{n-1} e^{\lambda t_k} |\widehat{X}_{t_k}|^2 h_k + 2 \sum_{k=0}^{n-1} e^{\lambda t_k} \langle \widehat{X}_{t_k} + f(\widehat{X}_{t_k}, \overline{X}_{t_k-\tau}) h_k, g(\widehat{X}_{t_k}, \overline{X}_{t_k-\tau}) \Delta W_k \rangle \Bigg\}. \tag{6.17}$$

Substituting Equation (6.10) (from Lemma 6.5) into (6.17) yields

$$\begin{aligned} e^{\lambda t_n}|\widehat{X}_{t_n}|^2 &\le |X_0|^2 + C + C\sum_{k=0}^{n-1} e^{\lambda t_k}|g(\widehat{X}_{t_k},\overline{X}_{t_k-\tau})|^2(|\Delta W_k|^2 - h_k d) \\ &\quad + C\sum_{k=0}^{n-1} e^{\lambda t_k}\langle \widehat{X}_{t_k} + f(\widehat{X}_{t_k},\overline{X}_{t_k-\tau})h_k, g(\widehat{X}_{t_k},\overline{X}_{t_k-\tau})\Delta W_k\rangle \Big\} \\ &\le C + C\{M_n + N_n\}, \end{aligned}$$

where:

- $M_n := \sum_{k=0}^{n-1} e^{\lambda t_k}|g(\widehat{X}_{t_k},\overline{X}_{t_k-\tau})|^2(|\Delta W_k|^2 - h_k d)$;
- $N_n := \sum_{k=0}^{n-1} e^{\lambda t_k}\langle \widehat{X}_{t_k} + f(\widehat{X}_{t_k},\overline{X}_{t_k-\tau})h_k, g(\widehat{X}_{t_k},\overline{X}_{t_k-\tau})\Delta W_k\rangle$;
- C is a positive constant (that changed from the second to the last line) dependent on $\omega \in \Omega$ and on the constants $\alpha_1, \alpha_2, h_{max}$ and λ, but not on t_n.

Taking logarithms and dividing by t_n, it follows that

$$\frac{1}{t_n}\log(e^{\lambda t_n}|X_{t_n}|^2) \le \frac{1}{t_n}\log\left(C + C\{M_n + N_n\}\right).$$

We observe that

$$\begin{aligned} \mathbb{E}[M_{n+1}|\mathcal{F}_{t_n}] &= \mathbb{E}[e^{\lambda t_n}|g(\widehat{X}_{t_n},\overline{X}_{t_n-\tau})|^2(|\Delta W_n|^2 - h_n d) + M_n|\mathcal{F}_{t_n}] \\ &= e^{\lambda t_n}|g(\widehat{X}_{t_n},\overline{X}_{t_n-\tau})|^2(\mathbb{E}[|\Delta W_n|^2] - h_n d) + M_n = M_n \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[N_{n+1}|\mathcal{F}_{t_n}] &= \mathbb{E}[2e^{\lambda t_n}\langle \widehat{X}_{t_n} + f(\widehat{X}_{t_n},\overline{X}_{t_n-\tau})h_n, g(\widehat{X}_{t_n},\overline{X}_{t_n-\tau})\Delta W_n\rangle + N_n|\mathcal{F}_{t_n}] \\ &= 2e^{\lambda t_n}\langle \widehat{X}_{t_n} + f(\widehat{X}_{t_n},\overline{X}_{t_n-\tau})h_n, g(\widehat{X}_{t_n},\overline{X}_{t_n-\tau})\mathbb{E}[\Delta W_n]\rangle + N_n = N_n. \end{aligned}$$

Hence $M + N$ is a local martingale with respect to $\{\mathcal{F}_{t_n}\}$. Thus by the discrete semimartingale convergence theorem (see lemma 2 in [16]), one can see that

$$\lim_{n\to\infty}(M_n + N_n) < \infty \quad \text{a.s.}$$

Therefore,

$$\limsup_{n\to\infty}\frac{1}{t_n}\log(e^{\lambda t_n}|\widehat{X}_{t_n}|^2) \le 0 \text{ a.s.}$$

