Rational Enriched Motivic Spaces

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Abstract

Rational enriched motivic spaces are introduced and studied in this thesis to provide new models for connective and very effective motivic spectra with rational coefficients. We first study homological algebra for Grothendieck categories of functors enriched in Nisnevich sheaves with specific transfers $\mathcal{A}$. Following constructions of Voevodsky for triangulated categories of motives and framed motivic Γ-spaces, we introduce and study motivic structures on unbounded chain complexes of enriched functors yielding two new models of the triangulated category of big motives with $\mathcal{A}$-tranfers $DM_{\mathcal{A}}$. We next define enriched motivic spaces as certain enriched functors of simplicial $\mathcal{A}$-sheaves. We then use the properties of enriched motivic spaces and the above reconstruction results to recover $SH(k)_{\geqslant 0,\mathbb{Q}}$ and $SH^{\mathsf{veff}}(k)_{\mathbb{Q}}$.

Declarations

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

[redacted].. (candidate)

Date ..... 19.10.2023 ....

This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by explicit references. A bibliography is appended.

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I hereby give consent for my thesis, if accepted, to be available for electronic sharing.

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The University's ethical procedures have been followed and, where appropriate, that ethical approval has been granted.

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Chapter 1

Introduction

In his celebrated paper [48] Segal introduced Γ-spaces and showed that they yield infinite loop spaces. In [5] Bousfield and Friedlander defined a model category structure for Γ-spaces and showed that its homotopy category recovers connective S^1-spectra. They also showed that fibrant objects in this model category are given by very special Γ-spaces.

Garkusha, Panin and Østvær [25] have recently introduced and studied motivic Γ-spaces. They are $\mathcal{M}$-enriched functors in two variables

$$\mathcal{X} : \Gamma^{op} \boxtimes \mathsf{Sm}_{k,+} \to \mathcal{M},$$

where $\mathcal{M}$ is the category of pointed motivic spaces and $\mathsf{Sm}_{k,+}$ is the $\mathcal{M}$-category of framed correspondences of level 0. Special and very special motivic Γ-spaces are defined in [25] as $\mathcal{M}$-enriched functors

$$\mathcal{X} : \Gamma^{op} \boxtimes \mathsf{Sm}_{k,+} \to \mathcal{M}^{\mathsf{fr}}$$

satisfying several axioms, where $\mathcal{M}^{\mathsf{fr}}$ is the $\mathcal{M}$-category of pointed motivic spaces with framed correspondences. The axioms are a combination of Segal's axioms and axioms reflecting basic properties of framed motives of algebraic varieties in the sense of Garkusha–Panin [23] (see [25] for details).

Inspired by [25] we introduce and study additive versions for motivic Γ-spaces. We start with a reasonable additive category of correspondences $\mathcal{A}$ and replace $\mathcal{M}$ by the closed symmetric monoidal Grothendieck category $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ of simplicial Nisnevich sheaves with $\mathcal{A}$-transfers. The $\mathcal{M}$-category $\mathsf{Sm}_{k,+}$ is replaced here by a $\Delta^{op}\mathsf{Shv}(\mathcal{A})$-category $\mathcal{S}m$ whose objects are those of Sm_k.

We define enriched motivic $\mathcal{A}$-spaces as objects of the Grothendieck category of $\Delta^{op}\mathsf{Shv}(\mathcal{A})$-enriched functors $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$. Special enriched motivic $\mathcal{A}$-spaces are defined similarly to special motivic Γ-spaces with slight modifications due to the additive context (see Definition 5.1.1 for the full list of axioms). In particular, the category Γ^{op} is redundant in this context (see Section 6.1). The category $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ comes equipped with a local and a motivic model structure. Denote the model categories by $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]_{\mathsf{nis}}$ and $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]_{\mathrm{mot}}$ respectively (see Section 7.2). Let $\mathcal{D}([\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})])$ be the homotopy category of $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]_{\mathsf{nis}}$. Define $\mathrm{Spc}_{\mathcal{A}}[\mathcal{S}m]$ as the full subcategory of $\mathcal{D}([\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})])$ consisting of special enriched motivic $\mathcal{A}$-spaces. It is worth mentioning that $\mathcal{D}([\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})])$ is equivalent to the full subcategory of connective chain complexes in the derived category $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$ of the Grothendieck category $[\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]$. Thus $\mathrm{Spc}_{\mathcal{A}}[\mathcal{S}m]$ can also be regarded as a full subcategory of $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$, so that it can be studied by methods of classical homological algebra.

The following result is reminiscent of Bousfield–Friedlander's theorem mentioned above for classical Γ-spaces (see Theorem 7.2.7).

Theorem. *Assume that the exponential characteristic p of k is invertible in $\mathcal{A}$. The category $\mathrm{Spc}_{\mathcal{A}}[\mathcal{S}m]$ is equivalent to the homotopy category of the model category $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]_{\mathrm{mot}}$. The fibrant objects of $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]_{\mathrm{mot}}$ are the pointwise locally fibrant special enriched motivic $\mathcal{A}$-spaces.*

As applications of the preceding theorem we recover connective motivic bispectra with rational coefficients $SH(k)_{\mathbb{Q},\geqslant 0}$ (respectively very effective motivic bispectra with rational coefficients $SH^{\mathsf{veff}}(k)_{\mathbb{Q}}$) from special rational enriched motivic $\mathcal{A}$-spaces $\mathrm{Spc}_{\mathcal{A}}[\mathcal{S}m]$ (respectively very effective rational enriched motivic $\mathcal{A}$-spaces $\mathrm{Spc}^{\mathsf{veff}}_{\mathcal{A}}[\mathcal{S}m]$) — see Theorems 7.4.2 and 7.4.4. Here we take $\mathcal{A}$ to be the category of finite Milnor–Witt correspondences with rational coefficients $\widetilde{Cor} \otimes \mathbb{Q}$.

Theorem. *The $(S^1, \mathbb{G}_m)$-evaluation functor induces equivalences of categories*

$$ev_{S^1,\mathbb{G}_m} : \mathrm{Spc}_{\widetilde{\mathrm{Cor}},\mathbb{Q}}[\mathcal{S}m] \to SH(k)_{\mathbb{Q},\geqslant 0}.$$

and

$$ev_{S^1,\mathbb{G}_m} : \mathrm{Spc}^{\mathsf{veff}}_{\widetilde{\mathrm{Cor}},\mathbb{Q}}[\mathcal{S}m] \to SH^{\mathsf{veff}}(k)_{\mathbb{Q}}.$$

In particular, the preceding theorem makes $SH(k)_{\mathbb{Q}}$ more amenable to methods of homological algebra.

To prove the above results we will first study the Grothendieck category of unbounded chain complexes of enriched functors $\mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$. We prove two reconstruction theorems recovering Voevodsky's fundamental triangulated category of big $\mathcal{A}$-motives $DM_{\mathcal{A}}$.

In more detail, let $\mathcal{A}$ be a symmetric monoidal category of correspondences that satisfies the strict V-property and cancellation, as defined in [19]. We recover the triangulated category of big $\mathcal{A}$-motives $DM_{\mathcal{A}}$ out of Grothendieck categories of enriched functors $[\mathcal{B}, \mathsf{Shv}(\mathcal{A})]$ in the sense of [1], where $\mathcal{B}$ is either the $\mathsf{Shv}(\mathcal{A})$-category $\mathcal{C}$ of the powers $\mathbb{G}_m^{\times n}$ or the $\mathsf{Shv}(\mathcal{A})$-category $\mathcal{S}m$ of all smooth k-schemes. To this end, we use homological algebra of enriched Grothendieck categories developed in [20, 21].

In our context we consider two types of the $\mathbb{A}^1$-locality of chain complexes in $\mathsf{Ch}([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])$: one for the contravariant $\mathbb{A}^1$-locality in the $\mathcal{A}$-direction (i.e. the usual one), denoted by $\mathbb{A}^1_1$, another for the covariant $\mathbb{A}^1$-locality in the $\mathcal{B}$-direction, denoted by $\mathbb{A}^1_2$. We also consider τ-locality in $\mathsf{Ch}([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])$ with respect to the family

$$\tau = \{[\mathbb{G}_m^{\wedge n+1}, -] \underset{\mathsf{Shv}(\mathcal{A})}{\otimes} \mathbb{G}_m^{\wedge 1} \to [\mathbb{G}_m^{\wedge n}, -] \mid n \geqslant 0\}$$

as well as Nis-locality in the covariant $\mathcal{B}$-direction associated to the elementary Nisnevich squares. As we work with Grothendieck categories of $\mathsf{Shv}(\mathcal{A})$-enriched functors here, we say that the relevant chain complexes are strictly local with respect to the specified family above. We refer the reader to Section 3.1 for details. The relations are also counterparts of the axioms (2)-(5) for special motivic Γ-spaces in the sense of [25] and framed spectral functors in the sense of [24, Section 6].

Our first reconstruction result states the following (see Theorem 3.1.8).

Theorem. *Let $\mathcal{C}$ be the natural $\mathsf{Shv}(\mathcal{A})$-category represented by the $\mathcal{A}$-sheaves $\mathcal{A}(-, \mathbb{G}_m^{\times n})_{\mathsf{nis}}$, $n \geqslant 0$. Let $DM_{\mathcal{A}}[\mathcal{C}]$ be the full triangulated subcategory of the derived category $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ consisting of the strictly $\mathbb{A}^1_1$-local and τ-local complexes. Then the canonical evaluation functor*

$$ev_{\mathbb{G}_m} : DM_{\mathcal{A}}[\mathcal{C}] \to DM_{\mathcal{A}}$$

is an equivalence of compactly generated triangulated categories.

Our second reconstruction result states the following (see Theorem 3.1.14).

Theorem. *Let $\mathcal{S}m$ be the natural $\mathsf{Shv}(\mathcal{A})$-category represented by the $\mathcal{A}$-sheaves $\mathcal{A}(-,X)_{\mathsf{nis}}$, $X \in \mathsf{Sm}_k$. Let $DM_{\mathcal{A}}[\mathcal{S}m]$ be the full triangulated subcategory of the derived category $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$ consisting of the strictly $\mathbb{A}^1_1$-, τ-, Nis- and $\mathbb{A}^1_2$-local complexes. Then the canonical evaluation functor*

$$ev_{\mathbb{G}_m} : DM_{\mathcal{A}}[\mathcal{S}m][1/p] \to DM_{\mathcal{A}}[1/p]$$

is an equivalence of compactly generated triangulated categories, where p is the exponential characteristic of the base field k.

It is worth mentioning that the latter result requires the recollement theorems of Garkusha–Jones [21] as well as a generalization of Röndigs–Østvær's Theorem [46] (see Section 4.2).

Outline

The thesis consists of two halves: The first half, consisting of Chapters 2, 3 and 4, deals with certain enriched functors of unbounded chain complexes of Nisnevich sheaves. The second half, consisting of Chapters 5, 6 and 7, deals with certain enriched functors of simplicial Nisnevich sheaves, which we call enriched motivic $\mathcal{A}$-spaces. In the second half we will always assume that the exponential characteristic p of k is invertible in $\mathcal{A}$.

In Chapter 2 we recall the definition of a category of correspondences. For a suitable category of correspondences $\mathcal{A}$, we construct a well-behaved model structure on the category of unbounded chain complexes of Nisnevich sheaves $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.

In Chapter 3 we state our two reconstruction results for $DM_{\mathcal{A}}$ and prove the first one. In Chapter 4 we prove the second reconstruction result.

In Chapter 5 we introduce enriched motivic $\mathcal{A}$-spaces, and construct a model structure on the category of simplicial Nisnevich sheaves $\Delta^{op}\mathsf{Shv}(\mathcal{A})$.

In Chapter 6 we study how enriched motivic $\mathcal{A}$-spaces are related to motivic Γ-spaces in the sense of [25], and how they are related to the enriched functors of unbounded chain complexes from the first half of the thesis.

In Chapter 7 we use enriched motivic $\mathcal{A}$-spaces to provide new models for the connective and very effective parts of the rational stable motivic homotopy category $SH(k)_{\mathbb{Q}}$.

Notation

Throughout the thesis we use the following notation.

k	field of exponential characteristic p
Sm_k	smooth separated schemes of finite type over k
$\mathcal{A}$	symmetric monoidal additive strict V-category of correspondences
$\mathsf{Psh}(\mathcal{A})$	presheaves of abelian groups on $\mathcal{A}$
$\mathsf{Shv}(\mathcal{A})$	Nisnevich sheaves of abelian groups on $\mathcal{A}$
$DM_{\mathcal{A}}$	triangulated category of big motives with $\mathcal{A}$-correspondences
$SH(k)$	stable motivic homotopy category over k
$\mathcal{S}m$	enriched category of smooth schemes (see Section 3.1)
$\mathcal{C}$	subcategory of $\mathcal{S}m$ on $\mathbb{G}_m^{\times n}$ for $n \in \mathbb{N}$ (see Section 3.1)
I	canonical embedding $\mathcal{S}m \to \mathsf{Shv}(\mathcal{A})$, $X \mapsto \mathcal{A}(-, X)_{\mathsf{nis}}$
$M_{\mathcal{A}}(X)$	$\mathcal{A}$-motive of $X \in \mathsf{Sm}_k$
$\mathcal{M}$	category of motivic spaces
$f\mathcal{M}$	category of finitely presented motivic spaces

Also, we assume that 0 is a natural number.

Chapter 2

Nisnevich sheaves with transfers

In this chapter we recall the definition of a category of correspondeces $\mathcal{A}$ and the construction of the triangulated category of big motives with $\mathcal{A}$-correspondences $DM_{\mathcal{A}}$ in the sense of Voevodsky [52].

After that we take an additive symmetric monoidal category of correspondences $\mathcal{A}$ that satisfies the strict V-property, and construct a model structure on the category $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ of unbounded chain complexes of Nisnevic sheaves on $\mathcal{A}$.

2.1 Categories of correspondences

The following definition is due to [19].

2.1.1 Definition. A *preadditive category of correspondences* $\mathcal{A}$ consists of

1. a preadditive category $\mathcal{A}$ whose objects are those of Sm_k, called the *underlying preadditive category*,
2. a functor $\Gamma : \mathsf{Sm}_k \to \mathcal{A}$, called the *graph functor*,
3. a functor $\boxtimes : \mathcal{A} \times \mathsf{Sm}_k \to \mathcal{A}$

such that the following axioms are satisfied:

1. the functor $\Gamma : \mathsf{Sm}_k \to \mathcal{A}$ is the identity on objects;

2. for every elementary Nisnevich square

$$\begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

the sequence of Nisnevich sheaves

$$0 \to \mathcal{A}(-, U')_{\mathsf{nis}} \to \mathcal{A}(-, U)_{\mathsf{nis}} \oplus \mathcal{A}(-, X')_{\mathsf{nis}} \to \mathcal{A}(-, X)_{\mathsf{nis}} \to 0$$

is exact. Moreover, we require $\mathcal{A}(-, \emptyset)_{\mathsf{nis}} = 0$;

3. for every $\mathcal{A}$-presheaf $\mathcal{F}$ (i.e. an additive contravariant functor from $\mathcal{A}$ to Abelian groups Ab) the associated Nisnevich sheaf $\mathcal{F}_{\mathsf{nis}}$ has a unique structure of an $\mathcal{A}$-presheaf for which the canonical morphism $\mathcal{F} \to \mathcal{F}_{\mathsf{nis}}$ is a morphism of $\mathcal{A}$-presheaves.

4. the functor $\boxtimes : \mathcal{A} \times \mathsf{Sm}_k \to \mathcal{A}$ sends an object $(X, U) \in \mathsf{Sm}_k \times \mathsf{Sm}_k$ to $X \times U \in \mathsf{Sm}_k$ and satisfies $1_X \boxtimes f = \Gamma(1_X \times f)$, $(u+v) \boxtimes f = u \boxtimes f + v \boxtimes f$ for all $f \in \mathrm{Mor}(Sm/k)$ and $u, v \in \mathrm{Mor}(\mathcal{A})$.

2.1.2 Definition. 1. A preadditive category of correspondences $\mathcal{A}$ is called an *additive category of correspondences* if its underlying preadditive category is an additive category.

2. A preadditive category of correspondences $\mathcal{A}$ is called a *symmetric monoidal category of correspondences* if its underlying preadditive category $\mathcal{A}$ is also equipped with an Ab-enriched symmetric monoidal structure, such that the graph functor $\Gamma : \mathsf{Sm}_k \to \mathcal{A}$ is a strong monoidal functor with respect to the cartesian monoidal structure on Sm_k. This means in particular that for $X, Y \in \mathsf{Sm}_k$ the tensor product $X \otimes Y$ in $\mathcal{A}$ is isomorphic to the usual product of schemes $X \times Y$.

3. A preadditive category of correspondences $\mathcal{A}$ is called a *V-category of correspondences* if it satisfies the *V-property*: The V-property says that for any $\mathbb{A}^1$-invariant $\mathcal{A}$-presheaf of abelian groups $\mathcal{F}$ the associated Nisnevich sheaf $\mathcal{F}_{\mathsf{nis}}$ is $\mathbb{A}^1$-invariant, in the sense that for all $X \in \mathsf{Sm}_k$ the map

$$\mathcal{F}_{\mathsf{nis}}(X) \to \mathcal{F}_{\mathsf{nis}}(X \times \mathbb{A}^1)$$

induced by the projection $X \times \mathbb{A}^1 \to X$ is an isomorphism.

4. Recall from [51] that a Nisnevich sheaf $\mathcal{F}$ of abelian groups is *strictly* $\mathbb{A}^1$-*invariant* if for any $X \in Sm/k$, the canonical morphism

$$H^*_{\mathsf{nis}}(X, \mathcal{F}) \to H^*_{\mathsf{nis}}(X \times \mathbb{A}^1, \mathcal{F})$$

is an isomorphism. A V-category of correspondences $\mathcal{A}$ is a *strict V-category of correspondences* if for any $\mathbb{A}^1$-invariant $\mathcal{A}$-presheaf of abelian groups $\mathcal{F}$ the associated Nisnevich sheaf $\mathcal{F}_{\mathsf{nis}}$ is strictly $\mathbb{A}^1$-invariant.

5. For $i \leqslant k+1 \in \mathbb{N}$ let $\iota_{i,k} : \mathbb{G}_m^{\times k} \to \mathbb{G}_m^{\times k+1}$ be the inclusion map in Sm_k sending $(x_1, \ldots, x_k)$ to $(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_k)$. For any scheme X let $\mathcal{A}(-, X)$ be the presheaf represented by X, and $\mathcal{A}(-, X)_{\mathsf{nis}}$ be the sheafification of $\mathcal{A}(-, X)$. The maps $\iota_{i,k} : \mathbb{G}_m^{\times k} \to \mathbb{G}_m^{\times k+1}$ induce maps $\iota_{i,k*} : \mathcal{A}(-, \mathbb{G}_m^{\times k})_{\mathsf{nis}} \to \mathcal{A}(-, \mathbb{G}_m^{\times k+1})_{\mathsf{nis}}$. In $\mathsf{Shv}(\mathcal{A})$ define

$$\mathbb{G}_m^{\wedge k} := \mathcal{A}(-, \mathbb{G}_m^{\times k})_{\mathsf{nis}} / \sum_{i=1}^{k} \mathrm{Im}(\iota_{i,k-1*}).$$

Furthermore, let $\Delta_k^n := \mathrm{Spec}(k[t_0, \ldots, t_n]/(t_0 + \cdots + t_n - 1))$. Similarly to [19, Definition 3.5] we can define bivariant $\mathcal{A}$-motivic cohomology groups by

$$H^{p,q}_{\mathcal{A}}(X, Y) := H^p_{\mathsf{nis}}(X, \mathcal{A}(- \times \Delta_k^\bullet, Y \wedge \mathbb{G}_m^{\wedge q})_{\mathsf{nis}}[-q]),$$

where the H^p_{nis} on the right hand side refers to Nisnevich hypercohomology groups. We say that a strict V-category of correspondences $\mathcal{A}$ *satisfies the cancellation property* if all the canonical maps

$$\beta^{p,q} : H^{p,q}_{\mathcal{A}}(X, Y) \to H^{p+1,q+1}_{\mathcal{A}}(X \wedge \mathbb{G}_m^{\wedge 1}, Y)$$

are isomorphisms.

From now on, $\mathcal{A}$ is an additive symmetric monoidal strict V-category correspondences. From Section 3.1 onwards we will furthermore assume that $\mathcal{A}$ satisfies the cancellation property. Non-trivial examples are given by finite correspondences Cor in the sense of Voevodsky [52], finite Milnor–Witt correspondences $\widetilde{Cor}$ in the sense of Calmès–Fasel [7] or $K_0^{\oplus}$ in the sense of Walker [54]. Given a ring R (not necessarily commutative) which is flat as a $\mathbb{Z}$-algebra and a category of correspondences $\mathcal{A}$, we can form an additive category of correspondences $\mathcal{A} \otimes R$ with coefficients in R, by defining $(\mathcal{A} \otimes R)(X, Y) := \mathcal{A}(X, Y) \otimes R$ for all $X, Y \in \mathsf{Sm}_k$.

We are now passing to the construction of Voevodsky's triangulated category of big motives with $\mathcal{A}$-correspondences $DM_{\mathcal{A}}$. Let $\mathsf{Shv}(\mathcal{A})$ be the Grothendieck category of Nisnevich sheaves on $\mathcal{A}$ with values in abelian groups. The category $\mathsf{Shv}(\mathcal{A})$ of Ab-valued Nisnevich sheaves on $\mathcal{A}$ is symmetric closed monoidal with the Day convolution product [10] that is induced by the monoidal structure of $\mathcal{A}$. The internal hom of $\mathsf{Shv}(\mathcal{A})$ will be denoted sometimes by $\underline{\mathrm{Hom}}_{\mathsf{Shv}(\mathcal{A})}(-,-)$, and sometimes by $[-,-]$ if there is no likelihood of confusion. Let $D(\mathsf{Shv}(\mathcal{A}))$ be the derived category of $\mathsf{Shv}(\mathcal{A})$. Consider the localizing subcategory $\mathcal{L}$ in $D(\mathsf{Shv}(\mathcal{A}))$ that is compactly generated by the shifts of the complexes

$$\cdots \to 0 \to \mathcal{A}(-, X \times \mathbb{A}^1)_{\mathsf{nis}} \to \mathcal{A}(-, X)_{\mathsf{nis}} \to 0 \to \cdots$$

for all $X \in \mathsf{Sm}_k$, where $\mathcal{A}(-, X)_{\mathsf{nis}} \in \mathsf{Shv}(\mathcal{A})$ is the sheaf represented by X.

By general localization theory for triangulated categories [45] we can form the quotient triangulated category $D(\mathsf{Shv}(\mathcal{A}))/\mathcal{L}$.

2.1.3 Definition. We call $DM^{\mathsf{eff}}_{\mathcal{A}} := D(\mathsf{Shv}(\mathcal{A}))/\mathcal{L}$ the *triangulated category of effective motives with $\mathcal{A}$-correspondences.* It can be identified with the full subcategory of $D(\mathsf{Shv}(\mathcal{A}))$ of those objects that have $\mathbb{A}^1$-invariant cohomology sheaves.

In $DM^{\mathsf{eff}}_{\mathcal{A}}$ we can $\otimes$-invert $\mathbb{G}_m^{\wedge 1}$ using a procedure similar to [36, 5.2]. Namely, we define a $\mathbb{G}_m^{\wedge 1}$-spectrum of chain complexes C to be a collection $(C_n, \sigma_n)_{n \in \mathbb{N}}$ consisting for each $n \in \mathbb{Z}_{\geqslant 0}$ of a chain complex $C_n \in \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, and a morphism of chain complexes $\sigma_n : C_n \otimes \mathbb{G}_m^{\wedge 1} \to C_{n+1}$. A morphism of $\mathbb{G}_m^{\wedge 1}$-spectra of chain complexes is a graded morphism of complexes respecting the structure maps σ_n. The category of $\mathbb{G}_m$-spectra of chain complexes is denoted $\mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))$.

2.1.4 Definition. 1. Let $I : \mathsf{Sm}_k \to \mathsf{Shv}(\mathcal{A})$ be the obvious inclusion functor $I(X) := \mathcal{A}(-, X)_{\mathsf{nis}}$. For any $\mathbb{G}_m^{\wedge 1}$-spectrum of chain complexes C we define *presheaves of homology groups* by assigning to each $U \in \mathsf{Sm}_k$ and $n, m \in \mathbb{Z}$ the group $\mathbb{H}_n(C)_m(U)$ as the colimit of the diagram

$$\cdots \to Hom_{DM^{\mathsf{eff}}_{\mathcal{A}}}(I(U)[n-m] \otimes \mathbb{G}_m^{\wedge m+r}, C_r) \to \ldots$$

ranging over $r \in \mathbb{N}$.

2. A morphism of $\mathbb{G}_m^{\wedge 1}$-spectra of chain complexes is called a *stable motivic equivalence* if it induces isomorphisms on these homology presheaves.

3. We define $DM_{\mathcal{A}}$ to be the category obtained from $\mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))$ by inverting the stable motivic equivalences. We call $DM_{\mathcal{A}}$ the *triangulated category of big* $\mathcal{A}$*-motives*.

2.2 A model structure on $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$

Let $\mathcal{A}$ be a symmetric monoidal category of correspondences satisfying the V-property. The goal of this section is to construct a monoidal model structure on $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ that is weakly finitely generated (Definition 2.2.9), satisfies the monoid axiom [47, Definition 3.3], and in which the weak equivalences are the quasi-isomorphisms. Once we have such a model structure we can use [20, Theorem 5.5] to construct the projective model structure on the category of chain complexes $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ of the Grothendieck category of enriched functors $[\mathcal{C}, \mathsf{Shv}(\mathcal{A})]$ for any small $\mathsf{Shv}(\mathcal{A})$-enriched category $\mathcal{C}$. The model structure will be useful for proving the reconstruction theorems of the next two chapters.

There is a finitely generated monoidal model structure on the category of unbounded chain compelxes of abelian groups $\mathsf{Ch}(\mathrm{Ab})$, where weak equivalences are quasi-isomorphisms and fibrations are epimorphisms [49]. This model structure also satisfies the monoid axiom in the sense of [47, Definition 3.3]. For any abelian group A, let $S^n A$ be the chain complex that is A in degree n and 0 everywhere else. Let $D^n A$ be the chain complex that is A in degree n and $n+1$, and 0 everywhere else, and where the differential from degree $n+1$ to degree n is the identity map on A. For every $n \in \mathbb{Z}$ there is a canonical map $S^n A \to D^n A$ which is id_A in degree n. A set of generating cofibrations of $\mathsf{Ch}(\mathrm{Ab})$ is $I_{\mathsf{Ch}} := \{S^n\mathbb{Z} \to D^n\mathbb{Z} \mid n \in \mathbb{Z}\}$, and a set of generating trivial cofibrations is $J_{\mathsf{Ch}} := \{0 \to D^n\mathbb{Z} \mid n \in \mathbb{Z}\}$.

Let $\mathsf{Psh}(\mathcal{A})$ be the category of Ab-enriched functors $\mathcal{A}^{op} \to \mathrm{Ab}$. We can then apply [20, Theorem 5.5] to get a weakly finitely generated monoidal model structure on $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$, where weak equivalences are sectionwise quasi-isomorphisms, and the fibrations are epimorphisms. We call it the *standard projective model* *structure on presheaves*, or sometimes just the *projective model structure on presheaves*. The proof of [14, Theorem 4.2] shows that the generating cofibrations and generating trivial cofibrations of this model structure are given by the sets

$$I_{\text{proj}} := \{\mathcal{A}(-,X) \otimes S^n\mathbb{Z} \to \mathcal{A}(-,X) \otimes D^n\mathbb{Z} | X \in \mathsf{Sm}_k, n \in \mathbb{Z}\}$$

$$J_{\text{proj}} := \{0 \to \mathcal{A}(-,X) \otimes D^n\mathbb{Z} | X \in \mathsf{Sm}_k, n \in \mathbb{Z}\}.$$

From [14, Theorem 4.4] it also follows that this model structure on $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ satisfies the monoid axiom.

2.2.1 Lemma. *Every cofibration in the projective model structure on* $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ *is a degreewise split monomorphism with degreewise projective cokernel.*

Proof. Take a cofibration $f : A \to B$ in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$. Take an arbitrary $n \in \mathbb{Z}$. Define a morphism of complexes $\varphi : A \to D^n(A_n)$ by means of the following diagram

$$\begin{array}{ccccccccccc}
\dots & \xrightarrow{\partial_A^{n+3}} & A_{n+2} & \xrightarrow{\partial_A^{n+2}} & A_{n+1} & \xrightarrow{\partial_A^{n+1}} & A_n & \xrightarrow{\partial_A^{n}} & A_{n-1} & \xrightarrow{\partial_A^{n-1}} & \dots \\
 & & \downarrow & & \downarrow{\scriptstyle \partial_A^{n+1}} & & \downarrow{\scriptstyle id} & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & A_n & \xrightarrow{id} & A_n & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

In the following commutative diagram in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A})))$ the right hand side morphism is a surjective quasi-isomorphism, i.e. a projective trivial fibration

$$\begin{array}{ccc}
A & \xrightarrow{\varphi} & D^n(A_n) \\
{\scriptstyle f}\downarrow & \overset{s}{\nearrow} & \downarrow \\
B & \longrightarrow & 0
\end{array}$$

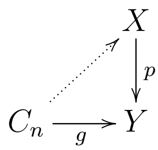
So we get a lift $s : B \to D^n(A_n)$ with $s \circ f = \varphi$. In particular $s_n \circ f_n = \varphi_n = id_{A_n}$. Since $n \in \mathbb{Z}$ was arbitrary, f is a degreewise split monomorphism.

We have a pushout diagram:

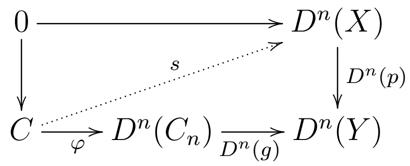
$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{Coker}(f)
\end{array}$$

Since the upper map is a cofibration, the lower map is a cofibration. So Coker(f) is a cofibrant object. To show that f is a degreewise split monomorphism with degreewise projective cokernel, we now just need to show that every cofibrant object in Ch(Psh($\mathcal{A}$)) is degreewise projective.

Let C be any cofibrant object in Ch(Psh($\mathcal{A}$)), and let $n \in \mathbb{Z}$. We claim that C_n is projective in Psh($\mathcal{A}$). Take an arbitrary epimorphism $p : X \to Y$ in Psh($\mathcal{A}$) and an arbitrary map $g : C_n \to Y$ in Psh($\mathcal{A}$). We need to find a lift in the diagram



Just like at the begining of the lemma we can construct a morphism of chain complexes $\varphi : C \to D^n(C_n)$ with $\varphi_n = id_{C_n}$, $\varphi_{n+1} = \partial_C^{n+1}$ and $\varphi_k = 0$ for $k \notin \{n, n+1\}$. In Ch(Psh($\mathcal{A}$)) we then have a diagram



We claim that in this diagram a lift $s : C \to D^n(X)$ exists. This is true for the following reason: Since p is an epimorphism, $D^n(p)$ is an epimorphism, so $D^n(p)$ is a projective fibration. Since $D^n(X)$ and $D^n(Y)$ are both acyclic, it follows that $D^n(p)$ is a quasi-isomorphism, so $D^n(p)$ is a trivial fibration. Since $0 \to C$ is a cofibration, it follows that the lift $s : C \to D^n(X)$ exists. Then $s_n : C_n \to X$ satisfies $p \circ s_n = g$, and this then shows that C_n is projective. □

2.2.2 Corollary. *The standard projective model structure on* Ch(Psh($\mathcal{A}$)) *is cellular, in the sense of [27, Definition 12.1.1]*

Proof. The domains and codomains from I_{proj} and J_{proj} are compact. By Lemma 2.2.1 every cofibration is a degreewise split monomorphism. Since Ch(Psh($\mathcal{A}$)) is an abelian category, every monomorphism is an effective monomorphism. So every cofibration is an effective monomorphism, and the projective model structure on Ch(Psh($\mathcal{A}$)) is cellular. □

We next apply a left Bousfield localization on the projective model structure on presheaves.

2.2.3 Definition. Let $\mathcal{Q}$ be the set of all elementary Nisnevich squares in Sm_k. We want to make the following class of maps in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ into weak equivalences:

1. The morphism $0 \to \mathcal{A}(-, \emptyset)$ will be a weak equivalence.

2. For every elementary Nisnevich square $Q \in \mathcal{Q}$ of the form

$$\begin{array}{ccc} U' & \xrightarrow{\beta} & X' \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} \\ U & \xrightarrow{\delta} & X \end{array}$$

we get a square

$$\begin{array}{ccc} \mathcal{A}(-,U') & \xrightarrow{\beta_*} & \mathcal{A}(-,X') \\ \downarrow{\scriptstyle\alpha_*} & & \downarrow{\scriptstyle\gamma_*} \\ \mathcal{A}(-,U) & \xrightarrow{\delta_*} & \mathcal{A}(-,X) \end{array}$$

in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ (we regard each entry of the square as a complex concentrated in zeroth degree). We take the mapping clyinder C of the map $\mathcal{A}(-,U') \to \mathcal{A}(-,X')$. So the map factors as a cofibration followed by a trivial fibration $\mathcal{A}(-,U') \rightarrowtail C \overset{\sim}{\twoheadrightarrow} \mathcal{A}(-,X')$, and C is finitely presented. Let $s_Q := \mathcal{A}(-,U) \coprod_{\mathcal{A}(-,U')} C$. Then s_Q is also finitely presented. Notice that s_Q is the homotopy pushout of $\mathcal{A}(-,U)$ and $\mathcal{A}(-,X')$ over $\mathcal{A}(-,U')$. Take the mapping cylinder t_Q of the map $s_Q = \mathcal{A}(-,U) \coprod_{\mathcal{A}(-,U')} C \to \mathcal{A}(-,X)$, so that it factors as $s_Q \overset{p_Q}{\rightarrowtail} t_Q \overset{\sim}{\twoheadrightarrow} \mathcal{A}(-,X)$, and t_Q is finitely presented.
For every $Q \in \mathcal{Q}$ this cofibration $p_Q : s_Q \to t_Q$ will be a weak equivalence.

Our notation here is similar to that of [15, Notation 2.13]. Denote the set of all the shifts of these morphisms by $S = \{0 \to \mathcal{A}(-,\emptyset)[n] \mid n \in \mathbb{Z}\} \cup \{p_Q[n] | Q \in \mathcal{Q}, n \in \mathbb{Z}\}$. We can apply [27, Theorem 4.11] to get the left Bousfied localization of the projective model structure of presheaves with respect to S. We call the resulting model structure the *local projective model structure on presheaves*. We write I_{local}, J_{local} for the generating cofibrations, generating trivial cofibrations and weakly generating trivial cofibrations of the local projective model structure on $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$.

We will say that an object $F \in \mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ is *locally fibrant*, if it is fibrant in the local projective model structure.

2.2.4 Lemma. *An object* $F \in \mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ *is locally fibrant if and only if* $F(\emptyset) \to 0$ *is a quasi-isomorphism in* $\mathsf{Ch}(\mathrm{Ab})$, *and* F *sends elementary Nisnevich squares to homotopy pullback squares.*

Proof. Let $\tau_{\geqslant 0} : \mathsf{Ch}(\mathsf{Psh}(\mathcal{A})) \to \mathsf{Ch}_{\geqslant 0}(\mathsf{Psh}(\mathcal{A}))$ be the good truncation functor, sending

$$\cdots \to A_1 \to A_0 \overset{\partial^0_A}{\to} A_{-1} \to \dots$$

to

$$\cdots \to A_1 \to \ker(\partial^0_A).$$

For $A, B \in \mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ let $\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))}(A, B)$ be the internal hom of $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ and let $\mathrm{map}^{\Delta^{op}\,\mathrm{Set}}(A, B) \in \Delta^{op}\,\mathrm{Set}$ be the derived simplicial mapping space. Define

$$\mathrm{map}^{\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})}(A, B) := \tau_{\geqslant 0}(\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))}(A, B)(pt)) \in \mathsf{Ch}_{\geqslant 0}(\mathrm{Ab}).$$

If A is cofibrant and B is fibrant, then for every $n \geqslant 0$ we have an isomorphism of abelian groups

$$H_n(\mathrm{map}^{\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})}(A, B)) \cong \pi_n(\mathrm{map}^{\Delta^{op}\,\mathrm{Set}}(A, B)).$$

By [27, Definition 3.1.4] an object $F \in \mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ is locally fibrant if and only if for every $s : A \to B$, with $s \in S$ the map

$$s^* : \mathrm{map}^{\Delta^{op}\,\mathrm{Set}}(B, F) \to \mathrm{map}^{\Delta^{op}\,\mathrm{Set}}(A, F)$$

is a weak equivalence of simplicial sets. Since s is a cofibration between cofibrant objects, and every object in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ in the standard projective model structure is fibrant, it follows that F is locally fibrant if and only if

$$s^* : \mathrm{map}^{\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})}(B, F) \to \mathrm{map}^{\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})}(A, F)$$

is a quasi-isomorphism in $\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})$. If s is of the form $0 \to \mathcal{A}(-,\emptyset)[n]$, this means that the map

$$\tau_{\geqslant 0}(F(\emptyset)[-n]) \to 0$$

is a quasi-isomorphism. This holds for every $n \in \mathbb{Z}$ if and only if $0 \to F(\emptyset)$ is a quasi-isomorphism. If s is of the form $p_Q : s_Q \to t_Q$ for an elementary Nisnevich square Q of the form

$$\begin{array}{ccc} U' & \xrightarrow{\beta} & X' \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} \\ U & \xrightarrow{\delta} & X \end{array}$$

then this means that the map

$$\tau_{\geqslant 0}(F(X)[-n]) \to \tau_{\geqslant 0}((F(X') \underset{F(U')}{\times^h} F(U))[-n])$$

is a quasi-isomorphism in $\mathsf{Ch}(\mathrm{Ab})$, where $F(X') \underset{F(U')}{\times^h} F(U)$ is the homotopy pullback of $F(U) \to F(U') \leftarrow F(X')$. This holds for every $n \in \mathbb{Z}$ if and only if

$$F(X) \to F(X') \underset{F(U')}{\times^h} F(U)$$

is a quasi-isomorphism in $\mathsf{Ch}(\mathrm{Ab})$, which is the case if and only if F sends Q to a homotopy pullback square. □

The property of sending elementary Nisnevich squares to homotopy pullback squares is also called the B.G.-property in [39]. We now prove basic facts about the local projective model structure.

2.2.5 Lemma. *A morphism $f : A \to B$ in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ is a weak equivalence in the local projective model structure if and only if it is a local quasi-isomorphism, in the sense that it is a stalkwise quasi-isomorphism with respect to the Nisnevich topology.*

Proof. This follows using a similar argument as in [31, C.2.1]. They use finite correspondences, but all the arguments of [31, §C.2] work for an arbitrary additive symmetric monoidal category of correspondences satisfying the strict V-property. □

2.2.6 Lemma. *Let* $C \in \mathsf{Psh}(\mathcal{A})$ *be projective. Then* C *is flat, in the sense that*

$$C \underset{\mathsf{Psh}}{\otimes} - : \mathsf{Psh}(\mathcal{A}) \to \mathsf{Psh}(\mathcal{A})$$

is an exact functor.

Proof. Since $\mathsf{Psh}(\mathcal{A})$ is an abelian category with enough projectives, we know that for every $A \in \mathsf{Psh}(\mathcal{A})$ the tensor product functor $A \underset{\mathsf{Psh}}{\otimes} -$ has left derived functors

$$\mathrm{Tor}_i^{\mathsf{Psh}}(A, -) : \mathsf{Psh}(\mathcal{A}) \to \mathsf{Psh}(\mathcal{A})$$

for $i \geqslant 0$. By [55, Corollary 2.4.2], if C is projective, then

$$\mathrm{Tor}_i^{\mathsf{Psh}}(A, C) = 0$$

for all $i \neq 0$ and all $A \in \mathsf{Psh}(\mathcal{A})$. Since $\mathrm{Tor}_i^{\mathsf{Psh}}$ is symmetric we therefore also get $\mathrm{Tor}_i^{\mathsf{Psh}}(C, A) = 0$. But this then means that the functor $C \underset{\mathsf{Psh}}{\otimes} - : \mathsf{Psh}(\mathcal{A}) \to \mathsf{Psh}(\mathcal{A})$ is exact. □

2.2.7 Lemma. *Let* $C \in \mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ *be a degreewise flat chain complex. Then* C *is a flat chain complex in the sense that*

$$C \otimes - : \mathsf{Ch}(\mathsf{Psh}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$$

is an exact functor.

Proof. Since the functor $C \otimes -$ is right exact, we just need to show that $C \otimes -$ preserves monomorphisms. Let $\iota : A \to B$ be a monomorphism in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$. For every $n \in \mathbb{Z}$ we have

$$(C \otimes \iota)_n = \bigoplus_{p+q=n} C_p \otimes \iota_q.$$

Since each C_p is flat and each ι_q is a monomorphism, each $C_p \otimes \iota_q$ is a monomorphism. Then $(C \otimes \iota)_n$ is a monomorphism because it is a direct sum of monomorphisms. So $C \otimes \iota$ is a monomorphism, and therefore C is flat in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$. □

There is an adjunction $L_{\mathsf{nis}} : \mathsf{Psh}(\mathcal{A}) \rightleftarrows \mathsf{Shv}(\mathcal{A}) : U_{\mathsf{nis}}$, where the left adjoint L_{nis} is Nisnevich sheafification and the right adjoint U_{nis} is the forgetful functor. The sheafification functor L_{nis} is well-defined because one of the axioms of the category of correspondences $\mathcal{A}$ states that for every $\mathcal{A}$-presheaf the associated sheaf with respect to the Nisnevich topology on Sm_k has a unique strucutre of an $\mathcal{A}$-presheaf. This adjunction extends to an adjunction on chain complexes

$$L_{\mathsf{nis}} : \mathsf{Ch}(\mathsf{Psh}(\mathcal{A})) \rightleftarrows \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) : U_{\mathsf{nis}}.$$

2.2.8 Lemma. *The local projective model structure on* $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ *is monoidal.*

Proof. We use [56, Theorem B]. Cofibrant objects in the local projective model structure are also cofibrant in the standard projective model structure. By Lemma 2.2.1 they are degreewise projective, and therefore degreewise flat by Lemma 2.2.6, and therefore flat by Lemma 2.2.7. We now need to show for every elementary Nisnevich square Q and cofibrant object K that the morphism

$$K \otimes p_Q : K \otimes s_Q \to K \otimes t_Q$$

is a local quasi-isomorphism. For this it suffices to show that the sheafification $L_{\mathsf{nis}}(K \otimes p_Q)$ is a local quasi-isomorphism. Since $L_{\mathsf{nis}} : \mathsf{Ch}(\mathsf{Psh}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is a strong monoidal functor we have

$$L_{\mathsf{nis}}(K \otimes p_Q) \cong L_{\mathsf{nis}}(K) \otimes L_{\mathsf{nis}}(p_Q).$$

Since K is a cofibrant object in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$, it follows that K is flat in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$. This then also implies that the sheafification $L_{\mathsf{nis}}K$ of K is flat in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, and this implies that the functor $L_{\mathsf{nis}}(K) \otimes - : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ preserves local quasi-isomorphisms. Since p_Q is a local quasi-isomorphism, it follows that $L_{\mathsf{nis}}(K) \otimes L_{\mathsf{nis}}(p_Q)$ is a local quasi-isomorphism. So $K \otimes p_Q$ is a local quasi-isomorphism. Similarly $0 \to K \otimes \mathcal{A}(-, \emptyset)$ is a local quasi-isomorphism. With this we have proved the lemma. □

We want to show that the local projective model structure is weakly finitely generated in the sense of [14, Definition 3.4]. For the convenience of the reader we recall this notion here.

2.2.9 Definition. A cofibrantly generated model category M is said to be *weakly finitely generated*, if it is cofibrantly generated and the generating cofibrations I and generating trivial cofibrations J can be chosen such that

1. The domains and codomains of maps in I are finitely presented.
2. The domains of maps in J are small.
3. There exists a subset $J' \subseteq J$ of maps with finitely presented domains and codomains, such that for every map $f : A \to B$, if B is fibrant and f has the right lifting property with respect to J', then f is a fibration.

We will call J' the set of *weakly generating trivial cofibrations*.

Let $I_{\mathsf{Ch}_{\geqslant 0}} = \{S^n\mathbb{Z} \to D^n\mathbb{Z} \mid n \geqslant 0\} \cup \{0 \to S^0\mathbb{Z}\}$ be a set of generating cofibrations for the standard projective model structure on the category of connective chain complexes $\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})$. Let $S\Box I_{\mathsf{Ch}_{\geqslant 0}}$ denote the set of all maps which are pushout-products of maps in S and $I_{\mathsf{Ch}_{\geqslant 0}}$.

2.2.10 Lemma. *An object* $F \in \mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ *is fibrant in the local projective model structure if and only if the map* $F \to 0$ *has the right lifting property with respect to* $S\Box I_{\mathsf{Ch}_{\geqslant 0}}$.

Proof. For $A, B \in \mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ let $\mathrm{map}^{\mathsf{Ch}(\mathrm{Ab})}(A, B) \in \mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})$ denote the good truncation of the chain complex of morphisms $A \to B$, just like in the proof of Lemma 2.2.4. An object $F \in \mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ is S-local if and only if for every $s : X \to Y$, $s \in S$ the map

$$s^* : \mathrm{map}^{\mathsf{Ch}(\mathrm{Ab})}(Y, F) \to \mathrm{map}^{\mathsf{Ch}(\mathrm{Ab})}(X, F)$$

is a quasi-isomorphism. Since s is a cofibration and F is fibrant, the map s^* is a fibration in $\mathsf{Ch}(\mathrm{Ab})$. So s^* is a quasi-isomorphism in $\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})$ if and only if s^* is trivial fibration in $\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})$, and that is the case if and only if s^* has the right lifting property with respect to $I_{\mathsf{Ch}_{\geqslant 0}}$. For every $\iota : A \to B$ in $I_{\mathsf{Ch}_{\geqslant 0}}$ we have that the following diagram has a lift

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{map}^{\mathsf{Ch}(\mathrm{Ab})}(Y, F) \\ {\scriptstyle\iota}\downarrow & \nearrow & \downarrow{\scriptstyle s^*} \\ B & \longrightarrow & \mathrm{map}^{\mathsf{Ch}(\mathrm{Ab})}(X, F) \end{array}$$

in $\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})$ if and only if the following diagram has a lift

$$\begin{array}{ccc} A \otimes Y \coprod_{A \otimes X} B \otimes X & \longrightarrow & F \\ {\scriptstyle \iota \square s} \downarrow & \nearrow & \downarrow \\ B \otimes Y & \longrightarrow & 0 \end{array}$$

in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$. So F is fibrant in the local projective model structure if and only if $F \to 0$ has the right lifting property with respect to $S \square I_{\mathsf{Ch}_{\geqslant 0}}$. $\square$

2.2.11 Lemma. *The local model structure on* $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ *is weakly finitely generated. A set of weakly generating trivial cofibrations is given by* $J'_{\mathrm{local}} := J_{\mathrm{proj}} \cup (S \square I_{\mathsf{Ch}_{\geqslant 0}})$.

Proof. The domains and codomains from J'_{local} are clearly finitely presented.

All morphisms from J_{proj} are local projective trivial cofibrations. Since S consists out of cofibrations that are S-local equivalences, it consists out of local projective trivial cofibrations. Since the local projective model structure is monoidal, it follows that $S \square I_{\mathsf{Ch}_{\geqslant 0}}$ consists out of local projective trivial cofibrations. So all morphisms from J'_{local} are trivial cofibrations in the local projective model structure, so $J'_{\mathrm{local}} \subseteq J_{\mathrm{local}}$ for a suitable choice of J_{local}.

