

RESEARCH ARTICLE

Correspondences and stable homotopy theory

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EP/W012030/1**Abstract**

A general method of producing correspondences and spectral categories out of symmetric ring objects in general categories is given. As an application, stable homotopy theory of spectra SH is recovered from modules over a commutative symmetric ring spectrum defined in terms of framed correspondences over an algebraically closed field. Another application recovers stable motivic homotopy theory $SH(k)$ from spectral modules over associated spectral categories.

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Contents

1. INTRODUCTION	124
2. GRADED SYMMETRIC SEQUENCES	126
3. SYMMETRIC SPECTRA	131
4. RECONSTRUCTING STABLE HOMOTOPY THEORY SH	135
5. SPECTRAL CATEGORIES ASSOCIATED TO SYMMETRIC RING SPECTRA	144
6. ENRICHED MOTIVIC HOMOTOPY THEORY	146
REFERENCES.	153

1 | INTRODUCTION

Algebraic Kasparov K -theory is stable homotopy theory of nonunital k -algebras Alg_k [9, 10]. In detail, we start with the category $U_*(\text{Alg}_k)$ of pointed simplicial functors from Alg_k to pointed simplicial sets, where each algebra $A \in \text{Alg}_k$ is regarded as the representable object

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$rA = \text{Hom}_{\text{Alg}_k}(A, -)$. The category $U_*(\text{Alg}_k)$ comes equipped with a motivic model structure. Let S^1 be the standard simplicial circle. Stabilization of $U_*(\text{Alg}_k)$ in the S^1 -direction leads to a stable motivic model category $Sp_{S^1}(\text{Alg}_k)$ of S^1 -spectra in $U_*(\text{Alg}_k)$. The S^1 -suspension spectrum $\Sigma_{S^1}^\infty rA$ of an algebra A is computed as the fibrant spectrum $\mathbb{K}(A, -)$ [10], where $\mathbb{K}(A, B)$ is the algebraic Kasparov KK -theory spectrum of $A, B \in \text{Alg}_k$ defined in [9]. A key nonunital homomorphism involved in the computation is $\sigma_A : JA \rightarrow \Omega A$, where $JA = \text{Ker}(TA \rightarrow A)$ with $TA = A \oplus A^{\otimes 2} \oplus \dots$ the algebraic tensor algebra and $\Omega A = (x^2 - x)A[x]$. The morphism $r(\sigma_A)$ is a motivic equivalence in $U_*(\text{Alg}_k)$.

In stable motivic homotopy theory the suspension \mathbb{P}^1 -spectrum $\Sigma_{\mathbb{P}^1}^\infty X_+$ of a smooth algebraic variety $X \in \text{Sm}/k$ is computed in [19, Theorem 4.1] as a (positively) fibrant spectrum $M_{\mathbb{P}^1}(X)_f$. Locally in the Nisnevich topology it is equal to the \mathbb{P}^1 -spectrum $(\text{Fr}(\Delta_k^* \times -, X), \text{Fr}(\Delta_k^* \times -, X \times T), \dots)$, where $T = \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$ and $\text{Fr}(-, X)$ is the sheaf of stable framed correspondences introduced by Voevodsky [35] in 2001. A key morphism involved in the computation is the canonical motivic equivalence $\sigma_X : X_+ \wedge \mathbb{P}^{\wedge 1} \rightarrow X_+ \wedge T$ in the category of pointed motivic spaces \mathcal{M} . Here $\mathbb{P}^{\wedge 1}$ is the pointed projective line (\mathbb{P}^1, ∞) .

The computations of $\Sigma_{S^1}^\infty rA$ and $\Sigma_{\mathbb{P}^1}^\infty X_+$ share lots of common properties [11]. Inspired by these computations, two categorical constructions are introduced in this paper. The first one produces correspondences associated with two objects $P, T \in \mathcal{C}$ and ring objects of the category of symmetric sequences \mathcal{C}^Σ , where \mathcal{C} is a symmetric monoidal category with finite colimits and zero object 0. The correspondences are constructed between objects of an arbitrary full subcategory \mathcal{B} of \mathcal{C} closed under monoidal product. See Theorem 2.4 for details. After Voevodsky, correspondences play a prominent role in motivic homotopy theory. In particular, they are necessary for computing motivic homotopy types as well as for producing triangulated categories of motives. For example, Voevodsky's fundamental graded category of framed correspondences $\text{Fr}_*(k)$ is recovered from Theorem 2.4 if we take $\mathcal{C} = \mathcal{M}$, $\mathcal{B} = \{X_+ \mid X \in \text{Sm}/k\}$, $P = (\mathbb{P}^1, \infty)$, $T = \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$, and the commutative ring object (S^0, T, T^2, \dots) in \mathcal{M}^Σ . Next, if $\sigma : P \rightarrow T$ is a morphism in \mathcal{C} then the second categorical construction produces spectral categories, that is, categories enriched over symmetric S^1 -spectra $Sp_{S^1}^\Sigma$, which are used for applications mentioned below. See Theorem 5.2 for details. Spectral categories are of great utility in classical and equivariant stable homotopy theory (see, e.g., [21, 33]) as well as in constructing triangulated categories of K -motives [16, 17].

