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Existence et Optimisation de la Vitesse Critique pour des Ondes Progressives avec un Terme de Convection dans le Cylindre Non-Borné

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Vianney Domenech

Existence and Optimization of the Critical Speed for Travelling Front Solutions with Convection in Unbounded Cylinders

Résumé

Pour $n > 1$, on considère une équation de réaction-diffusion

$$u_t = \Delta u + \alpha(y)\nabla \cdot G(u) + f(u), \tag{0.1}$$

dans un cylindre non borné $\Omega := \mathbb{R} \times D$, où $D \subset \mathbb{R}^{n-1}$ est un domaine borné lisse, avec la présence d'un terme de convection, sous les conditions au bord de Neumann et de Dirichlet sur $\partial\Omega$. Pour les deux types de conditions au bord, on considère deux différentes formes de terme de convection : $\alpha(y)\nabla \cdot G(u)$ et $\nabla \cdot (\alpha(y)G(u))$. Le terme de réaction f est "monostable". Dans les deux cas Neumann et Dirichlet, on prouve qu'il existe une vitesse critique $c^\star \in \mathbb{R}$ telle qu'il existe une onde progressive solution de la forme $u(x,t) = w(x_1 - ct, y)$ avec une vitesse c si et seulement si $c \geq c^\star$, où x_1 désigne la coordonnée de l'axe du cylindre. La vitesse critique $c^\star$ joue souvent un rôle important pour les problèmes monostables en caractérisant le comportement asymptotique du problème de valeur initiale. L'existence d'ondes progressives pour tout $c \geq c^\star$ est typique des problèmes monostables comme par exemple l'équation bien connue de Fisher-KPP.

On donne une formule min-max pour la vitesse critique $c^\star$. Dans les deux cas de conditions au bord, on prouve que $c^\star$ est minorée par une quantité c' qui est liée à un problème de valeurs propres associé au problème linéarisé autour de 0. Remarquons que sous les conditions au bord de Dirichlet, une hypothèse supplémentaire est nécessaire afin d'assurer l'existence de c'. Plus précisément, $f'(0)$ doit être plus grand que la valeur propre principale de l'opérateur linéarisé.

On présente deux cas particuliers où l'on a l'égalité $c^\star = c'$. Sous les conditions au bord de Neumann et Dirichlet, le premier cas particulier est lorsque $G = (G_1, 0, \cdots, 0)$, en supposant la condition de KPP pour f et que $\alpha(y)G_1'(u) \geq \alpha(y)G_1'(0)$, pour tout $y \in D$ et $u \in]0,1[$. Le second cas particulier est traité sous les conditions au bord de Neumann : lorsque $G_1'(0) = 0$, en supposant la condition de KPP pour f et que $\alpha(y)G_1'(u) \geq 0$, pour tout $y \in D$ et $u \in]0,1[$. Dans ce cas particulier, on obtient une formule explicite : $c^\star = c' = 2\sqrt{f'(0)}$.

Sous les conditions au bord de Dirichlet, on met en avant l'influence du domaine D, du terme de réaction f et du terme de convection $\alpha(y)\nabla \cdot G(u)$ sur la vitesse critique $c^\star$. Dans le cas particulier où $G = (G_1, 0, \cdots, 0)$, en utilisant l'égalité $c^\star = c'$, on utilise le problème de valeurs propres lié à c' afin d'obtenir des résultats d'optimisation pour $c^\star$.

Abstract

For $n > 1$, we consider a reaction-diffusion equation

$$u_t = \Delta u + \alpha(y)\nabla \cdot G(u) + f(u), \tag{0.2}$$

in an unbounded cylinder $\Omega := \mathbb{R} \times D$, where $D \subset \mathbb{R}^{n-1}$ is a smooth bounded domain, with a presence of a convection term, under both Neumann and Dirichlet boundary conditions on $\partial\Omega$. For both types of boundary condition, we consider two different forms of convection term, namely : $\alpha(y)\nabla \cdot G(u)$ and $\nabla \cdot (\alpha(y)G(u))$. The reaction term f is "monostable". In both Neumann and Dirichlet cases, we prove that there exists a critical speed $c^\star \in \mathbb{R}$ such that there exists a travelling front solution of the form $u(x,t) = w(x_1 - ct, y)$ with speed c if and only if $c \geq c^\star$, where x_1 is the coordinate corresponding to the axis of the cylinder. The critical speed $c^\star$ often plays an important role for monostable problems by characterizing the long-time behaviour of the initial value problem. The existence of travelling waves for all $c \geq c^\star$ is typical of monostable problems such as the prototype Fisher-KPP equation.

We give a min-max formula for the speed $c^\star$. For both types of boundary conditions, we prove that $c^\star$ is bounded below by a quantity c' which is related to a certain eigenvalue problem, associated with the linearized problem around 0. Note that under Dirichlet boundary conditions, an extra assumption is needed to ensure that c' exists, namely, $f'(0)$ has to be greater than the principal eigenvalue of the linearized operator.

We discuss two special cases where the equality $c^\star = c'$ holds. Under both Neumann and Dirichlet boundary conditions, the first special case is when $G = (G_1, 0, \cdots, 0)$, assuming the so-called KPP condition for f and that $\alpha(y)G_1'(u) \geq \alpha(y)G_1'(0)$, for all $y \in D$ and all $u \in (0,1)$. The second case is treated only under Neumann boundary conditions : when $G_1'(0) = 0$, assuming the KPP condition for f, and that $\alpha(y)G_1'(u) \geq 0$, for all $y \in D$ and $u \in (0,1)$. Note that in that case, we give an explicit formula : $c^\star = c' = 2\sqrt{f'(0)}$.

Under Dirichlet boundary conditions, we highlight the influence of the domain D, the reaction term f and the convection term $\alpha(y)\nabla \cdot G(u)$ on the critical speed $c^\star$. In the special case where $G = (G_1, 0, \cdots, 0)$, using that $c^\star = c'$, we use the eigenvalue problem related to c' to establish some optimization results for $c^\star$.

Contents

1 Introduction

1.1 Introduction (French version)

Les ondes progressives sont des ondes qui se propagent sans changement de forme. Autrement dit, si $u(x,t)$ est une onde progressive au temps t et de coordonnée spatiale x, la forme de la solution sera la même pour tout temps t, et sa vitesse de propagation c est constante. Plus précisément, si la solution $u(x,t) = w(x - ct)$, alors w est une onde progressive se déplaçant à vitesse constante c et dans la direction positive x si $c > 0$ et dans la direction négative x si $c < 0$. En remplaçant $u(x - ct)$ par $u(x + ct)$ on obtient une onde progressive qui se propage dans la direction opposée. Les ondes progressives apparaissent naturellement en biologie, par exemple comme onde progressive d'une concentration d'une espèce chimique ou d'une densité de population, voir [MUR, Section 11.1]. Afin d'obtenir des résultats physiquement réalistes, u doit être bornée et positive. Par exemple, l'équation suivante en dimension 1 peut modéliser un changement biochimique causé par la cinétique de réaction et la diffusion :

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u), \tag{1.3}$$

où u est la concentration, $f(u)$ la cinétique et D le coefficient de diffusion. Selon le profil du terme de réaction f dans (1.3), différentes propriétés d'ondes progressives peuvent survenir. C'est précisément ce qui a poussé les mathématiciens à étudier la branche de la théorie de la diffusion par réaction. Dans [FIS], Fisher a proposé l'équation (1.3) avec $f(u) = ku(1-u)$ où k est positive, pour modéliser la propagation d'un gène privilégié dans une population. Ce cas particulier a été beaucoup étudié par la suite, par plusieurs mathématiciens tels que Kolmogoroff, Petrovsky et Piskounoff, voir [KPP]. De telles équations apparaissent également en modélisation dans le domaine de la physique et de la chimie. Avec ce type de terme non linéaire f, la fonction u peut aussi représenter le profil de température normalisé d'un mélange de deux gaz, dans un modèle de combustion, voir [HAM, Introduction] pour d'autres références.

Pour être plus réaliste physiquement, il est nécessaire d'étudier ce type d'équations dans des dimensions supérieures. En 1937, avec l'intention de modéliser le processus de diffusion spatiale lorsque des individus mutants avec une adaptibilité supérieure se manifestent dans une population, Fisher proposa l'équation de diffusion bidimensionnelle suivante :

$$\frac{\partial u}{\partial t} = D\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) + (\varepsilon - \mu u)u, \tag{1.4}$$

où $u(x, y, t)$ représente la densité de population au temps t, en la coordonnée spatiale (x, y). Le premier terme du membre de droite illustre un phénomène de diffusion, tandis que le second exprime une croissance démographique locale. Ces deux termes entraînent des changements dans l'évolution de la densité de population modélisée par le terme $\frac{\partial u}{\partial t}$. Le coefficient de diffusion D indique la vitesse à laquelle la densité de population peut varier, tandis que ε représente le taux intrinsèque d'augmentation et la constante $\mu \geq 0$ prend en compte le taux de reproduction de l'espèce étudiée. Cette équation a été considérée par Kolmogorov en 1937 et de nombreux autres scientifiques dans différents domaines afin d'étudier l'expansion d'une bactérie, voire même la propagation de cultures humaines, voir [SK1, Chapter 3]. Par exemple, en 1951, Skellam appliqua cette équation avec $\mu = 0$ pour étudier l'évolution des rats musqués. En particulier, il a été montré que la densité de population croît exponentiellement lorsque t devient suffisamment grand, ce qui signifie que les effets de diffusion et de croissance conduisent à une expansion de la population. De plus, il a été prouvé que ce front d'ondes se déplace avec une vitesse constante $c = 2\sqrt{\varepsilon D}$.

Cependant, lorsque l'espèce étudiée est transportée par le vent ou un courant d'eau, l'équation (1.4) doit être modifiée pour prendre en compte ce phénomène, en ajoutant un terme supplémentaire dans (1.4) :

$$\frac{\partial u}{\partial t} = D\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) - c\frac{\partial u}{\partial x} + \varepsilon u, \tag{1.5}$$

où l'axe des x est aligné selon la direction du vent. Notons que la présence d'un terme de diffusion non linéaire pourrait également être considérée comme un effet de convection non linéaire. Dans le cas unidimensionnel, l'équation suivante apparaît dans de nombreux domaines différents comme dans les colonnes d'échanges d'ions ou la chromatographie :

$$\frac{\partial u}{\partial t} + \frac{\partial h(u)}{\partial x} = \frac{\partial^2 u}{\partial x^2} + f(u), \tag{1.6}$$

où $h'(u)$ est appelée vitesse de convection, voir [MUR, Section 13.4]. La présence d'un terme de convection non linéaire peut avoir un impact sur les solutions. En effet, en considérant ce type de terme, un processus de transport majeur, qui dépend non linéairement de u, joue un rôle important dans ce nouveau modèle. Notons que la présence d'un terme non linéaire de diffusion peut aussi être vu comme un effet non linéaire de convection.
De nombreux travaux ont été faits sur les solutions de front d'ondes progressives dans des cylindres non bornés, voir par exemple [VE2], [BN2], [VO1] et [HAM]. Dans [BN2] et [HAM], les auteurs ont considéré un terme $\beta(y)\partial_1 u$ dans l'équation qui peut représenter un terme de transport ou d'écoulement le long de la direction du cylindre. Ce terme ne dépend pas de la coordonnée de l'axe du cylindre. Dans les deux cas, les conditions au bord de Neumann ont été imposées au bord du cylindre, contrairement à [VE2] qui a considéré les conditions au bord de Dirichlet. Dans [VO1], Volpert

a traité les conditions au bord de Dirichlet, expliquant que les conditions au bord de Dirichlet sont par exemple utilisées dans le modèle d'explosion de chaleur de Frank-Kamenetskii qui étudia notamment l'équation de diffusion de réaction dans un domaine borné d'une réaction exothermique monomoléculaire. D'un autre côté, les conditions au bord de Neumann illustrent une sorte de flux nul signifiant que l'espèce ne peut s'échapper du domaine.
Une notion importante est la stabilité des solutions de modèles biologiques qui a été prise en compte dans une série d'articles couvrant une variété de scénarios, voir par exemple [MUR, Section 11.3], [ROQ], [VO3] et [BN2, Section 1]. Une solution de front d'onde est localement stable si une petite perturbation de cette solution converge dans un certain sens vers ce front lorsque $t \to +\infty$. Notons que différents types de convergence vers une onde peuvent être définis, voir par exemple [Section 5 - [VO2]], où l'approche en forme et l'approche uniforme d'une onde ont été définies. La stabilité des solutions est un phénomène qui est lié aux propriétés du spectre du problème linearisé, voir [VO1, Introduction]. Cependant, nous n'allons pas étudier le concept de stabilité ici.

Dans cette thèse, on étudie l'existence d'ondes progressives solutions d'une équation de réaction-diffusion avec la présence d'un terme de convection. Plus précisément, pour $n > 1$, on considère le problème suivant :

$$u_t = \Delta u + \alpha(y)\nabla \cdot G(u) + f(u), \tag{1.7}$$

où $t \in \mathbb{R}$ et $u(x,t) \in \mathbb{R}$, dans le cylindre non borné $\Omega = \mathbb{R} \times D$, où $D \subset \mathbb{R}^{n-1}$ est un domaine borné lisse. On note $x = (x_1, y) \in \mathbb{R} \times D$, où $y = (x_2, \cdots, x_n)$.
De plus, on s'intéresse aux solutions u satisfaisant les conditions au bord de Neumann sur la frontière du cylindre :

$$\frac{\partial u}{\partial \nu}(x,t) = 0, \text{ pour tout } x = (x_1, y) \in \mathbb{R} \times \partial D, \text{ et pour tout } t \in \mathbb{R}, \tag{N}$$

où ν désigne la dérivée normale extérieure à ∂D.
On considèrera également les conditions au bord de Dirichlet :

$$u(x,t) = 0 \text{ pour tout } x = (x_1, y) \in \mathbb{R} \times \partial D, \text{ et pour tout } t \in \mathbb{R}. \tag{D}$$

On suppose que la fonction de réaction $f : \mathbb{R} \to \mathbb{R}$ est C^1 et "monostable", ce qui signifie que

$$f(0) = f(1) = 0, \ f'(0) > 0, \ f'(1) < 0, \ f(u) > 0 \text{ if } 0 < u < 1. \tag{1.8}$$

Le terme $\alpha(y)\nabla \cdot G(u)$ dans (1.7) est un terme non-linéaire de convection. Ce drift non linéaire dépend de la fonction u, ce qui signifie que le terme de convection peut dépendre de la densité de population de l'espèce étudiée. A notre connaissance, le terme non-linéaire de convection $\alpha(y)\nabla \cdot G(u)$ n'a jamais été considéré dans de précédents travaux. On considère dans une première partie cette forme mais on étudiera plus tard la forme suivante : $\nabla \cdot (\alpha(y)G(u))$.

Dans le cas des conditions au bord de Neumann, on s'intéresse aux solutions de la forme $u(x,t) = w(x_1 - ct, y)$, où $c \in \mathbb{R}$ est la vitesse de propagation de l'onde,

$$w(\xi, y) \to 1 \text{ lorsque } \xi \to -\infty \text{ et } w(\xi, y) \to 0 \text{ lorsque } \xi \to +\infty,$$

uniformément avec $y \in D$.
Sous les conditions au bord de Dirichlet, puisque la fonction constante 1 ne satisfait pas les conditions au bord de Dirichlet, une onde progressive solution doit alors converger vers un autre état stationnaire en $-\infty$. C'est pourquoi sous les conditions au bord de Dirichlet nous supposerons qu'il existe une fonction $w_- \in C^{2,\lambda}(\overline{D})$ telle qu'une onde progressive solution de vitesse c doit satisfaire

$$w(\xi, y) \to w_-(y) \text{ lorsque } \xi \to -\infty \text{ et } w(\xi, y) \to 0 \text{ lorsque } \xi \to +\infty,$$

uniformément avec $y \in D$.

Sans terme de convection, autrement dit lorsque $\alpha = 0$, Berestycki et Nirenberg ont prouvé, voir [BN2], qu'il existe une vitesse critique $c^\star \in \mathbb{R}$ telle que : une onde progressive solution de vitesse c de (1.7) existe si et seulement si $c \geq c^\star$. L'existence d'une telle vitesse critique $c^\star$ est typique des problèmes monostables et caractérise le comportement asymptotique du problème initial. En effet, de manière générale, pour une certaine classe de conditions initiales, la solution du problème initial va converger vers l'onde progressive ayant une vitesse critique $c^\star$ lorsque $t \to +\infty$. De plus, Berestycki et Nirenberg ont également prouvé que cette vitesse minimale est strictement positive, ce qui signifie que toutes les ondes progressives se propagent dans la direction positive de l'axe du cylindre.
On montrera que sous certaines hypothèses concernant les fonctions f et G, une vitesse critique $c^\star$ existe également avec la présence d'un terme de convection, mais dans notre cas, cette vitesse minimale peut être positive ou négative. En particulier, si $c^\star$ est strictement négative, certaines ondes progressives ayant pour vitesse $0 > c \geq c^\star$ vont se propager selon l'axe négatif du cylindre. Berestycki et Nirenberg ont obtenu une formule explicite pour $c^\star$ dans [BN2]. Plus précisément, sans terme de convection, et si la fonction f satisfait la condition de KPP

$$f(u) \leq f'(0)u, \quad \forall u \in]0,1[, \tag{1.9}$$

les auteurs ont montré, voir [BN2, Théorème 1.5, Section 10], que $c^\star = c'$, où c' est une quantité liée à un problème de valeurs propres associé au problème linéarisé autour de 0.

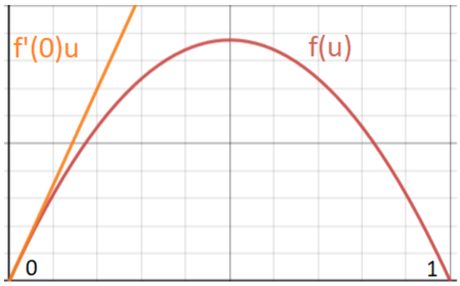

Figure 1: Illustration d'une fonction de réaction typique satisfaisant la condition KPP.

Comme nous l'avons mentionné précédemment, beaucoup de travaux ont été effectués concernant ce type d'équations dans des cylindres non bornés, sans terme de convection, voir [BN2], [BLL], [ROQ] et [VE1]. Certains articles considèrent un terme de convection en dimension 1, voir [CRO], [CRM] et [CRT]. Des résultats d'existence ont été prouvés dans ce cas, voir [CRM, Théorème 2.4], [CRO, Théorème 2.4] et [AKC, Théorème 3.6]. Dans cette thèse, nous étendons ces résultats. Dans le cas multidimensionnel, un grand drift à divergence nulle a été considéré dans [BHN], avec les conditions au bord de Dirichlet ou de Neumann, ou le cas périodique, où les auteurs ont étudié le comportement asymptotique de la valeur propre principale d'un opérateur elliptique.

Sous les conditions au bord de Neumann, et avec la première forme du terme de convection, $\alpha(y)\nabla \cdot G(u)$, si $u(x,t) = w(x_1 - ct, y)$, est une solution de (1.7), alors la fonction w vérifie :

$$\begin{cases} -c\partial_1 w = \Delta w + \alpha(y)\nabla \cdot G(w) + f(w) & \text{dans } \Omega, \\ w(-\infty, y) = 1, \quad w(+\infty, y) = 0 & \text{uniformément pour } y \in D, \\ w \geq 0, & \\ w_\nu = 0 & \text{sur } \mathbb{R} \times \partial D. \end{cases} \tag{1.10}$$

On prouve qu'il existe une vitesse critique $c^\star$ telle qu'une onde progressive solution de vitesse c de ce problème existe si et seulement si $c \geq c^\star$. Pour démontrer ce résultat, on suivra l'approche de [VO1, Chapitre 5, Section 4], tout en utilisant diverses idées de [BN2]. Plus précisément, on montre dans un premier temps qu'il existe une onde progressive solution de vitesse $c > c^\star$ sur le cylindre tronqué $\Omega_N :=] - N, N[\times D$, avec les conditions au bord de Dirichlet sur $\{\pm N\} \times D$ et les conditions au bord de Neumann sur $] - N, N[\times \partial D$, et ensuite on fait tendre N vers l'infini afin d'obtenir une onde progressive solution de vitesse $c > c^\star$ sur le cylindre infini. On étudiera ensuite un problème de valeurs propres lié au problème linéarisé autour de 0, en s'inspirant de [BN2]. On montrera qu'il existe une valeur critique $c' \in \mathbb{R}$ telle que ce problème de valeurs propres possède deux valeurs propres strictement positives si et seulement si $c > c'$. On compare par la suite ces deux valeurs critiques $c^\star$ et c', et en utilisant le [BN3, Théorème 2.1], on montre que $c^\star \geq c'$ sous certaines conditions sur f et G; voir Théorème 1.18. Soulignons que sans présence de convection, Berestycki et Nirenberg ont prouvé dans [BN2, Section 10], que $c^\star = c'$ si la fonction f satisfait la condition de KPP (1.9). Cependant, à cause du terme de convection et plus précisément des termes de dérivation $\partial_i u$ pour $2 \leq i \leq n$, la méthode de sous et sursolutions utilisée dans [BN2] pour démontrer que $c^\star \leq c'$ ne permet pas de conclure. En revanche, dans cette thèse on mettra en avant deux cas particuliers où cette égalité est satisfaite. Plus précisément, le premier cas particulier apparaît lorsque f satisfait la condition de KPP (1.9) et que

$$G = (G_1, 0, \cdots, 0), \quad \alpha(y)G_1'(u) \geq \alpha(y)G_1'(0) \text{ pour tout } y \in D \text{ et } u \in]0,1[. \tag{1.11}$$

Il n'y a alors pas de termes de dérivation $\partial_i u$ pour $2 \leq i \leq n$, et la méthode utilisée dans [BN2, Section 10] par Berestycki et Nirenberg donne l'égalité $c^\star = c'$. Le second cas particulier est lorsque

$$G_1'(0) = 0, \quad \alpha(y)G_1'(u) \geq 0 \text{ pour tout } y \in D \text{ et } u \in]0,1[, \tag{1.12}$$

et sous la condition de KPP (1.9) pour f. Dans ce cas précis, une formule explicite est obtenue pour la vitesse minimale $c^\star$, à savoir $c^\star = c' = 2\sqrt{f'(0)}$. Notons que cette formule a été obtenue pour $c^\star$ par Berestycki et Nirenberg dans [BN2], sans terme de convection, et lorsque f satisfait la condition de KPP (1.9).
Sous les conditions au bord de Neumann, et avec la première forme du terme de convection $\alpha(y)\nabla \cdot G(u)$, on suppose que la fonction f satisfait (1.8) et on note $L > 0$ sa constante de Lipschitz sur $]0,1[$. On considère $p > n$ et on suppose les conditions suivantes :

- **(GN1) :** La fonction $G : \mathbb{R} \to \mathbb{R}^n$ est C^2.
- **(GN2) :** La fonction G_1' est lipschitzienne sur $]0,1[$, de constante de Lipschitz $\tilde{L} > 0$.
- **(GN3) :** Pour tout $1 \leq i \leq n$, $G_i(0) = 0$.
- **(AlphaN1) :** La fonction $\alpha : D \to \mathbb{R}$ est dans $C^1(\overline{D})$.
- **(AlphaN2) :** La fonction α satisfait $\alpha \equiv 0$ sur ∂D.

  Comme dans [VO1, Section 4.1], une hypothèse d'unicité des solutions du problème sur la section transversale D est nécessaire :

- **(AN) :** Les seules solutions dans $W^{2,p}(D)$ du problème sur la section transversale sont 0 et 1. Plus précisément, pour $n < p < \infty$, si $z \in W^{2,p}(D) : D \to \mathbb{R}$ satisfait

$$\begin{cases} \Delta' z + \alpha(y) \sum_{i=2}^n G_i'(z) \frac{\partial z}{\partial x_i} + f(z) = 0 & \text{dans } D, \\ z_\nu = 0 & \text{sur } \partial D, \end{cases} \tag{1.13}$$

où

$$\Delta' = \sum_{i=2}^{n} \frac{\partial^2}{\partial x_i^2}, \text{ et } y = (x_2, x_3, \cdots, x_n).$$

alors

$$z \equiv 0 \text{ ou } z \equiv 1.$$

Remarque 1.1. Remarquons qu'avec la première forme du terme de convection $\alpha(y)\nabla \cdot G(u)$, on peut supposer l'hypothèse **(GN3)** sans aucune perte de généralité. En effet, si $G(0) = \beta \neq 0$, posons $\tilde{G}(u) = G(u) - \beta$. Puisque $\tilde{G}_i'(u) = G_i'(u)$ pour tout $1 \leq i \leq n$, il vient que

$$\nabla \cdot \tilde{G}(u) = \sum_{i=1}^{n} \tilde{G}_i'(u)\partial_i u = \sum_{i=1}^{n} G_i'(u)\partial_i u = \nabla \cdot G(u),$$

ce qui montre que le problème (1.32) est inchangé.

Remarque 1.2. Si $\alpha G'$ est un vecteur constant, alors l'hypothèse **(AN)** est satisfaite. En effet, en utilisant les [YIH, Théorèmes 4.9] et [YIH, Théorèmes 4.11] avec $\varepsilon \in]0, 1[$ en tant que sous-solution et 1 en tant que sursolution de (1.13), il existe une solution minimale $\underline{u}$ et une solution maximale $\overline{u}$ telles que $\varepsilon \leq \underline{u} \leq \overline{u} \leq 1$, dans le sens où si u est une solution de (1.13), alors u vérifie $\underline{u} \leq u \leq \overline{u}$. Notons que puisque 1 est une solution, il vient que $\overline{u} \equiv 1$, et alors, $\underline{u}$ vérifie $\varepsilon \leq \underline{u} \leq 1$. En multipliant l'équation (1.13) satisfaite par $\underline{u}$ par $e^{\alpha G' \cdot y}$ et en intégrant sur D, il vient :

$$\int_D e^{\alpha G' \cdot y}(\Delta' \underline{u} + \alpha G' \cdot \nabla' \underline{u}) + \int_D e^{\alpha G' \cdot y} f(\underline{u}) = 0. \tag{1.14}$$

Puisque

$$\nabla' \cdot \left(e^{\alpha G' \cdot y} \nabla' \underline{u} \right) = e^{\alpha G' \cdot y}(\Delta' \underline{u} + \alpha G' \cdot \nabla' \underline{u}),$$

et $\underline{u}$ satisfait les conditions au bord de Neumann sur ∂D, il vient que la première intégrale de (1.14) est nulle et $\int_D e^{\alpha G' \cdot y} f(\underline{u}) = 0$, ce qui donne $f(\underline{u}) \equiv 0$, par positivité de l'intégrande. Enfin, puisque $\underline{u} \geq \varepsilon > 0$, et que f ne s'annule qu'en 0 et 1, on obtient $\underline{u} \equiv 1$. On a donc prouvé que si z est une solution strictement positive de (1.13), en choisissant ε telle que $z > \varepsilon$ dans D, il vient que $z \equiv 1$.
Supposons maintenant qu'il existe une solution positive z de (1.13) telle que $z(y_0) = 0$ où $y_0 \in D$. Mais alors, puisque $p > n$, le Principe du Maximum Fort, assure que z est constante dans D, et par conséquent, $z \equiv 0$. Notons que le lemme de Hopf empêche la fonction z de s'annuler sur ∂D.

Avec la seconde forme du terme de convection, à savoir $\nabla \cdot (\alpha(y)G(u))$, et sous les conditions au bord de Neumann, on obtient les mêmes résultats sous des hypothèses légèrement différentes :

- **(GN1′) :** La fonction $G : \mathbb{R} \mapsto \mathbb{R}^n$ est C^2.
- **(GN2′) :** La fonction G_1' est lipschitzienne sur $[0,1]$, de constante de Lipschitz $\tilde{L} > 0$.
- **(GN3′) :** La fonction G_1 satisfait $G_1(0) = 0$.
- **(GN4′) :** Pour tout $2 \leq i \leq n$, la fonction G_i satisfait $G_i(0) = G_i(1) = 0$.
- **(AlphaN1′) :** La fonction $\alpha : D \to \mathbb{R}$ appartient à $C^1(\overline{D})$.
- **(AlphaN2′) :** La fonction α satisfait $\alpha \equiv 0$ sur ∂D.

  L'hypothèse d'unicité de solution sur la section transversale D devient :

- **(AN′) :** Les seules solutions dans $W^{2,p}(D)$ du problème sur la section transversale D sont 0 et 1. Plus précisement, pour $n < p < \infty$, si $z \in W^{2,p}(D) : D \to \mathbb{R}$ satisfait

$$\begin{cases} \Delta' z + \alpha(y) \sum_{i=2}^n G_i'(z) \frac{\partial z}{\partial x_i} + \sum_{i=2}^n G_i(z) \frac{\partial \alpha}{\partial x_i} + f(z) = 0 & \text{dans } D, \\ z_\nu = 0 & \text{sur } \partial D, \end{cases} \tag{1.15}$$

  alors

$$z \equiv 0 \text{ ou } z \equiv 1.$$

- **(EN′) :** La condition suivante est satisfaite :

$$f'(0) + \inf_{y \in D} \sum_{i=2}^n G_i'(0) \frac{\partial \alpha}{\partial x_i}(y) > 0. \tag{1.16}$$

  Cette condition est suffisante afin de garantir l'existence de la valeur critique c', mais aussi pour assurer l'existence de sous-solutions pour le problème défini sur le cylindre tronqué Ω_N, voir (4.122).

Remarque 1.3. L'hypothèse **(AN′)** implique en particulier que

$$\sum_{i=2}^n G_i(0) \frac{\partial \alpha}{\partial x_i} = \sum_{i=2}^n G_i(1) \frac{\partial \alpha}{\partial x_i} = 0,$$

ce qui est en particulier vérifié lorsque l'on suppose (GN4′).

Avec les conditions au bord de Dirichlet et avec la première forme de terme de convection $\alpha(y)\nabla\cdot G(u)$, si $u(x,t)=w(x_1-ct,y)$ est une solution de (1.7), la fonction w satisfait :

$$\begin{cases} -c\partial_1 w = \Delta w + \alpha(y)\nabla\cdot G(w) + f(w) & \text{dans } \Omega, \\ w(-\infty,y)=w_-(y), \quad w(+\infty,y)=0 & \text{uniformément pour } y\in D, \\ w\geq 0, & \\ w=0 & \text{sur } \mathbb{R}\times\partial D. \end{cases} \tag{1.17}$$

Avec les conditions au bord de Dirichlet, une différence importante est qu'une constante positive $\varepsilon\in]0,1[$ n'est plus une sous-solution du problème sur le cylindre tronqué Ω_N, nous empêchant donc d'utiliser la méthode de sous et sursolutions. C'est pourquoi, sous les conditions au bord de Dirichlet, il est nécessaire de faire une hypothèse supplémentaire. On suppose que f satisfait (1.8) et les conditions suivantes :

- **(GD) :** La fonction G est C^2 et satisfait $G_i(0)=0$ pour tout $2\leq i\leq n$.
- **(AlphaD) :** La fonction $\alpha: D\to\mathbb{R}$ est dans $C^1(\overline{D})$.
- **(AD) :** Soit $\lambda\in]0,1[$. Il existe une fonction positive $w_-\in C^{2,\lambda}(\overline{D})$ telle que les seules solutions dans $C^{2,\lambda}(\overline{D})$ du problème sur la section transversale D sont w_- et 0. Plus précisément, si $z\in C^{2,\lambda}(\overline{D}):\overline{D}\to\mathbb{R}$ satisfait

  $$\begin{cases} \Delta' z + \alpha(y)\sum_{i=2}^n G_i'(z)\frac{\partial z}{\partial x_i} + f(z) = 0 & \text{dans } D, \\ z=0 & \text{sur } \partial D, \end{cases} \tag{1.18}$$

  alors $z\equiv 0$ ou $z\equiv w_-$.
  Comme dans [VO1], afin d'assurer l'existence de sous-solutions du problème sur le cylindre tronqué Ω_N, voir (5.145), nous supposerons :

- **(BD) :** Il existe une suite de fonctions $\{v_k(y)\}_{k\in\mathbb{N}}$ uniformément bornée dans $C^{2,\lambda}(\overline{D})$, qui tend uniformément vers 0 lorsque k tend vers $+\infty$ et qui satisfait pour tout $k\in\mathbb{N}$

  $$0 < v_{k+1}(y) < v_k(y) < w_-(y), \text{ pour } y\in D,$$

  et

  $$\begin{cases} \Delta' v_k + \alpha(y)\sum_{i=2}^n G_i'(v_k)\frac{\partial v_k}{\partial x_i} + f(v_k) \geq 0 & \text{dans } D, \\ v_k=0 & \text{sur } \partial D. \end{cases} \tag{1.19}$$

Remarque 1.4. Egalement dans ce cas, nous pouvons supposer que $G_i(0)=0$ pour tout $1\leq i\leq n$, sans perte de généralité, voir Remarque 1.1.

Il est intéressant de savoir sous quelles conditions sur f, G ou encore le domaine D, ces hypothèses peuvent être satisfaites. On montrera que si le domaine D est suffisamment grand, et si le vecteur $\alpha G'(0)$ est assez petit en norme L^∞ alors les hypothèses **(AD)** et **(BD)** sont satisfaites, voir section 6.4.

Sous les conditions au bord de Dirichlet, et avec la seconde forme du terme de convection $\nabla \cdot (\alpha(y)G(u))$, on obtient les mêmes résultats sous l'hypothèse supplémentaire (1.16). Remarquons qu'il s'agit exactement de la même condition que dans le cas des conditions au bord de Neumann et de la seconde forme du terme de convection. Enfin, sous les conditions au bord de Dirichlet, on s'intéressera particulièrement à l'influence du domaine D, du terme de convection αG et de la fonction g sur la vitesse minimale $c^\star$, voir chapitre 7.
Enfin, sous les conditions de Dirichlet et avec la seconde forme de terme de convection $\nabla \cdot (\alpha(y)G(u))$, nous supposerons que la fonction f satisfait (1.8) et les conditions suivantes :

- **(GD1′) :** La fonction $G : \mathbb{R}^n \mapsto \mathbb{R}$ est C^2 et la fonctions G_1 satisfait $G_1(0) = 0$.
- **(GD2′) :** Pour tout $2 \leq i \leq n$, la fonction G_i satisfait $G_i(0) = 0$.
- **(AlphaD′) :** La fonction $\alpha : D \to \mathbb{R}$ est dans $C^1(\overline{D})$.
- **(AD′) :** Soit $\lambda \in (0, 1)$. Il existe une fonction positive $w_- \in C^{2,\lambda}(\overline{D})$ telle que les seules solutions dans $C^{2,\lambda}(\overline{D})$ du problème sur la section transversale D sont w_- et 0. Plus précisément, si $z \in C^{2,\lambda}(\overline{D}) : D \to \mathbb{R}$ satisfait

$$\begin{cases} \Delta' z + \alpha(y) \sum_{i=2}^n G_i'(z) \frac{\partial z}{\partial x_i} + \sum_{i=2}^n G_i(z) \frac{\partial \alpha}{\partial x_i} + f(z) = 0 & \text{dans } D, \\ z = 0 & \text{sur } \partial D, \end{cases} \tag{1.20}$$

  alors $z \equiv 0$ ou $z \equiv w_-$.

- **(BD′) :** Il existe une suite de fonctions $\{v_k(y)\}_{k \in \mathbb{N}}$ uniformément bornée dans $C^{2,\lambda}(\overline{D})$, qui tend uniformément vers 0 lorsque k tend vers $+\infty$ et qui vérifie pour tout $k \in \mathbb{N}$

$$0 < v_{k+1}(y) < v_k(y) < w_-(y), \text{ pour } y \in D,$$

  et

$$\begin{cases} \Delta' v_k + \alpha(y) \sum_{i=2}^n G_i'(v_k) \frac{\partial v_k}{\partial x_i} + \sum_{i=2}^n G_i(v_k) \frac{\partial \alpha}{\partial x_i} + f(v_k) \geq 0 & \text{dans } D, \\ v_k = 0 & \text{sur } \partial D. \end{cases} \tag{1.21}$$

- **(FD′) :** La condition suivante est satisfaite :

$$f'(0) + \inf_{y \in D} \sum_{i=2}^{n} G_i'(0) \frac{\partial \alpha}{\partial x_i}(y) > \lambda_1(-L), \tag{1.22}$$

où $-L := -\Delta' - \alpha(y) \sum_{i=2}^{n} G_i'(0) \frac{\partial}{\partial x_i}$.

Remarque 1.5. L'hypothèse **(AD′)** implique en particulier, puisque la fonction 0 satisfait (1.20),

$$\sum_{i=2}^{n} G_i(0) \frac{\partial \alpha}{\partial x_i} = 0,$$

ce qui est vérifié lorsque l'on suppose **(GD2′)**.

Cette thèse est organisée de la façon suivante.

Dans le chapitre 2, avec les conditions au bord de Neumann et la première forme du terme de convection, on commence par définir $c^\star$ via une formule min-max, puis on montre que cette quantité est strictement inférieur à $+\infty$. On prouve ensuite l'existence d'ondes progressives solutions de vitesse $c > c^\star$ sur le cylindre tronqué $\Omega_N :=]-N, N[\times D$, en utilisant la méthode des sous et sursolutions. En faisant tendre N vers $+\infty$, on obtient alors le résultat suivant :

Théorème 1.6. *Sous les conditions au bord de Neumann et la première forme de terme de convection, supposons* **(GN1)***,* **(GN2)***,* **(GN3)***,* **(AlphaN1)***,* **(AN)** *et que la fonction f est C^1 et vérifie* (1.8)*. Then for $c > c^\star$, il existe au moins une solution w de* (1.32)*.*
De plus, cette solution est strictement décroissante par rapport à x_1 et vérifie $\frac{\partial w}{\partial x_1} < 0$.

Nous nous inspirons de [VO1] et [BN2], mais plutôt que de construire une solution sur le demi-cylindre infini comme [VO1], nous la construisons sur un cylindre tronqué Ω_N avant de faire tendre N vers l'infini. Par ailleurs, lorsque $N \to +\infty$, il est possible que la suite des solutions construites tende vers 0. Afin d'éviter cela, nous montrons en amont qu'il existe une solution sur le cylindre tronqué Ω_N satisfaisant une condition de normalisation lorsque $x_1 = 0$, assurant ainsi que la solution ne s'écroule pas vers 0. Enfin, on démontre qu'il existe une onde progressive solution sur le cylindre infini Ω ayant pour vitesse exacte $c^\star$, et que pour $c < c^\star$, il n'existe pas d'onde progressive solution strictement décroissante.

Dans le chapitre 3 avec les conditions au bord de Neumann et la première forme du terme de convection, on introduit un problème de valeurs propres lié au problème linéarisé autour de 0, et on prouve l'existence de la valeur critique c'. Précisément, on démontre qu'il existe deux valeurs propres strictement positives du problème de valeurs propres si et seulement si $c > c'$. Il est alors naturel de comparer les deux valeurs critiques $c^\star$ et c', et nous obtenons l'un des résultats principaux de cette thèse :

Théorème 1.7. *Sous les conditions au bord de Neumann et la première forme de terme de convection, soit c' défini dans la Proposition 3.4. Supposons les hypothèses* **(GN1)**, **(GN3)**, **(AlphaN1)**, **(AlphaN2)** *et que f est C^1 et vérifie* (1.8). *Supposons également qu'il existe $s_0 \in]0,1[$ telle que la condition suivante soit satisfaite :*

$$f'(0) > k, \qquad \text{où } k := \sup_{(s,y)\in]0,s_0[\times \overline{D}} \left| \sum_{i=2}^{n} \frac{G_i(s)}{s} \frac{\partial \alpha}{\partial x_i}(y) \right|. \tag{1.23}$$

Alors,

$$c^\star \geq c'.$$

En particulier, $c^\star > -\infty$.

Nous présentons également deux cas particuliers où les valeurs critiques c' et $c^\star$ sont égales. Le premier cas particulier est lorsque le terme de convection est de la forme $G = (G_1, 0, \cdots, 0)$, que f satisfait la condition KPP (1.9) et sous la condition (1.11). Le second cas particulier est toujours sous la condition de KPP (1.9) pour f et la condition (1.12).

Le chapitre 4 présente une forme alternative de terme de convection : $\nabla \cdot (\alpha(y)G(u))$ au lieu de $\alpha(y)\nabla \cdot G(u)$. Tout au long de ce chapitre, nous établissons des résultats similaires, sous la contrainte supplémentaire (1.16) impliquant le terme de convection G et la fonction f. Cette condition assure non seulement qu'une constante $\varepsilon \in]0,1[$ est toujours une sous-solution du problème sur le domaine tronqué Ω_N, mais aussi l'existence de la valeur critique c'. Nous comparons alors les valeurs $c^\star$ et c' et obtenons :

Théorème 1.8. *Sous les conditions au bord de Neumann et avec la seconde forme de terme de convection, supposons les hypothèses* **(GN1′)**, **(GN3′)**, **(AlphaN1′)**, **(AlphaN2′)**, **(EN′)** *et qu'il existe $C > 0$ et que f est C^1 est satisfait* (1.8). *Then*

$$c^\star \geq c'.$$

Notont que dans la preuve du Théorème 1.8, nous avons besoin de la condition suivante : il existe $C > 0$ et $s_0 \in]0,1[$ telles que pour tout $s \in]0, s_0[$, $|G_1(s)| \leq Cs$, ce qui est une conséquence de **(GN3′)**. Avec cette nouvelle forme de terme de convection, l'égalité $c^\star = c'$ n'est montrée que dans le cas particulier où $G = (G_1, 0, \cdots, 0)$.

Dans le chapitre 5, on considère les conditions au bord de Dirichlet et la première forme de terme de convection. Dans ce cas, une constante $\varepsilon \in]0,1[$ n'est plus une sous-solution du problème sur le cylindre tronqué Ω_N. On fait alors une hypothèse supplémentaire **(BD)** afin de garantir l'existence de sous-solutions du problème sur Ω_N. Cette hypothèse n'est peut-être pas nécessaire, mais on montre qu'elle est suffisante. Comme précédemment, on montre l'existence d'une valeur critique $c^\star$ telle qu'il existe une onde progressive solution sur le cylindre non borné Ω, de vitesse c si et seulement si $c > c^\star$:

Théorème 1.9. *Sous les conditions au bord de Dirichlet et avec la première forme de terme de convection, supposons les conditions* **(GD)**, **(AlphaD)**, **(AD)**, **(BD)** *et que f est C^1 et vérifie* (1.8)*. Alors pour $c > c^\star$, il existe au moins une solution w de* (1.17)*.*
De plus, cette solution est strictement décroissante par rapport à x_1 et vérifie $\frac{\partial w}{\partial x_1} < 0$.

Comme précédemment, on montre également qu'il existe une solution avec une vitesse critique $c = c^\star$, mais que pour $c < c^\star$, il n'existe pas de solution strictement décroissante par rapport à x_1 de (1.17).

Dans le chapitre 6, on introduit le problème de valeurs propres associé au problème linéarisé autour de 0. Sous les conditions au bord de Dirichlet, une hypothèse supplémentaire est requise pour assurer l'existence de la valeur critique c'. A savoir, on demande que $f'(0)$ soit strictement supérieure à la valeur propre principale de l'opérateur linéarisé $-L := -\Delta + \alpha(y) \sum_{i=2}^n G_i'(0) \frac{\partial}{\partial x_i}$. Par ailleurs, on démontre que cette hypothèse est vérifiée dès lors que le domaine D est suffisamment grand et que le vecteur $\alpha(y)(G_2'(0), \cdots, G_n'(0))$ est assez petit en norme L^∞. Enfin, lorsque c' existe, on démontre le résultat suivant :

Théorème 1.10. *Sous les conditions au bord de Dirichlet et avec la première forme de terme de convection, soit c' défini dans la Proposition 6.6. On suppose* **(GD)**, **(AlphaD)**, **(FD)** *et que f est C^1 et vérifie* (1.8)*. Supposons également qu'il existe $s_0 \in]0,1[$ telle que la condition suivante soit satisfaite :*

$$f'(0) > k, \qquad \text{où } k := \sup_{(s,y)\in]0,s_0[\times \overline{D}} \left| \sum_{i=2}^n \frac{G_i(s)}{s} \frac{\partial \alpha}{\partial x_i}(y) \right|. \tag{1.24}$$

Alors,

$$c^\star \geq c'.$$

En particulier, $c^\star > -\infty$.

Notons que dans la preuve du Théorème 1.10, nous avons besoin de la condition suivante : il existe $C > 0$ et $s_0 \in]0,1[$ telles que $|G(s)| \le Cs$ pour tout $s \in]0, s_0[$, ce qui est une conséquence de l'hypothèse **(GD)**.
Sous les conditions au bord de Dirichlet et avec la première forme de terme de convection, l'égalité $c^\star = c'$ n'est montrée que dans le cas particulier où $G = (G_1, 0, \cdots, 0)$ la fonction f satisfait la condition KPP (1.9), que le domaine D est assez grand et sous la condition (1.11). A la fin du chapitre 6, nous montrons que les hypothèses **(AD)** et **(BD)** sont satisfaites lorsque le domaine D est assez grand et que la quantité $\|\alpha G'(0)\|_\infty$ est assez petite.

Dans le chapitre 7, sous les conditions au bord de Dirichlet et avec la première forme de terme de convection, on met en avant l'influence des fonctions f, G et α sur la vitesse minimale $c^\star$. Par exemple, on prouve que si α et G sont fixées, alors l'application $f \mapsto c^\star(f)$ est croissante. On montre des résultats similaires dans le cas particulier où $G = (G_1, 0, \cdots, 0)$, et sous la condition (1.11) en utilisant que sous ces hypothèses, l'égalité $c^\star = c'$ est valide. On se concentre ensuite sur l'influence du domaine D sur la vitesse critique $c^\star$. Puisque la fonction α est définie sur D, notons qu'il est nécessaire de la définir sur un autre domaine avant d'étudier l'influence de D sur $c^\star$. On prouve par exemple que si D_R est une homothétie du domaine D de coefficient $R > 0$, alors $c^\star(D) < c^\star(D_R)$ pour tout $R > 1$. On montre également un résultat similaire après avoir considéré $D^\star$, la boule centrée en 0 telle que $|D^\star| = |D|$, autrement dit le réarrangement symétrique du domaine D.

Dans le chapitre 8 on considère la seconde forme de terme de convection $\nabla \cdot (\alpha(y)G(u))$, sous les conditions au bord de Dirichlet. Sous l'hypothèse supplémentaire (1.16) qui permet d'assurer l'existence de c', on obtient les mêmes résultats. Notons que cette hypothèse garantit une nouvelle fois l'existence de la valeur critique c'. Nous présentons le résultat principal de ce chapitre :

Théorème 1.11. *Sous les conditions au bord de Dirichlet et la seconde forme du terme de convection, supposons* **(GD1′)**, **(AlphaD′)**, **(FD′)** *et que* f *est* C^1 *et satisfait* (1.8). *Alors,*

$$c^\star \ge c'.$$

Soulignons encore que dans la preuve du Théorème 1.11 nous avons besoin de la condition suivante : il existe $C > 0$ et $s_0 \in]0,1[$ telles que $|G_1(s)| \le Cs$ pour tout $s \in]0, s_0[$, qui est satisfaite puisque nous avons supposé **(GD1′)**.

Le chapitre 9 est la conclusion dans laquelle nous mettons en avant certaines questions naturelles qui peuvent se poser à la suite des résultats montrés dans cette thèse. Plus précisément, dans cette thèse nous nous sommes intéressés à un profil particulier concernant la fonction f, généralement appelé monostable. Il est alors naturel de se demander quelles auraient été les différences si nous avions considéré un autre profil, comme le cas bistable, et plus précisément quel aurait été le signe de la valeur critique $c^\star$, qui caractérise le comportement asymptotique d'une onde progressive solution puisqu'il détermine ce qu'un observateur verra lorsque $t \to +\infty$. En effet, rappelons que si $c^\star > 0$, alors toutes les ondes progressives solutions auront une vitesse strictement positive $c \geq c^\star > 0$ et donc se propageront dans une seule direction.