Let $f : A \to B$ be a map in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$, where B is fibrant in the local projective model structure and f satisfies the right lifting property with respect to $J'_{\mathrm{local}} = J_{\mathrm{proj}} \cup (S \square I_{\mathsf{Ch}_{\geqslant 0}})$. Then f satisfies the right lifting property with respect to J_{proj}, so f is a fibration in the standard projective model structure. Since $f : A \to B$ and $B \to 0$ satisfy the right lifting property with respect to $S \square I_{\mathsf{Ch}_{\geqslant 0}}$, also the composition $A \to 0$ satisfies the right lifting property with respect to $S \square I_{\mathsf{Ch}_{\geqslant 0}}$. By Lemma 2.2.10 it follows that A is fibrant in the local projective model structure. From [27, Proposition 3.3.16] it follows that f is a fibration in the local projective model structure. So the local projective model structure on $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ is weakly finitely generated with J'_{local} as the set of weakly generating trivial cofibrations. $\square$

We next want to transfer the local projective model structure along the adjunction

$$L_{\mathsf{nis}} : \mathsf{Ch}(\mathsf{Psh}(\mathcal{A})) \rightleftarrows \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) : U_{\mathsf{nis}}.$$

2.2.12 Definition. Given a model category M and an adjunction $L : M \rightleftarrows N : R$ we say that the *left transferred model structure along L exists* if there is a model structure on N such that a morphism f in N is a weak equivalence (resp. fibration) if and only if $R(f)$ is a weak equivalence (resp. fibration) in M.

2.2.13 Remark. Let M be a model category and $L : M \rightleftarrows N : R$ an adjunction. If the left transferred model structure along L exists, then the adjunction $L : M \rightleftarrows N : R$ is a Quillen adjunction. If M is cofibrantly generated with generating cofibrations I and generating trivial cofibrations J and if $L(I)$ and $L(J)$ permit the small object argument in N, then $L(I)$ is a set of generating cofibrations and $L(J)$ is a set of generating trivial cofibrations for N.

We next want to show that the left transferred model structure along L_{nis} : $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ exists.

2.2.14 Lemma. *The forgetful functor* $U_{\mathsf{nis}} : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ *preserves filtered colimits.*

Proof. This follows from the fact that every covering in the Nisnevich topology has a finite subcovering. To spell it out in more detail, let I be a filtered diagram and $A_{(-)} : I \to \mathsf{Shv}(\mathcal{A})$ a functor. Let $A := \operatorname{colim}_{i\in I} U_{\mathsf{nis}}(A_i)$. We need to show that the canonical map

$$A \to U_{\mathsf{nis}}(\operatorname{colim}_{i\in I} A_i)$$

is an isomorphism. If we apply L_{nis} to this map then it clearly becomes an isomorphism in $\mathsf{Shv}(\mathcal{A})$. Also the presheaf $U_{\mathsf{nis}}(\operatorname{colim}_{i\in I} A_i)$ is a sheaf. To prove the lemma, it now suffices to show that the presheaf A is a sheaf.

Take a Nisnevich covering $\{Y_j \to X\}_{j\in J}$, and compatible sections $s_j \in A(Y_j)$. Since every covering has a finite subcovering we can assume without loss of generality that the index set J is finite. Now for each $j \in J$, there exists some $i_j \in I$ so that s_j is the restriction of some section $t_{i,j} \in U_{\mathsf{nis}}(A_{i_j})(Y_j)$ along the canonical map $U_{\mathsf{nis}}(A_{i_j}) \to A$. Since I is a filtered category, we can find a single $k \in I$ such that every s_j is the restriction of some section $t_j \in U_{\mathsf{nis}}(A_k)(Y_j)$ along the map $U_{\mathsf{nis}}(A_k) \to A$. Since A_k is a sheaf we can glue together all the sections t_j into a single section $t \in U_{\mathsf{nis}}(A_k)(X)$. If we include t into the colimit $\operatorname{colim}_{i\in I} U_{\mathsf{nis}}(A_i)(Y_j)$ then we get a section $s \in A(X)$ which is a unique gluing of all the s_j. So A is a sheaf, and U_{nis} preserves filtered colimits. □

2.2.15 Corollary. *$L_{\mathsf{nis}} : \mathsf{Ch}(\mathsf{Psh}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ preserves finitely presented objects.*

Proof. Let $X \in \mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ be finitely presented. Let I be a filtered diagram, and let $A_{(-)} : I \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ be a functor. Then using Lemma 2.2.14 we get

$$\begin{aligned}&\mathrm{Hom}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(L_{\mathsf{nis}}X, \operatorname*{colim}_{i\in I} A_i) \cong \mathrm{Hom}_{\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))}(X, U_{\mathsf{nis}}\operatorname*{colim}_{i\in I} A_i) \overset{2.2.14}{\cong} \\ &\mathrm{Hom}_{\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))}(X, \operatorname*{colim}_{i\in I} U_{\mathsf{nis}}A_i) \cong \operatorname*{colim}_{i\in I} \mathrm{Hom}_{\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))}(X, U_{\mathsf{nis}}A_i) \cong \\ &\operatorname*{colim}_{i\in I} \mathrm{Hom}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(L_{\mathsf{nis}}X, A_i)\end{aligned}$$

so $L_{\mathsf{nis}}X$ is finitely presented. □

2.2.16 Lemma. *For the local projective model structure on $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$, the left transferred model structure along $L_{\mathsf{nis}} : \mathsf{Ch}(\mathsf{Psh}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ exists.*

Proof. We use [27, Theorem 11.3.2]. Since $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is a Grothendieck category [1, Proposition 3.4], every object is small, so I_{local} and J_{local} permit the small object argument.

Next, we need to show that U_{nis} takes relative $L_{\mathsf{nis}}(J_{\mathrm{local}})$-complexes to stalkwise quasi-isomorphisms in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$. Since U_{nis} preserves filtered colimits, it commutes with transfinite compositions. Also, stalkwise quasi-isomorphisms are closed under transfinite composition. It therefore suffices to show that U_{nis} takes any pushout of a map from $L_{\mathsf{nis}}(J_{\mathrm{local}})$ to a stalkwise quasi-isomorphism.

Let $f : A \to B$ be a map in J_{local}, and consider a pushout of the form

$$\begin{array}{ccc} L_{\mathsf{nis}}A & \xrightarrow{L_{\mathsf{nis}}f} & L_{\mathsf{nis}}B \\ \downarrow & & \downarrow \\ X & \xrightarrow{\quad g \quad} & Y \end{array}$$

We need to show that $U_{\mathsf{nis}}g$ is a stalkwise quasi-isomorphism. Since $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is an abelian category, this pushout gives rise to a short exact sequence in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$

$$0 \to L_{\mathsf{nis}}A \to L_{\mathsf{nis}}B \oplus X \to Y \to 0.$$

For every point x of the Nisnevich site, we get a short exact sequence on stalks

$$0 \to A_x \to B_x \oplus X_x \to Y_x \to 0$$

in $\mathsf{Ch}(\mathrm{Ab})$. This short exact sequence of chain complexes induces a long exact sequence on homology groups

$$\cdots \to H_{n+1}(Y_x) \to H_n(A_x) \to H_n(B_x) \oplus H_n(X_x) \to H_n(Y_x) \to H_{n-1}(A_x) \to \cdots$$

Since f is in J_{local}, it is a stalkwise quasi-isomorphism, so the map $H_n(A_x) \to H_n(B_x)$ is an isomorphism. This then implies that $H_n(X_x) \to H_n(Y_x)$ is also an isomorphism, so $g : X \to Y$ is a stalkwise quasi-isomorphism.

Therefore the transferred model structure on $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ exists, with generating cofibrations $L_{\mathsf{nis}}(I_{\mathrm{local}})$ and generating trivial cofibrations $L_{\mathsf{nis}}(J_{\mathrm{local}})$, and the adjunction $L_{\mathsf{nis}} : \mathsf{Ch}(\mathsf{Psh}(\mathcal{A})) \leftrightarrows \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) : U_{\mathsf{nis}}$ is a Quillen adjunction. $\square$

2.2.17 Lemma. *Let M be a model category that is weakly finitely generated with weakly generating trivial cofibrations J'_M, and let $L : M \rightleftarrows N : R$ be an adjunction, such that the left transferred model structure along L exists. Assume that L preserves small objects and finitely presented objects. Then the transferred model structure on N is weakly finitely generated, and $L(J'_M)$ is a set of weakly generating trivial cofibrations for N.*

Proof. Let I_M denote a set of generating cofibrations and J_M denote a set of generating trivial cofibrations for M. Then by definition of the transferred model structure, $L(I_M)$ is a set of generating cofibrations and $L(J_M)$ is a set of generating trivial cofibrations for N.

Since L preserves small objects and finitely presented objects, the domains and codomains from $L(I_M)$ and $L(J'_M)$ are finitely presented, and the domains from $L(J_M)$ are small.

Take $f : A \to B$ in N with B fibrant and f having the right lifting property with respect to $L(J'_M)$. To show the lemma we now just have to show that f is a fibration in N. By adjunction $R(f)$ has the right lifting property with respect to J'_M. Since $R : N \to M$ is a right Quillen functor and B is fibrant in N we know that $R(B)$ is fibrant in M. Since J'_M is a set of weakly generating trivial cofibrations for M it now follows that $R(f)$ is a fibration in M. From the definition of the transferred model structure it follows that f is a fibration in N. Therefore $L(J'_M)$ is a set of weakly generating trivial cofibrations for N. $\square$

2.2.18 Corollary. *The model category $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is weakly finitely generated, with $L_{\mathsf{nis}}(J'_{\mathrm{local}})$ as a set of weakly generating trivial cofibrations.*

Proof. By Lemma 2.2.15 we know that L_{nis} preserves finitely presented objects. It also preserves small objects, because all objects in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ are small. The result now follows from Lemma 2.2.17. $\square$

There is a symmetric monoidal structure on $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ defined by $X \otimes Y := L_{\mathsf{nis}}(U_{\mathsf{nis}}(X) \otimes U_{\mathsf{nis}}(Y))$. With respect to this monoidal structure the adjunction $L_{\mathsf{nis}} : \mathsf{Ch}(\mathsf{Psh}(\mathcal{A})) \rightleftarrows \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) : U_{\mathsf{nis}}$ is a monoidal adjunction. This means that the left adjoint L_{nis} is strong monoidal, while the right adjoint U_{nis} is lax monoidal. We use the following lemma to make $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ into a monoidal model category in the sense of [47, Definition 3.1].

2.2.19 Lemma. *Let M, N be closed symmetric monoidal categories, and let $L : M \leftrightarrows N : R$ be a monoidal adjunction. Let M be equipped with a cofibrantly generated monoidal model structure with generating cofibrations I and generating trivial cofibrations J. Assume that the left transferred model structure along $L : M \to N$ exists and that $L(I)$ and $L(J)$ permit the small object argument. Furthermore assume that the monoidal unit $\mathbb{1}_M$ is cofibrant in M. Then the left transferred model structure on N is a monoidal model structure and the unit $\mathbb{1}_N$ is cofibrant.*

Proof. Let I be the generating cofibrations of M, and let J be the generating trivial cofibrations of M. Then $L(I)$ is a set of generating cofibrations and $L(J)$ is a set of generating trivial cofibrations for N. Given two morphisms f, g, we write $f \square g$ to denote the pushout-product of f and g. To verify the pushout-product axiom for the transferred model structure on N, it suffices by [28, Corollary 4.2.5] to show that $L(I) \square L(I)$ consists out of cofibrations, and $L(J) \square L(I)$ consists out of trivial cofibrations.

Since L is a strong monoidal left adjoint functor, it preserves pushout products, in the sense that for all morphisms $f : A \to B$ and $g : C \to D$ in M we have a commutative diagram in which the vertical maps are isomorphisms:

$$\begin{array}{ccc}
L(A \otimes D \coprod_{A \otimes C} B \otimes C) & \xrightarrow{L(f \square g)} & L(B \otimes D) \\
\downarrow \sim & & \downarrow \sim \\
L(A) \otimes L(D) \coprod_{L(A) \otimes L(C)} L(B) \otimes L(C) & \xrightarrow{L(f) \square L(g)} & L(B) \otimes L(D)
\end{array}$$

This can also be expressed by saying that $L(f \Box g) \cong L(f) \Box L(g)$ in the arrow category $\mathsf{Arr}(N)$.

So any morphism in $L(I) \Box L(I)$, respectively $L(J) \Box L(I)$, is isomorphic to a morphism in $L(I \Box I)$, respectively $L(J \Box I)$, in the arrow category $\mathsf{Arr}(N)$. Since M is a monoidal model category, all morphisms from $I \Box I$, respectively $J \Box I$, are cofibrations, respectively trivial cofibrations. Since $L : M \to N$ is a left Quillen functor it preserves cofibrations and trivial cofibrations. Since cofibrations and trivial cofibrations are closed under isomorphisms in $\mathsf{Arr}(N)$ it follows that $L(I) \Box L(I)$ consists out of cofibrations and $L(J) \Box L(I)$ consists out of trivial cofibrations. So N satisfies the pushout-product axiom.

Since $\mathbb{1}_M$ is cofibrant in M and L is a left Quillen functor, $L(\mathbb{1}_M)$ is cofibrant in N. Since L is strong monoidal $L(\mathbb{1}_M) \cong \mathbb{1}_N$, so $\mathbb{1}_N$ is cofibrant in N. This in particular implies that N is a monoidal model category. □

We will now prove some lemmas to show that $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ satisfies the monoid axiom.

2.2.20 Lemma. *If* $f \in J'_{\text{local}}$ *then* $\operatorname{Coker}(f) \in \mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ *is a bounded chain complex and degreewise free.*

Proof. Take $f \in J'_{\text{local}}$. Then $f \in J_{\text{proj}}$ or $f \in S \Box I_{\mathsf{Ch}_{\geqslant 0}}$. If $f \in J_{\text{proj}}$, then

$$\operatorname{Coker}(f) = \mathcal{A}(-, X) \otimes D^n\mathbb{Z}$$

for some $X \in \mathsf{Sm}_k, n \in \mathbb{Z}$, and that is clearly bounded and free. If $f \in S \Box I_{\mathsf{Ch}_{\geqslant 0}}$, then $f = g \Box h$ for some $g \in I_{\mathsf{Ch}_{\geqslant 0}}$ and some $h \in S$. Since g is just a map of the form $S^n\mathbb{Z} \to D^n\mathbb{Z}$ for some $n \geqslant 0$, it suffices to show that h has a bounded and degreewise free cokernel. Up to a shift, h is either the morphism $0 \to \mathcal{A}(-, \emptyset)$ or h is a morphism of the form $s_Q \to t_Q$ for some Nisnevich square $Q \in \mathcal{Q}$. The cokernel of $0 \to \mathcal{A}(-, \emptyset)$ is clearly bounded and free. So assume now that h is of the form $s_Q \to t_Q$ for some Nisnevich square $Q \in \mathcal{Q}$, of the form

$$\begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

Recall from Definition 2.2.3 that s_Q is defined via the pushout square

$$\begin{array}{ccc} \mathcal{A}(-,U') & \longrightarrow & C \\ \downarrow & & \downarrow \\ \mathcal{A}(-,U) & \longrightarrow & s_Q \end{array}$$

where C is the mapping cylinder of $\mathcal{A}(-,U') \to \mathcal{A}(-,X')$. By the usual construction of mapping cylinders [55, 1.5.5] we have in each individual degree n an equality

$$C_n = \mathcal{A}(-,U')_n \oplus \mathcal{A}(-,U')_{n-1} \oplus \mathcal{A}(-,X')_n$$

and the canonical map $\mathcal{A}(-,U') \to C$ is in each individual degree n a coproduct inclusion.

Thus the pushout defining s_Q is a pushout of bounded and degreewise free complexes along a morphism which is degreewise a coproduct inclusion. This then implies that s_Q is bounded and degreewise free.

Next, recall that t_Q is defined as the mapping cylinder of $s_Q \to \mathcal{A}(-,X)$. Thus the canonical map $h : s_Q \to t_Q$ is also a degreewise coproduct inclusion between bounded and degreewise free objects. This then implies that $\mathrm{Coker}(h)$ is bounded and degreewise free.

And then it follows that $\mathrm{Coker}(f)$ is bounded and degreewise free. □

2.2.21 Lemma. *If $f \in J'_{\mathrm{local}}$ and $Z \in \mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$, then $f \otimes Z$ is a local quasi-isomorphism and a monomorphism in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$.*

Proof. We can calculate $f \otimes Z$ in degree $n \in \mathbb{Z}$ by

$$(f \otimes Z)_n = \bigoplus_{i+j=n} f_i \otimes Z_j.$$

By Lemma 2.2.1 each f_i is a split monomorphism. Then also every $f_i \otimes Z_j$ is a split monomorphism, so their direct sum is a split monomorphism. So $f \otimes Z$ is a monomorphism. We now just need to show that $f \otimes Z$ is a local quasi-isomorphism. Since it is already a monomorphism, we now just need to show that $\mathrm{Coker}(f \otimes Z)$ is locally acyclic. Let $C := \mathrm{Coker}(f)$. By Lemma 2.2.20 the complex C is bounded and degreewise free. Since f is a local quasi-isomorphism, we know that C is locally acyclic. Also we have an isomorphism $\mathrm{Coker}(f \otimes Z) \cong$

$\mathrm{Coker}(f) \otimes Z = C \otimes Z$. So to prove the lemma we now just need to show the following claim:

If $C \in \mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ is bounded, degreewise free and locally acyclic, then $C \otimes Z$ is locally acyclic.

We will first show this claim for the case where Z is concentrated in degree 0. So we assume $Z \in \mathsf{Psh}(\mathcal{A})$. We claim that $C \otimes Z$ is locally acyclic.

Take a free resolution of Z in $\mathsf{Psh}(\mathcal{A})$

$$\cdots \to F_2 \to F_1 \to F_0 \to Z \to 0.$$

We can tensor this resolution with C to get the following double complex

$$\begin{array}{ccccccc}
 & & \cdots & & \cdots & & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & F_1 \otimes C_1 & \longrightarrow & F_0 \otimes C_1 & \longrightarrow & Z \otimes C_1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & F_1 \otimes C_0 & \longrightarrow & F_0 \otimes C_0 & \longrightarrow & Z \otimes C_0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \cdots & & \cdots & & \cdots
\end{array}$$

Denote this double complex by $D_{\bullet,\bullet}$.

Since C is degreewise free, by Lemma 2.2.6 each C_i is also flat, so each row is exact. This then means that the horizontal homology of $D_{\bullet,\bullet}$ vanishes. So we have for all $q \in \mathbb{Z}$,

$$H_{\mathrm{hor},q}(D_{\bullet,\bullet}) = 0$$

in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$.

Associated to the double complex D we have a spectral sequence in $\mathsf{Psh}(\mathcal{A})$ computing the homology of the total complex [41].

$$E^2_{p,q} = H_{\mathrm{vert},p}(H_{\mathrm{hor},q}(D_{\bullet,\bullet})) \implies H_{p+q}(\mathrm{Tot}(D_{\bullet,\bullet}))$$

Since $H_{\mathrm{hor},q}(D_{\bullet,\bullet}) = 0$ it follows that $H_{p+q}(\mathrm{Tot}(D_{\bullet,\bullet})) = 0$.

If this homology vanishes, then it also locally vanishes. So if $L_{\mathsf{nis}}(D_{\bullet,\bullet})$ denotes the sheafification of $D_{\bullet,\bullet}$, and if H^{nis} denotes Nisnevich homology sheaves in $\mathsf{Shv}(\mathcal{A})$, then we have for all $p.q \in \mathbb{Z}$ that $H^{\mathsf{nis}}_{p+q}(\mathrm{Tot}(L_{\mathsf{nis}}(D_{\bullet,\bullet})) = 0$.

By mirroring the double complex $L_{\mathsf{nis}}(D_{\bullet,\bullet})$ and then using the double complex spectral sequence in the Grothendieck category $\mathsf{Shv}(\mathcal{A})$, we get another spectral sequence computing the same homology

$$E^2_{p,q} = H^{\mathsf{nis}}_{\mathrm{hor},p}(H^{\mathsf{nis}}_{\mathrm{vert},q}(L_{\mathsf{nis}}(D_{\bullet,\bullet}))) \implies H^{\mathsf{nis}}_{p+q}(\mathrm{Tot}(L_{\mathsf{nis}}(D_{\bullet,\bullet}))).$$

Since C is bounded, degreewise free and locally acyclic, and since each F_i is free, we can use an argument similar to [50, Corollary 2.3] to show for every $q \geqslant 0$ that

$$H^{\mathsf{nis}}(L_{\mathsf{nis}}(F_q \otimes C)) = 0.$$

This then means that the Nisnevich homology of all vertical columns of $L_{\mathsf{nis}}(D_{\bullet,\bullet})$ in positive degree vanishes. So for $q \neq 0$ and $p \in \mathbb{Z}$ we have

$$H^{\mathsf{nis}}_{\mathrm{vert},q}(L_{\mathsf{nis}}(D_{\bullet,\bullet}))_p = H^{\mathsf{nis}}_p(L_{\mathsf{nis}}(F_{q-1} \otimes C)) = 0.$$

Here we consider the $L_{\mathsf{nis}}(Z \otimes C_i)$ column of $L_{\mathsf{nis}}(D_{\bullet,\bullet})$ to be in degree 0.

Thus the spectral sequence $E^2_{p,q} = H^{\mathsf{nis}}_{\mathrm{hor},p}(H^{\mathsf{nis}}_{\mathrm{vert},q}(L_{\mathsf{nis}}(D_{\bullet,\bullet})))$ stabilizes at the second page, and consists only of a single column whose terms are $H^{\mathsf{nis}}_p(L_{\mathsf{nis}}(Z \otimes C))$. Since the spectral sequence converges against $H^{\mathsf{nis}}_{p+q}(\mathrm{Tot}(L_{\mathsf{nis}}(D_{\bullet,\bullet}))) = 0$ it follows that $H^{\mathsf{nis}}_p(L_{\mathsf{nis}}(Z \otimes C)) = 0$ for every p, so the chain complex $Z \otimes C$ is locally acyclic.

So we have now shown the lemma in the case where Z is concentrated in degree 0. Let us show the lemma in full generality. Namely, let C be bounded, degreewise free and locally acyclic, and let $Z \in \mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ be any chain complex. We claim that $C \otimes Z$ is locally acyclic.

For every $k \in \mathbb{Z}$, let $\tau_k(Z)$ denote the following truncated chain complex

$$\cdots \to Z_{k+3} \overset{\partial^{k+3}_Z}{\to} Z_{k+2} \overset{\partial^{k+2}_Z}{\to} Z_{k+1} \to \ker(\partial^k_Z) \to 0,$$

where $\ker(\partial^k_Z)$ is in degree k. The chain complex $\tau_k(Z)$ is k-connected.

For every $k \in \mathbb{Z}$ there is a canonical map $\varphi_k : \tau_k(Z) \to \tau_{k-1}(Z)$ with $\varphi_{k,i} = id_{Z_i}$ for all $i \geqslant k+1$, as shown in this diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & Z_{k+2} & \xrightarrow{\partial^{k+2}_Z} & Z_{k+1} & \xrightarrow{\partial^{k+1}_Z} & \ker(\partial^k_Z) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & Z_{k+2} & \xrightarrow{\partial^{k+2}_Z} & Z_{k+1} & \xrightarrow{\partial^{k+1}_Z} & Z_k & \xrightarrow{\partial^k_Z} & \ker(\partial^{k-1}_Z)
\end{array}$$

In $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ we can consider the $\mathbb{Z}$-indexed diagram

$$\cdots \to \tau_{k+1}(Z) \to \tau_k(Z) \to \tau_{k-1}(Z) \to \cdots$$

The colimit of this diagram is obviously Z. In particular $C \otimes Z \cong \operatorname{colim}_{k\in\mathbb{Z}}(C \otimes \tau_k(Z))$.

Since filtered colimits in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ preserve local quasi-isomorphisms, we know that filtered colimits of locally acyclic objects are locally acyclic. So to show that $C \otimes Z$ is locally acyclic, we now just need to show that each $C \otimes \tau_k(Z)$ is locally acyclic. Let $k \in \mathbb{Z}$ be arbitrary. We have a distinguished triangle in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$

$$\tau_{k+1}(Z)[-k] \to \tau_k(Z)[-k] \to H_k(Z) \to \tau_{k+1}(Z)[1-k]$$

where $H_k(Z) \in \mathsf{Psh}(\mathcal{A})$ is regarded as a chain complex concentrated in degree 0. So if we consider the following diagram in $D(\mathsf{Psh}(\mathcal{A}))$

$$\cdots \to \tau_{k+i}(Z)[-k] \to \cdots \to \tau_{k+1}(Z)[-k] \to \tau_k(Z)[-k]$$

then for every $i \in \mathbb{N}$, the i-th morphism in the sequence has a cofiber isomorphic to $H_{k+i}(Z)[i]$. Also the i-th term in the sequence $\tau_{k+i}(Z)[-k]$ is i-connected. By Lemma 2.2.6 we know that C is degreewise flat. So if we tensor the above diagram with C we get a diagram

$$\cdots \to C \otimes \tau_{k+i}(Z)[-k] \to \cdots \to C \otimes \tau_{k+1}(Z)[-k] \to C \otimes \tau_k(Z)[-k]$$

in which the i-th morphism has a cofiber isomorphic to $C \otimes H_{k+i}(Z)[i]$. From [18, Corollary 6.1.1] we get a strongly convergent spectral sequence

$$E^2_{pq} = H^{\mathsf{nis}}_{p+q}(C \otimes H_{k+q}(Z)[q]) \implies H^{\mathsf{nis}}_{p+q}(C \otimes \tau_k(Z)[-k]).$$

Since $H_{k+q}(Z)[q]$ is concentrated in a single degree, we know that $C \otimes H_{k+q}(Z)[q]$ is locally acyclic. So $H^{\mathsf{nis}}_{p+q}(C \otimes H_{k+q}(Z)[q]) = 0$, and then the spectral sequence implies that $H^{\mathsf{nis}}_{p+q}(C \otimes \tau_k(Z)[-k]) = 0$, hence $C \otimes \tau_k(Z)[-k]$ is locally acyclic. Then also $C \otimes \tau_k(Z)$ is locally acyclic, and then also the colimit $C \otimes Z \cong \operatorname{colim}_{k\in\mathbb{Z}}(C \otimes \tau_k(Z))$ is locally acyclic, which then proves the entire lemma. □

2.2.22 Lemma. *Let M be a monoidal model category that is weakly finitely generated. Denote the set of weakly generating trivial cofibrations by J'.*

Then the monoid axiom for M can be checked on J'. This means with the notations from [47], that if every element of $(J' \otimes M)-\mathrm{cof}_{\mathrm{reg}}$ is a weak equivalence then M satisfies the monoid axiom.

Proof. Before verifying the monoid axiom we first show that every trivial cofibration with fibrant codomain lies in $J'-\mathrm{cof}$.

Let $f : A \overset{\sim}{\rightarrowtail} B$ be a trivial cofibration with fibrant codomain B. We claim that f lies in $J'-\mathrm{cof}$. According to the small object argument [47, Lemma 2.1] we can factor f as $f = qi$ with $q \in \mathrm{RLP}(J')$ and $i \in J'-\mathrm{cof}_{\mathrm{reg}}$.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle q} & \\ & Z & \end{array}$$

Since q has a fibrant codomain and $q \in \mathrm{RLP}(J')$ it follows that q is a fibration. Then f has the left lifting property against q so by [28, Lemma 1.1.9] f is a retract of i. Since $i \in J'-\mathrm{cof}_{\mathrm{reg}}$ this implies $f \in J'-\mathrm{cof}$.

Now we start verifying the monoid axiom. Assume every element of $(J' \otimes M)-\mathrm{cof}_{\mathrm{reg}}$ is a weak equivalence. Let $f : A \overset{\sim}{\rightarrowtail} B$ be any trivial cofibration, let $Z \in M$ be any object and consider an arbitrary pushout diagram of the form

$$\begin{array}{ccc} A \otimes Z & \xrightarrow{f \otimes Z} & B \otimes Z \\ \downarrow & & \downarrow \\ X & \xrightarrow{h} & Y \end{array}$$

We claim that h is a weak equivalence. Since M is weakly finitely generated, we know by [14, Lemma 3.5] that transfinite compositions of weak equivalences are weak equivalences in M. So if we show that h is a weak equivalence, then this immediately implies the monoid axiom.

Denote the terminal object of M by 1. Factor the map $B \to 1$ into a trivial cofibration followed by a fibration. We then have a trivial cofibration $g : B \overset{\sim}{\rightarrowtail} B^f$ with B^f fibrant. Then both $g : B \to B^f$ and $gf : A \to B^f$ are trivial cofibrations with fibrant codomain. So g and gf both lie in $J'-\mathrm{cof}$. Then $Z \otimes g$ and $Z \otimes gf$ lie in $Z \otimes (J'-\mathrm{cof})$. By a simple argument using the adjunction $- \otimes Z \dashv \mathrm{Hom}(Z, -)$ one can show that $Z \otimes (J'-\mathrm{cof}) \subseteq (Z \otimes J')-\mathrm{cof}$. So $Z \otimes g$ and $Z \otimes gf$ lie in $(Z \otimes J')-\mathrm{cof}$, and thus also in $(M \otimes J')-\mathrm{cof}$.

Consider the pushout diagram

$$\begin{array}{ccccc} A \otimes Z & \xrightarrow{f\otimes Z} & B \otimes Z & \xrightarrow{g\otimes Z} & B^f \otimes Z \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{h} & Y & \xrightarrow{k} & (B^f \otimes Z) \coprod_{B\otimes Z} Y \end{array}$$

Since $g \otimes Z$ and $gf \otimes Z$ lie in $(J' \otimes M)-\mathrm{cof}$, and since $(J' \otimes M)-\mathrm{cof}$ is stable under pushouts, it follows that k and kh also lie in $(J' \otimes M)-\mathrm{cof}$. By [47, Lemma 2.1] this means that k and kh are retracts of morphisms from $(J' \otimes M)-\mathrm{cof}_{\mathrm{reg}}$. Since we assume that all morphisms from $(J' \otimes M)-\mathrm{cof}_{\mathrm{reg}}$ are weak equivalences, and since weak equivalences are stable under retracts, it follows that k and kh are weak equivalences. Then by 2-of-3 also h is a weak equivalence. This then proves the monoid axiom for M. □

2.2.23 Lemma. $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ *satisfies the monoid axiom in the sense of [47].*

Proof. By Lemmas 2.2.22 and 2.2.18 it suffices to check the monoid axiom on the set $L_{\mathsf{nis}}(J'_{\mathrm{local}})$.

Take $f : A \to B$, with $f \in L_{\mathsf{nis}}(J'_{\mathrm{local}})$ and take $Z \in \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. We claim that $f \underset{\mathrm{Shv}}{\otimes} Z$ is an injective quasi-isomorphism. Since $\mathsf{Shv}(\mathcal{A})$ is a Grothendieck category, we know that injective quasi-isomorphisms in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ are stable under pushouts and transfinite compositions. So if we show that $f \underset{\mathrm{Shv}}{\otimes} Z$ is an injective quasi-isomorphism, then this proves the entire monoid axiom.

If $f \in L_{\mathsf{nis}}(J'_{\mathrm{local}})$, then there exists $f' : A' \to B'$ with $f' \in J'_{\mathrm{local}}$ and $L_{\mathsf{nis}}(f') = f$. By Lemma 2.2.21 we know $f' \underset{\mathrm{Psh}}{\otimes} U_{\mathsf{nis}}Z$ is an injective local quasi-isomorphism in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$. Since L_{nis} is strongly monoidal we have an isomorphism of arrows

$$L_{\mathsf{nis}}(f' \underset{\mathrm{Psh}}{\otimes} U_{\mathsf{nis}}Z) \cong L_{\mathsf{nis}}(f') \underset{\mathrm{Shv}}{\otimes} L_{\mathsf{nis}}U_{\mathsf{nis}}Z \cong f \underset{\mathrm{Shv}}{\otimes} Z$$

So we just need to show that $L_{\mathsf{nis}}(f' \underset{\mathrm{Psh}}{\otimes} U_{\mathsf{nis}}Z)$ is an injective quasi-isomorphism.

Since $f' \underset{\mathrm{Psh}}{\otimes} U_{\mathsf{nis}}Z$ is injective, and the sheafification functor L_{nis} is exact, we know that $L_{\mathsf{nis}}(f' \underset{\mathrm{Psh}}{\otimes} U_{\mathsf{nis}}Z)$ is injective. So we now just need to show that $L_{\mathsf{nis}}(f' \underset{\mathrm{Psh}}{\otimes} U_{\mathsf{nis}}Z)$ is a quasi-isomorphism. By definition of the transferred model structure on $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ we thus need to show that $U_{\mathsf{nis}}L_{\mathsf{nis}}(f' \underset{\mathrm{Psh}}{\otimes} U_{\mathsf{nis}}Z)$ is a local quasi-isomorphism in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$.

We have a commutative diagram, where η is the unit of the adjunction $L_{\mathsf{nis}} \dashv U_{\mathsf{nis}}$:

$$\begin{array}{ccc} U_{\mathsf{nis}}L_{\mathsf{nis}}(A' \underset{\mathrm{Psh}}{\otimes} U_{\mathsf{nis}}Z) & \xrightarrow{U_{\mathsf{nis}}L_{\mathsf{nis}}(f' \underset{\mathrm{Psh}}{\otimes} U_{\mathsf{nis}}Z)} & U_{\mathsf{nis}}L_{\mathsf{nis}}(B' \underset{\mathrm{Psh}}{\otimes} U_{\mathsf{nis}}Z) \\ \eta\uparrow & & \eta\uparrow \\ A' \underset{\mathrm{Psh}}{\otimes} U_{\mathsf{nis}}Z & \xrightarrow{f' \underset{\mathrm{Psh}}{\otimes} U_{\mathsf{nis}}Z} & B' \underset{\mathrm{Psh}}{\otimes} U_{\mathsf{nis}}Z \end{array}$$

The diagram commutes by the naturality of η. Since η is stalkwise an isomorphism, it is by Lemma 2.2.5 in particular a local quasi-isomorphism in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$.

Since $f' \underset{\mathrm{Psh}}{\otimes} U_{\mathsf{nis}}Z$ is also a local quasi-isomorphism, it follows from the 2-of-3-property that $U_{\mathsf{nis}}L_{\mathsf{nis}}(f' \underset{\mathrm{Psh}}{\otimes} U_{\mathsf{nis}}Z)$ is a local quasi-isomorphism. So $f \otimes Z \cong L_{\mathsf{nis}}(f' \underset{\mathrm{Psh}}{\otimes} U_{\mathsf{nis}}Z)$ is an injective quasi-isomorphism, and this concludes the proof of the lemma. □

2.2.24 Lemma. $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ *is strongly left proper in the sense of [14, Definition 4.6].*

Proof. For any Grothendieck category $\mathcal{B}$, quasi-isomorphisms in $\mathsf{Ch}(\mathcal{B})$ are stable under pushouts along degreewise monomorphisms. So to show that $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is strongly left proper we just need to show that for any cofibration f and any object $Z \in \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ the map $Z \otimes f$ is a degreewise monomorphism. The set $L_{\mathsf{nis}}(I_{\mathrm{proj}})$ is a set of generating cofibrations for $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ so we have $f \in L_{\mathsf{nis}}(I_{\mathrm{proj}}) - \mathrm{cof}$. Then

$$Z \otimes f \in (Z \otimes L_{\mathsf{nis}}(I_{\mathrm{proj}})) - \mathrm{cof}\,.$$

All morphisms from $L_{\mathsf{nis}}(I_{\mathrm{proj}})$ are degreewise split monomorphisms, so all morphisms from $Z \otimes L_{\mathsf{nis}}(I_{\mathrm{proj}})$ are degreewise split monomorphisms, and this implies that all morphisms from $(Z \otimes L_{\mathsf{nis}}(I_{\mathrm{proj}})) - \mathrm{cof}$ are degreewise split monomorphisms. So $Z \otimes f$ is a degreewise split monomorphism. Therefore $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is strongly left proper. □

Chapter 3

First Reconstruction Theorem for $DM_{\mathcal{A}}$

In this chapter in Section 3.1 we state two reconstruction theorems that recover $DM_{\mathcal{A}}$ from certain derived categories of enriched functors. The first reconstruction theorem is Theorem 3.1.8 and recovers $DM_{\mathcal{A}}$ from enriched functors on an enriched category $\mathcal{C}$ whose objects are powers of $\mathbb{G}_m$. The second reconstruction theorem is Theorem 3.1.14. It requires inverting the exponential characteristic p of k, and recovers $DM_{\mathcal{A}}[1/p]$ from enriched functors on an enriched category $\mathcal{S}m$ whose objects are the smooth schemes. In this chapter we will also prove the first reconstruction theorem in Section 3.2. The proof of the second reconstruction theorem will be proven in Chapter 4.

3.1 Statements of the two reconstruction theorems

From now on we will additionally assume that $\mathcal{A}$ satisfies the cancellation property in the sense of Definition 2.1.2. We define a $\mathsf{Shv}(\mathcal{A})$-enriched category $\mathcal{S}m$, by letting the objects of $\mathcal{S}m$ be smooth schemes over k, and by defining

$$\mathcal{S}m(X,Y) := \underline{\mathrm{Hom}}_{\mathsf{Shv}(\mathcal{A})}(\mathcal{A}(-,X)_{\mathsf{nis}}, \mathcal{A}(-,Y)_{\mathsf{nis}}).$$

We have a $\mathsf{Shv}(\mathcal{A})$-enriched inclusion functor $I : \mathcal{S}m \to \mathsf{Shv}(\mathcal{A})$ defined on objects by $I(X) := \mathcal{A}(-,X)_{\mathsf{nis}}$, and which acts on morphism sets as the identity

$\mathcal{S}m(X,Y) = \underline{\mathrm{Hom}}_{\mathsf{Shv}(\mathcal{A})}(I(X), I(Y))$.

Let $\mathcal{C}$ be the full enriched subcategory of $\mathcal{S}m$ consisting of the objects $\mathbb{G}_m^{\times n}$ where $n \in \mathbb{Z}_{\geqslant 0}$.

We write $\underset{\mathrm{Shv}}{\otimes}$ for the tensor product of $\mathsf{Shv}(\mathcal{A})$, and $\underset{\mathrm{Day}}{\otimes}$ for the Day convolution product on $[\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]$ or $[\mathcal{C}, \mathsf{Shv}(\mathcal{A})]$, as defined in [10]:

$$(F \underset{\mathrm{Day}}{\otimes} G)(c) = \int^{(a,b)\in\mathcal{S}m\otimes\mathcal{S}m} \mathcal{S}m(a \times b, c) \underset{\mathrm{Shv}}{\otimes} F(a) \underset{\mathrm{Shv}}{\otimes} G(b).$$

The Grothendieck category of enriched functors $[\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]$ is tensored and cotensored over $\mathsf{Shv}(\mathcal{A})$ by $\underset{\mathrm{Shv}}{\otimes}$. Given any enriched functor $F : \mathcal{S}m \to \mathsf{Shv}(\mathcal{A})$ and $X \in \mathsf{Shv}(\mathcal{A})$ we can form an enriched functor $F \underset{\mathrm{Shv}}{\otimes} X$, given by

$$F \underset{\mathrm{Shv}}{\otimes} X(U) := F(U) \underset{\mathrm{Shv}}{\otimes} X.$$

If X is representable by a scheme U, so that $X = \mathcal{A}(-,U)_{\mathsf{nis}}$, then we write $F \underset{\mathrm{Shv}}{\otimes} U$ for $F \underset{\mathrm{Shv}}{\otimes} X$.

The monoidal structure on $\mathsf{Shv}(\mathcal{A})$ induces a monoidal structure on $\mathcal{S}m$ via the following easy lemma.

3.1.1 Lemma. *Let $\mathcal{V}$ be a symmetric monoidal closed category. Let $\mathcal{C}$ be a full $\mathcal{V}$-subcategory of $\mathcal{V}$, such that $\mathbb{1}_{\mathcal{V}}$ is isomorphic to an object of $\mathcal{C}$, and for every $X, Y \in \mathcal{C}$ the monoidal product $X \otimes Y$ is isomorphic to an object of $\mathcal{C}$. Then $\mathcal{C}$ can be made into a symmetric monoidal $\mathcal{V}$-category such that the inclusion functor $\mathcal{C} \to \mathcal{V}$ is strong monoidal.*

Proof. Let $\overline{\mathcal{C}}$ be the full $\mathcal{V}$-subcategory of $\mathcal{V}$ on all those objects which have the property of being isomorphic to some object of $\mathcal{C}$. Then $\mathbb{1} \in \overline{\mathcal{C}}$, and for all $X, Y \in \overline{\mathcal{C}}$ we have $X \otimes Y \in \overline{\mathcal{C}}$. So the functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ restricts to a functor $\otimes : \overline{\mathcal{C}} \times \overline{\mathcal{C}} \to \overline{\mathcal{C}}$. For all $X, Y, Z \in \overline{\mathcal{C}}$ we have coherence isomorphisms

$$\ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X$$

$$\rho_X : X \otimes \mathbb{1} \xrightarrow{\sim} X$$

$$\varphi_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$$

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$$

in $\mathcal{V}$. The domains and codomains of all these coherence isomorphisms lie in $\overline{\mathcal{C}}$. Since $\overline{\mathcal{C}}$ is a full subcategory of $\mathcal{V}$, all these coherence isomorphisms lie in $\overline{\mathcal{C}}$. Obviously these coherences isomorphisms in $\overline{\mathcal{C}}$ still make exactly the same diagrams commute as in $\mathcal{V}$. So $\overline{\mathcal{C}}$ is a symmetric monoidal $\mathcal{V}$-category, and the inclusion $\overline{\mathcal{C}} \to \mathcal{V}$ is a strict monoidal $\mathcal{V}$-functor.

We have an inclusion $\mathcal{V}$-functor $\mathcal{C} \to \overline{\mathcal{C}}$. This functor is essentially surjective, and it is the identity on morphism objects. This then implies that $\mathcal{C} \to \overline{\mathcal{C}}$ is an equivalence in the 2-category $\mathcal{V} - CAT$, and we then get an induced symmetric monoidal $\mathcal{V}$-category structure on $\mathcal{C}$. □

3.1.2 Corollary. *$\mathcal{S}m$ and $\mathcal{C}$ are symmetric monoidal* $\mathsf{Shv}(\mathcal{A})$*-categories.*

Proof. The unit of $\mathsf{Shv}(\mathcal{A})$ is isomorphic to $\mathcal{A}(-, pt)_{\mathsf{nis}}$. We claim that for all $X, Y \in \mathcal{S}m$ we have an isomorphism

$$\mathcal{A}(-, X)_{\mathsf{nis}} \underset{\mathrm{Shv}}{\otimes} \mathcal{A}(-, Y)_{\mathsf{nis}} \cong \mathcal{A}(-, X \times Y)_{\mathsf{nis}}.$$

This isomorphism is constructed as follows. The sheafification functor $(-)_{\mathsf{nis}} : \mathsf{Psh}(\mathcal{A}) \to \mathsf{Shv}(\mathcal{A})$ is strongly monoidal, so if $\underset{\mathrm{Psh}}{\otimes}$ denotes the presheaf tensor product, then we have a natural isomorphism $\mathcal{A}(-, X)_{\mathsf{nis}} \underset{\mathrm{Shv}}{\otimes} \mathcal{A}(-, Y)_{\mathsf{nis}} \cong (\mathcal{A}(-, X) \underset{\mathrm{Psh}}{\otimes} \mathcal{A}(-, Y))_{\mathsf{nis}}$. The presheaf tensor product $\underset{\mathrm{Psh}}{\otimes}$ is a Day convolution with respect to the monoidal structure on $\mathcal{A}$. The monoidal structure on $\mathcal{A}$ is given on objects by the cartesian product on Sm_k. By general properties of Day convolution we have an isomorphism of presheaves $\mathcal{A}(-, X) \underset{\mathrm{Psh}}{\otimes} \mathcal{A}(-, Y) \cong \mathcal{A}(-, X \times Y)$ and thus an isomorphism of sheaves $\mathcal{A}(-, X)_{\mathsf{nis}} \underset{\mathrm{Shv}}{\otimes} \mathcal{A}(-, Y)_{\mathsf{nis}} \cong \mathcal{A}(-, X \times Y)_{\mathsf{nis}}$. The previous lemma now implies that $\mathcal{S}m$ is a symmetric monoidal $\mathsf{Shv}(\mathcal{A})$-category. Since $\mathcal{A}(-, pt)_{\mathsf{nis}} = \mathcal{A}(-, \mathbb{G}_m^{\times 0})_{\mathsf{nis}}$ and $\mathcal{A}(-, \mathbb{G}_m^{\times n})_{\mathsf{nis}} \underset{\mathrm{Shv}}{\otimes} \mathcal{A}(-, \mathbb{G}_m^{\times m})_{\mathsf{nis}} \cong \mathcal{A}(-, \mathbb{G}_m^{\times n+m})_{\mathsf{nis}}$ it also follows that $\mathcal{C}$ is a symmetric monoidal $\mathsf{Shv}(\mathcal{A})$-category. □

Since $\mathsf{Shv}(\mathcal{A})$ is a closed symmetric monoidal Grothendieck category, and $\mathcal{S}m$ is a monoidal $\mathsf{Shv}(\mathcal{A})$-category, we can apply [20, Theorem 5.5] to get a weakly finitely generated monoidal model structure on $\mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$, where the weak equivalences are the pointwise quasi-isomorphisms and the fibrations are the pointwise fibrations. We will say that $F \in \mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$ is *locally fibrant* if it is fibrant in this model category. The homotopy category of this model category is the derived category $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$ of the Grothendieck category $[\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]$.

We write $\underset{\text{Day}}{\otimes}^{\mathbf{L}}$ for the derived tensor product on $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$. Since the model structure on $\mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$ is monoidal by [20, Theorem 5.5], we can compute this derived tensor product by using cofibrant replacements in $\mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$. Also note that every representable functor $\mathcal{S}m(X, -) : \mathcal{S}m \to \mathsf{Shv}(\mathcal{A})$ is cofibrant in $\mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$, because it is isomorphic to the cofibrant object $\mathcal{S}m(X, -) \underset{\text{Shv}}{\otimes} pt$. We similarly have a weakly finitely generated monoidal model structure on $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$, whose homotopy category is $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$.

We now define two families of morphisms in the enriched functor category $[\mathcal{C}, \mathsf{Shv}(\mathcal{A})]$. The first family of morphisms we call $\mathbb{A}^1_1$, and it consists of the morphisms

$$\mathcal{C}(\mathbb{G}_m^{\times n}, -) \underset{\text{Shv}}{\otimes} \mathbb{A}^1 \to \mathcal{C}(\mathbb{G}_m^{\times n}, -) \underset{\text{Shv}}{\otimes} pt$$

induced by the projection map $\mathbb{A}^1 \to pt$ for every $n \in \mathbb{Z}_{\geqslant 0}$.

The second family of morphisms, denoted by τ, consists for every $n \in \mathbb{N}$ of the morphism

$$\tau_n : [\mathbb{G}_m^{\wedge n+1}, I(-)] \underset{\text{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1} \to [\mathbb{G}_m^{\wedge n}, I(-)]$$

where for every $U \in \mathsf{Sm}_k$ the map $[\mathbb{G}_m^{\wedge n+1}, I(U)] \underset{\text{Shv}}{\otimes} \mathbb{G}_m^{\wedge n+1} \xrightarrow{\tau_n} I(U)$ in $\mathsf{Shv}(\mathcal{A})$ is given by the counit of the adjunction $- \underset{\text{Shv}}{\otimes} \mathbb{G}_m^{\wedge n+1} \dashv [\mathbb{G}_m^{\wedge n+1}, -]$. We also sometimes write $\mathcal{S}m(\mathbb{G}_m^{\wedge n+1}, -)$ or $\mathcal{C}(\mathbb{G}_m^{\wedge n+1}, -)$ for $[\mathbb{G}_m^{\wedge n+1}, I(-)]$, even though $\mathbb{G}_m^{\wedge n+1}$ is not in $\mathcal{S}m$ or $\mathcal{C}$ strictly speaking.

The domains and codomains of all these morphisms are compact in the derived category $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ according to [20, Theorem 6.2].

Let $\sim_{\mathcal{C}}$ be the union of both of these classes of morphisms

$$\sim_{\mathcal{C}} = \mathbb{A}^1_1 + \tau$$

considered as a class of morphisms in $[\mathcal{C}, \mathsf{Shv}(\mathcal{A})]$.