The spectral categories and symmetric spectra constructed in this paper lead to the following applications. We first introduce the stable homotopy category SH_k over an arbitrary field k in Section 4. It is defined as the homotopy category of S_k -modules, where S_k is a commutative symmetric ring spectrum defined over k . Then one reconstructs in Theorem 4.15 the stable homotopy theory of S^1 -spectra SH as SH_k if k is algebraically closed (we need to invert the exponential characteristic). Moreover, this reconstruction is given by a functor taking a symmetric S^1 -spectrum N to its symmetric framed motive $M_{fr}^\Sigma(N)$ introduced in this paper (see Definition 4.9). Another application gives yet another genuinely local model of stable motivic homotopy theory $SH(k)$ (in addition to [20]) and, more generally, a local model for the category of E -modules in $SH(k)$, where E is a symmetric Thom ring spectrum. See Theorem 6.7 and Corollary 6.9 for details. For the latter result, we apply Theorem 5.2 to produce a spectral category \mathcal{O}_Δ^E using data as above: $\mathcal{C} = \mathcal{M}$, $\mathcal{B} = \{X_+ \mid X \in \text{Sm}/k\}$, $P = (\mathbb{P}^1, \infty)$, $T = \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$. We also use the enriched motivic homotopy theory of motivic spectral categories developed in [16, 17]. The reader will also find reconstruction theorems for E -modules of $SH(k)$ in terms of ∞ -categories of "tangentially framed correspondences" in [7, 8]. The approach presented in Section 6 is combinatorial in the sense

that it is based on explicit spectral categories produced by Theorem 5.2 and modules over them defined in terms of original Voevodsky's framed correspondences [35]. This approach also produces triangulated categories of E -framed motives out of spectral categories of Theorem 5.2. They are constructed in a similar fashion as the classical Voevodsky category of motives or the category of K -motives in the sense of [16, 17].

The author also expects further applications of (spectral) categories of correspondences, constructed in this paper for quite general categories, in classical and algebraic Kasparov K -theory as well as in noncommutative algebraic geometry. This will be the material of subsequent papers. In this paper he has concentrated on applications in classical and motivic stable homotopy theory.

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Notation. Throughout the paper, we employ the following notation.

k and pt	A field of exponential characteristic e and $\text{Spec}(k)$
Sm/k	The category of smooth separated schemes of finite type
$\text{Fr}_0(k)$ or Sm/k_+	The category of framed correspondences of level zero
$\text{Shv}_*(\text{Sm}/k)$	The closed symmetric monoidal category of pointed Nisnevich sheaves
$\mathcal{M} = \Delta^{\text{op}}\text{Shv}_*(\text{Sm}/k)$	The category of pointed motivic spaces, also known as the category of pointed simplicial Nisnevich sheaves
\mathbf{S}	The category of pointed simplicial sets

2 | GRADED SYMMETRIC SEQUENCES

Let (C, \wedge, \vee, S) be a symmetric monoidal category with finite coproducts, unit object S and zero object 0 . We assume that a canonical morphism

$$v : \bigvee_{i \in I} (A_i \wedge B) \rightarrow \left(\bigvee_{i \in I} A_i \right) \wedge B$$

is an isomorphism for any finite set I and $A_i, C \in C^\dagger$. In particular, if $I = \emptyset$ then $0 \wedge B \cong B \wedge 0 \cong 0$.

In what follows, we shall also assume that C has finite colimits. By [22, section 7], the category of symmetric sequences C^Σ is symmetric monoidal with

$$(X \wedge Y)_n = \bigvee_{p+q=n} \Sigma_n \times_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q.$$

The symmetric sequence $(S, 0, 0, \dots)$ is a monoidal unit of C^Σ . This notation needs some explanation (we follow [22, section 7]). Given a finite set Γ and an object $A \in C$, $\Gamma \times A$ is the coproduct of $|\Gamma|$ copies of A . If Γ is a group, then $\Gamma \times A$ has an obvious left Γ -action; $\Gamma \times A$ is the free Γ -object on A . Note that a Γ -action on A is then equivalent to a map $\Gamma \times A \rightarrow A$ satisfying the usual unit and associativity conditions. Also, if Γ admits a right action by a group Γ' , and A is a left Γ' -object, then we can form $\Gamma \times_{\Gamma'} A$ as the colimit of the Γ' -action on $\Gamma \times A$, where $\alpha \in \Gamma'$ takes the copy of A corresponding to $\beta \in \Gamma$ to the copy of A corresponding to $\beta\alpha^{-1}$ by the action of α .

[†] Such a category C is also known as a distributive symmetric monoidal category.

