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Original Article

Eager Term Rewriting For The Fracterm Calculus Of Common Meadows

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Eager equality is a novel semantics for equality in the presence of partial operations. We consider term rewriting for eager equality for arithmetic in which division is a partial operator. We use common meadows which are essentially fields that contain an absorptive element ⊥. The idea is that term rewriting is supposed to be semantics preserving for non-⊥ terms only. We show soundness and adequacy results for eager term rewriting w.r.t. the class of all common meadows. However, we show that an eager term rewrite system which is complete for common meadows of rational numbers is not easy to obtain, if it exists at all.

Keywords: Eager equality; term rewriting; common meadow; fracterm calculus

1. INTRODUCTION

Calculation in arithmetic depends upon transforming expressions using rules R that rewrite expressions t, t' and preserve equality:

$$t \text{ rewrites to } t' \text{ implies } t = t'.$$

These transformations are taught in school arithmetic and algebra, if only by examples. This algorithmic methodology is recognizable as the genesis of *equational term rewriting*, wherein terms are rewritten into equivalent terms via the application of rules expressed as equations. Equational term rewriting plays a fundamental role in theoretical computer science and its contribution to software tools, e.g. in compiling and theorem proving.

However, these rules become more difficult to formulate and apply if the operations in the terms are partial. What to do with undefined terms such as $\frac{x}{0}$? In particular, equality can have several different meanings: the semantics of equality suddenly becomes a subject in itself. Each notion of equality may or may not have a satisfactory term writing theory.

The situation is simplified by the needs of computing. This is because in any practical computing system, invoking an operator *must* return some value or message: returning no value means an unacceptable indefinite wait for the user. This means that data types cannot have partial operations when they are to be implemented.

In this paper, we study a new semantics for the equality of partial terms, called *eager equality*, and develop a term rewriting theory for it. To do this we use an arithmetical data type that is total called a *common meadow*. Eager equality was introduced in [1]; the common meadows were introduced in [2, 3] and most recently revised in [4].

1.1. Eager equality

In [1] we have introduced and investigated in some detail *eager equality*, denoted ≈. The general form of eager equality is as follows:

Let $[\![t]\!]$ be the value of term t in a partial algebra A and write $t \downarrow$ if $[\![t]\!]$ is defined and $t \uparrow$ if $[\![t]\!]$ is not defined. Now, unsurprisingly, when the expressions are defined in A, eager equality of expressions is the same as equality = in A, roughly:

$$\text{if } t \downarrow \text{ and } t' \downarrow \text{ then: } [\![t]\!] = [\![t']\!] \iff t \approx t'.$$

However, what if one or other, or both, of t and t' are not defined? Eager equality ≈ then satisfies

$$t \uparrow \text{ or/and } t' \uparrow \implies t \approx t'.$$

Thus, eager equality means that if an expression is undefined then *all other other expressions*—defined or not—will be deemed equal to it. This motivates the description 'eager'. Finally, we also know that

$$t \not\approx t' \iff t \downarrow \text{ and } t' \downarrow \text{ and } t \neq t'.$$

Eager equality is symmetric and reflexive, though not transitive. Eager equality satisfies what we call *safe transitivity*:

$$x \approx y \wedge y \approx z \wedge y \not\approx u \to x \approx z.$$

It does satisfy the congruence property, i.e. operations preserve ≈. Eager equality contrasts with one of the oldest notions of partial equality, that of *Kleene equality*, which requires both the

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expressions to be either defined and equal, or both undefined [5]. As emphasized in our [1], eager equality is something rather different and unusual.

1.2. Eager rewriting and arithmetic

In [1], we began developing a theory of eager equality by applying it to arithmetical data types. We considered the notion in a number of algebras based on fields with division $x \div y$, which is partial for $y = 0$; these algebras are called meadows [6]. Eager equality allows equations to be true that are conventionally considered mistaken, such as $\frac{x}{x} \approx 1$. Specifically, we work with the rational numbers, which is arguably the most widespread of all data types: it is a number system that is the base of all practical computation by hand or machine. However, to qualify as a data type, the rational numbers must be conceived as a minimal algebra.[1] This means that in addition to the constants of 0, 1, and the operations of $x+y$, $-x$ and $x \cdot y$, we must add either x^{-1} or division $x \div y$. Let $\mathbb{Q}$ be any field of rational numbers with $\div$ added to make the partial meadow $\mathbb{Q}(\div)$.

