Stability analysis for nonlinear neutral stochastic functional differential equation *

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Abstract

In this paper, we provide some sufficient conditions for the existence and uniqueness, the stochastic stability for the global solution of nonlinear neutral stochastic functional differential equation. When the drift term and the diffusion term satisfy a locally Lipschitz condition, and the Lyapunov monotonicity condition has a sign-changed time-varying coefficient, the existence and uniqueness of the global solution for such equation will be studied by using the Lyapunov-Krasovskii function and the theory of stochastic analysis. The stability in pth($p \geq 2$)-moment, the asymptotical stability in pth($p \geq 2$)-moment, and the exponential stability in pth($p \geq 2$)-moment will be investigated. Three different characterizations for these three kinds of stochastic stability in moment will be established, which are presented in terms of integration conditions, respectively. These results have seldom been reported in the existing literature. In addition, the almost surely exponential stability for the global solution of such equation is also discussed. Some discussions and comparisons are provided. Two examples are given to illustrate the effectiveness of the theoretical results obtained.

Keywords. Nonlinear neutral stochastic functional differential equation; time-varying equation; global solution; existence and uniqueness; stochastic stability.

AMS subject classifications. 34D20, 34K20, 93E15, 34F05, 60H10.

1. Introduction

Neutral functional differential equation (NFDE) is a class of functional differential equation, in which the derivatives of the past history or derivatives of functionals of the past history are involved as well as the present state of the system [1]. Since NFDE has the extensive applications

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in chemical process, aeroelasticity, Lotka-Volterra systems, steam or water pipes, heat exchangers, partial element equivalent circuits and control of constrained manipulators with delay measurements, many papers, see, e.g. [2-7] and the references therein, have presented the stability analysis, boundedness, oscillation, and bifurcation of NFDE. When NFDE is subject to the environmental external disturbances, it can be characterized by neutral stochastic functional differential equation (NSFDE) [8-9]. Since NSFDE is used in the science and engineering, e.g. computer-chip interface circuitry, distributed networks, population dynamics, and chemical process control, the study on the dynamical properties was extensively developed. One fundamental issue in the dynamical properties of NSFDE is placed on the stochastic stability analysis. In [10, 11], when the drift term and the diffusion term satisfy a globally Lipschitz condition, the Lyapunov-Rauzmikhin theorem was established to discuss the exponential stability in moment and the asymptotical stability in moment for NSFDE, respectively. In [12, 13], when the global Lipschitz condition is satisfied for the drift term and the diffusion term, by using the Lyapunov-Krasovskii function approach and the theory of stochastic analysis, the exponential stability in moment for NSFDE was discussed under the Lyapunov monotonicity condition. Over past few decades, the stochastic stability analysis for NSFDE has been well developed, and some results have been presented in [14-21] and the references therein.

For many practical mathematical models such as stochastic logistic model and stochastic Lotka-Volterra model, the drift term and the diffusion term don't satisfy the globally Lipschitz condition, but the locally Lipschitz condition. Such models are usually characterized by highly nonlinear stochastic differential equation [22-27]. When the drift term and the diffusion term only satisfy the locally Lipschitz condition, up to now, there are some works on the existence and uniqueness, and the stochastic stability of highly nonlinear NSFDE, see [28-30] and the references therein. For example, in [28, 29], by using the Lyapunov function and the theory of stochastic analysis, the problems on the existence and uniqueness, and the robustness of general decay stability analysis of the global solution have been discussed.

On the other hand, in many realistic models, the time-varying system is very universal [31]. This system with its characteristics changing with time is also called variable coefficient system. For example, a rocket is a typical example of a time-varying system in which its mass will decrease with time due to fuel consumption; another common example is the manipulator. The moment of inertia of the joints around the corresponding axis is a complex function of time as an independent variable. The stochastic stability for time-varying stochastic functional differential equation (SFDE) was extensively considered in [32-37] and the references therein. For example, in [32], by using the Lyapunov-Razumikhin theorem, Peng and Zhang have studied the asymptotical stability in moment for time-varying SFDE. In [34], when the globally Lipschitz condition holds for the drift term and the diffusion term, by constructing an auxiliary delay differential equation and the comparison principle, the stochastic stability in moment of time-varying SFDE was investigated, which includes the stability in pth($p \geq 2$)-moment, the asymptotic stability in pth($p \geq 2$)-moment and the exponential stability in pth($p \geq 2$)-moment. In [34], different characterizations of sufficient condition to guarantee these three kinds of stochastic stability in pth($p \geq 2$)-moment for time-varying SFDE have been presented. In [37], when the drift term and the diffusion term satisfy the locally Lipschitz condition, by using the Lyapunov-Krasovskii function and the theory of stochastic analysis, the existence and uniqueness, and the stochastic stability for the global solution of time-varying SFDE were studied. Very recently, when the drift term and the diffusion term satisfy the globally Lipschitz condition, by using the comparison principle and the proof by contradiction, the exponential stability in mean square for time-varying NSFDE was investigated in [21], and one sufficient condition was given in terms of the algebraic inequality.

When the drift term and the diffusion term only satisfy the locally Lipschitz condition and the Lyapunov monotonicity condition, the existence and uniqueness for the global solution of time-varying NSFDE is seldom considered in the existing literature. In addition, for functional differential equation, the state space of the solution is infinite dimension. In the infinite space, in general, the exponential stability means the asymptotic stability, and vice versa. In [11], the stability in pth($p \geq 2$)-moment and the asymptotic stability in pth($p \geq 2$)-moment for NSFDE were analyzed by using the Lyapunov-Razumikhin theorem, but the sign changed time-varying coefficient is not embodied in the obtained results. In [36], the Lypunov-Razumikhin theorem and the Lyapunov-Krasovskii theorem were used to analyze the exponential stability in pth($p \geq 2$)-moment for time-varying SFDE when the Lyapunov monotonicity condition has a sign changed time-varying coefficient, but the obtained results cannot be used for time-varying NSFDE. To our knowledge, there are no available results on the existence and uniqueness, and the stability analysis for the global solution of time-varying NSFDE when the Lyapunov monotonicity condition has a sign changed time-varying coefficient, let alone have reported different characterizations of sufficient condition to guarantee these three kinds of stochastic stability in pth($p \geq 2$)-moment for time-varying NSFDE. Thus, solving these problems is the main motivation in this paper.

In this paper, when the drift term and the diffusion term satisfy the locally Lipschitz condition, and a sign changed time-varying coefficient exists in the Lyapunov monotonicity condition, we shall investigate the problems on the existence and uniqueness, the stochastic stability in pth($p \geq 2$)-moment for the global solution of time-varying NSFDE. By using the Lyapunov-Krasovskii function and the theory of stochastic analysis, different characterizations of sufficient conditions to guarantee the existence and uniqueness, the stability in pth($p \geq 2$)-moment, the asymptotic stability in pth($p \geq 2$)-moment and the exponential stability in pth($p \geq 2$)-moment for the global solution of time-varying NSFDE will be presented. The almost surely exponential stability will be also considered by the nonnegative semimartingale convergence theorem. Finally, two examples are provided to illustrate the effectiveness of the theoretical results derived.

The contributions in this paper are summarized as follows:

(1) The existence and uniqueness for the global solution of nonlinear NSFDE is investigated, when a locally Lipschitz condition is satisfied for the drift term and the diffusion term, and a sign-changed time-varying coefficient is permitted in the Lyapunov monotonicity condition. On this issue, the Lyapunov monotonicity condition seldom has a sign-changed time-varying coefficient in the existing literature.

(2) We provide three different characterizations for the stability in pth($p \geq 2$)-moment, the asymptotical stability in pth($p \geq 2$) -moment, and the exponential stability in pth($p \geq 2$)-moment of NSFDE. In this paper, apart from the Lyapunov monotonicity condition, all sufficient conditions on the existence and uniqueness as well as these three kinds of stochastic stability in pth($p \geq 2$)-moment are given in integral form.

(3) For time-varying NSFDE, if the time-varying coefficient in the Lyapunov monotonicity condition is estimated as the constant one, then its important information may be lost, and in particular,

the sign-changed coefficient is not easily embodied. In this paper, the characteristics of the sign-changed time-varying coefficient is reflected in the sufficient conditions.
(4) The technique used in this paper differs from the ones proposed for the stability analysis on time-varying SFDE in this existing literature. The methodology utilized for the latter is not easily used in this paper. The technique employed in this paper can be also used for analyzing the existence and uniqueness, the stochastic stability in moment for the global solution of time-varying NSFDE without the neutral term in our model.

The rest content of this paper is organized as follows. In Section 2, some problems and preliminaries are formulated. Section 3 presents the main results and their proofs. Discussion and comparison are provided in Section 4. Two examples are given in Section 5.

Notation: Let $|\cdot|$ be the norm of the n-dimensional real Euclidean space R^n. For an n-dimensional column vector $a = \mathrm{col}[a_1, a_2, \ldots, a_n] \in R^n$, $|a| = \sqrt{\sum_{i=1}^{n} |a_i|^2}$. If A is a matrix, its transpose is denoted by A^T. $|A| = \sqrt{\mathrm{trace}(A^T A)}$ is also the trace norm of the square matrix A. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P})$ represents a complete probability space, in which a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfies the usual conditions. $\mathbb{E}\{\cdot\}$ stands for the expectation operator. Let $C([-\tau, 0]; R^n)$ $(\tau > 0)$ be the family of all bounded continuous R^n-valued functions φ on $[-\tau, 0]$ with norm $\|\varphi\|_C = \sup\{|\varphi(\theta)| : -\tau \leq \theta \leq 0\}$. $\mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$ $(p \geq 2)$ denotes the family of all $\mathcal{F}_{t_0}$-measurable $C([-\tau, 0]; R^n)$-valued random variable with $\mathbb{E}\|\zeta\|^p_C < +\infty$ for any $\zeta \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$. $f(\cdot) \in \mathcal{L}^1([t_0, +\infty); [0, +\infty))$ implies that $\int_{t_0}^{+\infty} f(t)dt < +\infty$. For any $a, b \in R$, $a \vee b = \max\{a, b\}$. $\mathcal{L}^p([t_0, T]; \mathcal{Z})$ $(p \geq 1)$ represents the set of all $\{\mathcal{F}_t\}_{t\geq t_0}$-adapted $\mathcal{Z}$-valued processes $X(\cdot)$ such that $\int_{t_0}^{T} |X(t)|^p dt < +\infty$ *a.s.*, where $\mathcal{Z} = R^n$ or $R^{n\times m}$.

2. Problem formulation and preliminaries

Consider the following nonlinear NSFDE:

$$d[x(t) - \mathcal{D}(x_t)] = f(t, x(t), x_t)dt + g(t, x(t), x_t)d\mathcal{B}(t), \quad t \geq t_0, \tag{1}$$

with the initial value $x_{t_0} = \phi = \{\phi(\theta) : -\tau \leq \theta \leq 0\} \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$ $(p \geq 2)$. For any $t \in [t_0, +\infty)$, $x_t = \{x(t + \theta) : -\tau \leq \theta \leq 0\}$ can be considered as a $C([-\tau, 0]; R^n)$-valued stochastic process. $x(t) = \mathrm{col}[x_1(t),\ x_2(t),\ \ldots,\ x_n(t)] \in R^n$ denotes the state vector. $\mathcal{D}(\cdot) : C([-\tau, 0]; R^n) \to R^n$, $f(\cdot, \cdot, \cdot) : [t_0, +\infty) \times R^n \times C([-\tau, 0]; R^n) \to R^n$, and $g(\cdot, \cdot, \cdot) : [t_0, +\infty) \times R^n \times C([-\tau, 0]; R^n) \to R^{n\times m}$ are Borel measurable. Let $x(t, t_0, \phi)$ be the solution of NSFDE (1). For simplicity, $x(t) = x(t, t_0, \phi)$ and $\tilde{x}(t) = x(t) - \mathcal{D}(x_t)$.

Definition 2.1 ([9]): An R^n-valued stochastic process $x(t)$ on $t_0 - \tau \leq t \leq T$ is said to be a solution of NSFDE (1) with the initial value $x_{t_0} = \phi = \{\phi(\theta) : -\tau \leq \theta \leq 0\} \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$, if it has the following properties:
(i) it is continuous and $\{x_t\}_{t\in[t_0,T]}$ is $\mathcal{F}_t$-adapted;
(ii) $f(\cdot, x(\cdot), x_\cdot) \in \mathcal{L}^1([t_0, T]; R^n)$ and $g(\cdot, x(\cdot), x_\cdot) \in \mathcal{L}^2([t_0, T]; R^{n\times m})$;
(iii) $x_{t_0} = \phi$ and for any $t \in [t_0, T]$, $x(t)$ satisfies

$$x(t) - \mathcal{D}(x_t) = x(t_0) - \mathcal{D}(x_{t_0}) + \int_{t_0}^{t} f(s, x(s), x_s)ds + \int_{t_0}^{t} g(s, x(s), x_s)d\mathcal{B}(s).$$

A solution $x(t)$ is said to be unique if any other solution $\bar{x}(t)$ is indistinguishable from it, that is, $\mathcal{P}\{x(t) = \bar{x}(t), \text{ for all } t \in [t_0 - \tau, T]\} = 1$.

Definition 2.2 ([9]): Let $x(t)$ $(t \in [t_0-\tau, \varrho_\infty)$ be a continuous $\mathcal{F}_t$-adapted R^n-valued local stochastic process, where ϱ_∞ is a stopping time. It is said to be a local solution of NSFDE (1) with the initial value $x_{t_0} = \phi = \{\phi(\theta) : -\tau \leq \theta \leq 0\} \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$, if $x_{t_0} = \phi$ and for any $t \geq t_0$,

$$x(t \wedge \varrho_k) - \mathcal{D}(x_{t\wedge\varrho_k}) = x(t_0) - \mathcal{D}(x_{t_0}) + \int_{t_0}^{t\wedge\varrho_k} f(s, x(s), x_s)ds + \int_{t_0}^{t\wedge\varrho_k} g(s, x(s), x_s)d\mathcal{B}(s)$$

holds, where $\{\varrho_k\}_{k\geq1}$ is a nondecreasing sequence of finite stopping time such that $\varrho_k \uparrow \varrho_\infty$ a.s. Moreover, if $\limsup_{t\to\varrho_\infty} |x(t)| = \infty$ whenever $\varrho_\infty < \infty$ a.s., it is said to be maximal local solution and ϱ_∞ is the explosion time. A maximal local solution $x(t)$ $(t \in [t_0 - \tau, \varrho_\infty)$ is said to be unique if for any other maximal local solution $\bar{x}(t)$ $(t \in [t_0 - \tau, \bar{\varrho}_\infty)$, we have $\varrho_\infty = \bar{\varrho}_\infty$ *a.s.* and $x(t) = \bar{x}(t)$ for all $t \in [t_0 - \tau, \bar{\varrho}_\infty)$ *a.s.*

Denote $\Gamma([-\tau, 0]; (0, +\infty))$ by the family of all Borel measurable bounded nonnegative functions $\eta(\cdot)$ on $[-\tau, 0]$ such that $\int_{-\tau}^{0} \eta(\theta)d\theta = 1$ [9]. For NSFDE (1), three hypotheses are stated as follows:

Hypothesis I: There exists a function $K(\cdot) \in \mathcal{L}^1([t_0, +\infty); [0, +\infty))$ such that for any $t \geq t_0$, $p \geq 2, \phi, \varphi \in C([-\tau, 0]; R^n)$, $|f(t, \phi(0), \phi) - f(t, \varphi(0), \varphi)|^p \vee |g(t, \phi(0), \phi) - g(t, \varphi(0), \varphi)|^p \leq K(t)(|\phi(0) - \varphi(0)|^p + \|\phi - \varphi\|^p_C)$, $f(t, 0, 0) = 0$, and $g(t, 0, 0) = 0$.