Given two maps $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ in \mathcal{C}^Σ , the $\Sigma_p \times \Sigma_q$ -equivariant maps $f_p \wedge g_q : X_p \wedge Y_q \rightarrow X'_p \wedge Y'_q$ yield the *tensor product morphism* $f \wedge g : X \wedge Y \rightarrow X' \wedge Y'$ in \mathcal{C}^Σ .

The twist isomorphism $twist : X \wedge Y \rightarrow Y \wedge X$ for $X, Y \in \mathcal{C}^\Sigma$ is the natural map taking the summand $(\alpha, X_p \wedge Y_q)$ to the summand $(\alpha\chi_{q,p}, Y_q \wedge X_p)$ for $\alpha \in \Sigma_{p+q}$, where $\chi_{q,p} \in \Sigma_{p+q}$ is the (q, p) -shuffle given by $\chi_{q,p}(i) = i + p$ for $1 \leq i \leq q$ and $\chi_{q,p}(i) = i - q$ for $q < i \leq p + q$. It is worth noting that the map defined without the shuffle permutation is not a map of symmetric sequences.

Definition 2.1 (Ring objects). In what follows we shall refer to monoid objects in \mathcal{C}^Σ as *ring objects*. There is a standard description of a ring object E in \mathcal{C}^Σ that we need later:

- ◇ a sequence of objects $E_n \in \mathcal{C}$ for $n \geq 0$;
- ◇ a left action of the symmetric group Σ_n on E_n for each $n \geq 0$;
- ◇ $\Sigma_n \times \Sigma_m$ -equivariant *multiplication maps*

$$\mu_{n,m} : E_n \wedge E_m \rightarrow E_{n+m}$$

for $n, m \geq 0$, and

- ◇ a *unit map* $\iota_0 : S \rightarrow E_0$.

These data are subject to the following conditions:

(Associativity) The square

$$\begin{array}{ccc} E_n \wedge E_m \wedge E_p & \xrightarrow{\text{id} \wedge \mu_{m,p}} & E_n \wedge E_{m+p} \\ \mu_{n,m} \wedge \text{id} \downarrow & & \downarrow \mu_{n,m+p} \\ E_{n+m} \wedge E_p & \xrightarrow{\mu_{n+m,p}} & E_{n+m+p} \end{array}$$

commutes for all $n, m, p \geq 0$.

(Unit) The two composites

$$E_n \cong E_n \wedge S \xrightarrow{E_n \wedge \iota_0} E_n \wedge E_0 \xrightarrow{\mu_{n,0}} E_n$$

$$E_n \cong S \wedge E_n \xrightarrow{\iota_0 \wedge E_n} E_0 \wedge E_n \xrightarrow{\mu_{0,n}} E_n$$

are the identity for all $n \geq 0$.

A morphism $f : E \rightarrow E'$ of ring objects consists of Σ_n -equivariant maps $f_n : E_n \rightarrow E'_n$ for $n \geq 0$, which are compatible with the multiplication and unit maps in the sense that $f_{n+m} \circ \mu_{n,m} = \mu_{n,m} \circ (f_n \wedge f_m)$ for all $n, m \geq 0$, and $f_0 \circ \iota_0 = \iota_0$.

A ring object E is commutative if the square

$$\begin{array}{ccc} E_n \wedge E_m \wedge E_p & \xrightarrow{\text{id} \wedge \mu_{m,p}} & E_n \wedge E_{m+p} \\ \mu_{n,m} \wedge \text{id} \downarrow & & \downarrow \mu_{n,m+p} \\ E_{n+m} \wedge E_p & \xrightarrow{\mu_{n+m,p}} & E_{n+m+p} \end{array}$$

commutes for all $n, m \geq 0$.

Definition 2.2 (Modules). A *right module* M over a ring object $E \in \mathcal{C}^\Sigma$ is defined in a standard way. There is an equivalent definition that we need later:

- ◊ a sequence of objects $M_n \in \mathcal{C}$ for $n \geq 0$,
- ◊ a left action of the symmetric group Σ_n on M_n for each $n \geq 0$, and
- ◊ $\Sigma_n \times \Sigma_m$ -equivariant action maps $\alpha_{n,m} : M_n \wedge E_m \rightarrow M_{n+m}$ for $n, m \geq 0$.

The action maps have to be associative and unital in the sense that the following diagrams commute

$$\begin{array}{ccc}
 M_n \wedge E_m \wedge E_p & \xrightarrow{M_n \wedge \mu_{m,p}} & M_n \wedge E_{m+p} \\
 \alpha_{n,m} \wedge E_p \downarrow & & \downarrow \alpha_{n,m+p} \\
 M_{n+m} \wedge E_p & \xrightarrow{\alpha_{n+m,p}} & M_{n+m+p}
 \end{array}
 \qquad
 \begin{array}{ccc}
 M_n \cong M_n \wedge S & \xrightarrow{M_n \wedge \iota_0} & M_n \wedge E_0 \\
 \text{id}_{M_n} \searrow & & \downarrow \alpha_{n,0} \\
 & & M_n
 \end{array}$$

for all $n, m, p \geq 0$. A *morphism* $f : M \rightarrow N$ of right E -modules consists of Σ_n -equivariant maps $f_n : M_n \rightarrow N_n$ for $n \geq 0$, which are compatible with the action maps in the sense that $f_{n+m} \circ \alpha_{n,m} = \alpha_{n,m} \circ (f_n \wedge E_m)$ for all $n, m \geq 0$. We denote the category of right E -modules by $\text{Mod } E$.