As in [1], we will focus on division while working in a total algebra called a common meadow. In general, a common meadow takes a field F and adds a division operator $\div$ (or inverse operator $^{-1}$) to make a partial meadow $F(\div)$; then a new element $\perp$ is added for which the field operators return $\perp$ when they are undefined, so

$$x \div 0 = \perp,$$

and for which $\perp$ is *absorptive*, meaning when an operator has $\perp$ in its argument then it returns the value $\perp$. Understood as an element different from elements of the field, we refer to $\perp$ as a *peripheral element*. These steps create an algebra $\mathsf{Enl}_{\perp}(F(\div))$, called a *common meadow*.[2]

Applying this totalization process to the rational partial meadow $\mathbb{Q}(\div)$ creates a total algebra $\mathsf{Enl}_{\perp}(\mathbb{Q}(\div))$, a common meadow of rational numbers.

Here we refine the equational axiomatics of common meadows and study the role of equational term rewriting for eager equality. One of the many attractions of an equational axiomatization of a data type is the role that term rewriting plays in calculating and reasoning about the data. Indeed, term rewriting is both theoretically and practically fundamental in the creation of the many software tools that employ equations.

To the equations of common meadows, which are two-way rewrite rules, we add three one-way rewrite rules to create a term rewriting system $TRS_{cmfc/e}$; see Table 3. We prove the rewrite system is comprehensive or adequate in the following sense:

Theorem. Suppose that in any common meadow, and for all evaluations of the variables of t and r, if $t \neq \perp$ then $t = r$. Then, using rewrite system $R = TRS_{cmfc/e}$ it is the case that $t \Rightarrow^{\star}_{R} r$.

This is Proposition 4.3 below. Turning to the rational numbers, however, we prove these results:

Theorem. $TRS_{cmfc/e}$ is a sound but not comprehensive eager rewriting system w.r.t. native equality on $\mathsf{Enl}_{\perp}(\mathbb{Q}(\div))$. Furthermore, if there exists a finite set R of equational rewrite rules,

$$R = TRS_{cmfc/e:\mathbb{Q}} \supseteq TRS_{cmfc/e},$$

that is sound and comprehensive as an eager rewriting system for native equality on $\mathsf{Enl}_{\perp}(\mathbb{Q}(\div))$ then the solvability of Diophantine equations over $\mathbb{Q}$ is decidable.

This combines Propositions 4.4 and 4.5 below.

Some basic properties of term rewriting can be shown for our eager rewrite system such as

Theorem. Suppose $R = TRS_{cmfc/e}$ and consider terms t and r then the following are equivalent:

(i) $\mathsf{Enl}_{\perp}(\mathbb{Q}(\div)) \models t \approx r$,
(ii) for some term s, $t \Rightarrow^{\star}_{R} s$ and $r \Rightarrow^{\star}_{R} s$.

This is Proposition 4.2 below.

The decidability of solving Diophantine equations is number 10 in the list of Hilbert's Problems of 1900. For solutions in the integers it was finally proved undecidable in 1970 by combining results of Martin Davis, Yuri Matiyasevich, Hilary Putnam and Julia Robinson [9]. For solutions in the rationals it remains an important open problem [10].

1.3. Structure of the paper

In Section 2 we gather some preliminary concepts and results about rings, fields and common meadows. In Section 3 we introduce an axiomatization of common meadows sometimes called the *fracterm calculus*. In Section 4 we introduce eager rewriting system and prove the theorems. Although the proofs of our theorems are concise they depend upon some technically complicated earlier results. In the concluding Section 5 we discuss issues to do with rewriting.

2. RINGS, FIELDS AND COMMON MEADOWS

Let Σ be the signature of commutative rings and fields containing constants 0, 1, and the operations addition $x + y$, additive inverse $-x$, and multiplication $x \cdot y$. We add $\div$ to Σ to make a signature Σ_m of meadows. Since $\perp$ is not included in the signature Σ_m stating and proving results about syntactic expressions over Σ_m does not involve $\perp$.

We assume that $\mathbb{Q}$ is some fixed field of rational numbers and $\mathbb{Q}(\div)$ is the expansion of $\mathbb{Q}$ with a partial division operator, for which we will also use the horizontal bar notation $\frac{x}{y}$.