1.2 Introduction (English version)

Travelling waves are waves that propagate without change of shape. Namely, if $u(x,t)$ is a travelling wave at time t and spatial coordinate x, the shape of the solution will be the same for all time t, and the speed c of propagation is constant. Precisely, if the solution $u(x,t) = w(x - ct)$, then w is a travelling wave profile moving at constant speed c in the positive x-direction if $c > 0$ and in the negative x-direction if $c < 0$. Replacing $u(x - ct)$ by $u(x + ct)$ we obtain a travelling wave moving in the opposite direction. Travelling waves naturally appear in biology, for instance as travelling waves of chemical concentration or population density, see [MUR, Section 11.1]. In order to get physically realistic results, u has to be bounded and non-negative everywhere. For example, the following one-dimensional equation can model a biochemical change caused by reaction kinetics and diffusion :

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u), \tag{1.25}$$

where u is the concentration, $f(u)$ the kinetics and D the diffusion coefficient. Depending on the form of the reaction term f in (1.25), different properties of travelling wave solutions can arise. This is precisely what motivated mathematicians to start studying the field of reaction diffusion theory. In [FIS], Fisher proposed the equation (1.25) with $f(u) = ku(1 - u)$, where k is positive, to model the spread of a favoured gene in a population. This special case has been studied a lot by many mathematicians, starting with Kolmogoroff, Petrovsky and Piskounoff, see [KPP]. Such equations also arise widely in modeling in physics or chemistry. With this kind of non linear term f, the function u can also represent the normalized temperature profile of a mix of two gases in a combustion model, see [HAM, Introduction], for references.

Higher dimensional spaces are needed to be more realistic. In 1937, with the intention of modeling the process of spatial spread when mutant individuals with higher adaptability manifest in the population, Fisher suggested the two-dimensional diffusion equation :

$$\frac{\partial u}{\partial t} = D\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) + (\varepsilon - \mu u)u, \tag{1.26}$$

where $u(x,y,t)$ represents the population density at time t, and spatial coordinate (x,y), see [SK1, Section 3.9]. The first term of the right hand side illustrates a diffusion phenomenon and the second term a local population growth. Both of these terms lead to changes in the population density modeled by the term $\frac{\partial u}{\partial t}$. The diffusion coefficient D indicates how quickly the density population can vary, ε is the intrinsic rate of increase and the constant $\mu \geq 0$ takes into account the reproduction rate of the species. This equation has been considered by Kolmogorov in 1937 and many other mathematicians in various fields to study the expansion of a bacterium, or even the spread of human cultures, see [SK1, Chapter 3]. For instance, in 1951, Skellam applied this equation with $\mu = 0$ in order to study the evolution of muskrats. In particular, it was shown that the population density increases exponentially when t is large enough, which means that the diffusion and growth effects lead to an expansion of the population range. Moreover, it was proved that this wave front moves with a constant speed $c = 2\sqrt{\varepsilon D}$.

However, when the species is carried by wind or water flow, the equation (1.26) has to be modified to take this into account, for instance by adding an additional term in (1.26) :

$$\frac{\partial u}{\partial t} = D\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) - c\frac{\partial u}{\partial x} + \varepsilon u, \tag{1.27}$$

where the x-axis is aligned to the direction of the wind. In the one dimensional case, the following equation appears in many different fields such as in ion exchange columns or chromatography :

$$\frac{\partial u}{\partial t} + \frac{\partial h(u)}{\partial x} = \frac{\partial^2 u}{\partial x^2} + f(u), \tag{1.28}$$

where $h'(u)$ is called the convective velocity, see [MUR, Section 13.4]. The presence of a non linear convection term can have an impact on the solutions. Indeed, by considering this kind of term, a major transport process, which depends non linearly on u, plays an important role in this new model. Note that the presence of a non linear diffusion term could also be thought of a non linear convection effect.
A lot of work has been done regarding this kind of equation in unbounded cylinders, without any convection term, see [BN2], [BLL], [ROQ] and [VE1]. Many works have studied travelling front solutions in an unbounded cylinder, see for example [VE2], [BN2], [VO1] and [HAM]. In [BN2] and [HAM], the authors considered a term $\beta(y)\partial_1 u$ in the problem, which can represent a transport term or a driving flow along the direction of the cylinder. Note that this term does not depend on the coordinate of the axis of the cylinder. In both cases, Neumann boundary conditions are imposed on the edge of the cylinder, contrary to [VE2] who studied the problem with Dirichlet boundary conditions. In [VO1], Volpert considered Dirichlet conditions, explaining that Dirichlet boundary conditions are for instance used in the Frank-Kamenetskii model of heat explosion, which studies on the reaction diffusion equation in a bounded domain, of a one-step monomolecular exothermic reaction. On the other hand, Neumann boundary conditions illustrate a sort of zero flux at the boundary meaning that the species cannot escape from the domain.
An important notion is the stability of solutions of biological models which has been considered in a range of papers covering a variety of scenarios, see for instance, [MUR, Section 11.3], [ROQ], [VO3] and [BN2, Section 1]. A wave front solution is locally stable if a small perturbation of this solution converges in some sense to this front when $t \to +\infty$. Note that different kinds of convergence to a wave can be defined, see [VO2, Section 5] where the approach in form and the uniform approach to a wave were defined. The stability of solutions is related to spectral properties of the linearized problem, see [VO1, Introduction]. However, we will not focus on the

concept of stability here.

In this thesis, we study the existence of travelling front solutions for a certain reaction-diffusion equation with a convection term. Precisely, for $n > 1$, we consider the following problem :

$$u_t = \Delta u + \alpha(y)\nabla \cdot G(u) + f(u), \tag{1.29}$$

where $t \in \mathbb{R}$ and $u(x,t) \in \mathbb{R}$, in the unbounded cylinder $\Omega = \mathbb{R} \times D$, where $D \subset \mathbb{R}^{n-1}$ is a smooth bounded domain. We write $x = (x_1, y) \in \mathbb{R} \times D$, where $y = (x_2, \cdots, x_n)$. We first seek solutions u satisfying Neumann boundary conditions on the edge of the cylinder :

$$\frac{\partial u}{\partial \nu}(x,t) = 0, \quad \text{for all } x = (x_1, y) \in \mathbb{R} \times \partial D, \text{ and for all } t \in \mathbb{R}, \tag{N}$$

where ν is the normal derivative exterior to ∂D.
Then, we will consider Dirichlet boundary conditions

$$u(x,t) = 0 \text{ for all } x = (x_1, y) \in \mathbb{R} \times \partial D, \text{ and for all } t \in \mathbb{R}. \tag{D}$$

Throughout the thesis, the reaction function $f : \mathbb{R} \to \mathbb{R}$ is assumed to be C^1 and "monostable", in the sense that

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f'(1) < 0, \quad f(u) > 0 \text{ if } 0 < u < 1. \tag{1.30}$$

The term $\alpha(y)\nabla \cdot G(u)$ in (1.29) is a non linear convection term. This non linear drift is depending on u, which means that the drift can depend on the density of the species. To our knowledge, the non linear convection term $\alpha(y)\nabla \cdot G(u)$ has not been considered in previous work. We first consider this first form of convection term, and we will consider later the alternative form $\nabla \cdot (\alpha(y)G(u))$.

In the Neumann case, we are interested in travelling front solutions of the form $u(x,t) = w(x_1 - ct, y)$, where $c \in \mathbb{R}$ is the speed of the front, and

$$w(\xi, y) \to 1 \text{ as } \xi \to -\infty \text{ and } w(\xi, y) \to 0 \text{ as } \xi \to +\infty, \text{ uniformly in } y \in D.$$

In the Dirichlet case, since the constant function 1 does not satisfy Dirichlet boundary conditions, a travelling front solution has to converge to another steady state at $-\infty$. That is why, with Dirichlet boundary conditions, we assume that there exists a function $w_- \in C^{2,\lambda}(\overline{D})$ such that a travelling front solution $u(x,t) = w(x_1 - ct, y)$ of speed c has to satisfy

$$w(\xi, y) \to w_-(y) \text{ as } \xi \to -\infty \text{ and } w(\xi, y) \to 0 \text{ as } \xi \to +\infty, \text{ uniformly in } y \in D.$$

Without the convection term, which means for $\alpha = 0$, it was proved by Berestycki and Nirenberg in [BN2] that in the Neumann case, there exists a critical value $c^\star \in \mathbb{R}$ such that travelling front solutions with speed c of (1.29) exist if and only if $c \geq c^\star$. The existence of such a critical speed $c^\star$ is characteristic of monostable problems and often can be shown to play a key role in the long-time behaviour of the initial value problem. Indeed, the solution of the initial value problem for a certain class of initial conditions will typically converge to the travelling wave of speed $c^\star$ as t tends to infinity. Moreover, Berestycki and Nirenberg also proved that this critical value $c^\star$ is positive, which means that all the travelling waves propagate in the positive x_1-direction.

We will prove that under certain conditions on f and G, such a critical value $c^\star$ also exists in the presence of non linear drift, but in our case, the critical value can be positive or negative because of the convection term. In particular, if $c^\star$ is negative, some of the travelling waves with negative speed $0 > c \geq c^\star$ propagate in the negative x_1-direction. In [BN2], the authors obtained an explicit formula for the critical value $c^\star$ under an additional condition on f. Precisely, without any convection term and under the famous KPP condition

$$f(u) \leq f'(0)u, \quad \forall u \in (0,1), \tag{1.31}$$

it was proved in [BN2, Theorem 1.5 and Section 10], that $c^\star = c'$, where c' is related to a certain eigenvalue problem and the linearized travelling-front problem around 0.

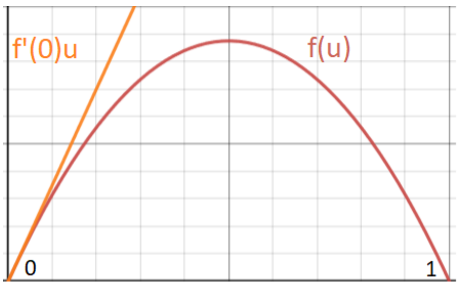

Figure 2: Illustration of a typical reaction function satisfying KPP condition.

As we mentioned earlier, a lot of work has been done regarding this kind of equation in unbounded cylinders, without any convection term, see [BN2], [BLL], [ROQ] and [VE1]. Some papers deal with the presence of convection term in dimension 1, see [CRO], [CRM] and [CRT]. Some existence results were proved in that case, see [CRM, Theorem 2.4], [CRO, Theorem 2.4] and [AKC, Theorem 3.6]. In this thesis, we extend these results to multi-dimensional cylinders . In the multidimensional case, a large divergence free drift was considered in [BHN], with both Neumann and

Dirichlet boundary conditions and also in the periodic case, where they studied the asymptotic behaviour of the principal eigenvalue of some linear elliptic equations.

Under Neumann boundary conditions and with the first form of the convection term $\alpha(y)\nabla \cdot G(u)$, if $u(x,t) = w(x_1 - ct, y)$, is a solution of (1.29), the function w satisfies :

$$\begin{cases} -c\partial_1 w = \Delta w + \alpha(y)\nabla \cdot G(w) + f(w) & \text{in } \Omega, \\ w(-\infty, y) = 1, \quad w(+\infty, y) = 0 & \text{uniformly in } y \in D, \\ w \geq 0, & \\ w_\nu = 0 & \text{on } \mathbb{R} \times \partial D. \end{cases} \tag{1.32}$$

We prove that there exists a critical speed $c^\star$ such that travelling front solutions exist if and only if $c \geq c^\star$. To do that, we are following the approach of [VO1, Chapter 5, Section 4] and using some ideas of [BN2]. Precisely, we first show that there exists a travelling front solution on a truncated cylinder $\Omega_N := (-N, N) \times D$ with Dirichlet boundary conditions on $\{\pm N\} \times D$ and Neumann boundary conditions on $(-N, N) \times \partial D$, and then let N tend to infinity. Then, as in [BN2], we study an eigenvalue problem related to the linearized problem around 0, and show that there exists a critical value c' such that for c greater than c', this eigenvalue problem has two positive eigenvalues. We then compare these two critical values $c^\star$ and c', and by using the key [BN3, Theorem 2.1], we show that $c^\star \geq c'$, under some assumptions on f and G; see Theorem 1.18. Note that without the convection term, it was proved in [BN2] that $c^\star = c'$ under the KPP condition (1.31) on f. Contrary to [BN2], the method of sub and supersolutions they used to prove that $c^\star \leq c'$ is difficult to use in general because of the non linear drift and specially the presence of the derivative terms $\partial_i u$ for $2 \leq i \leq n$. However, in this paper we highlight two special cases where if the convection term G has a specific form, then $c^\star = c'$. Note that this equality holds in two "opposite" cases. In the first case, assuming the KPP condition (1.31) for f, and that

$$G = (G_1, 0, \cdots, 0), \quad \alpha(y)G_1'(u) \geq \alpha(y)G_1'(0) \text{ for all } y \in D \text{ and } u \in (0,1), \tag{1.33}$$

there are no derivative terms $\partial_i u$ for $2 \leq i \leq n$ and then the method used in [BN2, Section 10] proves that $c^\star = c'$. In the second case, assuming the KPP condition (1.31) for f and that

$$G_1'(0) = 0, \quad \alpha(y)G_1'(u) \geq 0 \text{ for all } y \in D \text{ and } u \in (0,1), \tag{1.34}$$

the equality $c^\star = c'$ holds and we in fact obtain also the explicit formula $c^\star = c' = 2\sqrt{f'(0)}$. Note that this is the formula obtained for $c^\star$ by Berestycki and Nirenberg in [BN2] when f satisfies the KPP condition (1.31).

Under Neumann boundary conditions, with the first form of the convection term $\alpha(y)\nabla \cdot G(u)$, we assume the condition (1.30) for the function f, and denote by

$L > 0$ the Lipschitz constant on $(0, 1)$ of f. We consider $p > n$ and assume the following conditions :

- **(GN1) :** The function $G : \mathbb{R} \to \mathbb{R}^n$ is C^2.
- **(GN2) :** The function G_1' is Lipschitz continuous on $[0, 1]$, with constant $\tilde{L} > 0$.
- **(GN3) :** For all $1 \leq i \leq n$, $G_i(0) = 0$.
- **(AlphaN1) :** The function $\alpha : D \to \mathbb{R}$ belongs to $C^1(\overline{D})$.
- **(AlphaN2) :** The function α satisfies $\alpha \equiv 0$ on ∂D.

  As in [VO1, Section 4.1], we also need a uniqueness assumption of the solutions of the problem on the cross section D :

- **(AN) :** The only solutions in $W^{2,p}(D)$ of the problem on the cross section are 0 and 1. Precisely, for $n < p < \infty$, if $z \in W^{2,p}(D) : D \to \mathbb{R}$ satisfies

$$\begin{cases} \Delta' z + \alpha(y) \sum_{i=2}^n G_i'(z) \frac{\partial z}{\partial x_i} + f(z) = 0 & \text{in } D, \\ z_\nu = 0 & \text{on } \partial D, \end{cases} \tag{1.35}$$

  where

$$\Delta' = \sum_{i=2}^n \frac{\partial^2}{\partial x_i^2}, \text{ and } y = (x_2, x_3, \cdots, x_n).$$

  then

$$z \equiv 0 \text{ or } z \equiv 1.$$

Remark 1.12. Note that with the first form of the convection term $\alpha(y) \nabla \cdot G(u)$, we are not losing generality assuming **(GN3)**. Indeed, if $G(0) = \beta \neq 0$, define $\tilde{G}(u) = G(u) - \beta$. Since $\tilde{G}_i'(u) = G_i'(u)$ for all $1 \leq i \leq n$, it follows that

$$\nabla \cdot \tilde{G}(u) = \sum_{i=1}^n \tilde{G}_i'(u) \partial_i u = \sum_{i=1}^n G_i'(u) \partial_i u = \nabla \cdot G(u),$$

which shows that the problem (1.32) is unchanged.

Remark 1.13. If $\alpha G'$ is a constant vector, the assumption **(AN)** is satisfied. Indeed, by [YIH, Theorem 4.9] and [YIH, Theorem 4.11] applied with $\varepsilon \in (0, 1)$ as a subsolution and 1 as a supersolution of (1.35), there exist a minimal solution $\underline{u}$ and a maximal solution $\overline{u}$ of (1.35) such that $\varepsilon \leq \underline{u} \leq \overline{u} \leq 1$, in the sense where if u is a solution of (1.35), then u satisfies $\underline{u} \leq u \leq \overline{u}$. Note that, since 1 is in fact a solution, it follows that $\overline{u} \equiv 1$, and then, $\underline{u}$ satisfies $\varepsilon \leq \underline{u} \leq 1$. By multiplying the equation (1.35) satisfied by $\underline{u}$ by $e^{\alpha G' \cdot y}$ and integrating over D, it follows that

$$\int_D e^{\alpha G' \cdot y}(\Delta' \underline{u} + \alpha G' \cdot \nabla' \underline{u}) + \int_D e^{\alpha G' \cdot y} f(\underline{u}) = 0. \tag{1.36}$$

Since

$$\nabla' \cdot \left(e^{\alpha G' \cdot y} \nabla' \underline{u} \right) = e^{\alpha G' \cdot y}(\Delta' \underline{u} + \alpha G' \cdot \nabla' \underline{u}),$$

and $\underline{u}$ satisfies Neumann boundary conditions on ∂D, it then follows that the first integral in (1.36) is 0, and then $\int_D e^{\alpha G' \cdot y} f(\underline{u}) = 0$, which gives $f(\underline{u}) \equiv 0$, by positivity of the integrand. Finally, since $\underline{u} \geq \varepsilon > 0$, and since f only vanishes at 0 and 1, we obtain $\underline{u} \equiv 1$. We have then proved that if z is a strictly positive solution of (1.35) that is bounded away from 0, by choosing ε such that $z > \varepsilon$ in D, it follows that $z \equiv 1$.
Assume now that there exists a non negative solution z of (1.35) such that $z(y_0) = 0$ at an interior point $y_0 \in D$. Then, the Strong Maximum Principle ensures that z is constant in D, and hence $z \equiv 0$ on D. Note that because of the Hopf lemma, we cannot have $z = 0$ anywhere on ∂D.

With the second form of the convection term $\nabla \cdot (\alpha(y) G(u))$, and again under Neumann boundary conditions, we obtain the same results as for the first form of convection term provided we assume some slightly different assumptions :

- **(GN1′) :** The function $G : \mathbb{R} \mapsto \mathbb{R}^n$ is C^2.
- **(GN2′) :** The function G_1' is Lipschitz continuous on $[0, 1]$, with constant $\tilde{L} > 0$.
- **(GN3′) :** The function G_1 satisfies $G_1(0) = 0$.
- **(GN4′) :** For all $2 \leq i \leq n$, the function G_i satisfies $G_i(0) = G_i(1) = 0$.
- **(AlphaN1′) :** The function $\alpha : D \to \mathbb{R}$ belongs to $C^1(\overline{D})$.
- **(AlphaN2′) :** The function α satisfies $\alpha \equiv 0$ on ∂D.

The uniqueness assumption of the solutions of the problem on the cross section D becomes :

- **(AN′) :** The only solutions in $W^{2,p}(D)$ of the problem on the cross section are 0 and 1. Precisely, for $n < p < \infty$, if $z \in W^{2,p}(D) : D \to \mathbb{R}$ satisfies

$$\begin{cases} \Delta' z + \alpha(y) \sum_{i=2}^n G_i'(z) \frac{\partial z}{\partial x_i} + \sum_{i=2}^n G_i(z) \frac{\partial \alpha}{\partial x_i} + f(z) = 0 & \text{in } D, \\ z_\nu = 0 & \text{on } \partial D, \end{cases} \tag{1.37}$$

then

$$z \equiv 0 \text{ or } z \equiv 1.$$

- **(EN′) :** The following condition holds :

$$f'(0) + \inf_{y \in D} \sum_{i=2}^{n} G_i'(0) \frac{\partial \alpha}{\partial x_i}(y) > 0. \tag{1.38}$$

  This condition is sufficient to ensure the existence of c' and also to make sure that a small constant is still a subsolution of the problem (4.122) on the truncated cylinder Ω_N.

Remark 1.14. Assumption **(AN′)** implies in particular that

$$\sum_{i=2}^{n} G_i(0) \frac{\partial \alpha}{\partial x_i} = \sum_{i=2}^{n} G_i(1) \frac{\partial \alpha}{\partial x_i} = 0,$$

which holds in particular if we assume **(GN4′)**.

Under Dirichlet boundary conditions and with the first form of the convection term $\alpha(y)\nabla \cdot G(u)$, if $u(x,t) = w(x_1 - ct, y)$ is a solution of (1.29), the function w satisfies :

$$\begin{cases} -c\partial_1 w = \Delta w + \alpha(y)\nabla \cdot G(w) + f(w) & \text{in } \Omega, \\ w(-\infty, y) = w_-(y), \quad w(+\infty, y) = 0 & \text{uniformly in } y \in D, \\ w \geq 0, & \\ w = 0 & \text{on } \mathbb{R} \times \partial D. \end{cases} \tag{1.39}$$

With Dirichlet boundary conditions, a key difference with the Neumann case is that a small constant $\varepsilon \in (0,1)$ is no longer a subsolution of the problem on the truncated cylinder Ω_N anymore, which prevents us to use sub and supersolution method with a non-zero constant as the sub-solution. That is why under Dirichlet boundary conditions, we need slightly different assumptions. We assume that the function f satisfies (1.30), and the following conditions :

- **(GD) :** The function G is C^2 and satisfies $G_i(0) = 0$ for all $1 \leq i \leq n$.
- **(AlphaD) :** The function $\alpha : D \to \mathbb{R}$ belongs to $C^1(\overline{D})$.
- **(AD) :** Let $\lambda \in (0,1)$. There exists a non negative function $w_- \in C^{2,\lambda}(\overline{D})$ such that the only solutions in $C^{2,\lambda}(\overline{D})$ of the problem on the cross section D are w_- and 0. Precisely, if $z \in C^{2,\lambda}(\overline{D}) : D \to \mathbb{R}$ satisfies

$$\begin{cases} \Delta' z + \alpha(y) \sum_{i=2}^{n} G_i'(z) \frac{\partial z}{\partial x_i} + f(z) = 0 & \text{in } D, \\ z = 0 & \text{on } \partial D, \end{cases} \tag{1.40}$$

  then $z \equiv 0$ or $z \equiv w_-$.

  To make sure that there exist subsolutions of the problem (5.145) on the truncated cylinder Ω_N, as in [VO1], we also assume the following condition :

- **(BD) :** There exists a sequence of functions $(v_k(y))_{k\in\mathbb{N}}$ uniformly bounded in $C^{2,\lambda}(\overline{D})$, and which tends uniformly to 0 when k tends to $+\infty$ and which satisfies for every $k \in \mathbb{N}$

$$0 < v_{k+1}(y) < v_k(y) < w_-(y), \text{ for } y \in D,$$

and

$$\begin{cases} \Delta' v_k + \alpha(y) \sum_{i=2}^n G_i'(v_k) \frac{\partial v_k}{\partial x_i} + f(v_k) \geq 0 & \text{in } D, \\ v_k = 0 & \text{on } \partial D. \end{cases} \tag{1.41}$$

- **(FD) :** The following conditions holds :

$$f'(0) > \lambda_1(-L),$$

where $-L := -\Delta' - \alpha(y) \sum_{i=2}^n G_i'(0) \frac{\partial}{\partial x_i}$.

Remark 1.15. In this case also, we are not losing generality assuming that $G_i(0) = 0$ for all $1 \leq i \leq n$, see Remark 1.12.

We also give some sufficient conditions in section 6.4 under which assumptions **(AD)**, **(BD)** and **(FD)** are satisfied. Namely, if the measure of the domain D is big enough, and the vector $\alpha G'(0)$ is sufficiently small in the infinity norm, then those assumptions are satisfied.

Under Dirichlet boundary conditions, we then show the influence of the domain D, the convection term αG and the function f on the critical speed $c^\star$, see chapter 7.

Finally, with Dirichlet boundary conditions and the second form of the convection term $\nabla \cdot (\alpha(y) G(u))$, we assume that the function f satisfies (1.30), and the following conditions :

- **(GD1′) :** The function $G : \mathbb{R}^n \mapsto \mathbb{R}$ is C^2 and the function G_1 satisfies $G_1(0) = 0$.
- **(GD2′) :** For all $2 \leq i \leq n$, the function G_i satisfies $G_i(0) = 0$.
- **(AlphaD′) :** The function $\alpha : D \to \mathbb{R}$ belongs to $C^1(\overline{D})$.
- **(AD′) :** Let $\lambda \in (0,1)$. There exists a non negative function $w_- \in C^{2,\lambda}(\overline{D})$ such that the only solutions in $C^{2,\lambda}(\overline{D})$ of the problem on the cross section D are w_- and 0. Precisely, if $z \in C^{2,\lambda}(\overline{D}) : D \to \mathbb{R}$ satisfies

$$\begin{cases} \Delta' z + \alpha(y) \sum_{i=2}^n G_i'(z) \frac{\partial z}{\partial x_i} + \sum_{i=2}^n G_i(z) \frac{\partial \alpha}{\partial x_i} + f(z) = 0 & \text{in } D, \\ z = 0 & \text{on } \partial D, \end{cases} \tag{1.42}$$

then $z \equiv 0$ or $z \equiv w_-$.

- **(BD′) :** There exists a sequence of functions $\{v_k(y)\}_{k\in\mathbb{N}}$ uniformly bounded in $C^{2,\lambda}(\overline{D})$, and which tends uniformly to 0 when k tends to $+\infty$ and which satisfies for every $k \in \mathbb{N}$

$$0 < v_{k+1}(y) < v_k(y) < w_-(y), \text{ for } y \in D,$$

and

$$\begin{cases} \Delta' v_k + \alpha(y) \sum_{i=2}^n G_i'(v_k) \frac{\partial v_k}{\partial x_i} + \sum_{i=2}^n G_i(v_k) \frac{\partial \alpha}{\partial x_i} + f(v_k) \geq 0 & \text{in } D, \\ v_k = 0 & \text{on } \partial D. \end{cases} \tag{1.43}$$

- **(FD′) :** The following conditions holds :

$$f'(0) + \inf_{y\in D} \sum_{i=2}^n G_i'(0) \frac{\partial \alpha}{\partial x_i}(y) > \lambda_1(-L), \tag{1.44}$$

where $-L := -\Delta' - \alpha(y) \sum_{i=2}^n G_i'(0) \frac{\partial}{\partial x_i}$.

We will show in chapter 8 that if (1.38) holds, the measure of the domain D is big enough and $\|\alpha G'(0)\|_{L^\infty}$ small enough, then the inequality (1.44) holds.

Remark 1.16. Assumption **(AD′)** implies in particular, since the function 0 satisfies (1.42), that

$$\sum_{i=2}^n G_i(0) \frac{\partial \alpha}{\partial x_i} = 0,$$

which is satisfied if we assume **(GD2′)**.

This thesis is organized as follows.

In chapter 2, with Neumann boudary conditions and the first form of the convection term, we first define $c^\star$ by a min-max formula, and show that $c^\star < +\infty$. We prove the existence of travelling front solutions with speed $c \geq c^\star$ on a truncated cylinder $\Omega_N := (-N, N) \times D$, by using the method of sub and supersolutions. We then let N tend to infinity and obtain the following result :

Theorem 1.17. *With Neumann boundary conditions and the first form of convection term, assume* **(GN1)**, **(GN2)**, **(GN3)**, **(AlphaN1)**, **(AN)** *and that f is C^1 and satisfies* (1.30). *Then, for $c > c^\star$, there exists at least one solution w of* (1.32). *In addition this solution w is decreasing with respect to x_1 and satisfies $\frac{\partial w}{\partial x_1} < 0$.*

Our strategy is inspired by [VO1] and [BN2] but instead of constructing a solution on the half cylinder, we do this on a truncated cylinder Ω_N and then let N tend to infinity. When N tends to infinity, we have to avoid that the solution collapses to 0. To do that, we show that there exists a travelling front solution on the truncated cylinder Ω_N which satisfies a normalization condition when $x_1 = 0$, which ensures that the solution will not tend to 0 when N tends to infinity. We also show that there exists a decreasing (with respect to x_1) travelling front solution on the infinite cylinder Ω with a speed $c = c^\star$, and that for $c < c^\star$, there is no decreasing travelling front solution.

In chapter 3, again with Neumann boundary conditions and the first form of the convection term, we introduce an eigenvalue problem related to the linearized problem around 0, and show the existence of the critical value c'. Namely, we prove that there exist two positive eigenvalues of the generalized eigenvalue problem if and only if $c > c'$. We then compare $c^\star$ and c' and obtain one of the main results of this thesis :

Theorem 1.18. *With Neumann boundary conditions and the first form of convection term, let c' be as defined in Proposition 3.4 below. Assume* **(GN1)**, **(GN3)**, **(AlphaN1)**, **(AlphaN2)** *and that f is C^1 and satisfies* (1.30). *Assume also that there exists $s_0 \in (0, 1)$ such that the following condition holds :*

$$f'(0) > k, \qquad \text{where } k := \sup_{(s,y)\in(0,s_0)\times\overline{D}} \left| \sum_{i=2}^{n} \frac{G_i(s)}{s} \frac{\partial \alpha}{\partial x_i}(y) \right|. \tag{1.45}$$

Then one has :

$$c^\star \geq c'.$$

In particular, $c^\star > -\infty$.

We also discuss two special cases where $c^\star$ and c' are equal. The first special case is when the convection term has the form $G = (G_1, 0, \cdots, 0)$, if f satisfies KPP condition (1.31) and under the condition (1.33). The second special case is still under KPP condition (1.31) for f and also the condition (1.34).

Chapter 4 is devoted to the alternative form of convection term, still under Neumann boundary conditions, where we consider the form $\nabla \cdot (\alpha(y)G(u))$ instead of $\alpha(y)\nabla \cdot G(u)$. Throughout this chapter we establish similar results to before, but under the additional condition (1.38) involving the convection term G and the function f. This condition ensures that a small constant is still a subsolution of the problem on the truncated cylinder Ω_N, allowing us to use the sub and supersolutions method as before. Note that this condition ensures the existence of c' as well. We then compare the two critical values $c^\star$ and c' and claim :

Theorem 1.19. *With Neumann boundary conditions and the second form of the convection term, assume* **(GN1′)**, **(GN3′)**, **(GN4′)**, **(AlphaN1′)**, **(AlphaN2′)**, **(EN′)** *and that f is C^1 and satisfies* (1.30). *Then*

$$c^\star \geq c'.$$

Note that in the proof of Theorem 1.19, we need the following condition : there exist $C > 0$ and $s_0 \in (0, 1)$ such that for all $s \in (0, s_0)$, $|G_1(s)| \leq Cs$, which is satisfied since we assumed **(GN3′)**. With the second form of convection term, only the special case where $G = (G_1, 0, \cdots, 0)$ can be handled to show that $c^\star = c'$.

In chapter 5, we consider the Dirichlet boundary conditions and the first form of the convection term. In this case, a small constant is not a subsolution of the problem on the truncated cylinder Ω_N. That is why we need an extra assumption to make sure that such a subsolution does exist, namely (1.38). Similarly to before, we show the existence of a critical speed $c^\star$ which satisfies :

Theorem 1.20. *With Dirichlet boundary conditions and the first form of the convection term, assume the assumptions* **(GD)**, **(AlphaD)**, **(AD)**, **(BD)** *and that f is C^1 and satisfies* (1.30). *Then for $c > c^\star$, there exists at least one solution w of* (1.39).
In addition this solution w is decreasing with respect to x_1 and satisfies $\frac{\partial w}{\partial x_1} < 0$.

We also show that there exists a solution with a critical speed $c = c^\star$ and for $c < c^\star$ there is no decreasing solution with respect to x_1 of (1.39).

In chapter 6, we introduce the generalized eigenvalue problem under Dirichlet boundary conditions, which is associated with the linearized equation. Under Dirichlet boundary conditions, an extra assumption is needed to make sure that c' exists : $f'(0)$ has to be greater than the principal eigenvalue of the linearized operator $-L := -\Delta + \alpha(y) \sum_{i=2}^{n} G_i'(0) \frac{\partial}{\partial x_i}$. We show that this assumption can be satisfied if the domain D is big enough, and the vector $\alpha(y)(G_2'(0), \cdots G_n'(0))$ is sufficiently small in L^∞ norm. Knowing the existence of c', we prove the following result :

Theorem 1.21. *With Dirichlet boundary conditions and the first form of the convection term, let c' defined in Proposition 6.6. Assume* **(GD)**, **(AlphaD)**, **(FD)** *and that f is C^1 and satisfies* (1.30). *Assume also that there exists $s_0 \in (0, 1)$ such that the following condition holds :*

$$f'(0) > k, \qquad \text{where } k := \sup_{(s,y) \in (0,s_0) \times \overline{D}} \left| \sum_{i=2}^{n} \frac{G_i(s)}{s} \frac{\partial \alpha}{\partial x_i}(y) \right|. \tag{1.46}$$

Then

$$c^\star \geq c'.$$

In particular, $c^\star > -\infty$.

In the proof of Theorem 1.21 we need the following condition : there exist $C > 0$ and $s_0 \in (0,1)$ such that $|G(s)| \leq Cs$, for all $s \in (0, s_0)$, which is satisfied since we assumed **(GD)**.
Under Dirichlet boundary conditions and with the first form of the convection term, we show that the equality $c^\star = c'$ holds in the special case where $G = (G_1, 0, \cdots, 0)$, assuming KPP condition (1.31) for f and that $\alpha(y)G_1'(u) \geq \alpha(y)G_1'(0)$ for all $y \in D$ and all $u \in (0,1)$, and if the domain D is big enough. At the end of chapter 6, we show that the Assumptions **(AD)** and **(BD)** are satisfied if the domain D is big enough, and if the quantity $\|\alpha G'(0)\|_\infty$ is small enough.

In chapter 7, with Dirichlet boundary conditions and the first form of the convection term, we highlight the influence of the functions f, α and G on the critical speed $c^\star$. For instance, we prove that if we fix the functions α and G, then the map $f \mapsto c^\star(f)$ is increasing. We show similar results in the special case where $G = (G_1, 0, \cdots, 0)$, using that in that case, the equality $c^\star = c'$ holds. We then focus on the influence of the domain D on the speed $c^\star$. In that case, since the function α is defined on D, we need to define the function α on other domains. We prove for instance that if D_R is a rescaling domain of D, then $c^\star(D) < c^\star(D_R)$ for all $R > 1$. We also show a similar result, considering the symmetric rearrangement of the domain D instead of a rescaling.

In chapter 8, we consider the second form of the convection term : $\nabla \cdot (\alpha(y)G(u))$ for the Dirichlet problem. We show the existence of a critical speed $c^\star \in \mathbb{R}$ such that travelling front solution exists with speed c if and only if $c \geq c^\star$. We also show the existence of c' under the extra assumption (1.38) which ensures that c' does exist. We then show the main result of this chapter :

Theorem 1.22. *With Dirichlet boundary conditions and the second form of the convection term, assume* **(GD1′)**, **(AlphaD′)**, **(FD′)** *and that f is C^1 and satisfies* (1.30). *Then*

$$c^\star \geq c'.$$

Note again that in the proof of Theorem 1.22, the precise condition needed is that there exist $C > 0$ and $s_0 \in (0,1)$ such that $|G_1(s)| \leq Cs$, for all $s \in (0, s_0)$, which is satisfied since we assumed **(GD1′)**.

Chapter 9 consists of some conclusions, where we reflect on the results of the thesis discuss what further natural questions we could ask. Precisely, we focused on a specific profile of the function f called monostable case, but we can wonder what could happen under other profils for f for instance the bistable case, and specially what would be the sign of the critical speed $c^\star$, which characterizes the profile long-time behaviour of the travelling front solution since it determines what an observer will see when $t \to +\infty$. Indeed, recall that if $c^\star > 0$, then all the travelling front solutions will have a positive speed $c \geq c^\star > 0$ and hence will travel in only one direction.

2 Existence of a solution for Neumann boundary conditions

2.1 Solution on the truncated cylinder

Our strategy here is to construct a solution of (1.32) on a bounded domain $\Omega_N := (-N, N) \times D$, and then let N tend to infinity. To do that, we will use an iteration method. Note that in [VO1, Chapter 5, Section 4], a solution is first constructed on a half-cylinder of the form $(-\infty, N) \times D$, before passing to the limit. In their case, they have Dirichlet boundary conditions on the edge of the cylinder and they can use standard theorems on existence of solutions of the initial value problem for parabolic equations with Dirichlet data on the boundary. In our case, because of the Neumann conditions on the edge of the cylinder, there would be Neumann conditions on part of the boundary and Dirichlet conditions on other parts of the boundary. Since it is not straightforward to track down suitable results in the literature about existence of solutions for parabolic equations with such boundary conditions, we will first argue on a truncated cylinder of the form $\Omega_N = (-N, N) \times D$.

Denote by K the set of functions $\rho \in C^2(\mathbb{R} \times \overline{D})$ such that

$$\begin{cases} \partial_1 \rho < 0 & & \text{in } \mathbb{R} \times D, \\ \lim_{x_1 \to -\infty} \rho(x_1, y) = 1, & \lim_{x_1 \to +\infty} \rho(x_1, y) = 0, & \text{uniformly in } y \in D, \\ \rho_v = 0 & & \text{on } \mathbb{R} \times \partial D. \end{cases} \tag{2.47}$$

For $\rho \in K$, let

$$r(\rho)(x) := \frac{\Delta\rho(x) + \alpha(y)\nabla \cdot G(\rho(x)) + f(\rho(x))}{-\partial_1 \rho(x)},$$

and

$$c^\star := \inf_{\rho \in K} \sup_{x \in \Omega} r(\rho)(x). \tag{2.48}$$

Proposition 2.1. *Assume the conditions* **(GN1)**, **(GN2)**, **(AlphaN1)** *and that f is Lipschitz continuous, with constant $L > 0$. Then*

$$c^\star < +\infty.$$

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ be a C^2 decreasing function such that $h'(x_1) < 0$ for all $x_1 \in \mathbb{R}$,

$$h(x_1) = \begin{cases} 1 - e^{x_1} & \text{if } x_1 < -1 \\ e^{-x_1} & \text{if } x_1 > 1 \end{cases}, \tag{2.49}$$

and define

$$g(x_1, y) := h(x_1), \quad \text{for all } (x_1, y) \in \Omega. \tag{2.50}$$

We will show that the function g belongs to K. First, g is C^1, and one has $\partial_1 g < 0$, $g(x_1, y) \to 1$ as x_1 tends to $-\infty$, and $g(x_1, y) \to 0$ as x_1 tends to $+\infty$, uniformly with respect to $y \in D$. Furthermore, if $x_1 < -1$,

$$\begin{aligned} r(g)(x_1, y) &= \frac{-e^{x_1} + \alpha(y)G_1'(g(x_1, y))(-e^{x_1}) + f(1 - e^{x_1})}{e^{x_1}} \\ &= -1 - \alpha(y)G_1'(1 - e^{x_1}) + \frac{f(1 - e^{x_1})}{e^{x_1}} \\ &\leq -1 - \alpha(y)G_1'(1 - e^{x_1}) + L. \end{aligned}$$

Similarly, we obtain for $x_1 > 1$,

$$r(g)(x_1, y) \leq 1 - \alpha(y)G_1'(e^{-x_1}) + L.$$

Since $r(g)$ is a continuous function in $[-1, 1] \times \overline{D}$, α is bounded and G_1' is a locally Lipschitz function, it follows that

$$\sup_{x \in \Omega} r(g)(x_1, y) < +\infty,$$

and consequently, $c^\star < +\infty$.

$\square$

We present two sets of sufficient conditions that ensure that $c^\star > -\infty$.
Note that we will show later, see Theorem 1.18, that under certain assumptions on f, G and α, $c^\star > c'$, where c' is defined in Proposition 3.4 which will also imply in paritular that $c^\star > -\infty$.
We first show that in the special case $G = (G_1, \cdots, 0)$, then $c^\star > -\infty$.

Proposition 2.2. *Assume that* $G = (G_1, \cdots, 0)$, *and that the function* $(y, s) \mapsto \alpha(y)G_1'(s)$ *is bounded on* $\overline{D} \times [0, 1]$. *Then*

$$c^\star > -\infty.$$

Proof. Assume that $c^\star = -\infty$, and let $A > 0$. By definition of $c^\star$, there exists $\rho \in K$ such that

$$\frac{\Delta\rho + \alpha(y)G_1'(\rho)\partial_1\rho + f(\rho)}{-\partial_1\rho} < -A.$$

Since $\partial_1\rho < 0$ and $f > 0$ on $(0, 1)$, it follows that

$$\Delta\rho + \Big(\alpha(y)G_1'(\rho) - A\Big)\partial_1\rho < 0.$$

Let $k := \sup_{(y,s) \in \overline{D} \times [0,1]} \alpha(y)G_1'(s)$. By integrating on D and because of the Neumann boundary conditions, we get

$$\int_D \partial_{11}\rho + (k - A)\int_D \partial_1\rho < 0. \tag{2.51}$$

Denote $q(x_1) := \int_D \rho(x_1,y)dy$. Since $\rho \in C^2(\mathbb{R}\times\overline{D})$, the inequality (2.51) can be rewritten

$$q''(x_1) + (k-A)q'(x_1) < 0.$$

Multiplying by $e^{(k-A)x_1}$, it follows that for all $x_1 > x_0$,

$$e^{(k-A)x_1}q'(x_1) < e^{(k-A)x_0}q'(x_0).$$

By multiplying by $e^{-(k-A)x_1}$ and then integrating between x_0, and $x_2 > x_1$:

$$q(x_2) - q(x_0) < e^{(k-A)x_0}q'(x_0)\frac{e^{-(k-A)x_0} - e^{-(k-A)x_2}}{k-A},$$

which we can rewrite as

$$q(x_2) < q(x_0) + \frac{q'(x_0)}{k-A}\Big(1 - e^{(A-k)(x_2-x_0)}\Big). \tag{2.52}$$

Taking $A > k$ ensures that $q'(x_0)/(k-A) > 0$. Hence, the right hand side of (2.52) tends to $-\infty$ when $x_2 \to +\infty$, which is impossible since $q > 0$.

□

In the general case, we need more assumptions to make sure that $c^\star > -\infty$. Denote $\tilde{G} = (G_2, \cdots, G_n)$. We claim :

Proposition 2.3. *Assume that $\alpha \equiv 0$ on ∂D. Assume also that there exist $\varepsilon > 0$, $\beta > 0$, $\gamma > 0$ and $s_0 \in (0,1)$ such that $|\nabla'\alpha(y)| \le \varepsilon$, for all $y \in D$, $|\tilde{G}(s)| \le \beta s$ for all $s \in [0,1]$, and $f(s) \ge \gamma s$ for all $s \in [0, s_0]$, and that $\gamma - \varepsilon\beta \ge 0$. Then $c^\star > -\infty$.*

Proof. Assume that $c^\star = -\infty$, and let $A > 0$. By definition of $c^\star$, there exists $\rho \in K$ such that

$$\frac{\Delta\rho + \alpha(y)G_1'(\rho)\partial_1\rho + \alpha(y)\nabla'\cdot\tilde{G}(\rho) + f(\rho)}{-\partial_1\rho} < -A.$$

Again, using that $\partial_1\rho < 0$ and integrating on D, we obtain

$$\int_D \partial_{11}\rho + (k-A)\int_D \partial_1\rho < -\Big(\int_D \alpha(y)\nabla'\cdot\tilde{G}(\rho) + \int_D f(\rho)\Big), \tag{2.53}$$

where $k := \sup_{(y,s)\in\overline{D}\times[0,1]} \alpha(y)G_1'(s)$.
Using Green's formula, the fact that $\alpha \equiv 0$ on ∂D, and that ρ tends uniformly (with respect to $y \in D$) to 0 when $x_1 \to +\infty$, one has for x_1 large enough :

$$\begin{aligned}\int_D \alpha(y)\nabla'\cdot\tilde{G}(\rho) + \int_D f(\rho) &= -\int_D \nabla'\alpha(y)\cdot\tilde{G}(\rho) + \int_D f(\rho)\\ &\ge -\varepsilon\beta\int_D \rho + \gamma\int_D \rho\\ &= (\gamma - \varepsilon\beta)\int_D \rho \ge 0.\end{aligned}$$

Hence, the right hand side of (2.53) is negative and we conclude as in the proof of Proposition 2.2. □

Now, let $c > c^{\star}$. Then, by definition of the infimum, there exists a function $\rho \in K$, such that

$$\begin{cases} \Delta\rho + \alpha(y)\nabla \cdot G(\rho) + f(\rho) + c\partial_1\rho < 0 & \forall (x_1, y) \in \mathbb{R} \times D \\ \partial_1\rho < 0 & \text{in } \mathbb{R} \times D, \\ \rho(-\infty, y) = 1, \quad \rho(+\infty, y) = 0, & \text{uniformly in } y, \\ \rho_v = 0 & \text{on } \mathbb{R} \times \partial D. \end{cases} \tag{2.54}$$

Let $N \geq 1$ be an integer. Since the function $y \mapsto \rho(N, y)$ is continuous and $\overline{D}$ is compact, there exists $\varepsilon_N \in (0,1)$ such that

$$\rho(N, y) > \varepsilon_N, \ \forall y \in D. \tag{2.55}$$

Note that $f(\varepsilon_N) > 0$. We claim:

Proposition 2.4. *Let $N > 1$ and $\Omega_N = (-N, N) \times D \subset \Omega$. Assume* **(GN1)**, **(AlphaN1)** *and that f satisfies* (1.30). *Then, there exists a unique solution $u \in W^{2,p}_{loc}\Big((-N, N) \times \overline{D}\Big)$ which satisfies*

- *$\rho(-N, y) \geq u(x_1, y) \geq \varepsilon_N$ for all $(x_1, y) \in (-N, N) \times D$,*
- *for all $x_1 \in (-N, N)$ there exists $y \in D$ such that $\rho(-N, y) > u(x_1, y)$,*

of the following problem :

$$\begin{cases} \Delta u + c\partial_1 u + \alpha(y)\nabla \cdot G(u) + f(u) = 0 & \text{in } \Omega_N, \\ u_\nu = 0 & \text{for } -N < x_1 < N, \ y \in \partial D, \\ u(-N, y) = \rho(-N, y), \ \ u(N, y) = \varepsilon_N, & y \in D. \end{cases} \tag{2.56}$$

Remark 2.5. Note that the proof will show that u is well defined in $\{-N\} \times D$ and $\{N\} \times D$, which is not trivial since $u \in W^{2,p}_{loc}\Big((-N, N) \times \overline{D}\Big)$.