3.1.3 Definition. Let $\mathcal{B}$ be any small $\mathsf{Shv}(\mathcal{A})$-enriched category.

We can consider $\mathsf{Ch}([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])$ to be a $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$-enriched category, and denote the morphism objects by $\mathrm{map}^{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(A, B) \in \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. These morphism objects are defined on $Z \in \mathsf{Sm}_k$ by

$$\mathrm{map}^{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(A, B)(Z) := \mathrm{map}^{\mathrm{Ch(Ab)}}(A \underset{\mathrm{Shv}}{\otimes} Z, B) \in \mathsf{Ch}(\mathrm{Ab})$$

where $\mathrm{map}^{\mathrm{Ch(Ab)}}$ refers to morphism objects of the $\mathsf{Ch}(\mathrm{Ab})$-enriched category $\mathsf{Ch}([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])$. Given an object $F \in \mathsf{Ch}([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])$ and a class of morphisms S in $\mathsf{Ch}([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])$, we say that F is *enriched S-local* if for every $f : A \to B$ in S we have a quasi-isomorphism of complexes of sheaves

$$\mathrm{map}^{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(B, F) \to \mathrm{map}^{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(A, F)$$

in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. Furthermore say that $F \in \mathsf{Ch}([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])$ is *strictly S-local* if its pointwise locally fibrant replacement F^f in $\mathsf{Ch}([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])$ is enriched S-local.

3.1.4 Lemma. *Let $\mathcal{B}$ be a small monoidal $\mathsf{Shv}(\mathcal{A})$-enriched category, and S a set of morphisms in $\mathsf{Ch}([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])$. Define a new set of morphisms*

$$\widehat{S} := \{(f \underset{\mathrm{Shv}}{\otimes} Z)[n] \mid n \in \mathbb{Z}, Z \in \mathsf{Sm}_k, f \in S\}$$

in $D([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])$.

Let $F \in \mathsf{Ch}([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])$ be locally fibrant, and assume that all domains and codomains from S are cofibrant.in the local model structure. Then F is strictly S-local in the sense of Definition 3.1.3 if and only if F is $\widehat{S}$-local in $D([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])$ in the usual sense, i.e. if and only if for all $g : C \to D, g \in \widehat{S}$ we have an isomorphism of abelian groups

$$g^* : \mathrm{Hom}_{D([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])}(D, F) \to \mathrm{Hom}_{D([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])}(C, F).$$

Proof. Suppose F is strictly S-local. Then for every $f : A \to B$, $f \in S$ we have a quasi-isomorphism of complexes of sheaves

$$f^* : \mathrm{map}^{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(B, F) \to \mathrm{map}^{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(A, F)$$

in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.

We claim that $\mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(B, F)$ is locally fibrant. In fact if we have a local trivial cofibration $h : X \to Y$, then a diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(B, F) \\ {\scriptstyle h}\downarrow & \nearrow & \downarrow \\ Y & \longrightarrow & 0 \end{array}$$

has a lift, by adjunction if and only if

$$\begin{array}{ccc} B \underset{\mathrm{Shv}}{\otimes} X & \longrightarrow & F \\ {\scriptstyle B \underset{\mathrm{Shv}}{\otimes} h}\downarrow & \nearrow & \downarrow \\ B \underset{\mathrm{Shv}}{\otimes} Y & \longrightarrow & 0 \end{array}$$

has a lift. But since B is cofibrant, then $B \underset{\mathrm{Shv}}{\otimes} h$ is still a trivial cofibration. Since F is locally fibrant the map $F \to 0$ is a local fibration, so the lift exists. Therefore $\mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(B, F)$ and similarly $\mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(A, F)$ are locally fibrant. We see that the quasi-isomorphism

$$f^* : \mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(B, F) \to \mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(A, F)$$

is sectionwise a quasi-isomorphism.

This means that for every $n \in \mathbb{Z}$ we have an isomorphism of homology presheaves

$$H_n(\mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(B, F)) \to H_n(\mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(A, F)).$$

Therefore for every $Z \in \mathsf{Sm}_k$ one has

$$H_n(\mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(B, F))(Z) \cong \mathrm{Hom}_{D([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])}((B \underset{\mathrm{Shv}}{\otimes} Z)[-n], F).$$

It follows that F is $\widehat{S}$-local in $D([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])$.

Conversely, assume that F is $\widehat{S}$-local in $D([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])$. Then for every $f : A \to B$ in S the map

$$f^* : \mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(B, F) \to \mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(A, F)$$

is a sectionwise quasi-isomorphism, because for every $n \in \mathbb{Z}$ and $Z \in \mathsf{Sm}_k$ the map

$$H_n(f)(Z) : H_n(\mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(B, F))(Z) \to H_n(\mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(A, F))(Z)$$

is isomorphic to the map

$$(f \underset{\mathrm{Shv}}{\otimes} Z)[-n]^* : \mathrm{Hom}_{D([\mathcal{B},\mathsf{Shv}(\mathcal{A})])}((B \underset{\mathrm{Shv}}{\otimes} Z)[-n], F) \to \mathrm{Hom}_{D([\mathcal{B},\mathsf{Shv}(\mathcal{A})])}((A \underset{\mathrm{Shv}}{\otimes} Z)[-n], F)$$

and since $(f \underset{\mathrm{Shv}}{\otimes} Z)[-n] \in \widehat{S}$ and F is $\widehat{S}$-local this map is an isomorphism. So F is strictly S-local if and only if F is $\widehat{S}$-local in $D([\mathcal{B}, \mathsf{Shv}(\mathcal{A})])$. □

We can localize the compactly generated triangulated category $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ with respect to the family of morphisms between compact objects $\widehat{\sim_{\mathcal{C}}}$.

3.1.5 Definition. We write $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}$ for the localized compactly generated triangulated category. Furthermore we write $DM_{\mathcal{A}}[\mathcal{C}]$ for the full triangulated subcategory of $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ consisting of the strictly $\sim_{\mathcal{C}}$-local objects.

It follows from Lemma 3.1.4 that the category $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}$ is equivalent to $DM_{\mathcal{A}}[\mathcal{C}]$.

3.1.6 Definition. An enriched functor $F : \mathcal{C} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ or $F : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is said to *satisfy cancellation*, if for every $n \geqslant 0$ the canonical map $F(\mathbb{G}_m^{\wedge n}) \to [\mathbb{G}_m^{\wedge 1}, F(\mathbb{G}_m^{\wedge n+1})]$ is a local quasi-isomorphism.

Note that an enriched functor F satisfies cancellation if and only if it is enriched τ-local.

3.1.7 Definition. Let $F \in \mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$. We say that F is $\sim_{\mathcal{C}}$*-fibrant* if it is pointwise locally fibrant in $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ and strictly $\sim_{\mathcal{C}}$-local.

Note that F is strictly $\sim_{\mathcal{C}}$-local if and only if it is strictly $\mathbb{A}^1_1$-local and satisfies cancellation.

Our first theorem is that there is a canonical equivalence of compactly generated triangulated categories

$$D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}} \xrightarrow{\sim} DM_{\mathcal{A}}.$$

The equivalence is constructed as follows. For an enriched functor $F : \mathcal{C} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ and $k \in \mathbb{N}$ define

$$F(\mathbb{G}_m^{\wedge k}) := F(\mathbb{G}_m^{\times k}) / \sum_{i=0}^{k+1} \mathrm{Im}(F(\iota_{i,k})).$$

There is an isomorphism of categories $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})]) \cong [\mathcal{C}, \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))]$ by [20, Theorem 5.4]. For this reason we will often implicitly pass back and forth between those categories without mentioning it.

Let $\mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Shv}(\mathcal{A}))$ be the category of $\mathbb{G}_m^{\wedge 1}$-spectra in $\mathsf{Shv}(\mathcal{A})$. Define

$$ev_{\mathbb{G}_m} : \mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})]) \to \mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))$$

by taking $F \in \mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ (regarding it as an enriched functor $F : \mathcal{C} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$) to the $\mathbb{G}_m^{\wedge 1}$-spectrum $(F(\mathbb{G}_m^{\wedge n}))_{n \in \mathbb{N}}$. We construct the structure maps

$$F(\mathbb{G}_m^{\wedge k}) \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1} \to F(\mathbb{G}_m^{\wedge k+1})$$

by applying the tensor-hom adjunction to

$$\mathbb{G}_m^{\wedge 1} \to [\mathbb{G}_m^{\wedge n}, \mathbb{G}_m^{\wedge n+1}] \to [F(\mathbb{G}_m^{\wedge n}), F(\mathbb{G}_m^{\wedge n+1})].$$

This functor sends quasi-isomorphisms in $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ to stable motivic equivalences in $\mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))$, so it induces a functor $ev_{\mathbb{G}_m} : D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})]) \to DM_{\mathcal{A}}$. This functor can then be restricted to the full triangulated subcategory $DM_{\mathcal{A}}[\mathcal{C}] \subseteq D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$. We are now in a position to formulate the following theorem.

3.1.8 Theorem. *The functor*

$$ev_{\mathbb{G}_m} : DM_{\mathcal{A}}[\mathcal{C}] \to DM_{\mathcal{A}}$$

is an equivalence of compactly generated triangulated categories. In particular there is an equivalence

$$D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})]) / \sim_{\mathcal{C}} \xrightarrow{\sim} DM_{\mathcal{A}}.$$

The proof of this theorem is given in Section 3.2. To state our next result we now define some additional classes of morphisms in $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$. Firstly, in $\mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$ let $\mathbb{A}^1_1$ denote the class of morphisms

$$\mathcal{S}m(U, -) \underset{\mathrm{Shv}}{\otimes} \mathbb{A}^1 \to \mathcal{S}m(U, -)$$

for $U \in \mathcal{S}m$, and let τ denote the class of morphisms

$$\tau_n : [\mathbb{G}_m^{\wedge n+1}, I(-)] \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1} \to [\mathbb{G}_m^{\wedge n}, -]$$

just like in $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$. By $\mathbb{A}^1_2$ we mean the family consisting for every $Y \in \mathsf{Sm}_k$ of the morphism

$$\mathcal{S}m(Y, -) \to \mathcal{S}m(Y \times \mathbb{A}^1, -).$$

The family of morphisms Nis is defined as follows. For every elementary Nisnevich square

$$\begin{array}{ccc} U' & \xrightarrow{\beta} & X' \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} \\ U & \xrightarrow{\delta} & X \end{array}$$

in Sm_k, we have a square in $\mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$

$$\begin{array}{ccc} \mathcal{S}m(U', -) & \xleftarrow{\beta^*} & \mathcal{S}m(X', -) \\ \uparrow{\scriptstyle \alpha^*} & & \uparrow{\scriptstyle \gamma^*} \\ \mathcal{S}m(U, -) & \xleftarrow{\delta^*} & \mathcal{S}m(X, -) \end{array}$$

It induces a map of chain complexes $p : \mathrm{hocofib}(\gamma^*) \to \mathrm{hocofib}(\alpha^*)$, where hocofib refers to the naive mapping cone chain complex. The family Nis consists of all the morphisms p for every elementary Nisnevich square. Denote by $\sim$ the union of all the four morphism sets defined above. Namely,

$$\sim := \mathbb{A}^1_1 + \tau + \mathbb{A}^1_2 + Nis.$$

3.1.9 Definition. A functor $F \in \mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$ is said to *satisfy Nisnevich excision* if it sends elementary Nisnevich squares in Sm_k to homotopy cartesian squares in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.

Note that we consider here covariant Nisnevich excision in the $\mathcal{S}m$-variable, rather than contravariant Nisnevich excision in the $\mathcal{A}$-variable.

3.1.10 Lemma. *Let $F \in \mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$ be a functor. Then F satisfies Nisnevich excision if and only if F is enriched Nis-local.*

Proof. By the $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$-enriched Yoneda lemma there is a natural isomorphism in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$

$$F(X) \cong \mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathcal{S}m(X, -), F).$$

So

$$\begin{array}{ccc} F(U') & \xrightarrow[F(\beta)]{} & F(X') \\ \downarrow{\scriptstyle F(\alpha)} & & \downarrow{\scriptstyle F(\gamma)} \\ F(U) & \xrightarrow{F(\delta)} & F(X) \end{array}$$

is homotopy cartesian if and only if

$$\begin{array}{ccc} \mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathcal{S}m(U', -), F) & \xrightarrow[\beta^{**}]{} & \mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathcal{S}m(X', -), F) \\ \downarrow{\scriptstyle \alpha^{**}} & & \downarrow{\scriptstyle \gamma^{**}} \\ \mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathcal{S}m(U, -), F) & \xrightarrow{\delta^{**}} & \mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathcal{S}m(X, -), F) \end{array}$$

is homotopy cartesian. This is the case if and only if $\mathrm{hocofib}(\alpha^{**}) \to \mathrm{hocofib}(\gamma^{*})$ is a local quasi-isomorphism. The latter holds if and only if the induced morphism $p^* : \mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathrm{hocofib}(\alpha^*), F) \to \mathrm{map}^{\mathrm{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathrm{hocofib}(\gamma^*), F)$ is a local quasi-isomorphism, which means that F is enriched Nis-local. □

3.1.11 Definition. Let $F \in \mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$. We say that F is $\sim$-*fibrant* if it is pointwise locally fibrant in $\mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$ and strictly $\sim$-local.

3.1.12 Definition. Let $DM_{\mathcal{A}}[\mathcal{S}m]$ be the full subcategory of $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$ of those complexes which satisfy the following properties:

1. For every $U \in \mathcal{S}m$, the complex of sheaves $F(U)$ has $\mathbb{A}^1$-invariant cohomology sheaves.

2. F satisfies cancellation.

3. F is covariantly $\mathbb{A}^1$-invariant, in the sense that $F(U \times \mathbb{A}^1) \to F(U)$ is a local quasi-isomorphism.

4. F satisfies Nisnevich excision.

These properties are similar to the axioms (2)-(5) for special motivic Γ-spaces defined in [25] and axioms for framed spectral functors in the sense of [24, Section 6].

3.1.13 Proposition. *The category $DM_{\mathcal{A}}[\mathcal{S}m]$ is equal to the full subcategory of $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$ of those complexes F which are strictly $\sim$-local. In particular, the inclusion from $DM_{\mathcal{A}}[\mathcal{S}m]$ to $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$ induces an equivalence of triangulated categories*

$$DM_{\mathcal{A}}[\mathcal{S}m] \xrightarrow{\sim} D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim .$$

Proof. The proposition follows from the following four claims:

1. A functor F is strictly $\mathbb{A}^1_1$-local if and only if for every $U \in \mathsf{Sm}_k$, the complex $F(U)$ has $\mathbb{A}^1$-invariant cohomology sheaves.

2. A strictly $\mathbb{A}^1_1$-local functor F satisfies cancellation if and only if it is strictly τ-local.

3. A functor F is covariantly $\mathbb{A}^1$-invariant if and only if it is strictly $\mathbb{A}^1_2$-local.

4. A functor F satisfies Nisnevich excision if and only if it is strictly Nis-local.

Here are the proofs for those claims.

1. F is strictly $\mathbb{A}^1_1$-invariant if and only if for every $U \in \mathsf{Sm}_k$ the canonical map
$$F^f(U) \to F^f(U)(\mathbb{A}^1 \times -)$$
is a local quasi-isomorphism in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. Since $F^f(U)$ and $F^f(U)(\mathbb{A}^1 \times -)$ are locally fibrant in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, it follows that the above map is a local quasi-isomorphism if and only if it is a sectionwise quasi-isomorphism in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$. This is the case if and only if F^f has $\mathbb{A}^1$-invariant cohomology presheaves in the sense that for each $n \in \mathbb{Z}$ the map
$$H_n(F^f(U)) \to H_n(F^f(U)(\mathbb{A}^1 \times -)) = H_n(F^f(U))(\mathbb{A}^1 \times -)$$
is an isomorphism in $\mathsf{Psh}(\mathcal{A})$. This means that $F^f(U)$ is motivically fibrant, which is the case if and only if $F(U)$ is $\mathbb{A}^1$-local. By [38, Theorem 6.2.7] this is the case if and only if $F(U)$ has $\mathbb{A}^1$-invariant cohomology sheaves.

2. The Yoneda lemma implies that a functor F satisfies cancellation if and only if it is enriched τ-local. We now claim that a strictly $\mathbb{A}^1_1$-local functor F is enriched τ-local if and only if it is strictly τ-local. Let F be a strictly $\mathbb{A}^1_1$-local functor, and let F^f be its pointwise local fibrant replacement. For every $U \in \mathsf{Sm}_k$ and $n \in \mathbb{Z}$, consider the following diagram in $\mathsf{Shv}(\mathcal{A})$

$$\begin{array}{ccc} H_n^{\mathsf{nis}}(\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge 1}, F(U))) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathsf{Shv}(\mathcal{A})}(\mathbb{G}_m^{\wedge 1}, H_n^{\mathsf{nis}}(F(U))) \\ \downarrow & & \downarrow \\ H_n^{\mathsf{nis}}(\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge 1}, F^f(U))) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathsf{Shv}(\mathcal{A})}(\mathbb{G}_m^{\wedge 1}, H_n^{\mathsf{nis}}(F^f(U))) \end{array}$$

Since $F(U)$ and $F^f(U)$ have $\mathbb{A}^1$-invariant cohomology sheaves, it follows from [37, Lemma 4.3.11] that the two horizontal maps in the diagram are isomorphisms. Since the canonical map $F(U) \to F^f(U)$ is a local quasi-isomorphism, the map $H_n^{\mathsf{nis}}(F(U)) \to H_n^{\mathsf{nis}}(F^f(U))$ is an isomorphism in $\mathsf{Shv}(\mathcal{A})$, so the right vertical map in the above diagram is also an isomorphism. This implies the left vertical map in the diagram

$$H_n^{\mathsf{nis}}(\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge 1}, F(U))) \to H_n^{\mathsf{nis}}(\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge 1}, F^f(U)))$$

is an isomorphism in $\mathsf{Shv}(\mathcal{A})$. Hence

$$\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge 1}, F(U)) \to \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge 1}, F^f(U))$$

is a local quasi-isomorphism in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.

Now consider the diagram in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.

$$\begin{array}{ccc} F(\mathbb{G}_m^{\wedge n}) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge 1}, F(\mathbb{G}_m^{\wedge n+1}))) \\ \downarrow & & \downarrow \\ F^f(\mathbb{G}_m^{\wedge n}) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge 1}, F^f(\mathbb{G}_m^{\wedge n+1}))) \end{array}$$

The two vertical maps are local quasi-isomorphisms.

F is enriched τ-local if and only if the upper horizontal map is a local quasi-isomorphism. This is the case if and only if the lower horizontal map is a quasi-isomorphism, and that is true if and only if F is strictly τ-local.

3. From the Yoneda lemma it follows that a functor F is covariantly $\mathbb{A}^1$-invariant if and only if it is enriched $\mathbb{A}^1_2$-local. And every functor F is enriched $\mathbb{A}^1_2$-local if and only if it is strictly $\mathbb{A}^1_2$-local, because the relation $\mathbb{A}^1_2$ only affects the covariant $\mathcal{S}m$-variable and is thus not affected by pointwise local fibrant replacement. More precisely, consider the following diagram, in which the vertical maps are local quasi-isomorphisms:

$$\begin{array}{ccc} F(X \times \mathbb{A}^1) & \longrightarrow & F(X) \\ \sim\downarrow & & \downarrow\sim \\ F^f(X \times \mathbb{A}^1) & \longrightarrow & F^f(X) \end{array} .$$

   F is enriched $\mathbb{A}^1_2$-local if and only if the upper morphism is a local quasi-isomorphism, which is the case if and only if the lower morphism is a quasi-isomorphism, which is the case if and only if F^f is enriched $\mathbb{A}^1_2$-local, which means that F is strictly $\mathbb{A}^1_2$-local.

4. By Lemma 3.1.10 a functor F satisfies Nisnevich excision if and only if it is enriched Nis-local. Just like for $\mathbb{A}^1_2$, since the relation Nis only affects the covariant argument, it is not affected by pointwise local fibrant replacement, so that a functor F is enriched Nis-local if and only if it is strictly Nis-local.

This completes the proof. □

Next, the evaluation functor

$$ev_{\mathbb{G}_m} : D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim \to DM_{\mathcal{A}}$$

is defined as follows. We send $F \in D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim$ to $ev_{\mathbb{G}_m}(F')$, where $ev_{\mathbb{G}_m} : D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]) \to DM_{\mathcal{A}}$ is the evaluation functor defined just like the one in Theorem 3.1.8, and F' is a $\sim$-fibrant replacement of F in $\mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$.

When $ev_{\mathbb{G}_m}$ is restricted to the subcategory $DM_{\mathcal{A}}[\mathcal{S}m]$, it is the naive $\mathbb{G}_m$-evaluation functor

$$ev_{\mathbb{G}_m} : DM_{\mathcal{A}}[\mathcal{S}m] \to DM_{\mathcal{A}}$$

that sends F to the $\mathbb{G}_m$-spectrum $(F(\mathbb{G}_m^{\wedge k}))_{k \geqslant 0}$.

For any pre-additive category $\mathcal{B}$ we denote by $\mathcal{B}[1/p]$ the pre-additive category where all hom-sets get tensored with $\mathbb{Z}[1/p]$. Explicitly, for $x, y \in \mathcal{B}$ we define

$$\mathcal{B}[1/p](x,y) := \mathcal{B}(x,y) \otimes \mathbb{Z}[1/p].$$

Another main result of this thesis is as follows.

3.1.14 Theorem. *Let p be the exponential characteristic of k. After inverting p the functor $ev_{\mathbb{G}_m}$ is an equivalence of compactly generated triangulated categories*

$$ev_{\mathbb{G}_m} : (D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim)[1/p] \xrightarrow{\sim} DM_{\mathcal{A}}[1/p].$$

In particular the naive $\mathbb{G}_m$-evaluation functor

$$ev_{\mathbb{G}_m} : DM_{\mathcal{A}}[\mathcal{S}m][1/p] \to DM_{\mathcal{A}}[1/p]$$

is an equivalence of compactly generated triangulated categories.

The proof of this theorem is given at the end of Section 4.3.

3.2 Proof of Theorem 3.1.8

In this section we prove Theorem 3.1.8.

We will sometimes write $\mathcal{C}(\mathbb{G}_m^{\wedge k}, -)$ for $[\mathbb{G}_m^{\wedge k}, I(-)] = \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge k}, I(-))$.

3.2.1 Lemma. *In $\mathsf{Shv}(\mathcal{A})$ we have an isomorphism*

$$I(\mathbb{G}_m^{\times k}) \cong \bigoplus_{i=0}^{k} \binom{k}{i} \mathbb{G}_m^{\wedge i}$$

where $\binom{k}{i}$ is the binomial coefficient, and $\binom{k}{i}\mathbb{G}_m^{\wedge i} := \bigoplus_{j=1}^{\binom{k}{i}} \mathbb{G}_m^{\wedge i}$.

In particular we have an isomorphism in $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$

$$\mathcal{C}(\mathbb{G}_m^{\times k}, -) \cong \bigoplus_{i=0}^{k} \binom{k}{i} \mathcal{C}(\mathbb{G}_m^{\wedge k}, -).$$

Proof. First note that $\mathbb{G}_m^{\wedge k} \otimes \mathbb{G}_m^{\wedge 1} \cong \mathbb{G}_m^{\wedge k+1}$, so $\mathbb{G}_m^{\wedge k} \cong (\mathbb{G}_m^{\wedge 1})^{\otimes k}$. Also since the map $pt \xrightarrow{\iota_{1,1}} \mathbb{G}_m^{\times 1}$ splits, the splitting lemma for abelian categories implies $I(\mathbb{G}_m^{\times 1}) \cong \mathbb{G}_m^{\wedge 1} \oplus I(pt)$. The binomial theorem, applied to the semi-ring of isomorphism classes of the symmetric monoidal closed category $\mathsf{Shv}(\mathcal{A})$, then yields an isomorphism

$$I(\mathbb{G}_m^{\times k}) \cong (\mathbb{G}_m^{\wedge 1} \oplus pt)^{\otimes k} \cong \bigoplus_{i=0}^{k} \binom{k}{i} (\mathbb{G}_m^{\wedge 1})^{\otimes i} \otimes pt^{\otimes k-i} \cong \bigoplus_{i=0}^{k} \binom{k}{i} \mathbb{G}_m^{\wedge i}$$

as required. □

3.2.2 Definition. 1. We define the *Suslin complex functor*

$$C_* : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$$

by sending $F_\bullet \in \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ and $U \in \mathsf{Sm}_k$ to

$$C_*(F_\bullet)(U) := \mathrm{Tot}(F_\bullet(\Delta_k^\bullet \times U)) \in \mathsf{Ch}(\mathrm{Ab}).$$

Here Tot is the total complex functor and $\Delta_k^n = \mathrm{Spec}(k[t_0, \dots, t_n]/(t_0 + \cdots + t_n - 1))$ is the algebraic simplex.

2. For $X \in \mathsf{Sm}_k$ we define the *$\mathcal{A}$-motive of X* to be

$$M_{\mathcal{A}}(X) := C_*(I(X)) = C_*(\mathcal{A}(-, X)_{\mathsf{nis}})$$

in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.

3. The enriched functor $\mathcal{M}_{\mathcal{A}}(X) : \mathcal{C} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ defined by

$$\mathcal{M}_{\mathcal{A}}(X)(U) := M_{\mathcal{A}}(X \times U)$$

will be called the *enriched $\mathcal{A}$-motive of X*.

4. For $X \in \mathsf{Sm}_k$ we define its *$\mathbb{G}_m^{\wedge 1}$-suspension spectrum* $\Sigma^\infty_{\mathbb{G}_m} X_+ \in DM_{\mathcal{A}}$, by defining it in weight n as

$$(\Sigma^\infty_{\mathbb{G}_m} X_+)(n) := \mathbb{G}_m^{\wedge n} \underset{\mathrm{Shv}}{\otimes} I(X)$$

and equipping it with the obvious structure maps.

If $F : \mathcal{C} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is an enriched functor, then we define $C_*F : \mathcal{C} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ by $(C_*F)(U) := C_*(F(U))$. The endofunctor $C_* : \mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})]) \to \mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ preserves pointwise local quasi-isomorphisms, because $\mathcal{A}$ satisfies the strict V-property. Thus C_* induces an endofunctor on the derived category

$$C_* : D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})]) \to D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})]).$$

For $X \in \mathsf{Sm}_k$ we have the zero inclusion map $X \to \mathbb{A}^1_X$. Let $\mathbb{A}^1_X/X \in \mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ denote the cokernel of the induced morphism

$$\mathcal{A}(-, X) \to \mathcal{A}(-, \mathbb{A}^1_X).$$

Then $\mathbb{A}^1_X/X$ is cofibrant in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ because it is a direct summand of the cofibrant object $\mathcal{A}(-,\mathbb{A}^1_X)$. We write $(\mathbb{A}^1_X/X)_{\mathsf{nis}} \in \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ for the sheafification of $\mathbb{A}^1_X/X$. Let $T_{\mathbb{A}^1_1} = \langle \mathcal{C}(U,-) \underset{\text{Shv}}{\otimes} (\mathbb{A}^1_X/X)_{\mathsf{nis}} \mid U \in \mathcal{C}, X \in \mathsf{Sm}_k \rangle$ be the full triangulated subcategory of $D([\mathcal{C},\mathsf{Shv}(\mathcal{A})])$ that is compactly generated by $\mathcal{C}(U,-) \underset{\text{Shv}}{\otimes} (\mathbb{A}^1_X/X)_{\mathsf{nis}}$.

3.2.3 Lemma. *In $D([\mathcal{C},\mathsf{Shv}(\mathcal{A})])$ we have that* $\ker(C_*) = T_{\mathbb{A}^1_1}$.

Proof. Consider a generator $\mathcal{C}(U,-) \underset{\text{Shv}}{\otimes} (\mathbb{A}^1_X/X)_{\mathsf{nis}}$ of $T_{\mathbb{A}^1_1}$. We claim that it is in $\ker(C_*)$. For this we need to show for every $V \in \mathcal{C}$ that $C_*(\mathcal{C}(U,V) \underset{\text{Shv}}{\otimes} (\mathbb{A}^1_X/X)_{\mathsf{nis}})$ is locally quasi-isomorphic to 0. Take a free resolution of $\mathcal{C}(U,V)$ in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$:

$$\cdots \to F_1 \to F_0 \to \mathcal{C}(U,V) \to 0$$

The presheaf $\mathbb{A}^1_X/X$ is projective because it is a direct summand of $\mathcal{A}(-,\mathbb{A}^1_X)$, and hence it is also flat by Lemma 2.2.6. Thus the following sequence is exact

$$\cdots \to F_1 \underset{\text{Psh}}{\otimes} \mathbb{A}^1_X/X \to F_0 \underset{\text{Psh}}{\otimes} \mathbb{A}^1_X/X \to \mathcal{C}(U,V) \underset{\text{Psh}}{\otimes} \mathbb{A}^1_X/X \to 0.$$

It then also follows that the sequence is exact in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ after applying C_*

$$\cdots \to C_*(F_1 \underset{\text{Psh}}{\otimes} \mathbb{A}^1_X/X) \to C_*(F_0 \underset{\text{Psh}}{\otimes} \mathbb{A}^1_X/X) \to C_*(\mathcal{C}(U,V) \underset{\text{Psh}}{\otimes} \mathbb{A}^1_X/X) \to 0.$$

Since each individual entry of this sequence is a chain complex, we can regard it as a double complex. Let $D_{\bullet,\bullet}$ be the double complex

$$D_{p,q} := \begin{cases} C_*(F_{p-1} \underset{\text{Psh}}{\otimes} \mathbb{A}^1_X/X)_q & p > 0 \\ C_*(\mathcal{C}(U,V) \underset{\text{Psh}}{\otimes} \mathbb{A}^1_X/X)_q & p = 0 \\ 0 & p < 0 \end{cases}$$

Then all horizontal homology groups of $D_{\bullet,\bullet}$ are zero. The double complex spectral sequence

$$E^2_{p,q} = H_{\text{vert},p}(H_{\text{hor},q}(D_{\bullet,\bullet})) \implies H_{p+q}(\text{Tot}(D_{\bullet,\bullet}))$$

implies that $H_n(\text{Tot}(D_{\bullet,\bullet})) = 0$.

One can now check that $C_*(\mathbb{A}^1_X/X)$ is locally quasi-isomorphic to 0 similarly to [50, Proposition 1.11(1)]. It follows that every $C_*(F_q \underset{\mathrm{Psh}}{\otimes} \mathbb{A}^1_X/X)$ is locally quasi-isomorphic to 0, because the F_q are free and for all $Y \in \mathcal{C}$ we have $\mathcal{A}(-,Y) \underset{\mathrm{Psh}}{\otimes} \mathbb{A}^1_X/X \cong \mathbb{A}^1_{Y\times X}/Y \times X$.

By mirroring the double complex $D_{\bullet,\bullet}$, the double complex spectral sequence for sheaves and the fact that $H_n(\mathrm{Tot}(D_{\bullet,\bullet})) = 0$ imply that $C_*(\mathcal{C}(U,V) \underset{\mathrm{Psh}}{\otimes} \mathbb{A}^1_X/X)$ is locally quasi-isomorphic to 0. We argue here similarly to the proof of Lemma 2.2.21. Then $C_*((\mathcal{C}(U,V) \underset{\mathrm{Psh}}{\otimes} \mathbb{A}^1_X/X)_{\mathsf{nis}}) \cong C_*(\mathcal{C}(U,V) \underset{\mathrm{Shv}}{\otimes} (\mathbb{A}^1_X/X)_{\mathsf{nis}})$ is locally quasi-isomorphic to 0. So $\mathcal{C}(U,-) \underset{\mathrm{Shv}}{\otimes} (\mathbb{A}^1_X/X)_{\mathsf{nis}}$ is in $\ker(C_*)$, as claimed.

Since $\ker(C_*)$ is a full triangulated subcategory and $T_{\mathbb{A}^1_1}$ is compactly generated by the $\mathcal{C}(U,-) \underset{\mathrm{Shv}}{\otimes} (\mathbb{A}^1_X/X)_{\mathsf{nis}}$ it follows that $T_{\mathbb{A}^1_1} \subseteq \ker(C_*)$.

Now show the other inclusion. Let $X \in \ker(C_*)$. Using [33, Section 5.6] and [33, Proposition 4.9.1] we can construct a triangle in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$

$$Y \to X \to LX$$

with $Y \in T_{\mathbb{A}^1_1}$ and LX orthogonal to $T_{\mathbb{A}^1_1}$. Apply C_* to the triangle to get

$$C_*Y \to C_*X \to C_*LX.$$

Since $X, Y \in \ker(C_*)$, we see that $C_*X = C_*Y = 0$, hence $C_*LX = 0$.

Since LX is orthogonal to $T_{\mathbb{A}^1_1}$, we can deduce that LX is strictly $\mathbb{A}^1_1$-local, so that for all $U \in \mathcal{C}$ we have a quasi-isomorphism $LX(U)(\mathbb{A}^1 \times -) \to LX(U)$ in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. From this property it follows that the canonical map $LX(U) \to C_*LX(U)$ is a quasi-isomorphism in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. Since $C_*LX = 0$ this implies $LX = 0$ in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$. But if $LX = 0$, then the map $Y \to X$ is an isomorphism in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ and then $X \in T_{\mathbb{A}^1_1}$. So $T_{\mathbb{A}^1_1} = \ker(C_*)$. □

Let $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/T_{\mathbb{A}^1_1}$ denote the quotient of $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ by the triangulated subcategory $T_{\mathbb{A}^1_1}$. By Lemma 3.1.4 $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/T_{\mathbb{A}^1_1}$ is equivalent to the full subcategory of $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ consisting of strictly $\mathbb{A}^1_1$-local objects.

3.2.4 Lemma. *Let $L : D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})]) \to D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ be the $T_{\mathbb{A}^1_1}$-localization endofunctor, which is the composite of the quotient functor $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})]) \to D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/T_{\mathbb{A}^1_1}$ and the inclusion of $T_{\mathbb{A}^1_1}$-local objects $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/T_{\mathbb{A}^1_1} \to$*

$D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$. *Then the functor L is naturally isomorphic to the endofunctor $C_* : D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})]) \to D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$.*

Proof. For every $X \in D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ we have an exact triangle in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$

$$Y \to X \to LX$$

with $Y \in \ker(L) = T_{\mathbb{A}^1_1}$. We can apply C_* to this triangle to get another triangle in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$

$$C_*Y \to C_*X \to C_*LX.$$

Since $Y \in T_{\mathbb{A}^1_1}$ and by Lemma 3.2.3 $T_{\mathbb{A}^1_1} = \ker(C_*)$ we know that $C_*Y = 0$ in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$. So we get an isomorphism

$$C_*X \cong C_*LX$$

in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$. Since the map $X \to LX$ is functorial in $X \in D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$, it follows that also the map $C_*X \to C_*LX$ is functorial in X. Therefore the isomorphism $C_*X \cong C_*LX$ is functorial in X. Since LX is strictly $\mathbb{A}^1_1$-invariant we have a natural quasi-isomorphism $LX \cong LX(\mathbb{A}^1 \times -)$ in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. This then implies that for every $n \in \mathbb{N}$ we also have a natural quasi-isomorphism $LX \cong LX(\Delta^n_k \times -)$. It now follows from the definition of C_* that we have a natural isomorphism $LX \cong C_*LX$ in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$. And then we have isomorphisms

$$C_*X \cong C_*LX \cong LX$$

natural in X, which proves the lemma. □

3.2.5 Definition. We say that a morphism $f : X \to Y$ in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is a *motivic equivalence* if and only if f is an isomorphism in $DM^{\mathsf{eff}}_{\mathcal{A}}$. Note that f in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is a motivic equivalence if and only if $C_*(f)$ is a local quasi-isomorphism in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.

Similarly, we say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ is a *motivic equivalence* if it is an isomorphism in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/T_{\mathbb{A}^1_1}$.

From the previous lemma we can deduce:

3.2.6 Corollary. *A morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ is a motivic equivalence if and only if $C_*(f)$ is a pointwise local quasi-isomorphism in $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$.*

3.2.7 Lemma. *For every $X \in \mathsf{Sm}_k$ the canonical map $I(X \times -) \to \mathcal{M}_{\mathcal{A}}(X)$ is a motivic equivalence in $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$. This means it is an isomorphism in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/T_{\mathbb{A}^1_1}$. In particular it is also an isomorphism in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}$.*

Proof. By Corollary 3.2.6 we just need to show for every $U \in \mathsf{Sm}_k$ that $C_*(I(X \times U)) \to C_*(M_{\mathcal{A}}(X \times U))$ is a local quasi-isomorphism in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. From the definition of $M_{\mathcal{A}}$ we know that $M_{\mathcal{A}}(X \times U) = C_*(I(X \times U))$. So the above map is equal to the canonical map $C_*(I(X \times U)) \to C_*C_*(I(X \times U))$ and this is clearly an isomorphism. □

3.2.8 Lemma. *The enriched motive functor $\mathcal{M}_{\mathcal{A}}(X)$ is strictly $\mathbb{A}^1_1$-local and strictly τ-local. So $\mathcal{M}_{\mathcal{A}}(X)$ is an object of $DM_{\mathcal{A}}[\mathcal{C}]$.*

Proof. The strict $\mathbb{A}^1_1$-locality follows from the $\mathbb{A}^1$-invariance of $C_*(\mathcal{A}(-, X)_{\mathsf{nis}})$. The cancellation property of $\mathcal{A}$ (see Definition 2.1.2) implies that $M_{\mathcal{A}}(X \times -)$ satisfies cancellation. Similarly to item (2) of the proof of Proposition 3.1.13, this implies $M_{\mathcal{A}}(X \times -)$ is strictly τ-local. □

The previous two lemmas together imply that $\mathcal{M}_{\mathcal{A}}(X)$ is a strictly $\sim_{\mathcal{C}}$-local replacement of $I(X \times -)$ in $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$.

3.2.9 Lemma. *If $f : X \to Y$ is a local quasi-isomorphism in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, and $X, Y \in \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ have $\mathbb{A}^1$-invariant cohomology sheaves, then the map*

$$f_* : \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge k}, X) \to \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge k}, Y)$$

is also a local quasi-isomorphism in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. In particular, the functor

$$\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge k}, -) : \mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})]) \to \mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$$

preserves pointwise local quasi-isomorphisms between strictly $\mathbb{A}^1_1$-local objects.

Proof. It follows from [37, Lemma 4.3.11] that for every X with $\mathbb{A}^1$-invariant cohomology sheaves and for every $n \in \mathbb{Z}$, we have a natural isomorphism

$$H_n^{\mathsf{nis}}(\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge k}, X)) \cong \underline{\mathrm{Hom}}_{\mathsf{Shv}(\mathcal{A})}(\mathbb{G}_m^{\wedge k}, H_n^{\mathsf{nis}}(X))$$

in $\mathsf{Shv}(\mathcal{A})$. So if $f : X \to Y$ is a local quasi-isomorphism between objects with $\mathbb{A}^1$-invariant cohomology sheaves, then we have for every $n \in \mathbb{Z}$ a commutative diagram

$$\begin{array}{ccc} H_n^{\mathsf{nis}}(\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge k}, X)) & \xrightarrow{H_n^{\mathsf{nis}}(f_*)} & H_n^{\mathsf{nis}}(\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge k}, Y)) \\ \downarrow \sim & & \downarrow \sim \\ \underline{\mathrm{Hom}}_{\mathsf{Shv}(\mathcal{A})}(\mathbb{G}_m^{\wedge k}, H_n^{\mathsf{nis}}(X)) & \xrightarrow{H_n^{\mathsf{nis}}(f)_*} & \underline{\mathrm{Hom}}_{\mathsf{Shv}(\mathcal{A})}(\mathbb{G}_m^{\wedge k}, H_n^{\mathsf{nis}}(Y)) \end{array}$$

in $\mathsf{Shv}(\mathcal{A})$. Since f is a local quasi-isomorphism, the lower horizontal map is an isomorphism. Therefore the upper horizontal map is an isomorphism. Then $f_* : \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge k}, X) \to \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge k}, Y)$ is also a local quasi-isomorphism in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. □

3.2.10 Lemma. *The functors*

$$\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge k}, -) : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$$

and

$$\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\times k}, -) : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$$

preserve motivic equivalences.

Proof. Let $f : A \to B$ be a motivic equivalence in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. Consider the diagram

$$\begin{array}{ccc} C_*\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge k}, A) & \xrightarrow{C_*(f_*)} & C_*\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge k}, B) \, . \\ \downarrow \sim & & \downarrow \sim \\ \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge k}, C_*A) & \xrightarrow{(C_*f)_*} & \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge k}, C_*B) \end{array}$$

The vertical maps are isomorphisms. Since f is a motivic equivalence we know that $C_*(f)$ is a local equivalence. Since C_*A and C_*B have $\mathbb{A}^1$-invariant cohomology sheaves it follows by Lemma 3.2.9 that the bottom horizontal map $(C_*f)_*$ is a local equivalence. This implies that the upper horizontal map $C_*(f_*)$ is a local equivalence, and hence $f_* : \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge k}, A) \to \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge k}, B)$ is a motivic equivalence. The second claim for $\mathbb{G}_m^{\times k}$ can be deduced from the claim for $\mathbb{G}_m^{\wedge k}$ by using Lemma 3.2.1. □

Let $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\tau$ denote the localization of $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ at the family of morphisms $\widehat{\tau}$. By Lemma 3.1.4 it is equivalent to the full subcategory of $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ of those functors which are strictly τ-local.

We will now prove some lemmas about $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\tau$, which show that $\mathcal{C}(\mathbb{G}_m^{\wedge k}, -)$ is a strongly dualizable object.

The model category $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ can be Bousfield localized along the family of morphisms $\hat{\tau}$, where just like Lemma 3.1.4, the family $\hat{\tau}$ is defined as

$$\hat{\tau} := \{(f \otimes Z)[n] | f \in \tau, Z \in \mathsf{Sm}_k, n \in \mathbb{Z}\}.$$

The homotopy category of this Bousfield localization is the derived category $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\tau$.

3.2.11 Lemma. *The left Bousfield localization of* $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ *along* $\widehat{\tau}$ *is a monoidal model category. In particular, the category* $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\tau$ *is closed symmetric monoidal and its tensor product* $\underset{\text{Day}}{\otimes}^{\mathbf{L}}$ *coincides with the tensor product in* $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$.

Proof. We apply [56, Theorem B]. Cofibrant objects in $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ are flat, so the theorem is applicable. The domains and codomains of the generating cofibrations of $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ are of the form $\mathcal{C}(\mathbb{G}_m^{\times k}, -) \underset{\text{Shv}}{\otimes} X$ for $k \in \mathbb{N}, X \in \mathsf{Sm}_k$. For $n \in \mathbb{N}$, let τ_n be the morphism

$$\mathcal{C}(\mathbb{G}_m^{\wedge n+1}, -) \underset{\text{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1} \xrightarrow{\tau_n} C(\mathbb{G}_m^{\wedge n}, -).$$

We need to show that for every $n, m, k \in \mathbb{N}, X, Z \in \mathsf{Sm}_k$ that

$$(\tau_n \underset{\text{Shv}}{\otimes} Z)[m] \underset{\text{Day}}{\otimes}^{\mathbf{L}} (\mathcal{C}(\mathbb{G}_m^{\times k}, -) \underset{\text{Shv}}{\otimes} X)$$

is a $\widehat{\tau}$-local equivalence in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$.

Since all involved objects are cofibrant we have

$$(\tau_n \underset{\text{Shv}}{\otimes} Z)[m] \underset{\text{Day}}{\otimes}^{\mathbf{L}} (\mathcal{C}(\mathbb{G}_m^{\times k}, -) \underset{\text{Shv}}{\otimes} X) \cong (\tau_n \underset{\text{Shv}}{\otimes} Z)[m] \underset{\text{Day}}{\otimes} (\mathcal{C}(\mathbb{G}_m^{\times k}, -) \underset{\text{Shv}}{\otimes} X).$$

Also we have

$$(\tau_n \underset{\text{Shv}}{\otimes} Z)[m] \underset{\text{Day}}{\otimes} (\mathcal{C}(\mathbb{G}_m^{\times k}, -) \underset{\text{Shv}}{\otimes} X) \cong (\tau_n \underset{\text{Day}}{\otimes} (\mathcal{C}(\mathbb{G}_m^{\times k}, -) \underset{\text{Shv}}{\otimes} (X \times Z)))[m]$$

so it suffices to show for every $n, k \in \mathbb{N}, X \in \mathsf{Sm}_k$ that every shift of $\tau_n \underset{\text{Day}}{\otimes} (\mathcal{C}(\mathbb{G}_m^{\times k}, -) \underset{\text{Shv}}{\otimes} X)$ is a $\widehat{\tau}$-local equivalence. This morphism is then equal to the composite

$$(\mathcal{C}(\mathbb{G}_m^{\wedge n+1}, -) \underset{\text{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1}) \underset{\text{Day}}{\otimes} (\mathcal{C}(\mathbb{G}_m^{\times k}, -) \underset{\text{Shv}}{\otimes} X) \cong \mathcal{C}(\mathbb{G}_m^{\wedge n+1} \times \mathbb{G}_m^{\times k}, -) \underset{\text{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1} \underset{\text{Shv}}{\otimes} X \to$$
$$\to \mathcal{C}(\mathbb{G}_m^{\wedge n} \times \mathbb{G}_m^{\times k}, -) \underset{\text{Shv}}{\otimes} X.$$

To show that it is a $\widehat{\tau}$-local equivalence, let $F \in \mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ be a τ-fibrant object, i.e. a functor that is locally fibrant and satisfies cancellation in the sense that $F(\mathbb{G}_m^{\wedge n}) \to F(\mathbb{G}_m^{\wedge n+1})(\mathbb{G}_m^{\wedge 1} \times -)$ is a local quasi-isomorphism. Since both sides are locally fibrant, it is also a sectionwise quasi-isomorphism.

We now just need to show for all $m \in \mathbb{Z}$ that

$$\mathrm{Hom}_{D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])}(\mathcal{C}(\mathbb{G}_m^{\wedge n} \times \mathbb{G}_m^{\times k}, -) \underset{\text{Shv}}{\otimes} X, F[m]) \to$$
$$\to \mathrm{Hom}_{D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])}(\mathcal{C}(\mathbb{G}_m^{\wedge n+1} \times \mathbb{G}_m^{\times k}, -) \underset{\text{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1} \underset{\text{Shv}}{\otimes} X, F[m])$$

is an isomorphism in Ab.

Since $F[m]$ is locally fibrant and $\mathcal{C}(\mathbb{G}_m^{\wedge n} \times \mathbb{G}_m^{\times k}, -) \underset{\text{Shv}}{\otimes} X$ and $\mathcal{C}(\mathbb{G}_m^{\wedge n+1} \times \mathbb{G}_m^{\times k}, -) \underset{\text{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1} \underset{\text{Shv}}{\otimes} X$ are cofibrant, this is isomorphic to the arrow

$$\mathrm{Hom}_{K([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])}(\mathcal{C}(\mathbb{G}_m^{\wedge n} \times \mathbb{G}_m^{\times k}, -) \underset{\text{Shv}}{\otimes} X, F[m]) \to$$
$$\to \mathrm{Hom}_{K([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])}(\mathcal{C}(\mathbb{G}_m^{\wedge n+1} \times \mathbb{G}_m^{\times k}, -) \underset{\text{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1} \underset{\text{Shv}}{\otimes} X, F[m]).$$

And this is isomorphic to the following arrow between homology groups

$$H_m(F(\mathbb{G}_m^{\wedge n} \times \mathbb{G}_m^{\times k})(X)) \to H_m(F(\mathbb{G}_m^{\wedge n+1} \times \mathbb{G}_m^{\times k})(X \times \mathbb{G}_m^{\wedge 1})).$$

So we just need to show that the following arrow is a quasi-isomorphism.

$$F(\mathbb{G}_m^{\wedge n} \times \mathbb{G}_m^{\times k})(X) \to F(\mathbb{G}_m^{\wedge n+1} \times \mathbb{G}_m^{\times k})(X \times \mathbb{G}_m^{\wedge 1})$$

Lemma 3.2.1 implies $F(\mathbb{G}_m^{\wedge n} \times \mathbb{G}_m^{\times k}) \cong \bigoplus_{i=0}^{k} \binom{k}{i} F(\mathbb{G}_m^{\wedge n+i})$. We have to show that the map $\bigoplus_{i=0}^{k} \binom{k}{i} F(\mathbb{G}_m^{\wedge n+i})(X) \to \bigoplus_{i=0}^{k} \binom{k}{i} F(\mathbb{G}_m^{\wedge n+1+i})(X \times \mathbb{G}_m^{\wedge 1})$ is a quasi-isomorphism.