As mentioned in the Introduction, and following [4], the algebra

$$\mathsf{Enl}_{\perp}(\mathbb{Q}(\div))$$

is an algebraic enlargement of the partial algebra $\mathbb{Q}(\div)$ made by (i) introducing $\perp$ as a peripheral number to the domain of $\mathbb{Q}$ and setting $\frac{x}{0} = \perp$ for all rationals x; and (ii) requiring $\perp$ to be absorptive: any function produces $\perp$ whenever $\perp$ appears as an argument. Now, $\mathsf{Enl}_{\perp}(\mathbb{Q}(\div))$ is a primary example of a common meadow. Let $\Sigma_{cm} = \Sigma_m \cup \{\perp\}$ be its signature.

Starting with [6], we first used the term meadow for a field equipped with an inverse operator or with a division operator (as well as variations of such structures) and we began to develop equational axiomatizations. An equational axiomatization of common meadows first appeared in [2] and has been further developed, as we will see.

[1] An algebra is minimal if it is generated by its constants. Thus, every element of the algebra can be constructed by finitely many applications of the operations to the constants [7].
[2] Here we use a general notation $\mathsf{Enl}_{\perp}(A)$ for enlarging a partial algebra A by an absorptive element $\perp$, introduced in [8].

Table 1. E^b_{cr}: Balanced axioms for commutative rings.

$(x+y)+z = x+(y+z)$	(1)
$x+y = y+x$	(2)
$x+0 = x$	(3)
$x+(-x) = 0 \cdot x$	(4)
$x \cdot (y \cdot z) = (x \cdot y) \cdot z$	(5)
$x \cdot y = y \cdot x$	(6)
$1 \cdot x = x$	(7)
$x \cdot (y+z) = (x \cdot y) + (x \cdot z)$	(8)
$-(-x) = x$	(9)
$0 \cdot (x \cdot y) = 0 \cdot (x+y)$	(10)

Table 2. E_{cmfc}: Axioms for fracterm calculus for common meadows.

import $:E^b_{cr}$	(11)
$\frac{x}{y} = x \cdot \frac{1}{y}$	(12)
$x = \frac{x}{1}$	(13)
$-\frac{x}{y} = \frac{-x}{y}$	(14)
$\frac{x}{y} \cdot \frac{u}{v} = \frac{x \cdot u}{y \cdot v}$	(15)
$\frac{x}{y} + \frac{u}{v} = \frac{(x \cdot v)+(y \cdot u)}{y \cdot v}$	(16)
$\frac{1}{(\frac{1}{x})} = \frac{x \cdot x}{x}$	(17)
$\frac{1}{1+0 \cdot x} = 1 + 0 \cdot x$	(18)
$\frac{1}{0} = \bot$	(19)
$x + \bot = \bot$	(20)

Here we present a new modularization of the axioms for common meadows. These algebras build on commutative rings but the new totalizing absorptive element $\bot$ needs careful handing as familiar properties fail in $\mathsf{Enl}_\bot(\mathbb{Q}(\div))$: for example, because $\bot + (-\bot) = \bot$ and $\bot \neq 0$, we must accept that for some x,

$$x + (-x) \neq 0,$$

and because $\bot \cdot 0 = \bot$, we must also accept that for some x,

$$x \cdot 0 \neq 0.$$

So, first we adapt the axioms of commutative rings to accommodate a future use of $\bot$. We do this using the idea of balanced equations.

Definition 2.1. An equation $t = t'$ is a *balanced equation* if the terms t and t' have the same variables.

For example, $x + (-x) = 0 \cdot x$ is balanced, but $x + (-x) = 0$ is not. Their key property is this:

Lemma 2.1. In a common meadow, any balanced equation is true when one of its variables is assigned $\bot$.

Consider the set E^b_{cr} of equations for commutative rings in Table 1, each of which is balanced. The usual equations are intact, except for new equations $x + (-x) = 0 \cdot x$ and $0 \cdot (x \cdot y) = 0 \cdot (x + y)$. The equations of E^b_{cr} are derivable from the equations defining commutative rings.

We build the equations of common meadows by axiomatizing $\div$ on top of this set.

3. FRACTERM CALCULUS FOR COMMON MEADOWS

A *fracterm* is a formalization of the notion of a fraction: it is a term over the signature Σ_m of meadows whose leading function symbol is division. A fracterm is *flat* if it contains one and only one division symbol.