Proof. The proof relies on the theory of sub and super solutions.
The function ε_N (resp. ρ) is a subsolution (resp. a supersolution) of (2.56). Indeed, one has

$$\Delta\varepsilon_N + c\partial_1\varepsilon_N + \alpha(y)\nabla\cdot G(\varepsilon_N) + f(\varepsilon_N) = f(\varepsilon_N) > 0 > \Delta\rho + c\partial_1\rho + \alpha(y)\nabla\cdot G(\rho) + f(\rho),$$

$$\frac{\partial \varepsilon_N}{\partial \nu} = \frac{\partial \rho}{\partial \nu} \quad \forall (x_1, y) \in (-N, N) \times \partial D,$$

and, since $\partial_1\rho < 0$ and (2.55) holds,

$$\rho(x_1, y) > \varepsilon_N \quad \forall (x_1, y) \in [-N, N] \times D.$$

From now on, we will denote by $\underline{u}$ and $\overline{u}$ the sub and supersolutions $\underline{u} := \varepsilon_N$, and $\overline{u} := \rho$. We will also use the classical notation

$$G_i'(u)\frac{\partial u}{\partial x_i} := \sum_{i=1}^{n} G_i'(u)\frac{\partial u}{\partial x_i} = \nabla \cdot G(u).$$

We will construct a sequence of functions $(u_j)_{j\geq 0}$ on $[-N,N]\times D$, where u_j will belong to $C\Big([-N,N]\times\overline{D}\Big)\cap W^{2,p}_{loc}\Big(\overline{\Omega_N}\setminus\{-N,N\}\times\partial D\Big)$, with $u_0 := \underline{u}$, solving the following equation :

$$\Delta u_{j+1} + c\frac{\partial u_{j+1}}{\partial x_1} + \alpha(y)G_i'(u_j)\frac{\partial u_{j+1}}{\partial x_i} - k_0 u_{j+1} = -f(u_j) - k_0 u_j, \tag{2.57}$$

where k_0 is a constant which will be chosen large enough, as well as the boundary conditions

$$\begin{cases} u_{j+1} = \rho & \text{on } \{-N\}\times D, \\ u_{j+1} = \varepsilon_N & \text{on } \{N\}\times D, \\ \frac{\partial u_{j+1}}{\partial \nu} = 0 & \text{on } (-N,N)\times\partial D. \end{cases}$$

We start with $u_0 = \underline{u}$. The [BN1, Lemma 7.1] gives the existence of $u_1 \in C\Big([-N,N]\times\overline{D}\Big)\cap W^{2,p}_{loc}\Big(\overline{\Omega_N}\setminus\{-N,N\}\times\partial D\Big)$ which satisfies

$$\Delta u_1 + c\partial_1 u_1 + \alpha(y)G_i'(u_0)\frac{\partial u_1}{\partial x_i} - k_0 u_1 = -f(u_0) - k_0 u_0,$$

with the same boundary conditions. Note that [BN1, Lemma 7.1] is applied with $a_{ij} = \delta_i^j$, $b_1(x) = c + \alpha(y)G_1'(u_0)$, $b_i(x) = \alpha(y)G_i'(u_0)$ for $2\leq i\leq n$, and $c = -k_0$. Assume that for $j\geq 1$, u_j which was constructed solving (2.57) with u_{j-1} in the right hand side, belongs to $C\Big([-N,N]\times\overline{D}\Big)$, which implies that $u_j \mapsto f(u_j)+k_0u_j \in L^\infty\Big((-N,N)\times D\Big)$. Hence, [BN1, Lemma 7.1] gives the existence of the function $u_{j+1}\in C\Big([-N,N]\times\overline{D}\Big)\cap W^{2,p}_{loc}\Big(\overline{\Omega_N}\setminus\{-N,N\}\times\partial D\Big)$ which satisfies

$$\Delta u_{j+1} + c\frac{\partial u_{j+1}}{\partial x_1} + \alpha(y)G_i'(u_j)\frac{\partial u_{j+1}}{\partial x_1} - k_0 u_{j+1} = -f(u_j) - k_0 u_j,$$

with the boundary conditions.
Now, we will show by induction that for all $j\in\mathbb{N}$, one has

$$\underline{u} \leq u_j \leq \overline{u}. \tag{2.58}$$

For $j=0$, it is trivial. Let $j\geq 0$, and assume that $\underline{u}\leq u_j\leq\overline{u}$. First, we want to prove $\underline{u}\leq u_{j+1}$. The functions u_j and $\underline{u}$ satisfy

$$\begin{cases} \Delta u_{j+1} + c\frac{\partial u_{j+1}}{\partial x_1} + \alpha(y)G_i'(u_j)\frac{\partial u_{j+1}}{\partial x_i} - k_0 u_{j+1} & = -f(u_j) - k_0 u_j, \\ \Delta\underline{u} + c\frac{\partial \underline{u}}{\partial x_1} + \alpha(y)G_i'(\underline{u})\frac{\partial \underline{u}}{\partial x_i} & \geq -f(\underline{u}). \end{cases} \tag{2.59}$$

By subtraction, one has

$$
\begin{aligned}
\Delta(\underline{u}-u_{j+1}) + c\frac{\partial}{\partial x_1}(\underline{u}-u_{j+1}) + \alpha(y)\Big(G_i'(\underline{u}) - G_i'(u_j)\Big)\frac{\partial \underline{u}}{\partial x_i} + \alpha(y)G_i'(u_j)\frac{\partial}{\partial x_i}(\underline{u}-u_{j+1}) \\
- k_0(\underline{u}-u_{j+1}) \geq -f(\underline{u}) - k_0\underline{u} + f(u_j) + k_0 u_j,
\end{aligned}
$$

which gives

$$
\begin{aligned}
\Delta(\underline{u}-u_{j+1}) &+ c\frac{\partial}{\partial x_1}(\underline{u}-u_{j+1}) + \alpha(y)G_i'(u_j)\frac{\partial}{\partial x_i}(\underline{u}-u_{j+1}) - k_0(\underline{u}-u_{j+1}) \\
&\geq f(u_j) - f(\underline{u}) + k_0(u_j-\underline{u}) + \alpha(y)\Big(G_i'(u_j) - G_i'(\underline{u})\Big)\frac{\partial \underline{u}}{\partial x_i} \\
&\geq -L(u_j-\underline{u}) + k_0(u_j-\underline{u}) - \tilde{L}C(u_j-\underline{u}) \qquad \text{since } 0<\underline{u}\leq u_j\leq \overline{u}<1, \\
&\geq 0,
\end{aligned}
$$

where $C \geq 0$ is such that

$$
|\alpha(y)| \sup_{x\in\Omega_N} \max_{1\leq i\leq n} \left\{ \left|\frac{\partial \underline{u}}{\partial x_i}(x)\right|, \left|\frac{\partial \overline{u}}{\partial x_i}(x)\right| \right\} \leq C \text{ for all } y \in D,
$$

and we choose $k_0 \geq L + \tilde{L}C$.
Analogously, the functions u_{j+1} and $\overline{u}$ satisfy

$$
\left\{
\begin{array}{ll}
\Delta u_{j+1} + c\frac{\partial u_{j+1}}{\partial x_1} + \alpha(y)G_i'(u_j)\frac{\partial u_{j+1}}{\partial x_i} - k u_{j+1} & = -f(u_j) - k u_j, \\
\Delta \overline{u} + c\frac{\partial \overline{u}}{\partial x_1} + \alpha(y)G_i'(\overline{u})\frac{\partial \overline{u}}{\partial x_i} & \leq -f(\overline{u}).
\end{array}
\right. \tag{2.60}
$$

By subtraction, the same method shows that $\Delta(\overline{u}-u_{j+1}) \leq 0$, provided that k is chosen as before. By [GIL, Theorem 9.1], the function $\underline{u}-u_{j+1}$ reaches its maximum over $[-N,N]\times\overline{D}$ on the boundary $\partial\Omega_N$. Let P be a point of $\partial([-N,N]\times D)$ where $\underline{u}-u_{j+1}$ reaches its maximum :

$$
(\underline{u}-u_{j+1})(P) = \max_{[-N,N]\times\overline{D}} (\underline{u}-u_{j+1}).
$$

If P is on the part of the boundary where the Neumann boundary condition holds, then at P, one has

$$
\frac{\partial \underline{u}}{\partial \nu} - \frac{\partial u_{j+1}}{\partial \nu} = 0,
$$

but since $\underline{u}-u_{j+1}$ is not constant, the Hopf lemma, see [PW1, Chapter 3, Theorem 7], ensures that

$$
\frac{\partial \underline{u}}{\partial \nu} - \frac{\partial u_{j+1}}{\partial \nu} > 0,
$$

which is impossible. This shows that P is on the other part of the boundary where the Dirichlet conditions hold, one has $\underline{u}-u_{j+1}=0$ if $P\in\{N\}\times D$, or $\underline{u}-u_{j+1}<0$, if $P\in\{-N\}\times D$. Hence,

$$\max_{[-N,N]\times\overline{D}}(\underline{u}-u_{j+1})\leq 0,$$

and consequently : $\underline{u}\leq u_{j+1}$. An analogous argument shows that $u_{j+1}\leq\overline{u}$, and by induction, we proved that for all $j\in\mathbb{N}$, $\underline{u}\leq u_j\leq\overline{u}$.
Thus, one has for all $j\in\mathbb{N}$,

$$\begin{cases} \underline{u}\leq u_j\leq\overline{u}, \\ \Delta u_j+c\frac{\partial u_j}{\partial x_1}+\alpha(y)G_i'(u_{j-1})\frac{\partial u_j}{\partial x_i}-ku_j=-f(u_{j-1})-ku_{j-1}. \end{cases} \tag{2.61}$$

In addition, the inequalities (2.58) imply that the functions u_j are uniformly bounded in $L^\infty\Big((-N,N)\times D\Big)$ and because of the equations (2.61) that are satisfied by u_j, the functions u_j are uniformly bounded in $W^{2,p}_{loc}\Big((-N,N)\times\overline{D}\Big)$, for all $p\in(1,+\infty)$. Consequently, by taking a diagonal subsequence, $(u_j)_{j\geq 0}$ has a subsequence that converges strongly in $C^{1,\lambda}$, for all $\lambda\in(0,1)$, on compact subsets of $(-N,N)\times\overline{D}$ to a solution u^N of (2.56). Then, for each $N>0$, one has a solution u^N defined on $(-N,N)\times D$ which satisfies $\underline{u}\leq u^N\leq\overline{u}$ and belongs to $W^{2,p}_{loc}\Big((-N,N)\times\overline{D}\Big)$.

In order to know the behaviour of the solution u^N on the boundary of $(-N,N)\times D$, we will construct a barrier function.
The equation (2.60) can be rewritten in the form

$$Lu_{j+1}:=\Delta u_{j+1}+b_i(x)\frac{\partial u_{j+1}}{\partial x_i}+du_{j+1}=g(x),$$

with

$$b_i(x)=\begin{cases} c+\alpha(y)G_1'(u_j) & \text{if } i=1 \\ \alpha(y)G_i'(u_j) & \text{if } i>1 \end{cases},$$

$d=-k$, and $g(x)=-f(u_j)-ku_j$.
Let $b\geq\sqrt{\sum_{i=1}^n b_i^2(x)}$, for all $x\in\Omega_N$. As in the proof of [BN1, Lemma 7.1], we can define a concave and positive function h on $[-N,0]$ by

$$h(x_1)=\frac{1}{b^2}e^{bN}\big(1-e^{-b(x_1+N)}\big)-\frac{1}{b}(x_1+N). \tag{2.62}$$

One has

$$h'(x_1)=\frac{1}{b}\left(e^{-bx_1}-1\right) \text{ and } h''(x_1)=-e^{-bx_1}, \text{ for } -N\leq x_1\leq 0.$$

Thus, h is a concave positive function and satisfies $Lh\leq -1$ on $[-N,0]$.
We extend h on $[0,N]$ to be symmetric. Hence, the function h is concave, symmetric and defined on $[-N,N]$. In addition, one has : $h(-N)=h(N)=0$, and

$$Lh=h''+b_1h'+dh\leq h''+bh' \text{ on } [-N,0].$$

The change of functions in the proof of [BN1, Lemma 7.1] becomes

$$u_{j+1} = v + \frac{N - x_1}{2N}\rho(-N, y) + \frac{N + x_1}{2N}\varepsilon_N.$$

It follows that the function g defined by $Lv = g$ is bounded in $L^\infty\Big((-N, N) \times D\Big)$ independently of N.
Moreover, with the definition (2.62) of the barrier function h, by the computations in the proof of [BN1, Lemma 7.1], one has on $(-N, N) \times D$

$$\left| u_{j+1}(x_1, y) - \frac{N - x_1}{2N}\rho(-N, y) - \frac{N + x_1}{2N}\varepsilon_N \right| \leq Ch(x_1), \tag{2.63}$$

with C and h independent of N. Let j tend to $+\infty$, the function u^N has to satisfy (2.63) as well. This estimate gives the continuity of u^N on $\{N\} \times D$, and then, u^N satisfies the boundary condition

$$u^N(N, y) = \varepsilon_N \quad \forall y \in D.$$

Similarly, using that $h(-N) = 0$, and the continuity of u^N on $\{-N\} \times D$, it follows that the function u^N satisfies

$$u^N(-N, y) = \rho(-N, y), \quad \forall y \in D,$$

which ensures that u^N is continuous on $\overline{\Omega_N}$. Finally, since $(u_j)_{j\geq 0}$ converges uniformly on each compact subsets of $(-N, N) \times \overline{D}$ to $u^N \in W^{2,p}_{loc}\Big((-N, N) \times \overline{D}\Big)$ as $j \to +\infty$, and since for $-N < x_1 < N$ and $y \in \partial D$, the function u_j satisfies

$$\frac{\partial u_j}{\partial \nu} = 0,$$

it follows that the limit function u^N also satisfies

$$\frac{\partial u^N}{\partial \nu} = 0, \text{ for } -N < x_1 < N \text{ and } y \in \partial D.$$

□

The uniqueness of the function u^N will be proved in the next section.

2.2 Solution on the infinite cylinder and solution with a critical speed $c^\star$

Now that we have a solution u^N of (2.56) on a truncated cylinder Ω_N, the next goal is to let N tend to infinity to get a solution on the unbounded cylinder $\Omega = \mathbb{R} \times D$. To do that, we first show that the function u^N is monotone with respect to x_1.

Proposition 2.6. *Assume* **(GN1)**, **(GN3)**, **(AlphaN1)** *and that f satisfies* (1.30). *Assume also that u is a solution of the problem* (2.56) *which satisfies*

- $\rho(-N,y) \geq u(x_1,y) \geq \varepsilon_N$ *for all* $(x_1,y) \in (-N,N) \times D$,
- *for all* $x_1 \in (-N,N)$ *there exists* $y \in D$ *such that* $\rho(-N,y) > u(x_1,y)$.

Then u is decreasing with respect to x_1, and $\partial_1 u < 0$. Moreover, the solution u of (2.56) *is unique.*

Proof. Since for all $(x_1,y) \in (-N,N) \times D$, $\rho(-N,y) \geq u(x_1,y) \geq \varepsilon_N$, and since for all $x_1 \in (-N,N)$, there exists $y \in D$ such that $\rho(-N,y) > u(x_1,y)$, we can apply [BN1, Theorem 2.4] with

$$F(x,u,Du,D^2u) = \Delta u + \Big(c + \alpha(y)G_1'(u)\partial_1 u\Big) + \alpha(y)\sum_{i=2}^{n} G_i'(u)\frac{\partial u}{\partial x_i} + f(u),$$

and we get immediately that u is decreasing with respect to x_1. Note that [BN1, Theorem 2.4] gives the uniqueness of the solution u.

□

Now, we would like to obtain a solution u on $(-\infty,+\infty) \times D$. We need to ensure that the solution we obtain is neither identically 0, nor 1.

Proof of Theorem 1.17. The structure of the proof follows that in [BN2, Section 9.1]. Precisely, we consider a translation of the function ρ introduced in (2.54), in order to have a solution on Ω_N which satisfies $\max_{y\in\overline{D}} u(0,y) = 1/2$, thus avoiding that the solution collapses when we pass to the limit. Consider for all $r \in \mathbb{R}$, and all $(x_1,y) \in \Omega_N$,

$$\begin{cases} \rho^r(x_1,y) & := \rho(x_1+r,y), \\ h^r & := \min_{y\in\overline{D}} \rho(N+r,y). \end{cases}$$

By compactness of $\overline{D}$, and continuity of ρ, $r \mapsto h^r$ exists and is continuous on $\mathbb{R}$. As before, there exists a unique function $v^r \in W^{2,p}_{loc}\Big((-N,N)\times\overline{D}\Big) \cap C\Big([-N,N]\times\overline{D}\Big)$ with $h^r \leq v^r \leq \rho^r$ in $(-N,N) \times D$, satisfying

$$\begin{cases} \Delta v^r + c\partial_1 v^r + \alpha(y)G_i'(v^r)\frac{\partial v^r}{\partial x_i} + f(v^r) = 0 & \text{on } (-N,N) \times D, \\ v^r_\nu = 0 & \text{for } -N < x_1 < N,\ y \in \partial D, \\ v^r(-N,y) = \rho^r(-N,y), \quad v^r(N,y) = h^r, & y \in D. \end{cases} \tag{2.64}$$

Indeed, ρ^r and h^r are super and subsolutions of (2.64).
Using the same arguments as in [BN2, Section 9.1], and since ρ tends to 1 (respectively 0) when x_1 tends to $-\infty$ (respectively $+\infty$), we obtain a solution u which

satisfies

$$\begin{cases} \Delta u + c\partial_1 u + \alpha(y) G_i'(u) \frac{\partial u}{\partial x_i} + f(u) = 0 & \text{on } \mathbb{R}\times D, \\ u_\nu = 0 & \text{on } \mathbb{R}\times\partial D, \\ 0 \le u \le 1, \quad \partial_1 u \le 0 & \text{in } \mathbb{R}\times\overline{D}, \\ \max_{y\in\overline{D}} u(0,y) = \frac{1}{2}. \end{cases} \tag{2.65}$$

In particular, the last condition and the fact that $f > 0$ in $(0,1)$ show that the function u is not constant. Moreover, since u is bounded and non-increasing with respect to x_1, it follows that u has finite limits $\tilde{u}$ and $u^\star$ when x_1 goes to $+\infty$ and $-\infty$. We will show that this convergence is uniform with respect to $y \in D$. Let $\Omega_1 = (-1,1)\times D$, and define $z_m : \Omega_1 \to \mathbb{R}$, by

$$z_m(x_1, y) := u(x_1 + m, y).$$

Because of the equation satisfied by u, the family $(z_m)_m$ is bounded in $W^{2,p}(\Omega_1)$. Hence, there exists a subsequence $(z_{m_k})_k$ of $(z_m)_m$ which is weakly convergent in $W^{2,p}(\Omega_1)$ and strongly in $C^{1,\lambda}(\overline{\Omega_1})$. But since for $y \in D$,

$$\lim_{x_1\to+\infty} u(x_1, y) = \tilde{u}(y),$$

it follows that $(z_{m_k})_k$ converges to $\tilde{u}$ in $C^{1,\lambda}(\overline{\Omega_1})$. Thus, all the subsequences of $(z_m)_m$ converge to $\tilde{u}$ in $C^{1,\lambda}(\overline{\Omega_1})$, which implies that $(z_m)_m$ tends to $\tilde{u}$ in $C^{1,\lambda}(\overline{\Omega_1})$ when m tends to $+\infty$, which in turn gives that u converges in $C^{1,\lambda}(\overline{D})$ to $\tilde{u}$ when $x_1 \to +\infty$. Indeed, suppose u does not converge to $\tilde{u}$ in $C^{1,\lambda}(\overline{D})$ as $x_1 \to +\infty$. Then there exist $\varepsilon > 0$ and a sequence $(b_m)_{m\in\mathbb{N}}$, which tends to $+\infty$ as $m \to +\infty$, such that

$$\|u(b_{m_k}, \cdot) - \tilde{u}(\cdot)\|_{C^{1,\lambda}(\overline{D})} \ge \varepsilon. \tag{2.66}$$

But the sequence $(z_{b_m})_{m\in\mathbb{N}}$ is bounded in $W^{2,p}(\Omega_1)$ so there is a subsequence $(z_{b_{m_k}})_{k\in\mathbb{N}}$ that is weakly convergent in $W^{2,p}(\Omega_1)$ and strongly in $C^{1,\lambda}(\overline{\Omega_1})$ to a limit that must be $\tilde{u}$, since we know that $\lim_{x_1\to+\infty} u(x_1,y) = \tilde{u}(y)$ for each $y \in D$. This implies in particular that

$$\|u(b_m, \cdot) - \tilde{u}(\cdot)\|_{C^{1,\lambda}(\overline{D})} \to 0 \quad \text{ as } m_k \to +\infty,$$

which contradicts (2.66).
A similar argument shows that u converges weakly in $W^{2,p}(\Omega_1)$ and strongly in $C^{1,\lambda}(\overline{\Omega_1})$ to $u^\star$ when $x_1 \to -\infty$.
Now, we want to show that the limits $\tilde{u}$ and $u^\star$ of u as $x_1 \to \pm\infty$, which in principle depend on y, have to satisfy the problem (1.35) on the cross section. To do that, we will use the following lemma :

Lemma 2.7. *Let u be a solution of* (2.65). *Then, for all $y \in D$,*

$$\lim_{x_1\to+\infty} \frac{\partial u}{\partial x_1}(x_1, y) = 0.$$

Proof. Let $y \in D$. Assume that the partial derivative does not tend to 0 when x_1 goes to infinity. Since $\partial u/\partial x_1 < 0$, there exists a sequence $(x_1^n)_{n\geq 0}$ which tends to infinity when n tends to infinity, and $\varepsilon > 0$ such that

$$\frac{\partial u}{\partial x_1}(x_1^n, y) < -\varepsilon.$$

Since $u \in C^{1,\lambda}(\mathbb{R} \times \overline{D})$, the partial derivative $\partial u/\partial x_1$ is uniformly continous. Hence, there exists $\delta > 0$, such that

$$\frac{\partial u}{\partial x_1}(x_1, y) < -\frac{\varepsilon}{2} \text{ for all } |x_1 - x_1^n| \leq \delta, \text{ and for all } n \geq 0.$$

But this is impossible, because if $x_1 > x_1^n + \delta$, then

$$\begin{aligned}
u(x_1, y) &= u(x_1^1 - \delta, y) + \int_{x_1^1-\delta}^{x_1} \frac{\partial u}{\partial x_1}(s, y)ds \\
&= u(x_1^1 - \delta, y) + \sum_{i=1}^{n} \int_{x_1^i-\delta}^{x_1^i+\delta} \frac{\partial u}{\partial x_1}(s, y)ds + \int_{\{s \,|\, s > x_1^1-\delta\ ;\ |s-x_1^i| \geq \delta,\ s < x_1,\ i=1,\cdots,n\}} \frac{\partial u}{\partial x_1}(s, y)ds \\
&\leq u(x_1^1 - \delta, y) + n2\delta\Big(\frac{-\varepsilon}{2}\Big), \quad \text{since } \frac{\partial u}{\partial x_1} \leq 0 \text{ and } \frac{\partial u}{\partial x_1}(s, y) < -\frac{\varepsilon}{2} \text{ if } |s - x_1^i| < \delta.
\end{aligned}$$

Here, we assumed without loss of generality that $x_1^{i+1} - x_1^i > 2\delta$, for all $i = 1, \cdots, n-1$, which means that the intervals $(x_1^i - \delta, x_1^i + \delta)$ are disjoint. The last term $-n\delta\varepsilon$ tends to $-\infty$ when $n \to +\infty$, which contradicts the fact that $u(x_1, y) \geq 0$ for all $(x_1, y) \in \mathbb{R} \times D$. □

Now, let $v \in C_0^1(D)$ be a test function and $\tilde{x_1} \in \mathbb{R}$, one has,

$$\int_{\Omega_1} v\Delta u + c\int_{\Omega_1} v\frac{\partial u}{\partial x_1} + \int_{\Omega_1} v\alpha(y)\sum_{i=1}^{n} G_i'(u)\frac{\partial u}{\partial x_i} + \int_{\Omega_1} vf(u) = 0, \tag{2.67}$$

where $\Omega_1 = (\tilde{x_1} - 1, \tilde{x_1} + 1) \times D$. First, by integration by parts, one has

$$\int_{\Omega_1} v\Delta u = \int_{\Omega_1} v\frac{\partial^2 u}{\partial x_1^2} - \int_{\Omega_1} \nabla' v \cdot \nabla' u.$$

Using Lemma 2.7, one has

$$\int_{\Omega_1} v\frac{\partial^2 u}{\partial x_1^2} = \int_D v\Big(\frac{\partial u}{\partial x_1}(\tilde{x_1} + 1, y) - \frac{\partial u}{\partial x_1}(\tilde{x_1} - 1, y)\Big) \to 0, \text{ when } \tilde{x_1} \to +\infty.$$

Since $u(x_1, \cdot)$ converges to $\tilde{u}$ in $C^{1,\lambda}(\overline{D})$, it follows that

$$\int_{\Omega_1} \nabla' v(y) \cdot \nabla' u(x_1, y)dx_1dy = \int_{D\times(-1,1)} \nabla' v(y) \cdot \nabla' u(p + \tilde{x_1}, y)dpdy \to 2\int_D \nabla' v \cdot \nabla' \tilde{u}.$$

Similarly, one has

$$\int_{\Omega_1} v\frac{\partial u}{\partial x_1} = \int_D v(y)\Big(u(\tilde{x_1}+1,y) - u(\tilde{x_1}-1,y)\Big)dy \to 0 \text{ when } \tilde{x_1} \to +\infty,$$

$$\int_{\Omega_1} v\alpha \sum_{i=1}^{n} G_i'(u)\frac{\partial u}{\partial x_i} = \int_{D\times(-1,1)} v(y)\alpha(y)\sum_{i=1}^{n} G_i'(u(p+\tilde{x_1},y))\frac{\partial u}{\partial x_i}(p+\tilde{x_1},y)dpdy,$$

which tends to

$$\int_{D\times(-1,1)} v(y)\alpha(y)\sum_{i=2}^{n} G_i'(\tilde{u}(y))\frac{\partial \tilde{u}}{\partial x_i}(y)dpdy = 2\int_D v(y)\alpha(y)\sum_{i=2}^{n} G_i'(\tilde{u}(y))\frac{\partial \tilde{u}}{\partial x_i}(y)dy,$$

when $\tilde{x_1} \to +\infty$, by Lemma 2.7 again. Analogously, when $\tilde{x_1} \to +\infty$, the last integral of (2.67) tends to

$$\int_{\Omega_1} vf(u) = \int_{D\times(-1,1)} v(y)f(u(p+\tilde{x_1},y))dpdy \to \int_{D\times(-1,1)} v(y)f(\tilde{u}(y))dpdy = 2\int_D v(y)f(\tilde{u}(y))dy.$$

Hence, by passing to the limit when $\tilde{x_1} \to +\infty$ in (2.67), the limit function $\tilde{u}$ has to satisfy :

$$\int_D v\Delta'\tilde{u} + \int_D \alpha(y)\sum_{i=2}^{n} G_i'(\tilde{u})\frac{\partial \tilde{u}}{\partial x_i}v + \int_D f(\tilde{u})v = 0. \tag{2.68}$$

Since u tends to $\tilde{u}$ in $C^1(\overline{D})$ as $x_1 \to +\infty$, it follows that $\tilde{u}$ satisfies also the boundary condition

$$\frac{\partial \tilde{u}}{\partial \nu} = 0, \text{ on } \partial D.$$

Hence, since $\tilde{u} \in W^{2,p}(D)$ for all p, the limit function $\tilde{u}$ solves the problem on the cross section (1.35).
However, we made the assumption **(AN)** that the weak form of the problem on the cross section has no other solutions than 0 and 1. Consequently, due the normalization

$$\max_{y\in\overline{D}} u(0,y) = \frac{1}{2},$$

and the fact that u is decreasing in x_1, one has :

$$\lim_{x_1\to-\infty} u(x_1,y) = 1 \text{ and } \lim_{x_1\to+\infty} u(x_1,y) = 0.$$

□

We will show the last point of Theorem 1.17, namely that w is decreasing with respect to x_1 if the function G is C^2 after proving the following result which is a more general result than Lemma 2.7 :

Proposition 2.8. *Let $c > c^\star$ and let w the solution of* (1.32) *constructed in Proposition* (1.17). *Then,*

$$\lim_{x_1 \to \pm\infty} \|\nabla w(x_1, \cdot)\|_{L^\infty(D)} = 0.$$

Proof. By construction, we know that $w \in W^{2,p}(\Omega_1)$, where $\Omega_1 = (-1,1) \times D$. Now, consider all the possible translations of w in the x_1 direction, and define $z_N : \Omega_1 \to \mathbb{R}$, by

$$z_N(x_1, y) = w(x_1 + N, y).$$

The family $\{z_N\}_N$ is bounded [1] in $W^{2,p}(\Omega_1)$ by [YIH, Theorem A.26] and the fact that there exists $C > 0$ such that $\|w\|_{L^p(\Omega_2)} \le C$ and $\|f(w)\|_{L^p(\Omega_2)} \le C$. Hence there exists a subsequence (z_{N_j}) of $(z_N)_N$ which is weakly convergent in $W^{2,p}(\Omega_1)$ and strongly in $C^{1,\lambda}(\overline{\Omega_1})$. But since w tends uniformly to 0 (with respect to y) when x_1 goes to infinity, the limit has to be 0. Hence, all the subsequences of (z_N) converge to 0, in $C^{1,\lambda}$ which implies that z_N tends to 0 when N goes to infinity.
Thus, one has

$$\lim_{x_1 \to +\infty} \|w(x_1, \cdot)\|_{C^{1,\lambda}(\overline{D})} = 0. \tag{2.69}$$

The proof for $x_1 \to -\infty$ is very similar. Indeed, since w tends uniformly (with respect to y) to 1 when $x_1 \to -\infty$, it follows that all the subsequences of (z_N) converge to 1 in $C^{1,\lambda}$, which implies that z_N tends to 1 when $N \to -\infty$. Hence,

$$\lim_{x_1 \to -\infty} \|w(x_1, \cdot) - 1\|_{C^{1,\lambda}(\overline{D})} = 0, \tag{2.70}$$

[1] In fact, the estimate of [YIH, Theorem A.26] (or [ADN, Theorem 15.2]) is valid on any domain of the form $(-1,1) \times D'$ where $D' \subset\subset D$. The estimate of [YIH, Theorem A.26] up to the boundary on $\Omega_1 = (-1,1) \times D$ follows from an adaptation of the proof of [ADN, Theorem 15.2], taking into account the [ADN, Remark (a) - Section 14 - Chapter 5], saying that "we can obtain estimates near the boundary for solutions satisfying boundary conditions on merely a portion of the boundary". Indeed, with respect to the notations in [ADN], we write :

$$Lu = \Delta u + b_1 \frac{\partial u}{\partial x_1} + \sum_{i=2}^{n} b_i \frac{\partial u}{\partial x_i} = F,$$

where $b_1 := c + \alpha G_1'(u)$, for $2 \le i \le n$, $b_i := \alpha G_i'(u)$ and $F := -f(u)$, and where the boundary conditions $B_j u = \phi_j$ is only imposed on a part Γ of the boundary. We use for the norm of the boundary data :

$$\|\phi_j\|_{l-m_j-\frac{1}{p}} = \inf_{\{v \in H_{j,L^p},\, v=\phi_j \text{ on } \Gamma\}} \|v\|_{l-m_j,L^p}.$$

Under Neumann boundary conditions, one has $\frac{\partial u}{\partial \nu} = 0 := \phi$. Hence, using the fact that for all $1 \le i \le n$, the functions b_i are bounded and continuous, the proof of [ADN, Theorem 15.2] gives that there exists $C_1 > 0$ such that :

$$\|u\|_{W^{2,p}(\Omega_1)} \le C_1 \Big(\|u\|_{L^p(\Omega_2)} + \|f(u)\|_{L^p(\Omega_2)} \Big)$$

on a slightly bigger domain Ω_2. Moreover, both $\|u\|_{L^p(\Omega_2)}$ and $\|f(u)\|_{L^p(\Omega_2)}$ are bounded by a constant $C_2 > 0$ which depends on the size of Ω_2 and $\|u\|_{L^\infty} \le 1$.

and the conclusion follows.

$\square$

We now prove the last point of Theorem 1.17, namely that the solution constructed in the proof of Theorem 1.17 is decreasing with respect to x_1.

End of the proof of Theorem 1.17. Recall that we assume that the function G is C^2, and let w the solution of (1.32) we have constructed in the proof of Theorem 1.17. Since the function w belongs to $W^{2,p}_{loc}$, it follows from the equation (1.32) satisfied by w, bootstrapping and by standard regularity results, see [KRY, Chapter 9 - Section 4 - Theorem 1], and hence, we can differentiate the equation (1.32) satisfied by w with respect to x_1 to obtain

$$\Delta\Big(\frac{\partial w}{\partial x_1}\Big)+c\frac{\partial}{\partial x_1}\frac{\partial w}{\partial x_1}+\alpha(y)\sum_{i=1}^{n}G_i''(w)\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_1}+\alpha(y)\sum_{i=1}^{n}G_i'(w)\frac{\partial}{\partial x_i}\frac{\partial w}{\partial x_1}+f'(w)\frac{\partial w}{\partial x_1}=0. \tag{2.71}$$

It follows that $v(x,t) := -\frac{\partial}{\partial x_1}w(x_1-ct,y)$ solves

$$\begin{aligned} v_t &= \Delta v + \alpha(y)\sum_{i=1}^{n}G_i''(w)\frac{\partial w}{\partial x_i}v + \alpha(y)\sum_{i=1}^{n}G_i'(w)\frac{\partial v}{\partial x_i} + f'(w)v \\ &= \Delta v + b(x,t)\cdot\nabla v + c(x,t)v, \end{aligned}$$

where for all $1 \le i \le n$,

$$b_i(x,t) := \alpha(y)G_i'\Big(w(x_1-ct,y)\Big)$$

and

$$c(x,t) := \alpha(y)\sum_{i=1}^{n}G_i''\Big(w(x_1-ct,y)\Big)\frac{\partial w}{\partial x_i} + f'\Big(w(x_1-ct,y)\Big).$$

By construction, we know that $v(x,t) \ge 0$ for all $x \in \Omega$, $t \ge 0$. Using that $\|\nabla w(x_1,\cdot)\|_{L^\infty(D)} \to 0$ as $x_1 \to +\infty$ by Proposition 2.8, it follows that

$$c(x,0) := \alpha(y)\sum_{i=1}^{n}G_i''\Big(w(x_1,y)\Big)\frac{\partial w}{\partial x_i} + f'\Big(w(x_1,y)\Big) \not\equiv 0,$$

since $c(x,0) \to f'(0)$ as $x_1 \to +\infty$, uniformly with $y \in D$. By [VO1, Chapter 2 - Theorem 3.26], it follows that either $v \equiv 0$ or $v > 0$. Since w converges to different limits as $x_1 \to \pm\infty$, it follows that $v \not\equiv 0$, and hence $v > 0$, meaning that $\frac{\partial w}{\partial x_1} < 0$ for all $x \in \Omega$ and all $t \ge 0$, and that the constructed solution w is decreasing.

$\square$

Now that we know that if $c > c^\star > -\infty$, there exists a solution of (1.32), we will show that there exists a solution of (1.32) with a speed $c = c^\star$, and for $c < c^\star$, such solutions do not exist.

Proposition 2.9. *Suppose that $c^\star \in \mathbb{R}$, and assume* **(GN1)**, **(GN2)**, **(GN3)**, **(AlphaN1)**, **(AN)** *and that f is C^1 and satisfies* (1.30). *Then, for $c = c^\star$, there exists a decreasing solution (with respect to x_1) of* (1.32), *and there is no decreasing solution of* (1.32) *if $c < c^\star$.*

Proof. Let $m > 0$, and consider the family of solutions $(u_m)_{m>0}$ of (1.32) with speed $c_m := c^\star + 1/m$. We argue as in the proof of Proposition 1.17, using a diagonal argument, weak convergence in $W^{2,p}(\Omega_1)$ and then $C^{1,\lambda}(\overline{\Omega_1})$ convergence, there exists a subsequence of $(u_m)_{m>0}$ which converges to $\tilde{u}$ satisfying

$$\begin{cases} \Delta\tilde{u} + c^\star\partial_1\tilde{u} + \alpha(y)G_i'(\tilde{u})\frac{\partial\tilde{u}}{\partial x_i} + f(\tilde{u}) = 0 & \text{on } \mathbb{R}\times D, \\ \tilde{u}_\nu = 0 & \text{on } \mathbb{R}\times\partial D, \\ 0 \le \tilde{u} \le 1, \quad \partial_1\tilde{u} \le 0 & \text{in } \mathbb{R}\times\overline{D}, \\ \max_{y\in\overline{D}}\tilde{u}(0,y) = \frac{1}{2}. \end{cases} \tag{2.72}$$

Furthermore, $\tilde{u}$ is bounded and x_1-decreasing and then has finite limits when $x_1 \to \pm\infty$. Since $\partial\tilde{u}/\partial x_1$ tends to 0 when x_1 tends to infinity, the limit has to satisfy (1.35). Under assumption **(AN)**, the normalization $\max_{y\in\overline{D}}\tilde{u}(0,y) = \frac{1}{2}$, and that $\partial_1\tilde{u} \le 0$, it follows

$$\lim_{x_1\to-\infty}\tilde{u}(x_1,y) = 1 \text{ and } \lim_{x_1\to+\infty}\tilde{u}(x_1,y) = 0.$$

Thus, the function $\tilde{u}$ is a solution of (1.32) with speed $c^\star$.
Let $c < c^\star$, and assume that there exists a solution u of (1.32), decreasing in x_1, with a speed c. Then as argued previously, we must have $\frac{\partial u}{\partial x_1} < 0$, so the function u belongs to the set K defined in (2.47), and hence $c^\star > \inf_{\rho\in K}\sup_{x\in\Omega} r(\rho)(x)$, which is a contradiction. □

3 Lower bound for $c^\star$ for Neumann boundary conditions

Without any convection term, i.e when $G = 0$, it was shown in [BN2, Theorem 1.5 and Section 10], that if f satisfies the KPP condition (1.31) then $c^\star$ is determined by an eigenvalue problem related to the linearized equation around 0. In dimension 1, an explicit formula exists for $c^\star$ which only depends on f, namely,

$$c^\star = 2\sqrt{f'(0)}.$$

In our situation, we will show that that $c^\star$ is bounded from below by a quantity related to the eigenvalue problem (3.74), under some more general conditions involving both f and G. To do that, we will follow the approach of [BN2] but with some modifications due to the presence of the convection term.

3.1 Associated linearized operator and eigenvalue problem

In [BN2], Berestycki and Nirenberg consider the linearized problem around 0 and show that there exists a critical value γ such that a certain eigenvalue problem has two positive eigenvalues if $c > \gamma$. They then prove that under the KPP condition (1.31) on f, $c^\star = \gamma$. We will follow this method and prove that with a convection term in the equation, there still exists a critical value that we will call c'. To do that, consider the linearized system of (1.32) around 0

$$\begin{cases} \Delta w + \Big(c + \alpha(y)G_1'(0)\Big)\partial_1 w + \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial w}{\partial x_i} + f'(0)w = 0 & \text{in } \mathbb{R}\times D, \\ w_\nu = 0 & \text{on } \mathbb{R}\times\partial D. \end{cases} \tag{3.73}$$

If $w(x_1, y) = e^{-\lambda x_1}\varphi(y)$, the function φ has to satisfy the following problem :

$$\begin{cases} -\Delta'\varphi - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial \varphi}{\partial x_i} - f'(0)\varphi = \Big(\lambda^2 - \lambda(c + \alpha(y)G_1'(0))\Big)\varphi & \text{in } D, \\ \varphi_\nu = 0 & \text{on } \partial D. \end{cases} \tag{3.74}$$

We say that λ is a principal eigenvalue of (3.74), if there exists a positive function φ such that (3.74) holds.

Consider now the following eigenvalue problem

$$\begin{cases} -\Delta'\sigma - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial \sigma}{\partial x_i} - f'(0)\sigma = \mu_1\sigma & \text{in } D, \\ \sigma_\nu = 0 & \text{on } \partial D. \end{cases} \tag{3.75}$$

By [YIH, Theorem 1.3], problem (3.75) has a simple eigenvalue $\mu_1 \in \mathbb{R}$, which corresponds to a positive eigenfunction. However, we can not deduce immediately that the eigenvalue problem (3.74) has an eigenvalue λ because the right-hand side depends on y. This is why we first need to prove a continuity property.

Proposition 3.1. *Let $\omega \subset \mathbb{R}^n$ be a domain of class C^2, and*

$$L := a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x)$$

be a uniformly strongly elliptic operator with $a_{ij} = a_{ji} \in C(\omega)$, b_i and $c \in L^\infty(\omega)$ for all $1 \le i, j \le n$. Define for $p > n$:

$$\gamma_1 := \sup\Big\{\gamma \mid \exists\phi \in W^{2,p}_{loc}(\omega), \phi > 0 \text{ in } \omega,\ \phi_v = 0 \text{ on } \partial\omega,\ (L+\gamma)\phi \le 0\Big\}.$$

1. *With this definition of γ_1, one has $\gamma_1 = \sigma^\omega(-L)$, where $\sigma^\omega(-L)$ is defined in [YIH, Theorem 1.3] as the only eigenvalue which corresponds to a positive eigenfunction. Moreover, this eigenvalue is real.*

2. *The function $c \mapsto \gamma_1(c)$ is Lipschitz continuous, with Lipschitz constant* 1.

3. *The function $c \mapsto \gamma_1(c)$ is concave.*

Remark : This definition of γ_1 is analogous to the definition of the principal eigenvalue in [BNV] for the Dirichlet boundary conditions.

Proof. 1. Let φ be the principal eigenfunction, given by [YIH, Theorem 1.3], of the operator $-L$. The function φ satisfies

$$\begin{cases} -L\varphi = \sigma^\omega(-L)\varphi & \text{in } \omega, \\ \varphi_\nu = 0 & \text{on } \partial\omega. \\ \varphi > 0 & \text{in } \omega. \end{cases} \tag{3.76}$$

Hence, the definition of γ_1 as a supremum yields

$$\gamma_1 \geq \sigma^\omega(-L).$$

Suppose now that $\gamma_1 > \sigma^\omega(-L)$. Then, there exists a positive function $\psi_0 \in W^{2,p}_{loc}(\omega)$, $(\psi_0)_\nu = 0$ on $\partial\omega$, and $\gamma_0 \in (\sigma^\omega(-L), \gamma_1)$, such that $-L\psi_0 \geq \gamma_0\psi_0$, which implies

$$\Big(- L - \sigma^\omega(-L)\Big)\psi_0 \geq \Big(\gamma_0 - \sigma^\omega(-L)\Big)\psi_0 > 0.$$

In other words, the function ψ_0 is a strict supersolution. Furthermore, [AML, Theorem 2.4] yields $\sigma^\omega\Big(-L-\sigma^\omega(-L)\Big) > 0$. It follows that $\sigma^\omega(-L) > \sigma^\omega(-L)$, which is impossible. Thus, one has the equality

$$\gamma_1 = \sigma^\omega(-L).$$

2. Let $\gamma_1(c)$ be the principal eigenvalue $\sigma^\omega(-L)$ of $-L$ and φ be the corresponding eigenfunction. The function φ is positive in ω and satisfies

$$\begin{cases} \sum_{ij} a_{ij}\partial_i\partial_j\varphi + \sum_i b_i\partial_i\varphi + (c+\gamma_1(c))\varphi = 0 & \text{in } \omega, \\ \varphi_\nu = 0 & \text{on } \partial\omega. \end{cases} \tag{3.77}$$

Hence, one has, for $\tilde{c} \in L^\infty(\omega)$,

$$\sum_{ij} a_{ij}\partial_i\partial_j\varphi + \sum_i b_i\partial_i\varphi + \Big(\tilde{c} + \gamma_1(c) - \|\tilde{c}-c\|_\infty\Big)\varphi = \Big(\tilde{c} - c - \|\tilde{c}-c\|_\infty\Big)\varphi \leq 0,$$

which implies that $\gamma_1(\tilde{c}) \geq \gamma_1(c) - \|\tilde{c}-c\|_\infty$, and hence,

$$\gamma_1(c) - \gamma_1(\tilde{c}) \leq \|\tilde{c}-c\|_\infty.$$

Exchanging the roles of c and $\tilde{c}$ yields the conclusion of 2.

3. The concavity follows from the proof of [BNV, Proposition 2.1], which establishes concavity in the case of Dirichlet boundary conditions. This proof adapts to our situation and does not depend on the boundary conditions.

□

Now, one can show the existence of an eigenvalue λ of (3.74), with $\lambda \in \mathbb{R}$.

Proposition 3.2. *Assume* **(GN1)** *and* **(AlphaN1)**. *Then the principal eigenvalue* μ_1 *of* (3.75) *is negative :*

$$\mu_1 = -f'(0). \tag{3.78}$$

Proof. By [YIH, Theorem 1.3], the eigenvalue problem (3.75) has a simple eigenvalue $\mu_1 \in \mathbb{R}$ which corresponds to a positive eigenfunction φ. In addition, none of the other eigenvalues corresponds to a positive eigenfunction. Denote by $-L$ the operator

$$-L = -\Delta' - \alpha(y) \sum_{i=2}^{n} G_i'(0) \frac{\partial}{\partial x_i} - f'(0).$$

If ε is a positive constant, one has $-L\varepsilon = -f'(0)\varepsilon$. In other words, $-f'(0)$ is an eigenvalue of (3.75) which corresponds to a positive eigenfunction ε. By uniqueness, it follows that

$$-f'(0) = \mu_1,$$

and in particular, $\mu_1 < 0$.

□

For each $t \in \mathbb{R}$, let $\mu_1^c(t)$ denote the principal eigenvalue of the operator

$$-\Delta' - \alpha(y) \sum_{i=2}^{n} G_i'(0) \frac{\partial}{\partial x_i} - f'(0) + t\beta_c(y),$$

with Neumann boundary conditions, where

$$\beta_c(y) := c + \alpha(y) G_1'(0).$$

By [YIH, Theorem 1.3], this principal eigenvalue $\mu_1^c(t)$ is characterized by the existence of a unique $\varphi = \varphi(t) \in W^{1,2}(D)$, such that $\varphi(t)(y) > 0$, for all $y \in \overline{D}$, satisfying :

$$\begin{cases} -\Delta'\varphi - \alpha(y) \sum_{i=2}^{n} G_i'(0) \frac{\partial \varphi}{\partial x_i} - f'(0)\varphi + t\beta_c(y)\varphi = \mu_1^c(t)\varphi & \text{in } D, \\ \varphi_\nu = 0 & \text{on } \partial D. \end{cases} \tag{3.79}$$

Note that φ is bounded in D by elliptic estimates, see [YIH, Theorem A.29]. With this notation, $\lambda \in \mathbb{R}$ is an eigenvalue of (3.74) if and only if

$$\lambda^2 = \mu_1^c(\lambda).$$

Indeed, if $\lambda \in \mathbb{R}$ is an eigenvalue of (3.74), then there exists a positive function φ in D such that the couple (λ, φ) satisfies (3.74). But then, the couple (λ, φ) satisfies also (3.79) with $(t, \mu_1^c(t))$ replaced by (λ, λ^2). Using uniqueness of φ (by [YIH, Theorem 1.3], it follows that $\mu_1^c(\lambda) = \lambda^2$.
On the other hand, if $\lambda^2 = \mu_1^c(\lambda)$, then there exists a positive function ϕ in D which satisfies

$$\begin{cases} -\Delta'\phi - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial\phi}{\partial x_i} - f'(0)\phi = \Big(\lambda^2 - \lambda\beta_c(y)\Big)\phi & \text{in } D, \\ \phi_\nu = 0 & \text{on } \partial D, \end{cases} \tag{3.80}$$

which means that λ is an eigenvalue of (3.74). Once again, according to [YIH, Theorem 1.3], since none of the other eigenvalues corresponds to a positive eigenfunction, it follows that λ is a principal eigenvalue of (3.74) if and only if $\mu_1^c(\lambda) = \lambda^2$; in other words, if and only if λ is a root of the equation $\mu_1^c(t) = t^2$.

Proposition 3.3. *The eigenvalue $\mu_1^c(t)$ of problem* (3.79) *is concave with respect to $t \in \mathbb{R}$.*

Proof. Denote $\mu_1\Big(-f'(0) + t\beta(y)\Big) := \mu_1^c(t)$ where $\beta_c(y) = c + \alpha(y)G_1'(0)$. Let t_1 and $t_2 \in \mathbb{R}$, and $\gamma \in (0,1)$. Using the concavity of $s \mapsto \gamma_1(s)$, as in Proposition 3.1, one has

$$\begin{aligned} \mu_1^c\Big(\gamma t_1 + (1-\gamma)t_2\Big) &= \mu_1\Big(-f'(0) + \Big(\gamma t_1 + (1-\gamma)t_2\Big)\beta(y)\Big) \\ &= \mu_1\Big(\gamma\Big(-f'(0) + t_1\beta(y)\Big) + (1-\gamma)\Big(-f'(0) + t_2\beta(y)\Big)\Big) \\ &\geq \gamma\mu_1\Big(-f'(0) + t_1\beta(y)\Big) + (1-\gamma)\mu_1\Big(-f'(0) + t_2\beta(y)\Big) \\ &= \gamma\mu_1^c(t_1) + (1-\gamma)\mu_1^c(t_2). \end{aligned}$$

□

Now we will prove that there exist two critical values c' and $\hat{c}$, such that if $c > c'$, the eigenvalue problem (3.74) has exactly two positive eigenvalues which we will denote $0 < \lambda_1(c) < \lambda_2(c)$. Note that by concavity of $t \mapsto \mu_1^c(t)$, the equation $\mu_1^c(t) = t^2$ admits at most two roots.

Proposition 3.4. *Assume* **(GN1)**, **(AlphaN1)** *and* (1.30) *for f. Let $g_c(t) = \mu_1^c(t) - t^2$. Then there exist $\hat{c} < c'$ such that*

$$\begin{aligned} c < \hat{c} \quad &\Rightarrow \quad g_c(t) = 0 \textit{ has exactly 2 negative solutions} \\ c = \hat{c} \quad &\Rightarrow \quad g_c(t) = 0 \textit{ has exactly 1 negative solution} \\ \hat{c} < c < c' \quad &\Rightarrow \quad g_c(t) = 0 \textit{ has no solutions} \\ c = c' \quad &\Rightarrow \quad g_c(t) = 0 \textit{ has exactly 1 positive solution} \\ c' < c \quad &\Rightarrow \quad g_c(t) = 0 \textit{ has exactly 2 positive solutions.} \end{aligned}$$

The number of roots of the equation $g_c(t) = 0$ corresponds to the number of principal eigenvalue(s) of (3.74).