The following definition is motivated by the fundamental graded category of Voevodsky’s framed correspondences, in which the role of \mathcal{C} is played by the category of pointed motivic spaces \mathcal{M}, \mathcal{B} is given by pointed motivic spaces of the form X_+ with $X \in \text{Sm}/k, P$ is the pointed projective line (\mathbb{P}^1, ∞) , and $E = (\text{pt}_+, T, T^2, \dots)$ with $T = \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$.

In what follows, we shall tacitly use iterated monoidal products and coherence.

Definition 2.3. Suppose \mathcal{B} is a full subcategory of \mathcal{C} closed under \wedge and $P \in \text{Ob } \mathcal{C}$. Let E be a ring object of \mathcal{C}^Σ . We define the set of (E, P) -correspondences of level n between two objects $X, Y \in \mathcal{B}$ by

$$\text{Corr}_n^E(X, Y) := \text{Hom}_{\mathcal{C}}(X \wedge P^{\wedge n}, Y \wedge E_n).$$

This set is pointed at the zeroth map. By definition, $\text{Corr}_0^E(X, Y) := \text{Hom}_{\mathcal{C}}(X, Y \wedge E_0)$.

Define a pairing

$$\varphi_{X,Y,Z} : \text{Corr}_n^E(X, Y) \wedge \text{Corr}_m^E(Y, Z) \rightarrow \text{Corr}_{n+m}^E(X, Z)$$

by the rule: $\varphi_{X,Y,Z}(f : X \wedge P^{\wedge n} \rightarrow Y \wedge E_n, g : Y \wedge P^{\wedge m} \rightarrow Z \wedge E_m)$ is given by the composition

$$\begin{aligned}
 X \wedge P^{\wedge n} \wedge P^{\wedge m} &\xrightarrow{f \wedge P^{\wedge m}} Y \wedge E_n \wedge P^{\wedge m} \xrightarrow{tw} Y \wedge P^{\wedge m} \wedge E_n \xrightarrow{g \wedge E_n} Z \wedge E_m \wedge E_n \xrightarrow{tw} \\
 &\rightarrow Z \wedge E_n \wedge E_m \xrightarrow{Z \wedge \mu_{n,m}} Z \wedge E_{n+m}.
 \end{aligned}$$

Theorem 2.4. Let E be a ring object in \mathcal{C}^Σ and \mathcal{B} is a full subcategory of \mathcal{C} closed under monoidal product. Then \mathcal{B} can be enriched over the closed symmetric monoidal category of symmetric sequences

of pointed sets Sets_*^Σ . Namely, Sets_*^Σ -objects of morphisms are defined by

$$(\text{Corr}_0^E(X, Y), \text{Corr}_1^E(X, Y), \text{Corr}_2^E(X, Y), \dots), \quad X, Y \in \mathcal{B}. \quad (1)$$

Compositions are defined by pairings $\varphi_{X,Y,Z}$. The resulting Sets_*^Σ -category is denoted by $\text{Corr}_*^E(\mathcal{B})$. Moreover, if E is a commutative ring object then $\text{Corr}_*^E(\mathcal{B})$ is symmetric monoidal. The monoidal product of $X, Y \in \text{Ob}(\text{Corr}_*^E(\mathcal{B}))$ is defined as $X \wedge Y$, where \wedge is the monoidal product in \mathcal{B} .

Proof. The left action of the symmetric group Σ_n on $\text{Corr}_n^E(X, Y) = \text{Hom}_C(X \wedge P^{\wedge n}, Y \wedge E_n)$ for each $n \geq 0$ is given by conjugation. In detail, for each $f : X \wedge P^{\wedge n} \rightarrow Y \wedge E_n$ and each $\tau \in \Sigma_n$ the morphism $\tau \cdot f$ is defined as the composition

$$X \wedge P^{\wedge n} \xrightarrow{X \wedge \tau^{-1}} X \wedge P^{\wedge n} \xrightarrow{f} Y \wedge E_n \xrightarrow{Y \wedge \tau} Y \wedge E_n. \quad (2)$$

With this definition each $\text{Corr}_n^E(X, Y)$ becomes a pointed Σ_n -set. Here Σ_n acts on $P^{\wedge n}$ by permutations, using the commutativity and associativity isomorphisms.