Expressed using fracterms—i.e. as a fracterm calculus—the axiomatization is the set E_{cmfc} of equations given in Table 2. The axioms in Tables 1 and 2 are taken from [4], which improves on [3]. A detailed discussion of other axiomatizations of the common meadows is [4]. The equations listed in Tables 1 and 2 are designed to make flattening immediate—they are not logically independent. A set of logically independent axioms is given in [2, 3].

Proposition 3.1. $\mathsf{Enl}_\bot(\mathbb{Q}(\div)) \models E_{cmfc}$.

The axioms E_{cmfc} enable key calculations: rewriting division-free expressions into polynomials, and the following, which originates in [2]:

Proposition 3.2. (Fracterm flattening). For each open expression t over Σ_{cm} there are expressions r and s without an occurrence of division such that:

$$E_{cmfc} \vdash t = \frac{r}{s}.$$

Proof. Fracterm flattening can be defined by the following transformations ϕ, ϕ_n, and ϕ_d on expressions, where we use $\equiv$ for syntactic definition.

$\phi(t) \equiv \frac{\phi_n(t)}{\phi_d(t)}$,
$\phi_n(0) \equiv 0$ and $\phi_d(0) \equiv 1$,
$\phi_n(1) \equiv 1$ and $\phi_d(1) \equiv 1$,
$\phi_n(\bot) \equiv 1$ and $\phi_d(\bot) \equiv 0$,
$\phi_n(x) \equiv x$ and $\phi_d(x) \equiv 1$ (for a variable x),
$\phi_n(-t) \equiv -\phi_n(t)$ and $\phi_d(-t) \equiv \phi_d(t)$,
$\phi_n(t \cdot r) \equiv \phi_n(t) \cdot \phi_n(r)$ and $\phi_d(t \cdot r) \equiv \phi_d(t) \cdot \phi_d(r)$,
$\phi_n(t + r) \equiv \phi_n(t) \cdot \phi_d(r) + \phi_d(t) \cdot \phi_n(r)$ and $\phi_d(t + r) \equiv \phi_d(t) \cdot \phi_d(r)$,
$\phi_n(\frac{t}{r}) \equiv \phi_n(t) \cdot \phi_d(r)^2$ and $\phi_d(\frac{t}{r}) \equiv \phi_d(t) \cdot \phi_n(r) \cdot \phi_d(r)$.

The following is immediate by induction on the structure of terms: for each $\Sigma_m \cup \{\bot\}$ term t, $\phi(t)$ is a flat fracterm such that $E_{cmfc} \vdash t = \phi(t)$, and also $E_{cmfc} \vdash t \approx \phi(t)$. ■

In fact, the axioms of Table 2 have been chosen in such a way as to make this proof work easily, which comes with the price of some logical redundancy. Finding minimal sets of axioms is of no concern to us in this paper as we focus on intuitive appeal instead.

Proposition 3.3. $\mathsf{Enl}_\bot(\mathbb{Q}(\div))$ is a computable algebra.

Proof. A computable algebra has a computable carrier, operations and native equality. Using the standard formal definition of a computable data type, the proposition follows by routine arguments [11, 12]. ■

4. EAGER REWRITING FOR THE FRACTERM CALCULUS OF COMMON MEADOWS

Following [1], eager equality $t \approx r$ of two terms t and r is defined as follows:

$$t \approx r \iff t = \bot \vee r = \bot \vee t = r.$$

Table 3. $TRS_{cmfc/e}$: Eager rewriting for the fracterm calculus for common meadows.

import : E_{cmfc} (equations as bidirectional rewrite rules)	(21)
$\frac{x}{x} \Rightarrow 1$	(22)
$0 \cdot x \Rightarrow 0$	(23)
$\frac{x}{0} \Rightarrow x$	(24)

The nature of eager equality requires a notion of eager rewriting system:

Definition 4.1. Let K be a class of common meadows. A rewrite system R is *sound for eager equality* w.r.t. the class K if for all terms t, r:

$$\textit{if } t \Rightarrow_R^\star r \textit{ then } K \models t \approx r.$$

An *eager rewrite system* is a rewrite system that is sound for eager equality.

We are interested in eager rewrite systems that are in addition sound for equality that is native to the algebra.