This follows from the fact that $F(\mathbb{G}_m^{\wedge n}) \to F(\mathbb{G}_m^{\wedge n+1})(\mathbb{G}_m^{\wedge 1} \times -)$ is a sectionwise quasi-isomorphism for any $n \in \mathbb{Z}$. $\square$

3.2.12 Lemma. *The enriched functor $\mathcal{C}(\mathbb{G}_m^{\wedge 1}, -) : \mathcal{C} \to \mathsf{Shv}(\mathcal{A})$ is $\underset{\mathrm{Day}}{\otimes}^{\mathbf{L}}$-invertible in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\tau$ and its inverse is $I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1}$.*

Proof. The enriched functor $\mathcal{C}(\mathbb{G}_m^{\times 1}, -)$ is cofibrant in $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$, because it is representable. The enriched functor $\mathcal{C}(\mathbb{G}_m^{\wedge 1}, -)$ is a direct summand of $\mathcal{C}(\mathbb{G}_m^{\times 1}, -)$, so $\mathcal{C}(\mathbb{G}_m^{\wedge 1}, -)$ is also cofibrant. For every cofibrant $F \in \mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ we therefore have $\mathcal{C}(\mathbb{G}_m^{\wedge 1}, -) \underset{\mathrm{Day}}{\otimes}^{\mathbf{L}} F \cong \mathcal{C}(\mathbb{G}_m^{\wedge 1}, -) \underset{\mathrm{Day}}{\otimes} F$. Now let $F := I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1}$, i.e. F is the enriched functor defined by $F(X) := I(X) \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1}$. This functor F is cofibrant, because it is a direct summand of $\mathcal{C}(pt, -) \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\times 1}$.

We now show that there is an isomorphism

$$\mathcal{C}(\mathbb{G}_m^{\wedge 1}, -) \underset{\mathrm{Day}}{\otimes} (I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1}) \cong (\mathcal{C}(\mathbb{G}_m^{\wedge 1}, -) \underset{\mathrm{Day}}{\otimes} I) \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1}.$$

It explicitly looks as follows. $\mathbb{G}_m^{\wedge 1} \underset{\mathrm{Shv}}{\otimes} -$ is a left adjoint, so it preserves all coends, so

$$(\mathcal{C}(\mathbb{G}_m^{\wedge 1}, -) \underset{\mathrm{Day}}{\otimes} (I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1}))(c) = \int^{(a,b)\in\mathcal{C}\otimes\mathcal{C}} \mathcal{C}(a \times b, c) \underset{\mathrm{Shv}}{\otimes} \mathcal{C}(\mathbb{G}_m^{\wedge 1}, a) \underset{\mathrm{Shv}}{\otimes} I(b) \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1} \cong$$

$$\cong \mathbb{G}_m^{\wedge 1} \underset{\mathrm{Shv}}{\otimes} \int^{(a,b)\in\mathcal{C}\otimes\mathcal{C}} \mathcal{C}(a \times b, c) \underset{\mathrm{Shv}}{\otimes} \mathcal{C}(\mathbb{G}_m^{\wedge 1}, a) \underset{\mathrm{Shv}}{\otimes} I(b) \cong \mathbb{G}_m^{\wedge 1} \underset{\mathrm{Shv}}{\otimes} (\mathcal{C}(\mathbb{G}_m^{\wedge 1}, -) \underset{\mathrm{Day}}{\otimes} I)$$

Now I is the monoidal unit of $\underset{\mathrm{Day}}{\otimes}$, so $\mathcal{C}(\mathbb{G}_m^{\wedge 1}, -) \underset{\mathrm{Day}}{\otimes} F \cong \mathcal{C}(\mathbb{G}_m^{\wedge 1}, -) \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1}$. Finally, the morphism τ gives an isomorphism $\mathcal{C}(\mathbb{G}_m^{\wedge 1}, -) \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1} \to I$ in the derived category $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\tau$. So we ultimately get an isomorphism $\mathcal{C}(\mathbb{G}_m^{\wedge 1}, -) \underset{\mathrm{Day}}{\otimes}^{\mathbf{L}} F \cong I$ in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\tau$, which shows that $\mathcal{C}(\mathbb{G}_m^{\wedge 1}, -)$ is invertible. $\square$

Since $I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1}$ is invertible, we also have that $I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge k}$ is invertible, because due to the isomorphism $I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge k+1} \cong (I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge k}) \underset{\mathrm{Day}}{\otimes} (I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge 1})$ it is a product of invertible objects. The inverse of $I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge k}$ is $\mathcal{C}(\mathbb{G}_m^{\wedge k}, -)$. Also note that in every symmetric closed monoidal category, every $\otimes$-invertible object is strongly dualizable. So $\mathcal{C}(\mathbb{G}_m^{\wedge k}, -)$ is strongly dualizable in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\tau$.

Since finite sums of strongly dualizable objects are strongly dualizable, and since Lemma 3.2.1 says that $\mathcal{C}(\mathbb{G}_m^{\times k}, -)$ is a finite sum of $\mathcal{C}(\mathbb{G}_m^{\wedge i}, -)$, we get the following corollary.

3.2.13 Corollary. *For all $k \in \mathbb{N}$ the enriched functors $\mathcal{C}(\mathbb{G}_m^{\times k}, -)$ and $\mathcal{C}(\mathbb{G}_m^{\wedge k}, -)$ are strongly dualizable in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\tau$ with duals $I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\times k}$ and $I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge k}$ respectively.*

The model category $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ can be Bousfield localized along the family of morphisms $\widehat{\sim_{\mathcal{C}}}$. The homotopy category of this Bousfield localization is the derived category $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}$.

3.2.14 Lemma. *The left Bousfield localization of $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ along $\widehat{\sim_{\mathcal{C}}}$ is a monoidal model category. In particular, the category $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}$ is closed symmetric monoidal and its tensor product $\underset{\mathrm{Day}}{\otimes}^{\mathbf{L}}$ coincides with the tensor product in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$.*

Proof. Similarly to Lemma 3.2.11, we apply [56, Theorem B]. The domains and codomains of the generating cofibrations of $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ are of the form $\mathcal{C}(\mathbb{G}_m^{\times k}, -) \underset{\mathrm{Shv}}{\otimes} X$ for $k \in \mathbb{N}, X \in \mathsf{Sm}_k$. We need to show for f in $\widehat{\sim_{\mathcal{C}}}$ that all $f \underset{\mathrm{Day}}{\otimes}^{\mathbf{L}} \mathcal{C}(\mathbb{G}_m^{\times k}, -) \underset{\mathrm{Shv}}{\otimes} X$ are $\widehat{\sim_{\mathcal{C}}}$-local equivalences. If $f \in \widehat{\tau}$, then we know this from the proof of Lemma 3.2.11. So assume that $f \in \widehat{\mathbb{A}^1_1}$, so that f is of the form

$$(\mathcal{C}(U, -) \underset{\mathrm{Shv}}{\otimes} \mathbb{A}^1) \otimes Z[n] \to \mathcal{C}(U, -) \otimes Z[n]$$

for some $U \in \mathcal{C}$. Since all involved objects are cofibrant we know that

$$f \underset{\mathrm{Day}}{\otimes}^{\mathbf{L}} \mathcal{C}(\mathbb{G}_m^{\times k}, -) \underset{\mathrm{Shv}}{\otimes} X = f \underset{\mathrm{Day}}{\otimes} \mathcal{C}(\mathbb{G}_m^{\times k}, -) \underset{\mathrm{Shv}}{\otimes} X.$$

So f is isomorphic to

$$(\mathcal{C}(U \times \mathbb{G}_m^{\times k}, -) \underset{\mathrm{Shv}}{\otimes} \mathbb{A}^1) \otimes (X \times Z)[n] \to \mathcal{C}(U \times \mathbb{G}_m^{\times k}, -) \otimes (X \times Z)[n]$$

and this morphism lies again in $\widehat{\mathbb{A}^1_1}$. In particular it is a $\widehat{\mathbb{A}^1_1}$-local equivalence, and therefore also a $\widehat{\sim_{\mathcal{C}}}$-local equivalence. □

3.2.15 Lemma. *There is an isomorphism in* $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}$*:*

$$\mathcal{C}(\mathbb{G}_m^{\wedge k}, -) \underset{\mathrm{Shv}}{\otimes} X \cong [\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)].$$

Proof. We have $\mathcal{C}(\mathbb{G}_m^{\wedge k}, -) \underset{\mathrm{Shv}}{\otimes} X \cong \mathcal{C}(\mathbb{G}_m^{\wedge k}, -) \underset{\mathrm{Day}}{\otimes}^{\mathbf{L}} (I \underset{\mathrm{Shv}}{\otimes} X)$ Since $\mathcal{C}(\mathbb{G}_m^{\wedge k}, -)$ is strongly dual to $I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge k}$ with respect to $\underset{\mathrm{Day}}{\otimes}^{\mathbf{L}}$ in $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}$ we get that

$$\mathcal{C}(\mathbb{G}_m^{\wedge k}, -) \underset{\mathrm{Day}}{\otimes}^{\mathbf{L}} (I \underset{\mathrm{Shv}}{\otimes} X) \cong \underline{\mathrm{Hom}}_{D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}}(I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge k}, I \underset{\mathrm{Shv}}{\otimes} X).$$

By Lemma 3.2.8 the functor $M_{\mathcal{A}}(X \times -)$ is strictly $\sim_{\mathcal{C}}$-local. Since $I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge k}$ is cofibrant we can therefore compute the above internal hom as

$$\underline{\mathrm{Hom}}_{D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}}(I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge k}, I \underset{\mathrm{Shv}}{\otimes} X) \cong \underline{\mathrm{Hom}}_{D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])}(I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge k}, M_{\mathcal{A}}(X \times -)).$$

Let $M_{\mathcal{A}}(X \times -)^f$ be a pointwise local fibrant replacement of $M_{\mathcal{A}}(X \times -)$ in $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$. Then $M_{\mathcal{A}}(X \times -)^f$ is $\sim_{\mathcal{C}}$-fibrant and we have an isomorphism in $\mathsf{Ch}([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$.

$$\underline{\mathrm{Hom}}_{D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])}(I \underset{\mathrm{Shv}}{\otimes} \mathbb{G}_m^{\wedge k}, M_{\mathcal{A}}(X \times -)) \cong [\mathbb{G}_m^{\wedge k}, M_{\mathcal{A}}(X \times -)^f] \cong [\mathbb{G}_m^{\wedge k}, M_{\mathcal{A}}(X \times -)].$$

The last isomorphism follows from the fact that due to Lemma 3.2.9 the functor $[\mathbb{G}_m^{\wedge k}, -]$ preserves local quasi-isomorphisms between strictly $\mathbb{A}_1^1$-local objects. □

3.2.16 Lemma. $DM_{\mathcal{A}}[\mathcal{C}]$ *is compactly generated by the set* $\{[\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)] \mid k \in \mathbb{N}, X \in \mathsf{Sm}_k\}$.

Proof. Let us first show that $[\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)]$ is an object of $DM_{\mathcal{A}}[\mathcal{C}]$. By Lemma 3.2.8 the functor $\mathcal{M}_{\mathcal{A}}(X)$ is strictly $\mathbb{A}_1^1$-local and strictly τ-local. So if $\mathcal{M}_{\mathcal{A}}(X)^f$ is a locally fibrant replacement of $\mathcal{M}_{\mathcal{A}}(X)$, then $\mathcal{M}_{\mathcal{A}}(X)^f$ is enriched $\mathbb{A}_1^1$-local and satisfies cancellation. Since it is enriched $\mathbb{A}_1^1$-local, for every $U \in \mathsf{Sm}_k$ the complex $M_{\mathcal{A}}(X \times U)^f$ is motivically fibrant in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. Since $\mathbb{G}_m^{\wedge k}$ is cofibrant in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, it follows that $[\mathbb{G}_m^{\wedge k}, M_{\mathcal{A}}(X \times U)^f]$ is motivically fibrant in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. This then implies that $[\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)^f]$ is enriched $\mathbb{A}_1^1$-local. Since $\mathcal{M}_{\mathcal{A}}(X)^f$ satisfies cancellation, it also follows that $[\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)^f]$ satisfies cancellation.

By Lemma 3.2.9 the functor $[\mathbb{G}_m^{\wedge k}, -]$ preserves local equivalences between strictly $\mathbb{A}^1_1$-local objects. Hence it follows that $[\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)^f]$ is a local fibrant replacement of $[\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)]$. Thus $[\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)]$ is strictly $\mathbb{A}^1_1$-local and strictly τ-local. So $[\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)]$ is in $DM_{\mathcal{A}}[\mathcal{C}]$.

Let us now show that the objects $[\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)]$ compactly generate $DM_{\mathcal{A}}[\mathcal{C}]$. According to [20, Theorem 6.2] the category $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ is a compactly generated triangulated category, that is compactly generated by the set $\{\mathcal{C}(c, -) \underset{\mathrm{Shv}}{\otimes} g_i \mid c \in \mathcal{C}, i \in I\}$, where $\{g_i \mid i \in I\}$ is a set of compact generators of $D(\mathsf{Shv}(\mathcal{A}))$.

Since $\mathsf{Shv}(\mathcal{A})$ is compactly generated by sheaves of the form $I(X)$ for $X \in \mathsf{Sm}_k$, we conclude that $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$, and hence also $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}$, are compactly generated by the set $\{\mathcal{C}(\mathbb{G}_m^{\times k}, -) \underset{\mathrm{Shv}}{\otimes} I(X) \mid k \in \mathbb{N}, X \in \mathsf{Sm}_k\}$. By Lemma 3.2.1 the enriched functor $\mathcal{C}(\mathbb{G}_m^{\times k}, -)$ is a direct sum of functors of the form $\mathcal{C}(\mathbb{G}_m^{\wedge k}, -)$. So we conclude that $\{\mathcal{C}(\mathbb{G}_m^{\wedge k}, -) \underset{\mathrm{Shv}}{\otimes} I(X) \mid k \in \mathbb{N}, X \in \mathsf{Sm}_k\}$is a set of compact generators of $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}$.

Since $\{\mathcal{C}(\mathbb{G}_m^{\wedge k}, -) \underset{\mathrm{Shv}}{\otimes} I(X) \mid k \in \mathbb{N}, X \in \mathsf{Sm}_k\}$ is a set of compact generators of $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}$ we now get that by Lemma 3.2.15 that $\{[\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)] \mid k \in \mathbb{N}, X \in \mathsf{Sm}_k\}$ is a set of compact generators of $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}$.

Now each functor $[\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)]$ is in $DM_{\mathcal{A}}[\mathcal{C}]$. We remarked in Definition 3.1.5 that the canonical map $DM_{\mathcal{A}}[\mathcal{C}] \to D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}$ is an equivalence. Therefore it follows that $\{[\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)] \mid k \in \mathbb{N}, X \in \mathsf{Sm}_k\}$ is a set of compact generators of $DM_{\mathcal{A}}[\mathcal{C}]$. □

3.2.17 Lemma. *For every $k \in \mathbb{N}$ and $X \in \mathsf{Sm}_k$ the canonical map*

$$ev_{\mathbb{G}_m}([\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)]) \to \Omega^k_{\mathbb{G}_m} ev_{\mathbb{G}_m}(\mathcal{M}_{\mathcal{A}}(X)^f)$$

is a levelwise local quasi-isomorphism in $\mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))$, where $\mathcal{M}_{\mathcal{A}}(X)^f$ is a pointwise local fibrant replacement of $\mathcal{M}_{\mathcal{A}}(X)$.

Proof. Let $M_{\mathcal{A}}(X \times -)^f$ be a locally fibrant replacement of $M_{\mathcal{A}}(X \times -)$. By Lemma 3.2.8 we know that $M_{\mathcal{A}}(X \times -)^f$ is enriched $\mathbb{A}^1_1$-local and enriched τ-local. So $M_{\mathcal{A}}(X \times -)^f$ is pointwise $\mathbb{A}^1$-invariant and satisfies cancellation. Since $M_{\mathcal{A}}(X \times -)^f$ is pointwise $\mathbb{A}^1$-invariant it follows that $ev_{\mathbb{G}_m}(M_{\mathcal{A}}(X \times -)^f)$ is levelwise motivically fibrant. Since $M_{\mathcal{A}}(X \times -)^f$ satisfies cancellation, we see that $ev_{\mathbb{G}_m}(M_{\mathcal{A}}(X \times -)^f)$ is an $\Omega_{\mathbb{G}_m}$-spectrum. So $ev_{\mathbb{G}_m}(M_{\mathcal{A}}(X \times -)^f)$ is stably motivically fibrant in $\mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))$, and hence $\Omega^k_{\mathbb{G}_m} ev_{\mathbb{G}_m}(M_{\mathcal{A}}(X \times -)^f)$ can be computed in weight n as

$$\Omega^k_{\mathbb{G}_m} ev_{\mathbb{G}_m}(M_{\mathcal{A}}(X \times -)^f)(n) = [\mathbb{G}_m^{\wedge k}, M_{\mathcal{A}}(X \times \mathbb{G}_m^{\wedge n})^f].$$

But that is also the n-th weight of $ev_{\mathbb{G}_m}([\mathbb{G}_m^{\wedge k}, M_{\mathcal{A}}(X \times -)^f])$. So the canonical map

$$ev_{\mathbb{G}_m}([\mathbb{G}_m^{\wedge k}, M_{\mathcal{A}}(X \times -)]) \to \Omega^k_{\mathbb{G}_m} ev_{\mathbb{G}_m}(M_{\mathcal{A}}(X \times -))$$

is isomorphic to

$$ev_{\mathbb{G}_m}([\mathbb{G}_m^{\wedge k}, M_{\mathcal{A}}(X \times -)]) \to ev_{\mathbb{G}_m}([\mathbb{G}_m^{\wedge k}, M_{\mathcal{A}}(X \times -)^f]).$$

This is a levelwise local quasi-isomorphism in $\mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))$, because due to Lemma 3.2.9 the functor $[\mathbb{G}_m^{\wedge k}, -]$ preserves local quasi-isomorphisms between strictly $\mathbb{A}^1_1$-local objects. □

To prove Theorem 3.1.8 and show that the functor $ev_{\mathbb{G}_m} : DM_{\mathcal{A}}[\mathcal{C}] \to DM_{\mathcal{A}}$ is an equivalence, we will use [21, Lemma 4.8], which says the following:

3.2.18 Lemma. *Let A, B be compactly generated triangulated categories. Let Σ be a set of compact generators in A. Let $F : A \to B$ be a triangulated functor such that*

1. *The collection $\{F(X) | X \in \Sigma\}$ is a set of compact generators in B*
2. *For all $X, Y \in \Sigma$ and $n \in \mathbb{Z}$ the map*

$$F_{X,Y[n]} : \mathrm{Hom}_A(X, Y[n]) \to \mathrm{Hom}_B(F(X), F(Y)[n])$$

is an isomorphism.

Then F is an equivalence of triangulated categories.

We are now in a position to prove the main result of this section.

Proof of Theorem 3.1.8. We use Lemma 3.2.18. Here $A = DM_{\mathcal{A}}[\mathcal{C}]$ and $B = DM_{\mathcal{A}}$ are in fact compactly generated triangulated categories. One set of compact generators of $DM_{\mathcal{A}}$ is given by $\{\Omega^k_{\mathbb{G}_m} ev_{\mathbb{G}_m}(\mathcal{M}_{\mathcal{A}}(X)^f) | k \in \mathbb{N}, X \in \mathsf{Sm}_k\}$, where $\mathcal{M}_{\mathcal{A}}(X)^f$ is a pointwise local fibrant replacement of $\mathcal{M}_{\mathcal{A}}(X)$. By Lemma 3.2.16 the set

$$\Sigma := \{[\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)] | k \in \mathbb{N}, X \in \mathsf{Sm}_k\}$$

is a set of compact generators of $DM_{\mathcal{A}}[\mathcal{C}]$. This is the set of compact generators to which we want to apply Lemma 3.2.18. We now check the two conditions of that lemma.

To show the first condition we use Lemma 3.2.17: For every $A \in \Sigma$ we have an isomorphism

$$ev_{\mathbb{G}_m}(A) = ev_{\mathbb{G}_m}([\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)]) \overset{3.2.17}{\cong} \Omega^k_{\mathbb{G}_m} ev_{\mathbb{G}_m}(\mathcal{M}_{\mathcal{A}}(X)^f)$$

which is one of the compact generators of $DM_{\mathcal{A}}$. So

$$\{ev_{\mathbb{G}_m}(A) | A \in \Sigma\} = \{\Omega^k_{\mathbb{G}_m} ev_{\mathbb{G}_m}(\mathcal{M}_{\mathcal{A}}(X)^f) | k \in \mathbb{N}, X \in \mathsf{Sm}_k\}$$

which shows condition 1.

Let us now check condition 2. Take $\mathcal{X}, \mathcal{Y} \in DM_{\mathcal{A}}[\mathcal{C}]$ and $n \in \mathbb{Z}$. We have to show that $\mathrm{Hom}_{DM_{\mathcal{A}}[\mathcal{C}]}(\mathcal{X}, \mathcal{Y}[n]) \cong \mathrm{Hom}_{DM_{\mathcal{A}}}(ev_{\mathbb{G}_m}(\mathcal{X}), ev_{\mathbb{G}_m}(\mathcal{Y})[n])$. Since Σ compactly generates $DM_{\mathcal{A}}[\mathcal{C}]$ it suffices to show this for the case $\mathcal{X} \in \Sigma$. So assume without loss of generality that $\mathcal{X} \in \Sigma$ is of the form $[\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)]$ for some $X \in \mathsf{Sm}_k$ and $k \in \mathbb{N}$. Furthermore, we may assume without loss of generality that $\mathcal{Y}$ is $\sim_{\mathcal{C}}$-fibrant. So $\mathcal{Y}$ is pointwise motivically fibrant and satisfies cancellation. Then we have with Lemma 3.2.15 that

$$\begin{aligned}\mathrm{Hom}_{DM_{\mathcal{A}}[\mathcal{C}]}([\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)], \mathcal{Y}[n]) &\overset{3.2.15}{\cong} \mathrm{Hom}_{D([\mathcal{C},\mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}}(\mathcal{C}(\mathbb{G}_m^{\wedge k}, -) \underset{\mathrm{Shv}}{\otimes} X, \mathcal{Y}[n]) = \\ &= \mathrm{Hom}_{D([\mathcal{C},\mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}}(\mathcal{C}(\mathbb{G}_m^{\wedge k}, -), \underline{\mathrm{Hom}}_{\mathsf{Shv}(\mathcal{A})}(I(X), \mathcal{Y})[n]) = H_n(\mathcal{Y}(\mathbb{G}_m^{\wedge k})(X)).\end{aligned}$$

By Lemma 3.2.17 we have an isomorphism

$$ev_{\mathbb{G}_m}([\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)]) \overset{3.2.17}{\cong} \Omega^k_{\mathbb{G}_m} ev_{\mathbb{G}_m}(\mathcal{M}_{\mathcal{A}}(X)^f).$$

Since $\mathcal{Y}$ satisfies cancellation, $ev_{\mathbb{G}_m}(\mathcal{Y})$ is an $\Omega_{\mathbb{G}_m}$-spectrum, hence $ev_{\mathbb{G}_m}(\mathcal{Y}) \cong \Omega^k_{\mathbb{G}_m} ev_{\mathbb{G}_m}(\mathcal{Y})(k)$. Since $\mathcal{Y}$ is pointwise motivically fibrant, it follows that $ev_{\mathbb{G}_m}(\mathcal{Y})$ is stably motivically fibrant in $DM_{\mathcal{A}}$. Therefore,

$$\begin{aligned}\mathrm{Hom}_{DM_{\mathcal{A}}}(ev_{\mathbb{G}_m}([\mathbb{G}_m^{\wedge k}, \mathcal{M}_{\mathcal{A}}(X)]), ev_{\mathbb{G}_m}(\mathcal{Y})[n]) &\cong \\ \mathrm{Hom}_{DM_{\mathcal{A}}}(\Omega^k_{\mathbb{G}_m} ev_{\mathbb{G}_m}(\mathcal{M}_{\mathcal{A}}(X)^f), ev_{\mathbb{G}_m}(\mathcal{Y})[n]) &\cong \\ \mathrm{Hom}_{DM_{\mathcal{A}}}(\Omega^k_{\mathbb{G}_m} ev_{\mathbb{G}_m}(\mathcal{M}_{\mathcal{A}}(X)^f), \Omega^k_{\mathbb{G}_m} ev_{\mathbb{G}_m}(\mathcal{Y})(k)[n]) &\cong \\ \mathrm{Hom}_{DM_{\mathcal{A}}}(ev_{\mathbb{G}_m}(\mathcal{M}_{\mathcal{A}}(X)^f), ev_{\mathbb{G}_m}(\mathcal{Y})(k)[n]) &\cong \\ \mathrm{Hom}_{DM_{\mathcal{A}}}(\Sigma^\infty_{\mathbb{G}_m} X_+, ev_{\mathbb{G}_m}(\mathcal{Y})(k)[n]) \cong H_n(\mathcal{Y}(\mathbb{G}_m^{\wedge k})(X)).&\end{aligned}$$

We use here the fact that $ev_{\mathbb{G}_m}(\mathcal{M}_{\mathcal{A}}(X)^f)$ is a stably motivically fibrant replacement of $\Sigma^\infty_{\mathbb{G}_m} X_+$. We have verified all the conditions of Lemma 3.2.18. So $ev_{\mathbb{G}_m} : DM_{\mathcal{A}}[\mathcal{C}] \to DM_{\mathcal{A}}$ is an equivalence of triangulated categories. In particular, we have a zig-zag of equivalences

$$D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}} \overset{\sim}{\leftarrow} DM_{\mathcal{A}}[\mathcal{C}] \overset{\sim}{\to} DM_{\mathcal{A}}.$$

This completes the proof of Theorem 3.1.8. □

Chapter 4

Second Reconstruction Theorem for $DM_{\mathcal{A}}$

The goal of this chapter is to prove Theorem 3.1.14, which recovers $DM_{\mathcal{A}}[1/p]$ from $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim [1/p]$. For this we will need several lemmas. In Section 4.1 we take a $\sim$-fibrant enriched functor $\hat{F} : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ and extend it to a functor $\hat{F} : f\mathcal{M} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ on the category of finitely presented motivic spaces $f\mathcal{M}$, such that $\hat{F}$ sends motivic equivalences to local equivalences. See Theorem 4.1.1. This result will be important for proving Theorem 4.2.1 from Section 4.2, which states that for every $\sim$-fibrant enriched functor $F : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ and $U \in \mathcal{S}m$, we have an isomorphism in $DM_{\mathcal{A}}[1/p]$:

$$ev_{\mathbb{G}_m}(F(U \times -)) \cong ev_{\mathbb{G}_m}(F) \otimes \mathcal{L}(\Sigma^{\infty}_{S^1,\mathbb{G}_m} U_+).$$

Here $\mathcal{L} : SH(k) \to DM_{\mathcal{A}}$ is the left adjoint of the forgetful functor $\mathcal{U} : DM_{\mathcal{A}} \to SH(k)$, and the construction of those two functors is recalled in Section 4.2. We call this result the generalized Röndigs–Østvær theorem, because it is close to the original Röndigs–Østvær theorem of [46, p. 721]. The generalized Röndigs–Østvær theorem will be crucial for proving the Reconstruction Theorem 3.1.14 in Section 4.3.

4.1 From motivic to local equivalences

Let $\mathcal{M}$ be the category of motivic spaces and $f\mathcal{M}$ the category of finitely presented motivic spaces defined in [15]. Then $\mathcal{M}$ has a motivic model structure, as defined in [15, Theorem 2.12]. The weak equivalences in this model structure are called motivic equivalences.

Given a $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$-enriched functor $G : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, we can extend G to a (non-enriched) functor $\hat{G} : f\mathcal{M} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ in the following way. We can apply G levelwise to simplicial objects to get a functor

$$G^{\Delta^{op}} : \Delta^{op}\mathcal{S}m \to \Delta^{op}\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})).$$

For a finite pointed set $n_+ = \{0, \dots, n\}$ and $U \in \mathsf{Sm}_k$ we write $n_+ \otimes U$ for the n-fold coproduct $\coprod_{i=1}^{n} U$. We first extend it to $G : f\mathcal{M} \to \Delta^{op}\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ by

$$G(A) := \operatorname*{colim}_{(\Delta[n]\times U)_+ \to A^c} G^{\Delta^{op}}(\Delta[n]_+ \otimes U),$$

where A^c is a cofibrant replacement of A in $f\mathcal{M}$. We then compose it with the Dold-Kan correspondence

$$DK^{-1} : \Delta^{op}\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}_{\geqslant 0}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))$$

and the total complex functor

$$\mathrm{Tot} : \mathsf{Ch}_{\geqslant 0}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))) \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})), \quad \mathrm{Tot}(X)_n := \bigoplus_{k+l=n} X_{k,l},$$

to obtain a functor

$$\hat{G} : f\mathcal{M} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$$
$$\hat{G}(A) := \mathrm{Tot}(DK^{-1}(\operatorname*{colim}_{(\Delta[n]\times U)_+ \to A^c} G^{\Delta^{op}}(\Delta[n]_+ \otimes U))). \tag{4.1}$$

Note that for $U \in \mathsf{Sm}_k$ we have $\hat{G}(U_+) \cong G(U)$.

Throughout this section let $F : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ be an enriched functor that is $\sim$-fibrant in $\mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$. This means that F is pointwise locally fibrant, satisfies Nisnevich excision in the sense of Definition 3.1.9, and for every $X \in \mathsf{Sm}_k$ there are natural quasi-isomorphisms $F(X \times \mathbb{A}^1) \to F(X)$, $F(\mathbb{G}_m^{\wedge n}) \to [\mathbb{G}_m^{\wedge 1}, F(\mathbb{G}_m^{\wedge n+1})]$ in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, and for every $X, U \in \mathsf{Sm}_k$ a natural quasi-isomorphism

$$F(X)(U) \to F(X)(U \times \mathbb{A}^1)$$

in $\mathsf{Ch}(\mathrm{Ab})$. By the above construction we can extend F to a functor $\hat{F} : f\mathcal{M} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.

In this section we prove the following theorem.

4.1.1 Theorem. *$\hat{F}$ sends motivic equivalences in $f\mathcal{M}$ to local quasi-isomorphisms in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.*

The proof is like that of [25, Theorem 4.2] and requires several lemmas.

4.1.2 Lemma. *Let $H : \mathcal{S}m \to \mathsf{Shv}(\mathcal{A})$ be a $\mathsf{Shv}(\mathcal{A})$-enriched functor. Then $H(\emptyset) \cong 0$ and for all $U, V \in \mathcal{S}m$ $H(U \coprod V) \cong H(U) \oplus H(V)$ in $\mathsf{Shv}(\mathcal{A})$. In particular, if $G : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is a $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$-enriched functor we have $G(\emptyset) \cong 0$ and $G(U \coprod V) \cong G(U) \oplus G(V)$ in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.*

Proof. By the $\mathsf{Shv}(\mathcal{A})$-enriched co-Yoneda lemma we can write H as the following co-end: for $U \in \mathcal{S}m$ we have

$$H(U) \cong \int^{X \in \mathcal{S}m} H(X) \otimes \mathcal{S}m(X, U) \cong \int^{X \in \mathcal{S}m} H(X) \otimes \mathcal{A}(-, U)_{\mathsf{nis}}(X \times -).$$

By Definition 2.1.1 Axiom (3), we have $\mathcal{A}(-, \emptyset)_{\mathsf{nis}} = 0$ and for all $U, V \in \mathsf{Sm}_k$,

$$\mathcal{A}(-, U \coprod V)_{\mathsf{nis}} \cong \mathcal{A}(-, U)_{\mathsf{nis}} \oplus \mathcal{A}(-, V)_{\mathsf{nis}}.$$

This implies that

$$H(\emptyset) \cong \int^{X \in \mathcal{S}m} H(X) \otimes \mathcal{A}(-, \emptyset)_{\mathsf{nis}}(X \times -) = \int^{X \in \mathcal{S}m} H(X) \otimes 0 = 0$$

and for all $U, V \in \mathsf{Sm}_k$,

$$\begin{aligned} H(U \coprod V) \cong \int^{X \in \mathcal{S}m} H(X) \otimes \mathcal{A}(-, U \coprod V)_{\mathsf{nis}}(X \times -) \cong \\ \int^{X \in \mathcal{S}m} H(X) \otimes (\mathcal{A}(-, U)_{\mathsf{nis}}(X \times -) \oplus \mathcal{A}(-, V)_{\mathsf{nis}}(X \times -)) \cong \\ (\int^{X \in \mathcal{S}m} H(X) \otimes \mathcal{A}(-, U)_{\mathsf{nis}}(X \times -)) \oplus (\int^{X \in \mathcal{S}m} H(X) \otimes \mathcal{A}(-, V)_{\mathsf{nis}}(X \times -)) \cong \\ \cong H(U) \oplus H(V) \end{aligned}$$

as required. □

4.1.3 Corollary. *Let $G : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ be a $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$-enriched functor. Then for every $n \in \mathbb{N}, U \in \mathsf{Sm}_k$ the canonical map*

$$G(n_+ \otimes U) = G(\coprod_{i=1}^{n} 1_+ \otimes U) \to \bigoplus_{i=1}^{n} G(1_+ \otimes U) = \bigoplus_{i=1}^{n} G(U)$$

is an isomorphism.

Recall that $\Delta^{op}\,\mathrm{Ab}$ is monoidal with respect to the degreewise tensor product, and $\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})$ is monoidal with respect to the usual tensor product of chain complexes.

4.1.4 Lemma. *The Dold-Kan equivalence $DK^{-1} : \Delta^{op}\,\mathrm{Ab} \to \mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})$ preserves tensor products up to chain homotopy equivalence in the following sense. There are maps*

$$\nabla_{A,B} : DK^{-1}(A) \otimes DK^{-1}(B) \to DK^{-1}(A \otimes B)$$

$$\Delta_{A,B} : DK^{-1}(A \otimes B) \to DK^{-1}(A) \otimes DK^{-1}(B)$$

natural in A, B, such that $\Delta_{A,B} \circ \nabla_{A,B} = id_{DK^{-1}(A) \otimes DK^{-1}(B)}$, and there is a chain homotopy $\nabla_{A,B} \circ \Delta_{A,B} \sim id_{DK^{-1}(A \otimes B)}$. This chain homotopy is natural in the following sense: for all maps $f : A \to A'$, $g : B \to B'$ the chain homotopy between the maps $DK^{-1}(f \otimes g) \circ \nabla_{A,B} \circ \Delta_{A,B} \sim DK^{-1}(f \otimes g)$ encoded by the diagram

$$\begin{array}{ccccc} DK^{-1}(A \otimes B) & \xrightarrow{\Delta} & DK^{-1}(A) \otimes DK^{-1}(B) & \xrightarrow{\nabla} & DK^{-1}(A \otimes B) \\ \downarrow{\scriptstyle DK^{-1}(f\otimes g)} & & & & \downarrow{\scriptstyle DK^{-1}(f\otimes g)} \\ DK^{-1}(A' \otimes B') & \xrightarrow{\Delta} & DK^{-1}(A') \otimes DK^{-1}(B') & \xrightarrow{\nabla} & DK^{-1}(A' \otimes B') \end{array}$$

with the curved arrow $DK^{-1}(A' \otimes B') \xrightarrow[\simeq]{id_{DK^{-1}(A' \otimes B')}} DK^{-1}(A' \otimes B')$ below,

is equal to the chain homotopy between the maps $DK^{-1}(f \otimes g) \circ \nabla_{A,B} \circ \Delta_{A,B} \sim DK^{-1}(f \otimes g)$ encoded by the diagram

$$DK^{-1}(A \otimes B) \xrightarrow{\Delta} DK^{-1}(A) \otimes DK^{-1}(B) \xrightarrow{\nabla} DK^{-1}(A \otimes B) \xrightarrow{DK^{-1}(f\otimes g)} DK^{-1}(A' \otimes B')$$

with the curved arrow $DK^{-1}(A \otimes B) \xrightarrow[\simeq]{id_{DK^{-1}(A \otimes B)}} DK^{-1}(A \otimes B)$ below.

Proof. Everything except for the naturality of the chain homotopy follows from [42]. The functor DK^{-1} is the normalized Moore complex, the map $\Delta_{A,B}$ is the Alexander-Whitney map and $\nabla_{A,B}$ is the Eilenberg-Zilber map. In [26, page 7] one can find explicit formulas for both of these maps, and one can also find an explicit formula for the chain homotopy $\nabla_{A,B} \otimes \Delta_{A,B} \sim id_{DK^{-1}(A\otimes B)}$, which is called the Shih operator in that paper. Using that explicit formula one can easily verify the naturality of the chain homotopy. □

Given a simplicial set $K \in \Delta^{op}\,\mathrm{Set}$ we can form the free simplicial abelian group $\mathbb{Z}^{(K)} \in \Delta^{op}\,\mathrm{Ab}$ and then apply the Dold-Kan equivalence $DK^{-1} : \Delta^{op}\,\mathrm{Ab} \to \mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})$ to get a chain complex which we will denote by $\mathbb{Z}[K]$:

$$\mathbb{Z}[K] := DK^{-1}(\mathbb{Z}^{(K)}) \in \mathsf{Ch}(\mathrm{Ab}). \tag{4.2}$$

The chain complex $\mathbb{Z}[K]$ is degreewise free. For example, with this notation $\mathbb{Z}[S^n]$ is the chain complex that is $\mathbb{Z}$ concentrated in homological degree n.

4.1.5 Lemma. *Let $G : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ be a $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$-enriched functor. For every finite simplicial set K and every $A \in f\mathcal{M}$ we have a chain homotopy equivalence*

$$\hat{G}(K_+ \wedge A) \xrightarrow{\sim} \mathbb{Z}[K] \otimes \hat{G}(A)$$

in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ which is natural in K and A. The chain homotopies here are also also natural in K and A, just like the chain homotopy from Lemma 4.1.4.

Proof. Since $\hat{G}(A)$ depends only on the cofibrant replacement A^c of A, it suffices to show the claim for A^c. We can write A^c as a filtered colimit of simplicial schemes $A^c = \operatorname{colim}_{i\in I} X_i$ for some $X_i \in \Delta^{op}\mathsf{Sm}_k$, and some filtered diagram I. Then also $K_+ \wedge A^c$ is cofibrant and we have $K_+ \wedge A^c = \operatorname{colim}_{i\in I}(K_+ \wedge X_i)$. Let $G^{\Delta^{op}} : \Delta^{op}\mathcal{S}m \to \Delta^{op}\mathsf{Shv}(\mathcal{A})$ be the functor that applies G in each simplicial degree. It follows from Corollary 4.1.3 that for each $i \in I$ we have an isomorphism

$$G^{\Delta^{op}}(K_+ \otimes X_i) \xrightarrow{\sim} \mathbb{Z}^{(K)} \otimes G^{\Delta^{op}}(X_i)$$

in $\Delta^{op}\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, where $\mathbb{Z}^{(K)} \in \Delta^{op}\,\mathrm{Ab}$ is the simplicial free abelian group on K and where the tensor product on the right side is degreewise the tensor product of $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, i.e. for each $n \in \mathbb{N}$

$$(\mathbb{Z}^{(K)} \otimes G^{\Delta^{op}}(X_i))_n := \mathbb{Z}^{(K)}_n \otimes G^{\Delta^{op}}(X_i)_n \in \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})).$$

It follows from Lemma 4.1.4 that the Dold-Kan correspondence DK^{-1} : $\Delta^{op}\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}_{\geqslant 0}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))$ preserves tensor products up to chain homotopy equivalence, and this chain homotopy equivalence is functorial. So the above isomorphism then implies that we have a natural chain homotopy equivalence $\hat{G}(K_+ \wedge X_{i,+}) \to \mathbb{Z}[K] \otimes \hat{G}(X_{i,+})$ in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. Then we get a natural chain homotopy equivalence

$$\hat{G}(K_+ \wedge A^c) = \underset{i\in I}{\mathsf{colim}}\hat{G}(K_+ \wedge X_{i,+}) \to \underset{i\in I}{\mathsf{colim}}\mathbb{Z}[K] \otimes \hat{G}(X_{i,+}) \cong$$

$$\cong \mathbb{Z}[K] \otimes \underset{i\in I}{\mathsf{colim}}\hat{G}(X_{i,+}) = \mathbb{Z}[K] \otimes \hat{G}(A^c)$$

in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. □

4.1.6 Corollary. *Let $G : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ be a $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$-enriched functor. Let K be a finite simplicial set, and let $f : A \to B$ be a morphism in $f\mathcal{M}$ such that $\hat{G}(f)$ is a local quasi-isomorphism in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. Then the map $\hat{G}(K_+ \wedge f)$: $\hat{G}(K_+ \wedge A) \to \hat{G}(K_+ \wedge B)$ is also a local quasi-isomorphism in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.*

Proof. By Lemma 4.1.5 the map $\hat{G}(K_+ \wedge f) : \hat{G}(K_+ \wedge A) \to \hat{G}(K_+ \wedge B)$ is chain homotopic to the map $\mathbb{Z}[K] \otimes \hat{G}(f) : \mathbb{Z}[K] \otimes \hat{G}(A) \to \mathbb{Z}[K] \otimes \hat{G}(B)$ in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. If $\hat{G}(f)$ is also a local quasi-isomorphism, then since $\mathbb{Z}[K]$ is degreewise flat, it follows that $\mathbb{Z}[K] \otimes \hat{G}(f)$ is also a local quasi-isomorphism. So $\hat{G}(K_+ \wedge f)$ is a local quasi-isomorphism in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. □

4.1.7 Lemma. *Let $G : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ be a $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$-enriched functor. Let K, L be finite simplicial sets, $A \in f\mathcal{M}$ and let $e : K \to L$ be a weak equivalence of simplicial sets. Then $\hat{G}(e_+ \wedge A) : \hat{G}(K_+ \wedge A) \to \hat{G}(L_+ \wedge A)$ is a sectionwise quasi-isomorphism in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.*

Proof. If $e : K \to L$ is a weak equivalence of simplicial sets, then it follows from basic properties of the Dold-Kan equivalence that $\mathbb{Z}[e] : \mathbb{Z}[K] \to \mathbb{Z}[L]$ is a quasi-isomorphism in $\mathsf{Ch}(\mathrm{Ab})$. Let $C := \mathrm{Cone}(\mathbb{Z}[e]) \in \mathsf{Ch}(\mathrm{Ab})$ be the homological mapping cone of $\mathbb{Z}[e]$. Since $\mathbb{Z}[e]$ is a quasi-isomorphism, we know that C is acyclic. Since $\mathbb{Z}[K]$ and $\mathbb{Z}[L]$ are degreewise free, we know that C is degreewise free. So $0 \to C$ is a trivial cofibration in the projective model structure on $\mathsf{Ch}(\mathrm{Ab})$. Since the projective model structure on $\mathsf{Ch}(\mathrm{Ab})$ satisfies the monoid axiom, then for every $D \in \mathsf{Ch}(\mathrm{Ab})$ the chain complex $C \otimes D$ is acyclic. Since

$C \otimes D$ is the mapping cone of $\mathbb{Z}[e] \otimes D$, then for every $D \in \mathsf{Ch}(\mathrm{Ab})$ the map $\mathbb{Z}[e] \otimes D : \mathbb{Z}[K] \otimes D \to \mathbb{Z}[L] \otimes D$ is a quasi-isomorphism in $\mathsf{Ch}(\mathrm{Ab})$.

By Lemma 4.1.5 $\hat{G}(e_+ \wedge A) : \hat{G}(K_+ \wedge A) \to \hat{G}(L_+ \wedge A)$ is chain homotopic to the map $\mathbb{Z}[e] \otimes \hat{G}(A) : \mathbb{Z}[K] \otimes \hat{G}(A) \to \mathbb{Z}[L] \otimes \hat{G}(A)$ in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. But this is a sectionwise quasi-isomorphism, because for every $V \in \mathsf{Sm}_k$ the map

$$\mathbb{Z}[e] \otimes \hat{G}(A)(V) : \mathbb{Z}[K] \otimes \hat{G}(A)(V) \to \mathbb{Z}[L] \otimes \hat{G}(A)(V)$$

is a quasi-isomorphism in $\mathsf{Ch}(\mathrm{Ab})$, by the above argument with $D := \hat{G}(A)(V)$. □

4.1.8 Definition. 1. A map $e : A \to X$ in a category $\mathcal{D}$ is called a *coprojection* if it is isomorphic to the coproduct inclusion $A \to A \coprod Y$ for some $Y \in \mathcal{D}$.

2. A map $e : A \to X$ in $\Delta^{op}\mathcal{D}$ is called a *termwise coprojection*, if for every $n \in \mathbb{N}$, the map in the n-th simplicial degree $e_n : A_n \to X_n$ is a coprojection in $\mathcal{D}$.

3. A pushout square in $\Delta^{op}\mathcal{D}$

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{e'} & D \end{array}$$

is called an *elementary pushout square*, if e and e' are termwise coprojections.

Recall that throughout this section $F : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is a $\sim$-fibrant enriched functor, and that we have above constructed a non-enriched functor $\hat{F} : f\mathcal{M} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.

4.1.9 Lemma. *$\hat{F}$ takes elementary pushout squares in $\Delta^{op}\mathcal{S}m$ to homotopy pushout squares in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.*

Proof. Take a pushout square in $\mathcal{S}m$, along coprojections e, e' :

$$\begin{array}{ccc} A & \xrightarrow{e} & A \coprod X \\ \downarrow & & \downarrow \\ B & \xrightarrow{e'} & B \coprod X \end{array}$$

We can apply F to get a square in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$:

$$\begin{array}{ccc} F(A) & \longrightarrow & F(A\coprod X) \\ \downarrow & & \downarrow \\ F(B) & \longrightarrow & F(B\coprod X) \end{array}$$

According to Lemma 4.1.2 this square is isomorphic to

$$\begin{array}{ccc} F(A) & \longrightarrow & F(A)\oplus F(X) \\ \downarrow & & \downarrow \\ F(B) & \longrightarrow & F(B)\oplus F(X) \end{array}$$

By taking a local cofibrant replacement $F(X)^c$ of $F(X)$ we see that this square is locally equivalent to

$$\begin{array}{ccc} F(A) & \longrightarrow & F(A)\oplus F(X)^c \\ \downarrow & & \downarrow \\ F(B) & \longrightarrow & F(B)\oplus F(X)^c \end{array}$$

This square is a homotopy pushout, because it is a strict pushout and $F(A) \to F(A) \oplus F(X)^c$ is a cofibration. So F sends pushout squares along coprojections in $\mathcal{S}m$ to homotopy pushout squares in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.

If we have an elementary pushout square Q in $\Delta^{op}\mathcal{S}m$ then in every simplicial degree it will be a pushout along coprojections. Then $F(Q)$ will be a square in $\Delta^{op}\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ that is in every simplicial degree a homotopy pushout. After applying the Dold-Kan correspondence we will still have a degreewise homotopy pushout, and after applying the total complex functor we obtain a single homotopy pushout square in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. So $\hat{F}(Q)$ is a homotopy pushout square in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. □

The previous lemma immediately implies the following corollary.

4.1.10 Corollary. *If we have an elementary pushout square in $\Delta^{op}\mathcal{S}m$,*

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{e'} & D \end{array}$$

and $\hat{F}(e)$ is a local quasi-isomorphism, then $\hat{F}(e')$ is a local quasi-isomorphism in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.

With all of these lemmas established, we can now prove the main result of this section.

Proof of Theorem 4.1.1. Let Q be an elementary Nisnevich square of the form

$$\begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

In the category of pointed simplicial Nisnevich sheaves $\mathcal{M} = \mathsf{Shv}(\mathsf{Sm}_k, \Delta^{op}\,\mathrm{Set}_*)$ we can factor the morphism $U'_+ \to X'_+$ by using the mapping cylinder $C := (U'_+ \times \Delta[1]_+)\coprod_{U'_+} X'_+$ to get a factorization $U'_+ \rightarrowtail C \xrightarrow{\sim} X'_+$ where the left map is a cofibration and the right map is a simplicial homotopy equivalence. We define $s(Q) := U_+ \coprod_{U'_+} C$. We can similarly take a mapping cylinder $t(Q)$ of the map $s(Q) \to X_+$ to factor it into $s(Q) \rightarrowtail t(Q) \xrightarrow{\sim} X_+$ where the left map is a cofibration and the right map a simplicial homotopy equivalence. We also take the mapping cylinder C_X of $(\mathbb{A}^1 \times X)_+ \to X_+$ to factor it as $(\mathbb{A}^1 \times X)_+ \rightarrowtail C_X \xrightarrow{\sim} X_+$.