Proof. Consider the t-dependent eigenvalue problem (3.79). For each $c \in \mathbb{R}$, we know that $t \mapsto \mu_1^c(t)$ is continuous and concave by Proposition 3.1 (3) . Thanks to Proposition 3.1 (1), we will use the following characterisation of $\mu_1^c(t)$:

$$\mu_1^c(t) = \sup\Big\{\mu^c(t) \mid \exists \phi \in W^{2,p}_{loc}(D),\ \phi > 0 \text{ in } D,\ \phi_v = 0 \text{ on } \partial D,\ (L_1 + \mu^c(t))\phi \leq 0\Big\}, \tag{3.81}$$

where

$$-L_1 = -\Delta' - \alpha(y)\sum_{i=2}^{n} G_i'(0)\frac{\partial}{\partial x_i} - f'(0) + t\beta_c(y). \tag{3.82}$$

First, note that for $t > 0$, the function $c \mapsto \mu_1^c(t)$ is increasing. Indeed, let $\tilde{c} > c$ and denote by ϕ the eigenfunction corresponding to $\mu_1^c(t)$. One has :

$$\begin{aligned} -\Delta'\phi - \alpha(y)\sum_{i=2}^{n} G_i'(0)\frac{\partial \phi}{\partial x_i} - f'(0)\phi + t(\tilde{c} + \alpha(y)G_1'(0))\phi &= \mu_1^c(t)\phi + t(\tilde{c} - c)\phi \\ &= \Big(\mu_1^c(t) + t(\tilde{c} - c)\Big)\phi. \end{aligned}$$

Hence, by using the characterisation (3.81), it follows that for $t > 0$,

$$\mu_1^{\tilde{c}}(t) \geq \mu_1^c(t) + t(\tilde{c} - c).$$

In particular, $\mu_1^c(t)$ is increasing with respect to c.

Now, let ε be a positive constant, $k := \inf_{y \in D} \alpha(y)G_1'(0)$ and $K := \sup_{y \in D} \alpha(y)G_1'(0)$.
If $t > 0$, then

$$\begin{aligned} -L_1\varepsilon = \Big(-f'(0) + t\beta_c(y)\Big)\varepsilon &= \Big(-f'(0) + t(\alpha(y)G_1'(0) + c)\Big)\varepsilon \\ &\geq \Big(-f'(0) + t(k + c)\Big)\varepsilon. \end{aligned}$$

By definition of $\mu_1^c(t)$, it follows that for $t > 0$,

$$\mu_1^c(t) \geq -f'(0) + t(k + c).$$

We can deduce from this inequality that for $t > 0$,

$$\lim_{c \to +\infty} \mu_1^c(t) = +\infty.$$

If $t < 0$, then

$$\begin{aligned}-L_1\varepsilon = \Big(-f'(0) + t\beta_c(y)\Big)\varepsilon &= \Big(-f'(0) + t(\alpha(y)G_1'(0) + c)\Big)\varepsilon \\ &\geq \Big(-f'(0) + t(K + c)\Big)\varepsilon.\end{aligned}$$

The same argument then yields that for $t < 0$,

$$\lim_{c\to-\infty} \mu_1^c(t) = +\infty.$$

First, note that $g_c(t) \to -\infty$ when $t \to +\infty$. Indeed, since for $t > 0$, $\mu_1^c(t) \leq -f'(0) + t(K + c)$, it follows that $g_c(t) = \mu_1^c(t) - t^2 \leq -f'(0) + t(K + c) - t^2$, which tends to $-\infty$ when $t \to +\infty$.
Similarly, since for $t < 0$, $\mu_1^c(t) \leq -f'(0)+t(k+c)$, it follows that $g_c(t) = \mu_1^c(t)-t^2 \leq -f'(0) + t(k + c) - t^2$, which tends to $-\infty$ when $t \to -\infty$.
Now, since $\mu_1^c(t)$ is a strictly increasing function of c for each $t > 0$, $g_c(0) = \mu_1^c(0) - 0^2 = \mu_1^c(0) = -f'(0) < 0$ and $g_c(t) \to -\infty$ as $t \to +\infty$ for each c, it follows that if $g_c(t) = 0$ has a positive solution t_0 and $c_1 > c$, then $g_{c_1}(t) = 0$ has exactly two positive solutions since $\mu_1^c(t)$ is concave function of t which implies that g_c is also concave, and $g_{c_1}(t_0) > 0$. Likewise, $\mu_1^c(t)$ is a strictly decreasing function of c for each $t < 0$, so if $g_c(t) = 0$ has a negative solution t_0 and $c_1 < c$, then $g_{c_1}(t) = 0$ has exactly two negative solutions since μ_1^c is concave, $g_{c_1}(t_0) > 0$ and $g_{c_1}(t) \to -\infty$ as $t \to -\infty$. Also, for each $c \in \mathbb{R}$, $g_c(t)$ is a strictly concave function of t and $g_c(0) < 0$, so $g_c(t) = 0$ cannot have both a negative and a positive solution, and we know that $g_c(t) = 0$ has two negative solutions when c is sufficiently negative, and two positive solutions when c is sufficiently positive, because $\lim_{c\to+\infty} \mu_1^c(t) = +\infty$ for $t > 0$, and the analogous fact for $t < 0$.

Now define

$$c' := \inf\Big\{c : g_c(t) = 0 \ \text{ has 2 positive solutions}\Big\},$$

$$\hat{c} := \sup\Big\{\tau : g_c(t) = 0 \ \text{ has 2 negative solutions}\Big\}.$$

Then it follows from the properties of g above that $c' \in \mathbb{R}$, and for each $c > c'$, $g_c(t) = 0$ has 2 positive solutions, whereas for each $c < c'$, $g_c(t) = 0$ has no positive solutions, and $g_{c'}(t) = 0$ has exactly one positive solution. Indeed, if $t \geq 0$, $g_c(t) < 0$ for all $c < c'$, so $g_{c'}(t) \leq 0$, so $g_{c'}$ vanishes at most once by concavity. If $g_{c'}(t) < 0$ for all $t \geq 0$, then $\max_{[0,\infty)} g_{c'} < 0$ and this remains true for $c > c'$ close enough, which is false.

Likewise $\hat{c} \in \mathbb{R}$, and for each $c < \hat{c}$, $g_c(t) = 0$ has 2 negative solutions, whereas for each $c > \hat{c}$, $g_c(t) = 0$ has no negative solutions, and $g_{\hat{c}}(t) = 0$ has exactly one negative solution.

Moreover $\hat{c} < c'$, since otherwise there would exist some c such that $g_c(t) = 0$ has both positive and negative solutions, which is impossible.

□

Before comparing the two critical values $c^\star$ and c', we show the monotonicity of $c \mapsto \lambda_1(c)$ and $c \mapsto \lambda_2(c)$.

Proposition 3.5. *Let $c > c'$. Then the functions $c \mapsto \lambda_1(c)$ and $c \mapsto \lambda_2(c)$ are respectively decreasing and increasing.*

Proof. Recall that if $c > c'$, then there exist $0 < \lambda_1(c) < \lambda_2(c)$, φ_1 and φ_2 positive in D such that for $j = 1, 2$:

$$\begin{cases} -\Delta'\varphi_j - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial \varphi_j}{\partial x_i} - f'(0)\varphi_j = \Big(\lambda_j^2 - \lambda_j(c + \alpha(y)G_1'(0))\Big)\varphi_j & \text{in } D, \\ \frac{\partial \varphi_j}{\partial \nu} = 0 & \text{on } \partial D. \end{cases} \tag{3.83}$$

Since $\mu_1^c(t)$ is increasing with respect to c, (see the proof of Proposition 3.4), one has for $t = \lambda_2(c)$:

$$\mu_1^{\tilde{c}}(\lambda_2(c)) > \lambda_2^2(c).$$

Hence, it follows that $\lambda_2(\tilde{c}) > \lambda_2(c)$. A similar argument shows that $c \mapsto \lambda_1(c)$ is decreasing for $t > 0$ and $c > c'$. □

3.2 Comparison between $c^\star$ and c'

Now, we want to compare c' and $c^\star$. Precisely, we will show that under some assumptions on G and f, $c^\star \geq c'$, with equality in certain special cases. To do that, we first need to study the asymptotic behaviour of solutions of (1.32). We obtain an exponential asymptotic behaviour under a condition involving f, G and α, see (3.84).

Proposition 3.6. *Assume* **(GN1)**, **(AlphaN1)** *and* **(AlphaN2)** *and let w be a solution of* (1.32). *Assume also that*

- *there exist $C > 0$ and $s_0 \in (0, 1)$ such that $|G(s)| \leq Cs$ for all $s \in (0, s_0)$.*
- *the following condition holds :*

$$f'(0) > k, \qquad \text{where } k := \sup_{(s,y)\in(0,s_0)\times\overline{D}} \left| \sum_{i=2}^n \frac{G_i(s)}{s}\frac{\partial \alpha}{\partial x_i}(y) \right|. \tag{3.84}$$

Then, there exist two positive constants C and ε such that, for all R large enough,

$$\int_R^\infty \int_D w \leq Ce^{-\varepsilon R}.$$

Remark 3.7. Note that the assumptions **(GN3)** and **(AlphaN1)** ensure that $k < +\infty$, where k is defined in (3.84).

Proof. The proof is slightly different from the one of [BN2, Lemma 3.1] because of the extra term $\alpha(y)\nabla \cdot G(w)$.
Let $N > R > 0$ and define a smooth cut-off function ξ on $\mathbb{R}$ such that $0 \leq \xi \leq 1$ and

$$\xi(x_1) := \begin{cases} 0 & \text{if } x_1 \geq N+1, \\ 1 & \text{if } R \leq x_1 \leq N, \\ 0 & \text{if } x_1 \leq R-1. \end{cases}$$

Multiplying (1.32) by ξ, integrating on $\Omega = \mathbb{R} \times D$, using Green's formula and the fact that $\alpha \equiv 0$ on ∂D, by assumption **(AlphaN2)**, it follows that

$$\int_\Omega \Big[w\xi'' - w\xi'\Big(c + \alpha(y)\frac{G_1(w)}{w}\Big) - \xi w \sum_{i=2}^n \frac{G_i(w)}{w}\frac{\partial \alpha}{\partial x_i} + \xi f(w)\Big] = 0.$$

Since we assumed (3.84), we can take $\delta > 0$ such that $(1-\delta)f'(0) > k$. Since w tends uniformly to 0 with respect to $y \in D$ when x_1 tends to $+\infty$, there exists $R > 0$ sufficiently large such that $f(w(x_1, y)) \geq (1-\delta)f'(0)w(x_1, y)$, for all $x_1 > R-1$, $y \in D$, and we obtain

$$\int_\Omega (1-\delta)f'(0)w\xi - \int_\Omega \xi w \sum_{i=2}^n \frac{G_i(w)}{w}\frac{\partial \alpha}{\partial x_i} \leq \int_\Omega w\Big[\xi'\Big(c + \alpha(y)\frac{G_1(w)}{w}\Big) - \xi''\Big]. \quad (3.85)$$

Since

$$\begin{aligned} \int_\Omega (1-\delta)f'(0)w\xi &= \int_{R-1}^R \int_D (1-\delta)f'(0)w\xi + \int_R^N \int_D (1-\delta)f'(0)w + \int_N^{N+1} \int_D (1-\delta)f'(0)w\xi \\ &\geq \int_R^N \int_D (1-\delta)f'(0)w, \end{aligned}$$

the inequality (3.85) implies

$$(1-\delta)f'(0)\int_R^N \int_D w \leq \int_\Omega w\Big[\xi'\Big(c + \alpha(y)\frac{G_1(w)}{w}\Big) - \xi''\Big] + \int_\Omega \xi w \sum_{i=2}^n \frac{G_i(w)}{w}\frac{\partial \alpha}{\partial x_i}. \quad (3.86)$$

Then, since ξ is constant on $(-\infty, R-1] \cup [R, N] \cup [N+1, +\infty)$, it follows that

$$\begin{aligned} \int_\Omega w\Big[\xi'\Big(c + \alpha(y)\frac{G_1(w)}{w}\Big) - \xi''\Big] &= \Big(\int_{R-1}^R + \int_N^{N+1}\Big)\int_D w\Big[\xi'\Big(c + \alpha(y)\frac{G_1(w)}{w}\Big) - \xi''\Big] \\ &\leq K\Big[\int_{R-1}^R \int_D w + \int_N^{N+1}\int_D w\Big], \end{aligned}$$

where K is such that

$$\left|\xi'\Big(c+\alpha(y)\frac{G_1(w)}{w}\Big)-\xi''\right|\leq K, \qquad \text{for all } y\in D \text{ and } x_1\in[R-1,R]\cup[N,N+1].$$

Similarly, since $w\to 0$ when $x_1\to+\infty$ uniformly in y, the last integral of (3.86) satisfies for R large enough :

$$\int_\Omega \xi w\sum_{i=2}^n \frac{G_i(w)}{w}\frac{\partial\alpha}{\partial x_i}\leq k\Big(\int_{R-1}^R\int_D w+\int_R^N\int_D w+\int_N^{N+1}\int_D w\Big).$$

Finally, the inequality (3.86) gives

$$\Big((1-\delta)f'(0)-k\Big)\int_R^N\int_D w\leq (K+k)\Big(\int_{R-1}^R\int_D w+\int_N^{N+1}\int_D w\Big). \tag{3.87}$$

Since w tends to 0 uniformly in y when x_1 tends to $+\infty$, it follows that when $N\to+\infty$,

$$\int_N^{N+1}\int_D w\to 0.$$

Then (3.87) yields that $w\in L^1\Big([R,+\infty)\times D\Big)$ by the monotone convergence theorem. Now, letting $N\to+\infty$ in (3.87), we obtain

$$\Big((1-\delta)f'(0)-k\Big)\int_R^\infty\int_D w\leq (K+k)\int_{R-1}^R\int_D w. \tag{3.88}$$

Let

$$g(R):=\int_R^\infty\int_D w.$$

From inequality (3.88) it follows that

$$g(R)\leq\frac{g(R-1)}{1+a}, \qquad \text{where } a:=\frac{(1-\delta)f'(0)-k}{K+k}>0.$$

As a consequence, there exist $C>0$ and $\varepsilon>0$ such that for all $R>0$,

$$g(R)\leq Ce^{-\varepsilon R}.$$

□

Thanks to Proposition 3.6, we are able to prove the following theorem :

Theorem 3.8. *Under the same assumptions as Proposition 3.6, there exist two positive contants C_0 and ε such that the solution w of* (1.32) *satisfies*

$$w(x_1,y)+|\nabla w(x_1,y)|\leq C_0e^{-\varepsilon x_1}, \quad \textit{for all } x_1>0 \textit{ and } y\in D. \tag{3.89}$$

Proof. Let $x_1 > 1$, $\Omega_1 := (x_1 - 1, x_1 + 1) \times D$ and $\Omega_2 := (x_1 - 2, x_1 + 2) \times D$. By embedding and then using [YIH, Theorem A.26] [2], there exist two positive constants C_1 and C_2 independent of x_1 such that

$$\|w\|_{C^{1,\lambda}(\overline{\Omega_1})} \leq C_1 \|w\|_{W^{2,p}(\Omega_1)} \leq C_2 \Big(\|w\|_{L^p(\Omega_2)} + \|f(w)\|_{L^p(\Omega_2)} \Big),$$

where $p > n$ is fixed.

Since there exists $C_3 > 0$ such that $|f(w)| \leq C_3 w$ for all $w \in (0,1)$, it follows that $\|f(w)\|_{L^p(\Omega_2)} \leq C_3 \|w\|_{L^p(\Omega_2)}$. Hence, there exists $K > 0$ such that

$$\|w\|_{C^{1,\lambda}(\overline{\Omega_1})} \leq K \|w\|_{L^p(\Omega_2)}.$$

Since $0 < w < 1$, and according to Proposition 3.6 there exists $C' > 0$ such that,

$$\begin{aligned} w(x_1, y) + |\nabla w(x_1, y)| &\leq K \Big(\int_{x_1-2}^{x_1+2} \int_D w^p \Big)^{1/p} \\ &\leq K \Big(\int_{x_1-2}^{x_1+2} \int_D w \Big)^{1/p} \\ &\leq KC' \Big(e^{-\varepsilon x_1} \Big)^{1/p} \\ &= C_0 e^{-\frac{\varepsilon}{p} x_1}, \end{aligned}$$

with $C_0 := KC'$. □

Now we will prove Theorem 1.18. To do this, we will show that if there exists a travelling front solution w of (1.32) with speed $c > c^\star$, then there exists at least one real eigenvalue λ of (3.74), which implies that $c \geq c'$.

Proof of Theorem 1.18. Let $c > c^\star$. By definition of $c^\star$ and Theorem 1.17, there exists a decreasing solution w of

$$\begin{cases} \Delta w + \Big(c + \alpha(y) G_1'(w) \Big) \frac{\partial w}{\partial x_1} + \alpha(y) \sum_{i=2}^n G_i'(w) \frac{\partial w}{\partial x_i} + f(w) = 0 & \text{in } \Omega, \\ w(-\infty, y) = 1, \quad w(+\infty, y) = 0 & \text{uniformly in } y \in D, \\ w \geq 0, & \\ w_\nu = 0 & \text{on } \mathbb{R} \times \partial D. \end{cases} \tag{3.90}$$

[2] Note that in fact [YIH, Theorem A.26] gives the estimates for $y \in D'$ where D' is such that $D' \subset\subset D$. As in the proof of Proposition 2.8, the local up to the boundary $W^{2,p}$ estimate for $y \in D$ follows by adaptating the proof of [ADN, Theorem 15.2], see footnote 1.

We want to prove that there exists at least one real eigenvalue λ, associated to a positive eigenfunction φ satisfying

$$\begin{cases} -\Delta'\varphi - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial\varphi}{\partial x_i} - f'(0)\varphi = \Big(\lambda^2 - \lambda(c+\alpha(y)G_1'(0))\Big)\varphi & \text{in } D, \\ \varphi_\nu = 0 & \text{on } \partial D. \end{cases} \tag{3.91}$$

First, we will show that there exists a positive solution h of the linearized problem

$$\begin{cases} \Delta h + \Big(c+\alpha(y)G_1'(0)\Big)\partial_1 h + \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial h}{\partial x_i} + f'(0)h = 0 & \text{in } \mathbb{R}\times D, \\ h_\nu = 0 & \text{on } \mathbb{R}\times\partial D. \end{cases} \tag{3.92}$$

To do this, choose a sequence $(x_1^N)_{N\geq 2}$, which tends to $+\infty$ when N goes to infinity and satisfies

$$\sup_{y\in\overline{D}} w(x_1^N, y) = \frac{1}{N}.$$

Note that such a sequence exists since w is continuous and tends uniformly (with respect to y) to 0 when x_1 tends to infinity. Now define

$$h^N(x_1,y) := \frac{w(x_1^N + x_1, y)}{2\sup_{y\in\overline{D}} w(x_1^N, y)}.$$

Note that for all $N\geq 2$, the function h^N satisfies

$$\sup_{y\in\overline{D}} h^N(0,y) = \frac{1}{2}.$$

Furthermore, the function h^N also satisfies

$$\Delta h^N + \Big(c+\alpha(y)G_1'\Big(w(x_1^N+x_1,y)\Big)\Big)\partial_1 h^N \\ + \alpha(y)\sum_{i=2}^n G_i'\Big(w(x_1^N+x_1,y)\Big)\frac{\partial h^N}{\partial x_i} + \frac{f\Big(w(x_1^N+x_1,y)\Big)}{2\sup_{y\in\overline{D}} w(x_1^N,y)} = 0.$$

On each relatively compact subset $\Omega_a := (-a,a)\times D$ of Ω, the family h^N is bounded in $W^{2,p}(\Omega_a)$.
Note that as in the proof of Proposition 2.8, the $W^{2,p}$ local up to the boundary estimate follows from an adaptation of the proof of [AGN, Theorem 15.2], see also footnote 1.
Hence, there exists a subsequence h^{N_k} of h^N which is weakly convergent in $W^{2,p}(\Omega_a)$ and strongly in $C^{1,\lambda}(\overline{\Omega_a})$. Then, using continuity of G_i' and differentiability of f, it follows that h^N converges in $C^{1,\lambda}_{loc}$ and weakly in $W^{2,p}_{loc}$ to a function h, which satisfies

$$\Delta h + \Big(c+\alpha(y)G_1'(0)\Big)\partial_1 h + \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial h}{\partial x_i} + f'(0)h = 0 \text{ in } \Omega. \tag{3.93}$$

Moreover, since the function w is non negative, h has to be non negative, and satisfies

$$\begin{cases} \Delta h + \Big(c + \alpha(y) G_1'(0)\Big)\partial_1 h + \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial h}{\partial x_i} + f'(0)h = 0 & \text{on } \mathbb{R}\times D, \\ h_\nu = 0 & \text{on } \mathbb{R}\times\partial D, \\ 0 \le h \le 1, \ \ \partial_1 h \le 0 & \text{in } \mathbb{R}\times\overline{D}, \\ \max_{y\in\overline{D}} h(0,y) = \frac{1}{2}. \end{cases} \tag{3.94}$$

Note that the boundary condition $h_\nu = 0$ on $\mathbb{R}\times\partial D$ follows from the $W^{2,p}$ local up to the boundary estimate, see footnote 1. Indeed, this estimate gives that $(h^{N_k})_k$ converges in $C^{1,\lambda}\Big(\overline{(-a,a)\times D}\Big)$, from which it follows that $h_\nu^{N_k}$ converges, so $h_\nu = 0$ on $\mathbb{R}\times D$.

Now, we prove that $h > 0$, using the maximum principle and the Hopf lemma. Assume that for some $x_1 \in \mathbb{R}$, and $y \in D$, one has $h(x_1, y) = 0$. Then,

$$\Delta h + (c + \alpha(y)G_1'(0))\partial_1 h + \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial h}{\partial x_i} = -f'(0)h \le 0.$$

The Strong Maximum Principle gives that h is constant in Ω, therefore $h = 0$ in Ω, which is impossible due to the condition $\max_{y\in\overline{D}} h(0,y) = 1/2$. So $h > 0$ in Ω. Now, assume that there exists $x_1^0 \in \mathbb{R}$ and $y^0 \in \partial D$, such that $h(x_1^0, y^0) = 0$. Since $h > 0$ in Ω, the Hopf lemma ensures that

$$\frac{\partial h}{\partial \nu}(x_1^0, y^0) < 0,$$

which is impossible due to the Neumann boundary conditions.

We will now show that

$$\lim_{x_1\to+\infty} \|h(x_1,\cdot)\|_{C^{1,\lambda}(\overline{D})} = 0.$$

First, since h is decreasing with respect to x_1 and bounded, it follows that for each $y \in D$, $h(x_1, y)$ has a finite limit $\tilde{h}(y)$ when x_1 tends to $+\infty$. Using a translation $z_m : [-1,1]\times D \to \mathbb{R}$ as in the proof of Theorem 1.17, $z_m(x_1,y) := h(x_1+m, y)$, we obtain that $h(x_1,\cdot)$ tends to $\tilde{h}$ in $C^1(\overline{D})$. Take $v \in C_0^1(D)$ and using Lemma 2.7, we can argue as in the proof of Theorem 1.17 to show that $\tilde{h}$ satisfies

$$\begin{cases} -\Delta'\tilde{h} - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial \tilde{h}}{\partial x_i} - f'(0)\tilde{h} = 0 & \text{in } D, \\ \tilde{h}_\nu = 0 & \text{on } \partial D. \end{cases} \tag{3.95}$$

If $\tilde{h}$ reaches its minimum on ∂D, the Hopf lemma gives that $\partial_\nu \tilde{h} < 0$, which is a contradiction. Therefore, $\tilde{h}$ reaches its minimum inside D, and by the maximum principle, $\tilde{h}$ is constant in D. Finally, since $\tilde{h}$ satisfies $-\Delta'\tilde{h} - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial \tilde{h}}{\partial x_i} - f'(0)\tilde{h} =$

0 in D, it follows that $\tilde{h} \equiv 0$.

Note that by construction, $h \in W^{2,p}_{loc}(\Omega)$, which, together with the fact that $h > 0$ enables [BN2, Theorem 3.2] to be applied to obtain the existence of two positive constants C and a such that

$$h(x_1, y) \geq Ce^{-ax_1} \text{ in } \Omega. \tag{3.96}$$

Now, we want to show that when x_1 is sufficiently large, the function h can be written as

$$h(x_1, y) = \sum_{k=1}^{N} \phi_k(x_1, y)e^{-\lambda_k x_1} + b(x_1, y),$$

where the λ_k are complex eigenvalues of (3.91), the functions ϕ_k are polynomial in x_1, and

$$b(x_1, y) = o\Big(\sum_{k=1}^{N} \phi_k(x_1, y)e^{-\lambda_k x_1}\Big), \text{ when } x_1 \to +\infty.$$

To do that, we will apply [BN3, Theorem 2.1], with $\mu = 0$ and $q = +\infty$, where μ and q are defined in [BN3]. First, note that one has $\|h(x_1, \cdot)\|_{L^p(D)} = o(1)$ when x_1 tends to $+\infty$. Indeed, since $\overline{D}$ is compact and h is a continuous function which tends to 0 in $C^1(\overline{D})$, we can write

$$\lim_{x_1 \to +\infty} \int_D h^p(x_1, y)dy = \int_D \lim_{x_1 \to +\infty} h^p(x_1, y)dy = 0.$$

The equation (3.93) satisfied by h can be rewritten as

$$\partial_{11}h + A_1(y, \partial_y)\partial_1 h + A_2(y, \partial_y)h = 0 \text{ in } \Omega,$$

where

$$A_1 := c + \alpha(y)G_1'(0), \text{ and } A_2 := \Delta' + \alpha(y)\sum_{i=2}^{n} G_i'(0)\frac{\partial}{\partial x_i} + f'(0).$$

By [BN3, Section 2], there exists a countable family of eigenvalues of (3.91), and any strip $\beta_1 < \operatorname{Re}\lambda < \beta_2$ in the complex plane contains a finite number of such eigenvalues.

Let $a > 0$ as in (3.96) and fix $\varepsilon_0 > 0$. Denote by $\lambda_1, \cdots, \lambda_N$ the complex eigenvalues of (3.91) in the strip

$$0 < \operatorname{Re}\lambda_j < a_0 := a + \varepsilon_0, \text{ for } 1 \leq j \leq N.$$

Fix $\varepsilon > 0$ sufficiently small such that $\varepsilon < \varepsilon_0$ and $0 < \operatorname{Re}\lambda_j < a_0 - \varepsilon$ for $1 \leq j \leq N$. By [BN3, Theorem 2.1], there exists a constant $K > 0$, such that

$$\Big\|h(x_1, \cdot) - \sum_{k=1}^{N} \phi_k(x_1, \cdot)e^{-\lambda_k x_1}\Big\|_{W^{2,p}(D)} \leq Ke^{-(a_0-\varepsilon)x_1}, \tag{3.97}$$

where the functions ϕ_k are polynomial in x_1, with coefficients depending on y, and the λ_k, called generalized eigenvalues, are solutions of (3.74), in the sense that there exists a function $\varphi \in W^{2,p}(D)$, non identically 0, such that the pair (φ, λ) satisfies (3.74). We adopt the convention $\mathcal{C}$ that $\phi_k \equiv 0$ if the corresponding eigenvalue λ_k does not contribute a non zero term in the expansion in (3.97).

Now note that the function

$$b(x_1, y) := h(x_1, y) - \sum_{k=1}^{N} \phi_k(x_1, y) e^{-\lambda_k x_1} = O\Big(e^{-(a_0-\varepsilon)x_1}\Big) \text{ when } x_1 \to +\infty,$$

uniformly for $y \in D$, so at least one eigenvalue λ_k has to make a non-zero contribution to be expansion $\sum_{k=1}^{N} \phi_k(x_1, \cdot) e^{-\lambda_k x_1}$ in (3.97). Indeed, if all the $\phi_k \equiv 0$, then since $a_0 - \varepsilon > a$, the estimate (3.97) implies that h decays more rapidly than the lower bound Ce^{-ax_1} of (3.96), which is impossible. Thus, there exists at least one eigenvalue λ in the strip $0 < \operatorname{Re}\lambda < a_0$ which makes a non zero contribution in the estimate (3.97).

Knowing that near infinity, $h(x_1, y) \sim \sum_{k=1}^{N} \phi_k(x_1, y) e^{-\lambda_k x_1}$ uniformly in $y \in D$ and that h is a positive function, we will prove that at least one of the λ_k which makes a non zero contribution has to be real and positive.

Denote by J the non-empty set of all the eigenvalues λ in the strip $0 < \operatorname{Re}\lambda < a_0$ such that λ makes a non-zero contribution in (3.97). Let

$$\alpha := \max_{1 \le k \le N,\ \lambda_k \in J} \operatorname{Re}(-\lambda_k). \tag{3.98}$$

Since there is at least one eigenvalue in the strip $0 < \operatorname{Re}\lambda < a_0$ which makes a non zero contribution in (3.97), α is well defined.
By (3.97) and assuming that the eigenvalues with real part $-\alpha$ are contained in the set

$$\Big\{\lambda_1 = -\alpha,\ \lambda_2 = -\alpha + i\beta_2,\ \lambda_3 = -\alpha + i\beta_3,\ \cdots,\ \lambda_N = -\alpha + i\beta_N\Big\}, \tag{3.99}$$

we can write

$$h(x_1, y) = e^{\alpha x_1}\Big(q_p(y) x_1^p + \sum_{j=2}^{N} x_1^{s_j} \varphi_j(y) e^{i\beta_j x_1}\Big) + \tilde{b}_1(x_1, y), \tag{3.100}$$

where we denote

$$e^{-\lambda_1 x_1} \phi_1(x_1, y) = e^{\alpha x_1}\Big(q_p(y) x_1^p + \cdots + q_0(y)\Big),$$

and for $j = 2, \cdots, N$,

$$e^{-\lambda_j x_1} \phi_j(x_1, y) = e^{(\alpha + i\beta_j) x_1}\Big(\varphi_{j,s_j}(y) x_1^{s_j} + \cdots + \varphi_{j,0}(y)\Big),$$

where $\varphi_{j,s_j} \not\equiv, 0$ for each $j = 2, \cdots, N$ and $\tilde{b}_1(x_1, \cdot)$ contains the terms of the form $e^{(\alpha+i\beta_j)x_1}\varphi_{j,s_k}x_1^{s_k}$ for $s_k < s_j$, as well as the terms in $\sum_{k=1}^N \phi_k(x_1, \cdot)e^{-\lambda_k x_1}$ where $\mathrm{Re}(-\lambda) < \alpha$ and the term

$$b(x_1, y) = h(x_1, y) - \sum_{k=1}^{N} \phi_k(x_1, \cdot)e^{-\lambda_k x_1},$$

which by (3.97) decays at an exponential rate that is strictly faster than the terms in $\sum_{k=1}^N \phi_k(x_1, \cdot)e^{-\lambda_k x_1}$.
Note that $\lambda_1 = -\alpha$ is not assumed a priori to be an eigenvalue but we adopt the convention that $\phi_1 \equiv 0$ if it is not.

Now setting $s := \max\{s_2, \cdots, s_N\}$ and redefining $\tilde{b}_1(x_1, y)$ to include the terms $e^{(\alpha+i\beta_j)x_1}x_1^{s_j}\varphi_j(y)$ when $s_j < s$, we can write

$$h(x_1, y) = e^{\alpha x_1}\Big(q_p(y)x_1^p + x_1^s \sum_{j=2}^{\tilde{N}} \varphi_j(y)e^{i\beta_j x_1}\Big) + \tilde{b}_1(x_1, y). \tag{3.101}$$

Since we know that $h \in \mathbb{R}$, we have

$$\begin{aligned}
h(x_1, y) &= \mathrm{Re}\Big\{e^{\alpha x_1}\Big(q_p(y)x_1^p + x_1^s \sum_{j=2}^{\tilde{N}} \varphi_j(y)e^{i\beta_j x_1}\Big) + \tilde{b}_1(x_1, y)\Big\} \\
&= e^{\alpha x_1}\Big[x_1^p \,\mathrm{Re}\, q_p(y) + x_1^s \sum_{j=2}^{\tilde{N}} \mathrm{Re}\Big(\varphi_j(y)e^{i\beta_j x_1}\Big)\Big] + \mathrm{Re}\, \tilde{b}_1(x_1, y) \\
&= e^{\alpha x_1}\Big[\,\mathrm{Re}\, q_p(y)x_1^p + x_1^s \sum_{i=2}^{\tilde{N}} \Big(\cos(\beta_j x_1)\,\mathrm{Re}\,\varphi_j(y) - \sin(\beta_j x_1)\,\mathrm{Im}\,\varphi_j(y)\Big)\Big] + \hat{b}(x_1, y),
\end{aligned}$$

where $\hat{b}_1(x_1, y) := \mathrm{Re}\, \tilde{b}_1(x_1, y)$. In other words, the function h can be written :

$$h(x_1, y) = e^{\alpha x_1}\Big(\,\mathrm{Re}\, q_p(y)x_1^p + p(x_1, y)x_1^s\Big) + \hat{b}(x_1, y), \tag{3.102}$$

where $p(x_1, y) := \sum_{i=2}^{\tilde{N}} \Big(\cos(\beta_j x_1)\,\mathrm{Re}\,\varphi_j(y) - \sin(\beta_j x_1)\,\mathrm{Im}\,\varphi_j(y)\Big)$.

Using that $h > 0$, we will show that for all $y \in D$, the function $\mathrm{Re}\, q_p$ is non-negative. To do that, we will use some properties of uniformly almost periodic functions, defined in [BES, Chapter 1], which we recall here.
If a continuous function $F : \mathbb{R} \mapsto R$ satisfies : for all $x \in R$, and all $\varepsilon > 0$, there exists $L(\varepsilon) > 0$ such that in every interval of length $L(\varepsilon)$, there exists $t > 0$ such that

$$|F(x + t) - F(x)| < \varepsilon,$$

then F is a *uniformly almost periodic function.* Note that a finite sum of uniformly almost periodic functions is a uniformly almost periodic function, see [BES, Theorem 12 - Section 1 - Chapter 1]. Then for a continuous function f, denote by $\overline{M_f}$ and $\underline{M_f}$ the upper and lower limit when $T \to +\infty$ of

$$\frac{1}{T}\int_0^T f(x)dx.$$

When $\overline{M_f} = \underline{M_f}$, we call their common value the *mean value of the function f*, denoted M_f. By [BES, Theorem 2 - Section 3 - Chapter 1], the mean value of any uniformly almost periodic function exists.

Now note first that $\operatorname{Re} q_p$ cannot be identically 0. Indeed, if $\operatorname{Re} q_p \equiv 0$, we have :

$$e^{-\alpha x_1} x_1^{-s} h(x_1, y) = p(x_1, y) + e^{-\alpha x_1} x_1^{-s} \hat{b}(x_1, y). \tag{3.103}$$

Since $e^{-\alpha x_1} x_1^{-s} \hat{b}(x_1, y) \to 0$ as $x_1 \to +\infty$, and p is an uniformly almost periodic function of x_1 with mean value (in x_1) equal to zero, it follows that the right hand-side of (3.103) takes negative values for a sequence $(x_1^k)_{k\in\mathbb{N}}$, where $x_1^k \to +\infty$, which contradicts the fact that $h > 0$. A similar argument shows that $\operatorname{Re} q_p$ cannot be negative for any $y \in D$, so $\operatorname{Re} q_p(y) \geq 0$, and also that $p \geq s$, since if $s > p$, dividing (3.103) by $e^{\alpha x_1} x_1^s$ gives

$$\operatorname{Re} q_p(y) x_1 p - s + e^{-\alpha x_1} x_1^{-s} \hat{b}_1 \to 0$$

as $x_1 \to +\infty$, which again contradicts that $h > 0$ and p is an uniformly almost periodic function with mean value 0.

We will show now that the function $\operatorname{Re} q_p$ has to be an eigenfunction of (3.91), associated to $\lambda_5 = -\alpha > 0$.
Since $e^{-\lambda_5 x_1}\phi_5(x_1, y)$ is a solution of (3.91), where

$$\phi_5(x_1, y) = q_p(y)x_1^p + q_{p-1}(y)x_1^{p-1} + \cdots + q_0(y),$$

it follows that

$$\Delta' q_p + \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial q_p}{\partial x_i} + f'(0)q_p + \Big(\lambda^2 - \lambda(c + \alpha(y)G_i'(0))\Big)q_p = 0, \tag{3.104}$$

where $\lambda = \lambda_5 = -\alpha$, and so, taking real parts of both sides of (3.104), it follows that $\operatorname{Re} q_p$ satisfies

$$\Delta'\Big(\operatorname{Re} q_p\Big) + \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial \operatorname{Re} q_p}{\partial x_i} + f'(0)\operatorname{Re} q_p + \Big(\lambda^2 - \lambda(c + \alpha(y)G_i'(0))\Big)\operatorname{Re} q_p = 0.$$

Hence, the function $\operatorname{Re} q_p$ satisfies (3.91), with $\operatorname{Re} q_p \geq 0$ in D. If $\operatorname{Re} q_p$ vanishes at $y_0 \in D$, then the Strong Maximum Principle ensures that $\operatorname{Re} q_p$ is constant in D which is impossible. Consequently, the function $\operatorname{Re} q_p$ which is positive in D has to be an eigenfunction of (3.91) associated to the real eigenvalue λ_5, which is positive. Thus, since c is the speed of the travelling front w, so by definition of c', we must have $c \geq c'$, and hence $c^\star \geq c'$.

□

3.3 Special case where $G = (G_1, 0, \cdots, 0)$.

The equality $c^\star = c'$ does not necessarily hold in general. Indeed, in the one-dimensional case, take for instance

$$f(u) = u(1-u) \text{ and } G(u) = -\gamma u^2,$$

for all $u \in [0,1]$, and $\gamma > \sqrt{11/3}$. Then, the functions f and G satisfy the conditions [AK1, Proposition 2.3], which imply that $c^\star > c'$. Note that the proof of Proposition 2.3 in [AK1] is an adaptation to the case of no convection term of the approach of Berestycki and Nirenberg, see [BN2, Remark 10.2]. In [BN2, Remark 10.2] Berestycki and Nirenberg proved that the strict inequality $c^\star > c'$ holds, in a one dimensional special case, where f does not satisfy the KPP condition (1.31), and where the travelling front solution satisfies the equation $u'' - cu' + f(u) = 0$.

In the absence of convection term, it was proved in [BN2, Section 10] that the KPP condition (1.31) is a sufficient condition that ensures that the equality $c^\star = c'$ holds.

A natural question is whether there exist cases where under the KPP condition, the equality still holds in presence of the convection term when $G \not\equiv 0$. We will show that $c^\star = c'$ in the case where $G = (G_1, 0, \cdots, 0)$, under an additional condition on α and G_1' and the KPP condition (1.31) also holds for f. Indeed, assuming that $c^\star > c'$, we will construct a solution of (1.32) with speed $c \in (c', c^\star)$, following the approach of [BN2, Section 10], using the method of sub and supersolutions.

Proposition 3.9. *Assume that $G = (G_1, 0, \cdots, 0)$, with $G_1 \neq 0$. Assume also that for $y \in D$ and $u \in \mathbb{R}$,*

$$\alpha(y) G_1'(u) \geq \alpha(y) G_1'(0), \tag{3.105}$$

and the KPP condition, namely, that for all $u \in (0,1)$,

$$f(u) \leq f'(0)u.$$

Then

$$c' = c^\star.$$

Proof. Assume $c^\star > c'$. Choose c such that $c^\star > c > c'$. Then, by definition of c', there exist $0 < \lambda_1 < \lambda_2$ and positive functions φ_1, φ_2 such that, for $j = 1, 2$,

$$\begin{cases} -\Delta' \varphi_j - f'(0)\varphi_j = \Big(\lambda_j^2 - \lambda_j(c + \alpha(y)G_1'(0))\Big)\varphi_j & \text{in } D, \\ \frac{\partial \varphi_j}{\partial \nu} = 0 & \text{on } \partial D. \end{cases} \tag{3.106}$$

Note that in fact we will only use the eigenfunction φ_1 and the eigenvalue λ_1.

We will construct a solution of (1.32), with speed c, using the method of sub and supersolutions.
Let $z(x_1, y) := e^{-\lambda_1 x_1}\varphi_1(y)$. Using that λ_1 and φ_1 satisfy (3.106), we can write

$$\begin{aligned} -\Delta z &= -\lambda_1^2 e^{-\lambda_1 x_1}\varphi_1 + e^{-\lambda_1 x_1}\Big[\lambda_1^2 - \lambda_1\Big(c + \alpha(y)G_1'(0)\Big)\varphi_1 + f'(0)\varphi_1\Big] \\ &= -\lambda_1 z\Big(c + \alpha(y)G_1'(0)\Big) + f'(0)z. \end{aligned}$$

Hence, since we assumed (3.105) and the KPP condition (1.31) for f, the function $z = e^{-\lambda_1 x_1}\varphi_1(y)$ satisfies

$$\begin{cases} -\Delta z - (c + \alpha(y)G_1'(z))\partial_1 z - f(z) \geq 0 & \text{in } \Omega, \\ \frac{\partial z}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ z(-\infty, \cdot) = +\infty, \text{ and } z(+\infty, \cdot) = 0 & \text{uniformly in } y. \end{cases} \tag{3.107}$$

Note that the limit when $x_1 \to -\infty$ is uniform with $y \in D$ because the function φ_1 satisfies $\min_{y \in \overline{D}} \varphi_1(y) > 0$ because of the Neumann boundary conditions.

Let $N > 1$ be an integer, such that

$$z(-N, y) > 1, \text{ for all } y \in \overline{D}. \tag{3.108}$$

The function $y \mapsto z(N, y)$ is continuous and $\overline{D}$ is compact. Hence, there exists $\varepsilon_N \in (0, 1)$ such that

$$z(N, y) > \varepsilon_N, \text{ for all } y \in D. \tag{3.109}$$

Note that $f(\varepsilon_N) > 0$.
The function ε_N is a subsolution and z is a supersolution on $\Omega_N = (-N, N) \times D$, since both satisfy Neumann boundary conditions, and

$$-\Delta z - (c + \alpha(y)G_1'(z))\partial_1 z - f(z) \geq 0 \geq -\Delta\varepsilon_N - (c + \alpha(y)G_1'(\varepsilon_N))\partial_1\varepsilon_N - f(\varepsilon_N).$$

Note that the constant function 1 is also a (super)solution.
Now we will apply Proposition 2.4 with ρ replaced by the constant function 1, and with $\underline{u} = \varepsilon_N$ and $\overline{u} = 1$. Proposition 2.4 gives the existence of a solution $u \in W^{2,p}_{loc}\Big((-N, N) \times \overline{D}\Big)$ of

$$\begin{cases} -\Delta u - (c + \alpha(y)G_1'(u))\partial_1 u - f(u) = 0 & \text{in } \Omega_N, \\ \frac{\partial u}{\partial \nu} = 0 & \text{for } -N < x_1 < N, y \in \partial D, \\ u(-N, y) = 1 \ , \ u(N, y) = \varepsilon_N & \text{and } \varepsilon_N \leq u \leq 1. \end{cases} \tag{3.110}$$

Furthermore, the functions ε_N and 1 are not solutions of (3.110), since no constant function solves (3.110). Then, [BN1, Theorem 7.2] ensures that there exists only one solution $u \in W^{2,p}_{loc}\Big(\overline{\Omega_N} \setminus (\{\pm N\} \times \partial D)\Big) \cap C(\overline{\Omega_N})$ of (3.110). Moreover, since $c + \alpha(y)G_1'$ is Lipschitz continuous in x_1, the same theorem also yields that $\partial_1 u(x_1, y) < 0$ for $-N < x_1 < N$ and $y \in \overline{D}$.

Thus, for each N sufficiently large, there exists a unique solution $u^N \in W^{2,p}_{loc}\Big(\overline{\Omega_N} \setminus (\{\pm N\} \times \partial D)\Big) \cap C(\overline{\Omega_N})$ which satisfies

$$\begin{cases} -\Delta u^N - (c + \alpha(y)G_1'(u^N))\partial_1 u^N - f(u^N) = 0 & \text{in } \Omega_N, \\ \frac{\partial u^N}{\partial \nu} = 0 & \text{for } -N < x_1 < N, y \in \partial D, \\ u^N(-N, y) = 1 \ , \ u^N(N, y) = \varepsilon_N & \text{and } \varepsilon_N \le u^N \le 1, \\ \partial_1 u^N(x_1, y) < 0 & \text{for } -N < x_1 < N, y \in \overline{D}. \end{cases} \tag{3.111}$$

Now, we will apply [BN2, Lemma 5.1] to show that $u^N \le z$ in Ω_N. Note that the functions z and u^N belong to $W^{2,p}_{loc}\Big(\overline{\Omega_N} \setminus (\{\pm N\} \times \partial D)\Big) \cap C(\overline{\Omega_N})$ and satisfy

$$\begin{cases} -\Delta z - (c + \alpha(y)G_1'(z))\partial_1 z - f(z) \ge 0 = -\Delta u^N - (c + \alpha(y)G_1'(u^N))\partial_1 u^N - f(u^N) & \text{in } \Omega_N, \\ \frac{\partial u^N}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0 & \text{for } y \in \partial D, \end{cases} \tag{3.112}$$

and since z is decreasing in x_1 and satisfies (3.108) and (3.109), for N sufficiently large, $u^N(N, y) < z(x_1, y)$ and $u^N(x_1, y) < z(-N, y)$, for $-N < x_1 < N$. Then, [BN2, Lemma 5.1] ensures that

$$u^N \le z \text{ in } \Omega_N.$$

Once again, we want to let N tend to infinity, preventing the solution u^N from tending neither to 0 nor to 1. To do this, consider the supersolution, as minimum of two supersolutions, see [YIH, Theorem 4.12], which is based on results in [LE1] :

$$h^r(x_1, y) := \min\Big(1, z(x_1 + r, y)\Big).$$

Using the constant function $\min_{y \in \overline{D}} h^r(N, y)$ which is a subsolution of (3.111) for N sufficiently large, we argue as in the proof of Proposition 1.17 and get a sequence of solutions $(u^N)_{N>0}$ which has a subsequence that converges to a limit u that satisfies

$$\begin{cases} -\Delta u - \Big(c + \alpha(y)G_1'(u)\Big)\partial_1 u - f(u) = 0 & \text{in } \mathbb{R} \times D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{for } y \in \partial D, \\ 0 \le u \le 1, & \\ \partial_1 u(x_1, y) \le 0 & \text{for } y \in \overline{D}, \\ \max_{y \in \overline{D}} u(0, y) = \frac{1}{2}. & \end{cases} \tag{3.113}$$

Finally, since u is non-inscreasing with respect to x_1, it follows that u has finite limits when x_1 tends to $\pm\infty$. Moreover, $\lim_{x_1\to+\infty} u(x_1, y)$ and $\lim_{x_1\to-\infty} u(x_1, y)$ have to satisfy (1.35). Hence, those limits have to be 0 or 1 because of Assumption **(AN)**. But the normalization condition

$$\max_{y\in\overline{D}} u(0, y) = \frac{1}{2},$$

ensures that

$$\lim_{x_1\to+\infty} u(x_1, y) = 0 \text{ and } \lim_{x_1\to-\infty} u(x_1, y) = 1.$$

Similarly as in the proof of Theorem 1.17 we prove that $\frac{\partial u}{\partial x_1} < 0$ and that u is decreasing.
Thus, the function u is a solution of (1.32) with a speed $c < c^\star$, which is impossible by definition of $c^\star$. □

3.4 Special case where $G_1'(0) = 0$.

We will prove that $c^\star = c'$ also in a second special case, again using a similar approach to that in [BN2, Section 10]. Note that in this case we have an explicit formula which agrees with the corresponding formula for c' when there is no convection term and when KPP condition (1.31) holds.

Proposition 3.10. *Assume $G_1'(0) = 0$. Then*

$$c' = 2\sqrt{f'(0)}.$$

Proof. If $G_1'(0) = 0$, the eigenvalue problem (3.74) can be written in the following form

$$\begin{cases} -\Delta'\varphi - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial\varphi}{\partial x_i} - f'(0)\varphi = \Big(\lambda^2 - \lambda c\Big)\varphi & \text{in } D, \\ \varphi_\nu = 0 & \text{on } \partial D. \end{cases} \tag{3.114}$$

By [YIH, Theorem 1.3], there exists a unique $\beta \in \mathbb{R}$ for which there exists a positive function ϕ such that the following eigenvalue problem

$$\begin{cases} -\Delta'\phi - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial\phi}{\partial x_i} = \beta\phi & \text{in } D, \\ \phi_\nu = 0 & \text{on } \partial D, \end{cases} \tag{3.115}$$

is satisfied, and φ is unique up to a multiplicative constant. Hence, since a positive constant function φ satisfies (3.115) when $\beta = 0$, it follows that $\beta = 0$, and ϕ is a constant function.