Definition 4.2. Let K be a class of common meadows. An eager rewrite system R is *sound for native equality* w.r.t. the class K if for all terms t, r:

$$\textit{if } t \Rightarrow_R^\star r \textit{ then } K \models t \neq \bot \to t = r.$$

If R is sound for native equality then it is sound for eager. Recall this standard notion from term rewriting:

Definition 4.3. The rewrite system R is *confluent* if for all terms u, t, r, with $u \Rightarrow_R^\star t$ and $u \Rightarrow_R^\star r$ there is a term s such that $t \Rightarrow_R^\star s$ and $r \Rightarrow_R^\star s$.

Definition 4.4. Let K be a class of common meadows. An eager rewrite system R is *confluent for eager equality* w.r.t. to the class K if

$$K \models t \approx r$$

implies that for some term s, both

$$t \Rightarrow_R^\star s \text{ and } r \Rightarrow_R^\star s.$$

Note that taking K to be a class containing a single common meadow, or an isomorphism class of common meadows, enables us to examine term writing for specific algebras.

4.1. An eager rewriting system for common meadows

Table 3 provides rules for the eager rewriting system $TRS_{cmfc/e}$ for the fracterm calculus of common meadows. The following observation is immediate by inspection of the rewrite steps generated by each rule of $TRS_{cmfc/e}$.

Lemma 4.1. With $R = TRS_{cmfc/e}$, and A a common meadow, with σ a valuation for it: if $A, \sigma \models t \neq \bot$ and $t \Rightarrow_R^\star s$ then $A, \sigma \models s \neq \bot$.

From this observation it follows that $TRS_{cmfc/e}$ is an eager rewriting system, and moreover that $TRS_{cmfc/e}$ is sound for native equality on common meadows.

Proposition 4.1. Suppose with $R = TRS_{cmfc/e}$, that terms t, r, s are such that $t \Rightarrow_R^\star r$ and $t \Rightarrow_R^\star s$. Moreover, assume $\mathsf{Enl}_\bot(\mathbb{Q}(\div)) \models t \neq \bot$, then $\mathsf{Enl}_\bot(\mathbb{Q}(\div)) \models r \approx s$.

Proof. Let $\frac{t_1}{t_2}$, $\frac{r_1}{r_2}$ and $\frac{s_1}{s_2}$ be flat fracterms provably equal to t, r, and s, respectively. We may assume that t_2, r_2 and s_2 represent polynomials without 0-coefficients (otherwise, equation 18 can be used to move subterms of the form $0 \cdot x$ from the denominator to the numerator of the fraction operator).

Now, we know that t_2 is not the zero polynomial because otherwise t is provably equal to $\bot$. It follows that for a non-$\bot$ valuation σ, $\mathsf{Enl}_\bot(\mathbb{Q}(\div)), \sigma \models t \neq \bot$ so that also $\mathsf{Enl}_\bot(\mathbb{Q}(\div)), \sigma \models r \neq \bot$ and $\mathsf{Enl}_\bot(\mathbb{Q}(\div)), \sigma \models s \neq \bot$. So we find that $\mathsf{Enl}_\bot(\mathbb{Q}(\div)), \sigma \models t_2 \neq 0 \to r_1 \cdot s_2 = s_1 \cdot r_2$ so that $\mathbb{Q} \models t_2 \cdot r_1(r_1 \cdot s_2 - s_1 \cdot r_2) = 0$. t_2 not being the 0-polynomial it must be the case that $\mathbb{Q} \models r_1 \cdot s_2 - s_1 \cdot r_2 = 0$. From this fact it follows that for all valuations σ, $\mathsf{Enl}_\bot(\mathbb{Q}(\div)), \sigma \models \frac{t_1}{t_2} \approx \frac{r_1}{r_2}$. ■

Proposition 4.2. Suppose with $R = TRS_{cmfc/e}$ and consider terms t and r then the following are equivalent:
(i) $\mathsf{Enl}_\bot(\mathbb{Q}(\div)) \models t \approx r$,
(ii) for some term s, $t \Rightarrow_R^\star s$ and $r \Rightarrow_R^\star s$.

Proof. (ii) → (i): If $\mathsf{Enl}_\bot(\mathbb{Q}(\div)), \sigma \models t \neq \bot$ and $\mathsf{Enl}_\bot(\mathbb{Q}(\div)), \sigma \models r \neq \bot$ then both terms t and r have the same value, under valuation σ, as s so that both values are equal, which establishes $\mathsf{Enl}_\bot(\mathbb{Q}(\div)) \models t \approx r$.