Let $J_{mot} = J_{proj} \cup J_{\mathbb{A}^1} \cup J_{nis}$ where

$$J_{proj} = \{\Lambda^r[n]_+ \wedge U_+ \to \Delta[n]_+ \wedge U_+ \mid U \in \mathsf{Sm}_k, n > 0, 0 \leqslant r \leqslant n\}$$

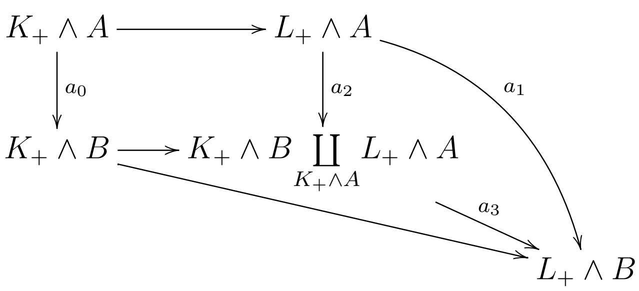
$$J_{\mathbb{A}^1} = \{\Delta[n]_+ \wedge U \times \mathbb{A}^1_+ \coprod_{\partial\Delta[n]_+ \wedge U \times \mathbb{A}^1} \partial\Delta[n]_+ \wedge C_U \to \Delta[n]_+ \wedge C_U \mid U \in \mathsf{Sm}_k\}$$

$$J_{nis} = \{\Delta[n]_+ \wedge s(Q) \coprod_{\partial\Delta[n]_+ \wedge s(Q)} \partial\Delta[n]_+ \wedge t(Q) \to \Delta[n]_+ \wedge t(Q) \mid Q \in \mathcal{Q}\}$$

where $\mathcal{Q}$ is the set of elementary Nisnevich squares. We claim that $\hat{F}$ sends all morphisms in J_{mot} to local quasi-isomorphisms. Since $\Lambda^r[n] \to \Delta[n]$ is a weak equivalence of simplicial sets it follows by Lemma 4.1.7 that $\hat{F}(\Lambda^r[n]_+ \wedge U_+) \to \hat{F}(\Delta[n]_+ \wedge U_+)$ is a local quasi-isomorphism, so $\hat{F}$ sends J_{proj} to local quasi-isomorphisms.

Note that $\hat{F}$ sends simplicial homotopy equivalences to chain homotopy equivalences, because $\hat{F}(\Delta[1]_+ \otimes A^c)$ is a cylinder object for $\hat{F}(A^c)$. Since we have a local quasi-isomorphism $\hat{F}(X \times \mathbb{A}^1) \to \hat{F}(X)$ and a simplicial homotopy equivalence $C_X \to X_+$ we have a local quasi-isomorphism $\hat{F}(X \times \mathbb{A}^1) \to \hat{F}(C_X)$.

Similarly, since F satisfies Nisnevich excision we have a local quasi-isomorphism $\hat{F}(s(Q)) \to \hat{F}(t(Q))$. Let $f : A \to B$ be a morphism either of the form $s(Q) \to t(Q)$ or $(X \times \mathbb{A}^1)_+ \to C_X$, and let $e : K \to L$ be a cofibration of simplicial sets. Then e is a termwise coprojection and $\hat{F}(f)$ is a local quasi-isomorphism. Consider the diagram



Since $\hat{F}(f)$ is a local quasi-isomorphism, by Lemma 4.1.6 also the maps $\hat{F}(a_0) = \hat{F}(K_+ \wedge f)$ and $\hat{F}(a_1) = \hat{F}(L_+ \wedge f)$ are local quasi-isomorphisms. By Corollary 4.1.10 also $\hat{F}(a_2)$ is a local quasi-isomorphism. By the 2-out-of-3-property this then implies that also $\hat{F}(a_3)$ is a local quasi-isomorphism. So $\hat{F}$ sends all morphisms from J_{mot} to local quasi-isomorphisms. Theorem 4.1.1 now follows by a simple small object argument, exactly like in the proof of Theorem 4.2 from [25]. □

4.2 The Generalized Röndigs–Østvær Theorem

Recall that the category of motivic spaces $\mathcal{M} = \mathsf{Shv}(\mathsf{Sm}_k, \Delta^{op}\,\mathrm{Set}_*)$ is equipped with a projective motivic model structure. See [15, Theorem 2.12] for details. This model structure induces a stable motivic model structure on the category of $(S^1, \mathbb{G}_m)$-bispectra of motivic spaces $\mathsf{Sp}_{S^1,\mathbb{G}_m}(\mathcal{M})$. We also have a motivic model structure on $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, given by taking the left Bousfield localization of the local model structure on $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ along the motivic equivalences from Definition 3.2.5. This motivic model structure induces a stable motivic model sturcture on the category of $\mathbb{G}_m$-spectra of chain complexes $\mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))$. The homotopy category of $\mathsf{Sp}_{S^1,\mathbb{G}_m}(\mathcal{M})$ is $SH(k)$. The homotopy category of $\mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))$ is $DM_{\mathcal{A}}$.

There is a forgetful functor $\mathcal{U} : DM_{\mathcal{A}} \to SH(k)$ with a left adjoint $\mathcal{L} : SH(k) \to DM_{\mathcal{A}}$. It can be described as follows. The functor $\mathcal{U}$ is the derived functor of the right Quillen functor

$$Sp_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))) \overset{J}{\to} Sp_{\mathbb{G}_m,S^1}(\mathrm{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))) \overset{DK}{\to}$$

$$Sp_{\mathbb{G}_m,S^1}(\Delta^{op}\mathsf{Shv}(\mathcal{A})) \overset{U}{\to} Sp_{\mathbb{G}_m,S^1}(\mathcal{M}).$$

Here $J : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Sp}_{S^1}(\mathrm{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A})))$ is the right Quillen equivalence that is called T in [30, Section 3]. If $\tau_{\geqslant 0} : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathrm{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ is the good truncation functor sending $A \in \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ to

$$\cdots \to A_2 \to A_1 \to \ker(A_0 \overset{\partial^0_A}{\to} A_{-1})$$

in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$, then J is defined on $A \in \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ by $J(A) = (\tau_{\geqslant 0}(A[n]))_{n\in\mathbb{N}} \in \mathsf{Sp}_{S^1}(\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A})))$.

The functor $DK : \mathrm{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A})) \to \Delta^{op}\mathsf{Shv}(\mathcal{A})$ is the Dold Kan equivalence, whose n-simplices are given by

$$DK(X)_n = \bigoplus_{\substack{[n]\to[k] \\ \text{surjective}}} X_k.$$

$U : \mathsf{Shv}(\mathcal{A}) \to \mathcal{M}$ is the functor that forgets transfers and the abelian group structure. We define $\hat{U} := U \circ DK \circ J$, so that $\mathcal{U}$ is the right derived functor of $\hat{U}$.

We write $\mathcal{L} : SH(k) \to DM_{\mathcal{A}}$ for the left adjoint of $\mathcal{U}$. The adjunction $\mathcal{L} : SH(k) \rightleftarrows DM_{\mathcal{A}} : \mathcal{U}$ is a monoidal adjunction, so that $\mathcal{U}$ is lax monoidal and $\mathcal{L}$ is strong monoidal. Furthermore $\mathcal{U}$ is a conservative functor. This means that if f is a morphism in $DM_{\mathcal{A}}$ such that $\mathcal{U}(f)$ is an isomorphism in $SH(k)$, then f is an isomorphism in $DM_{\mathcal{A}}$.

In this section we prove the following theorem, which is reminiscient of the Röndigs-Østvær theorem [46, Corollary 56].

4.2.1 Theorem. *Let* $F : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ *be an enriched functor that is* $\sim$*-fibrant in* $\mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim$. *Then for every* $X \in \mathsf{Sm}_k$*, the canonical morphism*

$$ev_{\mathbb{G}_m}(F) \otimes \mathcal{L}(\Sigma^\infty_{S^1,\mathbb{G}_m} X_+) \to ev_{\mathbb{G}_m}(F(X \times -))$$

is an isomorphism in $DM_{\mathcal{A}}[1/p]$*, which is natural in* X*.*

To prove 4.2.1 we will need several lemmas. The most important lemma we will need is the following one from [46, Corollary 56]:

4.2.2 Lemma. *Let* $X : f\mathcal{M} \to \mathcal{M}$ *be a motivic functor that sends motivic equivalences between cofibrant objects to motivic equivalences. Let* B *be a strongly dualizable object in* $SH(k)[1/p]$*. Then the canonical map of* $(S^1, \mathbb{G}_m)$*-bispectra*

$$ev_{S^1,\mathbb{G}_m}(X \wedge B) \to ev_{S^1,\mathbb{G}_m}(X \circ (- \wedge B))$$

is an isomoprhism in $SH(k)[1/p]$*.*

The following theorem by Riou can be found in [34, Appendix B, Corollary B.2].

4.2.3 Theorem. *If* $U \in \mathsf{Sm}_k$*, then* $\Sigma^\infty_{S^1,\mathbb{G}_m} U_+$ *is strongly dualizable in* $SH(k)[1/p]$*.*

To apply Lemma 4.2.2 in our situation, we have to convert $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$-enriched functors into motivic functors in the sense of [15]. We will now discuss how to do this.

We can consider the category of motivic spaces $\mathcal{M}$, the category of finitely presented motivic spaces $f\mathcal{M}$, the category of pointed smooth schemes $\mathsf{Sm}_{k,+}$ and the category of S^1-spectra of motivic spaces $\mathsf{Sp}_{S^1}(\mathcal{M})$ to all be $\mathcal{M}$-enriched categories. In the $\mathcal{M}$-enriched category $\mathsf{Sp}_{S^1}(\mathcal{M})$ the morphism objects $\mathrm{Map}_{\mathsf{Sp}_{S^1}(\mathcal{M})}(A, B) \in \mathcal{M}$, are defined for $A, B \in \mathsf{Sp}_{S^1}(\mathcal{M})$ via an equalizer diagram, like in [29, page 101]. So we have an equalizer diagram:

$$\mathrm{Map}_{\mathsf{Sp}_{S^1}(\mathcal{M})}(A, B) \longrightarrow \prod_{n\in\mathbb{N}} \underline{\mathrm{Hom}}_{\mathcal{M}}(A_n, B_n) \rightrightarrows \prod_{n\in\mathbb{N}} \underline{\mathrm{Hom}}_{\mathcal{M}}(S^1 \wedge A_n, B_{n+1})\,. \tag{4.3}$$

This makes $\mathsf{Sp}_{S^1}(\mathcal{M})$ into an $\mathcal{M}$-enriched category.

In order to relate $\mathcal{M}$-enriched categories and $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$-enriched categories, we need some lax monoidal functors between $\mathcal{M}$ and $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. We have a non-enriched forgetful functor $\hat{U} : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Sp}_{S^1}(\mathcal{M})$, and we have a functor $ev_0 : \mathsf{Sp}_{S^1}(\mathcal{M}) \to \mathcal{M}$ taking the 0-th weight of a S^1-spectrum. The functor $ev_0 \circ \hat{U} : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathcal{M}$ has a left adjoint $L : \mathcal{M} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.

4.2.4 Lemma. *The functor $ev_0 \circ \hat{U} : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathcal{M}$ and its left adjoint $L : \mathcal{M} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ are both lax monoidal functors.*

Proof. The functor $\hat{U}$ is the composite

$$\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \xrightarrow{J} \mathsf{Sp}_{S^1}(\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))) \xrightarrow{DK} \mathsf{Sp}_{S^1}(\Delta^{op}(\mathsf{Shv}(\mathcal{A}))) \xrightarrow{U} \mathsf{Sp}_{S^1}(\mathcal{M}).$$

Let $\tau_{\geqslant 0} : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ be the good truncation functor sending $A \in \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ to $\cdots \to A_2 \to A_1 \to \ker(A_0 \xrightarrow{\partial^0_A} A_{-1})$ in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. Then the following diagram commutes

$$\begin{array}{ccccccc}
\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) & \xrightarrow{J} & \mathsf{Sp}_{S^1}(\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))) & \xrightarrow{DK} & \mathsf{Sp}_{S^1}(\Delta^{op}\mathsf{Shv}(\mathcal{A})) & \xrightarrow{U} & \mathsf{Sp}_{S^1}(\mathcal{M}) \\
& \searrow^{\tau_{\geqslant 0}} & \downarrow{\scriptstyle ev_0} & & \downarrow{\scriptstyle ev_0} & & \downarrow{\scriptstyle ev_0} \\
& & \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A})) & \xrightarrow{DK} & \Delta^{op}\mathsf{Shv}(\mathcal{A}) & \xrightarrow{U} & \mathcal{M}
\end{array}.$$

To show that $ev_0 \circ \hat{U}$ is lax monoidal, we just have to show that U, DK and $\tau_{\geqslant 0}$ are lax monoidal, and to show that L is lax monoidal we just have to show that each of the left adjoints of U, DK and $\tau_{\geqslant 0}$ respectively is lax monoidal.

The left adjoint of $\tau_{\geqslant 0}$ is the inclusion functor $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. This inclusion is obviously strong monoidal. This then implies that $\tau_{\geqslant 0}$ is lax monoidal. See [43, Proposition 2.1] or [32, Theorem 1.2].

The quasi-inverse of the Dold–Kan correspondence $DK^{-1} : \Delta^{op}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ is the normalized Moore complex functor. It has a lax monoidal structure given by the Eilenberg–Zilber map and it has an oplax monoidal structure given by the Alexander–Whitney map. See [42] or [35, Definition 29.7]. Since DK^{-1} has an oplax monoidal structure it follows from [43, Proposition 2.1] that DK has a lax monoidal structure.

Finally, the forgetful functor $U : \Delta^{op}\mathsf{Shv}(\mathcal{A}) \to \mathcal{M}$ is clearly lax monoidal as its left adjoint is strong monoidal. So $ev_0 \circ \hat{U} : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathcal{M}$ and its left adjoint $L : \mathcal{M} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ are both lax monoidal functors. □

Let $F : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ be a $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$-enriched functor. We want to associate to F an $\mathcal{M}$-enriched functor

$$F^{\mathcal{M}} : f\mathcal{M} \to \mathsf{Sp}_{S^1}(\mathcal{M}).$$

To do this we will first construct a $\mathcal{M}$-enriched functor $\mathsf{Sm}_{k,+} \to \mathsf{Sp}_{S^1}(\mathcal{M})$ and then Kan extend it to $f\mathcal{M}$.

The $\mathcal{M}$-enriched functor $\mathsf{Sm}_{k,+} \to \mathsf{Sp}_{S^1}(\mathcal{M})$ is constructed as follows. On objects it sends $X_+ \in \mathsf{Sm}_{k,+}$ to $\hat{U}(F(X)) \in \mathsf{Sp}_{S^1}(\mathcal{M})$. To define it on morphisms we now need to define for each $X, Y \in \mathsf{Sm}_k$ a map in $\mathcal{M}$:

$$\mathsf{Sm}_{k,+}(X_+, Y_+) \to \mathrm{Map}_{\mathsf{Sp}_{S^1}(\mathcal{M})}(\hat{U}FX, \hat{U}FY).$$

This map is constructed in three steps. In the following construction $X, Y \in \mathsf{Sm}_k$ are smooth schemes. Recall that $L : \mathcal{M} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is the left adjoint of $ev_0 \circ \hat{U} : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathcal{M}$.

1. Since $L : \mathcal{M} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is lax monoidal, we have a map

   $$L\underline{\mathrm{Hom}}_{\mathcal{M}}(X_+, Y_+) \to \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(L(X_+), L(Y_+))$$

   in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. See [40, Example 3.1] for the construction of this map. By adjunction we get a map

   $$\underline{\mathrm{Hom}}_{\mathcal{M}}(X_+, Y_+) \to ev_0\hat{U}\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(L(X_+), L(Y_+))$$

   in $\mathcal{M}$. By construction, we have an isomorphism $L(X_+) \cong \mathcal{A}(-, X)_{\mathsf{nis}}$. Therefore $\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(L(X_+), L(Y_+)) \cong \mathcal{S}m(X, Y)$. Also $\mathsf{Sm}_{k,+}(X_+, Y_+) = \underline{\mathrm{Hom}}_{\mathcal{M}}(X_+, Y_+)$. We therefore get a map in $\mathcal{M}$.

   $$\mathsf{Sm}_{k,+}(X_+, Y_+) \to ev_0\hat{U}\mathcal{S}m(X, Y).$$

2. Since $F : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is a $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$-enriched functor we have a map $\mathcal{S}m(X, Y) \to \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(FX, FY)$ in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. We thus also get a map in $\mathcal{M}$.:

   $$ev_0\hat{U}\mathcal{S}m(X, Y) \to ev_0\hat{U}\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(FX, FY).$$

3. For every $n \in \mathbb{N}$, and every $A, B \in \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ the chain complex shift functor $[n]$ gives us an isomorphism

$$\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(A, B) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(A[n], B[n]).$$

   Since $ev_0\hat{U}$ is lax monoidal, we can use [40, Example 3.1] to get a canonical map

$$\begin{aligned} ev_0\hat{U}\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(A[n], B[n]) \to \underline{\mathrm{Hom}}_{\mathcal{M}}(ev_0\hat{U}A[n], ev_0\hat{U}B[n]) = \\ = \underline{\mathrm{Hom}}_{\mathcal{M}}((\hat{U}A)_n, (\hat{U}B)_n). \end{aligned}$$

All these maps $ev_0\hat{U}\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(A, B) \to \underline{\mathrm{Hom}}_{\mathcal{M}}((\hat{U}A)_n, (\hat{U}B)_n)$ yield a map

$$ev_0\hat{U}\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(A, B) \to \prod_{n\in\mathbb{N}} \underline{\mathrm{Hom}}_{\mathcal{M}}((\hat{U}A)_n, (\hat{U}B)_n).$$

We want to show that it factors over $\mathrm{Map}_{\mathsf{Sp}_{S^1}(\mathcal{M})}(\hat{U}A, \hat{U}B)$. Since $\hat{U}(A)$ is a S^1-spectrum we have for every $A \in \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ a map

$$S^1 \wedge ev_0\hat{U}(A[n]) \to ev_0\hat{U}(A[n+1])$$

in $\mathcal{M}$. Since $\hat{U}$ is a functor, this map is natural in A. Using this naturality one can check that for all $A, B \in \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ the following diagram commutes:

$$\begin{array}{ccc} ev_0\hat{U}\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(A[n], B[n]) & \xrightarrow{\sim} & ev_0\hat{U}\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(A[n+1], B[n+1]) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}_{\mathcal{M}}((\hat{U}A)_n, (\hat{U}B)_n) & & \underline{\mathrm{Hom}}_{\mathcal{M}}((\hat{U}A)_{n+1}, (\hat{U}B)_{n+1}) \\ {\scriptstyle S^1\wedge -}\downarrow & & \downarrow \\ \underline{\mathrm{Hom}}_{\mathcal{M}}(S^1 \wedge (\hat{U}A)_n, S^1 \wedge (\hat{U}B)_n) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{M}}(S^1 \wedge (\hat{U}A)_n, (\hat{U}B)_{n+1}) \end{array}$$

By the equalizer universal property of $\mathrm{Map}_{\mathsf{Sp}_{S^1}(\mathcal{M})}(\hat{U}A, \hat{U}B)$ from diagram (4.3) we get a dotted map like in the following diagram

$$\begin{array}{ccccc} & & ev_0\hat{U}\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(A, B) & & \\ & \swarrow\cdots & \downarrow & & \\ \mathrm{Map}_{\mathsf{Sp}_{S^1}(\mathcal{M})}(\hat{U}A, \hat{U}B) & \rightarrowtail & \prod_{n\in\mathbb{N}} \underline{\mathrm{Hom}}_{\mathcal{M}}((\hat{U}A)_n, (\hat{U}B)_n) & \rightrightarrows & \prod_{n\in\mathbb{N}} \underline{\mathrm{Hom}}_{\mathcal{M}}(S^1 \wedge (\hat{U}A)_n, (\hat{U}B)_{n+1}) \end{array}$$

In particular, we have a map

$$ev_0 \hat{U} \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(FX, FY) \to \mathrm{Map}_{\mathsf{Sp}_{S^1}(\mathcal{M})}(\hat{U}FX, \hat{U}FY).$$

And then we have maps

$$\mathsf{Sm}_{k,+}(X_+, Y_+) \to ev_0 \hat{U} \mathcal{S}m(X,Y) \to ev_0 \hat{U} \underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(FX, FY) \to \\ \to \mathrm{Map}_{\mathsf{Sp}_{S^1}(\mathcal{M})}(\hat{U}FX, \hat{U}FY).$$

By composing these three steps together we get a map

$$\mathsf{Sm}_{k,+}(X,Y) \to \mathrm{Map}_{\mathsf{Sp}_{S^1}(\mathcal{M})}(\hat{U}FX, \hat{U}FY)$$

in $\mathcal{M}$. This map preserves identity morphisms and is compatible with composition, so we get an $\mathcal{M}$-enriched functor $\mathsf{Sm}_{k,+} \to \mathsf{Sp}_{S^1}(\mathcal{M})$, sending X to $\hat{U}FX$.

We now define $F^{\mathcal{M}} : f\mathcal{M} \to \mathsf{Sp}_{S^1}(\mathcal{M})$ to be the $\mathcal{M}$-enriched Left Kan extension of this $\mathcal{M}$-enriched functor $\mathsf{Sm}_{k,+} \to \mathsf{Sp}_{S^1}(\mathcal{M})$ along the $\mathcal{M}$-enriched inclusion functor $\mathsf{Sm}_{k,+} \to f\mathcal{M}$.

$$\begin{array}{ccc} \mathsf{Sm}_{k,+} & \longrightarrow & \mathsf{Sp}_{S^1}(\mathcal{M}) \\ \downarrow & \overset{F^{\mathcal{M}}}{\nearrow} & \\ f\mathcal{M} & & \end{array}$$

The functor $F^{\mathcal{M}}$ can be explicitly computed on $A \in f\mathcal{M}$ as

$$F^{\mathcal{M}}(A) = \int^{X_+ \in \mathsf{Sm}_{k,+}} \hat{U}(F(X)) \wedge \underline{\mathrm{Hom}}_{\mathcal{M}}(X_+, A).$$

Note that $F^{\mathcal{M}}$ respects filtered colimits, because $X_+ \in \mathsf{Sm}_{k,+}$ is finitely presented in $\mathcal{M}$.

4.2.5 Lemma. *Let $F : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ be a $\sim$-fibrant functor. For every finitely presented motivic space $A \in f\mathcal{M}$ with cofibrant replacement A^c, we have a natural isomorphism $(\hat{U} \circ \hat{F})(A) \cong F^{\mathcal{M}}(A^c)$ in $\mathsf{Sp}_{S^1}(\mathcal{M})$. Here $\hat{U} : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Sp}_{S^1}(\mathcal{M})$ is the forgetful functor and $\hat{F} : f\mathcal{M} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is the extension of F defined by equation (4.1) in Section 4.1.*

Proof. If $A = X_+$ for some $X \in \mathsf{Sm}_k$ we have $\hat{U}(\hat{F}(X_+)) \cong \hat{U}(F(X))$ and by the $\mathcal{M}$-enriched co-Yoneda lemma we have

$$\hat{U}(F(X)) \cong \int^{Y_+ \in \mathsf{Sm}_{k,+}} \hat{U}(F(Y)) \wedge \underline{\mathrm{Hom}}_{\mathcal{M}}(Y_+, X_+) = F^{\mathcal{M}}(X_+).$$

So the claim is true for $A = X_+$. The claim then also follows for all other objects A in $f\mathcal{M}$, because A^c is a filtered colimit of simplicial schemes, and $F^{\mathcal{M}}$ respects filtered colimits. □

4.2.6 Lemma. *Let $F : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ be a pointwise locally fibrant functor, and let $A \in f\mathcal{M}$ be a finitely presented motivic space. Then $\hat{F}(A)$ is locally fibrant in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.*

Proof. For every scheme X we know that $F(X)$ is locally fibrant in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. If A is a finitely presented motivic space, then A^c is a filtered colimit of simplicial schemes. $A^c = \operatorname{colim}_{i \in I} X_i$ for some $X_i \in \Delta^{op}\mathsf{Sm}_k$ and filtered diagram I, and we have $\hat{F}(A) = \operatorname{colim}_{i \in I} \hat{F}(X_i)$. The fact that F is pointwise locally fibrant implies for each $i \in I$ that $\hat{F}(X_i)$ is locally fibrant in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. By Lemma 2.2.18 the model category $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is weakly finitely generated, so it follows by [14, Lemma 3.5] that filtered colimits of fibrant objects are fibrant. So $\hat{F}(A)$ is locally fibrant in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. □

For every $n \in \mathbb{N}$ we can take the n-th level of the functor $F^{\mathcal{M}} : f\mathcal{M} \to \mathsf{Sp}_{S^1}(\mathcal{M})$ to get an $\mathcal{M}$-enriched motivic functor

$$F_n^{\mathcal{M}} : f\mathcal{M} \to \mathcal{M}.$$

The functor $F_n^{\mathcal{M}}$ is then a motivic functor as defined in [15].

4.2.7 Lemma. *Let $F : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ be a $\sim$-fibrant enriched functor. For every $n \in \mathbb{N}$ the motivic functor $F_n^{\mathcal{M}} : f\mathcal{M} \to \mathcal{M}$ sends motivic equivalences between cofibrant objects to local equivalences.*

Proof. By Theorem 4.1.1 we know that $\hat{F} : f\mathcal{M} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ sends motivic equivalences to local quasi-isomorphisms. By Lemma 4.2.6 we know that $\hat{F}$ sends all objects of $f\mathcal{M}$ to locally fibrant objects. With respect to the S^1-stable local model structure on $\mathsf{Sp}_{S^1}(\mathcal{M})$ and the local model structure on $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, the functor $\hat{U} : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Sp}_{S^1}(\mathcal{M})$ is a right Quillen functor, so it preserves weak equivalences between fibrant objects. It then follows that $\hat{U} \circ \hat{F} : f\mathcal{M} \to \mathsf{Sp}_{S^1}(\mathcal{M})$ sends motivic equivalences to stable local equivalences between locally fibrant S^1-spectra in $\mathsf{Sp}_{S^1}(\mathcal{M})$. Hence $\hat{U} \circ \hat{F}$ sends motivic equivalences to levelwise local equivalences. By Lemma 4.2.5 this then means that $F^{\mathcal{M}} : f\mathcal{M} \to \mathsf{Sp}_{S^1}(\mathcal{M})$ sends motivic equivalences between cofibrant objects to levelwise local equivalences in $\mathsf{Sp}_{S^1}(\mathcal{M})$. So for every $n \in \mathbb{N}$ the motivic functor $F_n^{\mathcal{M}} : f\mathcal{M} \to \mathcal{M}$ sends motivic equivalences between cofibrant objects to local equivalences. □

Before proving the main theorem of this section, we need an additional lemma about $(S^1, S^1, \mathbb{G}_m)$-trispectra. To avoid confusion between the two S^1-directions we now introduce an extra notation. We write S^1_1 for the first S^1-direction and we write S^1_2 for the second S^1-direction. Therefore, whenever we discuss $(S^1, S^1, \mathbb{G}_m)$-spectra, we deal with $(S^1_1, S^1_2, \mathbb{G}_m)$-spectra following this notation. For every $F : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ we consider $F^{\mathcal{M}} : f\mathcal{M} \to \mathsf{Sp}_{S^1_2}(\mathcal{M})$ to be a functor landing in S^1_2-spectra.

Given a $\mathbb{G}_m$-spectrum of chain complexes $A \in \mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))$ we let $\mathbb{Z}[\mathbb{S}] \boxtimes A \in \mathsf{Sp}_{S^1_1,\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))$ refer to the $(S^1_1, \mathbb{G}_m)$-bispectrum of chain complexes that is given in S^1_1-weight n by

$$(\mathbb{Z}[\mathbb{S}] \boxtimes A)_n := \mathbb{Z}[S^n] \otimes A \in \mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))).$$

The definition of $\mathbb{Z}[S^n]$ is in Section 4.1, equation (4.2). It is the chain complex that is $\mathbb{Z}$ concentrated in homological degree n.

The functor $\hat{U} : \mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))) \to \mathsf{Sp}_{S^1_2,\mathbb{G}_m}(\mathcal{M})$ can naively be extended to a functor denoted by the same letter

$$\hat{U} : \mathsf{Sp}_{S^1_1,\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))) \to \mathsf{Sp}_{S^1_1,S^1_2,\mathbb{G}_m}(\mathcal{M})$$

by applying it S^1_1-levelwise.

4.2.8 Lemma. *Let $F : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ be a $\sim$-fibrant functor. For every $X \in \mathsf{Sm}_k$ we have a natural map of $(S^1_1, S^1_2, \mathbb{G}_m)$-trispectra*

$$ev_{S^1_1,\mathbb{G}_m}(F^{\mathcal{M}}(- \times X)) \to \hat{U}(\mathbb{Z}[\mathbb{S}] \boxtimes ev_{\mathbb{G}_m}(F(- \times X)))$$

in $\mathsf{Sp}_{S^1_1,S^1_2,\mathbb{G}_m}(\mathcal{M})$. This map is a S^1_1-levelwise $(S^1_2, \mathbb{G}_m)$-stable motivic equivalence.

Proof. Since we are only evaluating $F^{\mathcal{M}}$ on simplicial schemes, by Lemma 4.2.5 we just need to show that there is a S^1_1-levelwise $(S^1_2, \mathbb{G}_m)$-stable motivic equivalence

$$ev_{S^1_1,\mathbb{G}_m}((\hat{U} \circ \hat{F})(- \times X)) \to \hat{U}(\mathbb{Z}[\mathbb{S}] \boxtimes ev_{\mathbb{G}_m}(F(- \times X))).$$

And this follows from Lemma 4.1.5. □

We are now in a position to prove the main theorem of this section.

Proof of Theorem 4.2.1. Let $F : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ be a $\sim$-fibrant functor. Due to Lemma 4.2.7 and Lemma 4.2.3 we can apply Lemma 4.2.2 to get an isomorphism

$$ev_{S^1_1,\mathbb{G}_m}(F^{\mathcal{M}}_n) \wedge \Sigma^\infty_{S^1,\mathbb{G}_m} X_+ \xrightarrow{\sim} ev_{S^1_1,\mathbb{G}_m}(F^{\mathcal{M}}_n(- \times X))$$

in $SH(k)[1/p]$. These combine into a S^1_2-levelwise $(S^1_1, \mathbb{G}_m)$-stable motivic equivalence of $(S^1_1, S^1_2, \mathbb{G}_m)$-trispectra

$$ev_{S^1_1,\mathbb{G}_m}(F^{\mathcal{M}}) \wedge \Sigma^\infty_{S^1,\mathbb{G}_m} X_+ \xrightarrow{\sim} ev_{S^1_1,\mathbb{G}_m}(F^{\mathcal{M}}(- \times X))$$

in $\mathsf{Sp}_{S^1_1,S^1_2,\mathbb{G}_m}(\mathcal{M})[1/p]$. By Lemma 4.2.8 we have a commutative diagram

$$\begin{array}{ccc} ev_{S^1_1,\mathbb{G}_m}(F^{\mathcal{M}}) \wedge \Sigma^\infty_{S^1,\mathbb{G}_m} X_+ & \xrightarrow{\sim} & ev_{S^1_1,\mathbb{G}_m}(F^{\mathcal{M}}(- \times X)) \\ \downarrow \sim & & \downarrow \sim \\ \hat{U}(\mathbb{Z}[\mathbb{S}] \boxtimes ev_{\mathbb{G}_m}(F)) \wedge \Sigma^\infty_{S^1,\mathbb{G}_m} X_+ & \xrightarrow{\sim} & \hat{U}(\mathbb{Z}[\mathbb{S}] \boxtimes ev_{\mathbb{G}_m}(F(- \times X))) \end{array}$$

where the vertical maps are S^1_1-levelwise $(S^1_2, \mathbb{G}_m)$-stable equivalences. It follows that the bottom horizontal map is a $(S^1_1, S^1_2, \mathbb{G}_m)$-stable equivalence. By Lemma 4.2.3 we know that $\Sigma^\infty_{S^1,\mathbb{G}_m} X_+$ is strongly dualizable in $SH(k)[1/p]$. Since $\mathcal{L}$ and $\mathcal{U}$ are a monoidal adjunction, we can apply [3, Lemma 4.6] to get for every $n \in \mathbb{N}$ that

$$\mathcal{U}(\mathbb{Z}[S^n] \otimes ev_{\mathbb{G}_m}(F)) \wedge \Sigma^\infty_{S^1,\mathbb{G}_m} X_+ \cong \mathcal{U}(\mathbb{Z}[S^n] \otimes ev_{\mathbb{G}_m}(F) \otimes \mathcal{L}(\Sigma^\infty_{S^1,\mathbb{G}_m} X_+))$$

in $SH(k)[1/p]$. These assemble into a S^1_1-levelwise $(S^1_2, \mathbb{G}_m)$-stable equivalence of trispectra

$$\hat{U}(\mathbb{Z}[\mathbb{S}] \boxtimes ev_{\mathbb{G}_m}(F)) \wedge \Sigma^\infty_{S^1,\mathbb{G}_m} X_+ \to \hat{U}(\mathbb{Z}[\mathbb{S}] \boxtimes ev_{\mathbb{G}_m}(F) \otimes \mathcal{L}(\Sigma^\infty_{S^1,\mathbb{G}_m} X_+)).$$

We then have a commutative diagram

$$\begin{array}{ccc} \hat{U}(\mathbb{Z}[\mathbb{S}] \boxtimes ev_{\mathbb{G}_m}(F) \otimes \mathcal{L}(\Sigma^\infty_{S^1,\mathbb{G}_m} X_+)) & \longrightarrow & \hat{U}(\mathbb{Z}[\mathbb{S}] \boxtimes ev_{\mathbb{G}_m}(F(- \times X))) \\ \uparrow \sim & \nearrow_{\sim} & \\ \hat{U}(\mathbb{Z}[\mathbb{S}] \boxtimes ev_{\mathbb{G}_m}(F)) \wedge \Sigma^\infty_{S^1,\mathbb{G}_m} X_+ & & \end{array}$$

in $\mathsf{Sp}_{S^1_1,S^1_2,\mathbb{G}_m}(\mathcal{M})[1/p]$, where the two lower maps are $(S^1_1, S^1_2, \mathbb{G}_m)$-stable motivic equivalences. It follows that the upper horizontal map is a $(S^1_1, S^1_2, \mathbb{G}_m)$-stable motivic equivalence in $\mathsf{Sp}_{S^1_1,S^1_2,\mathbb{G}_m}(\mathcal{M})[1/p]$.

Since $\mathcal{U} : DM_{\mathcal{A}}[1/p] \to SH(k)[1/p]$ is conservative, we then get a $(S^1_1, \mathbb{G}_m)$-stable motivic equivalence

$$\mathbb{Z}[\mathbb{S}] \boxtimes ev_{\mathbb{G}_m}(F) \otimes \mathcal{L}(\Sigma^\infty_{S^1,\mathbb{G}_m} X_+) \xrightarrow{\sim} \mathbb{Z}[\mathbb{S}] \boxtimes ev_{\mathbb{G}_m}(F(- \times X))$$

in $\mathsf{Sp}_{S^1_1,\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))[1/p]$. Since the functor

$$\mathbb{Z}[S^1] \otimes - : \mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))[1/p] \to \mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))[1/p]$$

is an auto-equivalence, it follows from [29, Theorem 5.1] that

$$\mathbb{Z}[\mathbb{S}] \boxtimes - : \mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))[1/p] \to \mathsf{Sp}_{S^1_1,\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))[1/p]$$

is a Quillen equivalence, where $\mathsf{Sp}_{S^1_1,\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))$ is equipped with the stable model structure of $\mathbb{Z}[S^1]$-spectra in $\mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))$. Since $\mathbb{Z}[\mathbb{S}] \boxtimes -$ preserves weak equivalences between all objects from $\mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))[1/p]$, this then implies that

$$ev_{\mathbb{G}_m}(F) \otimes \mathcal{L}(\Sigma^\infty_{S^1,\mathbb{G}_m} X_+) \xrightarrow{\sim} ev_{\mathbb{G}_m}(F(- \times X))$$

is an isomorphism in $DM_{\mathcal{A}}[1/p]$. □

4.3 Proof of Theorem 3.1.14

In this section we will prove Theorem 3.1.14, but we first need a few lemmas.

4.3.1 Lemma. *The category $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim[1/p]$ is compactly generated by the set*

$$\{[\mathbb{G}_m^{\wedge n}, I(-)] \otimes Z \mid n \in \mathbb{N}, Z \in \mathsf{Sm}_k\}.$$

Proof. The objects $[\mathbb{G}_m^{\wedge n}, I(-)] \otimes Z$ are compact by [20, Theorem 6.2]. Let $F \in D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim[1/p]$ be an enriched functor such that for all $n \in \mathbb{N}, Z \in \mathsf{Sm}_k$

$$\mathrm{Hom}_{D([\mathcal{S}m,\mathsf{Shv}(\mathcal{A})])/\sim[1/p]}([\mathbb{G}_m^{\wedge n}, I(-)] \otimes Z, F) = 0.$$

Without loss of generality, F is $\sim$-fibrant. Then we get for all $n \in \mathbb{N}, Z \in \mathsf{Sm}_k$ that $F(\mathbb{G}_m^{\wedge n})(Z) \cong 0$ in $D(\mathrm{Ab})[1/p]$. This implies that $ev_{\mathbb{G}_m}(F) \cong 0$ in $DM_{\mathcal{A}}[1/p]$. It follows Theorem 4.2.1 that for every $U \in \mathsf{Sm}_k$

$$ev_{\mathbb{G}_m}(F(U \times -)) \cong ev_{\mathbb{G}_m}(F) \otimes \mathcal{L}(\Sigma^\infty_{S^1,\mathbb{G}_m} U_+) \cong 0$$

in $DM_{\mathcal{A}}[1/p]$. Since $F(U \times -)$ is $\sim$-fibrant, the $\mathbb{G}_m$-spectrum $ev_{\mathbb{G}_m}(F(U \times -))$ is motivically fibrant in $DM_{\mathcal{A}}[1/p]$. Then

$$F(U) \cong F(U \times pt) = ev_{\mathbb{G}_m}(F(U \times -))(0) \cong 0$$

in $D(\mathsf{Shv}(\mathcal{A}))[1/p]$. This means that $F \cong 0$ in $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim[1/p]$. So

$$\{[\mathbb{G}_m^{\wedge n}, I(-)] \otimes Z \mid n \in \mathbb{N}, Z \in \mathsf{Sm}_k\}$$

is a set of compact generators for $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim[1/p]$. □

4.3.2 Lemma. *The enriched functor $[\mathbb{G}_m^{\wedge n}, M_{\mathcal{A}}(-)] : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ satisfies Nisnevich excision in the sense of Definition 3.1.9.*

Proof. Take an elementary Nisnevich square:

$$\begin{array}{ccc} U' & \xrightarrow{\beta} & X' \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} \\ U & \xrightarrow{\delta} & X \end{array}$$

From Definition 2.1.1 it follows that there is an exact sequence

$$0 \to \mathcal{A}(-, U')_{nis} \to \mathcal{A}(-, U)_{nis} \oplus \mathcal{A}(-, X')_{nis} \to \mathcal{A}(-, X)_{nis} \to 0.$$

Since $\mathcal{A}$ is a strict V-category of correspondences, by applying C_* we get a triangle

$$M_{\mathcal{A}}(U') \to M_{\mathcal{A}}(U) \oplus M_{\mathcal{A}}(X') \to M_{\mathcal{A}}(X)$$

in $D(\mathsf{Shv}(\mathcal{A}))$. We can take local fibrant replacements $M_{\mathcal{A}}(X)^f$ of each of these terms $M_{\mathcal{A}}(X)$, and then apply $\Omega^n_{\mathbb{G}_m}$ to get a triangle of locally fibrant complexes

$$\Omega^n_{\mathbb{G}_m}(M_{\mathcal{A}}(U')^f) \to \Omega^n_{\mathbb{G}_m}(M_{\mathcal{A}}(U)^f) \oplus \Omega^n_{\mathbb{G}_m}(M_{\mathcal{A}}(X')^f) \to \Omega^n_{\mathbb{G}_m}(M_{\mathcal{A}}(X)^f)$$

in $D(\mathsf{Shv}(\mathcal{A}))$. Lemma 3.2.9 says that $\underline{\mathrm{Hom}}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\mathbb{G}_m^{\wedge 1}, -) : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ preserves local equivalences between $\mathbb{A}^1$-local complexes. This implies that $[\mathbb{G}_m^{\wedge n}, M_{\mathcal{A}}(X)]$ is locally equivalent to $[\mathbb{G}_m^{\wedge n}, M_{\mathcal{A}}(X)^f] = \Omega^n_{\mathbb{G}_m}(M_{\mathcal{A}}(X)^f)$. So we ultimately get a triangle

$$[\mathbb{G}_m^{\wedge n}, M_{\mathcal{A}}(U')] \to [\mathbb{G}_m^{\wedge n}, M_{\mathcal{A}}(U)] \oplus [\mathbb{G}_m^{\wedge n}, M_{\mathcal{A}}(X')] \to [\mathbb{G}_m^{\wedge n}, M_{\mathcal{A}}(X)]$$

in $D(\mathsf{Shv}(\mathcal{A}))$. This means that

$$\begin{array}{ccc} [\mathbb{G}_m^{\wedge n}, M_{\mathcal{A}}(U')] & \xrightarrow{\beta_*} & [\mathbb{G}_m^{\wedge n}, M_{\mathcal{A}}(X')] \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \gamma_*} \\ [\mathbb{G}_m^{\wedge n}, M_{\mathcal{A}}(U)] & \xrightarrow{\delta_*} & [\mathbb{G}_m^{\wedge n}, M_{\mathcal{A}}(X)] \end{array}$$

is homotopy cartesian, so $[\mathbb{G}_m^{\wedge n}, M_{\mathcal{A}}(-)] : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ satisfies Nisnevich excision. □

4.3.3 Lemma. *For every $Z \in \mathsf{Sm}_k$ the enriched functor $[\mathbb{G}_m^{\wedge n}, M_{\mathcal{A}}(- \times Z)] : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ satisfies Nisnevich excision in the sense of Definition 3.1.9.*

Proof. Take an elementary Nisnevich square

$$\begin{array}{ccc} U' & \xrightarrow{\beta} & X' \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} \\ U & \xrightarrow{\delta} & X \end{array}$$

Then the square

$$\begin{array}{ccc} U' \times Z & \xrightarrow{\beta\times 1} & X' \times Z \\ \downarrow{\scriptstyle \alpha\times 1} & & \downarrow{\scriptstyle \gamma\times 1} \\ U \times Z & \xrightarrow{\delta\times 1} & X \times Z \end{array}$$

is again an elementary Nisnevich square. The result now follows from Lemma 4.3.2. □

Proof of Theorem 3.1.14. Let

$$T_{\mathcal{C}} := \langle [\mathbb{G}_m^{\times n}, -] \otimes X \mid n \in \mathbb{N}, X \in \mathsf{Sm}_k \rangle$$

be the full triangulated subcategory of $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$ that is compactly generated by $[\mathbb{G}_m^{\times n}, -] \otimes X$ for all $n \in \mathbb{N}$ and $X \in \mathsf{Sm}_k$. According to [21, Lemma 4.10] the composite

$$T_{\mathcal{C}} \to D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]) \overset{res}{\to} D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$$

is an equivalence of triangulated categories, where the first map is the inclusion map and the second map is the map restricting functors from $\mathcal{S}m$ to $\mathcal{C}$.

Let $\widehat{\sim_{\mathcal{C}}}$ be the set of morphisms, following the notation from Lemma 3.1.4 by

$$\widehat{\sim_{\mathcal{C}}} := \{(f \otimes Z)[n] \mid f \in \sim_{\mathcal{C}}, Z \in \mathsf{Sm}_k, n \in \mathbb{N}\}.$$

Here $\sim_{\mathcal{C}}$ is defined in Section 3.1 on page 42. We can consider $\widehat{\sim_{\mathcal{C}}}$ to be a set of morphisms in $T_{\mathcal{C}}$. We write $T_{\mathcal{C}}/\sim_{\mathcal{C}}$ for the localization of $T_{\mathcal{C}}$ along the set of morphisms $\widehat{\sim_{\mathcal{C}}}$ between compact objects.

The equivalence $T_{\mathcal{C}} \to D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])$ then induces an equivalence of compactly generated triangulated categories

$$T_{\mathcal{C}}/\sim_{\mathcal{C}} \to D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}.$$

By Theorem 3.1.8 we have that

$$ev_{\mathbb{G}_m} : D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}} \to DM_{\mathcal{A}}$$

is an equivalence of compactly generated triangulated categories. So we have an equivalence of compactly generated triangulated categories

$$ev_{\mathbb{G}_m} : T_{\mathcal{C}}/\sim_{\mathcal{C}} \to DM_{\mathcal{A}}.$$

Next, the inclusion $T_{\mathcal{C}} \to D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$ induces a triangulated functor

$$\Phi : T_{\mathcal{C}}/\sim_{\mathcal{C}} \to D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim.$$

We will now use Lemma 3.2.18 to show that

$$\Phi[1/p] : T_{\mathcal{C}}/\sim_{\mathcal{C}} [1/p] \to D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim [1/p]$$

is an equivalence of triangulated categories. Following the notation of Lemma 3.2.18, here $A = T_{\mathcal{C}}/\sim_{\mathcal{C}} [1/p]$ and $B = D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim [1/p]$ are compactly generated triangulated categories.

Due to Lemma 3.2.1 and the definition of $T_{\mathcal{C}}$, the set

$$\Sigma := \{[\mathbb{G}_m^{\wedge n}, I(-)] \otimes X \mid n \in \mathbb{N}, X \in \mathsf{Sm}_k\}$$

is a set of compact generators for $T_{\mathcal{C}}/\sim_{\mathcal{C}} [1/p]$. This is the set of compact generators to which we apply Lemma 3.2.18. Due to Lemma 4.3.1, the functor $\Phi[1/p]$ sends Σ to a set of compact generators for $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim [1/p]$, so the first condition of Lemma 3.2.18 is satisfied. Let us check the second condition.

Since $T_{\mathcal{C}}/\sim_{\mathcal{C}}$ is equivalent to $D([\mathcal{C}, \mathsf{Shv}(\mathcal{A})])/\sim_{\mathcal{C}}$, by Lemma 3.2.15 we have an isomorphism

$$[\mathbb{G}_m^{\times n}, I(-)] \underset{\mathrm{Shv}}{\otimes} X \cong [\mathbb{G}_m^{\times n}, \mathcal{M}_{\mathcal{A}}(X)]$$

in $T_{\mathcal{C}}/\sim_{\mathcal{C}}$. From Lemma 4.3.3 it follows that the enriched functor $[\mathbb{G}_m^{\times n}, \mathcal{M}_{\mathcal{A}}(X)] : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ satisfies Nisnevich excision. Similarly to Lemma 3.2.8, it is also strictly local with respect to the relations $\mathbb{A}^1_1$, τ. The definitions of these relations is in Section 3.1, page 47. Since the map $M_{\mathcal{A}}(X \times \mathbb{A}^1) \to M_{\mathcal{A}}(X)$ is an isomorphism in $DM_{\mathcal{A}}^{\mathsf{eff}}$ between $\mathbb{A}^1$-local complexes, so it is also a local quasi-isomorphism. Since $[\mathbb{G}_m^{\wedge n}, -]$ preserves local quasi-isomorphisms between $\mathbb{A}^1$-local objects, it follows that $[\mathbb{G}_m^{\times n}, \mathcal{M}_{\mathcal{A}}(X)]$ is strictly local with respect to $\mathbb{A}^1_2$. So the enriched functor $[\mathbb{G}_m^{\times n}, \mathcal{M}_{\mathcal{A}}(X)] : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is strictly $\sim$-local. Also for every $d \in \mathbb{N}$ the shifted functor $[\mathbb{G}_m^{\times n}, \mathcal{M}_{\mathcal{A}}(X)][d] : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is strictly $\sim$-local.