As the multiplication maps $\mu_{*,*}$ for E are $\Sigma_n \times \Sigma_m$ -equivariant and the diagram

$$\begin{array}{ccccccc} X \wedge P^{\wedge n} \wedge P^{\wedge m} & & & & Z \wedge E_n \wedge E_m & \xrightarrow{Z \wedge \mu_{n,m}} & Z \wedge E_{n+m} \\ \uparrow \text{id}_X \wedge \alpha \wedge \beta & & & & \uparrow \text{id}_Y \wedge \alpha \wedge \beta & & \uparrow \text{id}_Y \wedge (\alpha \times \beta) \\ X \wedge P^{\wedge n} \wedge P^{\wedge m} & \xrightarrow{f \wedge P^{\wedge m}} & Y \wedge E_n \wedge P^{\wedge m} & \xrightarrow{\tau \omega} & Y \wedge P^{\wedge m} \wedge E_n & \xrightarrow{g \wedge E_n} & Z \wedge E_m \wedge E_n \\ & & & & \xrightarrow{\tau \omega} & & \xrightarrow{Z \wedge \mu_{n,m}} & Z \wedge E_n \wedge E_m & \xrightarrow{Z \wedge \mu_{n,m}} & Z \wedge E_{n+m} \end{array}$$

is commutative for any $\alpha \in \Sigma_n, \beta \in \Sigma_m$, it follows that the pairing $\varphi_{X,Y,Z}$ is $\Sigma_n \times \Sigma_m$ -equivariant.

If there is no likelihood of confusion, we shall sometimes write $(X \wedge P^{\wedge n}, Y \wedge E_n)$ to denote $\text{Hom}_C(X \wedge P^{\wedge n}, Y \wedge E_n)$. The ‘‘associativity square’’

$$\begin{array}{ccc} (X \wedge P^{\wedge n}, Y \wedge E_n) \wedge (Y \wedge P^{\wedge m}, Z \wedge E_m) \wedge (Z \wedge P^{\wedge p}, W \wedge E_p) & \xrightarrow{\text{id} \wedge \mu_{m,p}} & (X \wedge P^{\wedge n}, Y \wedge E_n) \wedge (Y \wedge P^{\wedge m+p}, W \wedge E_{m+p}) \\ \downarrow \mu_{n,m} \wedge \text{id} & & \downarrow \mu_{n,m+p} \\ (X \wedge P^{\wedge n+m}, Z \wedge E_{n+m}) \wedge (Z \wedge P^{\wedge p}, W \wedge E_p) & \xrightarrow{\mu_{n+m,p}} & (X \wedge P^{\wedge n+m+p}, W \wedge E_{n+m+p}) \end{array}$$

is commutative for all $n, m, p \geq 0$ due to the associativity of the multiplication maps $\mu_{*,*}$ for E , and so $\varphi_{X,Y,Z}$ is an associative pairing.

The identity morphism is defined by

$$u_X : X \cong X \wedge S \xrightarrow{\text{id}_X \wedge \iota_0} X \wedge E_0 \in \text{Corr}_0^E(X, X),$$

where $\iota_0 : S \rightarrow E_0$ is the unit map. We see that \mathcal{B} is enriched over Sets_*^Σ by means of (E, P) -correspondences.

Suppose E is a commutative ring object in C^Σ . If there is no likelihood of confusion, we shall sometimes write $[X, Y]$ to denote the Sets_*^Σ -object of morphisms (1). To show that $\text{Corr}_*^E(\mathcal{B})$ is a symmetric monoidal Sets_*^Σ -category, we need to define a Sets_*^Σ -functor

$$\psi : \text{Corr}_*^E(\mathcal{B}) \wedge \text{Corr}_*^E(\mathcal{B}) \rightarrow \text{Corr}_*^E(\mathcal{B}),$$

where $\text{Ob}(\text{Corr}_*^E(\mathcal{B}) \wedge \text{Corr}_*^E(\mathcal{B})) = \text{Ob} \text{Corr}_*^E(\mathcal{B}) \times \text{Ob} \text{Corr}_*^E(\mathcal{B})$ and $[(X, Y), (X', Y')] = [X, X'] \wedge [Y, Y']$ (see [1, p. 305]). By definition, $\psi(X, Y) = X \wedge Y$ for all $(X, Y) \in \text{Ob}(\text{Corr}_*^E(\mathcal{B}) \wedge \text{Corr}_*^E(\mathcal{B}))$.

Composition law in $\text{Corr}_*^E(\mathcal{B}) \wedge \text{Corr}_*^E(\mathcal{B})$ is given by (see [1, p. 305] for details)

$$[(X, Y), (X', Y')] \wedge [(X', Y'), (X'', Y'')] = [(X, X') \wedge [Y, Y'] \wedge [(X', X'')] \wedge [Y', Y'']] \xrightarrow{\text{twist}}$$

$$[(X, X') \wedge [(X', X'')] \wedge [Y, Y'] \wedge [Y', Y'']] \xrightarrow{\varphi_{X, X', X''} \wedge \varphi_{Y, Y', Y''}} [(X, X'')] \wedge [Y, Y''] = [(X, Y), (X'', Y'')].$$