(i) → (ii) Let $\frac{t_1}{t_2}$, $\frac{r_1}{r_2}$ and $\frac{s_1}{s_2}$ be flat fractems provably equal to t, r, and s, respectively. If t_2 takes value zero for all valuations, it is provably equal to 0, so that t is provably equal to $\bot$, which is provably equal to $\frac{r}{0}$. It follows that t can be rewritten to $\frac{r}{0}$ with the equations of fracterm calculus, and subsequently to r with the rule 24 of Table 3. A similar argument applies if r_2 takes value zero for all valuations. So, in both cases a common reduct is found and for the remaining cases it may be assumed that the numerators t_1, r_1 of both flat fracterms are nonzero polynomials. Now, from $\mathsf{Enl}_\bot(\mathbb{Q}(\div)) \models t \approx r$ we find that $\mathbb{Q} \models (t_2 \neq 0 \wedge r_2 \neq 0) \to t_1 \cdot r_2 = t_2 \cdot r_1$, and hence $\mathbb{Q} \models t_2 \cdot r_2 \cdot (t_1 \cdot r_2 - t_2 \cdot r_1) = 0$ from which it follows that $\mathbb{Q} \models t_1 \cdot r_2 - t_2 \cdot r_1 = 0$ because t_2 and r_2 are nonzero polynomials.

Now suppose that t_1 and t_2 share a factor f, say $t_1 = f \cdot t_1'$, $t_2 = f \cdot t_2'$, then

$$\mathbb{Q} \models 0 = t_1 \cdot r_2 - t_2 \cdot r_1 = f \cdot t_1' \cdot r_2 - f \cdot t_2' \cdot r_1 = f \cdot (t_1' \cdot r_2 - t_2' \cdot r_1).$$

As f cannot be zero everywhere, it follows that $\mathbb{Q} \models t_1' \cdot r_2 - t_2' \cdot r_1 = 0$ from which it follows that $\mathsf{Enl}_\bot(\mathbb{Q}(\div)) \models \frac{t_1'}{t_2'} \approx \frac{r_1}{r_2}$. Using rule 22 of Table 3 it follows that $\frac{t_1}{t_2} \Rightarrow_R \frac{t_1'}{t_2'}$ so that it now suffices to find a common reduct for $\frac{t_1'}{t_2'}$ and $\frac{r_1}{r_2}$.

In this way both terms can be reduced step-by-step until none has a common factor for its numerator and its denominator. Let the resulting pair of terms be $\frac{a_1}{a_2}$ and $\frac{b_1}{b_2}$ where still $\mathsf{Enl}_\bot(\mathbb{Q}(\div)) = \frac{a_1}{a_2} \approx \frac{b_1}{b_2}$ so that $\mathbb{Q} \models a_1 \cdot b_2 - a_2 \cdot b_1 = 0$.

Now let f be a factor of a_2 then it must be a factor of b_2 because it is not a factor of a_1. Let $a_2 = f \cdot a_2'$ and $b_2 = f \cdot b_2'$, then $\mathbb{Q} \models f \cdot (a_1 \cdot b_2' - a_2' \cdot b_1) = 0$ so that $\mathbb{Q} \models a_1 \cdot b_2' - a_2' \cdot b_1 = 0$. This argument can be repeated with the effect that a_2 and b_2 must have the same factors with the same powers, so that as polynomials these are equal. Then it follows that a_1 and b_1 represent the same polynomials; thus, $\frac{a_1}{a_2}$ and $\frac{b_1}{b_2}$ are provably equal in E_{cmfc} and each can be found as term reductions of the other. ■

4.2. An adequacy result for eager rewriting

Suppose that agent A asks agent B to calculate a term t. Which results r are valid for A? Now we assume that B may believe and trust that A's question was made in good faith and that in fact t does indeed have a value, i.e. $t \neq \bot$. So, it is a requirement on r that, under the assumption that $t \neq \bot$, $t = r$.

Definition 4.5. Let K be a class of common meadows. An eager and natively sound rewrite system R is said to be *comprehensive* for native equality relative to K, if $K \models t \neq \bot \to t = r$ implies $t \Rightarrow^{\star}_{R} r$.