Consequently, the principal eigenfunction φ of (3.114) is in fact a positive constant. From (3.114), it follows that the eigenvalue λ associated to this principal eigenfunction satisfies $\lambda^2 - \lambda c + f'(0) = 0$. Hence, there exists a positive real eigenvalue λ if and only if $c^2 - 4f'(0) \geq 0$. By definition of c', it follows that

$$c' = 2\sqrt{f'(0)}.$$

□

Now we will prove that $c^\star = c'$ in a second special case, assuming for contradiction $c^\star > c'$ and again using method of sub and supersolutions to construct a solution of (1.32) with speed c'.

Proposition 3.11. *Assume that*

$$G_1'(0) = 0, \quad \alpha(y)G_1'(u) \geq 0 \textit{ for all } u \in (0,1),$$

and that f satisfies the KPP condition (1.31). *Then, $c' = c^\star$.*

Proof. Assume that $c^\star > c'$. Take c such that $c^\star > c > c'$. Since $c > c'$, there exists a positive eigenvalue λ of (3.114).
Let $z := e^{-\lambda x_1}$. We claim that the function z is a supersolution of

$$\begin{cases} -\Delta u - (c + \alpha(y)G_1'(u))\partial_1 u - \alpha(y)\sum_{i=2}^n G_i'(u)\frac{\partial u}{\partial x_i} - f(u) = 0 & \text{in } \Omega, \\ u_\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.116}$$

Indeed, since z only depends on x_1, one has

$$\begin{aligned} -\Delta z - (c + \alpha(y)G_1'(z))\partial_1 z - f(z) &= -\lambda^2 z + \lambda(c + \alpha(y)G_1'(z))z - f(z) \\ &= \Big(-\lambda^2 + \lambda c - f'(0)\Big)z + \lambda\alpha(y)G_1'(z)z + f'(0)z - f(z) \\ &= \lambda\alpha(y)G_1'(z)z + f'(0)z - f(z), \end{aligned}$$

since λ satisfies $-\lambda^2 + \lambda c - f'(0) = 0$. Hence, z is a supersolution, assuming the KPP condition and that $\alpha(y)G_1'(z) \geq 0$, for all $z \in (0,1)$ and all $y \in D$.
Let $N > 1$. As before, there exists a subsolution $\varepsilon_N \in (0,1)$ such that $z(N,y) > \varepsilon_N$ for all $y \in D$. Arguing as in the proof of Proposition 3.9, it follows $u^N \leq z$ in Ω_N. By means of function $h^r(x_1) := \min\Big(1, z(x_1 + r)\Big)$, we can then rely on similar arguments to those in the proof of Proposition 3.9, to obtain a function u which is a solution of (1.32) with $\frac{\partial u}{\partial x_1} < 0$ and a speed $c < c^\star$, which is impossible.

□

4 Another form of convection term for Neumann boundary conditions

Different modeling approaches could give several forms of convection terms, so we also consider an alternative form of convection term.

Here we consider the case where the convection term has the form $\nabla \cdot (\alpha(y)G(u))$ instead of $\alpha(y)\nabla\cdot G(u)$. In this case, a travelling front solution $u(x,t) = w(x_1 - ct, y)$ will satisfy

$$\begin{cases} -c\partial_1 w = \Delta w + \nabla \cdot (\alpha(y)G(w)) + f(w) & \text{in } \Omega, \\ w(-\infty, y) = 1, \quad w(+\infty, y) = 0 & \text{uniformly in } y \in D, \\ w \geq 0, & \\ w_\nu = 0 & \text{on } \mathbb{R} \times \partial D. \end{cases} \tag{4.117}$$

Note that the first equation in (4.117) can be rewritten in the following form

$$\Delta w + \Big(c + \alpha(y)G_1'(w)\Big)\partial_1 w + \alpha(y)\sum_{i=2}^{n} G_i'(w)\frac{\partial w}{\partial x_i} + \sum_{i=2}^{n} G_i(w)\frac{\partial \alpha}{\partial x_i} + f(w) = 0. \tag{4.118}$$

Throughout this chapter, we make the following assumption :

$$f'(0) + \inf_{y\in D}\sum_{i=2}^{n} G_i'(0)\frac{\partial \alpha}{\partial x_i}(y) > 0. \tag{4.119}$$

Remark : Note that condition (3.84) which is the corresponding condition for the second form of convection term, implies condition (4.119).

With the inclusion new term $\sum_{i=2}^{n} G_i(w)\frac{\partial \alpha}{\partial x_i}$, we will now construct a solution on the truncated cylinder $\Omega_N = (-N, N) \times D$.

4.1 Existence of a solution on the unbounded cylinder

As before, for $\rho \in K$, defined in (2.47),

$$r(\rho)(x) := \frac{\Delta\rho(x) + \nabla \cdot \Big(\alpha(y)G(\rho(x))\Big) + f(\rho(x))}{-\partial_1\rho(x)},$$

and

$$c^\star := \inf_{\rho\in K}\sup_{x\in\Omega} r(\rho)(x). \tag{4.120}$$

We again have the existence of an upper bound for $c^\star$.

Proposition 4.1. *Assume the conditions* **(GN1′)**, **(GN2′)**, **(GN4′)**, **(AlphaN1′)** *and that f is Lipschitz continuous. Then,*

$$c^\star < +\infty.$$

Proof. To prove this, we will use the function g defined in (2.50). For $x_1 < -1$, one has

$$\begin{aligned} r(g) &= \frac{\Delta g + \nabla \cdot \Big(\alpha G(g)\Big) + f(g)}{-\partial_1 g} \\ &= \frac{h'' + \alpha G_1'(h)h' + \sum_{i=2}^n G_i(h)\frac{\partial \alpha}{\partial x_i} + f(h)}{-h'} \\ &= -\frac{h''}{h'} - \alpha G_1'(h) - \frac{1}{h'}\sum_{i=2}^n G_i(h)\frac{\partial \alpha}{\partial x_i} - \frac{f(h)}{1-h}\frac{1-h}{h'}. \end{aligned}$$

Since f satisfies $f(1) = 0$, it follows that $f(h)/(1-h)$ is bounded. The ratios $-h''/h'$ and $(1-h)/h$ are equal to -1, and by $\alpha G_1'(h)$ is bounded by **(AlphaN1′)**. Finally, using that **(GN4′)** and the mean value theorem, for all $2 \le i \le n$, there exists $\xi_i \in (h, 1)$ such that

$$\Big|\frac{1}{h'}\sum_{i=2}^n G_i(h)\frac{\partial \alpha}{\partial x_i}\Big| = \Big|\frac{1}{h'}\sum_{i=2}^n \Big(G_i(h) - G_i(1)\Big)\frac{\partial \alpha}{\partial x_i}\Big| = \Big|\frac{h-1}{h'}\sum_{i=2}^n G_i'(\xi_i)\frac{\partial \alpha}{\partial x_i}\Big|,$$

which is bounded. Hence, $r(g)$ is bounded for $x_1 < 1$.

For $x_1 > 1$, we write

$$\begin{aligned} r(g) &= \frac{h'' + \alpha G_1'(h)h' + \sum_{i=2}^n G_i(h)\frac{\partial \alpha}{\partial x_i} + f(h)}{-h'} \\ &= -\frac{h''}{h'} - \alpha G_1'(h) - \frac{1}{h'}\sum_{i=2}^n G_i(h)\frac{\partial \alpha}{\partial x_i} - \frac{f(h)}{h}\frac{h}{h'}. \end{aligned}$$

Since $f(0) = 0$, the ratio $f(h)/h$ is bounded. The ratios h''/h and h/h' are equal to -1, and $\alpha G_1'(h)$ is bounded by (AlphaN1′). Using (GN4′) and the mean value theorem, there exists $\xi \in (0, h)$ such that

$$\Big|\frac{1}{h'}\sum_{i=2}^n G_i(h)\frac{\partial \alpha}{\partial x_i}\Big| = \Big|\frac{1}{h'}\sum_{i=2}^n \Big(G_i(h) - G_i(0)\Big)\frac{\partial \alpha}{\partial x_i}\Big| = \Big|\frac{h}{h'}\sum_{i=2}^n G_i'(\xi_i)\frac{\partial \alpha}{\partial x_i}\Big|,$$

which is bounded. Hence, $r(g)$ is bounded for $x_1 > 1$.
Thus since $r(g)$ is a continuous function in Ω, it follows that $\sup_{x\in\Omega} r(g)(x_1, y) < +\infty$, and thus $c^\star < +\infty$.

□

We now provide some sufficient assumptions for the inequality $c^\star > -\infty$ to hold. Obviously, in the special case $G = (G_1, 0, \cdots, 0)$, we are back to the case of Proposition 2.2. In the general case, with this second form of convection term, we need fewer assumptions than in Proposition 2.3 :

Proposition 4.2. *Assume* **(GN1′)** *and* **(AlphaN2′)**. *Then* $c^\star > -\infty$.

Proof. Assume that $c^\star = -\infty$, and let $A > 0$ and $\tilde{G} := (G_2, \cdots, G_n)$. By definition of $c^\star$, there exists $\rho \in K$ such that

$$\frac{\Delta\rho + \alpha(y)G_1'(\rho)\partial_1\rho + \nabla' \cdot \Big(\alpha(y)\tilde{G}(\rho)\Big) + f(\rho)}{-\partial_1\rho} < -A.$$

As in the proof of Proposition 2.3, since $\partial_1\rho < 0$ we obtain, by integrating on D, :

$$\int_D \partial_{11}\rho + (k - A)\int_D \partial_1\rho < -\Big(\int_D \nabla' \cdot \Big(\alpha(y)\tilde{G}(\rho)\Big) + \int_D f(\rho)\Big), \tag{4.121}$$

where $k := \sup_{(y,s)\in\overline{D}\times[0,1]} \alpha(y)G_1'(s)$. Using that $\alpha \equiv 0$ on ∂D, one has :

$$\int_D \nabla' \cdot \Big(\alpha(y)\tilde{G}(\rho)\Big) + \int_D f(\rho) = \int_D f(\rho) \geq 0.$$

Hence, the right hand side of (4.121) is negative and we conclude as in the proof of Proposition 2.2.

□

Under certain assumptions, we will construct a solution of (4.117) in the truncated cylinder $\Omega_N = (-N, N) \times D$. Let $c > c^\star$. By definition of $c^\star$, there exists a supersolution ρ of (4.117). As in Proposition 2.4, let $N \geq 1$, and choose $\varepsilon_N \in (0, 1)$ such that $\rho(x_1, y) > \varepsilon_N$, for all $x_1 \in [-N, N]$ and $y \in \overline{D}$.

Proposition 4.3. *Assume the condition* (4.119), **(GN1′)** *and* **(AlphaN1′)**. *Assume also that the function f is C^1 and satisfies* (1.30). *Then, there exists a unique solution $u \in W^{2,p}_{loc}\Big((-N, N) \times \overline{D}\Big)$ which satisfies*

- $\rho(-N, y) \geq u(x_1, y) \geq \varepsilon_N$ *for all* $(x_1, y) \in (-N, N) \times D$,
- *for all* $x_1 \in (-N, N)$ *there exists* $y \in D$ *such that* $\rho(-N, y) > u(x_1, y)$,

of the following problem :

$$\begin{cases} \Delta u + \Big(c + \alpha(y)G_1'(u)\Big)\partial_1 u + \alpha(y)\sum_{i=2}^n G_i'(u)\frac{\partial u}{\partial x_i} + \sum_{i=2}^n G_i(u)\frac{\partial \alpha}{\partial x_i} + f(u) = 0 \ \text{in } \Omega_N \\ u_\nu = 0 & (x_1, y) \in (-N, N) \times \partial D, \\ u(-N, y) = \rho(-N, y), \quad u(N, y) = \varepsilon_N & y \in D. \end{cases} \tag{4.122}$$

Proof. The approach is similar as in chapter 2 and chapter 3, we focus on the places where the arguments are different. We start by proving the following lemma :

Lemma 4.4. *Asssume the condition* (4.119) *and that* $\varepsilon_N > 0$ *is sufficiently small. Then the following condition holds : for all* $y \in D$,

$$\sum_{i=2}^{n} G_i(\varepsilon_N)\frac{\partial \alpha}{\partial x_i}(y) + f(\varepsilon_N) > 0. \tag{4.123}$$

Proof. Fix y in D. Then

$$\begin{aligned}\sum_{i=2}^{n} G_i(\varepsilon_N)\frac{\partial \alpha}{\partial x_i}(y) + f(\varepsilon_N) &= \sum_{i=2}^{n} G_i(\varepsilon_N)\frac{\partial \alpha}{\partial x_i}(y) + f(\varepsilon_N) - \Big(\sum_{i=2}^{n} G_i(0)\frac{\partial \alpha}{\partial x_i}(y) + f(0)\Big)\\ &= \Big(\sum_{i=2}^{n} G_i'(\xi_i)\frac{\partial \alpha}{\partial x_i}(y) + f'(\xi)\Big)\varepsilon_N,\end{aligned}$$

for some $\xi \in (0, \varepsilon_N)$ and $\xi_i \in (0, \varepsilon_N)$.
Note that

$$\begin{aligned}\lim_{\varepsilon_N \to 0}\Big(\sum_{i=2}^{n} G_i'(\xi_i)\frac{\partial \alpha}{\partial x_i}(y) + f'(\xi)\Big) &= \sum_{i=2}^{n} G_i'(0)\frac{\partial \alpha}{\partial x_i}(y) + f'(0)\\ &\geq \inf_{z\in\overline{D}}\Big\{\sum_{i=2}^{n} G_i'(0)\frac{\partial \alpha}{\partial x_i}(z)\Big\} + f'(0) > 0.\end{aligned}$$

Hence, for ε_N sufficiently small (4.123) holds. □

As in the proof of Proposition 2.4, we use the method of sub and super solutions. First, note that the function ρ is a supersolution of (4.122), and under the condition (4.123), the function ε_N is a subsolution of (4.122). The proof is almost the same as the proof of Proposition 2.4. We detail some points that are different. Precisely, we will prove that $\underline{u} \leq u_j \leq \overline{u}$, where $\underline{u}$ and $\overline{u}$ are the sub and supersolutions ε_N and ρ, and where the sequence of functions $(u_j)_{j\geq 0}$ satisfies

$$\begin{cases} \Delta u_{j+1} + c\frac{\partial u_{j+1}}{\partial x_1} + \alpha(y)\sum_{i=1}^{n} G_i'(u_j)\frac{\partial u_{j+1}}{\partial x_i} + \sum_{i=2}^{n} G_i(u_j)\frac{\partial \alpha}{\partial x_i} - k_0 u_{j+1} & = -f(u_j) - k_0 u_j,\\ \frac{\partial u_{j+1}}{\partial \nu} = 0 & \text{on } (-N,N)\times\partial D,\end{cases} \tag{4.124}$$

and $\underline{u} = u_0 = \varepsilon_N$, $\overline{u} = \rho$.
First, we will prove that $\underline{u} \leq u_j$ by induction. It is obvious for $j = 0$. Assume that $\underline{u} \leq u_j$ for some $j \in \mathbb{N}$. The functions u_{j+1} and $\underline{u}$ satisfy

$$\begin{cases} \Delta u_{j+1} + c\frac{\partial u_{j+1}}{\partial x_1} + \alpha(y)\sum_{i=1}^{n} G_i'(u_j)\frac{\partial u_{j+1}}{\partial x_i} - k_0 u_{j+1} & = -f(u_j) - k_0 u_j - \sum_{i=2}^{n} G_i(u_j)\frac{\partial \alpha}{\partial x_i},\\ \Delta \underline{u} + c\frac{\partial \underline{u}}{\partial x_1} + \alpha(y)\sum_{i=1}^{n} G_i'(\underline{u})\frac{\partial \underline{u}}{\partial x_i} & \geq -f(\underline{u}) - \sum_{i=2}^{n} G_i(\underline{u})\frac{\partial \alpha}{\partial x_i}.\end{cases} \tag{4.125}$$

By subtraction, one has

$$\Delta(\underline{u}-u_{j+1})+c\frac{\partial}{\partial x_1}(\underline{u}-u_{j+1})+\alpha(y)\sum_{i=1}^{n}\Big(G_i'(\underline{u})-G_i'(u_j)\Big)\frac{\partial \underline{u}}{\partial x_i}+\alpha(y)\sum_{i=1}^{n}G_i'(u_j)\frac{\partial}{\partial x_i}(\underline{u}-u_{j+1})$$
$$-k_0(\underline{u}-u_{j+1})\geq -f(\underline{u})-k_0\underline{u}+f(u_j)+k_0u_j+\sum_{i=2}^{n}\Big(G_i(u_j)-G_i(\underline{u})\Big)\frac{\partial\alpha}{\partial x_i},$$

which gives

$$\begin{aligned}
&\Delta(\underline{u}-u_{j+1})+c\frac{\partial}{\partial x_1}(\underline{u}-u_{j+1})+\alpha(y)\sum_{i=1}^{n}G_i'(u_j)\frac{\partial}{\partial x_i}(\underline{u}-u_{j+1})-k_0(\underline{u}-u_{j+1})\\
&\quad\geq f(u_j)-f(\underline{u})+k_0(u_j-\underline{u})+\alpha(y)\sum_{i=1}^{n}\Big(G_i'(u_j)-G_i'(\underline{u})\Big)\frac{\partial \underline{u}}{\partial x_i}+\sum_{i=2}^{n}\Big(G_i(u_j)-G_i(\underline{u})\Big)\frac{\partial\alpha}{\partial x_i}\\
&\quad\geq -L(u_j-\underline{u})+k_0(u_j-\underline{u})-\tilde{L}C(u_j-\underline{u})-L'C'(u_j-\underline{u})\qquad\text{since } \underline{u}\leq u_j\leq\overline{u}\\
&\quad\geq 0,
\end{aligned}$$

where C' is such that

$$C'\geq\Big|\sum_{i=2}^{n}\frac{\partial\alpha}{\partial x_i}\Big|,$$

and we choose $k_0>L+\tilde{L}C+L'C'$.
Similarly, the functions u_{j+1} and $\overline{u}$ satisfy

$$\begin{cases}\Delta u_{j+1}+c\frac{\partial u_{j+1}}{\partial x_1}+\alpha(y)\sum_{i=1}^{n}G_i'(u_j)\frac{\partial u_{j+1}}{\partial x_i}-k_0u_{j+1} & =-f(u_j)-k_0u_j-\sum_{i=2}^{n}G_i(u_j)\frac{\partial\alpha}{\partial x_i},\\ \Delta\overline{u}+c\frac{\partial\overline{u}}{\partial x_1}+\alpha(y)\sum_{i=1}^{n}G_i'(\overline{u})\frac{\partial\overline{u}}{\partial x_i} & \leq -f(\overline{u})-\sum_{i=2}^{n}G_i(\overline{u})\frac{\partial\alpha}{\partial x_i}.\end{cases}\tag{4.126}$$

Again, the subtraction leads to

$$\begin{aligned}
&\Delta(\overline{u}-u_{j+1})+c\frac{\partial}{\partial x_1}(\overline{u}-u_{j+1})+\alpha(y)\sum_{i=1}^{n}G_i'(u_j)\frac{\partial}{\partial x_i}(\overline{u}-u_{j+1})-k_0(\overline{u}-u_{j+1})\\
&\quad\leq f(u_j)-f(\overline{u})+k_0(u_j-\overline{u})+\alpha(y)\sum_{i=1}^{n}\Big(G_i'(u_j)-G_i'(\overline{u})\Big)\frac{\partial \overline{u}}{\partial x_i}+\sum_{i=2}^{n}\Big(G_i(u_j)-G_i(\overline{u})\Big)\frac{\partial\alpha}{\partial x_i}\\
&\quad\leq -L(u_j-\overline{u})+k_0(u_j-\overline{u})-\tilde{L}C(u_j-\overline{u})-L'C'(u_j-\overline{u})\qquad\text{since } \underline{u}\leq u_j\leq\overline{u}\\
&\quad\leq 0.
\end{aligned}$$

As in the proof of Proposition 2.4, Maximum Principle and Hopf lemma arguments ensure that $\underline{u}\leq u_{j+1}\leq\overline{u}$. By induction, we proved that for all $j\in\mathbb{N}$,

$$\underline{u}\leq u_j\leq\overline{u}.$$

As before, we can take a diagonal subsequence of the sequence of functions $(u_j)_{j\geq 0}$. This sequence converges strongly in $C^{1,\lambda}$ on compact subsets of Ω_N to a function u which satisfies

$$\Delta u + (c + \alpha(y)G_1'(u))\partial_1 u + \alpha(y)\sum_{i=2}^{n} G_i'(u)\frac{\partial u}{\partial x_i} + \sum_{i=2}^{n} G_i(u)\frac{\partial \alpha}{\partial x_i} + f(u) = 0 \text{ in } \Omega_N.$$

Finally, we can use the exactly same barrier function (2.62) as in the proof of Proposition 2.4 to be sure that the solution u will satisfy the boundary conditions $u(N, y) = \varepsilon_N$, $u(-N, y) = \rho(-N, y)$ for $y \in D$, and $\frac{\partial u}{\partial \nu} = 0$ for $-N < x_1 < N$ and $y \in \partial D$.

Hence, for each N sufficiently large, we constructed a solution u of (4.122). Note that we have the monotonicity with respect to x_1 of the solution u of (4.122), as in Proposition 5.5. Indeed, [BN1, Theorem 2.4] yields the result again.

□

Now that we have a solution on a truncated cylinder Ω_N, we can argue as before to let N tend to infinity.

Theorem 4.5. *Assume that* (4.119) *holds. Assume also the assumptions* **(GN1′)**, **(GN4′)**, **(AlphaN1′)**, **(AN′)** *and that f is C^1 and satisfies* (1.30). *Then, for $c > c^\star$, there exists at least one solution of* (4.117).
In addition this solution is decreasing with respect to x_1.
Moreover, there exists also a solution of (4.117) *with speed $c = c^\star$.*

Proof. The proofs are analogous to those in the proof of Theorem 1.17 and the proof of Proposition 2.9.
We will focus on the point of Theorem 4.5 that a solution of (1.32) is decreasing with respect to x_1.
Let w be the solution of (4.117) we constructed. Note that the function w satisfies

$$\lim_{x_1 \to \pm\infty} \|\nabla w(x_1, .)\|_{L^\infty(D)} = 0. \tag{4.127}$$

Indeed, the proof of Proposition 2.8 is still valid, even with the second form of the convection term.
Using the fact that $w \in W^{2,p}_{loc}$, bootstrapping and standard regularity results, see [KRY, Chapter 9 - Section 4 - Theorem 1], it follows that we can differentiate with respect to x_1 the equation (4.117) satisfied by w as in the proof of Theorem 1.17

(see equation (2.71))to obtain that the function $v(x,t) := -\frac{\partial}{\partial x_1}w(x_1-ct,y)$ satisfies

$$\begin{aligned} v_t &= \Delta v + \alpha(y)\sum_{i=1}^n G_i''(w)\frac{\partial w}{\partial x_i}v + \alpha(y)\sum_{i=1}^n G_i'(w)\frac{\partial v}{\partial x_i} + \sum_{i=2}^n G_i'(w)\frac{\partial \alpha}{\partial x_i}v + f'(w)v \\ &= \Delta v + b(x,t)\cdot\nabla v + c(x,t)v, \end{aligned}$$

where for all $1 \leq i \leq n$,

$$b_i(x,t) := \alpha(y)G_i'\Big(w(x_1-ct,y)\Big)$$

and

$$c(x,t) := \alpha(y)\sum_{i=1}^n G_i''\Big(w(x_1-ct,y)\Big)\frac{\partial w}{\partial x_i}+\sum_{i=2}^n G_i'\Big(w(x_1-ct,y)\Big)\frac{\partial \alpha}{\partial x_i}+f'\Big(w(x_1-ct,y)\Big).$$

By construction, one has $v \geq 0$. Using (4.127), it follows that

$$c(x,0) := \alpha(y)\sum_{i=1}^n G_i''\Big(w(x_1,y)\Big)\frac{\partial w}{\partial x_i} + \sum_{i=2}^n G_i'\Big(w(x_1,y)\Big)\frac{\partial \alpha}{\partial x_i} + f'\Big(w(x_1,y)\Big) \not\equiv 0$$

since as $x_1 \to +\infty$, and uniformly in $y \in D$, one has

$$c(x,0) \to \sum_{i=2}^n G_i'(0)\frac{\partial \alpha}{\partial x_i} + f'(0)$$

which is positive by (4.119). It follows from [VO1, Chapter 2 - Theorem 3.26] that either $v \equiv 0$ or $v > 0$. Since w converges to different limits when $x_1 \to \pm\infty$, it follows that $v \not\equiv 0$ and hence $v > 0$, meaning that $\partial_1 w < 0$ and that w is decreasing with respect to x_1.

□

4.2 Linearized operator and eigenvalue problem

Now we will show the existence of a critical value c' as in Proposition 3.4, under some additional assumptions. To do this, consider the linearized problem of (4.117) around 0

$$\begin{cases} \Delta w + \Big(c+\alpha(y)G_1'(0)\Big)\partial_1 w + \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial w}{\partial x_i} + \Big(\sum_{i=2}^n G_i'(0)\frac{\partial \alpha}{\partial x_i} + f'(0)\Big)w = 0 & \text{in } \mathbb{R}\times D, \\ w_\nu = 0 & \text{on } \mathbb{R}\times\partial D. \end{cases} \tag{4.128}$$

If $w(x_1,y) = e^{-\lambda x_1}\varphi(y)$, then the function φ satisfies

$$\begin{cases} -\Delta'\varphi - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial \varphi}{\partial x_i} - \Big(\sum_{i=2}^n G_i'(0)\frac{\partial \alpha}{\partial x_i} + f'(0)\Big)\varphi = \Big(\lambda^2 - \lambda(c+\alpha(y)G_1'(0))\Big)\varphi & \text{in } D, \\ \varphi_\nu = 0 & \text{on } \partial D. \end{cases} \tag{4.129}$$

As before, [YIH, Theorem 1.3], the following eigenvalue problem

$$\begin{cases} -\Delta'\sigma - \alpha(y)\sum_{i=2}^{n} G_i'(0)\frac{\partial\sigma}{\partial x_i} - \Big(\sum_{i=2}^{n} G_i'(0)\frac{\partial\alpha}{\partial x_i} + f'(0)\Big)\sigma = \mu_2\sigma & \text{in } D, \\ \sigma_\nu = 0 & \text{on } \partial D, \end{cases} \tag{4.130}$$

has a simple eigenvalue $\mu_2 \in \mathbb{R}$, which corresponds to a positive eigenfunction. We claim :

Proposition 4.6. *Assume* **(GN1′)**, **(AlphaN1′)** *and* (4.119). *Then the principal eigenvalue* μ_2 *of* (4.130) *is strictly negative.*

Proof. Let

$$-L_1 := -\Delta' - \alpha(y)\sum_{i=2}^{n} G_i'(0)\frac{\partial}{\partial x_i} - f'(0),$$

and

$$-L_2 := -\Delta' - \alpha(y)\sum_{i=2}^{n} G_i'(0)\frac{\partial}{\partial x_i} - \Big(\sum_{i=2}^{n} G_i'(0)\frac{\partial\alpha}{\partial x_i} + f'(0)\Big).$$

Denote by μ_1 and μ_2 the principal eigenvalues of L_1 and L_2 characterised by, for $1 \le j \le 2$

$$\mu_j = \sup\Big\{\mu \mid \exists\phi > 0 \text{ in } D,\ \phi_\nu = 0 \text{ on } \partial D,\ (L_j + \mu)\phi \le 0\Big\}.$$

Let $\tilde{k} := \inf_{y\in D}\sum_{i=2}^{n} G_i'(0)\frac{\partial\alpha}{\partial x_i}(y)$. One has

$$\begin{aligned} \mu_1 &= \sup\Big\{\mu \mid \exists\phi > 0 \text{ in } D,\ \phi_\nu = 0 \text{ on } \partial D,\ (L_1 + \mu)\phi \le 0\Big\} \\ &= \sup\Big\{\mu + \tilde{k} \mid \exists\phi > 0 \text{ in } D,\ \phi_\nu = 0 \text{ on } \partial D,\ (L_1 + \mu + \tilde{k})\phi \le 0\Big\} \\ &= \tilde{k} + \sup\Big\{\mu \mid \exists\phi > 0 \text{ in } D,\ \phi_\nu = 0 \text{ on } \partial D,\ (L_1 + \mu + \tilde{k})\phi \le 0\Big\} \\ &\ge \tilde{k} + \sup\Big\{\mu \mid \exists\phi > 0 \text{ in } D,\ \phi_\nu = 0 \text{ on } \partial D,\ \Big(L_1 + \mu + \sum_{i=2}^{n} G_i'(0)\frac{\partial\alpha}{\partial x_i}\Big)\phi \le 0\Big\} \\ &= \tilde{k} + \sup\Big\{\mu \mid \exists\phi > 0 \text{ in } D,\ \phi_\nu = 0 \text{ on } \partial D,\ (L_2 + \mu)\phi \le 0\Big\} \\ &= \tilde{k} + \mu_2. \end{aligned}$$

Since $\mu_1 = -f'(0)$, we proved that $\mu_2 \le -f'(0) - \inf_{y\in D}\sum_{i=2}^{n} G_i'(0)\frac{\partial\alpha}{\partial x_i}$. Hence, $\mu_2 < 0$ if the condition (4.119) is satisfied.

□

Consider now the following eigenvalue problem depending on $t \in \mathbb{R}$:

$$\begin{cases} -\Delta'\varphi - \alpha(y)\sum_{i=2}^{n} G_i'(0)\frac{\partial\varphi}{\partial x_i} - \Big(\sum_{i=2}^{n} G_i'(0)\frac{\partial\alpha}{\partial x_i} + f'(0)\Big)\varphi + t\beta_c(y)\varphi = \mu_2^c(t)\varphi & \text{in } D, \\ \varphi_\nu = 0 & \text{on } \partial D, \end{cases} \tag{4.131}$$

where $\beta_c(y) = c + \alpha(y)G_1'(0)$.
As before, one has that λ is an eigenvalue of (4.129) if and only if λ satisfies

$$\mu_2^c(\lambda) = \lambda^2.$$

Proposition 4.7. *Assume* **(GN1′)***,* **(AlphaN1′)** *and* (4.119)*. Then the conclusion of Proposition 3.4 holds. In particular, there exists a critical value c' such that if $c > c'$, the eigenvalue problem* (4.129) *has two positive eigenvalues* $0 < \lambda_1 < \lambda_2$*, and only one positive for $c = c'$.*

Proof. The arguments are the same as in the proof of Proposition 3.4, using the concavity of $t \mapsto \mu_2^c(t)$, and the fact that $\mu_2^c(0) < 0$ by Proposition 4.6. □

4.3 Comparison between $c^\star$ and c'

Now that we have established the existence of the critical value c', we will compare $c^\star$ and c'. Precisely, we will prove Theorem 1.19 :

Proof of Theorem 1.19. We start by showing that we have the same conclusions as Proposition 3.6 and Theorem 3.8. Note that in this case, we only need that $\alpha \equiv 0$ on ∂D and that $G_1(u)/u$, is bounded, which is satisfied if $G_1(0) = 0$ and **(AlphaN2′)** holds.
We first show that we have the same conclusion as Proposition 3.6. Let w be a solution on Ω of

$$\Delta w + \Big(c + \alpha(y)G_1'(w)\Big)\partial_1 w + \alpha(y)\sum_{i=2}^{n} G_i'(w)\frac{\partial w}{\partial x_i} + \sum_{i=2}^{n} G_i(w)\frac{\partial \alpha}{\partial x_i} + f(w) = 0. \quad (4.132)$$

Let $N > R > 0$ and define the cut-off function

$$\xi(x_1) := \begin{cases} 0 & \text{if } x_1 \geq N+1, \\ 1 & \text{if } R \leq x_1 \leq N, \\ 0 & \text{if } x_1 \leq R-1. \end{cases}$$

By multiplying (4.132) by ξ, integrating over Ω and using Green's formula, it follows that

$$\int_\Omega w\xi'' - \int_\Omega \xi' w\Big(c + \alpha(y)\frac{G_1(w)}{w}\Big) + \int_\Omega \xi\alpha(y)\sum_{i=2}^{n} G_i'(w)\frac{\partial w}{\partial x_i} + \int_\Omega \xi\sum_{i=2}^{n} G_i(w)\frac{\partial \alpha}{\partial x_i} + \int_\Omega \xi f(w) = 0. \quad (4.133)$$

Using Green's formula on the third integral and that $\alpha \equiv 0$ on ∂D, one has :

$$\int_\Omega \xi\alpha(y)\sum_{i=2}^{n} G_i'(w)\frac{\partial w}{\partial x_i} = -\int_\Omega \xi\sum_{i=2}^{n} G_i(w)\frac{\partial \alpha}{\partial x_i}.$$

Thus, the equation (4.133) becomes

$$\int_\Omega \xi f(w) = \int_\Omega w\Big[\xi'\Big(c+\alpha(y)\frac{G_1(w)}{w}\Big) - \xi''\Big] \tag{4.134}$$

Note that

$$\int_\Omega \xi f(w) \geq \int_R^N \int_D f(w),$$

because $\xi = 1$ in $[R,N]$, $\xi \geq 0$ and $f \geq 0$. Since u tends uniformly to 0, with respect to y, when x_1 tends to infinity, we can take R sufficiently large such that there exists $\delta > 0$ such that $f(w) \geq (1-\delta)f'(0)w$ when $x_1 \geq R$, and we obtain

$$\int_R^N \int_D (1-\delta)f'(0)w \leq \int_\Omega w\Big[\xi'\Big(c+\alpha(y)\frac{G_1(w)}{w}\Big) - \xi''\Big].$$

Then, since ξ is constant on $(-\infty, R-1] \cup [R,N] \cup [N+1,+\infty)$, it follows that

$$\begin{aligned}\int_\Omega w\Big[\xi'\Big(c+\alpha(y)\frac{G_1(w)}{w}\Big) - \xi''\Big] &= \Big(\int_{R-1}^R + \int_N^{N+1}\Big)\int_D w\Big[\xi'\Big(c+\alpha(y)\frac{G_1(w)}{w}\Big) - \xi''\Big] \\ &\leq K\Big[\int_{R-1}^R \int_D w + \int_N^{N+1}\int_D w\Big],\end{aligned}$$

where K is such that

$$\Big|\xi'\Big(c+\alpha(y)\frac{G_1(w)}{w}\Big) - \xi''\Big| \leq K, \qquad \text{for all } y \in D.$$

Hence, we obtain

$$(1-\delta)f'(0)\int_R^N \int_D w \leq K\Big[\int_{R-1}^R \int_D w + \int_N^{N+1}\int_D w\Big]. \tag{4.135}$$

We argue as in the proof of Proposition 3.6 and let N tend to infinity to obtain :

$$(1-\delta)f'(0)\int_R^\infty \int_D w \leq K\int_{R-1}^R \int_D w,$$

and we conclude as in the proof of Proposition 3.6, using the function

$$g(R) := \int_R^\infty \int_D w,$$

that there exist $C > 0$ and $\varepsilon > 0$ such that for all $R > 0$,

$$\int_R^\infty \int_D w \leq Ce^{-\varepsilon R}.$$

Now we will prove the conclusion of Theorem 1.19, namely, that $c^\star \geq c'$.
Let $c > c^\star$, and consider a solution u of (4.117) with speed c.
As in the proof of Theorem 1.18, we will construct a positive solution of the linearized problem (4.128). To do this, define

$$h^N(x_1, y) := \frac{u(x_1^N + x_1, y)}{2\sup_{y\in\overline{D}} u(x_1^N, y)},$$

where $(x_1^N)_N$ is chosen as in the proof of Theorem 1.18. The function h^N satisfies

$$\Delta h^N + \Big(c + \alpha(y)G_1'\Big(u(x_1^N + x_1, y)\Big)\Big)\partial_1 h^N + \alpha(y)\sum_{i=2}^n G_i'\Big(u(x_1^N + x_1, y)\Big)\frac{\partial h^N}{\partial x_i}$$
$$+ \sum_{i=2}^n \frac{G_i(u(x_1^N + x_1, y))}{2\sup u(x_1^N, y)}\frac{\partial \alpha}{\partial x_i} + \frac{f\Big(u(x_1^N + x_1, y)\Big)}{2\sup_{y\in\overline{D}} u(x_1^N, y)} = 0.$$

As before, a compactness argument ensures that the sequence of functions $(h^N)_N$ converges weakly in $W^{2,p}_{loc}$ and strongly in $C^{1,\lambda}_{loc}$ to a function h when $N \to +\infty$. Moreover, when N tends to $+\infty$, the extra term will converge to

$$\lim_{N\to+\infty} \sum_{i=2}^n \frac{G_i(u(x_1^N + x_1, y))}{2\sup u(x_1^N, y)}\frac{\partial \alpha}{\partial x_i} = \sum_{i=2}^n \lim_{N\to+\infty} \frac{G_i(u(x_1^N + x_1, y))}{u(x_1^N + x_1, y)} h^N \frac{\partial \alpha}{\partial x_i} = \sum_{i=2}^n G_i'(0) h \frac{\partial \alpha}{\partial x_i}.$$

Hence, the function h satisfy the linearized problem (4.128) and the condition $\sup_{y\in\overline{D}} h(0, y) = \frac{1}{2}$. Furthermore, since we assumed the condition (4.119), one has

$$\begin{aligned}
\Delta h + \Big(c + \alpha(y)G_1'(0)\Big)\partial_1 h + \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial h}{\partial x_i} &= -\Big(\sum_{i=2}^n G_i'(0)\frac{\partial \alpha}{\partial x_i} + f'(0)\Big)h \\
&\leq -\Big(\inf_{y\in D}\Big\{\sum_{i=2}^n G_i'(0)\frac{\partial \alpha}{\partial x_i}\Big\} + f'(0)\Big)h \\
&\leq 0.
\end{aligned}$$

As in the proof of Theorem 1.18, note that in order to have that $h_\nu = 0$ on $\mathbb{R} \times \partial D$, we actually argue as in the proof of Proposition 2.8 to have the $C^{1,\lambda}$ convergence up to the boundary, see also Remark 1.

As before, Maximum Principle and Hopf lemma arguments ensure that $h > 0$ in $\mathbb{R} \times D$. The rest of the proof is identical to that of Theorem 1.18. □

4.4 Special case where $G = (G_1, 0, \cdots, 0)$.

With our alternative form of convection term, we can handle only one of the special cases. Note that in the special case where $G_1'(0) = 0$, the eigenvalue problem (4.129) becomes

$$\begin{cases} -\Delta'\varphi - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial\varphi}{\partial x_i} - \Big(\sum_{i=2}^n G_i'(0)\frac{\partial\alpha}{\partial x_i} + f'(0)\Big)\varphi = \Big(\lambda^2 - \lambda c\Big)\varphi & \text{in } D, \\ \varphi_\nu = 0 & \text{on } \partial D. \end{cases} \tag{4.136}$$

Then, since the extra term $\sum_{i=2}^n G_i'(0)\frac{\partial\alpha}{\partial x_i}$ depends on y, we cannot conclude as in the proof of Proposition 3.10 that $\lambda^2 - \lambda c = -f'(0)$. The extra term prevents us to get a supersolution as in the proof of Proposition 3.11. Thus, we cannot conclude that $c' = c^*$ in the case where $G_1'(0) = 0$. However, in the case where $G = (G_1, 0, \cdots, 0)$, we claim :

Proposition 4.8. *Assume that* $G = (G_1, 0, \cdots, 0)$, *with* $G_1 \neq 0$, *and that* f *satisfies the KPP condition* (1.31). *Assume also that for* $y \in D$ *and* $u \in \mathbb{R}$, $\alpha(y)G_1'(u) \geq \alpha(y)G_1'(0)$. *Then* $c' \geq c^\star$.

Proof. In this case, we obtain the same eigenvalue problem as (3.106). Hence, the proof is the same as the proof of Proposition 3.9. □

5 Existence of front solutions for Dirichlet boundary conditions

5.1 Introduction

Recall that under Dirichlet boundary conditions and with the first form of the convection term $\alpha(y)\nabla \cdot G(u)$, if $u(x,t) = w(x_1 - ct, y)$ is a solution of (1.29), the function w satisfies :

$$\begin{cases} -c\partial_1 w = \Delta w + \alpha(y)\nabla \cdot G(w) + f(w) & \text{in } \Omega, \\ w(-\infty, y) = w_-(y), \quad w(+\infty, y) = 0 & \text{uniformly in } y \in D, \\ w \geq 0, & \\ w = 0 & \text{on } \mathbb{R} \times \partial D. \end{cases} \tag{5.137}$$

Recall also that in that chapter, we assume that the function f satisfies (1.30), and the following conditions :

- **(GD) :** The function G is C^2 and satisfies $G_i(0) = 0$ for all $2 \leq i \leq n$.
- **(AlphaD) :** The function $\alpha : D \to \mathbb{R}$ is in $C^1(\overline{D})$.
- **(AD) :** Let $\lambda \in (0,1)$. There exists a non negative function $w_- \in C^{2,\lambda}(\overline{D})$ such that the only solutions in $C^{2,\lambda}(\overline{D})$ of the problem on the cross section D

are w_- and 0.
Precisely, if $z : D \to \mathbb{R}$ satisfies

$$\begin{cases} \Delta' z + \alpha(y) \sum_{i=2}^n G_i'(z) \frac{\partial z}{\partial x_i} + f(z) = 0 & \text{in } D, \\ z = 0 & \text{on } \partial D, \end{cases} \tag{5.138}$$

then $z \equiv 0$ or $z \equiv w_-$.

Since constants are no longer subsolutions of the problem (5.145) on the truncated cylinder Ω_N, we assume the following assumptions to ensure that there exist subsolutions of the problem (5.145) on the truncated cylinder Ω_N, :

- **(BD) :** There exists a sequence of functions $(v_k(y))_{k\in\mathbb{N}}$ uniformly bounded in $C^{2,\lambda}(\overline{D})$, and which tends uniformly to 0 when k tends to $+\infty$ and which satisfies for every $k \in \mathbb{N}$

$$0 < v_{k+1}(y) < v_k(y) < w_-(y), \text{ for } y \in D,$$

and

$$\begin{cases} \Delta' v_k + \alpha(y) \sum_{i=2}^n G_i'(v_k) \frac{\partial v_k}{\partial x_i} + f(v_k) \geq 0 & \text{in } D, \\ v_k = 0 & \text{on } \partial D. \end{cases} \tag{5.139}$$

- **(FD) :** The following conditions holds :

$$f'(0) > \lambda_1(-L),$$

where $-L := -\Delta' - \alpha(y) \sum_{i=2}^n G_i'(0) \frac{\partial}{\partial x_i}$.

We will prove later in section 6.4 a result about the existence of the functions $(v_k(y))_{k\in\mathbb{N}}$.

Remark 5.1. Finally, we also recall that in this case too, we are not losing generality assuming that $G_i(0) = 0$ for all $1 \leq i \leq n$, see Remark 1.12.

First, we will prove the existence of travelling front solutions on the truncated cylinder $\Omega_N := (-N, N) \times D$, and then pass to the limit using translates of these solutions problem on the half cylinder.

5.2 Existence of a solution on a truncated cylinder

Denote by $\tilde{K}$ the set of functions $\rho \in C^2(\mathbb{R} \times \overline{D})$ such that

$$\begin{cases} \partial_1 \rho < 0 & \text{in } \mathbb{R} \times D, \\ \lim_{x_1 \to -\infty} \rho(x_1, y) = w_-(y) & \text{uniformly in } y \in D, \\ \lim_{x_1 \to +\infty} \rho(x_1, y) = 0 & \text{uniformly in } y \in D, \\ \rho = 0 & \text{on } \mathbb{R} \times \partial D, \\ \frac{\partial \rho}{\partial \nu} < 0 & \text{for } (x_1, y) \in \mathbb{R} \times \partial D. \end{cases} \tag{5.140}$$

For $\rho \in \tilde{K}$, let

$$r(\rho)(x) := \frac{\Delta \rho(x) + \alpha(y) \nabla \cdot G(\rho(x)) + f(\rho(x))}{-\partial_1 \rho(x)}, \quad x \in \Omega, \tag{5.141}$$

and

$$c^\star := \inf_{\rho \in K} \sup_{x \in \Omega} r(\rho)(x). \tag{5.142}$$

Proposition 5.2. *Assume* **(GD)**, **(AlphaD)** *and that f is Lipschitz continuous. Then*

$$c^\star < +\infty.$$

Proof. Define the following function $h : \mathbb{R} \times D \to \mathbb{R}$

$$h(x_1, y) = \begin{cases} (1 - e^{x_1}) w_-(y) & \text{if } x_1 < -1, \\ e^{-x_1} w_-(y) & \text{if } x_1 > 1, \end{cases} \tag{5.143}$$

such that $\partial_1 h < 0$ for all $x \in \Omega$ and h is sufficiently smooth for $(x_1, y) \in (-1, 1) \times \mathbb{R}^{n-1}$ to ensure that $h \in C^2(\mathbb{R} \times \overline{D})$, and such that there exists $p \in C^2((-1, 1))$ such that we can write $h(x_1, y) := p(x_1) w_-(y)$ for $x_1 \in (-1, 1)$.
We will show that $h \in \tilde{K}$. First, one has $\partial_1 h < 0$, and

$$h(x_1, y) \to w_-(y) \text{ as } x_1 \to -\infty, \text{ and } h(x_1, y) \to 0 \text{ as } x_1 \to +\infty,$$

uniformly with respect to $y \in D$.
Furthermore, if $x_1 < -1$, using the equation satisfied by w_-, one has

$$
\begin{aligned}
r(h)(x_1,y) &= \frac{-e^{x_1}w_- + (1-e^{x_1})\Delta' w_- + \alpha(y)G_1'(h)(-e^{x_1}w_-) + \alpha(y)\sum_{i=2}^n G_i'(h)(1-e^{x_1})\frac{\partial w_-}{\partial x_i} + f(h)}{e^{x_1}w_-}\\
&= -1 - \alpha(y)G_1'(h) + (1-e^{x_1})\frac{\Delta' w_- + \alpha(y)\sum_{i=2}^n G_i'(h)\frac{\partial w_-}{\partial x_i}}{e^{x_1}w_-} + \frac{f(h)}{e^{x_1}w_-}\\
&= -1 - \alpha(y)G_1'(h) + \frac{1-e^{x_1}}{e^{x_1}w_-}\Big[\alpha(y)\sum_{i=2}^n \Big(G_i'(h) - G_i'(w_-)\Big)\frac{\partial w_-}{\partial x_i} - f(w_-)\Big] + \frac{f(h)}{e^{x_1}w_-}\\
&= -1 - \alpha(y)G_1'(h) + (1-e^{x_1})\alpha(y)\sum_{i=2}^n \frac{G_i'((1-e^{x_1})w_-) - G_i'(w_-)}{e^{x_1}w_-}\frac{\partial w_-}{\partial x_i}\\
&+ \frac{f\Big((1-e^{x_1})w_-\Big) - (1-e^{x_1})f(w_-)}{e^{x_1}w_-}.
\end{aligned}
$$

We want to show that this quantity is bounded for $x_1 < -1$. To do this, we will apply the mean value theorem : for $x_1 \leq 0$, since there exists $M > 0$ such that for all $y \in D$, $|w_-(y)| \leq M$, one has for $2 \leq i \leq n$:

$$
\Big|\frac{G_i'\Big((1-e^{x_1})w_-\Big) - G_i'(w_-)}{e^{x_1}w_-}\Big| \leq \sup_{(1-e^{x_1})w_- \leq z \leq w_-} |G_i''(z)| \leq \sup_{-2M \leq z \leq M} |G_i''(z)|,
$$

and

$$
\begin{aligned}
\Big|\frac{f\Big((1-e^{x_1})w_-\Big) - (1-e^{x_1})f(w_-)}{e^{x_1}w_-}\Big| &\leq \Big|\frac{f\Big((1-e^{x_1})w_-\Big) - f(w_-)}{e^{x_1}w_-}\Big| + \Big|\frac{f(w_-)}{w_-}\Big|\\
&\leq \Big|\sup_{(1-e^{x_1})w_- \leq z \leq w_-} |f'(z)| + \sup_{0 \leq s \leq w_-} |f'(s)|\\
&\leq \sup_{-2M \leq z \leq M} |f'(z)| + \sup_{0 \leq s \leq M} |f'(s)|.
\end{aligned}
$$

If $x_1 > 1$,

$$
\begin{aligned}
r(h)(x_1,y) &= \frac{e^{-x_1}w_- + e^{-x_1}\Delta' w_- + \alpha(y)G_1'(h)(-e^{-x_1}w_-) + \alpha(y)\sum_{i=2}^n G_i'(h)e^{-x_1}\frac{\partial w_-}{\partial x_i} + f(h)}{e^{-x_1}w_-}\\
&= 1 - \alpha(y)G_1'(h) + \frac{1}{w_-}\Big[\Delta' w_- + \alpha(y)\sum_{i=2}^n G_i'(h)\frac{\partial w_-}{\partial x_i}\Big] + \frac{f(h)}{e^{-x_1}w_-}\\
&= 1 - \alpha(y)G_1'(h) + \frac{1}{w_-}\Big[\alpha(y)\sum_{i=2}^n \Big(G_i'(h) - G_i'(w_-)\Big)\frac{\partial w_-}{\partial x_i} - f(w_-)\Big] + \frac{f(h)}{e^{-x_1}w_-}\\
&= 1 - \alpha(y)G_1'(h) + \alpha(y)\sum_{i=2}^n \frac{\Big(G_i'(h) - G_i'(w_-)\Big)}{w_-}\frac{\partial w_-}{\partial x_i} + \frac{1}{w_-}\Big(\frac{f(h)}{e^{-x_1}} - f(w_-)\Big).
\end{aligned}
$$

Using the mean value theorem and that there exists $M > 0$ such that for all $y \in D$, $|w_-(y)| \le M$, one has for $x_1 > 1$ and $2 \le i \le n$:

$$\begin{aligned}\Big|\frac{G_i'(e^{-x_1}w_-) - G_i'(w_-)}{w_-}\Big| &\le \Big|\frac{(e^{-x_1}-1)w_-}{w_-}\Big| \sup_{e^{-x_1}w_- \le z \le w_-} |G_i''(z)| \\ &\le (1-e^{-x_1}) \sup_{-Me^{-x_1} \le z \le M} |G_i''(z)| \\ &\le \sup_{-Me^{-1} \le z \le M} |G_i''(z)|,\end{aligned}$$

and

$$\Big|\frac{f(e^{-x_1}w_-)}{e^{-x_1}w_-} - \frac{f(w_-)}{w_-}\Big| \le \sup_{0 \le z \le e^{-x_1}w_-} |f'(z)| + \sup_{0 \le s \le w_-} |f'(s)| \le \sup_{0 \le z \le Me^{-1}} |f'(z)| + \sup_{0 \le s \le M} |f'(s)|.$$

Recall that for $x_1 \in (-1, 1)$, $h(x_1, y) = p(x_1)w_-(y)$, where p is a smooth function. Then,

$$\begin{aligned}r(h)(x_1,y) &= \frac{p''w_- + p\Delta'w_- + \alpha(y)G_1'(h)p'w_- + \alpha(y)\sum_{i=2}^n G_i'(h)p\frac{\partial w_-}{\partial x_i} + f(h)}{-p'w_-} \\ &= -\frac{p''}{p'} - \alpha(y)G_1'(h) - \frac{p}{p'w_-}\Big[\Delta'w_- + \alpha(y)\sum_{i=2}^n G_i'(h)\frac{\partial w_-}{\partial x_i}\Big] - \frac{f(h)}{p'w_-} \\ &= -\frac{p''}{p'} - \alpha(y)G_1'(h) - \frac{p}{p'w_-}\Big[\alpha(y)\sum_{i=2}^n \Big(G_i'(h) - G_i'(w_-)\Big)\frac{\partial w_-}{\partial x_i}\Big] + \frac{p}{p'}\Big(\frac{f(w_-)}{w_-} - \frac{f(pw_-)}{pw_-}\Big).\end{aligned}$$

Using mean value theorem again, one has for $2 \le i \le n$:

$$\begin{aligned}\Big|\frac{G_i'(pw_-) - G_i'(w_-)}{w_-}\Big| &\le |p-1| \sup_{z \in [pw_-, w_-]} |G_i''(z)| \\ &\le (\|p\|_\infty + 1) \sup_{z \in [-M\|p\|_\infty, M\|p\|_\infty]} |G_i''(z)|,\end{aligned}$$

and

$$\Big|\frac{f(w_-)}{w_-} - \frac{f(pw_-)}{pw_-}\Big| \le \sup_{-M \le z \le M} |f'(z)| + \sup_{-M\|p\|_\infty \le s \le M\|p\|_\infty} |f'(s)|.$$

Since $r(h)$ is a continuous function in Ω and α and G_1' are bounded it follows that

$$\sup_{x \in \Omega} r(h)(x_1, y) < +\infty,$$

and consequently, $c^\star < +\infty$.