It sends each quadruple

$$(f : X \wedge P^{\wedge p} \rightarrow X' \wedge E_p, g : Y \wedge P^{\wedge q} \rightarrow Y' \wedge E_q)$$

$$\wedge (f' : X' \wedge P^{\wedge s} \rightarrow X'' \wedge E_s, g' : Y' \wedge P^{\wedge t} \rightarrow Y'' \wedge E_t)$$

to

$$(f : X \wedge P^{\wedge p} \rightarrow X' \wedge E_p, (\chi_{s,q}, f' : X' \wedge P^{\wedge s} \rightarrow X'' \wedge E_s),$$

$$g : Y \wedge P^{\wedge q} \rightarrow Y' \wedge E_q, g' : Y' \wedge P^{\wedge t} \rightarrow Y'' \wedge E_t)$$

and then to the couple

$$(\tilde{\chi}_{s,q}, f \circ f' : X \wedge P^{\wedge p+s} \rightarrow X'' \wedge E_{p+s}, g \circ g' : Y \wedge P^{\wedge q+t} \rightarrow Y'' \wedge E_{q+t}),$$

where $\tilde{\chi}_{s,q} \in \Sigma_{p+s+q+t}$ is the permutation $\text{id} \times \chi_{s,q} \times \text{id} \in \Sigma_p \times \Sigma_{s+q} \times \Sigma_t$.

Define

$$\psi_{X',Y'}^{X,Y} : \text{Corr}_p^E(X, X') \wedge \text{Corr}_q^E(Y, Y') \rightarrow \text{Corr}_{p+q}^E(X \wedge Y, X' \wedge Y')$$

by sending $(f : X \wedge P^{\wedge p} \rightarrow X' \wedge E_p, g : Y \wedge P^{\wedge q} \rightarrow Y' \wedge E_q)$ to the composition

$$X \wedge Y \wedge P^{\wedge p} \wedge P^{\wedge q} \xrightarrow{tw} X \wedge P^{\wedge p} \wedge Y \wedge P^{\wedge q} \xrightarrow{f \wedge g} X' \wedge E_p \wedge Y' \wedge E_q \xrightarrow{tw} X' \wedge Y' \wedge E_p \wedge E_q$$

$$\xrightarrow{Y \wedge Y' \wedge \mu_{p,q}} X' \wedge Y' \wedge E_{p+q}.$$

The pairing $\psi_{X',Y'}^{X,Y}$ is plainly $\Sigma_p \times \Sigma_q$ -equivariant.

We have a commutative diagram

$$\begin{array}{ccccc}
 E_p \wedge E_s \wedge E_q \wedge E_t & \xrightarrow{\text{id} \wedge \text{twist} \wedge \text{id}} & E_p \wedge E_q \wedge E_s \wedge E_t & \xrightarrow{\mu_{p,q} \wedge \mu_{s,t}} & E_{p+q} \wedge E_{s+t} \\
 \text{id} \wedge \mu_{s,q} \wedge \text{id} \downarrow & & \downarrow \text{id} \wedge \mu_{q,s} \wedge \text{id} & & \downarrow \mu_{p+q,s+t} \\
 E_p \wedge E_{s+q} \wedge E_t & \xrightarrow{\text{id} \wedge \chi_{s,q} \wedge \text{id}} & E_p \wedge E_{q+s} \wedge E_t & & E_{p+q+s+t} \\
 \mu_{p,s+q} \wedge \text{id} \downarrow & & \downarrow \mu_{p,q+s} \wedge \text{id} & & \\
 E_{p+s+q} \wedge E_t & \xrightarrow{\text{id} \wedge \chi_{s,q} \wedge \text{id}} & E_{p+q+s} \wedge E_t & & \\
 \mu_{p+s+q,t} \downarrow & & \downarrow \mu_{p+q+s,t} & & \\
 E_{p+s+q+t} & \xrightarrow{\text{id} \wedge \chi_{s,q} \wedge \text{id}} & E_{p+q+s+t} & &
 \end{array}$$

Commutativity of the top square follows from commutativity of the multiplication maps, commutativity of the remaining squares follows from equivariancy of the multiplication maps. It follows that

$$\varphi_{X \wedge Y, X' \wedge Y', X'' \wedge Y''}(f \wedge g, f' \wedge g') = \varphi_{X, X', X''}(f, f') \wedge \varphi_{Y, Y', Y''}(g, g').$$

For $X, Y \in \mathcal{B}$, the corresponding unit map of $\text{Corr}_*^E(\mathcal{B}) \wedge \text{Corr}_*^E(\mathcal{B})$ is given by (see [1, p. 305])

$$S \xrightarrow{\rho^{-1}} S \wedge S \xrightarrow{u_X \wedge u_Y} [X, X] \wedge [Y, Y].$$

Clearly, $\psi_{X, Y}^{X, Y}$ takes it to the unit $u_{X \wedge Y} : S \rightarrow [X \wedge Y, X \wedge Y]$.