The (adequacy) result below indicates that rewriting with $TRS_{cmfc/e}$ can produce each result r which satisfies the criterion just stated.

Proposition 4.3. $TRS_{cmfc/e}$ is comprehensive for native equality relative to the class of all common meadows.

Proof. With t and r as given, first notice that in all common meadows A, and for all valuations σ, we have $A, \sigma \models \psi$ with $\psi \equiv r + 0 \cdot t = t$. To see this first consider the case that $A, \sigma \models t = \bot$. Now $A, \sigma \models \psi$ because $\bot$ is absorptive in A. Otherwise, if $A, \sigma \models t \neq \bot$ then it must be the case that $A, \sigma \models 0 \cdot t = 0$ because A is a common meadow, and all of its non-$\bot$ elements e satisfy $e \cdot 0 = 0$. Given that A is a common meadow the assumption of the proposition implies $A, \sigma \models t \neq \bot \to t = r$, so that, with $A, \sigma \models t \neq \bot$ it follows that $A, \sigma \models t = r$ whence $A, \sigma \models \psi$ is immediate from $A, \sigma \models 0 \cdot t = 0$.

Now we use the completeness result of [13] with the effect that $E_{cmfc} \vdash \psi$.[3]

It follows that using the equations of E_{cmfc} as rewrite rules in both directions with $TRS_{cmfc/e}$ one obtains $t \Rightarrow^{\star} r + 0 \cdot t$. Then using the rule $0 \cdot x \Rightarrow 0$ we have $r + 0 \cdot t \Rightarrow r + 0$ and finally with an equation of E_{cmfc} one finds $r + 0 \Rightarrow r$ so that with the rules of $TRS_{cmfc/e}$ one finds $t \Rightarrow^{\star} r$. ■

Concerning Proposition 4.3, we know that $0 \cdot \bot \neq 0$ by construction of a common meadow. Since we work in an arbitrary common meadow, we do not know for sure that an equation is true if all of its ground instances are true, but obviously it is certainly not true if some of its ground instances are false.

4.3. On lacking adequacy of eager rewriting for the rationals

There are more equations true in a single algebra (or isomorphism class) than that are true in all the algebras of a (non-isomorphism) class:

Proposition 4.4. $TRS_{cmfc/e}$ is sound but not comprehensive for native equality w.r.t. the singleton class of common meadows consisting of $\mathsf{Enl}_{\bot}(\mathbb{Q}(\div))$ only.

Proof. Soundness is obvious, incompleteness requires an argument. In the case of a common meadow of rationals $\mathsf{Enl}_{\bot}(\mathbb{Q}(\div))$ we have

$$\mathsf{Enl}_{\bot}(\mathbb{Q}(\div)) \models 1 + 0 \cdot x \neq \bot \to 1 + 0 \cdot x = \frac{x^2 - 2}{x^2 - 2}$$

We claim that rewriting $1 + 0 \cdot x \Rightarrow^{\star} \frac{x^2-2}{x^2-2}$ is not possible in $TRS_{cmfc/e}$. To see the latter notice that $TRS_{cmfc/e}$ is sound (understood as an eager rewrite system) for all common meadows, while $1 + 0 \cdot x = \frac{x^2-2}{x^2-2}$ fails in any common meadow that contains a square root of 2, and it fails on a substitution for x which keeps $1 + 0 \cdot x$ different from $\bot$. ■

Proposition 4.5. If there exists a finite set of rules

$$R = TRS_{cmfc/e:\mathbb{Q}} \supseteq TRS_{cmfc/e}$$

such that R is an eager rewrite system that is both sound and comprehensive for native equality rewriting on the singleton class containing $\mathsf{Enl}_{\bot}(\mathbb{Q}(\div))$, then solvability of Diophantine equations over $\mathbb{Q}$ is decidable.

Proof. To see this let p be a polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$. Then the equation $p = 0$ has no rational solution in $\mathbb{Q}$ if and only if the following holds:

$$\mathsf{Enl}_{\bot}(\mathbb{Q}(\div)) \models 1 + 0 \cdot (x_1 + \ldots + x_n) \neq \bot \to$$
$$1 + 0 \cdot (x_1 + \ldots + x_n) = \frac{p}{p},$$

which under the assumption made on $TRS_{cmfc/e:\mathbb{Q}}$ is equivalent to $1 + 0 \cdot (x_1 + \ldots + x_n) \Rightarrow^{\star}_{R} \frac{p}{p}$. ■

Given that the decidability of the solvability of Diophantine equations over the rationals is a long-standing open problem, it is not an easy matter, if at all possible, to find such rules R, or rather to determine whether or not these can be found.