Hypothesis II: For any integer $m \geq 1$, there exists a function $K_m(\cdot) \in \mathcal{L}^1([t_0, +\infty); [0, +\infty))$ such that for any $t \geq t_0$, $p \geq 2$, ϕ, $\varphi \in C([-\tau, 0]; R^n)$ with $|\phi(0)| \vee |\varphi(0)| \vee \|\phi\|_C \vee \|\varphi\|_C \leq m$, $|f(t, \phi(0), \phi) - f(t, \varphi(0), \varphi)|^p \vee |g(t, \phi(0), \phi) - g(t, \varphi(0), \varphi)|^p \leq K_m(t)(|\phi(0) - \varphi(0)|^p + \|\phi - \varphi\|^p_C)$, $f(t, 0, 0) = 0$, and $g(t, 0, 0) = 0$.

Hypothesis III: There exists a constant $\kappa \in (0, 1)$ and a function $\eta(\cdot) \in \Gamma([-\tau, 0]; (0, +\infty))$ such that for any ϕ, $\varphi \in C([-\tau, 0]; R^n)$, and $p \geq 2$, $|\mathcal{D}(\phi) - \mathcal{D}(\varphi)|^p \leq \kappa^p \int_{-\tau}^{0} \eta(\theta)|\phi(\theta) - \varphi(\theta)|^p d\theta$, and $\mathcal{D}(0) = 0$.

Remark 1: Under *Hypothesis II* and *Hypothesis III*, similar to *Lemma 3.2* in [38], the existence and uniqueness of the maximal trivial solution for NSFDE (1) can be guaranteed. The detailed methodology used in the proof can be also seen in [15, 16].

In [9], under *Hypothesis I* and *Hypothesis III*, the existence and uniqueness of the trivial solution for NSFDE (1) can be checked. The detailed proof refers to *Theorem 2.2* (pp. 204-209, [9]). Under *Hypothesis I* with $K(t) \equiv K$, and *Hypothesis III*, in [10-20] and their references therein, the stochastic stability in pth($p \geq 2$)-moment for NSFDE (1) has been discussed by using the Lyapunov Razumikhin theorem, the Lyapunov-Krasovskii functional and the Lyapunov-Krasovskii function, respectively. In [11], by using the Lyapunov Razumikhin theorem, the stability in pth($p \geq 2$)-moment and the asymptotic stability in pth($p \geq 2$)-moment for NSFDE have been considered. By using the comparison principle and the proof of contradiction, the exponential stability in mean square for NSFDE (1) was analyzed in [21], where *Hypothesis I* and *Hypothesis III* are satisfied for the drift term, the diffusion term and the neutral term, respectively. In [33, 34], the stability in pth($p \geq 2$)-moment, the asymptotic stability in pth($p \geq 2$)-moment and the exponential stability in pth($p \geq 2$)-moment for time-varying SFDE have been discussed. However, when a sign-changed time-varying coefficient is permitted in the Lyapunov monotonicity condition, and *Hypothesis II* and *Hypothesis III* are satisfied, the results derived in [10-20, 21, 33,

34] cannot be applied to guarantee the stability in pth($p \geq 2$)-moment, the asymptotic stability in pth($p \geq 2$)-moment and the exponential stability in pth($p \geq 2$)-moment for the global solution of NSFDE (1). The main reasons are stated as follows:
(i) in [10-20], the sign-changed time-varying coefficient is not embodied in the obtained results. In general, the methodology for analyzing the stability of autonomous differential equation is not easily generalized for non-autonomous differential equation, when there is a sign-changed time-varying coefficient. Besides, the neutral term exists in NSFDE (1), which can make this generalization more complicated.
(ii) in [21], the obtained results are only concerned with the exponential stability in mean square for NSFDE (1), and the Lyapunov monotonicity condition does not permit a sign-changed time-varying coefficient (see *Theorem 4.1 in* Section 4). The stability in pth($p \geq 2$)-moment and the asymptotic stability in pth($p \geq 2$)-moment are not discussed in [21]. In this paper, when the Lyapunov monotonicity condition has a sign-changed time-varying coefficient, we will give three different characterizations of sufficient conditions to guarantee the stability in pth($p \geq 2$)-moment, the asymptotic stability in pth($p \geq 2$)-moment and the exponential stability in pth($p \geq 2$)-moment for NSFDE (1), see *Theorems* 3.2-3.4 in Section 3;
(iii) Three different characterizations of sufficient conditions on the stability in pth($p \geq 2$)-moment, the asymptotic stability in pth($p \geq 2$)-moment and the exponential stability in pth($p \geq 2$)-moment for time-varying SFDE have been presented in [33, 34]. However, to our knowledge, since the neutral term and the stochastic perturbation coexist in NSFDE (1), the generalizations of those results and methodologies on analyzing the stochastic stability from SFDE to NSFDE are generally difficult. In this paper, the methodology used differs from the ones given in [33, 34].

Note that in this paper, $f(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot,\cdot)$ in NSFDE (1) don't satisfy *Hypothesis I*, but *Hypothesis II*. In order to discuss the existence and uniqueness, and the stochastic stability in moment of the global trivial solution for NSFDE (1), we need one Lyapunov monotonicity condition. To state this condition in our main theorem, one notation is presented as follows. Denote by $C^{1,2}([t_0,+\infty)\times R^n;[0,+\infty))$ the family of all continuous nonnegative functions $V(t,\phi(0))$ defined on $[t_0,+\infty)\times R^n$, such that they have continuously once and twice derivatives with respect to first variable and second variable, respectively. For any given $V(\cdot,\cdot)\in C^{1,2}([t_0,+\infty)\times R^n;[0,+\infty))$, $\mathcal{L}V(\cdot,\cdot,\cdot):\ [t_0,+\infty)\times R^n\times C([-\tau,0];R^n)\longrightarrow R$ is defined by

$$\begin{aligned}\mathcal{L}V(t,\phi(0),\phi) &= V_t(t,\phi(0)-\mathcal{D}(\phi))+V_x^T(t,\phi(0)-\mathcal{D}(\phi))f(t,\phi(0),\phi)\\ &\quad+\frac{1}{2}\text{trace}[g^T(t,\phi(0),\phi)V_{xx}(t,\phi(0)-\mathcal{D}(\phi))g(t,\phi(0),\phi)],\end{aligned}$$

where $V_t(t,\phi(0)-\mathcal{D}(\phi))$, $V_x(t,\phi(0)-\mathcal{D}(\phi))$ and $V_{xx}(t,\phi(0)-\mathcal{D}(\phi))$ are given in [9].

Definition 2.3 ([9, 11]): The trivial solution of NSFDE (1) is said to be:
(a) stable in pth($p \geq 2$)-moment, if for any $\epsilon > 0$, there exists $\delta = \delta(t_0,\epsilon) > 0$ satisfying that for $t \geq t_0$, $\mathbb{E}\{|x(t)|^p\} \leq \epsilon$, whenever $\phi \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau,0];R^n)$ and $\mathbb{E}\|\phi\|_C^p < \delta$;
(b) asymptotically stable in pth($p \geq 2$)-moment, if it is stable in pth($p \geq 2$)-moment, and for any $\epsilon' > 0$, there exists $T = T(\phi,\epsilon') > 0$ satisfying that for any $t \geq T$, $\phi \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau,0];R^n)$, $\mathbb{E}\{|x(t)|^p\} \leq \epsilon'$;
(c) exponentially stable in pth($p \geq 2$)-moment, if there exist two positive constants M and α such that for any $t \geq t_0$, and the initial value $\phi \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau,0];R^n)$ such that $\mathbb{E}\{|x(t)|^p\} \leq M\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}e^{-\alpha(t-t_0)}$;

(d) almost surely exponentially stable, if for any initial value $\phi \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$ such that $\limsup_{t\to+\infty} \frac{\log(|x(t)|)}{t} < 0$, *a.s.*

In particular, when $p = 2$, the trivial solution of NSFDE (1) is said to be stable in mean square, asymptotically stable in mean square and exponentially stable in mean square, respectively.

Here, we recall the four lemmas for the sequel use.

Lemma 2.4 ([9]): Let $p \geq 2$, then

$$|\phi(0)|^p \leq \frac{|\phi(0) - \mathcal{D}(\phi)|^p}{(1-\kappa)^{p-1}} + \frac{|\mathcal{D}(\phi)|^p}{\kappa^{p-1}},$$

for any $\phi \in C([-\tau, 0]; R^n)$.

Lemma 2.5 ([9]): Let $p \geq 2$, then

$$|\phi(0) - \mathcal{D}(\phi)|^p \leq (1+\kappa)^{p-1}(|\phi(0)|^p + |\mathcal{D}(\phi)|^p),$$

for any $\phi \in C([-\tau, 0]; R^n)$.

Lemma 2.6 ([39]): Let $\chi(\cdot)$ be a nonnegative function defined on $[t_0, \infty)$. If $\chi(\cdot)$ is Lebesgue integrable and uniformly continuous on $[t_0, \infty)$, then $\lim_{t\to\infty} \chi(t) = 0$.

Lemma 2.7 (*Convergence theorem of nonnegative semi-martingales* [9, 15-16]): Let $A_1(t)$ and $A_2(t)$ be two continuous adapted increasing processes on $t \geq t_0$ with $A_1(t_0) = A_2(t_0) = 0$ *a.s.* Let $\mathcal{M}(t)$ be a real-valued continuous local martingale with $\mathcal{M}(t_0) = 0$ *a.s.* Let ξ be a nonnegative $\mathcal{F}_{t_0}$ random variable such that $\mathbb{E}\{\xi\} < \infty$. Define

$$X(t) = \xi + A_1(t) - A_2(t) + \mathcal{M}(t), \quad \text{for} \ \ t \geq t_0.$$

If $X(t)$ is nonnegative, then $\{\lim_{t\to\infty} A_1(t) < \infty\} \subset \{\lim_{t\to\infty} X(t) < \infty\} \cap \{\lim_{t\to\infty} A_2(t) < \infty\}$ *a.s.*, where $C \subset D$ *a.s.* means $\mathbb{P}(C \cap D^c) = 0$. In particular, if $\lim_{t\to\infty} A_1(t) < \infty$ *a.s.*, then, with probability one, $\lim_{t\to\infty} X(t) < \infty$, $\lim_{t\to\infty} A_2(t) < \infty$, and $-\infty < \lim_{t\to\infty} \mathcal{M}(t) < \infty$. That is, all of these three processes $X(t)$, $A_2(t)$ and $\mathcal{M}(t)$ converge to finite random variables.

3. Main results

In this section, firstly, the existence and uniqueness of the global trivial solution for NSFDE (1) is discussed as follows.

Theorem 3.1: Assume that *Hypotheses II-III* hold. Suppose that there exist one Lyapunov-Krasovskii function $V(\cdot,\cdot) \in C^{1,2}([t_0, +\infty) \times R^n;\ [0, +\infty))$, two functions $\lambda_0(\cdot) : [t_0 - \tau, +\infty) \to R$, $\lambda_1(\cdot) : [t_0 - \tau, +\infty) \to [0, +\infty)$ with $\lambda_1(t) \leq \tilde{\lambda}_1$ $(t \geq t_0)$, a function $\tilde{\eta}(\cdot,\cdot) : [t_0, +\infty) \times [-\tau, 0] \to [0, +\infty)$ with $\tilde{\eta}(t, s) \leq \tilde{\eta}'$, and two constants $c_i > 0$ $(i = 1, 2)$ such that

(H_1) for any $t \geq t_0$, $x \in R^n$, and $p \geq 2$,

$$c_1|x|^p \leq V(t, x) \leq c_2|x|^p;$$

(H_2) for any $t \geq t_0$, $\phi \in C([-\tau, 0]; R^n)$, and $p \geq 2$,

$$\mathcal{L}V(t, \phi(0), \phi) \leq \lambda_0(t)V(t, \phi(0) - \mathcal{D}(\phi)) + \lambda_1(t)|\phi(0)|^p + \int_{-\tau}^{0} \tilde{\eta}(t, \theta)|\phi(\theta)|^p d\theta;$$

(H_3) there exists a constant $\bar{M}$ such that for any $t \geq t_0$,

$$\int_{t_0}^{t} \tilde{\lambda}(s)ds \leq \bar{M},$$

where $\tilde{\lambda}(t) = \lambda_0(t) + \frac{\lambda_1(t)+\int_{-\tau}^{0}\tilde{\eta}(t,\theta)d\theta e^{\varsigma}}{c_1(1-\kappa)^{p-1}(1-\kappa e^{\varsigma})}$, $\kappa e^{\varsigma} \in (0,1)$, and $\varsigma = \sup_{t\geq t_0}\sup_{\theta\in[-\tau,0]}\Big\{\int_{t+\theta}^{t}\Big[-\lambda_0(s) - \frac{\lambda_1(s)+\int_{-\tau}^{0}\tilde{\eta}(s,\theta)d\theta}{c_1(1-\kappa)^p}\Big] \times ds\Big\} < +\infty$. Then, for any initial value $\phi \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau,0];R^n)$, NSFDE (1) has a unique global trivial solution on $[t_0,+\infty)$.