This is

$$\limsup_{n\to\infty}\frac{\log|\widehat{X}_{t_n}|}{t_n} \le -\frac{\lambda}{2} \text{ a.s.}$$

The proof is therefore complete. □

Remark 6.1. In the Wei and Giles [2], the almost sure exponential stability of the approximate adaptive EM solution has not been investigated. Here we would like to point out that the adaptive EM solutions of SDEs also reproduce the almost sure exponential stability as SDDEs. A similar result is achieved in [6] by using the more computationally expensive BEM method. Let $\{W_t\}_{t\ge 0}$ be a d-dimensional Brownian motion. Consider the m-dimensional SDE

$$d\widetilde{Y}_t = f(\widetilde{Y}_t)dt + g(\widetilde{Y}_t)dW_t \tag{6.18}$$

for $t \ge 0$ where $f : \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^{m\times d}$ are Borel-measurable functions, and initial data $\widetilde{Y}_0 = \xi \in L^2_{\mathcal{F}_0}(\Omega;\mathbb{R}^m)$, i.e. ξ is a $\mathcal{F}_0$-measurable $\mathbb{R}^m$-valued random variable with $E|\xi|^2 < \infty$. In this case Assumption 6.1 can be written as

Assumption 6.6. The functions f and g satisfy the local Lipschitz condition: for every $R > 0$ there exists a positive constant C_R such that

$$|f(x) - f(y)| + ||g(x) - g(y)|| \le C_R(|x - y| \tag{6.19}$$

for all $x, y \in \mathbb{R}^m$ with $|x|, |y| \le R$. Furthermore, there exists a constant $\alpha \ge 0$ such that for all $x \in \mathbb{R}^m$, f and g satisfy

Table 1
Six simulations of the EM solution for $\Delta = 2e{-}3$.

Time	0	2e–3	4e–3	6e–3	8e–3	10e–3	12e–3	14e–3	16e–3	18e–3	20e–3
Sim 1	100	101.1	107.4	-141.1	418.1	-1.4e4	5.7e8	-3.7e22	1.1e64	-2.3e188	Inf
Sim 2	100	-98	88.97	-50.99	-24.51	-21.33	-19.37	-17.29	-16.15	-15.13	-14.87
Sim 3	100	-101.3	109.6	-150.1	525.68	-2.8e4	4.6e9	-2e25	1.6e72	-8.3e212	Inf
Sim 4	100	-101.9	108.5	-143.9	452.6	-1.8e4	1.2e9	-3.3e23	7.3e66	-7.9e196	Inf
Sim 5	100	-101.9	108.5	-143.9	452.6	-1.8e4	1.2e9	-3.3e23	7.3e66	-7.9e196	Inf
Sim 6	100	-99	91.8	-63.44	-11.65	-11.03	-10.87	-10.27	-10.17	-9.91	-10

$$\langle x, f(x)\rangle + \frac{1}{2}|g(x)|^2 \leq -\alpha|x|^2, \quad \alpha > 0. \tag{6.20}$$

Under the conditions (6.19) and (6.20), the SDE (6.18) has a unique solution (Theorem 2.3.6 in [10]).

In contrast to the EM solution, now we will see that the adaptive approximate solution of the SDE preserves the stability of the exact solution. We define the discrete-time adaptive approximate solution to the SDE (6.18) as

$$\widetilde{X}_0 := \widetilde{Y}_0, \quad h_n^\delta := h^\delta(\widetilde{X}_{t_n}), \quad t_{n+1} := t_n + h_n, \tag{6.21}$$

and

$$\widetilde{X}_{t_{n+1}} := \widetilde{X}_{t_n} + f(\widetilde{X}_{t_n})h_n^\delta + g(\widetilde{X}_{t_n})\Delta W_n, \tag{6.22}$$

where $\Delta W_n := W_{t_{n+1}} - W_{t_n}$. Now, Assumption 6.3 takes the form

Assumption 6.7. The time-step function h^δ satisfies

$$\langle x, f(x)\rangle + \frac{d}{2}|g(x)|^2 + \frac{1}{2}h^\delta(x)|f(x)|^2 \leq -\alpha|x|^2, \quad \alpha > 0 \tag{6.23}$$

for all $x \in \mathbb{R}$. Furthermore, h^δ is uniformly bounded by the real number $h^\delta_{\max} \in (0, \infty)$.