We see that $\psi : \text{Corr}_*^E(\mathcal{B}) \wedge \text{Corr}_*^E(\mathcal{B}) \rightarrow \text{Corr}_*^E(\mathcal{B})$ is a Sets_*^Σ -functor. It defines a symmetric monoidal Sets_*^Σ -category structure on $\text{Corr}_*^E(\mathcal{B})$ with the unit $S \in \text{Ob } \text{Corr}_*^E(\mathcal{B})$. Sets_*^Σ -natural associativity, symmetry isomorphisms and two Sets_*^Σ -natural unit isomorphisms are inherited from the same isomorphisms of the symmetric monoidal category structure on \mathcal{B} . This completes the proof of the theorem. \square

The proof of the theorem implies the following result.

Corollary 2.5. *Under the notation of Theorem 2.4 any morphism of two ring objects $\gamma : E \rightarrow E'$ in \mathcal{C}^Σ induces a morphism of Sets_*^Σ -categories $\gamma_* : \text{Corr}_*^E(\mathcal{B}) \rightarrow \text{Corr}_*^{E'}(\mathcal{B})$.*

Day's theorem [2] together with Theorem 2.4 also imply the following result.

Corollary 2.6. *Under the notation of Theorem 2.4 if E is a commutative ring object in \mathcal{C}^Σ then the category of $\text{Corr}_*^E(\mathcal{B})$ -modules is a closed symmetric monoidal category.*

3 | SYMMETRIC SPECTRA

Let \mathcal{C} be the category from the previous section and let T be an object of \mathcal{C} . Following [22, section 7] consider the free commutative monoid $\text{Sym}(T)$ on the object $(0, T, 0, 0, \dots)$ of \mathcal{C}^Σ . Then $\text{Sym}(T)$ is the symmetric sequence $(S, T, T^{\wedge 2}, T^{\wedge 3}, \dots)$, where Σ_n acts on $T^{\wedge n}$ by permutation, using the commutativity and associativity isomorphisms.

Definition 3.1.

- (1) Following [22, Definition 7.2], the *category of symmetric spectra* $\text{Sp}^\Sigma(\mathcal{C}, T)$ is the category of modules in \mathcal{C}^Σ over the commutative monoid $\text{Sym}(T)$ in \mathcal{C} . That is, a symmetric spectrum X is a sequence of Σ_n -objects $X_n \in \mathcal{C}$ and Σ_n -equivariant maps $X_n \wedge T \rightarrow X_{n+1}$, such that the composite

$$X_n \wedge T^{\wedge p} \rightarrow X_{n+1} \wedge T^{\wedge p-1} \rightarrow \dots \rightarrow X_{n+p}$$

is $\Sigma_n \times \Sigma_p$ -equivariant for all $n, p \geq 0$. A map of symmetric spectra is a collection of Σ_n -equivariant maps $X_n \rightarrow Y_n$ compatible with the structure maps of X and Y .

- (2) A *symmetric ring T -spectrum* is a ring spectrum $E \in \mathcal{C}^\Sigma$ such that there is another unit map $\iota_1 : T \rightarrow E_1$ subject to the following condition:

(Centrality) The diagram

$$\begin{array}{ccccc}
 E_n \wedge T & \xrightarrow{E_n \wedge t_1} & E_n \wedge E_1 & \xrightarrow{\mu_{n,1}} & E_{n+1} \\
 \text{twist} \downarrow & & & & \downarrow \chi_{n,1} \\
 T \wedge E_n & \xrightarrow{t_1 \wedge E_n} & E_1 \wedge E_n & \xrightarrow{\mu_{1,n}} & E_{1+n}
 \end{array}$$

commutes for all $n \geq 0$.

E is *commutative* if it is commutative as a ring object of C^Σ . A *morphism* $f : E \rightarrow E'$ of symmetric ring spectra is a morphism of ring objects in C^Σ such that $f_{1t_1} = t'_1$.

- (3) A *right module* M over a symmetric ring T -spectrum E is a symmetric right T -spectrum which is also a right E -module in the sense of Definition 2.2. We denote the category of right E -modules by $\text{Mod } E$ (its morphisms are morphisms of symmetric T -spectra satisfying Definition 2.2).

Because $\text{Sym}(T)$ is a commutative monoid, the category $Sp^\Sigma(C, T)$ is a symmetric monoidal category, with $\text{Sym}(T)$ itself as the unit. We denote the monoidal structure by $X \wedge Y = X \wedge_{\text{Sym}(T)} Y$, where $X \wedge_{\text{Sym}(T)} Y$ is defined similarly to [32, p. 499] as the coequalizer, in C^Σ , of the two maps $X \wedge \text{Sym}(T) \wedge Y \rightrightarrows X \wedge Y$ induced by the actions of $\text{Sym}(T)$ on X and Y , respectively.