4.4. A simpler term rewriting system

In Table 4 we have collected some slightly simplified rules for eager rewriting and made an attempt to make a simpler term rewriting system $TRS2_{cmfc/e}$. Notice that there is no mention of $\bot$. Indeed, under the assumption that one starts rewriting a non-$\bot$ term, it is not possible that rewriting ends up at $\bot$ at any stage, so $\bot$ is redundant.

$TRS2_{cmfc/e}$ is an eager rewriting system which is sound for native equality w.r.t. the class of all common meadows. From this observation together with Proposition 4.3 it follows that a reduct of t in $TRS2_{cmfc/e}$ is also a reduct of t in $TRS_{cmfc/e}$. Whether or not the converse is true we do not know. Let $R = TRS_{cmfc/e}$ and $R2 = TRS2_{cmfc/e}$.

Proposition 4.6. For all terms t, r with $t \Rightarrow^{\star}_{R} r$, there is a term s such that $r \Rightarrow^{\star}_{R2} s$ and $t \Rightarrow^{\star}_{R2} s$.

[3] *The finite equational axiomatization E_{cmfc} is sound and complete for the class* CM *of all common meadows w.r.t. equational logic, i.e. for any equation e over Σ_{cm}, $E_{cmfc} \vdash e$ if, and only if, $e \in EqnThy(\mathsf{CM})$.*

Table 4. $TRS2_{cmfc/e}$: simplified eager rewriting for the fracterm calculus for common meadows.

$(x+y)+z = x+(y+z)$	(25)
$x+y = y+x$	(26)
$x+0 = x$	(27)
$1+(-1) = 0$	(28)
$x\cdot(y\cdot z) = (x\cdot y)\cdot z$	(29)
$x\cdot y = y\cdot x$	(30)
$1\cdot x = x$	(31)
$x\cdot(y+z) = (x\cdot y)+(x\cdot z)$	(32)
$x = \frac{x}{1}$	(33)
$-\frac{x}{y} = \frac{-x}{y}$	(34)
$\frac{x}{y}\cdot\frac{u}{v} = \frac{x\cdot u}{y\cdot v}$	(35)
$\frac{x}{y}+\frac{u}{v} = \frac{(x\cdot v)+(y\cdot u)}{y\cdot v}$	(36)
$0\cdot x \Rightarrow 0$	(37)
$\frac{x}{x} \Rightarrow 1$	(38)
$\frac{x}{(\frac{y}{z})} \Rightarrow \frac{x\cdot z}{y}$	(39)

***Proof*.** From $t \Rightarrow^{\star}_{R} r$ it follows that $\mathsf{Enl}_{\bot}(Q(\div)) \models t \approx r$. Now, it suffices to notice that the proof of Proposition 4.2, part (ii) → (i), will also work for rewriting with $R2$. ■

5. CONCLUDING REMARKS

5.1. Term rewriting for arithmetic data types

Working with partiality using a total(ised) algebra allows the use of standard first order logical methods with native equality.

Eager rewriting makes sense for arithmetical data types that contain an absorptive element ⊥. Besides common meadows [2, 3], such data types are wheels [14, 15], and transrationals [16, 17]. If we totalize division on the rationals by simply taking $0^{-1} = 0$ then the native equality on the total algebra obtained is called the *Suppes-Ono equality* [18, 19]. For Suppes-Ono equality there is no distinction between conventional rewriting and eager rewriting.

5.2. Eager rewriting and school arithmetic

In school arithmetic it is often required of a student 'to compute or simplify an expression'. Now, in general, when *asked* to compute t the student may usually *assume* that $t \neq \bot$. So, for instance, calculating $\frac{x^2}{x}$ plausibly produces x, where the condition that x is non-zero is left implicit with the justification that otherwise the exercise would make no sense. This practice and implicit convention that $t \neq \bot$ draws attention to $TRS2_{cmfc/e}$ in Table 4.

In general, school arithmetic is calculational, and geared towards obtaining a result—rather than aiming at also finding a proof. Eager rewriting takes the calculational bias to its extreme by introducing asymmetric rules for arithmetical calculation.

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