Proof: From *Hypotheses II-III*, for any initial value $\phi \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau,0];R^n)$, *Lemma 3.2* [38] checks that there exists a unique maximal local trivial solution $x(t)$ of NSFDE (1) on $[t_0-\tau,\varrho_\infty)$, where ϱ_∞ is the explosion time. In order to guarantee the global trivial solution of NSFDE (1), we only need to demonstrate that $\varrho_\infty = \infty$ *a.s.*

For any $\phi \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau,0];R^n)$, there exists a sufficiently large integer $l_0 > 0$ satisfying $\|\phi\|_C < l_0$. For any $l \geq l_0$, define one stopping time as follows:

$$\rho_l = \inf\{t \in [t_0,\varrho_\infty) : \ |x(t)| \geq l\},$$

where $\inf\emptyset = \infty$, with $\emptyset$ being the empty set. It is seen that ρ_l is increasing as $l \to \infty$ and $\rho_l \to \rho_\infty \leq \varrho_\infty$ *a.s.* If $\rho_\infty = \infty$ *a.s.*, then $\varrho_\infty = \infty$ *a.s.*, which implies that the trivial solution $x(t)$ of NSFDE (1) is global. This is equivalent to prove that for any $t > t_0$, $\mathbb{P}\{\rho_l \leq t\} \to 0$ as $l \to \infty$.

By using the Itô formula to the Lyapunov-Krasovskii function $e^{-\int_{t_0}^{t}\tilde{\lambda}(s)ds}V(t,\tilde{x}(t))$ $(t \geq t_0)$, we have

$$\begin{aligned} e^{-\int_{t_0}^{t\wedge\rho_l}\tilde{\lambda}(s)ds}V(t\wedge\rho_l,\tilde{x}(t\wedge\rho_l)) &= V(t_0,\tilde{x}(t_0)) + \int_{t_0}^{t\wedge\rho_l} e^{-\int_{t_0}^{s}\tilde{\lambda}(s)ds}[\mathcal{L}V(s,x(s),x_s) - \tilde{\lambda}(s)V(s,\tilde{x}(s))]ds \\ &\quad + \int_{t_0}^{t\wedge\rho_l} e^{-\int_{t_0}^{s}\tilde{\lambda}(s)ds}V_x^T(s,\tilde{x}(s))g(s,x(s),x_s)d\mathcal{B}(s). \end{aligned}$$

Taking the expectation on the preceding equality, it yields from (H_1) and (H_2) that for any $t \geq t_0$,

$$\begin{aligned} &\mathbb{E}\{e^{-\int_{t_0}^{t\wedge\rho_l}\tilde{\lambda}(s)ds}|\tilde{x}(t\wedge\rho_l)|^p\} \\ \leq\ & \frac{\mathbb{E}\{V(t_0,\tilde{x}(t_0))\}}{c_1} + \frac{1}{c_1}\mathbb{E}\Big\{\int_{t_0}^{t\wedge\rho_l} e^{-\int_{t_0}^{s}\tilde{\lambda}(u)du}[\mathcal{L}V(s,x(s),x_s) - \tilde{\lambda}(s)V(s,\tilde{x}(s))]ds\Big\} \\ \leq\ & \frac{\mathbb{E}\{V(t_0,\tilde{x}(t_0))\}}{c_1} + \frac{1}{c_1}\mathbb{E}\Big\{\int_{t_0}^{t\wedge\rho_l} e^{-\int_{t_0}^{s}\tilde{\lambda}(u)du}[\lambda_1(s)|x(s)|^p + \int_{-\tau}^{0}\tilde{\eta}(s,\theta)|x(s+\theta)|^p d\theta]ds\Big\} \\ \leq\ & \frac{\mathbb{E}\{V(t_0,\tilde{x}(t_0))\}}{c_1} + \frac{1}{c_1}\mathbb{E}\Big\{\int_{t_0}^{t} e^{-\int_{t_0}^{s\wedge\rho_l}\tilde{\lambda}(u)du}[\tilde{\lambda}_1|x(s\wedge\rho_l)|^p + \tilde{\eta}'\int_{-\tau}^{0}|x(s\wedge\rho_l+\theta)|^p d\theta]ds\Big\} \\ =\ & \frac{\mathbb{E}\{V(t_0,\tilde{x}(t_0))\}}{c_1} + \frac{\tilde{\lambda}_1}{c_1}\int_{t_0}^{t}\mathbb{E}\{e^{-\int_{t_0}^{s\wedge\rho_l}\tilde{\lambda}(u)du}|x(s\wedge\rho_l)|^p\}ds \\ & + \frac{\tilde{\eta}'}{c_1}\int_{t_0}^{t}\int_{-\tau}^{0}\mathbb{E}\{e^{-\int_{t_0}^{s\wedge\rho_l}\tilde{\lambda}(u)du}|x(s\wedge\rho_l+\theta)|^p\}d\theta ds \\ \leq\ & \frac{\mathbb{E}\{V(t_0,\tilde{x}(t_0))\}}{c_1} + \frac{\tilde{\lambda}_1}{c_1}\int_{t_0}^{t}\mathbb{E}\{e^{-\int_{t_0}^{s\wedge\rho_l}\tilde{\lambda}(u)du}|x(s\wedge\rho_l)|^p\}ds \\ & + \frac{\tilde{\eta}'e^{\varsigma}}{c_1}\int_{t_0}^{t}\int_{-\tau}^{0}\mathbb{E}\{e^{-\int_{t_0}^{s\wedge\rho_l+\theta}\tilde{\lambda}(u)du}|x(s\wedge\rho_l+\theta)|^p\}d\theta ds. \end{aligned} \tag{2}$$

From *Lemma 2.4*, we have

$$|x(t\wedge\rho_l)|^p \le \kappa\int_{-\tau}^{0}\eta(\theta)|x(t\wedge\rho_l+\theta)|^p d\theta + \frac{|\tilde{x}(t\wedge\rho_l)|^p}{(1-\kappa)^{p-1}}. \tag{3}$$

From (2) and (3), it yields that for any $t \ge t_0$,

$$\begin{aligned}\mathbb{E}\{e^{-\int_{t_0}^{t\wedge\rho_l}\tilde{\lambda}(s)ds}|x(t\wedge\rho_l)|^p\} \le\ & \frac{c_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}}{c_1(1-\kappa)^{p-1}}\\ &+\kappa e^{\varsigma}\int_{-\tau}^{0}\eta(\theta)\mathbb{E}\{e^{-\int_{t_0}^{t\wedge\rho_l+\theta}\tilde{\lambda}(u)du}|x(t\wedge\rho_l+\theta)|^p\}d\theta\\ &+\frac{\tilde{\lambda}_1}{c_1(1-\kappa)^{p-1}}\int_{t_0}^{t}\mathbb{E}\{e^{-\int_{t_0}^{s\wedge\rho_l}\tilde{\lambda}(u)du}|x(s\wedge\rho_l)|^p\}ds\\ &+\frac{\tilde{\eta}' e^{\varsigma}}{c_1(1-\kappa)^{p-1}}\int_{t_0}^{t}\int_{-\tau}^{0}\mathbb{E}\{e^{-\int_{t_0}^{s\wedge\rho_l+\theta}\tilde{\lambda}(u)du}|x(s\wedge\rho_l+\theta)|^p\}d\theta ds. \end{aligned}\tag{4}$$

Besides, for any $t\in[t_0-\tau,t_0]$, $\mathbb{E}\{e^{-\int_{t_0}^{t\wedge\rho_l}\tilde{\lambda}(s)ds}|x(t\wedge\rho_l)|^p\} \le M\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}$, where $M\ge 1$.

Now, we will prove that for any $t\ge t_0-\tau$,

$$\mathbb{E}\{e^{-\int_{t_0}^{t\wedge\rho_l}\tilde{\lambda}(s)ds}|x(t\wedge\rho_l)|^p\} \le M'_\varepsilon e^{c(t-t_0)}, \tag{5}$$

where $M'_\varepsilon = \max\Big\{\frac{cc_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}+\varepsilon}{\tilde{\lambda}_1+\tilde{\eta}'\tau e^{\varsigma}}, (M\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}+\varepsilon)e^{\sup_{t\in[t_0-\tau,t_0]}\int_t^{t_0}\tilde{\lambda}(s)ds}\Big\}$ $(\varepsilon>0)$ and $c>0$ is suitably chosen such that $\kappa e^{\varsigma}+\frac{\tilde{\lambda}_1+\tilde{\eta}'\tau e^{\varsigma}}{cc_1(1-\kappa)^{p-1}}=1$. Obviously, for any $t\in[t_0-\tau,t_0]$, inequality (5) holds.

If inequality (5) is not satisfied for any $t\ge t_0$, then there exists $t>t_0$ such that $\mathbb{E}\{e^{-\int_{t_0}^{t\wedge\rho_l}\tilde{\lambda}(s)ds}|x(t\wedge\rho_l)|^p\} > M'_\varepsilon e^{c(t-t_0)}$. Letting $t^*=\inf\{t>t_0:\ \mathbb{E}\{e^{-\int_{t_0}^{t\wedge\rho_l}\tilde{\lambda}(s)ds}|x(t\wedge\rho_l)|^p\} > M'_\varepsilon e^{c(t-t_0)}\}$. Consequently, we have

$$\mathbb{E}\{e^{-\int_{t_0}^{t\wedge\rho_l}\tilde{\lambda}(s)ds}|x(t\wedge\rho_l)|^p\} \le M'_\varepsilon e^{c(t-t_0)},\quad t\in[t_0,t^*), \tag{6}$$

and

$$\mathbb{E}\{e^{-\int_{t_0}^{t^*\wedge\rho_l}\tilde{\lambda}(s)ds}|x(t^*\wedge\rho_l)|^p\} = M'_\varepsilon e^{c(t^*-t_0)}. \tag{7}$$

But, from (4) and (6), it follows

$$\begin{aligned}&\mathbb{E}\{e^{-\int_{t_0}^{t^*\wedge\tau_l}\tilde{\lambda}(s)ds}|x(t^*\wedge\rho_l)|^p\}\\ \le\ & \frac{c_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}}{c_1(1-\kappa)^{p-1}} + \kappa e^{\varsigma}\int_{-\tau}^{0}\eta(\theta)\mathbb{E}\{e^{-\int_{t_0}^{t^*\wedge\rho_l+\theta}\tilde{\lambda}(u)du}|x(t^*\wedge\rho_l+\theta)|^p\}d\theta\\ &+\frac{\tilde{\lambda}_1}{c_1(1-\kappa)^{p-1}}\int_{t_0}^{t^*}\mathbb{E}\{e^{-\int_{t_0}^{s\wedge\rho_l}\tilde{\lambda}(u)du}|x(s\wedge\rho_l)|^p\}ds\\ &+\frac{\tilde{\eta}' e^{\varsigma}}{c_1(1-\kappa)^{p-1}}\int_{t_0}^{t^*}\int_{-\tau}^{0}\mathbb{E}\{e^{-\int_{t_0}^{s\wedge\rho_l+\theta}\tilde{\lambda}(u)du}|x(s\wedge\rho_l+\theta)|^p\}d\theta ds\\ \le\ & \Big[\frac{c_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}}{c_1(1-\kappa)^{p-1}} - \frac{\tilde{\lambda}_1+\tilde{\eta}'\tau e^{\varsigma}}{cc_1(1-\kappa)^{p-1}}M'_\varepsilon\Big] + \Big[\kappa e^{\varsigma}+\frac{\tilde{\lambda}_1+\tilde{\eta}'\tau e^{\varsigma}}{cc_1(1-\kappa)^{p-1}}\Big]M'_\varepsilon e^{c(t^*-t_0)}.\end{aligned}$$

From the choices of M'_ε and c, we have $\frac{c_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}}{c_1(1-\kappa)^{p-1}} - \frac{\tilde{\lambda}_1+\tilde{\eta}'\tau e^{\varsigma}}{cc_1(1-\kappa)^{p-1}}M'_\varepsilon < 0$. Furthermore, $\mathbb{E}\{e^{-\int_{t_0}^{t^*\wedge\tau_l}\tilde{\lambda}(s)ds}|x(t^*\wedge\rho_l)|^p\} < M'_\varepsilon e^{c(t^*-t_0)}$, which contradicts with (7). Thus, for any $t \geq t_0-\tau$, inequality (5) holds. As $\varepsilon \to 0^+$ in (5), it gives that for any $t \geq t_0-\tau$,

$$\mathbb{E}\{e^{-\int_{t_0}^{t\wedge\rho_l}\tilde{\lambda}(s)ds}|x(t\wedge\rho_l)|^p\} \leq M'e^{c(t-t_0)}, \tag{8}$$

where $M' = \max\Big\{\frac{cc_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}}{\tilde{\lambda}_1+\tilde{\eta}'\tau e^{\varsigma}}, M\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}e^{\sup_{t\in[t_0-\tau,t_0]}\int_t^{t_0}\tilde{\lambda}(s)ds}\Big\}$.

By using the Chebyshev's inequality, from (8), we have

$$\begin{aligned}\mathbb{P}\{\rho_l \leq t\} &\leq \frac{\mathbb{E}\Big\{e^{-\int_{t_0}^{t\wedge\rho_l}\tilde{\lambda}(s)ds}|x(t\wedge\rho_l)|^p\Big\}}{\inf_{\{s\in[t_0,t]\}\times\{|x|\geq l\}}\Big[|x|^p e^{-\int_{t_0}^{s}\tilde{\lambda}(u)du}\Big]} \\ &\leq \frac{M'e^{\bar{M}+c(t-t_0)}}{\inf_{\{s\in[t_0,t]\}\times\{|x|\geq l\}}\{|x|^p\}},\end{aligned} \tag{9}$$

where (H_3) is used

As $l \to \infty$ in (9), we have $\mathbb{P}\{\rho_\infty \leq t\} = 0$. Since $t > t_0$ is arbitrary, $\mathbb{P}\{\rho_\infty < \infty\} = 0$, which implies that $\rho_\infty = \infty$ *a.s.*, that is, $\varrho_\infty = \infty$ *a.s.* It means that NSFDE (1) almost surely has a unique global solution $x(t)$ on $[t_0, \infty)$. □

Remark 2: (H_2) is a Lyapunov monotonicity condition. This condition can take many different forms, which have been commonly seen in [12-16, 19-21, 23, 27-30, 32-35] and the references therein. It is seen that the conditions satisfied for $\tilde{\lambda}(t)$ will be given in *Theorem 3.2*, *Theorem 3.3* and *Theorem 3.4*, respectively, which are also suitable in (H_3), see *Remark 3*.

Theorem 3.2: Suppose that all but condition (H_3) of *Theorem 3.1* are satisfied. Moreover, if $\pi = \sup_{t\geq t_0}\{\int_{t_0}^t\tilde{\lambda}(s)ds\} < +\infty$, where $\tilde{\lambda}(t)$ is given in *Theorem 3.1*, then the global trivial solution of NSFDE (1) is stable in pth($p \geq 2$)-moment.