The functor $\Phi : T_{\mathcal{C}}/\sim_{\mathcal{C}} \to D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim$ is by construction fully faithful on strictly $\sim$-local objects, in the sense that if $A, B \in T_{\mathcal{C}}/\sim_{\mathcal{C}}$ are strictly $\sim$-local then the map

$$\mathrm{Hom}_{T_{\mathcal{C}}/\sim_{\mathcal{C}}}(A, B) \to \mathrm{Hom}_{D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim}(\Phi(A), \Phi(B))$$

is a bijection of abelian groups. In particular Φ is fully faithful on all shifts of objects of the form $[\mathbb{G}_m^{\times n}, \mathcal{M}_{\mathcal{A}}(X)]$, where $n \in \mathbb{N}$, $X \in \mathsf{Sm}_k$. Since the objects $[\mathbb{G}_m^{\times n}, \mathcal{M}_{\mathcal{A}}(X)]$ are isomorphic to the objects $[\mathbb{G}_m^{\times n}, I(-)] \otimes X$ in $T_{\mathcal{C}}/\sim_{\mathcal{C}}$, it follows that Φ is fully faithful on all shifts of objects from the set of compact generators Σ.

This verifies the second condition from Lemma 3.2.18. It now follows that

$$\Phi : T_{\mathcal{C}}/\sim_{\mathcal{C}} [1/p] \to D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim [1/p]$$

is an equivalence of triangulated categories. Recall that by Lemma 3.1.13 we have a canonical equivalence of triangulated categories

$$DM_{\mathcal{A}}[\mathcal{S}m] \to D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim .$$

We then have a commutative diagram

$$\begin{array}{ccccc} DM_{\mathcal{A}}[\mathcal{S}m][1/p] & \xrightarrow{\sim} & D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])/\sim [1/p] & & \\ & & \Phi \uparrow \wr & \searrow^{ev_{\mathbb{G}_m}} & \\ & & T_{\mathcal{C}}/\sim_{\mathcal{C}} [1/p] & \xrightarrow[ev_{\mathbb{G}_m}]{\sim} & DM_{\mathcal{A}}[1/p] \end{array}$$

which shows that the evaluation functor

$$ev_{\mathbb{G}_m} : DM_{\mathcal{A}}[\mathcal{S}m][1/p] \to DM_{\mathcal{A}}[1/p]$$

is an equivalence of categories. This completes the proof of Theorem 3.1.14. □

Chapter 5

Enriched motivic spaces

So far we have studied the category $[\mathcal{S}m, \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))]$ of enriched functors of unbounded chain complexes. We are now passing to the study of the category $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ of enriched functors of simplicial sheaves.

Also from now on we will assume that the exponential characteristic p of k is invertible in $\mathcal{A}$. So $\mathcal{A}$ is an additive category of correspondences, that is symmetric monoidal, satisfies the strict V-property, the cancellation property, and the exponential characteristic p of k is invertible in $\mathcal{A}$. Note that for any additive category of correspondences $\mathcal{A}$ we can form an additive category of correspondences $\mathcal{A}[1/p]$ in which p is invertible by tensoring all morphism groups of $\mathcal{A}$ with $\mathbb{Z}[1/p]$.

In this chapter we introduce enriched motivic $\mathcal{A}$-spaces. In Section 5.2 we construct a model structure on the category $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ of simplicial Nisnevich sheaves.

5.1 Preliminaries

We shall adhere to the following notations from [19]. Let $\mathsf{Sp}_{S^1,\mathbb{G}_m}(k)$ denote the category of symmetric $(S^1, \mathbb{G}_m)$-bispectra, where the $\mathbb{G}_m$-direction is associated with the pointed motivic space $(\mathbb{G}_m, 1)$. It is equipped with a stable motivic model category structure. Denote by $SH(k)$ its homotopy category. The category $SH(k)$ has a closed symmetric monoidal structure with monoidal unit being the motivic sphere spectrum $\mathbb{S}$. Given $p > 0$, the category $\mathsf{Sp}_{S^1,\mathbb{G}_m}(k)$ has a further model structure whose weak equivalences are the maps of bispectra $f : X \to Y$ such that the induced map of bigraded Nisnevich sheaves $f_* : \underline{\pi}^{\mathbb{A}^1}_{*,*}(X) \otimes \mathbb{Z}[1/p] \to$

$\underline{\pi}^{\mathbb{A}^1}_{*,*}(Y) \otimes \mathbb{Z}[1/p]$ is an isomorphism. In what follows we denote its homotopy category by $SH(k)[1/p]$. The category $SH(k)_{\mathbb{Q}}$ is defined in a similar fashion.

Recall from Section 3.1 that there is a $\mathsf{Shv}(\mathcal{A})$-enriched category $\mathcal{S}m$, whose objects are those of Sm_k, and whose morphism sheaves are defined by

$$\mathcal{S}m(X,Y) := \underline{\mathrm{Hom}}_{\mathsf{Shv}(\mathcal{A})}(\mathcal{A}(-,X)_{\mathsf{nis}}, \mathcal{A}(-,Y)_{\mathsf{nis}}).$$

In Section 5.2 we will define a natural local model structure on $\Delta^{op}\mathsf{Shv}(\mathcal{A})$. Weak equivalences in this model structure are the local equivalences.

According to [9, Theorem 4.3.12], if $\mathcal{G}$ is a Grothendieck category with a generator G, then the category of simplicial objects $\Delta^{op}\mathcal{G}$ in $\mathcal{G}$ is also Grothendieck and the set $\{G \otimes \Delta[n] \mid n \geqslant 0\}$ is a family of generators for $\Delta^{op}\mathcal{G}$. In particular, a family of generators for the Grothendieck category $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ is given by the set

$$\{\mathcal{A}(-,X)_{\mathsf{nis}} \otimes \Delta[n] \mid X \in \mathsf{Sm}_k, n \geqslant 0\}.$$

Also, the category of enriched functors $[\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]$ is Grothendieck by [1]. Its family of generators is given by $\{\mathcal{S}m(X,-) \otimes_{\mathsf{Shv}(\mathcal{A})} \mathcal{A}(-,Y)_{\mathsf{nis}} \mid X, Y \in \mathsf{Sm}_k\}$. Hence $\Delta^{op}[\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]$ is Grothendieck by [9]. Its family of generators is given by $\{\mathcal{S}m(X,-) \otimes_{\mathsf{Shv}(\mathcal{A})} \mathcal{A}(-,Y)_{\mathsf{nis}} \otimes \Delta[n] \mid X, Y \in \mathsf{Sm}_k, n \geqslant 0\}$.

Note that $\Delta^{op}[\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]$ and $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ are equivalent, and we will freely pass back and forth between the two.

5.1.1 Definition. An *enriched motivic $\mathcal{A}$-space* is an object of the Grothendieck category $\Delta^{op}[\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]$. Similarly to [25, Axioms 1.1], an enriched motivic $\mathcal{A}$-space $\mathcal{X}$ is said to be *special* if it satisfies the following axioms:

1. For all $n \geqslant 0$ and $U \in \mathsf{Sm}_k$ the presheaf of homotopy groups $V \longmapsto \pi_n(\mathcal{X}(U))(V)$ is $\mathbb{A}^1$-invariant.

2. (Cancellation) Let $\mathbb{G}_m^{\wedge 1}$ denote the cokernel of the 1-section $\mathcal{A}(-,pt)_{\mathsf{nis}} \longrightarrow \mathcal{A}(-,\mathbb{G}_m)_{\mathsf{nis}}$ in $\mathsf{Shv}(\mathcal{A})$ and for $n \geqslant 1$ inductively define $\mathbb{G}_m^{\wedge n+1} := \mathbb{G}_m^{\wedge n} \otimes \mathbb{G}_m^{\wedge 1}$. For all $n \geqslant 0$ and $U \in \mathsf{Sm}_k$ the canonical map

$$\mathcal{X}(\mathbb{G}_m^{\wedge n} \times U) \longrightarrow \underline{\mathrm{Hom}}_{\Delta^{op}\mathsf{Shv}(\mathcal{A})}(\mathbb{G}_m^{\wedge 1}, \mathcal{X}(\mathbb{G}_m^{\wedge n+1} \times U))$$

   is a local equivalence.

3. ($\mathbb{A}^1$-invariance) For all $U \in \mathsf{Sm}_k$ the canonical map $\mathcal{X}(U \times \mathbb{A}^1) \longrightarrow \mathcal{X}(U)$ is a local equivalence.

4. (Nisnevich excision) For every elementary Nisnevich square in Sm_k

$$\begin{array}{ccc} U' & \longrightarrow & V' \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

   the induced square

$$\begin{array}{ccc} \mathcal{X}(U') & \longrightarrow & \mathcal{X}(V') \\ \downarrow & & \downarrow \\ \mathcal{X}(U) & \longrightarrow & \mathcal{X}(V) \end{array}$$

   is homotopy cartesian in the local model structure on $\Delta^{op}\mathsf{Shv}(\mathcal{A})$.

For $n \geqslant 0$ and every finitely generated field extension K/k, we have the standard algebraic n-simplex

$$\Delta^n_K = \mathrm{Spec}(K[x_0, \ldots, x_n]/(x_0 + \cdots + x_n - 1)).$$

For every $0 \leqslant i \leqslant n$ we define a closed subscheme v_i of Δ^n_K by the equations $x_j = 0$ for $j \neq i$. We write $\widehat{\Delta}^n_{K/k}$ for the semilocalization of the standard algebraic n-simplex Δ^n_K with closed points the vertices $v_0, \ldots, v_n \in \Delta^n_K$.

5.1.2 Definition. Similarly to [25, Axioms 1.1], we say that $\mathcal{X}$ is *very effective* or *satisfies Suslin's contractibility* if for every $U \in \mathcal{S}m$ and every finitely generated field extension K/k the diagonal of the bisimplicial abelian group $\mathcal{X}(\mathbb{G}_m^{\wedge 1} \times U)(\widehat{\Delta}^\bullet_{K/k})$ is contractible.

Since we assume that p is invertible in $\mathcal{A}$ the following lemma holds.

5.1.3 Lemma. *If $F : \mathcal{A} \to \mathrm{Ab}$ is an additive functor, then F factors over the full subcategory of $\mathbb{Z}[1/p]$-modules $\mathrm{Mod}_{\mathbb{Z}[1/p]} \subseteq \mathrm{Ab}$. In particular the inclusion functor $\mathrm{Mod}_{\mathbb{Z}[1/p]} \to \mathrm{Ab}$ induces an equivalence of categories $\mathsf{Shv}(\mathcal{A}, \mathrm{Ab}) \simeq \mathsf{Shv}(\mathcal{A}, \mathrm{Mod}_{\mathbb{Z}[1/p]})$.*

Proof. If $F : \mathcal{A} \to \mathrm{Ab}$ is additive, then F is an Ab-enriched functor. By the Ab-enriched co-Yoneda lemma we can write F as the following coend: for all $U \in \mathsf{Sm}_k$ we have an isomorphism in Ab,

$$F(U) \cong \int^{X \in \mathsf{Sm}_k} F(X) \otimes \mathcal{A}(U, X).$$

Since p is invertible in $\mathcal{A}(U,X)$ we have a canonical isomorphism $\mathcal{A}(U,X) \cong \mathcal{A}(U,X) \otimes \mathbb{Z}[1/p]$. Since the functor $- \otimes \mathbb{Z}[1/p] : \mathrm{Ab} \to \mathrm{Ab}$ is a left adjoint, it preserves coends, so we can compute

$$F(U) \cong \int^{X \in \mathsf{Sm}_k} F(X) \otimes \mathcal{A}(U,X) \cong \int^{X \in \mathsf{Sm}_k} F(X) \otimes (\mathcal{A}(U,X) \otimes \mathbb{Z}[1/p]) \cong$$

$$\mathbb{Z}[1/p] \otimes \int^{X \in \mathsf{Sm}_k} F(X) \otimes \mathcal{A}(U,X) \cong \mathbb{Z}[1/p] \otimes F(U)$$

which shows that $F(U)$ is a $\mathbb{Z}[1/p]$-module. □

For some of our results we will also have to make additional assumptions on the category of correspondences $\mathcal{A}$.

5.1.4 Definition. Let $\mathrm{Fr}_*(k)$ be the category of Voevodksy's framed correspondences (see [23, Definition 2.3]). For each $V \in \mathsf{Sm}_k$ let $\sigma_V : V \to V$ be the level 1 explicit framed correspondence $(\{0\} \times V, \mathbb{A}^1 \times V, \mathrm{pr}_{\mathbb{A}^1}, \mathrm{pr}_V)$.

1. We say that the category of correspondences $\mathcal{A}$ *has framed correspondences* if there is a functor $\Phi : \mathrm{Fr}_*(k) \to \mathcal{A}$ which is the identity on objects and which takes every σ_V to the identity of V.

2. We say that $\mathcal{A}$ *satisfies the* $\widehat{\Delta}$*-property* if for every $n > 0$ and for every finitely generated field extension K/k the diagonal of $M_{\mathcal{A}}(\mathbb{G}_m^{\wedge n})(\widehat{\Delta}^{\bullet}_{K/k})$ is quasi-isomorphic to 0. Here $M_{\mathcal{A}} : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is the enriched motive functor $M_{\mathcal{A}}(U) := C_*\mathcal{A}(-,U)_{\mathsf{nis}}$.

Basic examples satisfying both items are given by the categories of finite correspondences Cor or Milnor–Witt correspondences $\widetilde{Cor}$.

5.2 The local model structure

In Section 2.2 we constructed a model structure on $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ that is cellular, strongly left proper, weakly finitely generated, monoidal and satisfies the monoid axiom. In this section we construct a model structure on $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ that is cellular, strongly left proper, weakly finitely generated, monoidal, satisfies the monoid axiom, and in which weak equivalences are local quasi-isomorphisms.

We construct the model structure by taking the right transferred model structure along the inclusion $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. We then transfer the model structure along the Dold-Kan correspondence, to get a model structure on $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ that is cellular, strongly left proper, weakly finitely generated, monoidal, satisfies the monoid axiom, and in which weak equivalences are stalkwise weak equivalences of simplicial sets.

Let us now start by constructing the model structure on $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. We have an inclusion functor $\iota : \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. The inclusion functor ι has a left adjoint $\tau_{\text{naive}} : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$, called the naive truncation functor. It sends $\cdots \to A_1 \to A_0 \to A_{-1} \to \ldots$ to $\cdots \to A_1 \to A_0$. The inclusion functor ι also has a right adjoint τ_{good}, called the good truncation functor. It sends

$$\cdots \to A_1 \to A_0 \overset{\partial_A^0}{\to} A_{-1} \to \ldots$$

to $\cdots \to A_1 \to \ker(\partial_A^0)$. So we have $\tau_{\text{naive}} \dashv \iota \dashv \tau_{\text{good}}$.

5.2.1 Lemma. *The endofunctor* $\iota\tau_{naive} : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ *preserves cofibrations.*

Proof. Since $\iota\tau_{\text{naive}}$ is a left adjoint functor, it suffices to check it on the set of generating cofibrations

$$I_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))} = \{\mathcal{A}(-,X)_{\mathsf{nis}} \otimes S^n\mathbb{Z} \to \mathcal{A}(-,X)_{\mathsf{nis}} \otimes D^n\mathbb{Z} \mid n \in \mathbb{Z}, X \in \mathsf{Sm}_k\}.$$

So take $n \in \mathbb{Z}$, $X \in \mathsf{Sm}_k$ and consider the map

$$f_n : \mathcal{A}(-,X)_{\mathsf{nis}} \otimes S^n\mathbb{Z} \to \mathcal{A}(-,X)_{\mathsf{nis}} \otimes D^n\mathbb{Z}.$$

If $n \geqslant 0$ then $\iota\tau_{\text{naive}}(f_n) = f_n$ is a cofibration. If $n \leqslant -2$ then $\iota\tau_{\text{naive}}(f_n) = 0$ is a cofibration. If $n = -1$ then $\iota\tau_{\text{naive}}(f_{-1})$ is the map $0 \to \mathcal{A}(-,X)_{\mathsf{nis}} \otimes S^0\mathbb{Z}$ which is a cofibration, due to the following pushout square

$$\begin{array}{ccc}
\mathcal{A}(-,X)_{\mathsf{nis}} \otimes S^{-1}\mathbb{Z} & \longrightarrow & \mathcal{A}(-,X)_{\mathsf{nis}} \otimes D^{-1}\mathbb{Z} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{A}(-,X)_{\mathsf{nis}} \otimes S^0\mathbb{Z}
\end{array}$$

as required. ☐

5.2.2 Definition. Given a model category M and an adjunction $L : N \rightleftarrows M : R$, we say that the right transferred model structure along the adjunction $L \dashv R$ exists, if there exists a model structure on N, such that a morphism f is a weak equivalence (respectively cofibration) in N if and only if $L(f)$ is a weak equivalence (respectively cofibration) in M.

5.2.3 Lemma. *The left transferred model structure on* $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ *along the adjunction*

$$\iota : \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A})) \rightleftarrows \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) : \tau_{good}$$

exists. The resulting model structure on $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ *is cofibrantly generated.*

Proof. We use [4, Theorem 2.23]. All involved categories are locally presentable, and $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is cofibrantly generated, so the theorem is applicable. We now have to show that

$$\mathrm{RLP}(\iota^{-1}(\{\text{cofibrations}\})) \subseteq \iota^{-1}(\{\text{weak equivalences}\}).$$

So take $p : X \to Y$ with $p \in \mathrm{RLP}(\iota^{-1}(\{\text{cofibrations}\}))$. We want to show that $\iota(p)$ is a weak equivalence in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. We will show that $\iota(p)$ is a trivial fibration, by showing that it has the right lifting property with respect to cofibrations. Let $f : A \to B$ be a cofibration in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ and consider a lifting problem

$$\begin{array}{ccc} A & \longrightarrow & \iota X \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle \iota p} \\ B & \longrightarrow & \iota Y \end{array} .$$

By adjunction this diagram has a lift, if and only if the following diagram has a lift

$$\begin{array}{ccc} \tau_{\text{naive}} A & \longrightarrow & X \\ {\scriptstyle \tau_{\text{naive}} f}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \tau_{\text{naive}} B & \longrightarrow & Y \end{array} .$$

Since $p \in \mathrm{RLP}(\iota^{-1}(\{\text{cofibrations}\}))$, we can solve this lifting problem if $\tau_{\text{naive}} f \in \iota^{-1}(\{\text{cofibrations}\})$. So we have to show that $\iota\tau_{\text{naive}} f$ is a cofibration. As f is a cofibration, this follows from Lemma 5.2.1. □

We now have a model structure on $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$, in which a morphism f is weak equivalence (respectively cofibration) if and only if ιf is a weak equivalence (respectively cofibration) in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, and a morphism is a fibration in

$\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ if and only if it has the right lifting property with respect to all trivial cofibrations. Furthermore, the adjunction

$$\iota : \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A})) \rightleftarrows \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) : \tau_{\text{good}}$$

is a Quillen adjunction. Since weak equivalences in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ are the local quasi-isomorphisms, it follows that also weak equivalences in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ are the local quasi-isomorphisms.

5.2.4 Lemma. $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ *is a monoidal model category.*

Proof. Let us verify the pushout product axiom. Let f, g be two cofibrations in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$, and let $f \Box g$ be their pushout-product. Since $\iota : \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is a strong monoidal left adjoint functor, we have an isomorphism of arrows $\iota(f \Box g) \cong \iota(f) \Box \iota(g)$. As f, g are cofibrations in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$, we see that $\iota(f), \iota(g)$ are cofibrations in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. Since $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is a monoidal model category, $\iota(f) \Box \iota(g)$ is a cofibration in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. So $f \Box g$ is a cofibration in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. Also, if f or g is a trivial cofibration in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$, then $\iota(f)$ or $\iota(g)$ is a trivial cofibration in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. Thus $\iota(f) \Box \iota(g)$ is a trivial cofibration, hence $f \Box g$ is a trivial cofibration. Therefore $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ satisfies the pushout-product axiom.

Let us verify the unit axiom. If $\mathbb{1}_{\geqslant 0}$ is the monoidal unit of $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$, and $\mathbb{1}$ is the monoidal unit of $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, then since ι is strong monoidal we have an isomorphism $\iota \mathbb{1}_{\geqslant 0} \cong \mathbb{1}$. As $\mathbb{1} = \mathcal{A}(-, pt)_{\mathsf{nis}}$ is cofibrant in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ it follows that $\mathbb{1}_{\geqslant 0}$ is cofibrant in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. This implies the unit axiom. □

5.2.5 Lemma. $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ *satisfies the monoid axiom.*

Proof. Let $W_{\geqslant 0}$ denote the class of weak equivalences and $CW_{\geqslant 0}$ denote the class of trivial cofibrations in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. Let W denote the class of weak equivalences and CW denote the class of trivial cofibrations in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. We need to show that

$$((CW_{\geqslant 0}) \otimes \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))) - \text{cof} \subseteq W_{\geqslant 0}.$$

Since $W_{\geqslant 0} = \iota^{-1}(W)$, this means we have to show that

$$\iota(((CW_{\geqslant 0}) \otimes \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))) - \text{cof}) \subseteq W.$$

Since ι is a strong monoidal left adjoint functor we have

$$\iota(((CW_{\geqslant 0}) \otimes \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))) - \text{cof}) \subseteq (\iota(CW_{\geqslant 0}) \otimes \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))) - \text{cof}\,.$$

Since ι preserves trivial cofibrations we have $\iota(CW_{\geqslant 0}) \subseteq CW$. Since $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ satisfies the monoid axiom (see Lemma 2.2.23), it follows that

$$(\iota(CW_{\geqslant 0}) \otimes \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))) - \mathrm{cof} \subseteq (CW \otimes \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))) - \mathrm{cof} \subseteq W.$$

Hence $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ satisfies the monoid axiom. $\square$

5.2.6 Lemma. *Let $I_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}$ be a set of generating cofibrations for $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. Then the set $\tau_{naive}(I_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))})$ is a set of generating cofibrations of $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. In particular, $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ has a set of generating cofibrations with finitely presented domains and codomains.*

Proof. By Lemma 5.2.1 all morphisms from $\tau_{\mathrm{naive}}(I_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))})$ are cofibrations in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. Let f be a cofibration in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. We claim that $f \in (\tau_{\mathrm{naive}}(I_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))})) - \mathrm{cof}$. Since f is a cofibration in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$, also ιf is a cofibration in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. Since $I_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}$ is a set of generating cofibrations for $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, it follows that $\iota f \in I_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))} - \mathrm{cof}$. But then

$$f \cong \tau_{\mathrm{naive}}\iota f \in \tau_{\mathrm{naive}}(I_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))} - \mathrm{cof}) \subseteq (\tau_{\mathrm{naive}}(I_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))})) - \mathrm{cof}\,.$$

Therefore $\tau_{\mathrm{naive}}(I_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))})$ is a set of generating cofibrations for $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$.

Since the set

$$\{\mathcal{A}(-,X)_{\mathsf{nis}} \otimes S^n\mathbb{Z} \to \mathcal{A}(-,X)_{\mathsf{nis}} \otimes D^n\mathbb{Z} \mid X \in \mathsf{Sm}_k, n \in \mathbb{Z}\}$$

is a set of generating cofibrations with finitely presented domains and codomains for $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, it follows that the union of $\{\mathcal{A}(-,X)_{\mathsf{nis}} \otimes S^n\mathbb{Z} \to \mathcal{A}(-,X)_{\mathsf{nis}} \otimes D^n\mathbb{Z} \mid X \in \mathsf{Sm}_k, n \geqslant 0\}$ and $\{0 \to \mathcal{A}(-,X)_{\mathsf{nis}} \otimes S^0\mathbb{Z} \mid X \in \mathsf{Sm}_k\}$ together form a set of generating cofibrations with finitely presented domains and codomains of $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. $\square$

Next, we want to show that $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ is weakly finitely generated. To this end, we need to define a set of weakly generating trivial cofibrations J'. For this we need to construct a certain set of morphisms similar to Definition 2.2.3.

5.2.7 Definition. For every elementary Nisnevich square $Q \in \mathcal{Q}$ of the form

$$\begin{array}{ccc} U' & \xrightarrow{\beta} & X' \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} \\ U & \xrightarrow{\delta} & X \end{array}$$

we have a square

$$\begin{array}{ccc} \mathcal{A}(-,U')_{\mathsf{nis}} & \xrightarrow{\beta_*} & \mathcal{A}(-,X')_{\mathsf{nis}} \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \gamma_*} \\ \mathcal{A}(-,U)_{\mathsf{nis}} & \xrightarrow{\delta_*} & \mathcal{A}(-,X)_{\mathsf{nis}} \end{array}$$

in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. Take the homological mapping cyinder C of the map $\mathcal{A}(-,U')_{\mathsf{nis}} \to \mathcal{A}(-,X')_{\mathsf{nis}}$, so that the map factors as $\mathcal{A}(-,U')_{\mathsf{nis}} \to C \to \mathcal{A}(-,X')_{\mathsf{nis}}$. Let $s_Q := \mathcal{A}(-,U)_{\mathsf{nis}} \coprod_{\mathcal{A}(-,U')_{\mathsf{nis}}} C$. Next take the homological mapping cylinder t_Q of the map $s_Q = \mathcal{A}(-,U)_{\mathsf{nis}} \coprod_{\mathcal{A}(-,U')_{\mathsf{nis}}} C \to \mathcal{A}(-,X)_{\mathsf{nis}}$, so that it factors as $s_Q \overset{p_Q}{\to} t_Q \to \mathcal{A}(-,X)_{\mathsf{nis}}$. The map $p_Q : s_Q \to t_Q$ is a trivial cofibration between finitely presented objects of $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$.

Let $\mathcal{Q}$ be the set of all elementary Nisnevich squares. Define a set of morphisms $J_{\mathcal{Q}} := \{p_Q \mid Q \in \mathcal{Q}\}$. Let $I_{\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})}$ be a set of generating cofibrations with finitely presented domains and codomains for Quillen's standard projective model structure on $\mathsf{Ch}(\mathrm{Ab})_{\geqslant 0}$. We define sets of morphisms in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$

$$J_{\mathrm{proj}} := \{0 \to \mathcal{A}(-,X)_{\mathsf{nis}} \otimes D^n\mathbb{Z} \mid X \in \mathsf{Sm}_k, n \geqslant 0\}$$

and

$$J' := J_{\mathrm{proj}} \cup (J_{\mathcal{Q}} \Box I_{\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})}),$$

where $J_{\mathcal{Q}} \Box I_{\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})}$ is the set of all morphisms which are a pushout product of a morphisms from $J_{\mathcal{Q}}$ and $I_{\mathsf{Ch}(\mathrm{Ab})_{\geqslant 0}}$.

Note that all morphisms from $I_{\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})}$ are cofibrations and all morphisms from J_{proj} and $J_{\mathcal{Q}}$ are trivial cofibrations. Since $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ is a monoidal model category it follows that all morphisms from J' are trivial cofibrations.

5.2.8 Lemma. *A morphism $f : A \to B$ in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ has the right lifting property with respect to J_{proj} if and only if for every $n \geqslant 1$ the map $f_n : A_n \to B_n$ is sectionwise surjective.*

Proof. For every $n \geqslant 0, X \in \mathsf{Sm}_k$ we can solve the lifting problem

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \mathcal{A}(-,X)_{\mathsf{nis}} \otimes D^n\mathbb{Z} & \longrightarrow & B \end{array}$$

in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ if and only if $f_{n+1} : A(X)_{n+1} \to B(X)_{n+1}$ is surjective in Ab. $\Box$

5.2.9 Lemma. *For an object A in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ the following are equivalent:*

1. *$\iota(A)$ is fibrant in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$.*
2. *A is fibrant in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$.*
3. *$A \to 0$ has the right lifting property with respect to $J_{\mathcal{Q}} \square I_{\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})}$.*

Proof. (1) $\Longrightarrow$ (2). If $\iota(A)$ is fibrant in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, then $A \cong \tau_{\text{good}}(\iota(A))$ is fibrant in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ because τ_{good} is a right Quillen functor.

(2) $\Longrightarrow$ (3). If A is fibrant in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$, then $A \to 0$ has the right lifting property with respect to all trivial cofibrations, hence it has the right lifting property with respect to $J_{\mathcal{Q}} \square I_{\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})}$.

(3) $\Longrightarrow$ (1). Assume that $A \to 0$ has the right lifting property with respect to $J_{\mathcal{Q}} \square I_{\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})}$. We want to show that $\iota(A)$ is fibrant in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. By Lemma 2.2.4 we have to show that $A(\emptyset) \to 0$ is a quasi-isomorphism, and that A sends elementary Nisnevich squares to homotopy pullback squares. Since A is a chain complex of sheaves, we have $A(\emptyset) = 0$. Let us now show that A sends elementary Nisnevich squares to homotopy pullback squares. Let Q be an elementary Nisnevich square. For $X, Y \in \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ let $\underline{\mathrm{Hom}}_{\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))}(X, Y)$ be the internal hom of $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ and let

$$\mathrm{map}^{\mathsf{Ch}}(X, Y) \in \mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})$$

be defined by

$$\mathrm{map}^{\mathsf{Ch}}(X, Y) := \underline{\mathrm{Hom}}_{\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))}(X, Y)(pt).$$

The square $A(Q)$ will be a homotopy pullback square in $\mathsf{Ch}(\mathrm{Ab})$ if and only if the map

$$p_Q^* : \mathrm{map}^{\mathsf{Ch}}(t_Q, A) \to \mathrm{map}^{\mathsf{Ch}}(s_Q, A)$$

is a quasi-isomorphism in $\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})$. To show that p_Q^* is a quasi-isomorphism, it suffices to show that p_Q^* is a trivial fibration in $\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})$. For that we need to show that p_Q^* has the right lifting property with respect to $I_{\mathsf{Ch}(\mathrm{Ab})_{\geqslant 0}}$. Now for every map $f : M \to N$ in $I_{\mathsf{Ch}(\mathrm{Ab})_{\geqslant 0}}$ a square

$$\begin{array}{ccc} M & \longrightarrow & \mathrm{map}^{\mathsf{Ch}}(t_Q, A) \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle p_Q^*} \\ N & \longrightarrow & \mathrm{map}^{\mathsf{Ch}}(s_Q, A) \end{array}$$

has a lift in $\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})$ if and only if the square

$$\begin{array}{ccc} t_Q \otimes M \coprod_{s_Q \otimes M} s_Q \otimes N & \longrightarrow & A \\ {\scriptstyle p_Q \square f} \downarrow & \nearrow & \downarrow \\ t_Q \otimes N & \longrightarrow & 0 \end{array}$$

has a lift in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. This lift exists, because $A \to 0$ has the right lifting property with respect to $J_{\mathcal{Q}} \square I_{\mathsf{Ch}(\mathrm{Ab})_{\geqslant 0}}$. $\square$

In what follows, let $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))_{\mathrm{proj}}$ be the model category $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ with standard projective model structure. Let $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))_{\mathsf{nis}}$ be the model category $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))$ with local projective model structure. See Section 2.2 for details. Let $L_{\mathsf{nis}} : \mathsf{Ch}(\mathsf{Psh}(\mathcal{A})) \leftrightarrows \mathsf{Ch}(\mathsf{Shv})A : U_{\mathsf{nis}}$ be the adjunction consisting of the sheafification and the forgetful functors.

5.2.10 Proposition. *Let $f : A \to B$ be a morphism in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ such that B is fibrant and f has the right lifting property with respect to J'. Then f is a fibration in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$.*

Proof. Our first claim is that A is fibrant. Since B is fibrant, by Lemma 5.2.9 $B \to 0$ has the right lifting property with respect to $J_{\mathcal{Q}} \square I_{\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})}$. Since f has the right lifting property with respect to $J_{\mathcal{Q}} \square I_{\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})}$ it follows that $A \to 0$ has the right lifting property with resepct to $J_{\mathcal{Q}} \square I_{\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})}$. Lemma 5.2.9 implies A is fibrant.

Next, let $D^{-1}B_0 \in \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ denote the chain complex

$$\dots 0 \to 0 \to B_0 \xrightarrow{id} B_0 \to 0 \to \dots$$

that is B_0 in degree 0 and -1, and which is 0 everywhere else. We claim that $D^{-1}B_0$ is fibrant in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. Indeed, the map $U_{\mathsf{nis}} D^{-1}B_0 \to 0$ is a trivial fibration in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))_{\mathrm{proj}}$, hence it is also a trivial fibration in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))_{\mathsf{nis}}$. Therefore $D^{-1}B_0 \to 0$ is a trivial fibration in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, and so $D^{-1}B_0$ is fibrant. Note that $\tau_{\mathrm{good}}(D^{-1}B_0) = 0$ in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$.

In particular, $\iota(A) \oplus D^{-1}B_0$ is fibrant in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ and we have that

$$\tau_{\mathrm{good}}(\iota(A) \oplus D^{-1}B_0) \cong \tau_{\mathrm{good}}(\iota(A)) \oplus \tau_{\mathrm{good}}(D^{-1}B_0) \cong A \oplus 0 = A.$$

Define $g : D^{-1}B_0 \to \iota(B)$ in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ as the map

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & B_0 & \xrightarrow{id} & B_0 & \longrightarrow & 0 & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & B_1 & \longrightarrow & B_0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

Then $\iota(f) + g : \iota(A) \oplus D^{-1}B_0 \to \iota(B)$ is a map between fibrant objects, and we have a commutative diagram where the horizontal maps are isomorphisms

$$\begin{array}{ccccc}
\tau_{\text{good}}(\iota(A) \oplus D^{-1}B_0) & \xrightarrow{\sim} & A \oplus 0 & = & A \\
\downarrow{\scriptstyle \tau_{\text{good}}(\iota(f)+g)} & & \downarrow{\scriptstyle f+0} & & \downarrow{\scriptstyle f} \\
\tau_{\text{good}}(\iota(B)) & \xrightarrow{\sim} & B & = & B
\end{array}$$

We want to show that f is a fibration in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. Since τ_{good} is a right Quillen functor, we now just need to show that $\iota(f)+g$ is a fibration in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. For this it suffices to show that $U_{\mathsf{nis}}(\iota(f)+g)$ is a fibration in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))_{\mathsf{nis}}$. Since $U_{\mathsf{nis}}\iota(A \oplus D^{-1}B_0)$ and $U_{\mathsf{nis}}\iota(B)$ are fibrant in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))_{\mathsf{nis}}$, it suffices by [27, Proposition 3.3.16] to show that $U_{\mathsf{nis}}(\iota(f)+g)$ is a fibration in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))_{\text{proj}}$. So we have to show that the map $\iota(f)+g$ is sectionwise an epimorphism in $\mathsf{Ch}(\mathrm{Ab})$. In degree $n \geqslant 1$ the map $\iota(f) : \iota(A) \to \iota(B)$ is sectionwise surjective, because of Lemma 5.2.8 and the fact that f satisfies the right lifting property with respect to J_{proj}. In degree $n \leqslant -1$ the map $\iota(f) + g$ is sectionwise surjective, because $\iota(B)_n = 0$. Finally, in degree $n = 0$ the map $\iota(f) + g$ is sectionwise surjective, because $g : D^{-1}B_0 \to \iota(B)$ is sectionwise surjective in degree 0. So $U_{\mathsf{nis}}(\iota(f) + g)$ is a fibration in $\mathsf{Ch}(\mathsf{Psh}(\mathcal{A}))_{\text{proj}}$. Then $\iota(f) + g$ is a fibration in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$, and then $f \cong \tau_{\text{good}}(\iota(f) + g)$ is a fibration in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. □

5.2.11 Corollary. $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ *is weakly finitely generated and* J' *is a set of weakly generating trivial cofibrations for* $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$.

Proof. By Lemma 5.2.3 $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ is cofibrantly generated, so there exists a set J of generating trivial cofibrations. Since every object in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ is small, the domains and codomains from J are small. By Lemma 5.2.6 the category $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ has a set of generating cofibrations with finitely presented domains and codomains. All morphisms from J' are trivial cofibrations with finitely presented domains and codomains, so Proposition 5.2.10 implies that J' is set of weakly generating trivial cofibrations for $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. □

5.2.12 Lemma. *The model category* $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ *is cellular.*

Proof. Due to Corollary 5.2.11 we know that $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ is cofibrantly generated with a set of generating cofibrations with finitely presented domains and codomains. We now just need to show that cofibrations in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ are effective monomorphisms. If f is a cofibration in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$, then $\iota(f)$ is a cofibration in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. Then f is a monomorphism in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ and in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. Since $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ is an abelian category, every monomorphism is effective. Hence f is an effective monomorphism. □

5.2.13 Lemma. *The model category* $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ *is strongly left proper in the sense of [14, Definition 4.6]*

Proof. If we have a pushout square

$$\begin{array}{ccc} A \otimes Z & \xrightarrow{f} & B \\ {\scriptstyle g\otimes Z}\downarrow & & \downarrow \\ C \otimes Z & \xrightarrow[h]{} & D \end{array}$$

in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ with f a weak equivalence and $g : A \to C$ a cofibration, then the square

$$\begin{array}{ccccc} \iota(A) \otimes \iota(Z) & \xrightarrow{\sim} & \iota(A \otimes Z) & \xrightarrow{\iota(f)} & \iota(B) \\ {\scriptstyle \iota(g)\otimes\iota(Z)}\downarrow & & {\scriptstyle \iota(g\otimes Z)}\downarrow & & \downarrow \\ \iota(C) \otimes \iota(Z) & \xrightarrow{\sim} & \iota(C \otimes Z) & \xrightarrow[\iota(h)]{} & \iota(D) \end{array}$$

is a pushout square in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. Since $\iota(f)$ is a weak equivalence, $\iota(g)$ is a cofibration, and $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ is strongly left proper by Lemma 2.2.24, it follows that $\iota(h)$ is a weak equivalence in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. So h is a weak equivalence in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. □

In summary, we have a model category $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ that is cellular, weakly finitely generated and whose weak equivalences are the local quasi-isomorphisms. With respect to the usual tensor product of chain complexes $\otimes$ it is monoidal, strongly left proper and satisfies the monoid axiom.

We can transfer this model structure along the Dold-Kan correspondence

$$DK : \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A})) \overset{\sim}{\leftrightarrows} \Delta^{op}(\mathsf{Shv}(\mathcal{A})) : DK^{-1}.$$

So we define a model structure on $\Delta^{op}(\mathsf{Shv}(\mathcal{A}))$, where a morphism f is a weak equivalence (respectively fibration, cofibration), if and only if $DK^{-1}(f)$ is a weak equivalence (respectively fibration, cofibration) in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. Then weak equivalences in $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ are the stalkwise weak equivalences of simplicial sets. Furthermore $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ is weakly finitely generated and cellular. From now on, weak equivalences in $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ be called local equivalences, fibrations in $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ will be called local fibrations, and fibrant objects in $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ will be called locally fibrant objects.

Let $\otimes$ be the degreewise tensor product of $\Delta^{op}\mathsf{Shv}(\mathcal{A})$. We want to show that $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ is monoidal, strongly left proper and satisfies the monoid axiom with respect to $\otimes$. The Dold-Kan correspondence is unfortunately not strongly monoidal with respect to the degreewise tensor product $\otimes$ on $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ and the usual tensor product of chain complexes on $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. We define on $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ the *Dold-Kan twisted tensor product* $\underset{DK}{\otimes}$ by

$$A \underset{DK}{\otimes} B := DK^{-1}(DK(A) \otimes DK(B)).$$

Then the Dold-Kan correspondences is strongly monoidal with respect to the degreewise tensor product $\otimes$ on $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ and the Dold-Kan twisted tensor product $\underset{DK}{\otimes}$ on $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. So to show that $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ is monoidal, strongly left proper and satisfies the monoid axiom with respect to $\otimes$, we now just need to show that $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ is monoidal, strongly left proper and satisfies the monoid axiom with respect to $\underset{DK}{\otimes}$.

5.2.14 Lemma. *Let f be a cofibration and Z an object in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. Then $f \underset{DK}{\otimes} Z$ is a monomorphism.*

Proof. If $f : A \to B$ is a cofibration in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ then f is a degreewise split monomorphism. The functor $DK : \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A})) \to \Delta^{op}\mathsf{Shv}(\mathcal{A})$ can be explicitly computed in degree $n \geqslant 0$ by

$$DK(X)_n = \bigoplus_{\substack{[n]\to[k] \\ \text{surjective}}} X_k.$$

So $DK(f)$ is computed as the morphism

$$DK(f)_n = \bigoplus_{\substack{[n]\to[k] \\ \text{surjective}}} f_k : \bigoplus_{\substack{[n]\to[k] \\ \text{surjective}}} A_k \to \bigoplus_{\substack{[n]\to[k] \\ \text{surjective}}} B_k.$$

This is a direct sum of split monomorphisms. So $DK(f)$ is a degreewise split monomorphism in $\Delta^{op}\mathsf{Shv}(\mathcal{A})$. Hence, if Z is an object in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$, then the degreewise tensor product $DK(f) \otimes DK(Z)$ is again a split monomorphism in $\Delta^{op}\mathsf{Shv}(\mathcal{A})$. Since DK^{-1} preserves monomorphisms, this then implies that

$$f \underset{DK}{\otimes} Z = DK^{-1}(DK(f) \otimes DK(Z))$$

is a monomorphism in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. □

5.2.15 Lemma. $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ *satisfies the monoid axiom with respect to* $\underset{DK}{\otimes}$. *So* $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ *satisfies the monoid axiom with respect to* $\otimes$.

Proof. Since $\mathsf{Shv}(\mathcal{A})$ is a Grothendieck category, we know that injective quasi-isomorphisms in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ are stable under pushouts and transfinite compositions. So to prove the monoid axiom we just need to show that for every trivial cofibration $f : A \to B$ in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ the morphism $f \underset{DK}{\otimes} Z$ is an injective quasi-isomorphism. By Lemma 5.2.14 we know that it is injective. So we just need to show that it is a weak equivalence.

By [42] we have for all $X, Y \in \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ a natural chain homotopy equivalence

$$\nabla : X \otimes Y \to X \underset{DK}{\otimes} Y$$

between the usual tensor product of chain complexes and the Dold-Kan twisted tensor product. We then get a commutative diagram

$$\begin{array}{ccc} A \underset{DK}{\otimes} Z & \xrightarrow{f \underset{DK}{\otimes} Z} & B \underset{DK}{\otimes} Z \\ {\scriptstyle\nabla}\uparrow & & \uparrow{\scriptstyle\nabla} \\ A \otimes Z & \xrightarrow{f \otimes Z} & B \otimes Z \end{array}$$

where vertical maps are chain homotopy equivalences, and the lower horizontal map is a weak equivalence because $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ satisfies the monoid axiom with respect to $\otimes$. It follows that the upper horizontal map is a weak equivalence. So $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ satisfies the monoid axiom with respect to $\underset{DK}{\otimes}$. □

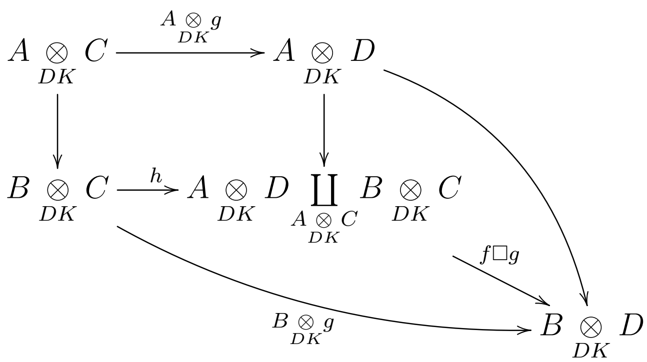
5.2.16 Lemma. $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ *is strongly left proper with respect to* $\underset{DK}{\otimes}$. *So* $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ *is strongly left proper with respect to* $\otimes$.

Proof. Since $\mathsf{Shv}(\mathcal{A})$ is a Grothendieck category, quasi-isomorphisms in the category $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ are stable under pushouts along monorphisms. For any cofibration f the map $f \underset{DK}{\otimes} Z$ is a monomorphism by Lemma 5.2.14. So $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ is strongly left proper with respect to $\underset{DK}{\otimes}$. □

5.2.17 Lemma. *$\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ is a monoidal model category with respect to $\underset{DK}{\otimes}$. So $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ is a monoidal model category with respect to $\otimes$.*

Proof. The unit for $\underset{DK}{\otimes}$ is the chain complex $\mathbb{Z}$ concentrated in degree 0. That is a cofibrant object, so $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ satisfies the unit axiom. Let us now show the pushout-product axiom. The category of simplicial abelian groups Δ^{op} Ab is monoidal and satisfies the monoid axiom with respect to the degree-wise tensor product of chain complexes $\otimes$. If we define a Dold-Kan twisted tensor product $\underset{DK}{\otimes}$ on chain complexes of abelian groups $\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})$ by $X \underset{DK}{\otimes} Y = DK^{-1}(DK(X) \otimes DK(Y))$ then $\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})$ with the standard projective model structure and tensor product $\underset{DK}{\otimes}$ is a monoidal model category satisfying the monoid axiom. Similarly, we can also define a Dold-Kan twisted tensor product $\underset{DK}{\otimes}$ on chain complexes of presheaves $\mathsf{Ch}_{\geqslant 0}(\mathsf{Psh}(\mathcal{A}))$, and it coincides with the Day convolution product induced by the Dold-Kan twisted tensor product on $\mathsf{Ch}_{\geqslant 0}(\mathrm{Ab})$ and the monoidal structure of $\mathcal{A}$. By [20, Theorem 5.5] it follows that $\mathsf{Ch}_{\geqslant 0}(\mathsf{Psh}(\mathcal{A}))$ with standard projective model structure and the Dold-Kan twisted tensor product $\underset{DK}{\otimes}$ is a monoidal model category. For $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ the set $\{\mathcal{A}(-,X)_{\mathsf{nis}} \otimes S^n\mathbb{Z} \to \mathcal{A}(-,X)_{\mathsf{nis}} \otimes D^n\mathbb{Z} \mid X \in \mathsf{Sm}_k, n \geqslant 0\} \cup \{0 \to \mathcal{A}(-,X)_{\mathsf{nis}} \otimes S^0\mathbb{Z} \mid X \in \mathsf{Sm}_k\}$ is a set of generating cofibrations. All these generating cofibrations are sheafifications of cofibrations from $\mathsf{Ch}_{\geqslant 0}(\mathsf{Psh}(\mathcal{A}))$. So if f and g are generating cofibrations in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Psh}(\mathcal{A}))$, and $f \square g$ is the pushout-product with respect to $\underset{DK}{\otimes}$, then we can find cofibrations f' and g' in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Psh}(\mathcal{A}))$ such that $f = L_{\mathsf{nis}}(f')$ and $g = L_{\mathsf{nis}}(g')$. Then $f \square g \cong L_{\mathsf{nis}}(f' \square g')$, where the pushout-product $f' \square g'$ in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Psh}(\mathcal{A}))$ is taken with respect to $\underset{DK}{\otimes}$. Since $\mathsf{Ch}_{\geqslant 0}(\mathsf{Psh}(\mathcal{A}))$ is a monoidal model category with respect to $\underset{DK}{\otimes}$ it follows that $f' \square g'$ is a cofibration in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Psh}(\mathcal{A}))$, and therefore $f \square g$ is a cofibration in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. All we need to show now is that a pushout-product of a cofibration with a trivial cofibration is a weak equivalence in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. So let $f : A \to B$ be a cofibration and $g : C \to D$ be a trivial cofibration in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. We need to show that the pushout-product $f \square g$ with respect to $\underset{DK}{\otimes}$ is a weak equivalence in

$\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. Consider the diagram



The morphism h is a base change of $A \underset{DK}{\otimes} g$. Since g is a trivial cofibration and $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ satisfies the monoid axiom with respect to $\underset{DK}{\otimes}$, this means that h is a weak equivalence in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. Similarly $B \underset{DK}{\otimes} g$ is a weak equivalence in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. So by 2-of-3 it follows that $f \Box g$ is a weak equivalence in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. So $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ is a monoidal model category. $\Box$

We document the above lemmas as follows.

5.2.18 Proposition. *The model category $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ with the usual degreewise tensor product is cellular, weakly finitely generated, monoidal, strongly left proper and satisfies the monoid axiom.*

From now on, weak equivalences in $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ be called local equivalences, fibrations in $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ will be called local fibrations, and fibrant objects in $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ will be called locally fibrant objects.

Chapter 6

Relation to framed motivic Γ-spaces

Recall that framed motivic Γ-spaces introduced in [25] model connective motivic spectra. They are a motivic counterpart of the celebrated Segal Γ-spaces [48].

In this chapter we associate framed motivic Γ-spaces to enriched motivic $\mathcal{A}$-spaces, when the category of correspondences $\mathcal{A}$ has framed correspondences. In Section 6.2 we also associate enriched functors of unbounded chain complexes to enriched motivic $\mathcal{A}$-spaces.