Theorem 6.8. *Consider the SDE* (6.18). *If f and g satisfy Assumption 6.6 and h^δ satisfies Assumption 6.7, then the adaptive approximate solution* (6.22) *is almost sure exponentially stable, i.e. there exists $\lambda > 0$ such that*

$$\limsup_{n\to\infty} \frac{\log|\widetilde{X}_{t_n}|}{t_n} \leq -\lambda \text{ a.s.}$$

7. Simulations

In this section we present simulations which illustrate the results discussed in Section 6. Consider the SDDE (1.1) with $\tau = 1$ and initial condition $Y(t) = 100, -1 \leq t \leq 0$. We simulated in Matlab paths of the EM solution of the SDDE (1.1) using different step sizes, Δ. As we have seen in section 6 there is a positive probability that the EM solution explodes. In Table 1 we present six different simulations of the EM solution for $\Delta = 2e - 3$. We observe in simulations 1,3,4 and 5 the EM solution explodes.

In Fig. 1, we graphed the logarithm of EM solution presented in Table 1.

Note: From Lemma 6.2 we know that as Δ decreases, the probability of explosion decreases. Thus, for "very small" Δ (say less than 10^{-4}) we couldn't find one explosion in 100,000 simulations.

In addition, we simulated the adaptive-EM solution of the SDDE (1.1) using the function h^δ defined in (6.9). As we proved in Section 6, the solution is a.s. exponentially stable. Fig. 2 shows 10,000 paths of the adaptive-EM solution.

Fig. 3 shows the first 10 values of $h^\delta(\widehat{X}_{t_n})$ for two different simulations. At the start, $\widehat{X}_0 = 100$, so the term $-\widehat{X}^3_{t_n}$ dominates the equation, making the diffusion term very "big" (in absolute value) in comparison with $\widehat{X}_{t_n}$. Therefore, the adaptive step is very "small" at the beginning and increases progressively as the ratio $f(\widehat{X}_{t_n}, \widehat{X}_{\hat{t}_n})/\widehat{X}_{t_n}$ decreases. This ensures all the simulated paths to decay exponentially in a "small" number of steps.

Data availability

Data will be made available on request.

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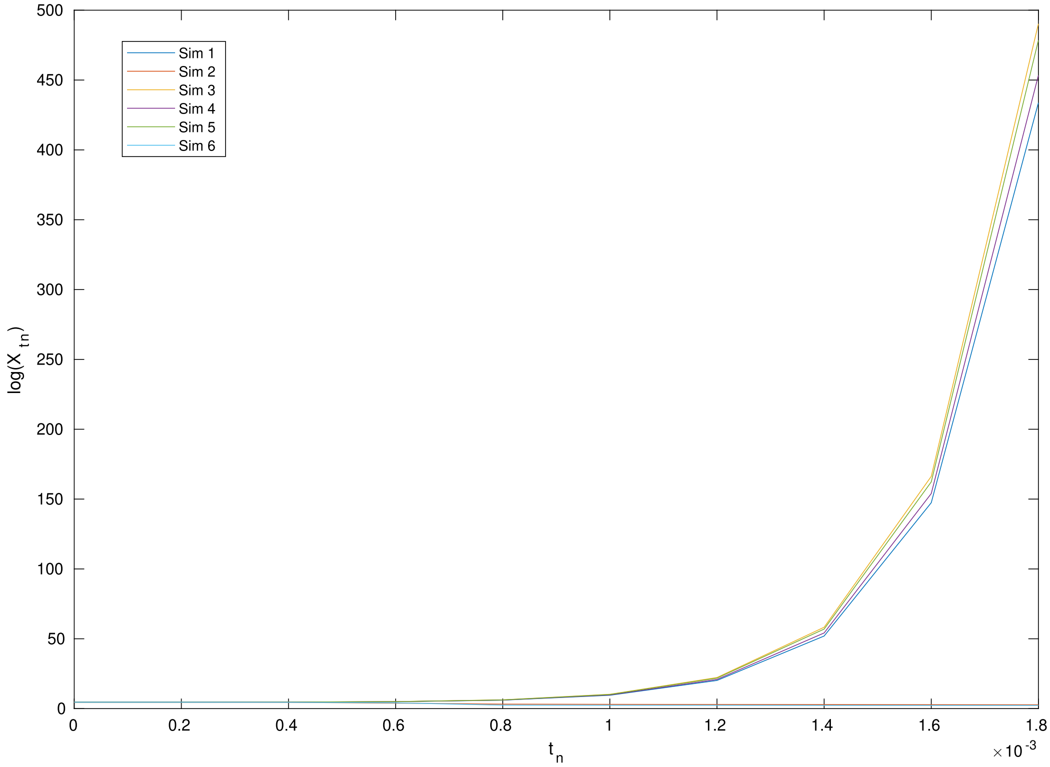