Proof: For NSFDE (1), by using the Itô formula to the Lyapunov-Krasovskii function $e^{-\int_{t_0}^t\lambda_0(s)ds}V(t,\tilde{x}(t))$, for any $t \geq t_0$, we have

$$\begin{aligned}e^{-\int_{t_0}^t\lambda_0(s)ds}V(t,\tilde{x}(t)) &= V(t_0,\tilde{x}(t_0)) + \int_{t_0}^t e^{-\int_{t_0}^s\lambda_0(u)du}[\mathcal{L}V(s,x(s),x_s) - \lambda_0(s)V(s,\tilde{x}(s))]ds \\ &\quad + \int_{t_0}^t e^{-\int_{t_0}^t\lambda_0(s)ds}V_x^T(s,\tilde{x}(s))g(s,x(s),x_s)d\mathcal{B}(s) \\ &\leq V(t_0,\tilde{x}(t_0)) + \int_{t_0}^t e^{-\int_{t_0}^s\lambda_0(u)du}[\lambda_1(s)|x(s)|^p + \int_{-\tau}^0\tilde{\eta}(s,\theta)|x(s+\theta)|^p d\theta]ds \\ &\quad + \int_{t_0}^t e^{-\int_{t_0}^s\lambda_0(u)du}V_x^T(s,\tilde{x}(s))g(s,x(s),x_s)d\mathcal{B}(s),\end{aligned}$$

where condition (H_2) is used.

Taking the mathematical expectation on the preceding inequality and using condition (H_1) in turn, it follows that for any $t \geq t_0$,

$$\mathbb{E}\{|\tilde{x}(t)|^p\} \leq \frac{c_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}}{c_1}e^{\int_{t_0}^t\lambda_0(s)ds}$$

$$+\frac{1}{c_1}\int_{t_0}^{t} e^{\int_s^t \lambda_0(u)du}\lambda_1(s)\mathbb{E}\{|x(s)|^p\}ds$$
$$+\frac{1}{c_1}\int_{t_0}^{t} e^{\int_s^t \lambda_0(u)du}\int_{-\tau}^{0}\tilde{\eta}(s,\theta)\mathbb{E}\{|x(s+\theta)|^p\}d\theta ds. \tag{10}$$

From *Lemma 2.4*, we have

$$\begin{aligned}\mathbb{E}\{|x(t)|^p\} &\le \frac{\mathbb{E}\{|\tilde{x}(t)|^p\}}{(1-\kappa)^{p-1}}+\frac{\mathbb{E}\{|\mathcal{D}(x_t)|^p\}}{\kappa^{p-1}}\\ &\le \frac{\mathbb{E}\{|\tilde{x}(t)|^p\}}{(1-\kappa)^{p-1}}+\kappa\int_{-\tau}^{0}\eta(\theta)\mathbb{E}\{|x(t+\theta)|^p\}d\theta. \end{aligned}\tag{11}$$

By substituting (10) into (11), it yields that for any $t \ge t_0$,

$$\begin{aligned}\mathbb{E}\{|x(t)|^p\} \le\ & \frac{c_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}}{c_1(1-\kappa)^{p-1}}e^{\int_{t_0}^t \lambda_0(s)ds}+\kappa\int_{-\tau}^{0}\eta(\theta)\mathbb{E}\{|x(t+\theta)|^p\}d\theta\\ &+\frac{1}{c_1(1-\kappa)^{p-1}}\int_{t_0}^{t} e^{\int_s^t \lambda_0(u)du}\lambda_1(s)\mathbb{E}\{|x(s)|^p\}ds\\ &+\frac{1}{c_1(1-\kappa)^{p-1}}\int_{t_0}^{t} e^{\int_s^t \lambda_0(u)du}\int_{-\tau}^{0}\tilde{\eta}(s,\theta)\mathbb{E}\{|x(s+\theta)|^p\}d\theta ds, \end{aligned}\tag{12}$$

and for any $t \in [t_0-\tau, t_0]$, $\mathbb{E}\{|x(t)|^p\} \le \bar{M}\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}$, where $\bar{M} \ge 1$.

For any $\varepsilon > 0$, define one constant

$$\hat{M}_\varepsilon = \max\left\{\frac{c_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}+\varepsilon}{c_1(1-\kappa)^{p-1}(1-\kappa e^{\vartheta})}, [\bar{M}\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}+\varepsilon]e^{\sup_{t\in[t_0-\tau,t_0]}\int_t^{t_0}\tilde{\lambda}(s)ds}\right\}. \tag{13}$$

In order to obtain the desired results, we only need to prove that for any $t \ge t_0 - \tau$,

$$\mathbb{E}\{|x(t)|^p\} \le \hat{M}_\varepsilon e^{\int_{t_0}^t \tilde{\lambda}(s)ds}, \tag{14}$$

where $\tilde{\lambda}(t) = \lambda_0(t) + \frac{\lambda_1(t)+\int_{-\tau}^0 \tilde{\eta}(t,\theta)d\theta e^{\varsigma}}{c_1(1-\kappa)^{p-1}(1-\kappa e^{\varsigma})}$.

Now, note that from the definition of $\hat{M}_\varepsilon$ in (13), it yields that for any $t \in [t_0-\tau, t_0]$, $(\bar{M}\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}$ $+\varepsilon)e^{\int_t^{t_0}\tilde{\lambda}(s)ds} \le \hat{M}_\varepsilon$. Moreover, $(\bar{M}\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}+\varepsilon)e^{\int_{t_0-\tau}^{t_0}\tilde{\lambda}(s)ds} \le \hat{M}_\varepsilon e^{\int_{t_0-\tau}^{t}\tilde{\lambda}(s)ds}$. Hence, $\bar{M}\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}+\varepsilon \le \hat{M}_\varepsilon e^{\int_{t_0}^t\tilde{\lambda}(s)ds}$. Then, inequality (14) is satisfied for any $t \in [t_0-\tau, t_0]$. If inequality (14) does not hold for all $t > t_0$, then there exists some t satisfying $t > t_0$ and $\mathbb{E}\{|x(t)|^p\} > \hat{M}_\varepsilon e^{\int_{t_0}^t\tilde{\lambda}(s)ds}$. Letting $\hat{t}^* = \inf\{t > t_0 : \mathbb{E}\{|x(t)|^p\} > \hat{M}_\varepsilon e^{\int_{t_0}^t\tilde{\lambda}(s)ds}\}$. Therefore,

$$\mathbb{E}\{|x(t)|^p\} \le \hat{M}_\varepsilon e^{\int_{t_0}^t \tilde{\lambda}(s)ds}, \quad t \in [t_0-\tau, \hat{t}^*), \tag{15}$$

and

$$\mathbb{E}\{|x(\hat{t}^*)|^p\} = \hat{M}_\varepsilon e^{\int_{t_0}^{\hat{t}^*} \tilde{\lambda}(s)ds}. \tag{16}$$

On the other hand, from (12) and (15), it gives

$$\begin{aligned}
\mathbb{E}\{|x(\hat{t}^*)|^p\} &\leq \frac{c_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}}{c_1(1-\kappa)^{p-1}}e^{\int_{t_0}^{\hat{t}^*}\lambda_0(s)ds}+\kappa\int_{-\tau}^{0}\eta(\theta)\mathbb{E}\{|x(\hat{t}^*+\theta)|^p\}d\theta \\
&\quad+\frac{1}{c_1(1-\kappa)^{p-1}}\int_{t_0}^{\hat{t}^*}e^{\int_s^{\hat{t}^*}\lambda_0(u)du}\lambda_1(s)\mathbb{E}\{|x(s)|^p\}ds \\
&\quad+\frac{1}{c_1(1-\kappa)^{p-1}}\int_{t_0}^{\hat{t}^*}e^{\int_s^{\hat{t}^*}\lambda_0(u)du}\int_{-\tau}^{0}\tilde{\eta}(s,\theta)\mathbb{E}\{|x(s+\theta)|^p\}d\theta ds \\
&\leq \frac{c_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}}{c_1(1-\kappa)^{p-1}}e^{\int_{t_0}^{\hat{t}^*}\lambda_0(s)ds}+\kappa e^{\vartheta}\hat{M}_\varepsilon e^{\int_{t_0}^{\hat{t}^*}\tilde{\lambda}(s)ds} \\
&\quad+\frac{\hat{M}_\varepsilon e^{\int_{t_0}^{\hat{t}^*}\lambda_0(s)ds}}{c_1(1-\kappa)^{p-1}}\int_{t_0}^{\hat{t}^*}\lambda_1(s)e^{\int_{t_0}^{s}[\tilde{\lambda}(u)-\lambda_0(u)]du}ds \\
&\quad+\frac{\hat{M}_\varepsilon e^{\vartheta}e^{\int_{t_0}^{\hat{t}^*}\lambda_0(s)ds}}{c_1(1-\kappa)^{p-1}}\int_{t_0}^{\hat{t}^*}\int_{-\tau}^{0}\tilde{\eta}(s,\theta)d\theta e^{\int_{t_0}^{s}[\tilde{\lambda}(u)-\lambda_0(u)]du}ds \\
&= \frac{c_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}}{c_1(1-\kappa)^{p-1}}e^{\int_{t_0}^{\hat{t}^*}\lambda_0(s)ds}+\kappa e^{\vartheta}\hat{M}_\varepsilon e^{\int_{t_0}^{\hat{t}^*}\tilde{\lambda}(s)ds} \\
&\quad+\hat{M}_\varepsilon(1-\kappa e^{\vartheta})e^{\int_{t_0}^{\hat{t}^*}\lambda_0(s)ds}\int_{t_0}^{\hat{t}^*}e^{\int_{t_0}^{s}[\tilde{\lambda}(u)-\lambda_0(u)]du}[\tilde{\lambda}(s)-\lambda_0(s)]ds \\
&= \left[\frac{c_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}}{c_1(1-\kappa)^{p-1}}-\hat{M}_\varepsilon(1-\kappa e^{\vartheta})\right]e^{\int_{t_0}^{\hat{t}^*}\lambda_0(s)ds} \\
&\quad+\hat{M}_\varepsilon e^{\int_{t_0}^{\hat{t}^*}\tilde{\lambda}(s)ds}. \qquad (17)
\end{aligned}$$

From the definition of $\hat{M}_\varepsilon$ in (13) again, we have

$$\begin{aligned}
&\frac{c_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}}{c_1(1-\kappa)^{p-1}}-\hat{M}_\varepsilon(1-\kappa e^{\vartheta}) \\
\leq\ &\frac{c_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}}{c_1(1-\kappa)^{p-1}}-\frac{c_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}+\varepsilon}{c_1(1-\kappa)^{p-1}} \\
<\ &0. \qquad (18)
\end{aligned}$$

Substituting (18) into (17) yields that $\mathbb{E}\{|x(\hat{t}^*)|^p\} < \hat{M}_\varepsilon e^{\int_{t_0}^{\hat{t}^*}\tilde{\lambda}(s)ds}$, which contradicts with (16). Therefore, inequality (14) holds for any $t \geq t_0-\tau$. As $\varepsilon \to 0^+$ in (14), we have

$$\mathbb{E}\{|x(t)|^p\} \leq \hat{M}e^{\int_{t_0}^{t}\tilde{\lambda}(s)ds}, \qquad (19)$$

for any $t \geq t_0-\tau$, where $\hat{M} = \max\left\{\frac{c_2(1+\kappa)^{p-1}(1+\kappa^p)\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}}{c_1(1-\kappa)^{p-1}(1-\kappa e^{\vartheta})}, \bar{M}\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\}e^{\sup_{t\in[t_0-\tau,t_0]}\int_t^{t_0}\tilde{\lambda}(s)ds}\right\}$.

For any $\epsilon > 0$, $\phi \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau,0];R^n)$, there exists $\delta > 0$ such that $\mathbb{E}\{\sup_{\theta\in[-\tau,0]}|\phi(\theta)|^p\} < \delta$ and $\max\left\{\frac{c_2(1+\kappa)^{p-1}(1+\kappa^p)}{c_1(1-\kappa)^{p-1}(1-\kappa e^{\vartheta})}, \bar{M}e^{\sup_{t\in[t_0-\tau,t_0]}\int_t^{t_0}\tilde{\lambda}(s)ds}\right\}\delta e^{\pi} < \epsilon$. Thus, from (19), for any $t \geq t_0$, we have $\mathbb{E}\{|x(t)|^p\} < \epsilon$, which implies that NSFDE (1) is stable in pth$(p \geq 2)$-moment. □

Theorem 3.3: Suppose that all but condition (H_3) of *Theorem 3.1* hold. Moreover, if $\int_{t_0}^{\infty}\tilde{\lambda}(s)ds = -\infty$, then the global trivial solution of NSFDE (1) is asymptotically stable in pth$(p \geq 2)$-moment.

Proof: If $\int_{t_0}^{\infty}\tilde{\lambda}(t)dt = -\infty$, then $\sup_{t\geq t_0}\int_{t_0}^{t}\tilde{\lambda}(s)ds < +\infty$, which means that NSFDE (1) is stable in pth($p \geq 2$)-moment.

In addition, for any $\epsilon' > 0$, there exists $T' = T'(\phi, \epsilon') > t_0$ such that for any $t > T'$, and $\phi \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$, $e^{\int_{t_0}^{t}\tilde{\lambda}(s)ds} < \frac{\epsilon'}{\hat{M}}$. Therefore, from (19), it gives that for any $t > T'$, $\mathbb{E}\{|x(t)|^p\} < \epsilon'$, which further means that NSFDE (1) is asymptotically stable in pth($p \geq 2$)-moment. □

Theorem 3.4: Suppose that all but condition (H_3) of *Theorem 3.1* hold. Moreover, if there exist two constants $\check{M} \in R$ and $\gamma > 0$ such that for any $t > t_0$, $\int_{t_0}^{t}\tilde{\lambda}(s)ds \leq \check{M} - \gamma(t - t_0)$, then the global trivial solution of NSFDE (1) is exponentially stable in pth($p \geq 2$)-moment.

Proof: From (19), we have

$$\mathbb{E}\{|x(t)|^p\} \leq \hat{M}e^{\check{M}}e^{-\gamma(t-t_0)}, \tag{20}$$

for any $t \geq t_0 - \tau$. Hence, NSFDE (1) is exponentially stable in pth($p \geq 2$)-moment. □

Remark 3: $\pi = \sup_{t\geq t_0}\{\int_{t_0}^{t}\tilde{\lambda}(s)ds\} < \infty$ in *Theorem 3.2* implies that for any $t \geq t_0$, $\int_{t_0}^{t}\tilde{\lambda}(s)ds \leq \pi$. When $\int_{t_0}^{\infty}\tilde{\lambda}(s)ds = -\infty$ in *Theorem 3.3* holds, $\sup_{t\geq t_0}\{\int_{t_0}^{t}\tilde{\lambda}(s)ds\} < +\infty$ is satisfied. If $\int_{t_0}^{t}\tilde{\lambda}(s)ds \leq \check{M} - \gamma(t - t_0)$ ($\check{M} \in R$, $\gamma > 0$) in *Theorem 3.4* holds, then we have $\check{M} = \sup_{t\geq t_0}\{\int_{t_0}^{t}\tilde{\lambda}(s)ds\} < \infty$. Therefore, when ($H_3$) in *Theorem 3.1* is replaced with $\pi = \sup_{t\geq t_0}\{\int_{t_0}^{t}\tilde{\lambda}(s)ds\} < \infty$ in *Theorem 3.2*, $\int_{t_0}^{\infty}\tilde{\lambda}(s)ds = -\infty$ in *Theorem 3.3*, and $\int_{t_0}^{t}\tilde{\lambda}(s)ds \leq \check{M} - \gamma(t - t_0)$ ($\check{M} \in R$, $\gamma > 0$) in *Theorem 3.4*, respectively, it is concluded that the existence and uniqueness of the global trivial solution for NSFDE (1) can be guaranteed.