For every natural number $n \geqslant 0$ let n_+ be the pointed set $\{0, \dots, n\}$ where 0 is the basepoint. We write Γ^{op} for the full subcategory of the category of pointed sets on the objects n_+. Γ^{op} is equivalent to the category of finite pointed sets. We write Γ for the opposite category of Γ^{op}. This category is equivalent to the category called Γ in Segal's original paper [48].

6.1 Relation to Γ-spaces

In the additive context we do not need the category Γ as a variable in contrast to framed motivic Γ-spaces in the sense of [25]. This section is to justify this fact (see Proposition 6.1.6). We also associate framed motivic Γ-spaces to enriched motivic $\mathcal{A}$-spaces (see Proposition 6.1.7).

Let $\mathcal{B}$ be an additive model category. By $\Gamma\mathrm{Spc}^{sp}(\mathcal{B})$ we denote the full subcategory of the functor category $\mathrm{Fun}(\Gamma^{op}, \mathcal{B})$ consisting of those functors $\mathcal{X} : \Gamma^{op} \to \mathcal{B}$ such that for every $n \in \mathbb{N}$ the canonical map $\mathcal{X}(n_+) \to \prod_{i=1}^{n} \mathcal{X}(1_+)$ is a weak equivalence in $\mathcal{B}$. This category is called the *category of special* Γ*-spaces in* $\mathcal{B}$.

We have a functor $\mathrm{EM} : \mathcal{B} \to \Gamma\mathrm{Spc}^{sp}(\mathcal{B})$ given by the Eilenberg Maclane construction $\mathrm{EM}(A)(n_+) := \bigoplus_{i=1}^{n} A$. If $f : m_+ \to n_+$ is a function between pointed finite sets, then f is a morphism in Γ^{op}, and we define

$$\mathrm{EM}(A)(f) : \bigoplus_{j=1}^{m} A \to \bigoplus_{i=1}^{n} A$$

as follows. For $0 \leqslant i \leqslant n$ the i-th component $\mathrm{EM}(A)(f)_i : \bigoplus_{j=1}^{m} A \to A$ is $\mathrm{EM}(A)(f)_i := \sum_{j \in f^{-1}(\{i\})} \pi_j$, where $\pi_j : \bigoplus_{i=1}^{m} A \to A$ is the j-th projection morphism. We have another functor $ev_1 : \Gamma\mathrm{Spc}^{sp}(\mathcal{B}) \to \mathcal{B}$ given by $ev_1(\mathcal{X}) := \mathcal{X}(1_+)$.

6.1.1 Lemma. *The functor* $ev_1 : \Gamma\mathrm{Spc}^{sp}(\mathcal{B}) \to \mathcal{B}$ *is left adjoint to* $\mathrm{EM} : \mathcal{B} \to \Gamma\mathrm{Spc}^{sp}(\mathcal{B})$*.*

Proof. Given a morphism $\varphi : \mathcal{X}(1_+) \to A$ in $\mathcal{B}$, we get for every $n \in \mathbb{N}$ a morphism

$$\mathcal{X}(n_+) \to \bigoplus_{i=1}^{n} \mathcal{X}(1_+) \to \bigoplus_{i=1}^{n} A = \mathrm{EM}(A)(n_+),$$

which together assemble into a morphism $\Phi(\varphi) : \mathcal{X} \to \mathrm{EM}(A)$ in $\Gamma\mathrm{Spc}^{sp}(\mathcal{B})$. Conversely, given a morphism $\psi : \mathcal{X} \to \mathrm{EM}(A)$ in $\Gamma\mathrm{Spc}^{sp}(\mathcal{B})$, we can evaluate it at 1_+ to get a morphism $\Psi(\psi) : \mathcal{X}(1_+) \to \mathrm{EM}(A)(1_+) = A$. For every $\varphi : \mathcal{X}(1_+) \to A$ we have $\Psi(\Phi(\varphi)) = \varphi$. Now take a morphism $\psi : \mathcal{X} \to \mathrm{EM}(A)$ in $\Gamma\mathrm{Spc}^{sp}(\mathcal{B})$. We claim that $\Phi(\Psi(\psi)) = \psi$. Take $n \in \mathbb{N}$ and show that $\psi(n_+) : \mathcal{X}(n_+) \to \mathrm{EM}(A) = \bigoplus_{i=1}^{n} A$ is equal to $\Phi(\Psi(\psi))(n_+) : \mathcal{X}(n_+) \to \bigoplus_{i=1}^{n} \mathcal{X}(1_+) \to \bigoplus_{i=1}^{n} A$. By the universal property of the product $\bigoplus$ we need to take i with $0 \leqslant i \leqslant n$ and show that the following diagram commutes

$$\begin{array}{ccc} \mathcal{X}(n_+) & \xrightarrow{\psi(n_+)} & \bigoplus_{i=1}^{n} A \\ \downarrow{\scriptstyle \mathcal{X}(\pi_i)} & & \downarrow{\scriptstyle \pi_i} \\ \mathcal{X}(1_+) & \xrightarrow{\psi(1_+)} & A \end{array}$$

But this just follows from the naturality of $\psi : \mathcal{X} \to \mathrm{EM}(A)$. □

6.1.2 Definition. (1) Let $\mathcal{B}$ be an additive model category. A morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\Gamma\mathrm{Spc}^{sp}(\mathcal{B})$ is called a *weak equivalence* if and only if for every $n \in \mathbb{N}$ the map $f(n_+) : \mathcal{X}(n_+) \to \mathcal{Y}(n_+)$ is a weak equivalence in the model category $\mathcal{B}$. We write W for the class of weak equivalences in $\Gamma\mathrm{Spc}^{sp}(\mathcal{B})$.

(2) We write $\mathrm{Ho}(\Gamma\mathrm{Spc}^{sp}(\mathcal{B}))$ for the localization of $\Gamma\mathrm{Spc}^{sp}(\mathcal{B})$ with respect to the class of weak equivalences W: $\mathrm{Ho}(\Gamma\mathrm{Spc}^{sp}(\mathcal{B})) := \Gamma\mathrm{Spc}^{sp}(\mathcal{B})[W^{-1}]$.

6.1.3 Remark. (1) All isomorphisms in $\Gamma\mathrm{Spc}^{sp}(\mathcal{B})$ are weak equivalences. Weak equivalences in $\Gamma\mathrm{Spc}^{sp}(\mathcal{B})$ satisfy the 2-out-of-3 property.

(2) The functors $\mathrm{EM} : \mathcal{B} \to \Gamma\mathrm{Spc}^{sp}(\mathcal{B})$ and $ev_1 : \Gamma\mathrm{Spc}^{sp}(\mathcal{B}) \to \mathcal{B}$ preserve all weak equivalences.

(3) It is a priori not obvious that the hom-sets of the category $\mathrm{Ho}(\Gamma\mathrm{Spc}^{sp}(\mathcal{B}))$ are small. However, Proposition 6.1.6 below implies that they are in fact small.

6.1.4 Lemma. *A morphism $\varphi : ev_1(\mathcal{X}) \to A$ is a weak equivalence in $\mathcal{B}$ if and only if its adjoint morphism $\Phi(\varphi) : \mathcal{X} \to \mathrm{EM}(A)$ is a weak equivalence in $\Gamma\mathrm{Spc}^{sp}(\mathcal{B})$.*

Proof. Let $\varphi : ev_1(\mathcal{X}) \to A$ be a weak equivalence. Take $n \in \mathbb{N}$. Then $\Phi(\varphi)$ evaluated at n_+ is defined as the composite

$$\mathcal{X}(n_+) \to \bigoplus_{i=1}^{n} \mathcal{X}(1_+) \to \bigoplus_{i=1}^{n} A = EM(A)(n_+).$$

The first map is a weak equivalence, because $\mathcal{X}$ is a special Γ-space. The second map is a weak equivalence, because $\varphi : \mathcal{X}(1_+) \to A$ is a weak equivalence. Therefore $\Phi(\varphi) : \mathcal{X} \to \mathrm{EM}(A)$ is a weak equivalence.

Conversely, let $\varphi : ev_1(\mathcal{X}) \to A$ be a map such that $\Phi(\varphi)$ is a weak equivalence in $\Gamma\mathrm{Spc}^{sp}(\mathcal{B})$. Then $\varphi = \Phi(\varphi)(1_+)$ is also a weak equivalence. □

The following lemma is folklore.

6.1.5 Lemma. *Let $\mathcal{C}, \mathcal{D}$ be categories, each equipped with a class of morphisms, called the weak equivalences, satisfying the 2-out-of-3-property. Let $\mathrm{Ho}(\mathcal{C})$, $\mathrm{Ho}(\mathcal{D})$ be the homotopy categories of $\mathcal{C}$, $\mathcal{D}$, i.e. the categories obtained by inverting the weak equivalences. Let $\ell_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ be the localization functor of $\mathcal{C}$, and $\ell_{\mathcal{D}} : \mathcal{D} \to \mathrm{Ho}(\mathcal{D})$ be the localization functor of $\mathcal{D}$. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors sending weak equivalences in $\mathcal{C}$ to weak equivalences in $\mathcal{D}$. Let $\tau : F \to G$ be a natural transformation. Then the functors F, G induce functors $\mathrm{Ho}(F), \mathrm{Ho}(G) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ satisfying $\mathrm{Ho}(F) \circ \ell_{\mathcal{C}} = \ell_{\mathcal{D}} \circ F$, $\mathrm{Ho}(G) \circ \ell_{\mathcal{C}} = \ell_{\mathcal{D}} \circ G$, and*

$\tau : F \to G$ induces a natural transformation $\mathrm{Ho}(\tau) : \mathrm{Ho}(F) \to \mathrm{Ho}(G)$ such that for every $A \in \mathcal{C}$, the component of $\mathrm{Ho}(\tau)$ at A is given by $\mathrm{Ho}(\tau)_A = \ell_{\mathcal{D}}(\tau_A)$.

The following statement informally says that Γ-spaces in an additive category $\mathcal{B}$ are entirely recovered by $\mathcal{B}$ itself (up to homotopy).

6.1.6 Proposition. *The adjunction $ev_1 \dashv \mathrm{EM}$ induces an equivalence of categories*

$$\mathrm{Ho}(ev_1) : \mathrm{Ho}(\Gamma\mathrm{Spc}^{sp}(\mathcal{B})) \overset{\sim}{\leftrightarrows} \mathrm{Ho}(\mathcal{B}) : \mathrm{Ho}(\mathrm{EM}).$$

Proof. Since ev_1 and EM preserve weak equivalences, they induce two functors $\mathrm{Ho}(ev_1) : \mathrm{Ho}(\Gamma\mathrm{Spc}^{sp}(\mathcal{B})) \to \mathrm{Ho}(\mathcal{B})$ and $\mathrm{Ho}(\mathrm{EM}) : \mathrm{Ho}(\mathcal{B}) \to \mathrm{Ho}(\Gamma\mathrm{Spc}^{sp}(\mathcal{B}))$ on the homotopy categories. For the adjunction $\mathsf{ev}_1 \dashv \mathrm{EM}$ there is a unit $\eta : \mathrm{Id}_{\Gamma\mathrm{Spc}^{\mathrm{sp}}(\mathcal{B})} \to \mathrm{EM} \circ ev_1$. By Lemma 6.1.5, applied to $F = \mathrm{Id}_{\Gamma\mathrm{Spc}^{sp}(\mathcal{B})}$, $G = \mathrm{EM} \circ ev_1$ and $\tau = \eta$, it induces a natural transformation $\mathrm{Ho}(\eta) : \mathrm{Id}_{\mathrm{Ho}(\Gamma\mathrm{Spc}^{\mathrm{sp}}(\mathcal{B}))} \to \mathrm{Ho}(\mathrm{EM}) \circ \mathrm{Ho}(ev_1)$.

For every $\mathcal{X} \in \Gamma\mathrm{Spc}^{sp}(\mathcal{B})$ the identity morphism $ev_1(\mathcal{X}) \to ev_1(\mathcal{X})$ is a weak equivalence, so by Lemma 6.1.4 applied to $A = ev_1(\mathcal{X})$, the adjunction unit map $\eta_{\mathcal{X}} : \mathcal{X} \to \mathrm{EM}(ev_1(\mathcal{X}))$ is a weak equivalence. This implies that the natural transformation $\mathrm{Ho}(\eta)$ is in fact a natural isomorphism of functors.

Furthermore we have a strict equality $ev_1 \circ \mathrm{EM} = \mathrm{Id}_{\mathcal{B}}$, which implies that $\mathrm{Ho}(ev_1) \circ \mathrm{Ho}(\mathrm{EM}) = \mathrm{Id}_{\mathrm{Ho}(\mathcal{B})}$. So $\mathrm{Ho}(ev_1)$ is an equivalence with pseudo-inverse $\mathrm{Ho}(\mathrm{EM})$. □

Let $\mathrm{Fr}_*(k)$ be the category of framed correspondences. For each $V \in \mathsf{Sm}_k$ let $\sigma_V : V \to V$ be the level 1 explicit framed correspondence $(\{0\} \times V, \mathbb{A}^1 \times V, \mathrm{pr}_{\mathbb{A}^1}, \mathrm{pr}_V)$. For the next result, assume that $\mathcal{A}$ has framed correspondences in the sense of Definition 5.1.4. So there is a functor $\Phi : \mathrm{Fr}_*(k) \to \mathcal{A}$ which takes every σ_V to the identity on V. Let $\mathcal{M}^{fr}$ be the category of pointed simplicial Nisnevich sheaves on $\mathrm{Fr}_*(k)$: $\mathcal{M}^{fr} := \Delta^{op}\mathsf{Shv}(\mathrm{Fr}_*(k), Set_*)$.

Φ induces a forgetful functor $U_\Phi : \Delta^{op}\mathsf{Shv}(\mathcal{A}) \to \mathcal{M}^{fr}$. The category $\mathcal{M}^{fr}$ is enriched in $\mathcal{M}$ where for $X, Y \in \mathcal{M}^{fr}$ the enriched morphism object $\mathcal{M}^{fr}(X, Y) \in \mathcal{M}$ is defined on $Z \in \mathsf{Sm}_k$ and $[n] \in \Delta^{op}$ by

$$\mathcal{M}^{fr}(X, Y)(Z)_n := \mathrm{Hom}_{M^{fr}}(X, Y(Z \times \Delta^n \times -)).$$

We have a monoidal adjunction $L_{\mathcal{M}} : \mathcal{M} \rightleftarrows \Delta^{op}\mathsf{Shv}(\mathcal{A}) : U_{\mathcal{M}}$, where the right adjoint $U_{\mathcal{M}}$ is the forgetful functor. For $X, Y \in \Delta^{op}\mathsf{Shv}(\mathcal{A})$ we have a canonical map

$$U_{\mathcal{M}}(\underline{\mathrm{Hom}}_{\Delta^{op}\mathsf{Shv}(\mathcal{A})}(X, Y)) \to \mathcal{M}^{fr}(U_\Phi(X), U_{\Phi(Y)})$$

defined on $Z \in \mathsf{Sm}_k$ and $[n] \in \Delta^{op}$ by the map

$$U_{\mathcal{M}}(\underline{\mathrm{Hom}}_{\Delta^{op}\mathsf{Shv}(\mathcal{A})}(X,Y))(Z)_n = \mathrm{Hom}_{\Delta^{op}\mathsf{Shv}(\mathcal{A})}(X, Y(Z \times \Delta^n \times -)) \xrightarrow{U_\Phi} \\ \to \mathrm{Hom}_{\mathcal{M}^{fr}}(U_\Phi(X), U_\Phi(Y)(Z \times \Delta^n \times -)).$$

Let Sm/k_+ be the category of framed correspondences of level 0 as defined in [25, Example 2.4]. Its morphism objects are defined by

$$\mathrm{Sm}/k_+(X,Y) := \underline{\mathrm{Hom}}_{\mathcal{M}}(X_+, Y_+).$$

Since $L_{\mathcal{M}}$ is lax monoidal, we have for every $X, Y \in \mathsf{Sm}_k$ a canonical map $L_{\mathcal{M}}(\mathrm{Sm}/k_+(X,Y)) \to \mathcal{S}m(X,Y)$ in $\Delta^{op}\mathsf{Shv}(\mathcal{A})$, which induces by adjunction a canonical map $\mathrm{Sm}/k_+(X,Y) \to U_{\mathcal{M}}(\mathcal{S}m(X,Y))$ in $\mathcal{M}$. For every enriched motivic $\mathcal{A}$-space $\mathcal{X}$ we can now define a $\mathcal{M}$-enriched functor

$$\mathrm{Sm}/k_+ \to \mathcal{M}^{fr}, \quad V \mapsto U_\Phi(\mathcal{X}(V)).$$

It acts on morphism sets via the composite

$$\mathrm{Sm}/k_+(X,Y) \to U_{\mathcal{M}}(\mathcal{S}m(X,Y)) \to U_{\mathcal{M}}(\underline{\mathrm{Hom}}_{\Delta^{op}(\mathsf{Shv}(\mathcal{A}))}(\mathcal{X}(X), \mathcal{X}(Y))) \to \\ \to \mathcal{M}^{fr}(U_\Phi(\mathcal{X}(X)), U_\Phi(\mathcal{X}(Y))).$$

With this enriched functor we can then also define a framed motivic Γ-space $\mathrm{EM}^{fr}(\mathcal{X})$ in the sense of [25, Definition 3.5] by defining

$$\mathrm{EM}^{fr}(\mathcal{X}) : \Gamma^{op} \times \mathrm{Sm}/k_+ \to \mathcal{M}^{fr}, \quad \mathrm{EM}^{fr}(\mathcal{X})(n_+, U) = U_\Phi(\mathcal{X}(U))^n.$$

6.1.7 Proposition. *Suppose that $\mathcal{A}$ has framed correspondences in the sense of Definition 5.1.4. For every special enriched motivic $\mathcal{A}$-space $\mathcal{X}$ the framed motivic Γ-space*

$$\mathrm{EM}^{fr}(\mathcal{X}) : \Gamma^{op} \times \mathrm{Sm}/k_+ \to \mathcal{M}^{fr}, \quad \mathrm{EM}^{fr}(\mathcal{X})(n_+, U) = U_\Phi(\mathcal{X}(U))^n,$$

is a very special framed motivic Γ-space in the sense of [25, Axioms 1.1].

Proof. We verify the axioms 1)-5) and 7) for very special motivic Γ-spaces from [25, Axioms 1.1]. For Axiom 1) we need to check that $\mathrm{EM}^{fr}(\mathcal{X})(0_+, U) = 0$, $\mathrm{EM}^{fr}(\mathcal{X})(n_+, \emptyset) = 0$ and that

$$\mathrm{EM}^{fr}(\mathcal{X})(n_+, U) \to \prod_{i=1}^{n} \mathrm{EM}^{fr}(\mathcal{X})(1_+, U)$$

is a local equivalence. We have that $\mathrm{EM}^{fr}(\mathcal{X})(0_+,U)=U_\Phi(\mathcal{X}(U))^0=0$, and

$$\mathrm{EM}^{fr}(\mathcal{X})(n_+,U)=U_\Phi(\mathcal{X}(U))^n\to\prod_{i=1}^{n}\mathrm{EM}^{fr}(\mathcal{X})(1_+,U)$$

is an isomorphism. According to Lemma 4.1.2 we have that $\mathcal{X}(\emptyset)=0$. This implies that $\mathrm{EM}^{fr}(\mathcal{X})(n_+,\emptyset)=0$, hence Axiom 1) holds. Axioms 2)-5) for motivic Γ-spaces follow directly from axioms 1)-4) of special enriched motivic $\mathcal{A}$-spaces, except for Axiom 2) we need to check that the presheaf of stable homotopy groups

$$V\mapsto\pi_n^s\mathrm{EM}^{fr}(\mathcal{X})(\mathbb{S},U)(V)$$

is radditive and σ-stable. The σ-stability follows from the fact that $\Phi:\mathrm{Fr}_*(k)\to\mathcal{A}$ sends σ_V to the identity. Let us now check that it is radditive. For every $U\in\mathsf{Sm}_k$, we have that $\mathcal{X}(U)$ is a sheaf of simplicial abelian groups. This implies that $\mathrm{EM}^{fr}(\mathcal{X})(\mathbb{S},U)$ is a sheaf of S^1-spectra. So we have isomorphisms of S^1-spectra $\mathrm{EM}^{fr}(\mathcal{X})(\mathbb{S},U)(\emptyset)=0$ and

$$\mathrm{EM}^{fr}(\mathcal{X})(\mathbb{S},U)(V_1\coprod V_2)\cong\mathrm{EM}^{fr}(\mathcal{X})(\mathbb{S},U)(V_1)\times\mathrm{EM}^{fr}(\mathcal{X})(\mathbb{S},U)(V_2).$$

Since stable homotopy groups π_n^s preserve products and zero objects, we get that

$$V\mapsto\pi_n^s\mathrm{EM}^{fr}(\mathcal{X})(\mathbb{S},U)(V)$$

is radditive. Axiom 7) follows from the fact that $\mathcal{X}$ lands in sheaves of abelian groups. □

6.1.8 Lemma. *Suppose that $\mathcal{A}$ has framed correspondences in the sense of Definition 5.1.4. Let $\mathcal{X}$ be an enriched motivic $\mathcal{A}$-space and let $\mathrm{EM}^{fr}(\mathcal{X})$ be its associated framed motivic Γ-space from Proposition 6.1.7. Then $\mathcal{X}$ is very effective in the sense of Definition 5.1.2 if and only if $\mathrm{EM}^{fr}(\mathcal{X})$ is very effective in the sense of [25, Axioms 1.1].*

Proof. This follows from the definitions of effectiveness for $\mathcal{X}$ and $\mathrm{EM}^{fr}(\mathcal{X})$. □

6.2 Enriched functors of chain complexes

In the previous section we associated framed motivic Γ-spaces to enriched motivic $\mathcal{A}$-spaces. In this section we associate $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$-enriched functors in

$[\mathcal{S}m, \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))]$ to enriched motivic $\mathcal{A}$-spaces. There is a canonical isomorphism of categories $\mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]) \cong [\mathcal{S}m, \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))]$ constructed in [20]. Likewise, there is a canonical isomorphism of categories $\Delta^{op}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]) \cong [\mathcal{S}m, \Delta^{op}(\mathsf{Shv}(\mathcal{A}))]$. In what follows we shall freely use these isomorphisms.

6.2.1 Definition. Let $\mathcal{X}$ be an special enriched motivic $\mathcal{A}$-space and let

$$DK^{-1} : \Delta^{op}[\mathcal{S}m, \mathsf{Shv}(\mathcal{A})] \to \mathrm{Ch}_{\geqslant 0}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$$

be the normalized Moore complex functor from the Dold-Kan correspondence. Denote by Λ the composite functor

$$\Delta^{op}[\mathcal{S}m, \mathsf{Shv}(\mathcal{A})] \xrightarrow{DK^{-1}} \mathrm{Ch}_{\geqslant 0}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]) \to \mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]).$$

6.2.2 Proposition. *Let $\mathcal{X} \in \Delta^{op}[\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]$ be an enriched motivic $\mathcal{A}$-space. Then $\mathcal{X}$ is special if and only if $\Lambda(\mathcal{X})$ is in $DM_{\mathcal{A}}[\mathcal{S}m]$, where the latter category is defined in Section 3.1.*

Proof. Four axioms defining special enriched motivic $\mathcal{A}$-spaces correspond to four properties of functors in $DM_{\mathcal{A}}[\mathcal{S}m]$. More precisely, the following four properties are true.

(1) $\mathcal{X}$ satisfies axiom (1) of special enriched motivic $\mathcal{A}$-spaces if and only if for every $U \in \mathsf{Sm}_k$ the complex of sheaves $\Lambda(\mathcal{X})(U)$ has $\mathbb{A}^1$-invariant cohomology sheaves.

(2) $\mathcal{X}$ satisfies the cancellation axiom (2) if and only if $\Lambda(\mathcal{X})$ satisfies cancellation in the sense of Definition 3.1.6.

(3) $\mathcal{X}$ satisfies the $\mathbb{A}^1$-invariance axiom (3) if and only if $\Lambda(\mathcal{X})$ is covariantly $\mathbb{A}^1$-invariant in the sense that $\Lambda(\mathcal{X})(U \times \mathbb{A}^1) \to \Lambda(\mathcal{X})(U)$ is a local quasi-isomorphism.

(4) $\mathcal{X}$ satisfies the Nisnevich excision axiom (4) if and only if $\Lambda(\mathcal{X})$ satisfies Nisnevich excision in the sense of Definition 3.1.9. Here the functor $DK^{-1} : \Delta^{op}\mathsf{Shv}(\mathcal{A}) \to \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$ preserves homotopy cartesian squares for the following reason: Since DK^{-1} preserves all weak equivalences, it is naturally weakly equivalent to its right derived functor $\mathbf{R}DK^{-1}$, and by [2, Proposition 4.10] the right derived functor $\mathbf{R}DK^{-1}$ preserves all homotopy limits, including homotopy pullback squares. □

Chapter 7

Reconstructing $SH(k)_{\geqslant 0,\mathbb{Q}}$

Based on the material and techniques developed in the previous chapters, we prove four reconstruction theorems in this chapter. Firstly we prove Theorem 7.3.3 and Theorem 7.3.11 which recover $DM_{\mathcal{A},\geqslant 0}$ and $DM^{\mathsf{eff}}_{\mathcal{A},\geqslant 0}$ from special enriched motivic $\mathcal{A}$-spaces. Secondly we prove Theorem 7.4.2 and Theorem 7.4.4 which recover $SH(k)_{\geqslant,\mathbb{Q}}$ and $SH^{\mathsf{veff}}(k)_{\mathbb{Q}}$ from rational special enriched motivic $\mathcal{A}$-spaces.

7.1 The Röndigs–Østvær Theorem for enriched motivic spaces

Throughout this section $\mathcal{X}$ is a pointwise locally fibrant special enriched motivic $\mathcal{A}$-space.

7.1.1 Definition. We can extend $\mathcal{X}$ to an enriched functor

$$\mathrm{EM}(\mathcal{X}) : \Gamma^{op} \times \mathcal{S}m \to \Delta^{op}\mathsf{Shv}(\mathcal{A}) \quad (n_+, U) \mapsto \mathcal{X}(U)^n.$$

We can take the $(S^1, \mathbb{G}_m)$-evaluation of $\mathrm{EM}(\mathcal{X})$ to get a motivic bispectrum $ev_{S^1,\mathbb{G}_m}(\mathrm{EM}(\mathcal{X})) \in SH(k)$. We define the *bispectrum associated to* $\mathcal{X}$ to be this bispectrum $ev_{S^1,\mathbb{G}_m}(\mathcal{X}) := ev_{S^1,\mathbb{G}_m}(\mathrm{EM}(\mathcal{X}))$. If $\mathcal{A}$ has framed correspondences, then $ev_{S^1,\mathbb{G}_m}(\mathcal{X})$ is also the evaluation of the framed motivic Γ-space $\mathrm{EM}^{fr}(\mathcal{X})$ from Proposition 6.1.7. Then by [25, Section 2.7] the bispectrum $ev_{S^1,\mathbb{G}_m}(\mathcal{X}) = ev_{S^1,\mathbb{G}_m}(\mathrm{EM}^{fr}(\mathcal{X}))$ is a framed bispectrum in the sense of [24, Definition 2.1]. In this case we say that $ev_{S^1,\mathbb{G}_m}(\mathcal{X})$ is the *framed bispectrum associated to* $\mathcal{X}$.

In this section we prove the following theorem extending Röndigs–Østvær's Theorem [46].

7.1.2 Theorem. *For every* $U \in \mathsf{Sm}_k$ *we have a natural isomorphism*

$$ev_{S^1,\mathbb{G}_m}(\mathcal{X}) \wedge \Sigma^{\infty}_{S^1,\mathbb{G}_m} U_+ \xrightarrow{\sim} ev_{S^1,\mathbb{G}_m}(\mathcal{X}(U \times -))$$

in $SH(k)[1/p]$*, where* p *is the exponential characteristic of* k*.*

To prove it we will need a few lemmas. For a finite pointed set $n_+ = \{0, \dots, n\}$ and $U \in \mathsf{Sm}_k$ let $n_+ \otimes U$ be the n-fold coproduct $\coprod_{i=1}^{n} U$. Let $f\mathcal{M}$ be the category of finitely presented motivic spaces in the sense of [15]. Given an enriched motivic $\mathcal{A}$-space $\mathcal{X}$ we can define an extended functor $\hat{\mathcal{X}} : f\mathcal{M} \to \Delta^{op}\mathsf{Shv}(\mathcal{A})$ by

$$\hat{\mathcal{X}}(A)_n := \operatorname*{colim}_{(\Delta[m]\times U)_+ \to A^c} \mathcal{X}(\Delta[m]_{n,+} \otimes U)_n$$

where A^c is a cofibrant replacement of A in $f\mathcal{M}$. We have for all $U \in \mathcal{S}m$ that $\hat{\mathcal{X}}(U) \cong \mathcal{X}(U)$ in $\Delta^{op}\mathsf{Shv}(\mathcal{A})$. Let $ev_{S^1,\mathbb{G}_m}(\hat{\mathcal{X}})$ be the $(S^1, \mathbb{G}_m)$-evaluation bispectrum of the extended functor $\hat{\mathcal{X}} : f\mathcal{M} \to \Delta^{op}\mathsf{Shv}(\mathcal{A})$.

7.1.3 Lemma. *We have a canonical isomorphism of motivic* $(S^1, \mathbb{G}_m)$*-bispectra* $ev_{S^1,\mathbb{G}_m}(\hat{\mathcal{X}}) \cong ev_{S^1,\mathbb{G}_m}(\mathcal{X})$ *between the* $(S^1, \mathbb{G}_m)$*-evaluation of the extended functor* $\hat{\mathcal{X}}$*, and the bispectrum associated with* $\mathcal{X}$ *in the sense of Definition 7.1.1.*

Proof. By Lemma 4.1.2 we have for all $U, V \in \mathsf{Sm}_k$ an isomorphism $\mathcal{X}(U \coprod V) \cong \mathcal{X}(U) \oplus \mathcal{X}(V)$ in $\Delta^{op}\mathsf{Shv}(\mathcal{A})$. This implies that we have for all $U \in \mathsf{Sm}_k, n \geqslant 0$ an isomorphism $\mathcal{X}(n_+ \otimes U) \cong \bigoplus_{i=1}^{n} \mathcal{X}(U) = \mathrm{EM}(\mathcal{X})(n_+, U)$ in $\Delta^{op}\mathsf{Shv}(\mathcal{A})$. We then compute for $A \in f\mathcal{M}$ that

$$\hat{\mathcal{X}}(A)_n = \operatorname*{colim}_{(\Delta[k]\times U)_+ \to A^c} \mathcal{X}(\Delta[k]_{n,+} \otimes U)_n \cong \operatorname*{colim}_{(\Delta[k]\times U)_+ \to A^c} \mathrm{EM}(\mathcal{X})(\Delta[k]_{n,+}, U)_n.$$

So $\hat{\mathcal{X}}$ naturally extends $\mathrm{EM}(\mathcal{X})$ from $\Gamma^{op} \times \mathrm{Sm}/k_+$ to $f\mathcal{M}$. This then implies that

$$ev_{S^1,\mathbb{G}_m}(\hat{\mathcal{X}}) \cong ev_{S^1,\mathbb{G}_m}(\mathrm{EM}(\mathcal{X})) = ev_{S^1,\mathbb{G}_m}(\mathcal{X})$$

as required. □

Proof of Theorem 7.1.2. Using Definition 6.2.1 we can associate to $\mathcal{X}$ an enriched functor $\Lambda(\mathcal{X}) : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. By Proposition 6.2.2 the functor $\Lambda(\mathcal{X})$ is in $DM_{\mathcal{A}}[\mathcal{S}m]$. By Proposition 3.1.13 this implies that $\Lambda(\mathcal{X})$ is strictly $\sim$-local in the sense of 3.1.3. Since $\mathcal{X}$ is pointwise locally fibrant, it follows that $\Lambda(\mathcal{X})$ is $\sim$-fibrant in the sense of Definition 3.1.11.

Using Section 4.2 we can associate to $\Lambda(\mathcal{X})$ an $\mathcal{M}$-enriched functor $\Lambda(\mathcal{X})^{\mathcal{M}} : f\mathcal{M} \to \mathsf{Sp}_{S^1}(\mathcal{M})$. We can take the 0-th level of this functor to get a motivic functor $\Lambda(\mathcal{X})^{\mathcal{M}}_0 : f\mathcal{M} \to \mathcal{M}$. By Lemma 4.2.7 the motivic functor $\Lambda(\mathcal{X})^{\mathcal{M}}_0$ preserves motivic equivalences between cofibrant objects. By [34, Appendix B, Corollary B.2] the suspension bispectrum $\Sigma^\infty_{S^1,\mathbb{G}_m} U_+$ is strongly dualizable in $SH(k)[1/p]$. From Lemma 4.2.2 it follows that we have an isomorphism

$$ev_{S^1,\mathbb{G}_m}(\Lambda(\mathcal{X})^{\mathcal{M}}_0) \wedge \Sigma^\infty_{S^1,\mathbb{G}_m} U_+ \cong ev_{S^1,\mathbb{G}_m}(\Lambda(\mathcal{X})^{\mathcal{M}}_0(U \times -))$$

in $SH(k)[1/p]$. To prove the theorem, we now just need to show that there is a natural isomorphism

$$ev_{S^1,\mathbb{G}_m}(\Lambda(\mathcal{X})^{\mathcal{M}}_0) \to ev_{S^1,\mathbb{G}_m}(\mathcal{X})$$

in $SH(k)$. For this we need some intermediate steps. Firstly, by Lemma 7.1.3 we have an isomorphism

$$ev_{S^1,\mathbb{G}_m}(\hat{\mathcal{X}}) \to ev_{S^1,\mathbb{G}_m}(\mathcal{X}).$$

So we now just need to find an isomorphism

$$ev_{S^1,\mathbb{G}_m}(\Lambda(\mathcal{X})^{\mathcal{M}}_0) \to ev_{S^1,\mathbb{G}_m}(\hat{\mathcal{X}})$$

in $SH(k)$.

In what follows, we let

$$DK^{-1} : \Delta^{op}\mathsf{Shv}(\mathcal{A}) \to \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$$

be the Dold-Kan equivalence, i.e. the normalized Moore complex functor, for the Grothendieck category $\mathsf{Shv}(\mathcal{A})$. We let

$$DK^{-1}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))} : \Delta^{op}\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Ch}_{\geqslant 0}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})))$$

be the Dold-Kan correspondence for the Grothendieck category $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$. And we let

$$DK^{-1}_{\mathrm{double}} : \Delta^{op}\Delta^{op}\mathsf{Shv}(\mathcal{A}) \to \mathsf{Ch}_{\geqslant 0}(\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A})))$$

be the Dold-Kan correspondence applied twice, so that it takes bisimplicial objects to double complexes.

Using Section 4.1, equation (4.1) we can extend $\Lambda(\mathcal{X})$ to a functor

$$\widehat{\Lambda(\mathcal{X})} : f\mathcal{M} \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})),$$

$$\widehat{\Lambda(\mathcal{X})}(A) := \mathrm{Tot}(DK^{-1}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\underset{(\Delta[k]\times U)_+\to A^c}{\mathsf{colim}} \Lambda(\mathcal{X})^{\Delta^{op}}(\Delta[k]_+ \otimes U))).$$

Now for every $A \in f\mathcal{M}$ we have a natural quasi-isomorphism

$$DK^{-1}(\hat{\mathcal{X}}(A)) \to \widehat{\Lambda(\mathcal{X})}(A)$$

in $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ for the following reason: $\hat{\mathcal{X}}(A)$ is the diagonal of the bisimplicial sheaf

$$\underset{(\Delta[m]\times U)_+\to A^c}{\mathsf{colim}} \mathcal{X}(\Delta[m]_+ \otimes U).$$

By [8, page 37, equation 24], or [12, Theorem 2.9], for every bisimplicial object $S \in \Delta^{op}\Delta^{op}\mathsf{Shv}(\mathcal{A})$ there is a quasi-isomorphism

$$DK^{-1}(\mathrm{diag}(S)) \to \mathrm{Tot}(DK^{-1}_{\mathrm{double}}(S))$$

in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. So for every $A \in f\mathcal{M}$ there is a quasi-isomorphism

$$DK^{-1}(\hat{\mathcal{X}}(A)) \to \mathrm{Tot}(DK^{-1}_{\mathrm{double}}(\underset{(\Delta[m]\times U)_+\to A^c}{\mathsf{colim}} \mathcal{X}(\Delta[m]_+ \otimes U))) \cong$$

$$\cong \mathrm{Tot}(DK^{-1}_{\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))}(\underset{(\Delta[m]\times U)_+\to A^c}{\mathsf{colim}} DK^{-1}(\mathcal{X}(\Delta[m]_+ \otimes U)))) \cong \widehat{\Lambda(\mathcal{X})}(A).$$

By construction $\widehat{\Lambda(\mathcal{X})}$ lands in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$, so we can take the functor

$$DK \circ \widehat{\Lambda(\mathcal{X})} : f\mathcal{M} \to \Delta^{op}\mathsf{Shv}(\mathcal{A})$$

and form the naive $(S^1, \mathbb{G}_m)$-evaluation bispectrum

$$ev_{S^1,\mathbb{G}_m}(DK \circ \widehat{\Lambda(\mathcal{X})}) \in SH(k).$$

The above quasi-isomorphism, then induces an isomorphism

$$ev_{S^1,\mathbb{G}_m}(DK \circ \widehat{\Lambda(\mathcal{X})}) \to ev_{S^1,\mathbb{G}_m}(\hat{\mathcal{X}})$$

in $SH(k)$. So to prove the theorem we now just need an isomorphism

$$ev_{S^1,\mathbb{G}_m}(\Lambda(\mathcal{X})_0^{\mathcal{M}}) \to ev_{S^1,\mathbb{G}_m}(DK \circ \widehat{\Lambda(\mathcal{X})})$$

in $SH(k)$.

By Lemma 4.2.5, for every $A \in f\mathcal{M}$ with cofibrant replacement A^c we have an isomorphism

$$\hat{U} \circ \widehat{\Lambda(\mathcal{X})}(A) \to \Lambda(\mathcal{X})^{\mathcal{M}}(A^c)$$

in $\mathsf{Sp}_{S^1}(\mathcal{M})$, where $\hat{U} : \mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \to \mathsf{Sp}_{S^1}(\mathcal{M})$ is the canonical functor defined in Section 4.2. Let $ev_0 : \mathsf{Sp}_{S^1}(\mathcal{M}) \to \mathcal{M}$ be the functor taking the 0-th level of a S^1-spectrum. So $\Lambda(\mathcal{X})_0^{\mathcal{M}} = ev_0 \circ \Lambda(\mathcal{X})^{\mathcal{M}}$. By the proof of Lemma 4.2.4, the functor $ev_0 \circ \hat{U}$ is isomorphic to the composite

$$\mathsf{Ch}(\mathsf{Shv}(\mathcal{A})) \overset{\tau_{\geqslant 0}}{\to} \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A})) \overset{DK}{\to} \Delta^{op}\mathsf{Shv}(\mathcal{A}) \overset{U}{\to} \mathcal{M},$$

where $\tau_{\geqslant 0}$ is the good truncation functor and U is the forgetful functor. Since $\widehat{\Lambda(\mathcal{X})}$ lands in $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$, it does not get changed by truncation. So we get that

$$ev_0 \circ \hat{U} \circ \widehat{\Lambda(\mathcal{X})} \cong U \circ DK \circ \widehat{\Lambda(\mathcal{X})}.$$

So for every $A \in f\mathcal{M}$ we have a natural isomorphism

$$(U \circ DK \circ \widehat{\Lambda(\mathcal{X})})(A) \to \Lambda(\mathcal{X})_0^{\mathcal{M}}(A^c)$$

in $\mathcal{M}$. Since S^1 and $\mathbb{G}_m$ are cofibrant in $f\mathcal{M}$, we get an isomorphism

$$ev_{S^1,\mathbb{G}_m}(\Lambda(\mathcal{X})_0^{\mathcal{M}}) \to ev_{S^1,\mathbb{G}_m}(DK \circ \widehat{\Lambda(\mathcal{X})})$$

in $SH(k)$, as claimed.

Putting it all together, we get a commutative diagram

$$\begin{array}{ccc}
ev_{S^1,\mathbb{G}_m}(\Lambda(\mathcal{X})_0^{\mathcal{M}}) \wedge \Sigma^\infty_{S^1,\mathbb{G}_m} U_+ & \xrightarrow{\sim} & ev_{S^1,\mathbb{G}_m}(\Lambda(\mathcal{X})_0^{\mathcal{M}}(U \times -)) \\
\downarrow \sim & & \downarrow \sim \\
ev_{S^1,\mathbb{G}_m}(DK \circ \widehat{\Lambda(\mathcal{X})}) \wedge \Sigma^\infty_{S^1,\mathbb{G}_m} U_+ & \longrightarrow & ev_{S^1,\mathbb{G}_m}(DK \circ \widehat{\Lambda(\mathcal{X})}(U \times -)) \\
\downarrow \sim & & \downarrow \sim \\
ev_{S^1,\mathbb{G}_m}(\hat{\mathcal{X}}) \wedge \Sigma^\infty_{S^1,\mathbb{G}_m} U_+ & \longrightarrow & ev_{S^1,\mathbb{G}_m}(\hat{\mathcal{X}}(U \times -)) \\
\downarrow \sim & & \downarrow \sim \\
ev_{S^1,\mathbb{G}_m}(\mathcal{X}) \wedge \Sigma^\infty_{S^1,\mathbb{G}_m} U_+ & \longrightarrow & ev_{S^1,\mathbb{G}_m}(\mathcal{X}(U \times -))
\end{array}$$

in which all the vertical maps and the top horizontal map are isomorphisms in $SH(k)[1/p]$. It follows that the bottom horizontal map is also an isomorphism in $SH(k)[1/p]$. This completes the proof. □

7.2 A motivic model structure for enriched motivic $\mathcal{A}$-spaces

In Section 5.2 we showed that $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ with the degreewise tensor product $\otimes$ has a model structure that is cellular, weakly finitely generated, monoidal, strongly left proper and satisfies the monoid axiom (see Proposition 5.2.18). We can apply [14, Theorem 4.2] to this model structure to get a weakly finitely generated model structure on the category of enriched functors $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ in which the weak equivalences, respectively fibrations, are the $\mathcal{S}m$-pointwise local equivalences, respectively $\mathcal{S}m$-pointwise local fibrations. We call this the *local model structure* on $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$. By [14, Theorem 4.4] the local model structure on $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ is monoidal with the usual Day convolution product. By [14, Corollary 4.8] the local model structure on $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ is left proper. Since $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ is weakly finitely generated, and all cofibrations in $\Delta^{op}\mathsf{Shv}(\mathcal{A})$ are monomorphisms, it follows that $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ is cellular. Note that for every $U \in \mathsf{Sm}_k$ the representable functor $\mathcal{S}m(U,-) \cong \mathcal{S}m(U,-) \otimes pt$ is cofibrant in $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$.

In this section we define another model structure on $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ such that the fibrant objects are the pointwise locally fibrant special enriched motivic $\mathcal{A}$-spaces.

7.2.1 Definition. Similarly to Section 3.1 we define four families of morphisms in $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$.

1. We let $\mathbb{A}^1_1$ be the family of morphisms consisting of

$$\mathcal{S}m(U,-) \otimes \mathbb{A}^1 \to \mathcal{S}m(U,-)$$

   for every $U \in \mathsf{Sm}_k$.

2. We let τ be the family of morphisms consisting of the evaluation map

$$\mathcal{S}m(\mathbb{G}_m^{\wedge n+1} \times U, -) \otimes \mathbb{G}_m^{\wedge 1} \to \mathcal{S}m(\mathbb{G}_m^{\wedge n} \times U, -)$$

   for every $n \geqslant 0$ and $U \in \mathsf{Sm}_k$.

3. We let $\mathbb{A}^1_2$ be the family of morphisms consisting of

$$\mathcal{S}m(U,-) \to \mathcal{S}m(U \times \mathbb{A}^1, -)$$

for every $U \in \mathsf{Sm}_k$.

4. We let Nis be the following family of morphisms: For every elementary Nisnevich square Q of the form

$$\begin{array}{ccc} U' & \xrightarrow{\beta} & X' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\gamma} \\ X & \xrightarrow[\delta]{} & X \end{array}$$

in Sm_k we have a square

$$\begin{array}{ccc} \mathcal{S}m(U',-) & \xleftarrow{\beta^*} & \mathcal{S}m(X',-) \\ {\scriptstyle\alpha^*}\uparrow & & \uparrow{\scriptstyle\gamma^*} \\ \mathcal{S}m(U,-) & \xleftarrow[\delta^*]{} & \mathcal{S}m(X,-) \end{array}$$

in $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$, which induces a map on homotopy fibers

$$p_Q : \text{hofib}(\gamma^*) \to \text{hofib}(\alpha^*).$$

We let Nis be the family of morphisms consisting of p_Q for every elementary Nisnevich square Q.

Finally, we let $\sim$ denote the union of all these four classes of morphisms.

$$\sim := \mathbb{A}^1_1 + \tau + \mathbb{A}^1_2 + Nis.$$

7.2.2 Definition. For $X, Y \in [\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ let

$$\text{map}^{\Delta^{op}\mathsf{Shv}(\mathcal{A})}(X,Y) \in \Delta^{op}\mathsf{Shv}(\mathcal{A})$$

be the simplicial sheaf of morphisms from X to Y. It is defined by taking the internal hom $\underline{\text{Hom}}_{[\mathcal{S}m,\Delta^{op}\mathsf{Shv}(\mathcal{A})]}(X,Y)$ and evaluating it at the point $pt \in \mathcal{S}m$.

$$\text{map}^{\Delta^{op}\mathsf{Shv}(\mathcal{A})}(X,Y) := \underline{\text{Hom}}_{[\mathcal{S}m,\Delta^{op}\mathsf{Shv}(\mathcal{A})]}(X,Y)(pt).$$

For $U \in \mathsf{Sm}_k$ and $n \geqslant 0$ we have

$$\mathrm{map}^{\Delta^{op}\mathsf{Shv}(\mathcal{A})}(X,Y)(U)_n = \mathrm{Hom}_{[\mathcal{S}m,\Delta^{op}\mathsf{Shv}(\mathcal{A})]}(X \otimes U \otimes \Delta[n], Y)$$

in Ab.

Similarly to Definition 3.1.3, given a class of morphisms S in $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ and an object $X \in [\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ with pointwise locally fibrant replacement X^f we say that X is *strictly S-local* if for every $s : A \to B$ with $s \in S$ the morphism

$$s^* : \mathrm{map}^{\Delta^{op}\mathsf{Shv}(\mathcal{A})}(B, X^f) \to \mathrm{map}^{\Delta^{op}\mathsf{Shv}(\mathcal{A})}(B, X^f)$$

is a local quasi-isomorphism of sheaves.

7.2.3 Lemma. *A enriched motivic $\mathcal{A}$-space $\mathcal{X} : \mathcal{S}m \to \mathsf{Shv}(\mathcal{A})$ is special if and only if it is strictly $\sim$-local.*

Proof. By Lemma 6.2.2 we know that $\mathcal{X}$ is special if and only if $\Lambda(\mathcal{X})$ lies in $DM_{\mathcal{A}}[\mathcal{S}m]$. By Proposition 3.1.13 this is the case if and only if $\Lambda(\mathcal{X})$ is strictly $\sim$-local in the sense of Definition 3.1.3, and this is the case if and only if $\mathcal{X}$ is strictly $\sim$-local in the sense of Definition 7.2.2. □

7.2.4 Definition. Given a class of morphisms S in $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$, we write $\widehat{S}$ for the class of morphisms

$$\widehat{S} := \{s \otimes Z \mid s \in S, Z \in \mathsf{Sm}_k\}.$$

We define the *enriched motivic model structure* on $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ to be the left Bousfield localization of the local model structure on $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ with respect to the class of morphisms $\widehat{\sim}$. This model category will be denoted by $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]_{\mathrm{mot}}$.