$\square$

We will show later, see Theorem 1.21, that under some assumptions on f, G and α, the minimal speed $c^\star$ is also bounded from below.

Now let $c > c^\star$. Then, by definition of the infimum, there exists a function $\rho \in \tilde{K}$, such that

$$\begin{cases} \Delta\rho + \alpha(y)\nabla\cdot G(\rho) + f(\rho) + c\partial_1\rho < 0 & \forall (x_1, y) \in \mathbb{R}\times D, \\ \partial_1\rho < 0 & \text{in } \mathbb{R}\times D, \\ \rho(-\infty, y) = w_-(y) & \text{uniformly in } y \in D, \\ \rho(+\infty, y) = 0 & \text{uniformly in } y \in D, \\ \frac{\partial\rho}{\partial\nu} < 0 & \text{for } (x_1, y) \in \mathbb{R}\times\partial D, \\ \rho = 0 & \text{on } \mathbb{R}\times\partial D. \end{cases}$$

Let $N > 1$ be an integer. Since the function $y \mapsto \rho(N, y)$ is continuous and the sequence of functions $\{v_k\}$ tends uniformly to 0 when k tends to infinity, there exists $k \in \mathbb{N}$ such that

$$\rho(N, y) > v_k(y),\ \forall y \in D. \tag{5.144}$$

We claim:

Proposition 5.3. *Let $N > 1$ and $\Omega_N = (-N, N)\times D \subset \Omega$. Assume* **(GD)**, **(AlphaD)** *and that f is C^1 and satisfies* (1.8)*. Then, there exists a unique solution $u \in W^{2,p}_{loc}\Big((-N, N)\times \overline{D}\Big)$ which satisfies*

- *$\rho(-N, y) \geq u(x_1, y) \geq v_k(y)$ for all $(x_1, y) \in (-N, N)\times D$,*
- *for all $x_1 \in (-N, N)$ there exists $y \in D$ such that $\rho(-N, y) > u(x_1, y)$,*

of the following problem :

$$\begin{cases} \Delta u + c\partial_1 u + \alpha(y)\nabla\cdot G(u) + f(u) = 0 & \text{in } \Omega_N, \\ u = 0 & \text{for } -N < x_1 < N,\ y \in \partial D, \\ u(-N, y) = \rho(-N, y),\quad u(N, y) = v_k(y) & y \in D. \end{cases} \tag{5.145}$$

Proof. The proof relies on the theory of sub and super solutions. The function $(x_1, y) \mapsto v_k(y)$ (resp. ρ) is a subsolution (resp. a supersolution) of (5.145). Indeed, one has

$$\Delta v_k + c\partial_1 v_k + \alpha(y)\nabla\cdot G(v_k) + f(v_k) \geq 0 > \Delta\rho + c\partial_1\rho + \alpha(y)\nabla\cdot G(\rho) + f(\rho),$$

$$v_k(y) = \rho(x_1, y) = 0 \quad \forall (x_1, y) \in (-N, N)\times\partial D,$$

and

$$\rho(x_1, y) > v_k(y) \quad \forall (x_1, y) \in [-N, N]\times D.$$

We will use the following adaptation of [BN1, Lemma 7.1], with Dirichlet boundary conditions instead of Neumann boundary conditions considered in [BN1]. Consider

$$\begin{cases} Lu := Mu + cu = a_{ij}(x)u_{ij} + b_i(x)u_i + c(x)u = g(x) & \text{in } \Omega_N, \\ u(x_1, y) = 0 & \text{for } -N < x_1 < N,\ y \in \partial D, \\ u(-N, y) = \psi_1(y), \quad u(N, y) = \psi_2(y) & y \in D. \end{cases} \tag{5.146}$$

Here, ψ_1 and ψ_2 belong to $W^{2,\infty}(D)$, and satisfy $\psi_1(y) = \psi_2(y) = 0$ for $y \in \partial D$. The function g belongs to $L^\infty(\Omega_N)$ and there exists a constant $C > 0$ such that $|b_i|, |c| \leq C$, where $c \leq 0$.

Lemma 5.4. *Under the above conditions, with $a_{ij} := \delta_{ij}$, the problem* (5.146) *has a solution $u \in W^{2,p}\Big(\overline{\Omega_N} \setminus (\{\pm N\} \times \partial D)\Big) \cap C(\overline{\Omega})$. Moreover, there exists a constant $C_1 > 0$ such that*

$$\max |u| \leq C_1\Big(\|g\|_{L^\infty(\Omega_N)} + \sum_j \|\psi_j\|_{W^{2,\infty}(D)}\Big).$$

Proof. Let

$$u = v + \frac{N - x_1}{2N}\psi_1(y) + \frac{N + x_1}{2N}\psi_2(y).$$

Then the function v has to satisfy

$$\begin{cases} Lv = \tilde{g}(x) & \text{in } \Omega_N, \\ v = 0 & \text{on } \partial\Omega_N, \end{cases} \tag{5.147}$$

where $\tilde{g} := g - \frac{N-x_1}{2N}L\psi_1 - \frac{N+x_1}{2N}L\psi_2$.
However, the existence of a solution v of the problem (5.147) is not straightforward because the domain Ω_N is not $C^{2,\alpha}$.
As in [BN1, Lemma 7.1], consider an approximate problem in $\Omega_N^\varepsilon \subset \Omega_N$ with smooth boundary. In Ω_N^ε, there exists a solution $v_\varepsilon \in W^{2,p}(\Omega_N^\varepsilon)$ of

$$\begin{cases} Lv_\varepsilon = \tilde{g} & \text{in } \Omega_N^\varepsilon, \\ v_\varepsilon = 0 & \text{on } \partial\Omega_N^\varepsilon. \end{cases} \tag{5.148}$$

We want to let ε tend to 0 and obtain a limit function v that is a solution of (5.147). Using the same barrier function h (2.62) as in the proof of [BN1, Lemma 7.1] and Chapter 2 , we will show that

$$|v_\varepsilon| \leq \|\tilde{g}\|_{L^\infty} h \text{ in } \Omega_N^\varepsilon. \tag{5.149}$$

As in (2.62), define the concave and positive function h on $[-N, 0]$ by

$$h(x_1) = \frac{1}{b^2}e^{bN}\big(1 - e^{-b(x_1+N)}\big) - \frac{1}{b}(x_1 + N), \tag{5.150}$$

where $b \geq \sqrt{\sum_{i=1}^n b_i^2}$.
We extend h on $[0, N]$ to be symmetric. As in the proof of Proposition 2.4, we show that the function h is a concave positive function and satisfies $Lh \leq -1$ on $(-N, N)$ provided that $b \geq \sqrt{\sum_{i=1}^n b_i^2}$.
Assume that $\|\tilde{g}\|_{L^\infty} = 1$. Then one has in Ω_N^ε :

$$\begin{aligned} L(v_\varepsilon - h) = Lv_\varepsilon - Lh &= \tilde{g} - Lh \\ &\geq \tilde{g} + 1 \\ &\geq 0. \end{aligned}$$

Hence, by Maximum Principle, if $v_\varepsilon - h$ is positive somewhere, it achieves its maximum at the boundary. But at the boundary, $v_\varepsilon = 0$ and then the maximum of $v_\varepsilon - h$ has to be negative, which is impossible. Hence, the function $v_\varepsilon - h$ is always negative, and we obtain

$$v_\varepsilon \leq h,$$

which ensures (5.149).
Now, as in the proof of [BN1, Lemma 7.1], using standard local $W^{2,p}$ estimates and diagonal arguments, we let ε tend to 0 and we obtain a limit function v solution of (5.147) and which satisfies

$$|v| \leq \|\tilde{g}\|_{L^\infty} h.$$

Hence, the function v is continuous in $\overline{\Omega_N}$ and $v \in W^{2,p}_{loc}\Big(\overline{\Omega_N} \setminus (\{\pm N\} \times \partial D)\Big)$.

□

From now on, we will denote by $\underline{u}$ and $\overline{u}$ the sub and supersolutions $\underline{u} := v_k$, where v_k is as chosen in (5.144) and $\overline{u} := \rho$. We will also use the classical notation

$$G_i'(u)\frac{\partial u}{\partial x_i} := \sum_{i=1}^n G_i'(u)\frac{\partial u}{\partial x_i} = \nabla \cdot G(u).$$

We will construct a sequence of functions $(u_j)_{j\geq 0}$ on $[-N, N] \times D$, with $u_0 := \underline{u}$, solving the following equation :

$$\Delta u_{j+1} + c\frac{\partial u_{j+1}}{\partial x_1} + \alpha(y) G_i'(u_j)\frac{\partial u_{j+1}}{\partial x_i} - k_0 u_{j+1} = -f(u_j) - k_0 u_j,$$

where k_0 will be chosen later, as well as the boundary conditions

$$\begin{cases} u_{j+1} = \rho \text{ on } \{-N\} \times D, \\ u_{j+1} = v_k \text{ on } \{N\} \times D, \\ u_{j+1} = 0 \text{ on } (-N, N) \times \partial D. \end{cases}$$

We start with $u_0 = \underline{u}$. Lemma 5.4 applied with $a_{ij} = \delta_i^j$, $b_1(x) = c + \alpha(y)G_1'(u_0)$, $b_i(x) = \alpha(y)G_i'(u_0)$ for $2 \le i \le n$, and $c = -k_0$ gives the existence of $u_1 \in C\Big([-N,N]\times\overline{D}\Big)\cap W^{2,p}_{loc}\Big(\overline{\Omega_N}\setminus\{-N,N\}\times\partial D\Big)$ which satisfies

$$\Delta u_1 + c\partial_1 u_1 + \alpha(y)G_i'(u_0)\frac{\partial u_1}{\partial x_i} - k_0 u_1 = -f(u_0) - k_0 u_0,$$

with the same boundary conditions. Assume that for $j\in\mathbb{N}$, $u_j \in C\Big([-N,N]\times\overline{D}\Big)$, which implies that $u_j \mapsto f(u_j)+k_0u_j \in L^\infty\Big((-N,N)\times D\Big)$. Hence, Lemma 5.4 gives the existence of the function $u_{j+1}\in C\Big([-N,N]\times\overline{D}\Big)\cap W^{2,p}_{loc}\Big(\overline{\Omega_N}\setminus\{-N,N\}\times\partial D\Big)$ which satisfies

$$\Delta u_{j+1} + c\frac{\partial u_{j+1}}{\partial x_1} + \alpha(y)G_i'(u_j)\frac{\partial u_{j+1}}{\partial x_1} - ku_{j+1} = -f(u_j) - ku_j.$$

Now, similarly to the proof of Proposition 2.4 in Chapter 2 under Neumann boundary conditions, we will show by induction that for all $j\in\mathbb{N}$, one has

$$\underline{u} \le u_j \le \overline{u}. \tag{5.151}$$

For $j=0$, (5.151) is trivial. Let $j\ge 1$, and assume that $\underline{u}\le u_j\le\overline{u}$. First, we want to prove $\underline{u}\le u_{j+1}$. The functions u_j and $\underline{u}$ satisfy

$$\Delta u_{j+1} + c\frac{\partial u_{j+1}}{\partial x_1} + \alpha(y)G_i'(u_j)\frac{\partial u_{j+1}}{\partial x_i} - k_0u_{j+1} = -f(u_j) - k_0u_j,$$

and

$$\Delta\underline{u} + c\frac{\partial\underline{u}}{\partial x_1} + \alpha(y)G_i'(\underline{u})\frac{\partial\underline{u}}{\partial x_i} \ge -f(\underline{u}).$$

By subtraction, one has

$$\begin{aligned}\Delta(\underline{u}-u_{j+1}) + c\frac{\partial}{\partial x_1}(\underline{u}-u_{j+1}) + \alpha(y)\Big(G_i'(\underline{u})-G_i'(u_j)\Big)\frac{\partial\underline{u}}{\partial x_i} + \alpha(y)G_i'(u_j)\frac{\partial}{\partial x_i}(\underline{u}-u_{j+1})\\ - k(\underline{u}-u_{j+1}) \ge -f(\underline{u}) - k_0\underline{u} + f(u_j) + k_0u_j,\end{aligned}$$

which gives

$$\begin{aligned}\Delta(\underline{u}-u_{j+1}) + c\frac{\partial}{\partial x_1}(\underline{u}-u_{j+1}) &+ \alpha(y)G_i'(u_j)\frac{\partial}{\partial x_i}(\underline{u}-u_{j+1}) - k(\underline{u}-u_{j+1})\\ &\ge f(u_j)-f(\underline{u}) + k(u_j-\underline{u}) + \alpha(y)(G_i'(u_j)-G_i'(\underline{u}))\frac{\partial\underline{u}}{\partial x_i}\\ &\ge -L(u_j-\underline{u}) + k(u_j-\underline{u}) - \tilde{L}C(u_j-\underline{u}) \qquad \text{since } \underline{u}\le u_j\le\overline{u}\\ &\ge 0,\end{aligned}$$

where $C \geq 0$ is such that

$$|\alpha(y)| \sup_{x\in\Omega_N} \max_{1\leq i\leq n} \left\{ \left|\frac{\partial \underline{u}}{\partial x_i}\right|, \left|\frac{\partial \overline{u}}{\partial x_i}\right| \right\} \leq C \text{ for all } y \in D,$$

and we choose k large enough such that $k \geq L + \tilde{L}C$.

We now prove (5.151). The functions u_{j+1} and $\overline{u}$ satisfy

$$\Delta u_{j+1} + c\frac{\partial u_{j+1}}{\partial x_1} + \alpha(y)G_i'(u_j)\frac{\partial u_{j+1}}{\partial x_i} - ku_{j+1} = -f(u_j) - ku_j, \tag{5.152}$$

and

$$\Delta \overline{u} + c\frac{\partial \overline{u}}{\partial x_1} + \alpha(y)G_i'(\overline{u})\frac{\partial \overline{u}}{\partial x_i} \leq -f(\overline{u}).$$

By subtraction, the same method shows that $\Delta(\overline{u} - u_{j+1}) \leq 0$, provided that k is chosen as before. By [GIL, Theorem 8.1], the function $\underline{u} - u_{j+1}$ reaches its maximum on the boundary $\partial\Omega_N$. Let P be a point of $\partial\Big([-N,N]\times D\Big)$ where $\underline{u} - u_{j+1}$ reaches its maximum. If $P \in \{-N\} \times D$,

$$\underline{u} - u_{j+1} < 0.$$

If $P \in \{N\} \times D$, or $(-N,N) \times \partial D$, then

$$\underline{u} - u_{j+1} = 0.$$

Hence, in all cases,

$$\max_{[-N,N]\times\overline{D}} (\underline{u} - u_{j+1}) \leq 0,$$

and consequently : $\underline{u} \leq u_{j+1}$. An analogous argument shows that $u_{j+1} \leq \overline{u}$, and by induction, we proved that for all $j \in \mathbb{N}$, $\underline{u} \leq u_j \leq \overline{u}$.

Thus, one has for all $j \in \mathbb{N}$,

$$\left\{ \begin{array}{l} \underline{u} \leq u_j \leq \overline{u}, \\ \Delta u_j + c\frac{\partial u_j}{\partial x_1} + \alpha(y)G_i'(u_{j-1})\frac{\partial u_j}{\partial x_i} - ku_j = -f(u_{j-1}) - ku_{j-1} \end{array} \right. . \tag{5.153}$$

In addition, the inequalities (5.151) imply that the functions u_j are uniformly bounded in $L^\infty\Big((-N,N)\times D\Big)$ and because of the equations (5.153) that are satisfied by u_j, the functions u_j are uniformly bounded in $W^{2,p}_{loc}\Big((-N,N)\times\overline{D}\Big)$ for all $p \in (1,+\infty)$. Hence, by taking a diagonal subsequence, $(u_j)_{j\in\mathbb{N}}$ has a subsequence that converges strongly in $C^{1,\lambda}$ for all $\lambda \in (0,1)$, on compact subsets of $\Omega_N = (-N,N)\times D$ to a solution u^N of (5.145). Then, for each $N > 0$, one has a solution u^N defined on $(-N,N)\times D$ which belongs to $W^{2,p}_{loc}\Big((-N,N)\times\overline{D}\Big)$.

In order to know the behaviour of the solution u^N on $\{-N,N\}\times D$, we will first use the barrier function h.
The change of variables in the proof of Lemma 5.4 becomes

$$u = v + \frac{N-x_1}{2N}\rho(-N,y) + \frac{N+x_1}{2N}v_k.$$

It follows that the function g defined by $Lv = g$ is actually bounded in $L^\infty\Big((-N,N)\times D\Big)$ independently of N.
Moreover, with the definition (5.150) of the barrier function h, and by [BN1, Lemma 7.1] one can deduce that on $(0,N)\times D$

$$\left|u^N(x_1,y) - \frac{N-x_1}{2N}\rho(-N,y) - \frac{N+x_1}{2N}v_k(y)\right| \le Ch(x_1), \tag{5.154}$$

with C and h independent of N. In particular, u^N is continuous on $\{N\}\times\partial D$, and satisfies the boundary condition

$$u^N(N,y) = v_k(y) \;\; \forall y\in D.$$

Due to the symmetry of h, one has the estimate (5.154) on $(-N,N)\times D$, and since $h(-N)=0$, the function u^N satisfies

$$u^N(-N,y) = \rho(-N,y), \;\; \forall y\in D,$$

which ensures that u^N is continuous on $\overline{\Omega_N}$. Finally, since local up to the boundary estimates (see footnote 1) ensure that $(u_j)_{j\in\mathbb{N}}$ converges uniformly on each compact subset of Ω_N to $u^N\in W^{2,p}\Big((-N,N)\times\overline{D}\Big)$, as $j\to+\infty$, and since for $-N<x_1<N$ and $y\in\partial D$, the function u_j satisfies

$$u_j = 0,$$

it follows that the limit function u^N also satisfies

$$u^N = 0, \text{ for } -N<x_1<N \text{ and } y\in\partial D.$$

Uniqueness in Proposition 5.3 will be addressed in the next section.

$\square$

5.3 Solution on the unbounded cylinder and solution with critical speed $c^\star$

In order to let N tend to infinity, we first show that u is monotone with respect to x_1. Recall that for $N > 1$, there exists k large enough such that (5.144) holds. We claim :

Proposition 5.5. *Assume* **(GD)**, **(AlphaD)** *and that f is C^1 and satisfies* (1.8). *Assume also that u is a solution of the problem* (5.145) *which satisfies*

- *$\rho(-N, y) \geq u(x_1, y) \geq v_k(y)$ for all $(x_1, y) \in (-N, N) \times D$,*
- *for all $x_1 \in (-N, N)$ there exists $y \in D$ such that $\rho(-N, y) > u(x_1, y)$.*

Then u is decreasing with respect to x_1, and one has $u_{x_1} < 0$. In addition, the solution u is unique.

Proof. Since for all $(x_1, y) \in (-N, N) \times D$, $\rho(-N, y) \geq u(x_1, y) \geq v_k(y)$, and since for all $x_1 \in (-N, N)$, there exists $y \in D$ such that $\rho(-N, y) > u(x_1, y)$, we can apply [BN1, Theorem 2.4] with

$$F(x, u, Du, D^2u) = \Delta u + \Big(c + \alpha(y) G_1'(u) \partial_1 u\Big) + \alpha(y) \sum_{i=2}^{n} G_i'(u) \frac{\partial u}{\partial x_i} + f(u),$$

and we obtain as before that u is decreasing with respect to x_1. Note that [BN1, Theorem 2.4] gives the uniqueness of the solution u.

□

Now, we would like to obtain a solution u on $\Omega = (-\infty, +\infty) \times D$. As in the proof of Theorem 1.17 under Neumann boundary conditions, we will let N tend to infinity, we will prove that there exists a solution u of (5.138). Note that we need to make sure that this solution u is neither w_- nor identically 0.

Proof of Theorem 1.20. To prove the result, let $r \in \mathbb{R}$ and consider

$$\begin{cases} \rho^r(x_1, y) & := \rho(x_1 + r, y), \\ h^r(y) & := \sup_k \Big\{ v_k(y) \quad | \quad v_k(y) \leq \rho(N + r, y), \quad \forall y \in D \Big\}. \end{cases}$$

By compactness of $\overline{D}$, the uniform convergence of $(v_k)_k$ to 0 on $\overline{D}$ and continuity of ρ, $r \mapsto h^r$ exists and is continuous on $\mathbb{R}$.
As before, there exists a unique function $v^r \in W^{2,p}\Big((-N, N) \times \overline{D}\Big) \cap C\Big([-N, N] \times \overline{D}\Big)$ with $h^r < v^r < \rho^r$ in $(-N, N) \times D$, satisfying

$$\begin{cases} \Delta v^r + c\partial_1 v^r + \alpha(y) G_i'(v^r)\frac{\partial v^r}{\partial x_i} + f(v^r) = 0 & \text{on } (-N,N)\times D, \\ v^r = 0 & \text{for } -N < x_1 < N,\ y \in \partial D, \\ v^r(-N,y) = \rho^r(-N,y),\ \ v^r(N,y) = h^r(y), & y \in D. \end{cases} \tag{5.155}$$

Indeed, ρ^r is a supersolution of (5.155) since

$$\Delta \rho^r + c\partial_1 \rho^r + \alpha(y) G_i'(\rho^r)\frac{\partial \rho^r}{\partial x_i} + f(\rho^r) \le 0,$$

and

$$\rho^r(N,y) = \rho(N+r,y) \ge h^r(y),$$

by definition of h^r. And h^r is a subsolution of (5.155) since

$$\Delta h^r + c\partial_1 h^r + \alpha(y) G_i'(h^r)\frac{\partial h^r}{\partial x_i} + f(h^r) \ge 0,$$

and

$$h^r \le \rho(N+r,y) \le \rho(-N+r,y) = \rho^r(-N,y).$$

In addition, since $v^r < \rho^r$ on $(-N,N)\times D$ and since ρ^r is decreasing with respect to x_1, it follows that for all x_1 in $(-N,N)$, there exists $y \in D$ such that $\rho^r(-N,y) > v^r(x_1,y)$. Hence, [BN1, Theorem 2.4] gives the uniqueness of solution of (5.155).
Since h^r and $\rho(N+r,y)$ vary continously with $r \in \mathbb{R}$, the uniqueness of solutions of (5.155) implies that v^r depends continously in the $C^0\Big([-N,N]\times\overline{D}\Big)$ - topology with respect to r. Also, since ρ is decreasing in x_1, the function v^r is a supersolution of (5.155) corresponding to any $r' > r$:

$$\begin{cases} \Delta v^{r'} + c\partial_1 v^{r'} + \alpha(y) G_i'(v^{r'})\frac{\partial v^{r'}}{\partial x_i} + f(v^{r'}) = 0 \text{ on } (-N,N)\times D \\ v^{r'} = 0 \text{ for } -N < x_1 < N,\ y \in \partial D \\ v^{r'}(-N,y) = \rho^{r'}(-N,y),\ \ v^{r'}(N,y) = h^{r'},\ \ y \in D \end{cases} \tag{5.156}$$

Indeed, v^r is a supersolution of (5.156) since

$$\Delta v^r + c\partial_1 v^r + \alpha(y) G_i'(v^r)\frac{\partial v^{r'}}{\partial x_i} + f(v^r) = 0,$$

and

$$v^r(-N,y) = \rho^r(-N,y) = \rho(-N+r,y) \ge \rho(-N+r',y) = \rho^{r'}(-N,y),$$

$$\begin{aligned} v^r(N,y) = h^r &= \sup_k\{v_k(y) \quad | \quad v_k(y) \le \rho(N+r,y), \quad \forall y \in D\} \\ &\ge \sup_k\{v_k(y) \quad | \quad v_k(y) \le \rho(N+r',y), \quad \forall y \in D\} \\ &= h^{r'}. \end{aligned}$$

The function v^r is decreasing in x_1 and in r, and bounded since $h^r \leq v^r \leq \rho^r$. Since ρ tends uniformly to 0 when x_1 tends to $+\infty$, it follows that

$$\lim_{r \to +\infty} v^r = 0, \text{ uniformly in } y \in D.$$

Furthermore, since ρ tends to w_- when x_1 tends to $-\infty$, and since

$$\lim_{r \to -\infty} h^r(y) = \sup_k \{v_k(y)\} = v_1(y),$$

we can deduce that $\lim_{r \to -\infty} v^r$ exists and satisfies

$$0 < v_1 \leq \lim_{r \to -\infty} v^r \leq w_-, \text{ for all } y \in D.$$

Since the function $r \mapsto \max_{y \in \overline{D}} v^r(0, y)$ is continuous, there exists a value r_0 such that

$$\max_{y \in \overline{D}} v^{r_0}(0, y) = \frac{\sup_{y \in \overline{D}} v_1(y)}{2}.$$

We denote by u^N the corresponding solution v^r. Actually, the function u^N satisfies :

$$\begin{cases} \Delta u^N + c\partial_1 u^N + \alpha(y) G_i'(u^N) \frac{\partial u^N}{\partial x_i} + f(u^N) = 0 & \text{on } (-N, N) \times D, \\ u^N = 0 \text{ for } -N < x_1 < N, & y \in \partial D, \\ 0 < u^N < w_-, \;\; \partial_1 u^N < 0 & \text{in } (-N, N) \times \overline{D}, \\ \max_{y \in \overline{D}} u^N(0, y) = \frac{\sup_{y \in \overline{D}} v_1(y)}{2}. \end{cases} \tag{5.157}$$

For any $p > 1$, given any compact subset of $\mathbb{R} \times \overline{D}$, the family $(u^N)_N$ is bounded in the $W^{2,p}$ norm as N goes to infinity. Hence, there exists a sequence $N_j \to +\infty$ such that $u^{N_j} \to u$ uniformly on compact sets of $\mathbb{R} \times \overline{D}$. Furthermore, since for all $N_j \in \mathbb{N}$ the function u^{N_j} satisfies

$$u^{N_j} = 0$$

on $\mathbb{R} \times \partial D$, it follows that the limit function u satisfies

$$\begin{cases} \Delta u + c\partial_1 u + \alpha(y) G_i'(u) \frac{\partial u}{\partial x_i} + f(u) = 0 & \text{on } \mathbb{R} \times D, \\ u = 0 & \text{on } \mathbb{R} \times \partial D, \\ 0 \leq u \leq w_-, \;\; \partial_1 u \leq 0 & \text{in } \mathbb{R} \times \overline{D}, \\ \max_{y \in \overline{D}} u(0, y) = \frac{\sup_{y \in \overline{D}} v_1(y)}{2}. \end{cases} \tag{5.158}$$

In particular, the last condition shows that the function u is neither 0 nor w_-. Moreover, since u is bounded and non-increasing with respect to x_1, it follows that u has finite limits $\tilde{u}$ and $u^\star$ when x_1 tends to $+\infty$ and $-\infty$.
We now argue as in the proof of Theorem 1.17 to show that this convergence is uniform with respect to $y \in D$.
Let $\Omega_1 = (-1, 1) \times D$, and define $z_m : \Omega_1 \to \mathbb{R}$, by

$$z_m(x_1, y) := u(x_1 + m, y).$$

As a result of the equation satisfied by u, the family $(z_m)_m$ is bounded in $W^{2,p}(\Omega_1)$. Hence, there exists a subsequence $(z_{m_k})_k$ of $(z_m)_m$ which is weakly convergent in $W^{2,p}(\Omega_1)$ and strongly in $C^{1,\lambda}(\overline{\Omega_1})$. But since for $y \in D$,

$$\lim_{x_1 \to +\infty} u(x_1, y) = \tilde{u}(y),$$

it follows that $(z_{m_k})_k$ converges to $\tilde{u}$ in $C^{1,\lambda}(\overline{\Omega_1})$. Thus, all the convergent subsequences of $(z_m)_m$ converge to $\tilde{u}$ in $C^{1,\lambda}(\overline{\Omega_1})$, which implies that $(z_m)_m$ tends to $\tilde{u}$ in $C^{1,\lambda}(\overline{\Omega_1})$ when m tends to $+\infty$, which gives that u converges in $C^1(\overline{D})$ to $\tilde{u}$ when $x_1 \to +\infty$. A similar argument shows that u converges also in $C^1(\overline{D})$ to $u^\star$ when $x_1 \to -\infty$.
Now, we want to show that the limits $\tilde{u}$ and $u^\star$ of u as $x_1 \to \pm\infty$, which in principle depend on y, have to satisfy the problem on the cross section D, namely (5.138). To do that, we will use the following lemma :

Lemma 5.6. *Let u be a solution of* (5.158)*. Then,*

$$\lim_{x_1 \to \pm\infty} \frac{\partial u}{\partial x_1} = 0.$$

Proof. The proof is exactly the same as the proof of Lemma 2.7. □

Now, let $v \in C_0^1(D)$ be a test function and $\tilde{x_1} \in \mathbb{R}$. Then

$$\int_{\Omega_1} v\Delta u + c\int_{\Omega_1} v\frac{\partial u}{\partial x_1} + \int_{\Omega_1} v\alpha(y)\sum_{i=1}^{n} G_i'(u)\frac{\partial u}{\partial x_i} + \int_{\Omega_1} vf(u) = 0, \tag{5.159}$$

where $\Omega_1 = (\tilde{x_1} - 1, \tilde{x_1} + 1) \times D$. First, by integration by parts, one has

$$\int_{\Omega_1} v\Delta u = \int_{\Omega_1} v\frac{\partial^2 u}{\partial x_1^2} - \int_{\Omega_1} \nabla' v \cdot \nabla' u.$$

Using Lemma 5.6,

$$\int_{\Omega_1} v\frac{\partial^2 u}{\partial x_1^2} = \int_D v\Big(\frac{\partial u}{\partial x_1}(\tilde{x_1}+1, y) - \frac{\partial u}{\partial x_1}(\tilde{x_1}-1, y)\Big) \to 0, \text{ when } \tilde{x_1} \to +\infty.$$

Since u converges to $\tilde{u}$ in $C^{1,\lambda}(D)$, it follows that

$$\int_{\Omega_1} \nabla' v(y)\cdot\nabla' u(x_1, y)dx_1dy = \int_{(-1,1)\times D} \nabla' v(y)\cdot\nabla' u(p+\tilde{x_1}, y)dpdy \to 2\int_D \nabla' v\cdot\nabla'\tilde{u}.$$

Similarly, one has

$$\int_{\Omega_1} v\frac{\partial u}{\partial x_1} = \int_D v(y)\Big(u(\tilde{x_1}+1, y) - u(\tilde{x_1}-1, y)\Big)dy \to 0 \text{ when } \tilde{x_1} \to +\infty,$$

$$\int_{\Omega_1} v\alpha \sum_{i=1}^{n} G_i'(u)\frac{\partial u}{\partial x_i} = \int_{(-1,1)\times D} v(y)\alpha(y)\sum_{i=1}^{n} G_i'(u(p+\tilde{x_1},y))\frac{\partial u}{\partial x_i}(p+\tilde{x_1},y)dpdy,$$

which tends to

$$\int_{(-1,1)\times D} v(y)\alpha(y)\sum_{i=2}^{n} G_i'(\tilde{u}(y))\frac{\partial \tilde{u}}{\partial x_i}(y)dpdy = 2\int_D v(y)\alpha(y)\sum_{i=2}^{n} G_i'(\tilde{u}(y))\frac{\partial \tilde{u}}{\partial x_i}(y)dy,$$

when $\tilde{x_1} \to +\infty$. Analogously, when $\tilde{x_1} \to +\infty$, the last integral of (5.159) tends to

$$\begin{aligned}\int_{\Omega_1} vf(u) = \int_{(-1,1)\times D} v(y)f(u(p+\tilde{x_1},y))dpdy \to \int_{(-1,1)\times D} v(y)f(\tilde{u}(y))dpdy \\ = 2\int_D v(y)f(\tilde{u}(y))dy.\end{aligned}$$

Hence, by passing to the limit when $\tilde{x_1} \to +\infty$ in (5.159), the limit function $\tilde{u}$ has to satisfy :

$$\int_D v\Delta'\tilde{u} + \int_D \alpha(y)\sum_{i=2}^{n} G_i'(\tilde{u})\frac{\partial \tilde{u}}{\partial x_i}v + \int_D f(\tilde{u})v = 0. \tag{5.160}$$

Since u tends to $\tilde{u}$ in $C^1(\overline{D})$ as $x_1 \to +\infty$, it follows that $\tilde{u}$ also satisfies the boundary condition

$$\tilde{u} = 0 \text{ on } \partial D.$$

Hence, the limit function $\tilde{u}$ satisfies the problem on the cross section (5.138). In addition, the function z_{m_k} converges weakly in $W^{2,p}(D)$ and strongly in $C^{1,\lambda}(\overline{D})$ to $\tilde{u}$, which ensures that $\tilde{u} \in W^{2,p}(D)$.

However, we assumed **(AD)** : the weak form of the problem on the cross section has no other solutions than 0 and w_-. Hence, due to the condition

$$\max_{y\in\overline{D}} u(0,y) = \frac{\sup_{y\in\overline{D}} v_1(y)}{2}$$

and the fact that u is non-increasing in x_1, one has :

$$\lim_{x_1\to-\infty} u(x_1,y) = w_-(y) \text{ and } \lim_{x_1\to+\infty} u(x_1,y) = 0.$$

In order to prove the last point of Theorem 1.20, we need the following result :

Proposition 5.7. *Let $c > c^\star$ and let w the solution of* (5.138) *constructed in the proof of Theorem* (1.20). *Then,*

$$\lim_{x_1\to+\infty} \|\nabla w(x_1,\cdot)\|_{L^\infty(D)} = 0.$$

Proof. The proof is exactly the same as the one of Proposition 2.8 under Neumann boundary conditions.

$\square$

Using Proposition 5.7 we are now able to prove that the constructed solution w of (4.117) is decreasing with respect to x_1. In fact, the proof is exactly the same as the proof of Theorem 1.17 under Neumann boundary conditions.

$\square$

Now that we know that if $c > c^\star > -\infty$, there exists a solution of (5.138), we will show that there exists a solution of (5.138) with a speed $c = c^\star$.

Proposition 5.8. *Assume* **(GD)**, **(AlphaD)** *and that f is C^1 and satisfies* (1.8). *Suppose that $c^\star \in \mathbb{R}$. Then, for $c = c^\star$, there exists a solution of* (5.138).

Proof. The proof is the same as in Neumann's boundary conditions case, provided that the constant function 1 is replaced by w_- in the proof of Proposition 2.9.

$\square$

6 Lower bound for $c^\star$ for Dirichlet boundary conditions

In the case of Neumann boundary conditions and without any convection term (that is, when $G \equiv 0$), it was shown in [BN1, Theorem 1.5 and Section 10] that if we assume the KPP condition, namely

$$f(u) \leq f'(0)u, \tag{6.161}$$

for $0 \leq u \leq 1$, then there exists an explicit formula for $c^\star$ which only depends on f namely

$$c^\star = \sqrt{2f'(0)}.$$

With the convection term $G \not\equiv 0$, and in the Dirichlet boundary condition case, we will obtain a lower bound for the critical speed $c^\star$, under a set of conditions involving both f and G.

We will establish a lower bound for $c^\star$ involving the principal eigenvalue of a linearized operator. Precisely, we will show that under the assumption **(FD)**, namely : $f'(0) > \lambda_1(-L)$, where the operator $-L := -\Delta' - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial}{\partial x_i}$, there exists a critical value c' which a lower bound for $c^\star$.

6.1 Existence of c' and comparison with $c^\star$

Recall that in [BN2], in the Neumann boundary condition case, Berestycki and Nirenberg consider the linearized problem around 0 to show that there exists a critical value γ such that a certain eigenvalue problem has two positive eigenvalues if $c > \gamma$. Then they prove that if f satisfies the KPP condition (6.161), the equality $c^\star = \gamma$ holds. We adapted their method in Chapter 4, under Neumann boundary conditions.

Under Dirichlet boundary conditions, and in order to obtain an explicit formula for $c^\star$ we will follow the approach in [BN1] but with some modifications due to the presence of the extra term $G \neq 0$ and the Dirichlet boundary conditions.

Consider the linearized system of (5.138) around 0 :

$$\begin{cases} \Delta w + \Big(c + \alpha(y) G_1'(0)\Big)\partial_1 w + \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial w}{\partial x_i} + f'(0)w = 0 & \text{in } \mathbb{R}\times D, \\ w = 0 & \text{on } \mathbb{R}\times\partial D. \end{cases} \tag{6.162}$$

If $w(x_1, y) = e^{-\lambda x_1}\varphi(y)$, the function φ has to satisfy the following problem

$$\begin{cases} -\Delta'\varphi - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial \varphi}{\partial x_i} - f'(0)\varphi = \Big(\lambda^2 - \lambda(c + \alpha(y)G_1'(0))\Big)\varphi & \text{in } D, \\ \varphi = 0 & \text{on } \partial D. \end{cases} \tag{6.163}$$

We say that λ is a principal eigenvalue of (6.163), if there exists a positive function φ such that (6.163) holds. Consider now the following eigenvalue problem

$$\begin{cases} -\Delta'\sigma - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial \sigma}{\partial x_i} - f'(0)\sigma = \mu_1\sigma & \text{in } D, \\ \sigma = 0 & \text{on } \partial D. \end{cases} \tag{6.164}$$

By [YIH, Theorem 1.3], this problem has a simple eigenvalue $\mu_1 \in \mathbb{R}$, which corresponds to a positive eigenfunction. However, as in Chapter 3, we can not deduce immediately that the eigenvalue problem (6.163) has an eigenvalue λ because the right-hand side term depends on y. We first need to prove a version of Proposition 3.1 with Dirichlet boundary conditions :

Proposition 6.1. *Let $\omega \subset \mathbb{R}^n$ a bounded domain of class C^2, and*

$$L := a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x)$$

be a uniformly strongly elliptic operator with $a_{ij} = a_{ji} \in C(\omega)$, b_i and $c \in L^\infty(\omega)$ for all $1 \le i, j \le n$. We define

$$\gamma_1 := \sup\Big\{\gamma \mid \exists\phi > 0 \text{ in } \omega,\ \phi = 0 \text{ on } \partial\omega,\ \mid (L+\gamma)\phi \le 0\Big\}.$$

1. *With this definition of γ_1, one has $\gamma_1 = \sigma^\omega(-L)$, where $\sigma^\omega(-L)$ is defined in [YIH, Theorem 1.3] as the only eigenvalue which corresponds to a positive eigenfunction. Moreover, this eigenvalue is real.*

2. *The function $c \mapsto \gamma_1(c)$ is Lipschitz continous, with Lipschitz constant* 1.

3. *The function $c \mapsto \gamma_1(c)$ is concave.*

Remark : This definition of γ_1 coincides with the definition of the principal eigenvalue in [BNV].

Proof. The proof of the two first points is analogous to the proof of Proposition 3.1, and the concavity is proved in [BNV, Proposition 2.1]. □

Now, one can show the existence of an eigenvalue λ of (6.163), with $\lambda \in \mathbb{R}$, under an additional condition on $f'(0)$.

Proposition 6.2. *Assume (GD), (AlphaD) and that*

$$f'(0) > \lambda_1(-L),$$

where the operator $-L := -\Delta' - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial}{\partial x_i}$. Then, the principal eigenvalue μ_1 of (6.164) *is strictly negative.*

Proof. By [YIH, Theorem 1.3], the eigenvalue problem (6.164) has a simple eigenvalue $\mu_1 \in \mathbb{R}$ which corresponds to a positive eigenfunction φ.
Consider $\lambda_1(-L)$ the principal eigenvalue of the operator $-L := -\Delta' - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial}{\partial x_i}$.
Recall that $\lambda_1(-L)$ is characterised by

$$\lambda_1(-L) := \sup\Big\{\lambda \mid \exists \phi > 0 \text{ in } D,\ \phi = 0 \text{ on } \partial D,\ \mid (L+\lambda)\phi \leq 0\Big\}. \tag{6.165}$$

By taking φ the positive eigenfunction associated to μ_1 in (6.164) as a test function in (6.165), one has

$$\begin{aligned} -L\varphi &= -\Delta'\varphi - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial \varphi}{\partial x_i} \\ &= \Big(f'(0) + \mu_1\Big)\varphi. \end{aligned}$$

Hence, by definition of $\lambda_1(-L)$, it follows that

$$\lambda_1(-L) \geq f'(0) + \mu_1,$$

and hence, a sufficient condition to ensure $\mu_1 < 0$ is

$$f'(0) > \lambda_1(-L). \tag{6.166}$$

□

We next show that the inequality (6.166) can be satisfied under certain conditions on Ω and the function αG.

Proposition 6.3. *Let $\Omega \subset \mathbb{R}^n$ be a C^2 bounded domain. For all $v \in L^\infty(\Omega, \mathbb{R}^n)$, let $\lambda_1(\Omega, v)$ be the principal eigenvalue of $-L := -\Delta + v \cdot \nabla$ in Ω, under Dirichlet boundary conditions. Then, there exist $C_1, C_2, > 0$ such that, for all $R > 0$ such that there exists $x_0 \in \Omega$ with $B(x_0, R) \subset \Omega$ and all $v \in L^\infty(\Omega, \mathbb{R}^n)$ with $\||v|\| =: \tau$,*

$$\lambda_1(\Omega, v) \leq \frac{C_1}{R^2} + \frac{1}{\sqrt{R}}\tau^{3/2} + C_2\tau^2. \tag{6.167}$$

Proof. First, we apply [BNV, Proposition 5.1] with their notation : for $1 \leq i \leq n$, $b_i' = 0$, so $L' = -\Delta$, $\delta = b = \tau$, and $c_0 = 1$. Hence, the condition $\sum_{i=1}^n b_i^2 \leq b^2$ is satisfied since

$$\sum_{i=1}^n b_i^2 = \sum_{i=1}^n v_i^2 = \|v\|^2 = \tau^2 = b^2,$$

and the condition $\sum_{i=1}^n (b_i' - b_i)^2 = \delta^2 \leq bc_0$ is valid for $\tau \leq 1$. Indeed,

$$\sum_{i=1}^n (b_i' - b_i)^2 = \sum_{i=1}^n b_i^2 = \tau^2 \leq \tau$$

if and only if $0 \leq \tau \leq 1$. Hence, for $0 \leq \tau \leq 1$, [BNV, Proposition 5.1] gives

$$\lambda_1(L) \leq \lambda_1(-\Delta) + \tau^{3/2}. \tag{6.168}$$

Now consider the case where $\Omega = B(0,1)$. It was proved in [HNR, Section 1] that, when $\Omega = B(0,1)$, the principal eigenvalue attains its supremum among all the drifts $v \in L^\infty$ with $\|v\|_\infty \leq \tau$ if and only if $v(x) = -\tau\frac{x}{|x|}$. Consider $\phi \in W_0^{1,2}(B(0,1))$ the principal eigenfunction of L which is positive in $B(0,1)$ and satisfies

$$-\Delta\phi(x) - \tau\frac{x}{|x|} \cdot \nabla\phi(x) = \lambda_1(B(0,1), \tau)\phi(x).$$

For $y \in B(0,\tau)$, define $\psi(y) := \phi(y/\tau)$. Then, the function ψ satisfies for $x \in B(0,1)$,

$$-\tau^2\Delta\psi(\tau x) - \tau^2\frac{x}{|x|} \cdot \nabla\psi(\tau x) = \lambda_1(B(0,1), \tau)\psi(\tau x).$$

In other words, the function $\psi \in H^1(B(0,\tau))$ satisfies $\psi > 0$ in $B(0,\tau)$ and

$$-\Delta\psi(y) - \frac{y}{|y|} \cdot \nabla\psi(y) = \frac{1}{\tau^2}\lambda_1(B(0,1), \tau)\psi(y),$$

for all $y \in B(0,\tau)$. By definition of $\lambda_1(B(0,\tau),1)$, it follows that for all $\tau \geq 1$,

$$\frac{1}{\tau^2}\lambda_1(B(0,1),\tau) = \lambda_1(B(0,\tau),1) \leq \lambda_1(B(0,1),1),$$

and so for all $\tau \geq 1$,

$$\lambda_1(B(0,1),\tau) \leq \tau^2\lambda_1(B(0,1),1).$$

Hence, using (6.168), it follows that for all $\tau > 0$, there exist $C_1 > 0$, and $C_2 > 0$, such that

$$\lambda_1(B(0,1),\tau) \leq C_1 + \tau^{3/2} + C_2\tau^2. \tag{6.169}$$

Consider now the case where $\Omega := B(0,R)$, with $R > 0$. The principal eigenfunction $\varphi \in H_0^1(B(0,R))$ of L is positive in $B(0,R)$ and satisfies

$$-\Delta\varphi(x) - \tau\frac{x}{|x|}\cdot\nabla\varphi(x) = \lambda_1(B(0,R),\tau)\varphi(x).$$

For all $x \in B(0,1)$, now define $\psi(x) := \varphi(Rx)$. Then,

$$-\frac{1}{R^2}\Delta\psi\Big(\frac{x}{R}\Big) - \frac{\tau}{R}\frac{x}{|x|}\cdot\nabla\psi\Big(\frac{x}{R}\Big) = \lambda_1(B(0,R),\tau)\psi\Big(\frac{x}{R}\Big).$$

In other words, the function $\psi \in H^1(B(0,1))$, $\psi > 0$ in $B(0,1)$ and

$$-\Delta\psi(y) - \tau R\frac{y}{|y|}\cdot\nabla\psi(y) = R^2\lambda_1(B(0,R),\tau)\psi(y),$$

for all $y \in B(0,1)$. This means that

$$\lambda_1(B(0,R),\tau) = \frac{1}{R^2}\lambda_1(B(0,1),\tau R).$$

Using (6.169), we obtain

$$\lambda_1(B(0,R),\tau) \leq \frac{1}{R^2}\Big(C_1 + (\tau R)^{3/2} + C_2(\tau R)^2\Big),$$

and finally,

$$\lambda_1(B(0,R),\tau) \leq \frac{C_1}{R^2} + \frac{\tau^{3/2}}{\sqrt{R}} + C_2\tau^2.$$

Finally, if Ω is such that $B(0,R) \subset \Omega$, we conclude using the fact that

$$\lambda_1(\Omega') \leq \lambda_1(\Omega) \text{ if } \Omega \subset \Omega',$$

which follows from the definition of the principal eigenvalue of λ_1, see for example [BNV, Chapter 1].
The general case where $B(x_0,R) \subset \Omega$ follows by translation.