Remark 4: In *Theorem 3.2*, *Theorem 3.3* and *Theorem 3.4*, we provide different characterizations for the stability in pth($p \geq 2$)-moment, the asymptotical stability in pth($p \geq 2$) -moment, and the exponential stability in pth($p \geq 2$)-moment of the global trivial solution for NSFDE (1), respectively. $\int_{t_0}^{t}\tilde{\lambda}(s)ds \leq \check{M} - \gamma(t - t_0)$ implies that $\int_{t_0}^{\infty}\tilde{\lambda}(s)ds = -\infty$, and $\int_{t_0}^{\infty}\tilde{\lambda}(s)ds = -\infty$ means that $\sup_{t\geq t_0}\{\int_{t_0}^{t}\tilde{\lambda}(s)ds\} < +\infty$, respectively. However, they don't work in reverse. Besides, the sign-changed time-varying coefficient $\lambda_0(t)$ is well embodied in these three different sufficient conditions.

Lemma 3.5: Suppose that all conditions in *Theorem 3.4* are satisfied with $|\lambda_0(t)| \leq \tilde{\lambda}_0$, then we have

$$\int_{t_0}^{\infty} e^{\varepsilon(t-t_0)}|x(t)|^p dt < +\infty,\ a.s., \tag{21}$$

$$\int_{t_0}^{\infty} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(\theta)|x(t+\theta)|^p d\theta dt < +\infty, a.s., \tag{22}$$

and

$$\int_{t_0}^{\infty} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} |x(t+\theta)|^p d\theta dt < +\infty, a.s., \tag{23}$$

where $\varepsilon \in (0, \gamma \wedge \frac{1}{\tau}\log(1/\kappa))$ with $\gamma > 0$ being the exponential decay determined in *Theorem 3.4*.

Proof: For any $T \geq t_0 + \tau$ and $\theta \in [-\tau, 0]$, it implies from (H_1) that

$$\begin{aligned}
&\int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)}\mathbb{E}\{V(t+\theta, \tilde{x}(t+\theta))\}dt \\
\leq\ & \frac{c_2}{(1-\kappa)^{p-1}}\int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)}\mathbb{E}\{|x(t+\theta)|^p\}dt + c_2\kappa\int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)}\int_{-\tau}^{0}\eta(u)\mathbb{E}\{|x(t+\theta+u)|^p\}dudt \\
\leq\ & \frac{c_2\hat{M}e^{\check{M}}e^{\varepsilon\tau}}{\gamma-\varepsilon}\Big[\frac{1}{(1-\kappa)^{p-1}} + \kappa e^{\gamma\tau}\Big],
\end{aligned}$$

where *Lemma 2.4* and inequality (20) are used.

Consequently, when $T \to \infty$ in the preceding inequality, we have

$$\int_{t_0+\tau}^{\infty} e^{\varepsilon(t-t_0)}\mathbb{E}\{V(t+\theta, \tilde{x}(t+\theta))\}dt \leq \frac{c_2\hat{M}e^{\check{M}}e^{\varepsilon\tau}}{\gamma-\varepsilon}\Big[\frac{1}{(1-\kappa)^{p-1}} + \kappa e^{\gamma\tau}\Big] < \infty. \tag{24}$$

By utilizing the Itô formula, it follows from (H_2) and (20) that for any $t_1,\ t_2 \in [t_0+\tau, \infty)$ $(t_2 > t_1)$, and $\theta \in [-\tau, 0]$,

$$\begin{aligned}
&\mathbb{E}\{V(t_2+\theta, \tilde{x}(t_2+\theta))\} - \mathbb{E}\{V(t_1+\theta, \tilde{x}(t_1+\theta))\} \\
=\ & \int_{t_1+\theta}^{t_2+\theta}\mathbb{E}\{\mathcal{L}V(s, x(s), x_s)\}ds \\
\leq\ & \hat{M}e^{\check{M}}[\tilde{\lambda}_0 c_2(1+\kappa)^{p-1}(1+\kappa^p) + \tilde{\lambda}_1 + \tilde{\eta}e^{\gamma\tau}]\int_{t_1+\theta}^{t_2+\theta} e^{-\gamma(s-t_0)}ds \\
\leq\ & \hat{M}e^{\check{M}}[\tilde{\lambda}_0 c_2(1+\kappa)^{p-1}(1+\kappa^p) + \tilde{\lambda}_1 + \tilde{\eta}e^{\gamma\tau}]e^{\gamma\tau}e^{-\gamma(t_1-t_0)}(t_2-t_1),
\end{aligned}$$

where *Lemma 2.5* is used.

Furthermore, for any $\theta \in [-\tau, 0]$, we have

$$\begin{aligned}
&|e^{\varepsilon(t_2-t_0)}\mathbb{E}\{V(t_2+\theta, \tilde{x}(t_2+\theta))\} - e^{\varepsilon(t_1-t_0)}\mathbb{E}\{V(t_1+\theta, \tilde{x}(t_1+\theta))\}| \\
\leq\ & |e^{\varepsilon(t_2-t_0)} - e^{\varepsilon(t_1-t_0)}|\mathbb{E}\{V(t_2+\theta, \tilde{x}(t_2+\theta))\} + e^{\varepsilon(t_1-t_0)} \\
& \times|\mathbb{E}\{V(t_2+\theta, \tilde{x}(t_2+\theta))\} - \mathbb{E}\{V(t_1+\theta, \tilde{x}(t_1+\theta))\}| \\
\leq\ & c_2\varepsilon e^{\varepsilon(\zeta_{t_1,t_2}-t_0)}|t_2-t_1|\mathbb{E}\{|\tilde{x}(t_2+\theta)|^p\} + \hat{M}e^{\check{M}}[\tilde{\lambda}_0 c_2(1+\kappa)^{p-1}(1+\kappa^p) + \tilde{\lambda}_1 + \tilde{\eta}\tau e^{\gamma\tau}]|t_2-t_1| \\
\leq\ & \Big[\Big(\kappa + \frac{1}{(1-\kappa)^{p-1}}\Big)c_2\varepsilon e^{2\gamma\tau} + \hat{M}e^{\check{M}}[\tilde{\lambda}_0 c_2(1+\kappa)^{p-1}(1+\kappa^p) + \tilde{\lambda}_1 + \tilde{\eta}\tau e^{\gamma\tau}]\Big]|t_2-t_1|,
\end{aligned}$$

where $\zeta_{t_1,t_2} \in (t_1, t_2)$, and *Lemma 2.5* is used.

Therefore, it implies that

$$\lim_{t_2\to t_1} e^{\varepsilon(t_2-t_0)}\mathbb{E}\{V(t_2+\theta, \tilde{x}(t_2+\theta))\} = e^{\varepsilon(t_1-t_0)}\mathbb{E}\{V(t_1+\theta, \tilde{x}(t_1+\theta))\},$$

is satisfied for any $\theta \in [-\tau, 0]$.

Then, by using *Lemma 2.6*, it yields that for any $\theta \in [-\tau, 0]$,

$$\lim_{t\to\infty} e^{\varepsilon(t-t_0)}\mathbb{E}\{V(t+\theta, \tilde{x}(t+\theta))\} = 0,$$

holds.

By using the Fubini theorem, it follows from (24) that for any $\theta \in [-\tau, 0]$, $\mathbb{E}\Big\{\int_{t_0+\tau}^{\infty} e^{\varepsilon(t-t_0)} V(t+\theta, \tilde{x}(t+\theta))dt\Big\} < \infty$, which implies that

$$\int_{t_0+\tau}^{\infty} e^{\varepsilon(t-t_0)} V(t+\theta, \tilde{x}(t+\theta))dt < \infty \quad a.s.$$

which follows from (H_1) that

$$\int_{t_0+\tau}^{\infty} e^{\varepsilon(t-t_0)} |\tilde{x}(t+\theta)|^p dt < \infty \quad a.s. \tag{25}$$

Therefore, from (25), we have

$$\int_{t_0+\tau}^{\infty} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(\theta)|\tilde{x}(t+\theta)|^p d\theta dt < \infty, \quad a.s. \tag{26}$$

For any $T > t_0 + \tau$ and $\theta \in [-\tau, 0]$, from *Lemma 2.4*, it yields that

$$\begin{aligned}
&\int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)} |x(t+\theta)|^p dt \\
\leq\;& \frac{1}{(1-\kappa)^{p-1}} \int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)} |\tilde{x}(t+\theta)|^p dt + \frac{1}{\kappa^{p-1}} \int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)} |\mathcal{D}(x_{t+\theta})|^p dt \\
\leq\;& \frac{1}{(1-\kappa)^{p-1}} \int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)} |\tilde{x}(t+\theta)|^p dt + \kappa \int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(u)|x(t+\theta+u)|^p du dt \\
\leq\;& \frac{1}{(1-\kappa)^{p-1}} \int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)} |\tilde{x}(t+\theta)|^p dt + \kappa e^{\varepsilon\tau} \int_{t_0}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(u)|x(t+u)|^p du dt.
\end{aligned} \tag{27}$$

Multiplying $\eta(\theta)$ on both sides of inequality (27), and then integrating from $-\tau$ to 0, it follows which further obtains

$$\begin{aligned}
&\int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(\theta)|x(t+\theta)|^p d\theta dt \\
\leq\;& \frac{1}{(1-\kappa)^{p-1}} \int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(\theta)|\tilde{x}(t+\theta)|^p d\theta dt + \kappa e^{\varepsilon\tau} \int_{t_0}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(\theta)|x(t+\theta)|^p d\theta dt.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(\theta)|x(t+\theta)|^p d\theta dt \;\leq\;& \frac{1}{(1-\kappa)^{p-1}(1-\kappa e^{\varepsilon\tau})} \int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(\theta)|\tilde{x}(t+\theta)|^p d\theta dt \\
&+ \frac{\kappa e^{\varepsilon\tau}}{1-\kappa e^{\varepsilon\tau}} \int_{t_0}^{t_0+\tau} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(\theta)|x(t+\theta)|^p d\theta dt,
\end{aligned}$$

where $\kappa e^{\varepsilon\tau} \in (0, 1)$ is used.

Consequently, it yields

$$\int_{t_0}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(\theta)|x(t+\theta)|^p d\theta dt$$

$$
\begin{aligned}
&= \int_{t_0}^{t_0+\tau} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(\theta)|x(t+\theta)|^p d\theta dt + \int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(\theta)|x(t+\theta)|^p d\theta dt \\
&\leq \frac{1}{(1-\kappa)^{p-1}(1-\kappa e^{\varepsilon\tau})} \int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(\theta)|\tilde{x}(t+\theta)|^p d\theta dt \\
&\quad + \frac{1}{1-\kappa e^{\varepsilon\tau}} \int_{t_0}^{t_0+\tau} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(\theta)|x(t+\theta)|^p d\theta dt, \quad a.s. \qquad (28)
\end{aligned}
$$

When $T \to \infty$ in (28), inequality (26) implies that the inequality (22) is obtained.

Similar to the derivation process of inequality (25), we can obtain

$$
\int_{t_0}^{\infty} e^{\varepsilon(t-t_0)}|\tilde{x}(t)|^p dt < \infty, \quad a.s. \qquad (29)
$$

By using *Lemma 2.4* again, it yields from (22) and (29) that

$$
\begin{aligned}
&\int_{t_0}^{\infty} e^{\varepsilon(t-t_0)}|x(t)|^p dt \\
&\leq \frac{1}{(1-\kappa)^{p-1}} \int_{t_0}^{\infty} e^{\varepsilon(t-t_0)}|\tilde{x}(t)|^p dt + \frac{1}{\kappa^{p-1}} \int_{t_0}^{\infty} e^{\varepsilon(t-t_0)}|\mathcal{D}(x_t)|^p dt \\
&\leq \frac{1}{(1-\kappa)^{p-1}} \int_{t_0}^{\infty} e^{\varepsilon(t-t_0)}|\tilde{x}(t)|^p dt + \kappa \int_{t_0}^{\infty} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(\theta)|x(t+\theta)|^p d\theta dt < \infty, \quad a.s.
\end{aligned}
$$

which means that the inequality (21) is also satisfied.

Taking the integration from $-\tau$ to 0 on both sides of inequalities (25) and (27), respectively, we have

$$
\int_{t_0+\tau}^{\infty} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} |\tilde{x}(t+\theta)|^p d\theta dt < \infty, \quad a.s. \qquad (30)
$$

and

$$
\begin{aligned}
&\int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} |x(t+\theta)|^p d\theta dt \\
&\leq \frac{1}{(1-\kappa)^{p-1}} \int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} |\tilde{x}(t+\theta)|^p d\theta dt + \kappa\tau e^{\varepsilon\tau} \int_{t_0}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(u)|x(t+u)|^p du dt, \quad a.s. \qquad (31)
\end{aligned}
$$

From (31), it follows from *Lemma 2.4* that $T > t_0 + \tau$,

$$
\begin{aligned}
&\int_{t_0}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} |x(t+\theta)|^p d\theta dt \\
&= \int_{t_0}^{t_0+\tau} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} |x(t+\theta)|^p d\theta dt + \int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} |x(t+\theta)|^p d\theta dt \\
&\leq \int_{t_0}^{t_0+\tau} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} |x(t+\theta)|^p d\theta dt + \frac{1}{(1-\kappa)^{p-1}} \int_{t_0+\tau}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} |\tilde{x}(t+\theta)|^p d\theta dt \\
&\quad + \kappa\tau e^{\varepsilon\tau} \int_{t_0}^{T} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} \eta(u)|x(t+u)|^p du dt, \quad a.s. \qquad (32)
\end{aligned}
$$

which implies from (22) and (30) that as $T \to \infty$ in (32),

$$\int_{t_0}^{\infty} e^{\varepsilon(t-t_0)} \int_{-\tau}^{0} |x(t+\theta)|^p d\theta dt < \infty, \quad a.s.$$

The inequality (23) can also be obtained. □

Theorem 3.6: Assume that all conditions of *Lemma 3.5* hold, then the global trivial solution of NSFDE (1) is almost surely exponentially stable.