Fig. 1. Simulations of the logarithm of the EM solution for $\Delta = 2e{-}3$.

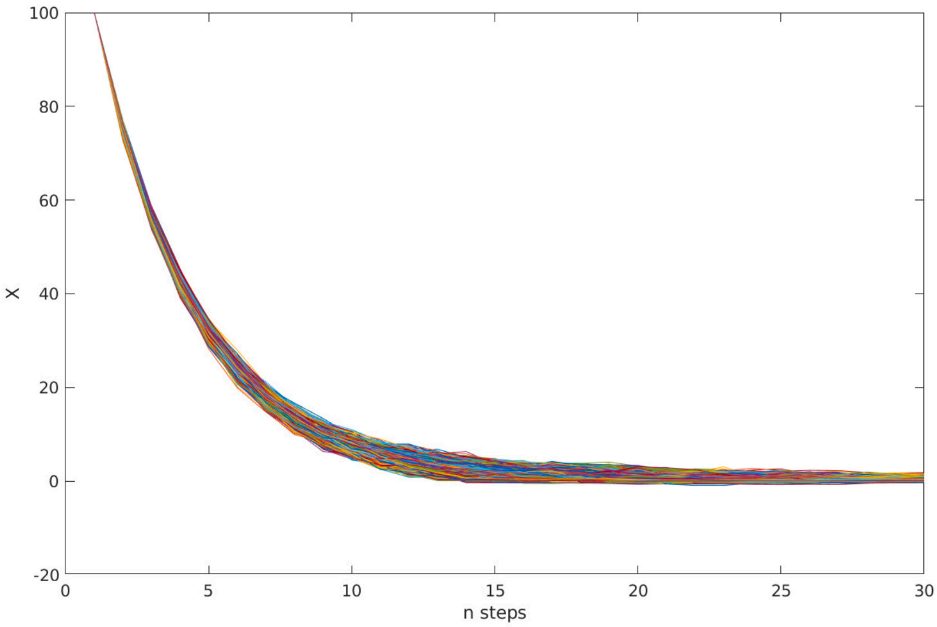

Fig. 2. Simulations of adaptive-EM solution.

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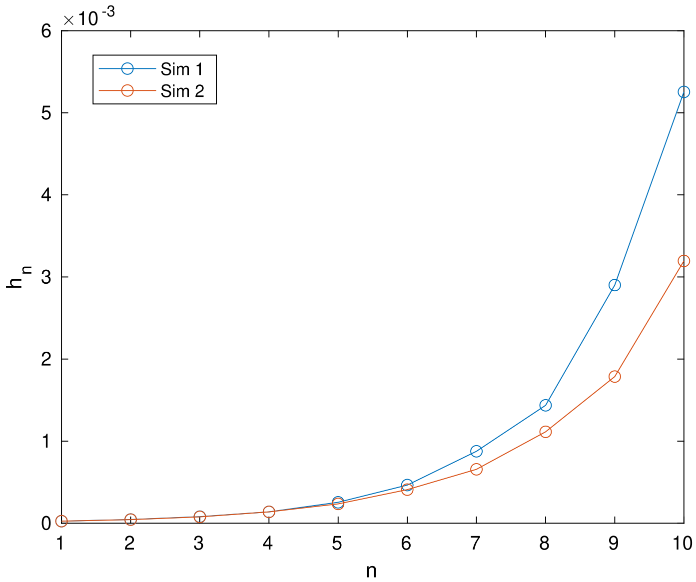

Fig. 3. The first ten adaptive steps for two different simulations.

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