$\square$

Corollary 6.4. *If the measure of the domain D is sufficiently large, and the supremum of $\alpha G'$ is sufficiently small, the inequality*

$$f'(0) > \lambda_1(-L) \tag{6.170}$$

holds.

Proof. It is an immediate consequence of Proposition 6.3 with

$$v := \alpha(y)\Big(G_2'(0), \cdots, G_n'(0)\Big).$$

$\square$

Now, for each $t \in \mathbb{R}$, let $\mu_1^c(t)$ denote the principal eigenvalue of the operator

$$-\Delta' - \alpha(y) \sum_{i=2}^{n} G_i'(0) \frac{\partial}{\partial x_i} - f'(0) + t\beta_c(y),$$

where

$$\beta_c(y) = c + \alpha(y) G_1'(0).$$

As before, according to [YIH, Theorem 1.3], the principal eigenvalue $\mu_1^c(t)$ is characterized by the existence of a unique $\varphi = \varphi(t) \in W_0^{1,2}(D)$, on the domain D with Dirichlet boundary conditions, such that $\varphi(t)(y) > 0$, for all $y \in D$, normalized by $\|\varphi(t)\|_{L^\infty(D)} = 1$. This unique function satisfies

$$\begin{cases} -\Delta'\varphi - \alpha(y) \sum_{i=2}^{n} G_i'(0) \frac{\partial \varphi}{\partial x_i} - f'(0)\varphi + t\beta_c(y)\varphi = \mu_1^c(t)\varphi & \text{in } D, \\ \varphi = 0 & \text{on } \partial D. \end{cases} \tag{6.171}$$

Denote by $\mu_1^c(t)$ the principal eigenvalue of this problem (6.171). With this notation, $\lambda \in \mathbb{R}$ is an eigenvalue of (6.163) if and only if

$$\lambda^2 = \mu_1^c(\lambda).$$

Indeed, $\lambda^2 = \mu_1^c(\lambda)$ if and only if there exists a positive function ϕ in D which satisfies

$$\begin{cases} -\Delta'\phi - \alpha(y) \sum_{i=2}^{n} G_i'(0) \frac{\partial \phi}{\partial x_i} - f'(0)\phi + \lambda\beta_c(y)\phi = \lambda^2\phi & \text{in } D, \\ \phi = 0 & \text{on } \partial D, \end{cases} \tag{6.172}$$

which is equivalent to

$$\begin{cases} -\Delta'\phi - \alpha(y) \sum_{i=2}^{n} G_i'(0) \frac{\partial \phi}{\partial x_i} - f'(0)\phi = \Big(\lambda^2 - \lambda\beta_c(y)\Big)\phi & \text{in } D, \\ \phi = 0 & \text{on } \partial D, \end{cases} \tag{6.173}$$

which means that λ is an eigenvalue of (6.163). Once again, according to [YIH, Theorem 1.3], none of the other eigenvalues corresponds to a positive eigenfunction. Hence, it follows that λ is a principal eigenvalue of (6.163) if and only if $\mu_1^c(\lambda) = \lambda^2$, in other words, if and only if λ is a root of the equation $\mu_1^c(t) = t^2$.
In addition, we claim :

Proposition 6.5. *The eigenvalue* $\mu_1^c(t)$ *of problem* (6.171) *is concave with respect to* $t \in \mathbb{R}$.

Proof. The proof is exactly the same as the proof of Proposition 3.3 since the proof does not depend on the boundary conditions.

□

Now we are able to prove that there exist two critical values c' and $\hat{c}$ such that if $c > c'$ the eigenvalue problem (6.163) has exactly two positive eigenvalues, and if $c < \hat{c}$, it has exactly two negative eigenvalues, similarly to the earlier results, namely Proposition 3.4 and Proposition 4.7 in the case of Neumann boundary conditions.

Proposition 6.6. *Assume* **(GD)**, **(AlphaD)** *and that*

$$f'(0) > \lambda_1(-L),$$

where the operator $-L := -\Delta' - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial}{\partial x_i}$. *Let* $g_c(t) = \mu_1^c(t) - t^2$. *Then, there exist* $\hat{c} < c'$ *such that*

$$\begin{aligned}
c < \hat{c} \quad &\Rightarrow \quad g_c(t) = 0 \text{ has 2 negative solutions}\\
c = \hat{c} \quad &\Rightarrow \quad g_c(t) = 0 \text{ has 1 negative solution}\\
\hat{c} < c < c' \quad &\Rightarrow \quad g_c(t) = 0 \text{ has no solutions}\\
c = c' \quad &\Rightarrow \quad g_c(t) = 0 \text{ has 1 positive solution}\\
c' < c \quad &\Rightarrow \quad g_c(t) = 0 \text{ has 2 positive solutions.}
\end{aligned}$$

The number of roots of the equation $g_c(t) = 0$ *corresponds to the number of principal eigenvalue(s) of* (6.163).

Proof. The proof is similar to the one of Proposition 3.4 under Neumann boundary conditions. We will details the points which are different.
Consider the t-dependent eigenvalue problem (6.171). For each $c \in \mathbb{R}$, we know that $t \mapsto \mu_1^c(t)$ is continuous and concave. In particular, by concavity of $t \mapsto \mu_1^c(t)$, the equation $\mu_1^c(t) = t^2$ admits at most two roots.
We are going to use the following characterisation of $\mu_1^c(t)$:

$$\mu_1^c(t) = \sup\Big\{\mu^c(t) \mid \exists\phi > 0 \text{ in } D,\ \phi = 0 \text{ on } \partial D,\ (L_1 + \mu^c(t))\phi \le 0\Big\}, \qquad (6.174)$$

where

$$-L_1 = -\Delta' - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial}{\partial x_i} - f'(0) + t\beta_c(y). \qquad (6.175)$$

Consider the eigenvalue problem

$$\begin{cases} -\Delta'\phi - \alpha(y)\sum_{i=2}^{n} G_i'(0)\frac{\partial\phi}{\partial x_i} = \mu\phi & \text{in } D, \\ \phi = 0 & \text{on } \partial D. \end{cases} \tag{6.176}$$

By [YIH, Theorem 1.3], this problem has a simple eigenvalue $\mu \in \mathbb{R}$ which corresponds to a positive eigenfunction. Denote by ϕ_0 this positive eigenfunction, and take it as a test function in the definition (6.174) of $\mu_1^c(t)$:

$$\begin{aligned} -L_1\phi_0 &= -\Delta'\phi_0 - \alpha(y)\sum_{i=2}^{n} G_i'(0)\frac{\partial\phi_0}{\partial x_i} - f'(0)\phi_0 + t\beta_c(y)\phi_0 \\ &= \mu\phi_0 - f'(0)\phi_0 + t\beta_c(y)\phi_0 \\ &= \Big(\mu - f'(0)\Big)\phi_0 + t\Big(c + \alpha(y)G_1'(0)\Big)\phi_0. \end{aligned}$$

Denote $k := \inf_{\overline{D}} \alpha(y)G_1'(0)$ and $K := \sup_{\overline{D}} \alpha(y)G_1'(0)$. Then, for $t > 0$, one has

$$-L_1\phi_0 \geq \Big(\mu - f'(0) + t(c+k)\Big)\phi_0,$$

and for $t < 0$,

$$-L_1\phi_0 \geq \Big(\mu - f'(0) + t(c+K)\Big)\phi_0.$$

Hence, by definition of $\mu_1^c(t)$, it follows that for $t > 0$,

$$\mu_1^c(t) \geq \mu - f'(0) + t(c+k),$$

and for $t < 0$, that

$$\mu_1^c(t) \geq \mu - f'(0) + t(c+K).$$

We can deduce from those inequalities that

$$\lim_{c\to+\infty} \mu_1^c(t) = +\infty \text{ for } t > 0, \quad \lim_{c\to-\infty} \mu_1^c(t) = +\infty \text{ for } t < 0.$$

The rest of the proof is similar to the one of Proposition 3.4 under Neumann boundary conditions.

$\square$

6.2 Comparison between $c^\star$ and c'

Throughout this section, we assume the condition **(FD)**, namely that

$$f'(0) > \lambda_1(-L), \tag{6.177}$$

where L is defined in Proposition 6.6, which ensures that the critical value c' defined in Proposition 6.6 does exist. Recall that this condition holds if the measure of the domain Ω is sufficiently big, and if the supremum of the function αG is sufficiently small, see Corollary 6.4.

Now, we want to compare c' and $c^\star$. Precisely, we will show that $c^\star \geq c'$ and that under certain assumptions on G and f, $c^\star = c'$.

Proposition 6.7. *Assume* **(GD)**, **(AlphaD)** *and that f is C^1 and satisfies* (1.8). *Let w be a solution of* (5.138) *and assume also that the following condition holds :*

$$f'(0) > k, \qquad \text{where } k := \sup_{(s,y)\in(0,s_0)\times\overline{D}} \left| \sum_{i=2}^{n} \frac{G_i(s)}{s} \frac{\partial \alpha}{\partial x_i}(y) \right|. \tag{6.178}$$

Then, there exist two positive constants C and ε such that, for all R large enough,

$$\int_R^\infty \int_D w \leq C e^{-\varepsilon R}.$$

Remark 6.8. Note that under Dirichlet boundary conditions, we do not need to assume that $\alpha \equiv 0$ on ∂D, in contrast to the corresponding result Proposition 3.6 in the Neumann boundary conditions case.
Note also that the assumptions **(GD)** and **(AlphaD)** ensure that $k < +\infty$, where k is defined in (6.178).

Proof. The proof is very similar to the one of Proposition 3.6 but not exactly the same because of the Dirichlet boundary conditions. We will stress the only point that differs. As before, let $N > R > 0$ and define a smooth cut-off function ξ on $\mathbb{R}$ such that $0 \leq \xi \leq 1$ and

$$\xi(x_1) := \begin{cases} 0 & \text{if } x_1 \geq N+1, \\ 1 & \text{if } R \leq x_1 \leq N, \\ 0 & \text{if } x_1 \leq R-1. \end{cases} \tag{6.179}$$

Multiplying (5.138) by ξ, integrating on $\Omega = \mathbb{R} \times D$, and using Green's formula it follows that

$$\int_{\Omega} \Big[w\xi'' - w\xi' \Big(c + \alpha(y) \frac{G_1(w)}{w} \Big) - \xi w \sum_{i=2}^{n} \frac{G_i(w)}{w} \frac{\partial \alpha}{\partial x_i} + \xi f(w) \Big] = 0. \tag{6.180}$$

Indeed, (6.180) holds because w satisfies Dirichlet boundary conditions on ∂D, which ensures that $G(w) \equiv 0$ on ∂D because $G(0) = 0$. In the proof of Proposition 3.6, we needed to assume that $\alpha \equiv 0$ on ∂D in order to have (6.180). The rest of the proof is exactly the same as the one of Proposition 3.6. □

Thanks to Proposition 6.7, we are able to prove the following theorem :

Theorem 6.9. *Under the assumptions of Proposition 6.7, there exist two positive contants C_0 and ε such that the solution w of* (5.138) *satisfies*

$$w(x_1, y) + |\nabla w(x_1, y)| \le C_0 e^{-\varepsilon x_1}, \text{ for all } (x_1, y) \in \mathbb{R} \times D. \tag{6.181}$$

Proof. Let $x_1 > 1$, $\Omega_1 := (x_1 - 1, x_1 + 1) \times D$, and $\Omega_2 := (x_1 - 2, x_1 + 2) \times D$. By embedding and then using [GIL, Theorem 9.13], there exist two positive constants C_1 and C_2 independent of x_1 such that

$$\|w\|_{C^{1,\lambda}(\overline{\Omega_1})} \le C_1 \|u\|_{W^{2,p}(\Omega_1)} \le C_2 \Big(\|w\|_{L^p(\Omega_2)} + \|f(w)\|_{L^p(\Omega_2)} \Big),$$

where $p > n$ is fixed.

Remark 6.10. Recall that the local up to the boundary $W^{2,p}$ estimate for $y \in D$ follows by adaptating the proof of [ADN, Theorem 15.2], see footnote 1.

Since there exists $C_3 > 0$ such that $f(u) \le C_3 u$ for all $u \in (0, 1)$, it follows that $\|f(w)\|_{L^p(\Omega_2)} \le C_3 \|w\|_{L^p(\Omega_2)}$. Hence, there exists $K > 0$ such that

$$\|w\|_{C^{1,\lambda}(\overline{\Omega_1})} \le K \|w\|_{L^p(\Omega_2)}.$$

Since $0 < w < 1$, and according to Proposition 6.7,

$$\begin{aligned} w(x_1, y) + |\nabla w(x_1, y)| &\le K \Big(\int_{x_1 - 2}^{x_1 + 2} \int_D w^p \Big)^{1/p} \\ &\le K \Big(\int_{x_1 - 2}^{x_1 + 2} \int_D w \Big)^{1/p} \\ &\le KC \Big(e^{-\varepsilon x_1} \Big)^{1/p} \\ &= C_0 e^{-\frac{\varepsilon}{p} x_1}, \end{aligned}$$

with $C_0 := KC$. □

We are now able to prove Theorem 1.21. As previously in the proof of Theorem 1.18, our strategy is to consider a travelling front solution w with speed $c > c^\star$ which is a solution of (5.138), and show that there must exist at least one real eigenvalue λ of (6.163).

Proof of Theorem 1.21. Let $c > c^\star$. By definition of $c^\star$ and Theorem 1.20 there exists a solution w of

$$\begin{cases} \Delta w + \Big(c + \alpha(y) G_1'(w)\Big) \frac{\partial w}{\partial x_1} + \alpha(y) \sum_{i=2}^{n} G_i'(w) \frac{\partial w}{\partial x_i} + f(w) = 0 & \text{in } \Omega, \\ w(-\infty, y) = w_-(y), \quad w(+\infty, y) = 0 & \text{uniformly in } y \in D, \\ w \geq 0, & \\ w = 0 & \text{on } \mathbb{R} \times \partial D. \end{cases} \tag{6.182}$$

We want to prove that there exists at least one real eigenvalue λ associated to a positive eigenfunction φ, of

$$\begin{cases} -\Delta' \varphi - \alpha(y) \sum_{i=2}^{n} G_i'(0) \frac{\partial \varphi}{\partial x_i} - f'(0)\varphi = \Big(\lambda^2 - \lambda(c + \alpha(y) G_1'(0))\Big)\varphi & \text{in } D, \\ \varphi = 0 & \text{on } \partial D. \end{cases} \tag{6.183}$$

First, we will show that there exists a positive solution h of the linearized problem

$$\begin{cases} \Delta h + \Big(c + \alpha(y) G_1'(0)\Big) \partial_1 h + \alpha(y) \sum_{i=2}^{n} G_i'(0) \frac{\partial h}{\partial x_i} + f'(0) h = 0 & \text{in } \mathbb{R} \times D, \\ h = 0 & \text{on } \mathbb{R} \times \partial D. \end{cases} \tag{6.184}$$

To do this, for $N \geq 2$, choose a sequence $x_1^{(N)}$, which tends to $+\infty$ when N goes to infinity and satisfies

$$\sup_{y \in \overline{D}} w(x_1^N, y) = \frac{1}{N}.$$

Note that such a sequence exists since w is continuous and tends uniformly (with respect to y) to 0 when x_1 tends to infinity. Now, define

$$h^N(x_1, y) := \frac{w(x_1^N + x_1, y)}{2 \sup_{y \in \overline{D}} w(x_1^N, y)}.$$

Exactly as in the proof of Theorem 1.18, we show that the function $h^{(N)}$ converges in $C^{1,\lambda}_{loc}$ to a non negative function h which satisfies the following problem :

$$\begin{cases} \Delta h + (c + \alpha(y) G_1'(0)) \partial_1 h + \alpha(y) \sum_{i=2}^{n} G_i'(0) \frac{\partial h}{\partial x_i} + f'(0) h = 0 & \text{on } \mathbb{R} \times D, \\ h = 0 & \text{on } \mathbb{R} \times \partial D, \\ 0 \leq h \leq 1, \;\; \partial_1 h \leq 0 & \text{in } \mathbb{R} \times \overline{D}, \\ \max_{y \in \overline{D}} h(0, y) = \frac{1}{2}. \end{cases} \tag{6.185}$$

Now, we want to prove that $h > 0$, in $\mathbb{R} \times D$ using the Strong Maximum Principle. Observe that

$$\Delta h + (c + \alpha(y)G_1'(0))\partial_1 h + \alpha(y)\sum_{i=2}^{n} G_i'(0)\frac{\partial h}{\partial x_i} = -f'(0)h \leq 0.$$

If h vanishes in Ω, the Strong Maximum Principle gives that h is constant in Ω, therefore $h = 0$ in Ω, which is impossible due to the condition $\max_{y\in\overline{D}} h(0, y) = 1/2$, hence, $h > 0$ in Ω.

We will now show that when x_1 tends to $+\infty$, h tends to 0 in $C^1(\overline{D})$. The same arguments as in the proof of Theorem 1.20 ensure that the function h tends to a function $\tilde{h}$ in $C^1(\overline{D})$ which satisfies :

$$\begin{cases} -\Delta'\tilde{h} - \alpha(y)\sum_{i=2}^{n} G_i'(0)\frac{\partial \tilde{h}}{\partial x_i} - f'(0)\tilde{h} = 0 & \text{in } D, \\ \tilde{h} = 0 & \text{on } \partial D. \end{cases} \tag{6.186}$$

First, recall that under the assumption **(FD)**, the principal eigenvalue μ_1 of (6.164) is negative. If $\tilde{h}$ does not vanish in D, then $\tilde{h} > 0$ in D, but then, $\tilde{h}$ is a positive eigenfunction of (6.164) associated to the eigenvalue 0, which contradicts the result in [YIH, Theorem 1.3] saying that this eigenvalue problem has a simple eigenvalue $\mu_1 < 0$ corresponding to a positive eigenfunction, and that none of the other eigenvalues corresponds to a positive eigenfunction. Hence $\tilde{h}$ vanishes at some point in D and by the Strong Maximum Principle, $\tilde{h}$ is constant, and because of the equation (6.186), $\tilde{h} \equiv 0$.

In the proof of Theorem 1.17 in the Neumann case, we used the lower bound (3.96) to show that at least one eigenvalue must make a non-zero contribution to the expansion in (3.97). The estimate (3.96) is based on the proof of [BN2, Theorem 3.2] which relies on the Krylov-Safonov-Harnack inequality :

$$\sup_S u \leq C_0 \inf_S u,$$

where $S = [\alpha - 1, \alpha] \times \overline{D}$, based on e.g, [BCN, Theorem 2.1]. However, this estimate up to the boundary that holds because of the Neumann boundary conditions on part of ∂S but cannot hold in the case of zero Dirichlet boundary conditions.
We will use a local Harnack inequality, see [GIL, Corollary 9.25] which ensures that for any ball $B(x_0, 2R) \subset \mathbb{R} \times D$ there exists $C > 0$ such that

$$\sup_{B(x_0,R)} h \leq C \inf_{B(x_0,R)} h. \tag{6.187}$$

Let $\overline{D'} \subset\subset D$ be sufficiently small that there exists $R > 0$ such that for each $\delta \in \mathbb{R}$, there exists $x_0 := (\tilde{x_1}, \tilde{y})$ such that

$$S := \left[\delta - \frac{1}{2}, \delta + \frac{1}{2}\right] \times \overline{D'} \subset B(x_0, R) \subset B(x_0, 2R) \subset \mathbb{R} \times D. \tag{6.188}$$

Then by [GIL, Corollary 9.25] there exists $C > 0$ independant of δ such that

$$\sup_S h \le C \inf_S h.$$

Then using that $\|h\|_{C^{1,\lambda}(\overline{S})} \le C\|h\|_{W^{2,p}(S)} \le \tilde{C}\|h\|_{L^p} \le \hat{C}\|h\|_{L^\infty}$ and arguing as in [BN2, Theorem 3.2], there exists $C_5 > 0$ such that for $x_1 \ge 1$ and $y \in \overline{D'}$

$$\left|\frac{\partial h}{\partial x_1}\right|(x_1, y) \le C_5 \sup_{|\xi - x_1| \le \frac{1}{2}\ y \in \overline{D'}} h(x_1, y),$$

and hence there exists $a > 0$ such that

$$-ah(x_1, y) \le \frac{\partial h}{\partial x_1}(x_1, y) \le ah(x_1, y)$$

which implies that

$$\frac{\partial h}{\partial x_1} + ah(x_1, y) \ge 0,$$

which is equivalent to

$$\frac{\partial}{\partial x_1}\left(e^{ax_1} h\right) \ge 0.$$

and then

$$e^{ax_1} h(x_1, y) \ge e^a h(1, y),$$

Finally, denote by $\gamma := e^a \inf_{y \in \overline{D'}} h(1, y)$, one has

$$h(x_1, y) \ge \gamma e^{-ax_1} \tag{6.189}$$

for $x_1 \ge 1$ and $y \in \overline{D'}$. Thus, we obtained the analogue of (3.96) for $y \in \overline{D'}$ instead of for $y \in \overline{D}$ in (3.96) under Neumann boundary conditions. However, the estimate (6.189) for $y \in \overline{D'}$ is enough to ensure that the expansion corresponding to (3.97) in the Dirichlet case cannot be identically zero. Indeed, if the expansion in (3.97) (under Dirichlet boundary conditions) is identically zero, that would then imply that

$$\|h\|_{L^\infty(\{x_1\} \times D)} \le Ce^{-(a+\varepsilon')x_1},$$

where $\varepsilon' = \varepsilon_0 - \varepsilon > 0$ with the notation in (3.97). Thus, that would contradict the fact that $h(x_1, y) \ge \gamma e^{-ax_1}$ for $x_1 \ge 1$ and $y \in \overline{D'}$.

The rest of the proof is similar to the one of Theorem 1.18 under Neumann boundary conditions. The only difference is that the function q_p (with the same notation as in the proof of Theorem 1.18) satisfies Dirichlet boundary conditions. The Strong Maximum Principle yields that q_p is positive in D, which implies that q_p is an eigenfunction of (6.183), associated to the eigenvalue $\lambda_5 > 0$. Hence, $c \ge c'$, which implies that $c^\star \ge c'$, as required.

$\square$

6.3 Special case where $G = (G_1, 0, \cdots, 0)$.

Recall that in the one dimensional case the equality $c^\star = c'$ does not necessarily hold. Indeed, [AK1, Proposition 2.3] with $f(u) = u(1-u)$ and $G(u) = -\gamma u^2$ for $u \in [0,1]$ and where $\gamma > \sqrt{11/3}$ ensures that $c^\star > c'$.

Recall also that under Neumann boundary conditions, and in a special case where $G = (G_1, 0, \cdots, 0)$, we showed in Proposition 3.9 that the equality $c^\star = c'$ holds by adapting the approach of [BN2, Section 10].

We now show that under Dirichlet boundary conditions, in that special case, the equality $c^\star = c'$ still holds with the same conditions on the functions f, G and α,.

Proposition 6.11. *Assume that $G = (G_1, 0, \cdots, 0)$, with $G_1 \neq 0$. Assume also that the measure of the domain D is sufficiently big and that for $y \in D$ and $u \in \mathbb{R}$,*

$$\alpha(y) G_1'(u) \geq \alpha(y) G_1'(0),$$

as well as the KPP condition, for all $u \in (0,1)$,

$$f(u) \leq f'(0) u.$$

Then

$$c' = c^\star.$$

Proof. The proof is very similar to the one of Proposition 3.9, using the method of sub and super solutions. But under Dirichlet boundary conditions, we will use a function v_k as a subsolution, instead of a small positive constant ε_N under Neumann boundary conditions.
First, by Corollary 6.4, when the measure of the domain D is big enough, we have $f'(0) > \lambda_1(-\Delta')$, which ensures that c' exists by Proposition 6.6. Assume $c^\star > c'$. Choose c such that $c^\star > c > c'$. Then, by definition of c', and since $f'(0) > \lambda(-\Delta')$ whenever $|D|$ is large enough, there exist $0 < \lambda_1 < \lambda_2$ and φ_1, φ_2 positive functions such that, for $j = 1, 2$

$$\begin{cases} -\Delta' \varphi_j - f'(0)\varphi_j = \Big(\lambda_j^2 - \lambda_j (c + \alpha(y) G_1'(0))\Big)\varphi_j & \text{in } D, \\ \varphi_j = 0 & \text{on } \partial D. \end{cases} \tag{6.190}$$

We will construct a solution of (5.138), using method of sub and supersolutions.
The function $z := e^{-\lambda_1 x_1}\varphi_1(y)$ satisfies

$$\begin{cases} -\Delta z - (c + \alpha(y) G_1'(z))\partial_1 z - f(z) \geq 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \\ z(+\infty, \cdot) = 0 & \text{uniformly in } y. \end{cases} \tag{6.191}$$

Note that contrary to the Neumann case (see the proof of Proposition 3.9), we do not have that $z(-\infty,\cdot) = +\infty$ uniformly in $y \in \overline{D}$, but in the Dirichlet case, we have that since $\varphi_1 \equiv 0$ on ∂D and $\partial_\nu \varphi_1 > 0$ by the Hopf lemma, it follows that there exists $N > 1$ sufficiently large such that $e^{-\lambda_1 x_1}\varphi_1(y) > w_-(y)$ for all $y \in D$ and $x_1 \leq -N$.
Then, fix an integer $N > 1$ such that $z(-N, y) > w_-(y)$, for all $y \in D$. Using again the Hopf which ensures that $\partial_\nu \varphi_1 > 0$ and that the sequence of functions $(v_k)_k$ tends uniformly to 0 when $k \to +\infty$, we can choose k large enough such that $z(N, y) > v_k(y)$, for all $y \in D$.
The function v_k is a subsolution and z is a supersolution on $\Omega_N = (-N, N) \times D$.
Now we will apply Proposition 5.3 with $\underline{u} = v_k$ and $\overline{u} = h(x_1, y) := \min(w_-(y), z(x_1, y))$. Proposition 5.3 gives the existence of a solution $u \in C(\overline{\Omega_N}) \cap W^{2,p}_{loc}\Big(\overline{\Omega_N} \setminus (\{\pm N\} \times \partial D)\Big)$ of

$$\begin{cases} -\Delta u - \Big(c + \alpha(y) G_1'(u)\Big)\partial_1 u - f(u) = 0 & \text{in } \Omega_N, \\ u = 0 & \text{for } -N < x_1 < N, y \in \partial D, \\ u(-N, y) = h(-N, y)\ ,\ u(N, y) = v_k(y) & \text{and } v_k \leq u \leq h. \end{cases} \tag{6.192}$$

[BN1, Theorem 7.2] ensures that there exists only one solution $u \in W^{2,p}_{loc}\Big(\overline{\Omega_N} \setminus (\{\pm N\} \times \partial D)\Big) \cap C(\overline{\Omega_N})$ of (6.192). Moreover, since $c + \alpha(y) G_1'$ is Lipschitz continuous in x_1, the [BN1, Theorem 7.2] gives also that $\partial_1 u(x_1, y) < 0$ for $-N < x_1 < N$ and $y \in \overline{D}$.
Thus, for each N sufficiently large, one has a unique solution $u^N \in W^{2,p}_{loc}\Big(\overline{\Omega_N} \setminus (\{\pm N\} \times \partial D)\Big) \cap C(\overline{\Omega_N})$ which satisfies

$$\begin{cases} -\Delta u^N - (c + \alpha(y) G_1'(u^N))\partial_1 u^N - f(u^N) = 0 & \text{in } \Omega_N, \\ u^N = 0 & \text{for } -N < x_1 < N, y \in \partial D, \\ u^N(-N, y) = h(-N, y)\ ,\ u^N(N, y) = v_k(y) & \text{and } v_k \leq u^N \leq h, \\ \partial_1 u^N(x_1, y) < 0 & \text{for } -N < x_1 < N, y \in D. \end{cases} \tag{6.193}$$

Now we want to let N tend to infinity, but we have to prevent the solution u^N from tending to w_- or 0 when N tends to infinity. To do this, consider a shift of h :

$$h^r(x_1, y) := \min\Big(w_-(y), z(x_1 + r, y)\Big),$$

and

$$\beta^r(y) := \sup_k \Big\{v_k(y), \quad v_k(y) \leq z(N + r, y), \quad \forall y \in D\Big\}.$$

By compactness of $\overline{D}$ and continuity of z, the function $r \mapsto h^r$ exists and is continuous on $\mathbb{R}$. As before, there exists a unique function $v^r \in W^{2,p}_{loc}\Big(\overline{\Omega_N} \setminus \{\pm N\} \times \partial D\Big) \cap$

$C\Big([-N,N]\times\overline{D}\Big)$ with $h^r\le v^r\le w_-$ in Ω_N satisfying

$$\begin{cases}\Delta v^r+(c+\alpha(y)G_1'(v^r))\partial_1 v^r+f(v^r)=0 & \text{in } \Omega_N,\\ v^r=0 & \text{for } -N<x_1<N, y\in\partial D,\\ v^r(-N,y)=h^r(-N,y)\ ,\ v^r(N,y)=\beta^r(y), & \text{for all } y\in D.\end{cases} \tag{6.194}$$

Indeed, the function h^r is a minimum of two supersolutions, and consequently h^r is a supersolution, according to [YIH, Lemma 5.1], which is based on results in [LE1], and β^r is a subsolution.
Moreover, [BN1, Theorem 2.4] gives that v^r is decreasing with respect to x_1 and the uniqueness of solution v^r satisfying (6.194). Since h^r vary continuously with $r\in\mathbb{R}$, the uniqueness of solution gives that $r\mapsto v^r$ is also continuous in $\overline{\Omega_N}$.
Also, since the function v^r satisfies $\beta^r(y)\le v^r(x_1,y)\le \min\Big(w_-(y),z(x_1+r,y)\Big)$, and that $\partial_1 v^r<0$, it follows that v^r tends uniformly to 0 when r tends to $+\infty$, and the limit of v^r when r tends to $-\infty$ exists and satisfies

$$v_1(y)\le \lim_{r\to-\infty} v^r(x_1,y)\le w_-(y),$$

where v_1 is one of the family $\{v_k\}_{k\in\mathbb{N}}$ which is assumed to exist in condition (BD). Hence, by continuity of $r\mapsto \max_{y\in\overline{D}} v^r(0,y)$, there exists a value of r such that

$$\max_{y\in\overline{D}} v^r(0,y)=\frac{\sup_{y\in D} v_1(y)}{2}.$$

Let u^N denote the corresponding solution v^r. Then, as in the proof of Proposition 3.9 under Neumann boundary conditions, using the boundedness of $(u^N)_N$ in $W^{2,p}_{loc}\Big(\overline{\Omega_N}\backslash\{\pm N\}\times\partial D\Big)$ and a diagonal argument there exists a sequence $(N_j)_j$ such that $u^{N_j}\to u$ uniformly on compact sets of $\mathbb{R}\times\overline{D}$. Furthermore, the limit function u satisfies

$$\begin{cases}-\Delta u-\Big(c+\alpha(y)G_1'(u)\Big)\partial_1 u-f(u)=0 & \text{in } \mathbb{R}\times D,\\ u=0 & \text{for } y\in\partial D,\\ 0\le u\le w_-, & \\ \partial_1 u(x_1,y)\le 0 & \text{for } y\in\overline{D},\\ \max_{y\in\overline{D}} u(0,y)=\frac{\sup_{y\in D} v_1(y)}{2}. & \end{cases} \tag{6.195}$$

Finally, since u is non-increasing with respect to x_1, it follows that u has a finite limit when x_1 tends to $\pm\infty$. Moreover, $\lim_{x_1\to+\infty} u(x_1,y)$ and $\lim_{x_1\to-\infty} u(x_1,y)$ have to satisfy the problem on the cross section D (5.138) which can be shown using arguments similar to those in the proof of Proposition 3.9 in the Neumann case.

Hence, by assumption **(AD)**, these limits have to be 0 or w_-. But the normalization condition

$$\max_{y\in\overline{D}} u(0,y) = \frac{\sup_{y\in D} v_1(y)}{2},$$

ensures that

$$\lim_{x_1\to+\infty} u(x_1,y) = 0 \text{ and } \lim_{x_1\to-\infty} u(x_1,y) = w_-.$$

Similarly as in Proposition 3.9 under Neumann boundary conditions, we show that the function u is decreasing and satisfies $\frac{\partial u}{\partial x_1} < 0$. Hence, u belongs to the set $\tilde{K}$ and is a solution of (5.137) with a speed $c < c^\star$, which is impossible by definition of $c^\star$.

□

6.4 Existence of the $\{v_k\}_k$ and w_-

Recall that we assumed **(AD)**, namely that there exists a function $w_- \in C^{2,\lambda}(\overline{D})$ such that if $z \in C^{2,\lambda}(\overline{D}) : D \to \mathbb{R}$ satisfies (5.138) then $z \equiv 0$ or $z \equiv w_-$. Hence, the function w_- satisfies

$$\begin{cases} \Delta' w_- + \alpha(y)\sum_{i=2}^n G_i'(w_-)\frac{\partial w_-}{\partial x_i} + f(w_-) = 0 & \text{in } D, \\ w_- = 0 & \text{on } \partial D, \\ w_- > 0 & \text{in } D, \end{cases} \tag{6.196}$$

and the existence of a sequence of subsolutions $\{v_k(y)\}_k$ uniformly bounded in $C^{2,\lambda}(\overline{D})$, which satisfies for every $k \in \mathbb{N}$

$$0 < v_{k+1}(y) < v_k(y) < w_-(y), \text{ for } y \in D,$$

and (5.139).

It is interesting to consider under which conditions on f and G these assumptions can be satisfied.

Proposition 6.12. *Assume the measure of the domain D to be large enough, and that $\|\alpha G'(0)\|_\infty$ is small enough. Then there exists a sequence of functions $\{v_k(y)\}_k \in C^{2,\lambda}(\overline{D})$, which satisfies* (5.139).

Proof. Consider the operator $-L = -\Delta' - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial}{\partial x_i}$. By [YIH, Theorem 1.3], this operator has a simple eigenvalue $\lambda \in \mathbb{R}$ which corresponds to a positive eigenfunction φ, under Dirichlet boundary conditions :

$$\begin{cases} -\Delta'\varphi - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial\varphi}{\partial x_i} = \lambda\varphi & \text{in } D, \\ \varphi = 0 & \text{on } \partial D. \end{cases} \tag{6.197}$$

Let $\delta \in (0,1)$, and compute

$$\begin{aligned}\Delta'(\delta\varphi) + \alpha(y)\sum_{i=2}^{n} G_i'(\delta\varphi)\frac{\partial(\delta\varphi)}{\partial x_i} + f(\delta\varphi) &= \delta\Big[\Delta'\varphi + \alpha(y)\sum_{i=2}^{n} G_i'(\delta\varphi)\frac{\partial\varphi}{\partial x_i} + \frac{f(\delta\varphi)}{\delta}\Big] \\ &= \delta\Big[\alpha(y)\sum_{i=2}^{n}\Big(G_i'(\delta\varphi) - G_i'(0)\Big)\frac{\partial\varphi}{\partial x_i} + \frac{f(\delta\varphi)}{\delta} - \lambda\varphi\Big].\end{aligned}$$

Using the mean value theorem, for each $2 \le i \le n$, there exists $\xi_i \in (0, \delta\varphi)$ such that $G_i'(\delta\varphi) - G_i'(0) = \delta\varphi G_i''(\xi_i)$. Hence, after dividing by φ, the function $\delta\varphi$ is a subsolution if and only if

$$\delta\alpha(y)\sum_{i=2}^{n} G_i''(\xi_i)\frac{\partial\varphi}{\partial x_i} + \frac{f(\delta\varphi)}{\delta\varphi} - \lambda \ge 0.$$

Letting δ tend to 0, the term $\delta\alpha(y)\sum_{i=2}^{n} G_i''(\xi_i)\frac{\partial\varphi}{\partial x_i}$ tends to 0, and the term $\frac{f(\delta\varphi)}{\delta\varphi} - \lambda$ tends to $f'(0) - \lambda$. Using Proposition 6.3, we know that when $|D|$ is large enough and $\|\alpha G'(0)\|_\infty$ small enough,

$$f'(0) > \lambda.$$

Thus, for a decreasing sequence $(\delta_k)_{k\in\mathbb{N}}$ with $\lim_{k\to+\infty}\delta_k = 0$ and δ_1 sufficiently small, the sequence of functions $\{\delta_k\varphi\}_{k\in\mathbb{N}}$ satisfies (5.139).

□

Corollary 6.13. *Under the same assumptions as Proposition 6.12, there exists a function $w_- \in C^{2,\lambda}(\overline{D})$ which satisfies* (6.196).

Proof. Now that we have the existence of a sequence of functions $\{v_k\}_k$ which satisfies (5.139). Hence, since the constant function 1 does not satisfy the Dirichlet boundary conditions, we can use a v_k as a subsolution, and 1 as a supersolution to construct a function $w_- \in C^{2,\lambda}(\overline{D})$ which satisfies (6.196).

□

We will show, building an idea in the proof of [ALL, Theorem 6.1], that in a special case where $\alpha G'$ is a constant, Assumption **(AD)** is satisfied.

Proposition 6.14. *Assume the measure of the domain D large enough, $\alpha G'$ is a constant vector the supremum of which is sufficiently small, and that the function $f(u)/u$ is non increasing. Then, Assumption* **(AD)** *is satisfied.*

Proof. First, let z be a positive solution of (5.138). There exist $k \ge 1$ and $C > 1$ such that $v_k \le z \le C$ in D.

By [YIH, Theorem 4.9 and Theorem 4.11] applied with v_k as a subsolution and a constant $C > 1$ as a supersolution of (5.138), there exist a minimal solution $\underline{u}$ and a maximal solution $\overline{u}$ of (5.138) (in the sense where if $\tilde{u}$ is a solution of (5.138) then $\underline{u} \leq \tilde{u} \leq \overline{u}$) which satisfy $v_k \leq \underline{u} \leq \overline{u} \leq C$, and

$$\begin{cases} \Delta'\overline{u} + \alpha G' \cdot \nabla\overline{u} + f(\overline{u}) = 0 & \text{in } D, \\ \Delta'\underline{u} + \alpha G' \cdot \nabla\underline{u} + f(\underline{u}) = 0 & \text{in } D. \end{cases} \tag{6.198}$$

Multiplying by $e^{\alpha G' \cdot y}\underline{u}$ the first equation and by $e^{\alpha G' \cdot y}\overline{u}$ the second one, subtracting and then integrating over D, we obtain

$$\int_D e^{\alpha G' \cdot y}\underline{u}\Big(\Delta'\overline{u} + \alpha G' \cdot \nabla\overline{u}\Big) - e^{\alpha G' \cdot y}\overline{u}\Big(\Delta'\underline{u} + \alpha G' \cdot \nabla\underline{u}\Big) + e^{\alpha G' \cdot y}\underline{u}\overline{u}\Big(\frac{f(\overline{u})}{\overline{u}} - \frac{f(\underline{u})}{\underline{u}}\Big) = 0. \tag{6.199}$$

Note that, after integration by parts, the following equality holds

$$\int_D e^{\alpha G' \cdot y}\underline{u}\Big(\Delta'\overline{u} + \alpha G' \cdot \nabla\overline{u}\Big) = -\int_D \nabla\underline{u} \cdot e^{\alpha G' \cdot y}\nabla\overline{u} = \int_D \overline{u}e^{\alpha G' \cdot y}\Big(\Delta'\underline{u} + \alpha G' \cdot \nabla\underline{u}\Big).$$

Hence, the equation (6.199) becomes

$$0 = \int_D e^{\alpha G' \cdot y}\underline{u}\overline{u}\Big(\frac{f(\overline{u})}{\overline{u}} - \frac{f(\underline{u})}{\underline{u}}\Big).$$

Since $f(u)/u$ is a non-increasing function and $\overline{u} \geq \underline{u} \geq 0$ in D,

$$\frac{f(\overline{u})}{\overline{u}} - \frac{f(\underline{u})}{\underline{u}} \leq 0 \text{ in } D.$$

Then it follows that

$$\frac{f(\overline{u})}{\overline{u}} = \frac{f(\underline{u})}{\underline{u}} \text{ in } D. \tag{6.200}$$

By subtraction, the function $\overline{u} - \underline{u}$ therefore satisfies

$$\Delta'(\overline{u} - \underline{u}) + \alpha G' \cdot \nabla(\overline{u} - \underline{u}) + \frac{f(\overline{u})}{\overline{u}}(\overline{u} - \underline{u}) = 0 \text{ in } D.$$

If there exists an interior point where $\overline{u} - \underline{u} = 0$, then by Strong Maximum Principle, $\overline{u} - \underline{u} \equiv 0$. Assume there is no such interior point, then, $\overline{u} - \underline{u} > 0$ in D. But since $f(u)/u$ is non increasing, it has to be a constant because of (6.200) for all $u \in (\min_{\overline{D}} \underline{u}, \max_{\overline{D}} \overline{u}) = (0, \max_{\overline{D}} \overline{u})$. Hence, the constant is equal to $\lim_{u \to 0} f(u)/u = f'(0)$, and thus, the function $\overline{u} - \underline{u}$ satisfies

$$\Delta'(\overline{u} - \underline{u}) + \alpha G' \cdot \nabla(\overline{u} - \underline{u}) + f'(0)(\overline{u} - \underline{u}) = 0 \text{ in } D$$

and Dirichlet boundary conditions on ∂D, which is impossible since

$$f'(0) > \lambda_1\Big(-\Delta' - \alpha G' \cdot \nabla\Big),$$

for $|D|$ sufficiently big and $\alpha G'$ small enough, by Corollary 6.4. As a consequence, $\overline{u} = \underline{u}$.

□

7 Optimization of the critical speed $c^\star$

The minimal front speed $c^\star$ typically characterises the longtime behaviour of solutions of the initial value problem of (1.29) with Dirichlet boundary conditions for initial conditions with compact support, so is very important from the point of view of applications. Hence it is interesting to investigate how this important value $c^\star$ depends on the various ingredients in the problem i.e f, G, α and D.

In this chapter, under Dirichlet boundary conditions, we will show the influence of α, G and f on the critical speed $c^\star$ in some special cases. Precisely, we will use the min-max formula (5.142) and the definition (5.141) of r to show how $c^\star$ depends on α, f and G.

Remark 7.1. Note that the results of Proposition 7.2 and Proposition 7.3 still hold under Neumann boundary conditions. We consider the Dirichlet boundary conditions in this chapter in order to use certain results about the optimisation of the principal eigenvalue under Dirichlet boundary conditions as Proposition 6.3 and the Faber-Krahn inequality.

Proposition 7.2. *Fix α and G. Then the map $f \mapsto c^\star(f)$ is nondecreasing, in the sense that if $\tilde{f} \in C\Big([0,1],\mathbb{R}\Big)$ satisfies*

$$f(u) \leq \tilde{f}(u)$$

for all $u \in [0,1]$, then

$$c^\star(f) \leq c^\star(\tilde{f}).$$

Proof. First, fix α, f and G, and consider $\tilde{w}$ a decreasing (with respect to x_1) travelling front solution of (5.137) with speed $c^\star(\tilde{f})$ associated to a function $\tilde{f}$ instead of f. We will use the function $\tilde{w}$ as a test function in the min-max formula (5.142). Note that $\frac{\partial \tilde{w}}{\partial x_1} < 0$, so $\tilde{w} \in \tilde{K}$. We obtain

$$r_f(\tilde{w})(x) = c^\star(\tilde{f}) + \frac{f(\tilde{w}) - \tilde{f}(\tilde{w})}{-\partial_1 \tilde{w}},$$

where r_f is defined in (5.141).
By definition of $c^\star(f)$ in (5.142) it follows that

$$c^\star(f) \leq c^\star(\tilde{f}) + \sup_{x\in\Omega} \frac{f(\tilde{w}) - \tilde{f}(\tilde{w})}{-\partial_1 \tilde{w}}.$$

Hence, if $f(u) - \tilde{f}(u) \leq 0$ for all $u \in (0,1)$, then

$$c^\star(f) \leq c^\star(\tilde{f})$$

since $-\partial_1 \tilde{w} < 0$.

$\square$

Recall that in the special case where $G = (G_1, 0, \cdots, 0)$, and under the assumptions of Proposition 6.11, the equality $c^\star = c'$ holds. It is interesting to use this equality to show how $c^\star$ depend on the functions f, G, and α.

Proposition 7.3. *Consider the special case where $G = (G_1, 0, \cdots, 0)$.*

1. *If $\alpha \leq 0$ in D (respectively $\alpha \geq 0$ in D) then the map $G_1' \mapsto c^\star(G_1')$ is nondecreasing (respectively nonincreasing).*
2. *If $G_1' \leq 0$ (respectively $G_1' \geq 0$) then the map $\alpha \mapsto c^\star(\alpha)$ is nondecreasing (respectively nonincreasing).*
3. *Assume that f satisfies the KPP condition (1.31) and that for $y \in D$ and $u \in \mathbb{R}$,*
$$\alpha(y) G_1'(u) \geq \alpha(y) G_1'(0).$$
Then if $G_1'(0) < 0$ (respectively $G_1'(0) > 0$) then the map $\alpha \mapsto c^\star(\alpha)$ is nondecreasing (respectively nonincreasing).

Proof. 1. Fix α and f and let $\tilde{w}$ be a decreasing (with respect to x_1) travelling front solution with speed $c^\star(\tilde{G})$ of (5.137) associated to $\tilde{G} = (\tilde{G}_1, 0, \cdots, 0)$ with $G_1' \leq \tilde{G}_1'$. Then

$$r_G(\tilde{w})(x) = c^\star(\tilde{G}) + \alpha(y)\Big(\tilde{G}_1'(\tilde{w}) - G_1'(\tilde{w})\Big),$$

where r_G is defined in (5.141).
By definition of $c^\star(G)$, we obtain

$$c^\star(G) \leq c^\star(\tilde{G}) + \sup_{x \in \Omega} \alpha(y)\Big(\tilde{G}_1'(\tilde{w}) - G_1'(\tilde{w})\Big).$$

Hence, if $\alpha \leq 0$ in D, and if for all $u \in [0, 1]$, $\tilde{G}_1'(u) \geq G_1'(u)$, then

$$c^\star(G) \leq c^\star(\tilde{G}).$$

2. Fix G and f, and let $\tilde{w}$ be a decreasing (with respect to x_1) travelling front solution with speed $c^\star(\tilde{\alpha})$ of (5.138) associated to $\tilde{\alpha}$. Using a similar argument to above, we obtain

$$c^\star(\alpha) \leq c^\star(\tilde{\alpha}) + \sup_{x \in \Omega} (\tilde{\alpha}(y) - \alpha(y)) G_1'(\tilde{w}).$$

Thus, if $G_1' \leq 0$, and if $\tilde{\alpha} \geq \alpha$ in D, then $c^\star(\alpha) \leq c^\star(\tilde{\alpha})$.