Proof: By using the Itô formula to the Lyapunov-Krasovskii function $e^{\varepsilon(t-t_0)}V(t, \tilde{x}(t))$, where ε is determined in *Lemma 3.5*, for any $t \geq t_0$, we have

$$\begin{aligned} e^{\varepsilon(t-t_0)}V(t, \tilde{x}(t)) &= V(t_0, \tilde{x}(t_0)) + \int_{t_0}^{t} e^{\varepsilon(s-t_0)}[\varepsilon V(s, \tilde{x}(s)) + \mathcal{L}V(s, x(s), x_s)]ds + \mathcal{M}(t) \\ &\leq V(t_0, \tilde{x}(t_0)) + [c_2(\varepsilon + \tilde{\lambda}_0)(1+\kappa)^{p-1} + \tilde{\lambda}_1] \int_{t_0}^{t} e^{\varepsilon(s-t_0)}|x(s)|^p ds \\ &\quad + c_2(\varepsilon + \tilde{\lambda}_0)(1+\kappa)^{p-1}\kappa^p \int_{t_0}^{t} e^{\varepsilon(s-t_0)} \int_{-\tau}^{0} \eta(\theta)|x(s+\theta)|^p d\theta ds \\ &\quad + \tilde{\eta}' \int_{t_0}^{t} e^{\varepsilon(s-t_0)} \int_{-\tau}^{0} |x(s+\theta)|^p d\theta ds + \mathcal{M}(t) \\ &= \xi_0 + \mathcal{A}(t) + \mathcal{M}(t), \end{aligned} \tag{33}$$

where *Lemma 2.5* is used, $\xi_0 = V(t_0, \tilde{x}(t_0))$, $\mathcal{A}(t) = [c_2(\varepsilon + \tilde{\lambda}_0)(1+\kappa)^{p-1} + \tilde{\lambda}_1] \int_{t_0}^{t} e^{\varepsilon(s-t_0)}|x(s)|^p ds + c_2(\varepsilon + \tilde{\lambda}_0)(1+\kappa)^{p-1}\kappa^p \int_{t_0}^{t} e^{\varepsilon(s-t_0)} \int_{-\tau}^{0} \eta(\theta)|x(s+\theta)|^p d\theta ds + \tilde{\eta}' \int_{t_0}^{t} e^{\varepsilon(s-t_0)} \int_{-\tau}^{0} |x(s+\theta)|^p d\theta ds$, and $\mathcal{M}(t) = \int_{t_0}^{t} e^{\varepsilon(s-t_0)} V_x^T(s, \tilde{x}(s)) g(s, x(s), x_s) d\mathcal{B}(s)$.

Note that ξ_0 is a nonnegative bounded $\mathcal{F}_{t_0}$-measurable random variable, $\mathcal{A}(t) < \infty$ *a.s.* is guaranteed from (21)-(23), and $\mathcal{M}(t)$ is a local continuous martingale with $\mathcal{M}(t_0) = 0$. By using *Lemma 2.7*, it implies from (33) that for any $t \geq t_0$,

$$\limsup_{t\to\infty} e^{\varepsilon(t-t_0)}V(t, \tilde{x}(t)) < \infty \quad a.s.$$

Thus, there exists a finite positive random variable ζ' satisfying

$$V(t, \tilde{x}(t)) \leq \zeta' e^{-\varepsilon(t-t_0)} \quad a.s. \quad \text{on} \quad t \geq t_0. \tag{34}$$

Furthermore, from (H_1) and (34), it follows that for any $t \geq t_0$,

$$|\tilde{x}(t)|^p \leq \frac{\zeta'}{c_1} e^{-\varepsilon(t-t_0)} \quad a.s.$$

The remaining proof can be seen in *Theorem 3.3* in [14] and *Theorem 3.2* in [15]. The detailed derivation process is omitted for brevity. □

When (H_1) holds and $\tilde{\eta}(t, \theta) = \lambda_2(t)\eta(\theta)$ in (H_2), for NSFDE (1), one result is presented as follows:

Corollary 3.7: Assume that *Hypotheses II-III* hold. Suppose that there exist one Lyapunov function $V(\cdot,\cdot) \in C^{1,2}([t_0, +\infty) \times R^n;\ [0, +\infty))$, three functions $\lambda_0(\cdot) :\ [t_0 - \tau, +\infty) \to R$, $\lambda_i(\cdot) :\ [t_0 - \tau, +\infty) \to [0, +\infty)$ with $|\lambda_0(t)| \leq \tilde{\lambda}_0$ $(\tilde{\lambda}_0 > 0)$, $\lambda_i(t) \leq \tilde{\lambda}_i$ $(t \geq t_0,\ \tilde{\lambda}_i > 0,\ i = 1, 2)$, a function

$\eta(\cdot) \in \Gamma([-\tau, 0]; (0, +\infty))$, and two constants $c_i > 0$ $(i = 1, 2)$ such that (H_1) in *Theorem 3.1* holds, and (H_2) in *Theorem 3.1* is replaced by
(H_2') for any $t \geq t_0$, $\phi \in C([-\tau, 0]; R^n)$, and $p \geq 2$,

$$\mathcal{L}V(t, \phi(0), \phi) \leq \lambda_0(t)V(t, \phi(0) - \mathcal{D}(\phi)) + \lambda_1(t)|\phi(0)|^p + \lambda_2(t)\int_{-\tau}^{0} \eta(\theta)|\phi(\theta)|^p d\theta.$$

Then, for any initial value $\phi \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$, we have
(i) if there exists a constant $\breve{M}$ such that for any $t \geq t_0$, $\int_{t_0}^{t} \hat{\lambda}(s)ds \leq \breve{M}$, where $\hat{\lambda}(t) = \lambda_0(t) + \frac{\lambda_1(t)+\lambda_2(t)e^{\hat{\varsigma}}}{c_1(1-\kappa)^{p-1}(1-\kappa e^{\hat{\varsigma}})}$, $\kappa e^{\hat{\varsigma}} \in (0, 1)$, and $\hat{\varsigma} = \sup_{t\geq t_0} \sup_{\theta\in[-\tau,0]} \left\{\int_{t+\theta}^{t}\left[-\lambda_0(s) - \frac{\lambda_1(s)+\lambda_2(t)}{c_1(1-\kappa)^p}\right]ds\right\} < +\infty$, then NSFDE (1) has a unique global solution on $[t_0, +\infty)$;
(ii) if $\hat{\pi} = \sup_{t\geq t_0}\{\int_{t_0}^{t} \hat{\lambda}(s)ds\} < +\infty$, then the global trivial solution of NSFDE (1) is stable in pth$(p \geq 2)$-moment;
(iii) if $\int_{t_0}^{\infty} \hat{\lambda}(s)ds = -\infty$, then the global trivial solution of NSFDE (1) is asymptotically stable in pth$(p \geq 2)$-moment;
(iv) if there exist two constants $\hat{M} \in R$ and $\hat{\beta} > 0$ such that for any $t \geq t_0$, $\int_{t_0}^{t} \hat{\lambda}(s)ds \leq \hat{M} - \hat{\beta}(t - t_0)$, then the global trivial solution of NSFDE (1) is exponentially stable in pth$(p \geq 2)$-moment and almost surely exponentially stable.

Corollary 3.8: Assume that *Hypothesis I* and *Hypothesis III* hold. Suppose that there exist one Lyapunov function $V(\cdot, \cdot) \in C^{1,2}([t_0, +\infty) \times R^n;\ [0, +\infty))$, three functions $\lambda_0(\cdot) :\ [t_0 - \tau, +\infty) \to R$, $\lambda_i(\cdot) :\ [t_0 - \tau, +\infty) \to [0, +\infty)$ $(i = 1, 2)$, and two constants $c_i > 0$ $(i = 1, 2)$ such that (H_1) in *Theorem 3.1* holds, and (H_2) in *Theorem 3.1* is replaced by
(H_2'') for any $t \geq t_0$, $\phi \in C([-\tau, 0]; R^n)$, and $p \geq 2$,

$$\mathbb{E}\{\mathcal{L}V(t, \phi(0), \phi)\} \leq \lambda_0(t)\mathbb{E}\{V(t, \phi(0) - \mathcal{D}(\phi))\} + \lambda_1(t)\mathbb{E}\{|\phi(0)|^p\} + \lambda_2(t) \sup_{\theta\in[-\tau,0]} \mathbb{E}\{|\phi(\theta)|^p\}.$$

Then, for any initial value $\phi \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$, we have
(i) if $\hat{\pi} = \sup_{t\geq t_0}\{\int_{t_0}^{t} \hat{\lambda}(s)ds\} < +\infty$, where $\hat{\lambda}(t)$ is given in *Corollary 3.7*, then the global trivial solution of NSFDE (1) is stable in pth$(p \geq 2)$-moment;
(ii) if $\int_{t_0}^{\infty} \hat{\lambda}(s)ds = -\infty$, then the global trivial solution of NSFDE (1) is asymptotically stable in pth$(p \geq 2)$-moment;
(iii) if there exist two constants $\hat{M} \in R$ and $\hat{\beta} > 0$ such that for any $t \geq t_0$, $\int_{t_0}^{t} \hat{\lambda}(s)ds \leq \hat{M} - \hat{\beta}(t - t_0)$, then the global trivial solution of NSFDE (1) is exponentially stable in pth$(p \geq 2)$-moment;
(iv) moreover, if there exist three constants $\tilde{\lambda}_0 > 0$, $\tilde{\lambda}_i > 0$ $(i = 1, 2)$ such that $|\lambda_0(t)| \leq \tilde{\lambda}_0$ and $\lambda_i(t) \leq \tilde{\lambda}_i$, then the global trivial solution of NSFDE (1) is almost surely exponentially stable.

When $\mathcal{D}(\cdot) = 0$, NSFDE (1) becomes the following SFDE:

$$dx(t) = f(t, x(t), x_t)dt + g(t, x(t), x_t)d\mathcal{B}(t), \quad t \geq t_0, \tag{35}$$

with the initial value $x_{t_0} = \phi = \{\phi(\theta) :\ -\tau \leq \theta \leq 0\} \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$.

Similar to the proofs given in *Theorem 3.1*, *Theorem 3.2*, *Theorem 3.3*, *Theorem 3.4* and *Theorem 3.6*, for SFDE (35), we have

Corollary 3.9: Assume that *Hypothesis II* holds. Suppose that there exist one Lyapunov function $V(\cdot, \cdot) \in C^{1,2}([t_0, +\infty) \times R^n;\ [0, +\infty))$, one function $\lambda_0(\cdot) :\ [t_0 - \tau, +\infty) \to R$, with $|\lambda_0(t)| \leq \tilde{\lambda}_0$

($\tilde{\lambda}_0 > 0$), a function $\bar{\eta}(\cdot,\cdot): [t_0,+\infty)\times[-\tau,0] \to [0,+\infty)$ with $\bar{\eta}(t,s) \le \bar{\eta}'$ ($\bar{\eta}' > 0$), and two constants $c_i > 0$ ($i = 1,2$) such that
($\tilde{H}_1$) for any $t \ge t_0$, $x \in R^n$, and $p \ge 2$,

$$c_1|x|^p \le V(t,x) \le c_2|x|^p;$$

($\tilde{H}_2$) for any $t \ge t_0$, $\phi \in C([-\tau,0];R^n)$, and $p \ge 2$,

$$\mathcal{L}V(t,\phi(0),\phi) \le \lambda_0(t)V(t,\phi(0)) + \int_{-\tau}^{0} \bar{\eta}(t,\theta)V(t+\theta,\phi(\theta))d\theta.$$

Then, for any initial value $\phi \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau,0];R^n)$, we have
(i) if there exists a constant $\check{M}$ such that for any $t \ge t_0$, $\int_{t_0}^{t}\bar{\lambda}(s)ds \le \check{M}$, where $\bar{\lambda}(t) = \lambda_0(t) + \int_{-\tau}^{0}\bar{\eta}(t,\theta)d\theta e^{\bar{\varsigma}}$, and $\bar{\varsigma} = \sup_{t\ge t_0}\sup_{\theta\in[-\tau,0]}\left\{\int_{t+\theta}^{t}[-\lambda_0(s) - \int_{-\tau}^{0}\bar{\eta}(s,\theta)d\theta]ds\right\} < +\infty$, then SFDE (35) has a unique global solution on $[t_0,+\infty)$;
(ii) if $\bar{\pi} = \sup_{t\ge t_0}\{\int_{t_0}^{t}\bar{\lambda}(s)ds\} < +\infty$, then the global trivial solution of SFDE (35) is stable in pth($p \ge 2$)-moment;
(iii) if $\int_{t_0}^{\infty}\bar{\lambda}(s)ds = -\infty$, then the global trivial solution of SFDE (35) is asymptotically stable in pth($p \ge 2$)-moment;
(iv) if there exist two constants $\bar{M} \in R$ and $\bar{\beta} > 0$ such that for any $t \ge t_0$, $\int_{t_0}^{t}\bar{\lambda}(s)ds \le \bar{M} - \bar{\beta}(t-t_0)$, then the global trivial solution of SFDE (35) is exponentially stable in pth($p \ge 2$)-moment and almost surely exponentially stable.

Corollary 3.10: Assume that all conditions of *Corollary 3.9* hold with $\bar{\eta}(t,\theta) = \lambda_1(t)\eta(\theta)$, $\lambda_1(\cdot): [t_0-\tau,+\infty) \to [0,+\infty)$ and $\lambda_1(t) \le \tilde{\lambda}_1'$ ($\tilde{\lambda}_1' > 0$) in ($\tilde{H}_2$), then for SFDE (35), then the results given in *Corollary 3.9* can be guaranteed with $\bar{\lambda}(t) = \lambda_0(t) + \lambda_1(t)e^{\tilde{\varsigma}}$ and $\tilde{\varsigma} = \sup_{t\ge t_0}\sup_{\theta\in[-\tau,0]}\{\int_{t+\theta}^{t}[-\lambda_0(s) - \lambda_1(s)]ds\} < +\infty$.