7.2.5 Lemma. *Let S be a class of morphisms with cofibrant domains and codomains in $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$. Then an object $F \in [\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ is strictly S-local if and only if its local fibrant replacement F^f is $\widehat{S}$-local in the usual model category theoretic sense of [27, Definition 3.1.4].*

Proof. Let F^f be a pointwise locally fibrant replacement of F. For every $s : A \to B, s \in \widehat{S}$ let $s^c : A^c \to B^c$ be a cofibrant replacement of s. This means we have a commutative square

$$\begin{array}{ccc} A^c & \xrightarrow{s^c} & B^c \\ \downarrow & & \downarrow \\ A & \xrightarrow{s} & B \end{array}$$

such that the vertical maps are trivial fibrations, A^c and B^c are cofibrant and s^c is a cofibration.

Note that for every $s \in \widehat{S}$ the domain A and codomain B are already cofibrant, but s is not neccessarily a cofibration.

For $X, Y \in [\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ let $\mathrm{map}^{\Delta^{op}\,\mathrm{Set}}(X,Y) \in \Delta^{op}\,\mathrm{Set}$ denote the non-derived simplicial mapping space. It can be defined by

$$\mathrm{map}^{\Delta^{op}\,\mathrm{Sets}}(X,Y) := \underline{\mathrm{Hom}}_{[\mathcal{S}m,\Delta^{op}\mathsf{Shv}(\mathcal{A})]}(X,Y)(pt)(pt).$$

Now F^f is $\widehat{S}$-local in the usual model category theoretic sense if and only if for every $s \in \widehat{S}$ the map

$$s^{c,*} : \mathrm{map}^{\Delta^{op}\,\mathrm{Sets}}(B^c, F^f) \to \mathrm{map}^{\Delta^{op}\,\mathrm{Sets}}(A^c, F^f)$$

is a weak equivalence. We have a commutative square

$$\begin{array}{ccc} \mathrm{map}^{\Delta^{op}\,\mathrm{Set}}(B, F^f) & \xrightarrow{s^*} & \mathrm{map}^{\Delta^{op}\,\mathrm{Set}}(A, F^f) \\ \downarrow & & \downarrow \\ \mathrm{map}^{\Delta^{op}\,\mathrm{Set}}(B^c, F^f) & \xrightarrow{s^{c,*}} & \mathrm{map}^{\Delta^{op}\,\mathrm{Set}}(A^c, F^f) \end{array}$$

Since the functor $\mathrm{map}^{\Delta^{op}\,\mathrm{Sets}}(-, F^f)$ sends trivial cofibrations to trivial fibrations, it follows by Ken Brown's lemma [28, Lemma 1.1.12], that the functor $\mathrm{map}^{\Delta^{op}\,\mathrm{Sets}}(-, F^f)$ sends weak equivalences between cofibrant objects to weak equivalences. Since the maps $A^c \to A$ and $B^c \to B$ are weak equivalences between cofibrant objects, it follows that the vertical maps in the above commutative diagram are weak equivalences. Therefore F^f is $\widehat{S}$-local if and only if for every $s \in \widehat{S}$ the map

$$s^* : \mathrm{map}^{\Delta^{op}\,\mathrm{Sets}}(B, F^f) \to \mathrm{map}^{\Delta^{op}\,\mathrm{Sets}}(A, F^f)$$

is a weak equivalence. Every $s \in \widehat{S}$ is of the form $t \otimes Z$ for some $Z \in \mathsf{Sm}_k$ and $t : C \to D$ with $t \in S$. We have a commutative diagram in which the vertical maps are isomorphisms:

$$\begin{array}{ccc} \mathrm{map}^{\Delta^{op}\,\mathrm{Sets}}(D \otimes Z, F^f) & \xrightarrow{(t \otimes Z)^*} & \mathrm{map}^{\Delta^{op}\,\mathrm{Sets}}(C \otimes Z, F^f) \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{map}^{\Delta^{op}\mathsf{Shv}(\mathcal{A})}(D, F^f)(Z) & \xrightarrow{t^*} & \mathrm{map}^{\Delta^{op}\mathsf{Shv}(\mathcal{A})}(C, F^f)(Z) \end{array}$$

So F^f is $\widehat{S}$-local if and only if for every $t : C \to D, t \in S$ the map

$$t^* : \mathrm{map}^{\Delta^{op}\mathsf{Shv}(\mathcal{A})}(D, F^f) \to \mathrm{map}^{\Delta^{op}\mathsf{Shv}(\mathcal{A})}(C, F^f)$$

is a sectionwise weak equivalence in $\Delta^{op}\mathsf{Shv}(\mathcal{A})$. Since C, D are cofibrant and F^f is locally fibrant, the domain and codomain of t^* are fibrant. So t^* is a sectionwise weak equivalence if and only if it is a local weak equivalence. Therefore F^f is $\widehat{S}$-local if and only if F is strictly S-local. □

So the fibrant objects of $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]_{\mathrm{mot}}$ are the pointwise locally fibrant special enriched motivic $\mathcal{A}$-spaces.

7.2.6 Definition. Let $\mathcal{D}([\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})])$ be the homotopy category of the category $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ with respect to the pointwise local model structure. Define $\mathrm{Spc}_{\mathcal{A}}[\mathcal{S}m]$ as the full subcategory of $\mathcal{D}([\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})])$ consisting of special enriched motivic $\mathcal{A}$-spaces.

We document above lemmas as follows.

7.2.7 Theorem. *The category* $\mathrm{Spc}_{\mathcal{A}}[\mathcal{S}m]$ *is equivalent to the homotopy category of the model category* $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]_{\mathrm{mot}}$. *The fibrant objects of the model category* $[\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]_{\mathrm{mot}}$ *are the pointwise locally fibrant special enriched motivic* $\mathcal{A}$*-spaces.*

The preceding theorem is also reminiscent of Bousfield–Friedlander's theorem [5] stating that fibrant objects in the model category of classical Γ-spaces are given by very special Γ-spaces.

7.3 Reconstructing $DM^{\mathsf{eff}}_{\mathcal{A},\geqslant 0}$

7.3.1 Definition. For $U \in \mathsf{Sm}_k$ define $M^{\mathbb{G}_m}_{\mathcal{A}}(U) \in DM_{\mathcal{A}}$ by

$$M^{\mathbb{G}_m}_{\mathcal{A}}(U) := (M_{\mathcal{A}}(U \times \mathbb{G}_m^{\wedge n}))_{n \geqslant 0},$$

where $M_{\mathcal{A}}(X) := C_*\mathcal{A}(-, X)_{\mathsf{nis}}$ is the $\mathcal{A}$-motive of X. We call $M^{\mathbb{G}_m}_{\mathcal{A}}(U)$ the *big* $\mathcal{A}$*-motive* of U.

Let $\mathcal{U} : DM_{\mathcal{A}} \to SH(k)$ be the forgetful functor, and let $\mathcal{L} : SH(k) \to DM_{\mathcal{A}}$ be its left adjoint.

7.3.2 Lemma. *The natural morphism*

$$\mathcal{L}(\Sigma^{\infty}_{S^1,\mathbb{G}_m} U_+) \to M^{\mathbb{G}_m}_{\mathcal{A}}(U)$$

is an isomorphism in $DM_{\mathcal{A}}$.

Proof. In weight n this morphism is the motivic equivalence

$$\mathcal{A}(-,U)_{\mathsf{nis}} \to C_*\mathcal{A}(-,U)_{\mathsf{nis}} = M_{\mathcal{A}}(U).$$

So the map $\mathcal{L}(\Sigma^{\infty}_{S^1,\mathbb{G}_m} U_+) \to M^{\mathbb{G}_m}_{\mathcal{A}}(U)$ is a levelwise motivic equivalence, and therefore an isomorphism in $DM_{\mathcal{A}}$. □

Let $DM_{\mathcal{A},\geqslant 0}$ be the full subcategory of $DM_{\mathcal{A}}$ consisting of those $\mathbb{G}_m$-spectra of chain complexes which are connective chain complexes in each weight. Note that by construction, for every $U \in \mathsf{Sm}_k$ we have $M^{\mathbb{G}_m}_{\mathcal{A}}(U) \in DM_{\mathcal{A},\geqslant 0}$.

7.3.3 Theorem. *The naive* $\mathbb{G}_m$*-evaluation functor is an equivalence of categories*

$$ev_{\mathbb{G}_m} : \mathrm{Spc}_{\mathcal{A}}[\mathcal{S}m] \to DM_{\mathcal{A},\geqslant 0}.$$

Proof. Since the exponential characteristic p of k is invertible in $\mathcal{A}$, it follows from 3.1.14 that the naive $\mathbb{G}_m$-evaluation functor is an equivalence of categories

$$ev_{\mathbb{G}_m} : DM_{\mathcal{A}}[\mathcal{S}m] \to DM_{\mathcal{A}}.$$

Here $DM_{\mathcal{A}}[\mathcal{S}m]$ consists of those enriched functors $F : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ which satisfy contravariant $\mathbb{A}^1$-invariance, cancellation, covariant $\mathbb{A}^1$-invariance and Nisnevich excision (see 3.1 for details).

Let $DM_{\mathcal{A}}[\mathcal{S}m]_{\geqslant 0}$ be the full subcategory of $DM_{\mathcal{A}}[\mathcal{S}m]$ on those functors $F : \mathcal{S}m \to \mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ which factor over $\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))$. The equivalence $ev_{\mathbb{G}_m}$ restricts to a fully faithful functor on connective chain complexes

$$ev_{\mathbb{G}_m,\geqslant 0} : DM_{\mathcal{A}}[\mathcal{S}m]_{\geqslant 0} \to DM_{\mathcal{A},\geqslant 0}.$$

The functor $ev_{\mathbb{G}_m} : \mathrm{Spc}_{\mathcal{A}}[\mathcal{S}m] \to DM_{\mathcal{A},\geqslant 0}$ of the theorem will factor through $ev_{\mathbb{G}_m,\geqslant 0}$. We claim that this restricted $\mathbb{G}_m$-evaluation functor $ev_{\mathbb{G}_m,\geqslant 0}$ is an equivalence. Since it is fully faithful we only need to show essential surjectivity.

Take $F \in DM_{\mathcal{A},\geqslant 0}$. Since $ev_{\mathbb{G}_m}$ is essentially surjective on non-connective chain complexes, there exists $G \in DM_{\mathcal{A}}[\mathcal{S}m]$ such that $ev_{\mathbb{G}_m}(G) \cong F$. Let

$$\tau_{\geqslant 0} : \mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]) \to \mathsf{Ch}_{\geqslant 0}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$$

be the good truncation functor for chain complexes of the Grothendieck category of enriched functors $[\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]$. Also denote by $\tau_{\geqslant 0} : \mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))) \to \mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A})))$ the good truncation functor of $\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))$ applied in each weight. Consider the commutative diagram

$$\begin{array}{ccc} ev_{\mathbb{G}_m}(\tau_{\geqslant 0}(G)) & \longrightarrow & \tau_{\geqslant 0}(F) \\ \downarrow & & \downarrow \wr \\ ev_{\mathbb{G}_m}(G) & \xrightarrow{\sim} & F \end{array}$$

We know the bottom horizontal map and the right vertical map are isomorphisms in $DM_{\mathcal{A}}$. We claim that $\tau_{\geqslant 0}(G) \to G$ is an isomorphism in $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$. For this it suffices to show that for every $U \in \mathsf{Sm}_k$ the negative homology sheaves of $G(U)$ are zero. We have a chain of isomorphisms in $D(\mathsf{Shv}(\mathcal{A}))$

$$G(U) \cong G(U \times pt) = ev_{\mathbb{G}_m}(G(U \times -))(0)$$

By Theorem 4.2.1 we have isomorphisms in $DM_{\mathcal{A}}$

$$ev_{\mathbb{G}_m}(G(U \times -)) \cong ev_{\mathbb{G}_m}(G) \wedge M_{\mathcal{A}}^{\mathbb{G}_m}(U) \cong F \wedge M_{\mathcal{A}}^{\mathbb{G}_m}(U).$$

Since $DM_{\mathcal{A},\geqslant 0}$ is closed under the smash product of $DM_{\mathcal{A}}$, we have that $F \wedge M_{\mathcal{A}}^{\mathbb{G}_m}(U) \in DM_{\mathcal{A},\geqslant 0}$. Therefore $G(U) = ev_{\mathbb{G}_m}(G(U \times -))(0)$ has vanishing negative homology sheaves. So $\tau_{\geqslant 0}(G) \to G$ is an isomorphism in $D([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})])$, and then it follows that the composite map

$$ev_{\mathbb{G}_m}(\tau_{\geqslant 0}(G)) \to ev_{\mathbb{G}_m}(G) \to F$$

is an isomorphism in $DM_{\mathcal{A}}$. So

$$ev_{\mathbb{G}_m,\geqslant 0} : DM_{\mathcal{A}}[\mathcal{S}m]_{\geqslant 0} \to DM_{\mathcal{A},\geqslant 0}$$

is essentially surjective, and hence an equivalence.

Let $\mathcal{D}([\mathcal{S}m, \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))])$ be the homotopy category of $[\mathcal{S}m, \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))]$ with respect to the local model structure. The Dold-Kan correspondence induces an equivalence of categories $\Lambda : \mathcal{D}([\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]) \to \mathcal{D}([\mathcal{S}m, \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))])$. From Proposition 6.2.2 it now follows that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{D}([\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]) & \xrightarrow{\Lambda} & \mathcal{D}([\mathcal{S}m, \mathsf{Ch}_{\geqslant 0}(\mathsf{Shv}(\mathcal{A}))]) \\ \uparrow & & \uparrow \\ \mathrm{Spc}_{\mathcal{A}}[\mathcal{S}m] & \longrightarrow & DM_{\mathcal{A}}[\mathcal{S}m]_{\geqslant 0} \end{array}$$

where the vertical maps are the inclusion maps. Proposition 6.2.2 implies that the bottom horizontal arrow is essentially surjective. Since the the vertical maps and the top horizontal map are also fully faithful, we know that the bottom horizontal map is fully faithful, so it is an equivalence of categories. So we get an equivalence of categories $ev_{\mathbb{G}_m} : \mathrm{Spc}_{\mathcal{A}}[\mathcal{S}m] \to DM_{\mathcal{A},\geqslant 0}$ as was to be shown. □

From now on assume that $\mathcal{A}$ has framed correspondences in the sense of Definition 5.1.4.

7.3.4 Proposition. *Let $\mathcal{X}$ be a special enriched motivic $\mathcal{A}$-space with associated framed bispectrum $ev_{S^1,\mathbb{G}_m}(\mathcal{X}) \in SH(k)^{fr}_{\mathsf{nis}}$ as in Definition 7.1.1. Then the framed bispectrum $ev_{S^1,\mathbb{G}_m}(\mathcal{X})$ is effective, in the sense of [24, Definition 3.5] if and only if $\mathcal{X}$ is very effective, in the sense of Definition 5.1.2.*

Proof. Suppose that $\mathcal{X}$ is very effective. By Lemma 6.1.8 the enriched motivic $\mathcal{A}$-space $\mathcal{X}$ is very effective if and only if the associated framed motivic Γ-space $\mathrm{EM}(\mathcal{X})$ is very effective. If $\mathrm{EM}(\mathcal{X})$ is very effective, then this clearly implies that the framed bispectrum $ev_{S^1,\mathbb{G}_m}(\mathcal{X})$, from Definition 7.1.1, is very effective in the sense of [24, Definition 3.5].

Now let us prove the other direction. Assume that $ev_{S^1,\mathbb{G}_m}(\mathcal{X})$ is very effective in the sense of [24, Definition 3.5]. Then for every $n > 0$ the diagonal of the bisimplicial abelian group $\mathcal{X}(\mathbb{G}_m^{\wedge n})(\widehat{\Delta}^{\bullet}_{K/k})$ is contractible.

We need to show that $\mathcal{X}$ satisfies Suslin's contractibility, i.e. that for every $U \in \mathcal{S}m$, the diagonal of $\mathcal{X}(\mathbb{G}_m^{\wedge 1} \times U)(\widehat{\Delta}^{\bullet}_{K/k})$ is contractible. So take $U \in \mathcal{S}m$. Then the functor $\mathcal{X}(U \times -) : \mathcal{S}m \to \Delta^{op}\mathsf{Shv}(\mathcal{A})$ is again a special enriched motivic $\mathcal{A}$-space, so we can form the framed bispectrum $ev_{S^1,\mathbb{G}_m}(\mathcal{X}(U \times -))$. Let $ev_{S^1,\mathbb{G}_m}(\mathcal{X}(U\times -))^f$ be a levelwise local fibrant replacement of $ev_{S^1,\mathbb{G}_m}(\mathcal{X}(U\times -))$. From [24, Lemma 2.8] it follows that $ev_{S^1,\mathbb{G}_m}(\mathcal{X}(U \times -))^f$ is motivically fibrant.

By Theorem 7.1.2 we have an isomorphism

$$ev_{S^1,\mathbb{G}_m}(\mathcal{X}) \wedge \Sigma^{\infty}_{S^1,\mathbb{G}_m} U_+ \cong ev_{S^1,\mathbb{G}_m}(\mathcal{X}(U \times -))$$

in $SH(k)[1/p]$. So after inverting p, the bispectrum $ev_{S^1,\mathbb{G}_m}(\mathcal{X}(U \times -))^f$ is a motivically fibrant replacement of $ev_{S^1,\mathbb{G}_m}(\mathcal{X}) \wedge \Sigma^{\infty}_{S^1,\mathbb{G}_m} U_+$.

Since both $ev_{S^1,\mathbb{G}_m}(\mathcal{X})$ and $\Sigma^{\infty}_{S^1,\mathbb{G}_m} U_+$ are very effective, this implies that $ev_{S^1,\mathbb{G}_m}(\mathcal{X}(U \times -))^f$ is very effective in $SH(k)[1/p]$.

From Lemma 5.1.3 it now follows that $ev_{S^1,\mathbb{G}_m}(\mathcal{X}(U \times -))^f$ is very effective when regarded as an object in $SH(k)$. With [24, Lemma 3.2] it follows that the diagonal of $\mathcal{X}(\mathbb{G}_m^{\wedge 1} \times U)(\widehat{\Delta}^{\bullet}_{K/k})$ is contractible, so $\mathcal{X}$ satisfies Suslin's contractibility. □

The proof of Proposition 7.3.4 also implies the following corollary.

7.3.5 Corollary. *Let $\mathcal{X}$ be a special enriched motivic $\mathcal{A}$-space. Then $\mathcal{X}$ is very effective in the sense of Defintion 5.1.2 if and only if for every $n \geqslant 1$ the diagonal of $\mathcal{X}(\mathbb{G}_m^{\wedge n})(\widehat{\Delta}^\bullet_{K/k})$ is contractible.*

Let $\mathcal{U} : DM_{\mathcal{A}} \to SH(k)$ be the canonical forgetful functor, and let $\mathcal{L} : SH(k) \to DM_{\mathcal{A}}$ be its left adjoint. Let $DM_{\mathcal{A}}^{\mathsf{eff}}$ be the full triangulated subcategory of $DM_{\mathcal{A}}$ compactly generated by the set $\{M_{\mathcal{A}}^{\mathbb{G}_m}(U) \mid U \in \mathsf{Sm}_k\}$. See 7.3.1 for the definition of $M_{\mathcal{A}}^{\mathbb{G}_m}(U)$. Recall that $SH^{\mathsf{eff}}(k)$ is the full subcategory of $SH(k)$ generated by the suspension bispectra $\Sigma^\infty_{S^1,\mathbb{G}_m} U_+$ for $U \in \mathsf{Sm}_k$.

7.3.6 Lemma. *Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a triangulated functor. Assume that F preserves small coproducts. Let $S_{\mathcal{C}}$ be a full triangulated subcategory of $\mathcal{C}$ compactly generated by a set $\Sigma_{\mathcal{C}}$. Let $S_{\mathcal{D}}$ be a full triangulated subcategory of $\mathcal{D}$ closed under small coproducts. Assume that for every $A \in \Sigma_{\mathcal{C}}$ we have $F(A) \in S_{\mathcal{D}}$. Then for every $A \in S_{\mathcal{C}}$ we have $F(A) \in S_{\mathcal{D}}$. In particular F restricts to a triangulated functor $F : S_{\mathcal{C}} \to S_{\mathcal{D}}$.*

Proof. Consider the full subcategory $F^{-1}(S_{\mathcal{D}})$ in $\mathcal{C}$ consisting of all those objects $A \in \mathcal{C}$ for which $F(A) \in S_{\mathcal{D}}$. We need to show that $S_{\mathcal{C}} \subseteq F^{-1}(S_{\mathcal{D}})$. Since $\Sigma_{\mathcal{C}} \subseteq F^{-1}(S_{\mathcal{D}})$, it suffices due to [44, Theorem 2.1] to show that the subcategory $F^{-1}(S_{\mathcal{D}})$ is a triangulated subcategory closed under triangles and small coproducts in $\mathcal{C}$.

If we have a triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{C}$ with $X, Y \in F^{-1}(S_{\mathcal{D}})$, then

$$F(X) \to F(Y) \to F(Z) \to \Sigma F(X)$$

is a triangle in $\mathcal{D}$ with $F(X), F(Y) \in S_{\mathcal{D}}$. Since $S_{\mathcal{D}}$ is closed under triangles it follows that $F(Z) \in S_{\mathcal{D}}$, so $Z \in F^{-1}(S_{\mathcal{D}})$, so $F^{-1}(S_{\mathcal{D}})$ is closed under triangles. Since F preserves small coproducts and $S_{\mathcal{D}}$ is closed under small coproducts, it follows that $F^{-1}(S_{\mathcal{D}})$ is closed under small coproducts. Therefore $F^{-1}(S_{\mathcal{D}})$ is closed under triangles and small coproducts. We get that $S_{\mathcal{C}} \subseteq F^{-1}(S_{\mathcal{D}})$, which proves the lemma. □

7.3.7 Lemma. *If $X \in SH^{\mathsf{eff}}(k)$, then $\mathcal{L}(X) \in DM_{\mathcal{A}}^{\mathsf{eff}}$. So the functor $\mathcal{L} : SH(k) \to DM_{\mathcal{A}}$ restricts to a functor*

$$\mathcal{L}^{\mathsf{eff}} : SH^{\mathsf{eff}}(k) \to DM_{\mathcal{A}}^{\mathsf{eff}}.$$

Proof. By Lemma 7.3.2 we have $\mathcal{L}(\Sigma^{\infty}_{S^1,\mathbb{G}_m}U_+) \cong M^{\mathbb{G}_m}_{\mathcal{A}}(U) \in DM^{\mathsf{eff}}_{\mathcal{A}}$. Since the $\Sigma^{\infty}_{S^1,\mathbb{G}_m}U_+$ compactly generate $SH^{\mathsf{eff}}(k)$ the result now follows from Lemma 7.3.6. □

7.3.8 Lemma. *The triangulated functor* $\mathcal{U} : DM_{\mathcal{A}} \to SH(k)$ *preserves small coproducts.*

Proof. Let I be a set, and $\{A_i \mid i \in I\}$ a family of objects. We want to show that the canonical morphism

$$\coprod_{i\in I}\mathcal{U}(A_i) \to \mathcal{U}(\coprod_{i\in I}A_i)$$

is an isomorphism in $SH(k)$. The triangulated category $SH(k)$ is compactly generated by the set $\Sigma_{SH(k)} := \{\Sigma^{\infty}_{S^1,\mathbb{G}_m}U_+ \wedge \mathbb{G}_m^{\wedge n} \mid U \in \mathsf{Sm}_k, n \in \mathbb{Z}\}$. Thus to show that the above morphism is an isomorphism, it suffices to show that for all $G \in \Sigma_{SH(k)}$ that the map

$$\mathrm{Hom}_{SH(k)}(G, \coprod_{i\in I}\mathcal{U}(A_i)) \to \mathrm{Hom}_{SH(k)}(G, \mathcal{U}(\coprod_{i\in I}A_i))$$

is an isomorphism of abelian groups.

The objects $\Sigma^{\infty}_{S^1,\mathbb{G}_m}U_+\wedge\mathbb{G}_m^{\wedge n}$ are compact in $SH(k)$, and also each $\mathcal{L}(\Sigma^{\infty}_{S^1,\mathbb{G}_m}U_+\wedge \mathbb{G}_m^{\wedge n})$ is compact in $DM_{\mathcal{A}}$. So for all $G \in \Sigma_{SH(k)}$ we get a chain of bijections

$$\begin{aligned}\mathrm{Hom}_{SH(k)}(G, \coprod_{i\in I}\mathcal{U}(A_i)) &\cong \coprod_{i\in I}\mathrm{Hom}_{SH(k)}(G, \mathcal{U}(A_i)) \cong \coprod_{i\in I}\mathrm{Hom}_{SH(k)}(\mathcal{L}(G), A_i) \cong \\ &\cong \mathrm{Hom}_{SH(k)}(\mathcal{L}(G), \coprod_{i\in I}A_i) \cong \mathrm{Hom}_{SH(k)}(G, \mathcal{U}(\coprod_{i\in I}A_i)).\end{aligned}$$

Therefore

$$\coprod_{i\in I}\mathcal{U}(A_i) \to \mathcal{U}(\coprod_{i\in I}A_i)$$

is an isomorphism in $SH(k)$, and $\mathcal{U}$ preserves small coproducts. □

7.3.9 Lemma. *Assume that* $\mathcal{A}$ *satisfies the* $\widehat{\Delta}$*-property in the sense of Definition 5.1.4. Then for all* $X \in DM_{\mathcal{A}}$ *we have* $X \in DM^{\mathsf{eff}}_{\mathcal{A}}$ *if and only if* $\mathcal{U}(X) \in SH^{\mathsf{eff}}(k)$*.*

Proof. Our first claim is that $\mathcal{U}(M^{\mathbb{G}_m}_{\mathcal{A}}(U)) \in SH^{\mathsf{eff}}(k)$ for every $U \in \mathsf{Sm}_k$.

Let $\mathbb{1}_{\mathcal{A}} \in DM_{\mathcal{A}}$ be the monoidal unit. Then

$$\mathcal{U}(M_{\mathcal{A}}^{\mathbb{G}_m}(U)) \cong \mathcal{U}(M_{\mathcal{A}}^{\mathbb{G}_m}(U) \wedge \mathbb{1}_{\mathcal{A}}).$$

We can regard $SH^{\mathsf{eff}}(k)[1/p]$ as a full subcategory of $SH^{\mathsf{eff}}(k)$. From Lemma 5.1.3 it follows that the adjunction $\mathcal{U} : DM_{\mathcal{A}} \leftrightarrows SH(k) : \mathcal{L}$ restricts to an adjunction

$$\mathcal{U} : DM_{\mathcal{A}} \leftrightarrows SH(k)[1/p] : \mathcal{L}.$$

By [34, Appendix B, Corollary B.2] the suspension spectrum $\Sigma_{S^1,\mathbb{G}_m} U_+$ is strongly dualizable in $SH(k)[1/p]$. So we can apply [3, Lemma 4.6] to get an isomorphism

$$\mathcal{U}(M_{\mathcal{A}}^{\mathbb{G}_m}(U) \wedge \mathbb{1}_{\mathcal{A}}) \cong \mathcal{U}(\mathcal{L}(\Sigma^{\infty}_{S^1,\mathbb{G}_m} U_+) \wedge \mathbb{1}_{\mathcal{A}}) \cong \Sigma_{S^1,\mathbb{G}_m} U_+ \wedge \mathcal{U}(\mathbb{1}_{\mathcal{A}})$$

in $SH(k)[1/p]$. Now $\Sigma_{S^1,\mathbb{G}_m} U_+$ is effective, and $SH^{\mathsf{eff}}(k)$ is closed under the $\wedge$ product, so to show that $\mathcal{U}(M_{\mathcal{A}}^{\mathbb{G}_m}(U)) \in SH^{\mathsf{eff}}(k)$, we now just need to show that $\mathcal{U}(\mathbb{1}_{\mathcal{A}}) \in SH^{\mathsf{eff}}(k)$. The bispectrum $\mathcal{U}(\mathbb{1}_{\mathcal{A}})$ is isomorphic to the bispectrum $M_{\mathcal{A}}^{\mathbb{G}_m}(pt) = (M_{\mathcal{A}}(\mathbb{G}_m^{\wedge j}))_{j \geqslant 0}$. By construction, the latter bispectrum is a framed bispectrum in the sense of [24], because $\mathcal{A}$ has framed correspondences. Since $\mathcal{A}$ also has the $\widehat{\Delta}$-property, the bispectrum $M_{\mathcal{A}}^{\mathbb{G}_m}(pt)$ is effective in the sence of [24, Definition 3.5]. And by [24, Theorem 3.6] this implies that $\mathcal{U}(\mathbb{1}_{\mathcal{A}}) \in SH^{\mathsf{eff}}(k)$. So we now have for every $U \in \mathsf{Sm}_k$ that $\mathcal{U}(M_{\mathcal{A}}^{\mathbb{G}_m}(U)) \in SH^{\mathsf{eff}}(k)$.

Due to Lemma 7.3.8 we can now apply Lemma 7.3.6 to get for every $E \in DM_{\mathcal{A}}^{\mathsf{eff}}$ that $\mathcal{U}(E) \in SH^{\mathsf{eff}}(k)$ (This argument is similar to an argument used in the proof of [3, Corollary 5.4]). So the functor $\mathcal{U} : DM_{\mathcal{A}} \to SH(k)$ restricts to a functor $\mathcal{U}^{\mathsf{eff}} : DM_{\mathcal{A}}^{\mathsf{eff}} \to SH^{\mathsf{eff}}(k)$. This shows one direction of the lemma. Let us now show the other direction of the lemma. According to Lemma 7.3.7 the functor $\mathcal{L} : SH(k) \to DM_{\mathcal{A}}$ restricts to a functor $\mathcal{L}^{\mathsf{eff}} : SH^{\mathsf{eff}}(k) \to DM_{\mathcal{A}}^{\mathsf{eff}}$. The functor $\mathcal{L}^{\mathsf{eff}}$ is left adjoint to $\mathcal{U}^{\mathsf{eff}}$.

By [53, Remark 2.1] the inclusion functors $\iota : DM_{\mathcal{A}}^{\mathsf{eff}} \to DM_{\mathcal{A}}$ and $\iota : SH^{\mathsf{eff}}(k) \to SH(k)$ have right adjoints $r_0 : DM_{\mathcal{A}} \to DM_{\mathcal{A}}^{\mathsf{eff}}$ and $r_0 : SH(k) \to SH^{\mathsf{eff}}(k)$.

The following diagrams commute:

$$\begin{array}{ccc} DM_{\mathcal{A}}^{\mathsf{eff}} & \xleftarrow{\mathcal{L}^{\mathsf{eff}}} & SH^{\mathsf{eff}}(k) \\ \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} \\ DM_{\mathcal{A}} & \xleftarrow{\mathcal{L}} & SH(k) \end{array} \qquad \begin{array}{ccc} DM_{\mathcal{A}}^{\mathsf{eff}} & \xrightarrow{\mathcal{U}^{\mathsf{eff}}} & SH^{\mathsf{eff}}(k) \\ \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} \\ DM_{\mathcal{A}} & \xrightarrow{\mathcal{U}} & SH(k) \end{array}$$

From the commutativity of the left diagram it follows by adjunction that also the following diagram commutes:

$$\begin{array}{ccc} DM^{\mathsf{eff}}_{\mathcal{A}} & \xrightarrow{\mathcal{U}^{\mathsf{eff}}} & SH^{\mathsf{eff}}(k) \\ {\scriptstyle r_0}\uparrow & & \uparrow{\scriptstyle r_0} \\ DM_{\mathcal{A}} & \xrightarrow{\mathcal{U}} & SH(k) \end{array}$$

Take $X \in DM_{\mathcal{A}}$ such that $\mathcal{U}(X) \in SH^{\mathsf{eff}}(k)$. We need to show that $X \in DM^{\mathsf{eff}}_{\mathcal{A}}$. Since $\mathcal{U}(X) \in SH^{\mathsf{eff}}(k)$ the counit ε of the adjunction $\iota : SH^{\mathsf{eff}}(k) \rightleftarrows SH(k) : r_0$ is an isomorphism at $\mathcal{U}(X)$. So $\varepsilon_{\mathcal{U}(X)} : \iota(r_0(\mathcal{U}(X))) \xrightarrow{\sim} \mathcal{U}(X)$ is an isomorphism in $SH(k)$. By the commutativity of the above diagram this implies that the composite

$$\mathcal{U}(\iota(r_0(X))) = \iota(\mathcal{U}^{\mathsf{eff}}(r_0(X))) \cong \iota(r_0(\mathcal{U}(X))) \xrightarrow{\sim} \mathcal{U}(X)$$

is an isomorphism in $SH(k)$. But this composite is equal to $\mathcal{U}(\varepsilon_X)$ where $\varepsilon_X : \iota(r_0(X)) \to X$ is the counit map of the adjunction $\iota : DM^{\mathsf{eff}}_{\mathcal{A}} \rightleftarrows DM_{\mathcal{A}} : r_0$. Now the forgetful functor $\mathcal{U} : DM_{\mathcal{A}} \to SH(k)$ is conservative, so if $\mathcal{U}(\varepsilon_X)$ is an isomorphism in $SH(k)$, then also ε_X is an isomorphism in $DM_{\mathcal{A}}$. But this then implies that X lies in $DM^{\mathsf{eff}}_{\mathcal{A}}$, which proves the lemma. □

We have an evaluation functor

$$ev_{\mathbb{G}_m} : \mathsf{Ch}([\mathcal{S}m, \mathsf{Shv}(\mathcal{A})]) \to \mathsf{Sp}_{\mathbb{G}_m}(\mathsf{Ch}(\mathsf{Shv}(\mathcal{A}))).$$

For $\mathcal{X} \in [\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})]$ we define $ev_{\mathbb{G}_m}(\mathcal{X}) := ev_{\mathbb{G}_m}(\Lambda(\mathcal{X}))$.

7.3.10 Lemma. *For $\mathcal{X} \in \mathrm{Spc}_{\mathcal{A}}[\mathcal{S}m]$ we have a canonical isomorphism in $SH(k)$*

$$\mathcal{U}(ev_{\mathbb{G}_m}(\mathcal{X})) \xrightarrow{\sim} ev_{S^1,\mathbb{G}_m}(\mathcal{X}).$$

Proof. Let $\mathbb{Z}^{S^n}$ be the reduced free simplicial abelian group on the pointed simplicial set S^n. The bispectrum $ev_{S^1,\mathbb{G}_m}(\mathcal{X}) = ev_{S^1,\mathbb{G}_m}(\mathrm{EM}(\mathcal{X}))$ can be computed in the (n,m)-th level as

$$ev_{S^1,\mathbb{G}_m}(\mathcal{X})[n](m) = \mathbb{Z}^{S^n} \otimes \mathcal{X}(\mathbb{G}_m^{\wedge m})$$

in $\mathcal{M}$. The bispectrum $\mathcal{U}(ev_{\mathbb{G}_m}(\mathcal{X})) = \mathcal{U}(ev_{\mathbb{G}_m}(\Lambda(\mathcal{X})))$ can be computed in the (n,m)-th level as

$$\mathcal{U}(ev_{\mathbb{G}_m}(\mathcal{X}))[n](m) = DK(DK^{-1}(\mathcal{X})(\mathbb{G}_m^{\wedge m})[m])$$

in $\mathcal{M}$. We claim that there is a natural homotopy equivalence

$$DK(DK^{-1}(\mathcal{X})(\mathbb{G}_m^{\wedge m})[m]) \to \mathbb{Z}^{S^n} \otimes \mathcal{X}(\mathbb{G}_m^{\wedge m})$$

in $\mathcal{M}$. The chain complex $DK^{-1}(\mathbb{Z}^{S^n})$ is $\mathbb{Z}$ in degree n and 0 in all other degrees. It follows for every chain complex A that $A[n] \cong A \otimes DK^{-1}(\mathbb{Z}^{S^n})$. According to [42] the Dold-Kan correspondence preserves tensor products up to homotopy equivalence. We then get a homotopy equivalence

$$DK(\Lambda(\mathcal{X})(\mathbb{G}_m^{\wedge m})[m]) \cong DK(DK^{-1}(\mathcal{X})(\mathbb{G}_m^{\wedge m}) \otimes DK^{-1}(\mathbb{Z}^{S^n})) \to \\ DK(DK^{-1}(\mathcal{X}(\mathbb{G}_m^{\wedge m}) \otimes \mathbb{Z}^{S^n})) \cong \mathcal{X}(\mathbb{G}_m^{\wedge m}) \otimes \mathbb{Z}^{S^n}.$$

These maps assemble together into an isomorphism $\mathcal{U}(ev_{\mathbb{G}_m}(\mathcal{X})) \xrightarrow{\sim} ev_{S^1,\mathbb{G}_m}(\mathcal{X})$ in $SH(k)$. □

Let $\mathrm{Spc}_{\mathcal{A}}^{\mathsf{veff}}[\mathcal{S}m]$ be the full subcategory of $\mathrm{Spc}_{\mathcal{A}}[\mathcal{S}m]$ consisting of the very effective special enriched motivic $\mathcal{A}$-spaces. By definition it is then also full subcategory of $\mathcal{D}([\mathcal{S}m, \Delta^{op}\mathsf{Shv}(\mathcal{A})])$ consisting of the very effective special enriched motivic $\mathcal{A}$-spaces.

7.3.11 Theorem. *Assume that $\mathcal{A}$ satisfies the $\widehat{\Delta}$-property in the sense of Definition 5.1.4. Then the naive $\mathbb{G}_m$-evaluation functor induces an equivalence of categories*

$$ev_{\mathbb{G}_m} : \mathrm{Spc}_{\mathcal{A}}^{\mathsf{veff}}[\mathcal{S}m] \to DM_{\mathcal{A},\geqslant 0}^{\mathsf{eff}}.$$

Proof. By Theorem 7.3.3 we have an equivalence

$$ev_{\mathbb{G}_m} : \mathrm{Spc}_{\mathcal{A}}[\mathcal{S}m] \to DM_{\mathcal{A},\geqslant 0}.$$

So we just need to show for $\mathcal{X} \in \mathrm{Spc}_{\mathcal{A}}[\mathcal{S}m]$ that $\mathcal{X} \in \mathrm{Spc}_{\mathcal{A}}^{\mathsf{veff}}[\mathcal{S}m]$ if and only if $ev_{\mathbb{G}_m}(\mathcal{X}) \in DM_{\mathcal{A},\geqslant 0}^{\mathsf{eff}}$. By Proposition 7.3.4 we know that $\mathcal{X} \in \mathrm{Spc}_{\mathcal{A}}^{\mathsf{veff}}[\mathcal{S}m]$ if and only if $ev_{S^1,\mathbb{G}_m}(\mathcal{X}) \in SH_{\mathsf{nis}}^{fr}(k)$ is effective. By [24, Theorem 3.6] this is the case if and only if $ev_{S^1,\mathbb{G}_m}(\mathcal{X})$ lies in $SH^{\mathsf{eff}}(k)$. By Lemma 7.3.10 we have a canonical isomorphism

$$ev_{S^1,\mathbb{G}_m}(\mathcal{X}) \cong \mathcal{U}(ev_{\mathbb{G}_m}(\mathcal{X}))$$

in $SH(k)$. So $ev_{S^1,\mathbb{G}_m}(\mathcal{X}) \in SH^{\mathsf{eff}}(k)$ if and only if $\mathcal{U}(ev_{\mathbb{G}_m}(\mathcal{X})) \in SH^{\mathsf{eff}}(k)$ and by Lemma 7.3.9 this is the case if and only if $ev_{\mathbb{G}_m}(\mathcal{X}) \in DM_{\mathcal{A}}^{\mathsf{eff}}$, which proves the theorem. □

7.4 Reconstructing $SH^{\mathsf{veff}}(k)_{\mathbb{Q}}$

In this section we apply the techniques and results from the previous sections to give new models for the stable motivic homotopy category of effective and very effective motivic bispectra with rational coefficients. It also requires the reconstruction theorem by [19] and the theory Milnor-Witt correspondences [3, 6, 7, 11, 16, 17].

Let $\widetilde{\mathrm{Cor}}$ be the category of finite Milnor-Witt correspondences in the sense of [7]. Then $\widetilde{\mathrm{Cor}}$ is a strict V-category of correspondences satisfying the cancellation property (See [16] for details). Furthermore it has framed correspondences by [11]. It also satisfies the $\widehat{\Delta}$-property by [3].

Denote by $SH(k)_{\mathbb{Q}}$ the category of motivic bispectra E whose sheaves of stable motivic homotopy groups $\pi^{\mathbb{A}^1}_{*,*}(E)$ are sheaves of rational vector spaces. The category $SH(k)_{\mathbb{Q}}$ is also called the *rational stable motivic homotopy category*. It is the homotopy category of a stable model structure in which weak equivalences are those morphisms of bispectra $f : E \to E'$ for which $\pi^{\mathbb{A}^1}_{*,*}(f) \otimes \mathbb{Q}$ is an isomorphism. Let $SH(k)_{\mathbb{Q},\geqslant 0}$ be the full subcategory of $SH(k)_{\mathbb{Q}}$ on the connective objects. Here a bispectrum object $X \in SH(k)_{\mathbb{Q}}$ with rational stable $\mathbb{A}^1$-homotopy groups $\underline{\pi}^{\mathbb{A}^1}_{p,q}(X) \otimes \mathbb{Q}$ is called *connective*, if $\underline{\pi}^{\mathbb{A}^1}_{p,q}(X) \otimes \mathbb{Q} \cong 0$ for all $p < q$.

Throughout this section we assume the base field k to be perfect of characteristic different from 2. The assumption on the characteristic is typical when working with finite Milnor–Witt correspondences. A theorem of Garkusha [19, Theorem 5.5] states that the forgetful functor $\mathcal{U} : DM_{\widetilde{\mathrm{Cor}},\mathbb{Q}} \to SH(k)_{\mathbb{Q}}$ is an equivalence of categories. This theorem was actually proven under the assumption that k is also infinite. The latter assumption is redundant due to [13, A.27] saying that the main result of [22] about strict invariance for Nisnevich sheaves with framed transfers is also true for finite fields.

7.4.1 Definition. We define $\mathrm{Spc}_{\widetilde{\mathrm{Cor}},\mathbb{Q}}[\mathcal{S}m]$, respectively $DM_{\widetilde{\mathrm{Cor}},\mathbb{Q},\geqslant 0}$ to be the category $\mathrm{Spc}_{\mathcal{A}}[\mathcal{S}m]$, respectively $DM_{\mathcal{A},\geqslant 0}$, associated to the category of correspondences $\mathcal{A} = \widetilde{\mathrm{Cor}} \otimes \mathbb{Q}$. We call $\mathrm{Spc}_{\widetilde{\mathrm{Cor}},\mathbb{Q}}[\mathcal{S}m]$ the category of *rational enriched motivic* $\widetilde{\mathrm{Cor}}$*-spaces*.

The following theorem says that the special rational enriched motivic $\widetilde{\mathrm{Cor}}$-spaces recover $SH(k)_{\mathbb{Q},\geqslant 0}$.

7.4.2 Theorem. *The* $(S^1, \mathbb{G}_m)$*-evaluation functor is an equivalence of categories*

$$ev_{S^1,\mathbb{G}_m} : \mathrm{Spc}_{\widetilde{\mathrm{Cor}},\mathbb{Q}}[\mathcal{S}m] \to SH(k)_{\mathbb{Q},\geqslant 0}.$$

Proof. By Theorem 7.3.3 the $\mathbb{G}_m$-evalulation functor is an equivalence of categories

$$ev_{\mathbb{G}_m} : \mathrm{Spc}_{\widetilde{\mathrm{Cor}},\mathbb{Q}}[\mathcal{S}m] \to DM_{\widetilde{\mathrm{Cor}},\mathbb{Q},\geqslant 0}.$$

By [19, Theorem 5.5] the forgetful functor $\mathcal{U} : DM_{\widetilde{\mathrm{Cor}},\mathbb{Q}} \to SH(k)_{\mathbb{Q}}$ is an equivalence of categories, and this implies that the forgetful functor $\mathcal{U} : DM_{\widetilde{\mathrm{Cor}},\mathbb{Q},\geqslant 0} \to SH(k)_{\mathbb{Q},\geqslant 0}$ is an equivalence of categories. So by Lemma 7.3.10 the $(S^1, \mathbb{G}_m)$-evaluation functor

$$ev_{S^1,\mathbb{G}_m} : \mathrm{Spc}_{\widetilde{\mathrm{Cor}},\mathbb{Q}}[\mathcal{S}m] \to SH(k)_{\mathbb{Q},\geqslant 0}$$

is an equivalence of categories. □

Let $SH^{\mathsf{veff}}(k)_{\mathbb{Q}}$ be the full subcategory of $SH(k)_{\mathbb{Q}}$ on the very effective bispectra. Here an object $X \in SH(k)_{\mathbb{Q}}$ is said to be *very effective* if it is both effective and connective:

$$SH^{\mathsf{veff}}(k)_{\mathbb{Q}} = SH^{\mathsf{eff}}(k)_{\mathbb{Q}} \cap SH(k)_{\mathbb{Q},\geqslant 0}.$$

7.4.3 Definition. We define $\mathrm{Spc}^{\mathsf{veff}}_{\widetilde{\mathrm{Cor}},\mathbb{Q}}[\mathcal{S}m]$, respectively $DM^{\mathsf{eff}}_{\widetilde{\mathrm{Cor}},\mathbb{Q},\geqslant 0}$, to be the category $\mathrm{Spc}^{\mathsf{veff}}_{\mathcal{A}}[\mathcal{S}m]$, respectively $DM^{\mathsf{eff}}_{\mathcal{A},\geqslant 0}$, associated to the category of correspondences $\mathcal{A} = \widetilde{\mathrm{Cor}} \otimes \mathbb{Q}$. We call $\mathrm{Spc}^{\mathsf{veff}}_{\widetilde{\mathrm{Cor}},\mathbb{Q}}[\mathcal{S}m]$ the category of *very effective rational enriched motivic* $\widetilde{\mathrm{Cor}}$*-spaces.*

We finish with the following result stating that very effective rational enriched motivic $\widetilde{\mathrm{Cor}}$-spaces recover $SH^{\mathsf{veff}}(k)_{\mathbb{Q}}$.

7.4.4 Theorem. *The* $(S^1, \mathbb{G}_m)$*-evaluation functor is an equivalence of categories*

$$ev_{S^1,\mathbb{G}_m} : \mathrm{Spc}^{\mathsf{veff}}_{\widetilde{\mathrm{Cor}},\mathbb{Q}}[\mathcal{S}m] \to SH^{\mathsf{veff}}(k)_{\mathbb{Q}}.$$

Proof. By Theorem 7.4.2 the $(S^1, \mathbb{G}_m)$-evaluation functor is an equivalence of categories

$$ev_{S^1,\mathbb{G}_m} : \mathrm{Spc}_{\widetilde{\mathrm{Cor}},\mathbb{Q}}[\mathcal{S}m] \to SH(k)_{\mathbb{Q},\geqslant 0}.$$

We want to show that it restricts to an equivalence of categories

$$ev_{S^1,\mathbb{G}_m} : \mathrm{Spc}^{\mathsf{veff}}_{\widetilde{\mathrm{Cor}},\mathbb{Q}}[\mathcal{S}m] \to SH^{\mathsf{veff}}(k)_{\mathbb{Q}}.$$

For this we just need to show that a special enriched motivic $\mathcal{A}$-space $\mathcal{X}$ is very effective if and only if $ev_{S^1,\mathbb{G}_m}(\mathcal{X})$ is very effective in $SH(k)$.

According to Proposition 7.3.4 the special enriched motivic $\mathcal{A}$-space $\mathcal{X}$ is very effective if and only if the framed bispectrum $ev_{S^1,\mathbb{G}_m}(\mathcal{X})$ is effective in $SH(k)^{fr}_{\mathsf{nis}}$. By [24, Theorem 3.6] this is the case if and only if $ev_{S^1,\mathbb{G}_m}(\mathcal{X})$ is effective in $SH(k)$. This concludes the proof of the theorem. □

We conclude this project with the following remarks. This project provides new models for Voevodsky's fundamental categories of big motives $DM_{\mathcal{A}}$, $DM_{\mathcal{A},\geqslant 0}$ and $DM^{\mathsf{eff}}_{\mathcal{A},\geqslant 0}$ as well as for the categories $SH(k)_{\mathbb{Q},\geqslant 0}$ and $SH^{\mathsf{veff}}(k)_{\mathbb{Q}}$. In the future we expect the techniques developed in this project to be applicable to non-linear categories of motives. Other applications are expected in equivariant motivic homotopy theory. Our approach also demonstrates the importance of enriched categories in motivic homotopy theory.

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