3. Recall that in this special case, the equality $c^\star = c'$ holds. We will use the characterisation, for $t > 0$,

$$\tilde{\mu}_1^c(t) = \sup\Big\{\mu^c(t) \mid \exists \phi > 0 \text{ in } D,\ \phi = 0 \text{ on } \partial D,\ \Big(\Delta' + f'(0) - t(c+\tilde{\alpha}(y)G_1'(0)) + \mu^c(t)\Big)\phi \leq 0\Big\},$$

already used in the proof of Proposition 6.6. Denote by ϕ_0 the positive eigenfunction that satisfies

$$-\Delta'\phi_0 - f'(0)\phi_0 + t(c + \alpha(y)G_1'(0))\phi_0 = \mu_1^c(t)\phi_0,$$

and with Dirichlet boundary conditions on ∂D. Using ϕ_0 as a test function, we obtain if $t > 0$, that

$$\begin{aligned} -\Delta'\phi_0 - f'(0)\phi_0 + t(c + \tilde{\alpha}(y)G_1'(0))\phi_0 &= \Big(\mu_1^c(t) + tG_1'(0)(\tilde{\alpha}(y) - \alpha(y))\Big)\phi_0 \\ &\geq \Big(\mu_1^c(t) + t\inf_{y\in D}\Big[G_1'(0)(\tilde{\alpha}(y) - \alpha(y))\Big]\Big)\phi_0. \end{aligned}$$

Hence, it follows that

$$\tilde{\mu}_1^c(t) \geq \mu_1^c(t) + t\inf_{y\in D}\Big[G_1'(0)(\tilde{\alpha}(y) - \alpha(y))\Big],$$

and then, if $\tilde{\alpha}(y)G_1'(0) \geq \alpha(y)G_1'(0)$, we obtain $\tilde{\mu}_1^c(t) \geq \mu_1^c(t)$. Recall that c' is the only positive value such that the equation $\mu_1^c(t) = t^2$ for each $t > 0$ has only one positive root. It follows that $c'(\tilde{\alpha}) \leq c'(\alpha)$, if $\tilde{\alpha}(y)G_1'(0) \geq \alpha(y)G_1'(0)$. Since $c' = c^\star$ in this special case, the proof is complete.

□

Now, we will investigate the influence of the domain D on $c^\star$ in the special case where $G = (G_1, 0, \cdots, 0)$ and α is constant.

Proposition 7.4. *Consider the special case where $G = (G_1, 0, \cdots, 0)$ and α is constant. Assume that $\alpha G_1'(u) \geq \alpha G_1'(0)$ for all u and the KPP condition (1.31). Assume also that $f'(0) > \lambda_1^D(-\Delta')$ where $\lambda_1^D(-\Delta')$ is the principal eigenvalue of the Laplacian on the domain D, with Dirichlet boundary conditions on ∂D. Then*

$$c^\star(D) \leq c^\star(B),$$

where B is the ball centered at 0 with the same measure as D, $|D| = |B|$.

Proof. Recall that in this case one has $c^\star = c'$ by Proposition 6.11. Consider the following eigenvalue problem :

$$\begin{cases} -\Delta'\phi - f'(0)\phi = \Big(\mu_1^c(t, D) - t(c + \alpha G_1'(0))\Big)\phi & \text{in } D, \\ \phi = 0 & \text{on } \partial D. \end{cases} \tag{7.201}$$

Recall that since we assume that $f'(0) > \lambda_1^D(-\Delta')$, by Proposition 6.6 there exists c' such that if $c > c'$, there exist $0 < \lambda_1 < \lambda_2$ eigenvalues of (7.201), which are roots of the equation $\mu_1^c(t, D) = t^2$, and for $c = c'$, this equation has only one root $\lambda_0 > 0$ which is an eigenvalue of (7.201). By the Faber-Krahn inequality, it is well known that the principal eigenvalue $\lambda_1^D(-\Delta')$ of the Laplacian

$$\begin{cases} -\Delta'\varphi = \lambda_1\varphi & \text{in } D, \\ \varphi = 0 & \text{on } \partial D, \end{cases} \tag{7.202}$$

reaches its minimum among all bounded C^2 domains with fixed measure when $D = B$. Moreover we know that if $\lambda_1^D(\Delta') = \lambda_1^B(\Delta')$, then $D = B$ up to translation. Hence, since we assumed that $f'(0) > \lambda_1^D(\Delta')$, it follows that the inequality $f'(0) > \lambda_1^B(\Delta')$ also holds. Consequently when D is the ball B, the quantity $\mu_1^c(t, D) - t(c + \alpha G_1'(0))$ reaches its minimum. Since $t(c + \alpha G_1'(0))$ does not depend on D it follows that $\mu_1^c(t, D)$ reaches its minimum when $D = B$.
Let $c = c'(B)$. By definition of c', the equation $\mu_1^{c'(B)}(t, B) = t^2$ has only one root, $\lambda_0 > 0$. But at this point λ_0, one has

$$\mu_1^{c'(B)}(\lambda_0, D) > \mu_1^{c'(B)}(\lambda_0, B) = \lambda_0^2,$$

if $D \neq B$ up to translation.
Recall that since $f'(0) > \lambda_1^D(-\Delta')$, one has $\mu_1^c(0) < 0$. By concavity and continuity of $t \mapsto \mu_1^c(t)$, using that $\mu_1^c(t) - t^2 \to -\infty$ when $t \to +\infty$, it follows that the equation $\mu_1^{c'(B)}(t, D) = t^2$ has two positive roots, which means that $c'(B) > c'(D)$. Since $c' = c^\star$ in this special case, the proof is complete. $\square$

In the case where α is not constant, we first need to define properly the function α on other domains. To do this, define for $R > 0$

$$\psi(x) := \phi(x/R)$$

for all x in

$$D_R := \{Rx, x \in D\}, \tag{7.203}$$

where ϕ is the principal eigenfunction of

$$\begin{cases} -\Delta'\phi + \Big[t(c + \alpha(y)G_1'(0)) - f'(0)\Big]\phi = \mu_1^c(t, D)\phi & \text{in } D, \\ \phi = 0 & \text{on } \partial D. \end{cases} \tag{7.204}$$

It follows that the function ψ satisfies

$$\begin{cases} -\Delta'\psi + \frac{1}{R^2}\Big[t(c + \alpha_R(y)G_1'(0)) - f'(0)\Big]\psi = \frac{\mu_1^c(t,D)}{R^2}\psi & \text{in } D_R, \\ \psi = 0 & \text{on } \partial D_R, \end{cases} \tag{7.205}$$

where

$$\alpha_R(y) := \alpha(y/R) \text{ for all } y \in D_R. \tag{7.206}$$

Proposition 7.5. *Consider the case $G = (G_1, 0, \cdots, 0)$. Assuming the KPP condition* (1.31) *for f and that $\alpha(y)G_1'(u) \geq \alpha(y)G_1'(0)$ for all $y \in D$, and all $u \in (0,1)$, then with the definitions* (7.203) *of D_R and* (7.206) *of α_R, one has*

$$c^\star(D) < c^\star(D_R),$$

for all $R > 1$.

Proof. If we denote by $\mu_1^c(t, R, D_R)$ the principal eigenvalue of (7.205), one has

$$\mu_1^c(t, R, D_R) = \frac{\mu_1^c(t, D)}{R^2} < \mu_1^c(t, D) \text{ for all } R > 1. \tag{7.207}$$

Hence, taking $c = c'(D)$, it follows that for $t > 0$ and $R > 1$, the equation $\mu_1^{c'(D)}(t, D) = t^2$ has exactly one root λ_0, whereas the equation $\mu_1^{c'(D)}(t, R, D_R) = t^2$ has no solution for $R > 1$ because of the strict inequality (7.207), which means that $c'(D) < c'(D_R)$. Recall that in this special case, the equality $c^\star = c'$ allows us to conclude.

□

Another way to define α on other domains is to use the Schwarz rearrangement $\alpha^\star$ of the function α. Before defining the Schwarz rearrangement $\alpha^\star$ of α, note that if D is a measurable domain of finite measure $|D|$ in $\mathbb{R}^n$, we denote by $D^\star$ the open ball centered at 0 of measure $|D|$. If α is a non negative measurable function on D, we define the Schwarz rearrangement $\alpha^\star$ on $D^\star$ of the function α by :

$$\alpha^\star(x) := \sup\Big\{t \in \mathbb{R},\ x \in \{\alpha > t\}^\star\Big\}. \tag{7.208}$$

We mention three properties of the Schwarz rearrangement that we will use, see [KAW] :

Proposition 7.6.

1. *The Schwarz rearrangement preserves the L^p norm. Namely, if $\alpha \in L^p(D)$, then*

$$\|\alpha^\star\|_{L^p(D)} = \|\alpha\|_{L^p(D)},\ \textit{for } 1 \leq p < +\infty. \tag{7.209}$$

2. *Let α_1 and α_2 be two measurable non negative functions on D. The Hardy-Littlewood inequality holds :*

$$\int_D \alpha_1\alpha_2 \leq \int_D \alpha_1^\star\alpha_2^\star. \tag{7.210}$$

3. *If $\alpha \in W_0^{1,p}(D)$, then $\alpha^\star \in W_0^{1,p}(D)$ and one has the Pólya-Szegö inequality :*

$$\|\nabla\alpha^\star\|_p \leq \|\nabla\alpha\|_p \textit{ for all } 1 \leq p \leq +\infty. \tag{7.211}$$

Proposition 7.7. *In the special case $G = (G_1, 0, \cdots, 0)$, assuming the KPP condition* (1.31) *for f, $\alpha \geq 0$ in D, and $G_1'(0) < 0$ and that for all $u \in (0,1)$, $G_1'(u) \geq G_1'(0)$, then*

$$c^\star(D) \leq c^\star(D^\star),$$

where $D^\star$ is the ball centered at 0 *with the same measure as D.*

Proof. Consider the following two eigenvalue problems, defined in D and $D^\star$:

$$\begin{cases} -\Delta'\phi + \Big[t(c + \alpha(y)G_1'(0)) - f'(0)\Big]\phi = \mu_1^c(t, D)\phi & \text{in } D, \\ \phi = 0 & \text{on } \partial D, \end{cases} \tag{7.212}$$

and

$$\begin{cases} -\Delta'\phi + \Big[t(c + \alpha^\star(y)G_1'(0)) - f'(0)\Big]\phi = \mu_1^c(t, D^\star)\phi & \text{in } D^\star, \\ \phi = 0 & \text{on } \partial D^\star, \end{cases} \tag{7.213}$$

Let $\varphi > 0$ in D be the principal eigenfunction of (7.212) associated with $\mu_1^c(t, D)$, which satisfies the normalisation condition $\|\varphi\|_{L^2(D)} = 1$. Using the Hardy-Littlewood inequality (7.210) and the fact that, for $1 < p < \infty$, the Schwarz rearrangement preserves the L^p norm (see (7.209)) and the Pólya-Szegö inequality (7.211) it follows that for $t > 0$

$$\begin{aligned}
\mu_1^c(t, D) &= tc - f'(0) + \int_D |\nabla\varphi(y)|^2 dy + tG_1'(0)\int_D \alpha(y)\varphi^2(y)dy \qquad \text{by (7.211)} \\
&\geq tc - f'(0) + \int_{D^\star} |\nabla\varphi^\star(x)|^2 dx + tG_1'(0)\int_{D^\star} \alpha^\star(x)(\varphi^2)^\star(x)dx \\
&= tc - f'(0) + \int_{D^\star} |\nabla\varphi^\star(x)|^2 dx + tG_1'(0)\int_{D^\star} \alpha^\star(x)(\varphi^\star(x))^2 dx, \qquad \text{because } (\varphi^2)^\star = (\varphi^\star)^2, \\
&\geq \inf_{\phi\in H_0^1(D^\star), \|\phi\|_2=1} \frac{\int_{D^\star} |\nabla\phi|^2 + \int_{D^\star}\Big(t(c + \alpha^\star(y)G_1'(0)) - f'(0)\Big)\phi^2}{\int_{D^\star}\phi^2} = \mu_1^c(t, D^\star).
\end{aligned}$$

The second line relies on the fact that $G_1'(0) < 0$ and the Hardy-Littlewood inequality (7.210). As in the proof of Proposition 7.5, it follows from $\mu_1^c(t, D) \geq \mu_1^c(t, D^\star)$ that $c'(D) \leq c'(D^\star)$, which concludes since $c^\star = c'$ in that special case.

□

8 Another form of convection term for Dirichlet boundary conditions

Different modelling approaches could give several forms of convection term, that is why we have considered a different form of convection term under Neumann boundary conditions in Chapter 4. A natural question is to explore what happens with this alternative form of convection term under Dirichlet boundary conditions.

Here we consider the case where the convection term has the form $\nabla \cdot (\alpha(y)G(u))$ instead of $\alpha(y)\nabla \cdot G(u)$.

With this form of convection term, a travelling front solution $u(x,t) = w(x_1 - ct, y)$ satisfies

$$\begin{cases} -c\partial_1 w = \Delta w + \nabla \cdot (\alpha(y)G(w)) + f(w) & \text{in } \Omega, \\ w(-\infty, y) = w_-(y), \quad w(+\infty, y) = 0 & \text{uniformly in } y \in D, \\ w \geq 0 & \text{in } \Omega, \\ w = 0 & \text{on } \mathbb{R} \times \partial D. \end{cases} \tag{8.214}$$

Note that the first equation can be rewritten in the following form

$$\Delta w + \Big(c + \alpha(y)G_1'(w)\Big)\partial_1 w + \alpha(y)\sum_{i=2}^{n} G_i'(w)\frac{\partial w}{\partial x_i} + \sum_{i=2}^{n} G_i(w)\frac{\partial \alpha}{\partial x_i} + f(w) = 0. \tag{8.215}$$

Throughout this chapter, we make the following assumptions :

- **(GD1′) :** The function $G : \mathbb{R}^n \mapsto \mathbb{R}$ is C^2 and the function G_1 satisfy $G_1(0) = 0$.
- **(GD2′) :** For all $2 \leq i \leq n$, the function G_i satisfies $G_i(0) = 0$.
- **(AlphaD′) :** The function $\alpha : D \to \mathbb{R}$ is in $C^1(\overline{D})$.
- **(AD′) :** Let $\lambda \in (0,1)$. There exists a non negative function $w_- \in C^{2,\lambda}(\overline{D})$ such that the only solutions in $C^{2,\lambda}(\overline{D})$ of the problem on the cross section D are w_- and 0.
  Precisely, if $z \in C^{2,\lambda}(D) : D \to \mathbb{R}$ satisfies

  $$\begin{cases} \Delta' z + \alpha(y)\sum_{i=2}^{n} G_i'(z)\frac{\partial z}{\partial x_i} + \sum_{i=2}^{n} G_i(z)\frac{\partial \alpha}{\partial x_i} + f(z) = 0 & \text{in } D, \\ z = 0 & \text{on } \partial D, \end{cases} \tag{8.216}$$

  then $z \equiv 0$ or $z \equiv w_-$.

- **(BD′) :** There exists a sequence of functions $\{v_k(y)\}_{k\in\mathbb{N}}$ uniformly bounded in $C^{2,\lambda}(\overline{D})$, and which tends uniformly to 0 when k tends to $+\infty$ and which satisfies for every $k \in \mathbb{N}$

  $$0 < v_{k+1}(y) < v_k(y) < w_-(y), \text{ for } y \in D,$$

  and

  $$\begin{cases} \Delta' v_k + \alpha(y)\sum_{i=2}^{n} G_i'(v_k)\frac{\partial v_k}{\partial x_i} + \sum_{i=2}^{n} G_i(v_k)\frac{\partial \alpha}{\partial x_i} + f(v_k) \geq 0 & \text{in } D, \\ v_k = 0 & \text{on } \partial D. \end{cases} \tag{8.217}$$

- **(FD′) :** The following condition holds :

$$f'(0) + \inf_{y\in D} \sum_{i=2}^{n} G_i'(0)\frac{\partial \alpha}{\partial x_i}(y) > \lambda_1(-L), \tag{8.218}$$

where $-L := -\Delta' - \alpha(y)\sum_{i=2}^{n} G_i'(0)\frac{\partial}{\partial x_i}$.

With the new term $\sum_{i=2}^{n} G_i(w)\frac{\partial \alpha}{\partial x_i}$, we will construct a solution on the truncated cylinder $\Omega_N = (-N, N) \times D$.

8.1 Existence of a solution on the unbounded cylinder

As before, for $\rho \in \tilde{K}$, defined in (5.140),

$$\tilde{r}(\rho)(x) := \frac{\Delta\rho(x) + \nabla\cdot\Big(\alpha(y)G(\rho(x))\Big) + f(\rho(x))}{-\partial_1\rho(x)}, \text{ and}$$

$$c^\star := \inf_{\tilde{\rho}\in K} \sup_{x\in\Omega} r(\rho)(x). \tag{8.219}$$

As before, we have the existence of an upper bound for $c^\star$.

Proposition 8.1. *Assume* **(GD1′)**, **(GD2′)**, **(AlphaD′)** *and that f is C^1 and satisfies* (1.30)*. Then,*

$$c^\star < +\infty.$$

Proof. To prove this, we will use the function $g(x_1, y) := h(x_1)w_-(y)$, where h is defined in (2.49).
For $x_1 < -1$, using the equation (8.216) satisfied by w_-, we obtain

$$\begin{aligned}
\tilde{r}(g) &= \frac{\Delta g + \nabla\cdot\Big(\alpha G(g)\Big) + f(g)}{-\partial_1 g}\\
&= \frac{h''w_- + h\Delta' w_- + \alpha G_1'(g)h'w_- + \alpha\sum_{i=2}^{n} G_i'(g)h\frac{\partial w_-}{\partial x_i} + \sum_{i=2}^{n} G_i(g)\frac{\partial\alpha}{\partial x_i} + f(hw_-)}{-h'w_-}\\
&= -\frac{h''}{h'} - \alpha G_1'(hw_-) + \frac{h\alpha\sum_{i=2}^{n}\Big(G_i'(hw_-) - G_i'(w_-)\Big)\frac{\partial w_-}{\partial x_i}}{-h'w_-}\\
&+ \frac{\sum_{i=2}^{n}\Big(G_i(hw_-) - hG_i(w_-)\Big)\frac{\partial\alpha}{\partial x_i}}{-h'w_-} + \frac{f(hw_-) - hf(w_-)}{-h'w_-}.
\end{aligned}$$

The terms $-h''/h'$ and $-\alpha G_1'(hw_-)$ are bounded. We are going to deal with the three other terms.

Using the mean value theorem, for all $2 \leq i \leq n$, there exists $\xi_i \in (hw_-, w_-)$ such that

$$\frac{h\alpha \sum_{i=2}^n \Big(G_i'(hw_-) - G_i'(w_-)\Big) \frac{\partial w_-}{\partial x_i}}{-h'w_-} = \frac{h(h-1)w_-\alpha \sum_{i=2}^n G_i''(\xi_i) \frac{\partial w_-}{\partial x_i}}{-h'w_-},$$

which is bounded because $h(h-1)/h'$ is bounded.
Using again the mean value theorem, for all $2 \leq i \leq n$, there exists $\xi_i \in (hw_-, w_-)$ such that :

$$\begin{aligned}
\frac{\sum_{i=2}^n \Big(G_i(hw_-) - hG_i(w_-)\Big) \frac{\partial \alpha}{\partial x_i}}{-h'w_-} &= \frac{\sum_{i=2}^n \Big(G_i(hw_-) - G_i(w_-) + G_i(w_-) - hG_i(w_-)\Big) \frac{\partial \alpha}{\partial x_i}}{-h'w_-} \\
&= \sum_{i=2}^n \Big[\frac{G_i'(\xi_i)(h-1)w_-}{-h'w_-} + \frac{(1-h)G_i(w_-)}{-h'w_-}\Big] \frac{\partial \alpha}{\partial x_i} \\
&= \sum_{i=2}^n \Big[\frac{h-1}{-h'} G_i'(\xi_i) + \frac{1-h}{-h'} \frac{G_i(w_-)}{w_-}\Big] \frac{\partial \alpha}{\partial x_i},
\end{aligned}$$

which is bounded because $(h-1)/h'$ is bounded and because $G_i(0) = 0$ for all $2 \leq i \leq n$ by **(GD2′)**, which ensures that the ratio $G_i(w_-)/w_-$ is bounded.
It can be shown similarly that the term

$$\frac{f(hw_-) - hf(w_-)}{-h'w_-}$$

is bounded, using the fact that $f(0) = 0$ and that the ratio $(1-h)/h'$ is bounded for $x_1 < -1$.
Hence, $\tilde{r}(g)$ is bounded for $x_1 < -1$.
For $x_1 > 1$, the ratio is $(1-h)/h'$ is not bounded anymore, but using that $G_i(0) = 0$ for all $2 \leq i \leq n$ and the mean value theorem, for all $2 \leq i \leq n$ there exist $\xi_i \in (0, hw_-)$ and $\hat{\xi}_i \in (0, w_-)$ such that

$$\begin{aligned}
\frac{\sum_{i=2}^n \Big(G_i(hw_-) - hG_i(w_-)\Big) \frac{\partial \alpha}{\partial x_i}}{-h'w_-} &= \frac{\sum_{i=2}^n \Big(G_i(hw_-) - G_i(0) + hG_i(0) - hG_i(w_-)\Big) \frac{\partial \alpha}{\partial x_i}}{-h'w_-} \\
&= \sum_{i=2}^n \Big[\frac{G_i'(\xi_i)hw_-}{-h'w_-} + \frac{hw_-G_i'(\hat{xi}_i)}{-h'w_-}\Big] \frac{\partial \alpha}{\partial x_i} \\
&= \sum_{i=2}^n \frac{h}{-h'} \Big[G_i'(\xi_i) + G_i(\hat{\xi}_i)\Big] \frac{\partial \alpha}{\partial x_i},
\end{aligned}$$

which is bounded, since for $x_1 > 1$ the ratio h/h' is bounded.
We can show similarly that the ratio

$$\frac{f(hw_-) - hf(w_-)}{-h'w_-}$$

is bounded, and hence, the function $\tilde{r}(g)$ is bounded for $x_1 > 1$.
We conclude using the continuity of $\tilde{r}(g)$, as in Proposition 5.2, that

$$\sup_{x \in \Omega} \tilde{r}(g)(x_1, y) < +\infty.$$

Thus $c^\star < +\infty$.

$\square$

Under certain assumptions, we will construct a solution of (8.214) on the truncated cylinder $\Omega_N = (-N, N) \times D$. Let $c > c^\star$. By definition of $c^\star$, there exists a supersolution ρ of (8.214). Assumption (BD$'$) ensures the existence of subsolutions of (8.214).

Recall that since $(v_k)_k$ tends uniformly to 0 when $k \to +\infty$, it follows that for $N > 1$, there exists k large enough such that $\rho(-N, y) > v_k(y)$ for all $y \in D$. We claim :

Proposition 8.2. *Let $N > 1$ and $\Omega_N = (-N, N) \times D \subset \Omega$. Assume* **(GD1$'$)**, **(GD2$'$)**, **(AlphaD$'$)**, **(AD$'$)**, **(BD$'$)** *and that f is C^1 and satisfies* (1.30). *Then, there exists a unique solution $u \in W^{2,p}_{loc}\Big((-N, N) \times \overline{D}\Big)$ which satisfies*

- *$\rho(-N, y) \geq u(x_1, y) \geq v_k(y)$ for all $(x_1, y) \in (-N, N) \times D$,*
- *for all $x_1 \in (-N, N)$ there exists $y \in D$ such that $\rho(-N, y) > u(x_1, y)$,*

of the following problem :

$$\begin{cases} \Delta u + (c + \alpha(y) G_1'(u))\partial_1 u + \alpha(y) \sum_{i=2}^n G_i'(u) \frac{\partial u}{\partial x_i} + \sum_{i=2}^n G_i(u) \frac{\partial \alpha}{\partial x_i} & + f(u) = 0 \text{ in } \Omega_N \\ u = 0 & (x_1, y) \in (-N, N) \times \partial D, \\ u(-N, y) = \rho(-N, y), \quad u(N, y) = v_k(y) & y \in D. \end{cases} \tag{8.220}$$

For $c > c^\star$, there exists at least one solution of (8.214).
Moreover, there exists also a solution of (8.214) *with speed $c = c^\star$.*

Proof. The proof of the existence of a solution on the truncated cylinder Ω_N is exactly the same as the proof of Proposition 5.3, using method of sub and super solutions with the functions v_k and ρ.

$\square$

Theorem 8.3. *Assume* **(GD1$'$)**, **(GD2$'$)**, **(AlphaD$'$)**, **(AD$'$)**, **(BD$'$)** *and that f is C^1 and satisfies* (1.30). *Then, for $c > c^\star$, there exists at least one solution of* (4.117).
In addition, if we also assume that the following condition holds :

$$\sum_{i=2}^n G_i'(0) \frac{\partial \alpha}{\partial x_i}(y) + f'(0) \not\equiv 0. \tag{8.221}$$

Then this solution is decreasing with respect to x_1.

Proof. The proofs of the existence of solution on the unbounded cylinder Ω with a speed $c > c^\star$ and a speed $c = c^\star$ are analogous to the proof of Theorem 1.20 and the proof of Proposition 5.8.
The proof of the monotonicity of the function w is very similar to the one of Theorem 4.5. The only difference is that we assume that (8.221) to make sure that $c(x, 0)$ does not tend to 0 when $x_1 \to +\infty$.

$\square$

8.2 Linearized operator and eigenvalue problem

Now we will show the existence of a critical value c' as in Proposition 6.6, under some additional assumptions.
To do this, consider the linearized problem of (8.214) around 0

$$\begin{cases} \Delta w + \Big(c + \alpha(y) G_1'(0)\Big)\partial_1 w + \alpha(y) \sum_{i=2}^n G_i'(0) \frac{\partial w}{\partial x_i} + \Big(\sum_{i=2}^n G_i'(0) \frac{\partial \alpha}{\partial x_i} + f'(0)\Big) w = 0 & \text{in } \mathbb{R} \times D, \\ w = 0 & \text{on } \mathbb{R} \times \partial D. \end{cases} \tag{8.222}$$

If $w(x_1, y) = e^{-\lambda x_1} \varphi(y)$, then the function φ has to satisfy

$$\begin{cases} -\Delta' \varphi - \alpha(y) \sum_{i=2}^n G_i'(0) \frac{\partial \varphi}{\partial x_i} - \Big(\sum_{i=2}^n G_i'(0) \frac{\partial \alpha}{\partial x_i} + f'(0)\Big) \varphi = \Big(\lambda^2 - \lambda(c + \alpha(y) G_1'(0))\Big) \varphi & \text{in } D, \\ \varphi = 0 & \text{on } \partial D. \end{cases} \tag{8.223}$$

As before, by [YIH, Theorem 1.3], the following eigenvalue problem

$$\begin{cases} -\Delta' \sigma - \alpha(y) \sum_{i=2}^n G_i'(0) \frac{\partial \sigma}{\partial x_i} - \Big(\sum_{i=2}^n G_i'(0) \frac{\partial \alpha}{\partial x_i} + f'(0)\Big) \sigma = \mu_2 \sigma & \text{in } D, \\ \sigma = 0 & \text{on } \partial D, \end{cases} \tag{8.224}$$

has a simple eigenvalue $\mu_2 \in \mathbb{R}$, which corresponds to a positive eigenfunction. We claim :

Proposition 8.4. *Assume that $G : \mathbb{R}^n \mapsto \mathbb{R}$ is C^1,* **(AlphaD′)** *and* (8.218)*. Then the principal eigenvalue μ_2 of* (8.224) *is negative.*

Proof. Denote $-L := -\Delta' - \alpha \sum_{i=2}^n G_i'(0) \frac{\partial \alpha}{\partial x_i}$.
We will use the following characterisation of $\lambda_1(-L)$:

$$\lambda_1(-L) := \sup \Big\{\lambda, \exists \phi > 0 \in D, \phi = 0 \text{ on } \partial D, (L + \lambda)\phi \leq 0\Big\}.$$

Taking φ the principal eigenfunction of (8.224) as a test function, it follows that

$$-L\varphi = \Big(\sum_{i=2}^n G_i'(0) \frac{\partial \alpha}{\partial x_i} + f'(0) + \mu_2\Big)\varphi \geq \Big(\inf_D \sum_{i=2}^n G_i'(0) \frac{\partial \alpha}{\partial x_i} + f'(0) + \mu_2\Big)\varphi,$$

which implies that, by definition of $\lambda_1(-L)$,

$$\lambda_1(-L) \geq \inf_D \sum_{i=2}^n G_i'(0)\frac{\partial \alpha}{\partial x_i} + f'(0) + \mu_2.$$

Hence, if

$$f'(0) + \inf_D \sum_{i=2}^n G_i'(0)\frac{\partial \alpha}{\partial x_i} > \lambda_1(-L), \tag{8.225}$$

then $\mu_2 < 0$.

□

Corollary 8.5. *Assume that G is C^1 and* **(AlphaD′)** *holds. Assume also that the measure of the domain D is large enough, that $\|\alpha G'(0)\|_\infty$ is small enough and that the following condition holds :*

$$f'(0) + \inf_D \sum_{i=2}^n G_i'(0)\frac{\partial \alpha}{\partial x_i} > 0. \tag{8.226}$$

Then the principal eigenvalue μ_2 of (8.224) *is negative.*

Proof. Using Proposition 6.3, we know that for $|D|$ large enough, and $\|\alpha G'(0)\|_\infty$ small enough, $\lambda_1(-L)$ can be as small as we want. Hence, if (8.226) holds, and if $|D|$ is big enough and $\|\alpha G'(0)\|_\infty$ small enough, then (8.225) holds, (in other words, assumption **(FD′)** is satisfied), which ensures that $\mu_2 < 0$.

□

Consider now the eigenvalue problem depending on $t \in \mathbb{R}$:

$$\begin{cases} -\Delta'\varphi - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial \varphi}{\partial x_i} - \Big(\sum_{i=2}^n G_i'(0)\frac{\partial \alpha}{\partial x_i} + f'(0)\Big)\varphi + t\beta_c(y)\varphi = \mu_2^c(t)\varphi & \text{in } D, \\ \varphi = 0 & \text{on } \partial D, \end{cases} \tag{8.227}$$

where $\beta_c(y) = c + \alpha(y)G_1'(0)$.
As before, λ is an eigenvalue of (8.223) if and only if λ satisfies

$$\mu_2^c(\lambda) = \lambda^2.$$

Proposition 8.6. *Assume that G is C^2,* **(AlphaD′)** *and* **(FD′)**. *Then the conclusion of Proposition 6.6 holds. In particular, there exists a critical value c' such that if $c > c'$, the eigenvalue problem* (8.223) *has two positive eigenvalues $0 < \lambda_1 < \lambda_2$, and only one positive for $c = c'$.*

Proof. The arguments are the same as in the proof of Proposition 6.6, using the concavity of $t \mapsto \mu_2^c(t)$, and the fact that $\mu_2^c(0) < 0$ by Proposition 8.4. □

8.3 Comparison between $c^\star$ and c'

Now that we have the existence of the critical value c', we will compare $c^\star$ and c'. Precisely, one has

Theorem 8.7. *Assume* **(GD1′)**, **(AlphaD′)** *and that f satisfies* (1.30). *Assume either* **(GD2′)** *or that $\alpha \equiv 0$ on ∂D. Let w be the solution of* (8.214). *Then*

$$w(x_1, y) + |\nabla w(x_1, y)| \leq C_0 e^{-\varepsilon x_1}, \text{ for all } (x_1, y) \in \mathbb{R} \times D. \tag{8.228}$$

Remark 8.8. Note that contrary to Theorem 6.9, in the proof of Theorem 8.7, we actually need that there exist $C > 0$ and $s_0 \in (0, 1)$ such that $|G_1(s)| \leq Cs$ for all $s \in (0, s_0)$, which is satisfied since $G_1(0) = 0$ (assumption **(GD1′)**).

Proof. We will first show that the conclusion of Proposition 6.7 holds. To do this, let w be a solution, with Dirichlet boundary conditions on $\partial\Omega$, of

$$\Delta w + \Big(c + \alpha(y) G_1'(w)\Big)\partial_1 w + \alpha(y) \sum_{i=2}^{n} G_i'(w) \frac{\partial w}{\partial x_i} + \sum_{i=2}^{n} G_i(w) \frac{\partial \alpha}{\partial x_i} + f(w) = 0. \tag{8.229}$$

Let $N > R > 0$ and consider the cut-off function ξ on $\mathbb{R}$ defined in (6.179).
By multiplying (8.229) by ξ, integrating over Ω and using Green's formula, it follows that

$$\int_\Omega w\xi'' - \int_\Omega \xi' w \Big(c + \alpha(y) \frac{G_1(w)}{w}\Big) + \int_\Omega \xi \alpha(y) \sum_{i=2}^{n} G_i'(w) \frac{\partial w}{\partial x_i} + \int_\Omega \xi \sum_{i=2}^{n} G_i(w) \frac{\partial \alpha}{\partial x_i} + \int_\Omega \xi f(w) = 0. \tag{8.230}$$

The main difference with the proof of Proposition 6.7 is that after using Green's formula on the third integral, under both assumption **(GD2′)** and that $\alpha \equiv 0$ on ∂D, one has

$$\int_\Omega \xi \alpha(y) \sum_{i=2}^{n} G_i'(w) \frac{\partial w}{\partial x_i} = -\int_\Omega \xi \sum_{i=2}^{n} G_i(w) \frac{\partial \alpha}{\partial x_i}.$$

Thus, the equation (8.230) becomes

$$\int_\Omega \xi f(w) = \int_\Omega w \Big[\xi' \Big(c + \alpha(y) \frac{G_1(w)}{w}\Big) - \xi''\Big]. \tag{8.231}$$

Note that

$$\int_\Omega \xi f(w) \geq \int_R^N \int_D f(w),$$

because $\xi = 1$ in $[R, N]$. Since w tends uniformly to 0, with respect to y, when x_1 tends to $+\infty$, we can take R sufficiently large such that there exists $\delta > 0$ such that $f(w) \geq (1 - \delta) f'(0) w$, and we obtain

$$\int_R^N \int_D (1 - \delta) f'(0) w \leq \int_\Omega w \Big[\xi' \Big(c + \alpha(y) \frac{G_1(w)}{w}\Big) - \xi''\Big].$$

Then, since ξ is constant on $(-\infty, R-1] \cup [R, N] \cup [N+1, +\infty)$, it follows that

$$\begin{aligned}\int_\Omega w\Big[\xi'\Big(c+\alpha(y)\frac{G_1(w)}{w}\Big)-\xi''\Big] &= \Big(\int_{R-1}^{R}+\int_N^{N+1}\Big)\int_D w\Big[\xi'\Big(c+\alpha(y)\frac{G_1(w)}{w}\Big)-\xi''\Big] \\ &\leq K\Big[\int_{R-1}^{R}\int_D w+\int_N^{N+1}\int_D w\Big],\end{aligned}$$

where K is such that

$$\Big|\xi'\Big(c+\alpha(y)\frac{G_1(w)}{w}\Big)-\xi''\Big| \leq K, \qquad \text{for all } y \in D.$$

Hence, we obtain

$$(1-\delta)f'(0)\int_R^N\int_D w \leq K\Big[\int_{R-1}^{R}\int_D w+\int_N^{N+1}\int_D w\Big]. \tag{8.232}$$

We argue as in the proof of Proposition 6.7 and let N tends to infinity :

$$(1-\delta)f'(0)\int_R^\infty\int_D w \leq K\int_{R-1}^{R}\int_D w,$$

and we conclude as in the proof of Proposition 3.6 under Neumann boundary conditions and the first form of convection term using the function

$$g(R) := \int_R^\infty\int_D w,$$

that there exist $C>0$ and $\varepsilon>0$ such that for all $R>0$,

$$\int_R^\infty\int_D w \leq Ce^{-\varepsilon R}.$$

The rest of the proof is identical to the one of Theorem 6.9. □

We now prove Theorem 1.22.

Proof of Theorem 1.22. Let $c > c^\star$, and consider a solution u of (8.214) with speed c.
As in the proof of Theorem 1.21, we will construct a positive solution of the linearized problem (8.222). To do that, define

$$h^N(x_1, y) := \frac{u(x_1^N + x_1, y)}{2\sup_{y\in\overline{D}} u(x_1^N, y)},$$

where $(x_1^N)_N$ is chosen as in the proof of Theorem 1.21. The function h^N satisfies

$$\Delta h^N + \Big(c + \alpha(y) G_1'\Big(u(x_1^N + x_1, y)\Big)\Big)\partial_1 h^N + \alpha(y)\sum_{i=2}^n G_i'\Big(u(x_1^N + x_1, y)\Big)\frac{\partial h^N}{\partial x_i}$$
$$+ \sum_{i=2}^n \frac{G_i(u(x_1^N + x_1, y))}{2\sup u(x_1^N, y)}\frac{\partial \alpha}{\partial x_i} + \frac{f\Big(u(x_1^N + x_1, y)\Big)}{2\sup_{y\in\overline{D}} u(x_1^N, y)} = 0.$$

As before, a compactness argument ensures that the sequence of functions $(h^N)_N$ converges weakly in $W^{2,p}_{loc}$ and strongly in $C^{1,\lambda}_{loc}$ to a function h when $N \to +\infty$. Moreover, when N tends to $+\infty$, the extra term will converge to

$$\lim_{N\to+\infty}\sum_{i=2}^n \frac{G_i(u(x_1^N + x_1, y))}{2\sup u(x_1^N, y)}\frac{\partial \alpha}{\partial x_i} = \sum_{i=2}^n \lim_{N\to+\infty}\frac{G_i(u(x_1^N + x_1, y))}{u(x_1^N + x_1, y)} h^N \frac{\partial \alpha}{\partial x_i} = \sum_{i=2}^n G_i'(0) h \frac{\partial \alpha}{\partial x_i}.$$

Hence, the function h satisfies the linearized problem (8.222) and the condition $\sup_{y\in\overline{D}} h(0, y) = \frac{1}{2}$. Furthermore, since we assumed the condition (8.218), one has

$$\begin{aligned}\Delta h + \Big(c + \alpha(y) G_1'(0)\Big)\partial_1 h + \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial h}{\partial x_i} &= -\Big(\sum_{i=2}^n G_i'(0)\frac{\partial \alpha}{\partial x_i} + f'(0)\Big)h \\ &\le -\Big(\inf_{y\in D}\Big\{\sum_{i=2}^n G_i'(0)\frac{\partial \alpha}{\partial x_i}\Big\} + f'(0)\Big)h \\ &\le 0.\end{aligned}$$

As before, Maximum Principle argument ensures that $h > 0$ in $\mathbb{R} \times D$. The rest of the proof is identical to that of Theorem 1.21. □

Remark 8.9. Note that in the special case where $G = (G_1, 0, \cdots, 0)$, the extra term is 0, and Proposition 6.11 gives that $c^\star = c'$.

8.4 Existence of the $\{v_k\}_k$ and w_-

Recall that in assumptions **(AD′)** and **(BD′)** we assumed that there exists a non negative function $w_- \in C^{2,\lambda}(\overline{D})$ which satisfies

$$\begin{cases} \Delta' w_- + \alpha(y)\sum_{i=2}^n G_i'(w_-)\frac{\partial w_-}{\partial x_i} + \sum_{i=2}^n G_i(w_-)\frac{\partial \alpha}{\partial x_i} + f(w_-) = 0 & \text{in } D, \\ w_- = 0 & \text{on } \partial D, \end{cases} \tag{8.233}$$

and that there exists a sequence of functions $\{v_k(y)\}_{k\in\mathbb{N}}$ uniformly bounded in $C^{2,\lambda}(\overline{D})$, and which tends uniformly to 0 when k tends to $+\infty$ and which satisfies for every $k \in \mathbb{N}$

$$0 < v_{k+1}(y) < v_k(y) < w_-(y), \text{ for } y \in D,$$

and

$$\begin{cases} \Delta' v_k + \alpha(y)\sum_{i=2}^n G_i'(v_k)\frac{\partial v_k}{\partial x_i} + \sum_{i=2}^n G_i(v_k)\frac{\partial \alpha}{\partial x_i} + f(v_k) \geq 0 & \text{in } D, \\ v_k = 0 & \text{on } \partial D. \end{cases} \tag{8.234}$$

Proposition 8.10. *Assume that the following inequality*

$$f'(0) + \sum_{i=2}^n G_i'(0)\frac{\partial \alpha}{\partial x_i} > \lambda \tag{8.235}$$

holds, where λ is the principal eigenvalue of the operator $-L = -\Delta' - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial}{\partial x_i}$. Then, there exists a sequence of functions $\{v_k\}_k \in C^{2,\lambda}(\overline{D})$, which satisfies (8.234)*.*

Proof. Consider the operator $-L = -\Delta' - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial}{\partial x_i}$. By [YIH, Theorem 1.3], this operator has a simple eigenvalue $\lambda \in \mathbb{R}$ which corresponds to a positive eigenfunction φ, under Dirichlet boundary conditions :

$$\begin{cases} -\Delta'\varphi - \alpha(y)\sum_{i=2}^n G_i'(0)\frac{\partial \varphi}{\partial x_i} = \lambda\varphi & \text{in } D, \\ \varphi = 0 & \text{on } \partial D. \end{cases} \tag{8.236}$$

Let $\delta \in (0,1)$, and compute

$$\begin{aligned} &\Delta'(\delta\varphi) + \alpha(y)\sum_{i=2}^n G_i'(\delta\varphi)\frac{\partial(\delta\varphi)}{\partial x_i} + \sum_{i=2}^n G_i(\delta\varphi)\frac{\partial \alpha}{\partial x_i} + f(\delta\varphi) \\ &\quad = \delta\Big[\alpha(y)\sum_{i=2}^n \Big(G_i'(\delta\varphi) - G_i'(0)\Big)\frac{\partial\varphi}{\partial x_i} + \frac{f(\delta\varphi)}{\delta} - \lambda\varphi\Big] + \sum_{i=2}^n G_i(\delta\varphi)\frac{\partial \alpha}{\partial x_i}. \end{aligned}$$

Using the mean value theorem, for all $2 \leq i \leq n$, there exists $\xi_i \in (0, \delta\varphi)$ such that $G_i'(\delta\varphi) - G_i'(0) = \delta\varphi G_i''(\xi_i)$. Hence, after dividing by φ, the function $\delta\varphi$ is a subsolution of (8.233) if and only if

$$\delta\alpha(y)\sum_{i=2}^n G_i''(\xi_i)\frac{\partial\varphi}{\partial x_i} + \frac{f(\delta\varphi)}{\delta\varphi} - \lambda + \sum_{i=2}^n \frac{G_i(\delta\varphi)}{\delta\varphi}\frac{\partial\alpha}{\partial x_i} \geq 0.$$

If δ tends to 0, the term $\delta\alpha(y)\sum_{i=2}^n G_i''(\xi_i)\frac{\partial\varphi}{\partial x_i}$ tends to 0, and the term $\frac{f(\delta\varphi)}{\delta\varphi} - \lambda + \sum_{i=2}^n \frac{G_i(\delta\varphi)}{\delta\varphi}\frac{\partial\alpha}{\partial x_i}$ tends to $f'(0) - \lambda + \sum_{i=2}^n G_i'(0)\frac{\partial\alpha}{\partial x_i}$ which is positive since we assumed (8.235).
Thus, for δ sufficiently small, the sequence of functions $\{\delta_k\varphi\}_k$ satisfies (8.234).

$\square$

Remark 8.11. If the measure of the domain D is sufficiently big, and if $\|\alpha G'(0)\|_\infty$ is small enough, then the equation (8.235) holds by Proposition 6.3 and hence there exists a sequence of functions $\{v_k\}_k \in C^{2,\lambda}(\overline{D})$, which satisfies (8.234).

Corollary 8.12. *Under the same assumptions of Proposition 8.10 there exists a function* $w_- \in C^{2,\lambda}(\overline{D})$ *which satisfies* (8.233).

Proof. Under the assumptions of Proposition 8.10, we know that there exists a subsolution of (8.233). Taking the constant function 1 as a supersolution, we construct a function w_- which satisfies all the required properties. □

9 Conclusion

Under both Neumann and Dirichlet boundary conditions, and with the two forms of the convection term, namely $\alpha(y)\nabla \cdot G(u)$ and $\nabla \cdot \Big(\alpha(y)G(u)\Big)$, we proved that under the assumptions of Theorem 1.17 and the assumptions of Theorem 4.3 (for Neumann boundary conditions), and of Theorem 1.20 and Theorem 8.2 (for Dirichlet boundary conditions), that there exists a critical speed $c^\star$ such that travelling front solutions of (1.35) and (4.117) exist with speed c if and only if $c \geq c^\star$. Due to the presence of the convection term, in both cases, this critical speed $c^\star$ might be negative. This is an important difference with the case without convection term in [BN2] where the critical speed $c^\star$ was positive. Indeed, if a travelling front solution has a negative speed $c < 0$, then the wave will go from 1 to 0 by moving to the right. Hence, what you see as an observer will depend on the sign of the speed c : if $c < 0$, the density of the population will converge to 0, meaning the extinction of the species.
Under both Neumann and Dirichlet we also proved that there exists a critical speed c' related to a certain eigenvalue problem associated to the linearized problem around 0. We proved that, for the two forms of the convection term, the inequality $c^\star \geq c'$ holds under the assumptions of Theorem 1.18, Theorem 1.19 (for Neumann boundary conditions) and Theorem 1.21 and Theorem 1.22 (for Dirichlet boundary conditions). Recall that in [BN2], Berestycki and Nirenberg proved that under the KPP condition (1.31) for f, the equality $c^\star = c'$ holds. With the presence of convection, the derivative terms $\partial_i u$ for $2 \leq i \leq n$, were complicated to handle and prevented us to adapt the method of sub and supersolution used by Berestycki and Nirenberg to prove the equality except in some special cases. However, under Neumann boundary conditions, we highlighted two cases where the equality $c^\star = c'$ holds. The first special case is when the convection term has the form $G = (G_1, 0, \cdots, 0)$ see Proposition 3.9 and the second one is when $G_1'(0) = 0$, see Proposition 3.10. Under Dirichlet boundary conditions, we also proved this equality holds in the case

where the convection term has the form $G = (G_1, 0, \cdots, 0)$. The equality $c^\star = c'$ does not hold in general, even when no convection terms are present, but it would be interesting to find sufficient conditions for equality for a wider class of convection terms G.

In [BN2], Berestycki and Nirenberg explained that the term $\alpha(y)\partial_1 u$ in the equation is a transport term or a driving flow along the direction of the cylinder. This flow is represented by the term $\alpha(y)$ which does not depend on x_1. In our special case when the convection term has the form $G = (G_1, 0, \cdots, 0)$, the equation is similar to the one in [BN2], except that the coefficient term in front of the first derivative $\partial_1 u$ depends on u, which means that this diving flow can depend on the density of the species. Note also that when the convection term G has the form $G = (G_1, 0, \cdots, 0)$, the equation is the same as the one in the one-dimensional case, and that is what motivated us to study this special case. Indeed, the one-dimensional theoretical model has sometimes been relevant to practical observations. For instance, in [MUR, Section 13.8], Murray explained that after a near extinction of the otter population in the early 1900s, the population followed a growth that was very close to the one-dimensional model.

Here we have only studied the case where f is monostable (see (1.30)). An interesting question is to ask what the results would have been if the function f was bistable, which means that there exists $s \in (0, 1)$ such that $f < 0$ on $(0, s)$, and $f > 0$ on $(s, 1)$, instead of monostable. For example, the bistable case could arise in the field of combustion, see [BN2, Introduction]. Without any convection term, it was proved in [BN2] that there exists a single speed $c^\star$, which means that all the travelling wave solutions move with the same speed $c^\star$. In the bistable case, the sign of $c^\star$ is again important, because it typically determines what an observer will see as $t \to +\infty$, and except in some simple cases, the sign of $c^\star$ is difficult to determine even without convection term, and will clearly be affected by the presence of convection.

The convection terms we have studied in this thesis have been prototype terms that were not motivated by particular applications. It would be very interesting to identify concrete applications where non-linear convection is important, which could motivate both specific forms of convection term and application-inspired research questions.

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