Corollary 3.11: Assume that *Hypothesis I* holds. Suppose that there exist one Lyapunov function $V(\cdot,\cdot) \in C^{1,2}$ ($[t_0,+\infty)\times R^n$; $[0,+\infty)$), two functions $\lambda_0(\cdot): [t_0-\tau,+\infty) \to R$, and $\lambda_1(\cdot)$: $[t_0-\tau,+\infty) \to [0,+\infty)$, and two constants $c_i > 0$ ($i = 1,2$) such that ($\tilde{H}_1$) in *Corollary 3.9* holds and ($\tilde{H}_2$) in *Corollary 3.9* is replaced by
($\tilde{H}_2'$) for any $t \ge t_0$, $\phi \in C([-\tau,0];R^n)$, and $p \ge 2$,

$$\mathbb{E}\{\mathcal{L}V(t,\phi(0),\phi)\} \le \lambda_0(t)\mathbb{E}\{|\phi(0)|^p\} + \lambda_1(t)\sup_{\theta\in[-\tau,0]}\mathbb{E}\{|\phi(\theta)|^p\},$$

Then, for any initial value $\phi \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau,0];R^n)$, we have
(i) if $\bar{\pi} = \sup_{t\ge t_0}\{\int_{t_0}^{t}\bar{\lambda}(s)ds\} < +\infty$, where $\bar{\lambda}(t)$ is given in *Corollary 3.10*, then the global trivial solution of SFDE (35) is stable in pth($p \ge 2$)-moment;
(ii) if $\int_{t_0}^{\infty}\bar{\lambda}(s)ds = -\infty$, then the global trivial solution of SFDE (35) is asymptotically stable in pth($p \ge 2$)-moment;
(iii) if there exist two constants $\bar{M} \in R$ and $\bar{\beta} > 0$ such that for any $t \ge t_0$, $\int_{t_0}^{t}\bar{\lambda}(s)ds \le \bar{M} - \bar{\beta}(t-t_0)$, then the global trivial solution of SFDE (35) is exponentially stable in pth($p \ge 2$)-moment;
(iv) moreover, if there exist two positive constants $\tilde{\lambda}_0'$, $\tilde{\lambda}_1' > 0$ such that $|\lambda_0(t)| \le \tilde{\lambda}_0'$ and $\lambda_1(t) \le \tilde{\lambda}_1'$, then the global trivial solution of SFDE (35) is almost surely exponentially stable.

4. Discussion and comparison

Note that when $f(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot,\cdot)$ satisfy the globally Lipschitz condition, the results given in *Theorem 3.2*, *Theorem 3.3*, *Theorem 3.4*, *Theorem 3.6*, and the results (ii)-(iv) presented in *Corollary 3.7* also hold.

For NSFDE (1), in [21], when the globally Lipschitz condition holds for $f(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot,\cdot)$, Ngoc has obtained one valuable result presented as follows:

Theorem 4.1 ([21]): Assume that there exist $\kappa \in (0,1)$, a Borel measurable function $\eta(\cdot) \in \Gamma([-\tau,0];(0,+\infty))$, two locally bounded Borel-measurable functions $\gamma(\cdot):\ [t_0,+\infty) \to R$, $\zeta(\cdot,\cdot):\ [t_0,+\infty)\times[-\tau,0] \to [0,+\infty)$ such that for any $t \geq t_0$ and $\phi \in C([-\tau,0];R^n)$,

$$|\mathcal{D}(\phi)|^2 \leq \kappa^2 \int_{-\tau}^{0} \eta(s)|\phi(s)|^2 ds \tag{36}$$

and

$$2(\phi(0)-\mathcal{D}(\phi))^T f(t,\phi(0),\phi) + |g(t,\phi(0),\phi)|^2 \leq \gamma(t)|\phi(0)-\mathcal{D}(\phi)|^2 + \int_{-\tau}^{0}\zeta(t,s)|\phi(s)|^2 ds. \tag{37}$$

If for any $t \geq t_0$, inequality

$$\gamma(t) + \frac{1}{(1-\kappa e^{\frac{\beta\tau}{2}})^2}\int_{-\tau}^{0}\zeta(t,\theta)e^{-\beta\theta}d\theta \leq -\beta, \tag{38}$$

holds for some $\beta \in (0, -\frac{2}{\tau}\ln\kappa)$, then the trivial solution of NSFDE (1) is exponentially stable in mean square.

Inequality (36) is *Hypothesis III* with $p=2$. When $V(t,\phi(0)-\mathcal{D}(\phi)) = |\phi(0)-\mathcal{D}(\phi)|^2$ in *Theorem 3.1*, condition (H_2) with $\lambda_0(t)=\gamma(t)$ and $\lambda_1(t)=0$ is inequality (37). In order to guarantee that inequality (38) holds, the value of $\gamma(t)$ is less than zero. Hence, in *Theorem 4.1*, $\gamma(t) \in R$ cannot be guaranteed. In [21], the exponential stability in mean square for NSFDE (1) was only discussed. In *Theorems 3.1-3.4*, when $f(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot,\cdot)$ satisfy *Hypothesis II*, the existence and uniqueness, the stability in pth($p \geq 2$)-moment, the asymptotic stability in pth($p\geq 2$)-moment and the exponential stability in pth($p\geq 2$)-moment for the global solution of NSFDE (1) have been considered, respectively, and three different characterizations of the sufficient conditions on these three kinds of stochastic stability in moment have been provided.

When for any $\phi \in C([-\tau,0];R^n)$, $f(t,\phi(0),\phi) = f_0(t,\phi(0)) + f_1(t,\phi)$ and $g(t,\phi(0),\phi) = g(t,\phi)$ in NSFDE (1) with $f_0(\cdot,\cdot):\ [t_0,+\infty)\times R^n \to R^n$ and $f_1(\cdot,\cdot):\ [t_0,+\infty)\times C([-\tau,0];R^n) \to R^n$ satisfying the globally Lipschitz condition, in [10, 13], Mao have taken the lead study on the stochastic stability in moment for NSFDE (1), and developed two pioneering works as follows.

Theorem 4.2 ([10]): Suppose that there exists $\kappa \in (0,1)$ such that for any $\phi \in C([-\tau,0];R^n)$,

$$\mathbb{E}|\mathcal{D}(\phi)|^2 \leq \kappa^2 \sup_{\theta\in[-\tau,0]}\mathbb{E}|\phi(\theta)|^2. \tag{39}$$

Assume that there exists two constants $\lambda_1 > 0$, $\lambda_2 > 0$ such that for any $t \geq t_0$, and $\phi \in C([-\tau,0];R^n)$,

$$2\mathbb{E}\{(\phi(0)-\mathcal{D}(\phi))^T[f_0(t,\phi(0)) + f_1(t,\phi)] + |g(t,\phi)|^2\} \leq -\lambda_1\mathbb{E}\{|\phi(0)|^2\} + \lambda_2 \sup_{\theta\in[-\tau,0]}\mathbb{E}\{|\phi(\theta)|^2\}. \tag{40}$$

Then, the trivial solution of NSFDE (1) is exponentially stable in means square provided that $\kappa \in (0, \frac{1}{2})$ and $\lambda_1 > \frac{\lambda_2}{(1-2\kappa)^2}$.

Theorem 4.3 ([13]): Assume that inequality (36) in *Theorem 4.1* is satisfied, and there exist two constants $\lambda_1 > \lambda_2 \geq 0$, and a Borel measurable function $\eta(\cdot) \in \Gamma$ such that for any $t \geq t_0$, and $\phi \in C([-\tau, 0]; R^n)$,

$$2(\phi(0) - \mathcal{D}(\phi))^T[f_0(t, \phi(0)) + f_1(t, \phi)] + |g(t, \phi)|^2 \leq -\lambda_1|\phi(0)|^2 + \lambda_2 \int_{-\tau}^{0} \eta(s)|\phi(s)|^2 ds. \tag{41}$$

Then the trivial solution of NSFDE (1) is exponentially stable in mean square.

Since $\lambda_1 > \lambda_2$ and inequality (41) are sharper than $\lambda_1 > \frac{\lambda_2}{(1-2\kappa)^2}$ and inequality (40), respectively, compared with the results in [10], the results obtained in [13] are conservative. If condition (H_2'') in *Corollary 3.8* is further estimated as inequality (40) or inequality (41), then the important information of time-varying parameters $\lambda_i(t)$ $(i = 0, 1, 2)$ in *Corollary 3.8* may be lost. On the other hand, since the time-varying parameter $\lambda_0(t)$ is sign-changed, it is very difficult that inequality (40) and inequality (41) are estimated from condition (H_2') in *Corollary 3.8*, respectively.

For SFDE (35), in [34], when the globally Lipschitz condition holds for $f(\cdot, \cdot, \cdot)$ and $g(\cdot, \cdot, \cdot)$, Li has obtained the following useful results:

Theorem 4.4 (see *Theorems 3.1-3.3* in [34]): Assume that there exist a Lyapunov-Krasovskii function $V(\cdot, \cdot) \in C^{1,2}([t_0, +\infty) \times R^n; [0, +\infty))$, two constants $c_1 > 0$, $c_2 > 0$, two functions $\lambda_0(\cdot) : [t_0, +\infty) \to [-a, +\infty)$ $(a > 0)$, $\lambda_1(\cdot) : [t_0, +\infty) \to [0, +\infty)$ such that
(A_1) for any $t \geq t_0$, and $x \in R^n$, $c_1|x|^p \leq V(t, x) \leq c_2|x|^p$;
(A_2) for any $t \geq t_0$, and $\phi \in C([-\tau, 0]; R^n)$, $\mathbb{E}\{\mathcal{L}V(t, \phi(0), \phi)\} \leq \lambda_0(t)\mathbb{E}\{V(t, \phi(0))\} + \lambda_1(t) \sup_{\theta \in [-\tau, 0]} \mathbb{E}\{V(t + \theta, \phi(\theta))\}$.

Then, for any initial value $\phi \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$, we have
(i) if $\sup_{t \geq t_0} \int_{t_0}^{t} \tilde{\lambda}(s)ds < +\infty$, where $\tilde{\lambda}(t) = \lambda_0(t) + \lambda_1(t)e^{a\tau}$, then the trivial solution of SFDE (35) is stable in pth$(p \geq 2)$-moment;
(ii) if $\lim_{t \to \infty} \int_{t_0}^{t} \tilde{\lambda}(s)ds = -\infty$, then the trivial solution of SFDE (35) is asymptotically stable in pth$(p \geq 2)$-moment;
(iii) if there exist two positive constants σ and ν such that for any $k \in \mathbb{N}$, $\int_{t^*+k\nu}^{t^*+(k+1)\nu} [\lambda_0(s) + \lambda_1(s)]ds \leq -\sigma$ holds for some $t^* \geq t_0$, then the trivial solution of SFDE (35) is exponentially stable in pth$(p \geq 2)$-moment.

Since $\lambda_0(\cdot) : [t_0, +\infty) \to [-a, +\infty)$ $(a > 0)$ and $\lambda_1(\cdot) : [t_0, +\infty) \to [0, +\infty)$, $\sup_{t \geq t_0} \sup_{\theta \in [-\tau, 0]} \{\int_{t+\theta}^{t} [-\lambda_0(s) - \lambda_1(s)]ds\} \leq a\tau$. In *Theorem 4.4*, when the condition in (iii) holds, there exist two constants $\bar{M} \in R$ and $\bar{\beta} > 0$ such that for any $t \geq t_0$, $\int_{t_0}^{t} \tilde{\lambda}(s)ds \leq \bar{M} - \bar{\beta}(t - t_0)$. The detailed reasoning process is given with *Theorem 3.3* in [34]. Thus, the sufficient conditions in *Theorem 4.4* are more conservative than ones given in *Corollary 3.11*.

For SFDE (35), in [33], Wu et al have derived the following valuable results:

Theorem 4.5 (see *Theorems 1-2* in [33]): Assume that there exist a Lyapunov-Krasovskii function $V(\cdot, \cdot) \in C^{1,2}([t_0, +\infty) \times R^n; [0, +\infty))$, two constants $c_1 > 0$, $c_2 > 0$, two functions $\lambda_0(\cdot) : [t_0, +\infty) \to R$, $\lambda_1(\cdot) : [t_0, +\infty) \to [0, +\infty)$ such that
(A_1) for any $t \geq t_0$, and $x \in R^n$, $c_1|x|^p \leq V(t, x) \leq c_2|x|^p$;
(A_2) for any $t \geq t_0$, and $\phi \in C([-\tau, 0]; R^n)$, $\mathbb{E}\{\mathcal{L}V(t, \phi(0), \phi)\} \leq \lambda_0(t)\mathbb{E}\{V(t, \phi(0))\} + \lambda_1(t) \sup_{\theta \in [-\tau, 0]} \mathbb{E}\{V(t + \theta, \phi(\theta))\}$.

Then, for any initial value $\phi \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$, we have

(i) if $\sup_{t\geq t_0+\tau}\int_{t_0+\tau}^t \hat{\lambda}(s)ds < +\infty$, where $\hat{\lambda}(t) = \lambda_0(t) + \lambda_1(t)e^{\varsigma}$ and $\sup_{\theta\in[-\tau,0]}\sup_{t\geq t_0+\tau}\int_{t+\theta}^t[-\lambda_0(u) - \lambda_1(u)]du \leq \varsigma$, then the trivial solution of SFDE (35) is stable in pth($p \geq 2$)-moment;

(ii) if $\int_{t_0+\tau}^{\infty} \hat{\lambda}(s)ds = -\infty$, then the trivial solution of SFDE (35) is asymptotically stable in pth($p \geq 2$)-moment;

(iii) if there exists a positive constant ϵ such that $\sup_{t\geq t_0+\tau}\int_{t_0+\tau}^t[\hat{\lambda}(s) + \epsilon]ds < +\infty$, then the trivial solution of SFDE (35) is exponentially stable in pth($p \geq 2$)-moment.

It is seen that if conditions (i), (ii) and (iii) in *Theorem 4.5* hold, then all conditions in *Corollary 3.11* are also satisfied, respectively. Besides, *Theorem 4.5* was obtained by using the comparison principle and the Lyapunov-Krasovskii function, when the globally Lipschitz condition is satisfied for $f(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot,\cdot)$. In this paper, one new methodology is provided, which can analyze the stability in pth($p \geq 2$)-moment, asymptotical stability in pth($p \geq 2$)-moment and exponential stability in pth($p \geq 2$)-moment for time-varying SFDE (35), and present three different characterizations of the sufficient conditions on these three kinds of stochastic stability in moment when the Lyapunov monotonicity condition also has a time-varying sign-changed coefficient.

5. Examples

Example 5.1: Let $\mathcal{B}(t)$ be a scalar Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Consider one dimensional NSFDE:

$$d[x(t) - \mathcal{D}(x_t)] = f(t, x(t), x_t)dt + g(t, x(t), x_t)d\mathcal{B}(t), \quad t \geq 0, \tag{42}$$

with the initial value $x_0(\theta) = \phi(\theta)$ ($\theta \in [-1, 0]$), $\phi \in \mathcal{L}^2_{\mathcal{F}_0}([-1, 0]; R^n)$. For any $\theta \in [-1, 0]$, $x_t(\theta) = x(t + \theta)$. In (42), it is assumed that $\mathcal{D}(\cdot): C([-1, 0]; R) \to S$, $f(\cdot,\cdot,\cdot): [0, +\infty) \times R \times C([-1, 0]; R) \to R$, and $g(\cdot,\cdot,\cdot): [0, +\infty) \times R \times C([-1, 0]; R) \to R$ with

$$\mathcal{D}(\phi) = 0.1\int_{-1}^{0}\eta(\theta)\phi(\theta)d\theta,$$

$$\begin{aligned} f(t, \phi(0), \phi) \;=\;& (0.75\sin^2(t) - 0.6)\phi(0) - 0.1\sin^2(t)\phi^3(0) \\ & +\cos^2(t)\mathcal{D}(\phi) + 0.1\sin^2(t)\phi^2(0)\mathcal{D}(\phi), \end{aligned}$$

and

$$g(t, \phi(0), \phi) = 0.2\sin(t)\phi(0),$$

where $\eta(\cdot) \in \Gamma([-1, 0]; (0, +\infty))$.

For NSFDE (42), one Lyapunov function is $V(t, x) = x^2$. Then, the Lyapunov monotonicity condition is estimated as

$$\begin{aligned} \mathcal{L}V(t, \phi(0), \phi) \;=\;& 2[\phi(0) - \mathcal{D}(\phi)][(0.75\sin^2(t) - 0.6)\phi(0) - 0.1\sin^2(t)\phi^3(0) + \cos^2(t)\mathcal{D}(\phi) \\ & +0.1\sin^2(t)\phi^2(0)\mathcal{D}(\phi)] + 0.04\sin^2(t)\phi^2(0) \\ \leq\;& [1.5\sin^2(t) - 1.2]|\phi(0) - \mathcal{D}(\phi)|^2 + [0.19 + 0.21\cos^2(t)]|\phi(0)|^2 \\ & +[0.15 + 0.25\cos^2(t)]|\mathcal{D}(\phi)|^2 \\ =\;& \lambda_0(t)|\phi(0) - \mathcal{D}(\phi)|^2 + \lambda_1(t)|\phi(0)|^2 + \lambda_2(t)\int_{-1}^{0}\eta(\theta)|\phi(\theta)|^2d\theta, \end{aligned}$$

where $\lambda_0(t) = -0.45-0.75\cos(2t)$, $\lambda_1(t) = 0.295+0.105\cos(2t)$, and $\lambda_2(t) = 0.00275+0.00125\cos(2t)$.

Hence, we have $\hat{\varsigma} = \sup_{t\geq 0}\sup_{\theta\in[-1,0]}\left\{\int_{t+\theta}^{t}\left[-\lambda_0(s)-\frac{\lambda_1(s)+\lambda_2(t)}{(1-\kappa)^p}\right]ds\right\} = \sup_{t\geq 0}\sup_{\theta\in[-1,0]}\{\int_{t+\theta}^{t}[-0.0824\theta+0.6188\cos(2s+\frac{\theta}{2})\sin(\frac{\theta}{2})]ds\} = 0.3790$, and $0.1e^{\hat{\varsigma}} = 0.14608$. Thus, $\hat{\lambda}(t) = \lambda_0(t) + \frac{\lambda_1(t)+\lambda_2(t)e^{\hat{\varsigma}}}{(1-\kappa)^{p-1}(1-\kappa e^{\hat{\varsigma}})} = -0.061 - 0.611\cos(2t)$. Furthermore, for any $t \geq 0$, $\int_0^t \hat{\lambda}(s)ds = -0.061t - 0.305\sin(2t) \leq 0.305 - 0.061t$. In addition, $\sup_{t\geq 0}\{\int_0^t \hat{\lambda}(s)ds\} \leq 0.305$ and $\int_0^\infty \hat{\lambda}(t)dt = -\infty$. Thus, all conditions in *Corollary 3.7* are satisfied. Hence, It is concluded from *Corollary 3.7* that NSFDE (42) has a unique global solution $x(t)$ on $[0, +\infty)$, which is stable in mean square, asymptotically stable in mean square, exponentially stable in mean square and almost surely exponentially stable, respectively.

Example 5.2: One coupled system consisting of a mass-spring-damper (MSD) model and a pendulum was analyzed in [40]. The pendulum is taken to be physical structure, and the MSD is numerically modeled. An actuator is taken to a transfer system. The mathematical expression of the system is NDDEs, which are written as

$$M\ddot{z}(t) + c\dot{z}(t) + kz(t) + m\ddot{z}(t-\tau_0) = 0 \tag{43}$$

on $t \geq 0$, where M, c, k are the mass, stiffness and damping of a mass-spring-damper model, m is the mass of a pendulum with $M = 1$, $c = 2$, $k = 3$ and $m = 0.05$, $\tau_0 = 0.05$ is the constant delay, and $z(t)$, $\dot{z}(t)$, $\ddot{z}(t)$ denote the position, velocity and acceleration of MSD model at time t. If this physical model is affected by the external force, then Eq. (43) is further described as

$$M\ddot{z}(t) + c\dot{z}(t) + kz(t) + m\ddot{z}(t-\tau_0) + F(t) = 0 \tag{44}$$

on $t \geq 0$, where $F(t)$ denotes the external force subject to the environmental noise, which is characterized by

$$F(t) = \sigma_1(t, z(t), z(t-\tau_0), z(t), z(t-\tau_0), z_t, \dot{z}_t) + \sigma_2(t, z(t), \dot{z}(t))\dot{\mathscr{B}}(t)$$

where $\dot{\mathscr{B}}(t)$ is a scalar white noise (i.e. $\mathscr{B}(t)$ is a scalar Brownian motion),

$$\begin{aligned}
&\sigma_1(t, z(t), z(t-\tau_0), \dot{z}(t), \dot{z}(t-\tau_0), z_t, \dot{z}_t)\\
= \;& a_1\cos^2(t)z(t) + a_2\sin^2(t)\int_{-\tau_1}^{0}\eta(\theta)\dot{z}(t+\theta)d\theta + \sin^2(t)[3z(t)+2\dot{z}(t)]^3\int_{-\tau_1}^{0}\eta(\theta)|z(t+\theta)|^2 d\theta\\
&+0.2\sin^2(t)[3z(t)+2\dot{z}(t)]^2\dot{z}(t-\tau_0)\int_{-\tau_1}^{0}\eta(\theta)|z(t+\theta)|^2 d\theta\\
&+\cos^2(t)[3z(t)+2\dot{z}(t)]\dot{z}^2(t-\tau_0)\int_{-\tau_1}^{0}\eta(\theta)|\dot{z}(t+\theta)|^2 d\theta + 0.2\cos^2(t)\dot{z}^3(t-\tau_0)\int_{-\tau_1}^{0}\eta(\theta)|\dot{z}(t+\theta)|^2 d\theta,
\end{aligned}$$

and

$$\sigma_2(t, z(t), \dot{z}(t)) = a_3\sin(t)z(t) + a_4\sin(t)\dot{z}(t),$$

where $\eta(\cdot) \in \Gamma([-\tau_1, 0]; (0, +\infty))$ $(\tau_1 > 0)$.

Let $x_1(t) = z(t)$ and $x_2(t) = \dot{z}(t)$, Eq. (44) can be written as a two-dimensional NSFDE:

$$d[x(t) - \mathscr{D}(x_t)] = f(t, x(t), x_t)dt + g(t, x(t), x_t)d\mathscr{B}(t), \tag{45}$$

where $x(t) = \mathrm{col}[x_1(t), x_2(t)]$,

$$\mathcal{D}(x_t) = Dx_t(-\tau) = Dx(t-\tau_0),$$

$$f(t, x(t), x_t) = \begin{bmatrix} f_1(t, x(t), x_t) \\ f_2(t, x(t), x_t) \end{bmatrix}, \quad g(t, x(t), x_t) = \begin{bmatrix} g_1(t, x(t), x_t) \\ g_2(t, x(t), x_t) \end{bmatrix},$$

with $f_1(t, x(t), x_t) = x_2(t)$, $g_1(t, x(t), x_t) = 0$,

$$D = \begin{bmatrix} 0 & 0 \\ 0 & -0.05 \end{bmatrix},$$

$$\begin{aligned} f_2(t, x(t), x_t) &= -2x_1(t) - 3x_3(t) - a_1\cos^2(t)x_1(t) - 1.5a_2\sin^2(t)\int_{-\tau_1}^{0}\eta(\theta)x_1(t+\theta)d\theta \\ &\quad -a_2\sin^2(t)\int_{-\tau_1}^{0}\eta(\theta)x_2(t+\theta)d\theta - \sin^2(t)[3x_1(t)+2x_2(t)]^3\int_{-\tau_1}^{0}\eta(\theta)|x_1(t+\theta)|^2d\theta \\ &\quad -0.02\sin^2(t)[3x_1(t)+2x_2(t)]^2x_2(t-\tau_0)\int_{-\tau_1}^{0}\eta(\theta)|x_1(t+\theta)|^2d\theta \\ &\quad -\cos^2(t)[3x_1(t)+2x_2(t)]x_2^2(t-\tau_0)\int_{-\tau_1}^{0}\eta(\theta)|x_2(t+\theta)|^2d\theta \\ &\quad -0.02\cos^2(t)x_2^3(t-\tau_0)\int_{-\tau_1}^{0}\eta(\theta)|x_2(t+\theta)|^2d\theta, \end{aligned}$$

and

$$g_2(t, x(t), x_t) = -a_3\sin(t)x_1(t) - a_4\sin(t)x_2(t).$$

Since matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ can be diagonalized, there exists a matrix $G = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ with $G^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ such that $G^{-1}AG = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, and $Q = (G^{-1})^TG^{-1} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$. Then, for NSFDE (45), the Lyapunov-Krasovskii function is chosen as $V(t, x) = x^TQx$ ($x \in R^2$), and the Lyapunov monotonicity condition is estimated as

$$\begin{aligned} \mathcal{L}V(t, x(t), x_t) &\leq [-2 + 4|a_1|\cos^2(t)]V(t, x(0) - Dx(t-\tau_0)) + [0.25 + |a_2|\sin^2(t) \\ &\quad + \sqrt{[4(a_3-a_4)^2 + 9(a_3-2a_4)^2][(a_3-a_4)^2 + (a_3-2a_4)^2]}\sin^2(t)]|x(t)|^2 \\ &\quad +[0.25 + 0.3162|a_2|\sin^2(t)]|x(t-\tau_0)|^2 + 0.3162|a_2|\sin^2(t)\int_{-\tau_1}^{0}\eta(\theta)|x(t+\theta)|^2d\theta, \end{aligned}$$

where and in the sequel, $|x|^2 = x^TQx$.

When $a_1 = 0.55$, $a_2 = -0.1$, $a_3 = 0.2$, and $a_4 = 0.1$, we have

$$\begin{aligned} \mathcal{L}V(t, x(t), x_t) &\leq [-2 + 2.2\cos^2(t)]V(t, x(t) - Dx(t-\tau_0)) + [0.25 + 0.12\sin^2(t)]|x(t)|^2 \\ &\quad +[0.25 + 0.0316\sin^2(t)]|x(t-\tau_0)|^2 + 0.0316\sin^2(t)\int_{-\tau_1}^{0}\eta(\theta)|x(t+\theta)|^2d\theta \\ &= \lambda_0(t)V(t, x(t) - Dx(t-\tau_0)) + \lambda_1(t)|x(t)|^2 + \lambda_2(t)|x(t-\tau_0)|^2 + \lambda_3(t)\int_{-\tau_1}^{0}\eta(\theta)|x(t+\theta)|^2d\theta. \end{aligned}$$

where $\lambda_0(t) = -0.9 + 1.1\cos(2t)$, $\lambda_1(t) = 0.31 - 0.06\cos(2t)$, $\lambda_2(t) = 0.2658 - 0.0158\cos(2t)$, and $\lambda_3(t) = 0.0158 - 0.0158\cos(2t)$.

Then, when $\tau_1 = 0.09$, $\tau = \max\{\tau_0, \tau_1\} = 0.09$. Furthermore, we have $\hat{\varsigma} = \sup_{t\geq 0}\sup_{u\in[-0.09,0]}\{\int_{t+u}^{t}[-\lambda_0(\upsilon) - \frac{\sum_{l=1}^{3}\lambda_l(\upsilon)}{(1-\kappa)^p}]d\upsilon\} = \sup_{t\geq 0}\sup_{u\in[-0.09,0]}[1.0534\cos(2t+u)\sin(u) - 0.0653(-u)] \approx 0.1006$, where $\kappa = |Q^{\frac{1}{2}}DQ^{-\frac{1}{2}}| = 0.1581$, and for any $t \geq 0$, $\int_0^t \hat{\lambda}(s)ds \leq 0.9633 - 0.0055t$ holds, with $\kappa e^{\hat{\varsigma}} = 0.1748 \in (0,1)$ and $\hat{\lambda}(t) = \lambda_0(t) + \frac{\lambda_1(t)+\sum_{l=2}^{3}\lambda_l(t)e^{\hat{\varsigma}}}{c_1(1-\kappa)(1-\kappa e^{\hat{\varsigma}})}$. Besides, $\sup_{t\geq 0}\{\int_0^t \hat{\lambda}(s)ds\} \leq 0.9633$ and $\int_0^{\infty}\hat{\lambda}(t)dt = -\infty$. Therefore, all conditions in *Corollary 3.7* hold. Hence, it is concluded from *Corollary 3.7* that NSFDE (45) has a unique global solution $x(t)$ on $[0, +\infty)$, which is stable in mean square, asymptotically stable in mean square, exponentially stable in mean square and almost surely exponentially stable, respectively.

6. Conclusion

In this paper, we investigated the problems on the existence and uniqueness, the stability in pth($p \geq 2$)-moment, the asymptotic stability in pth($p \geq 2$)-moment, the exponential stability in pth($p \geq 2$)-moment and the almost surely exponential stability of the global solution for highly nonlinear neutral stochastic functional differential equation, when the drift term and the diffusion term satisfy the locally Lipschitz condition and the Lyapunov monotonicity condition, respectively. The Lyapunov monotonicity condition has a sign-changed time-varying coefficient. The methodology proposed is the Lyapunov-Krasovskii function and the theory of stochastic analysis. We provided different characterizations of sufficient conditions on the stability in pth($p \geq 2$)-moment, the asymptotic stability in pth($p \geq 2$)-moment, and the exponential stability in pth($p \geq 2$)-moment for highly nonlinear neutral stochastic functional differential equation. These results have not been reported in the available literature. The almost surely exponential stability for the global solution of such equation was analyzed by using the nonnegative semimartingale convergence theorem. Some discussions and comparisons on the main results between some related works and this paper have been presented. Two examples have been provided to illustrate the effectiveness of the theoretical results obtained.

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