Given $X \in C$ and a pointed set $(K, *)$, we shall write $X \wedge K$ to denote $\bigvee_{K \setminus *} X$. Let $P \in C$ and $\sigma : P \rightarrow T$ be a morphism in C . Denote by $T^n := T \wedge \dots \wedge T$ (respectively, $P^{\wedge n} = P \wedge \dots \wedge P$). This notation is inherited from the standard notation for pointed motivic spaces T^n and $\mathbb{P}^{\wedge n}$ associated with pointed motivic spaces $T = \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$ and (\mathbb{P}^1, ∞) , where $\sigma : (\mathbb{P}^1, \infty) \rightarrow T$ is the canonical motivic equivalence given by the level 1 framed correspondence $(\{0\}, \mathbb{A}^1, t) \in \text{Fr}_1(\text{pt}, \text{pt})$.

In what follows S^1 is the standard simplicial circle with n -simplices being $n_+ = \{0, 1, \dots, n\}$ and $S^n := S^1 \wedge \dots \wedge S^1$. Given a symmetric right T -spectrum E , let $\mathbf{1}^E$ denote the following S^1 -spectrum in \mathbf{S}_* :

$$\mathbf{1}^E := (\text{Hom}_C(S, E_0), \text{Hom}_C(P, E_1 \wedge S^1), \text{Hom}_C(P^{\wedge 2}, E_2 \wedge S^2), \dots).$$

Here each $E_n \wedge S^n$ is performed in every degree to produce a simplicial object in C and a simplicial Hom-set. We also call $\mathbf{1}^E$ the *PT-spectrum* of E .

Each simplicial Hom-set is pointed at the zero morphism. Each structure map

$$u_n : \text{Hom}_C(P^{\wedge n}, E_n \wedge S^n) \wedge S^1 \rightarrow \text{Hom}_C(P^{\wedge n+1}, E_{n+1} \wedge S^{n+1})$$

coincides termwise with the natural morphisms

$$\bigvee \text{Hom}_C(P^{\wedge n}, E_n \wedge S^n) \rightarrow \text{Hom}_C(P^{\wedge n+1}, \bigvee (E_{n+1} \wedge S^n)),$$

where coproducts are indexed by nonbasepoint elements of $S^n_+ = n_+ = \{0, 1, \dots, n\}$. They take an element $f : P^{\wedge n} \rightarrow E_n \wedge S^n$ of the k th summand to the composition

$$P^{\wedge n+1} \xrightarrow{f \wedge \sigma} (E_n \wedge S^n) \wedge T \xrightarrow{v^{-1}} (E_n \wedge T) \wedge S^n \rightarrow E_{n+1} \wedge S^n \xrightarrow{\iota_k} \bigvee E_{n+1} \wedge S^n.$$

Here ι_k is the inclusion into the k th summand. If $E = \text{Sym}(T)$ then $\mathbf{1}^E$ will be denoted by $\mathbf{1}^\Sigma$.

Though the author was unable to find the following result in the literature, he does not pretend to originality.

Theorem 3.2. *Given a symmetric right T -spectrum $E \in Sp^\Sigma(C, T)$, the following statements are true:*

- (1) *the spectrum $\mathbf{1}^E$ is a symmetric S^1 -spectrum;*
- (2) *if E is a (commutative) symmetric ring T -spectrum, then $\mathbf{1}^E$ is a (commutative) symmetric ring S^1 -spectrum.*

Proof.

- (1.) We follow [31, section I.1] to verify the relevant conditions for symmetric S^1 -spectra. The left action of the symmetric group Σ_n on $\text{Hom}_C(P^{\wedge n}, E_n \wedge S^n)$ for each $n \geq 0$ is given by conjugation. In detail, for each $f : P^{\wedge n} \rightarrow E_n \wedge S^n$ and each $\tau \in \Sigma_n$ the morphism $\tau \cdot f$ is defined as the composition

$$P^{\wedge n} \xrightarrow{\tau^{-1}} P^{\wedge n} \xrightarrow{f} E_n \wedge S^n \xrightarrow{\tau \wedge \tau} E_n \wedge S^n. \quad (3)$$

With this definition each $\text{Hom}_C(P^{\wedge n}, E_n \wedge S^n)$ becomes a Σ_n -simplicial set. Here Σ_n acts on $P^{\wedge n}$ and S^n by permutations.

There are natural morphisms

$$\bigvee \text{Hom}_C(P^{\wedge n}, E_n \wedge S^n) \rightarrow \text{Hom}_C(P^{\wedge(n+k)}, \bigvee (E_{n+k} \wedge S^n)),$$

where coproducts are indexed by nonbasepoint elements of $S_\ell^k = \ell^{\wedge k} = (\{1, \dots, \ell\}^{\times k})_+$. They take an element $f : P^{\wedge n} \rightarrow E_n \wedge S^n$ of the (j_1, \dots, j_k) th summand to the composition

$$P^{\wedge n+k} \xrightarrow{f \wedge \sigma^{\wedge k}} (E_n \wedge S^n) \wedge T^k \xrightarrow{\cong} (E_n \wedge T^k) \wedge S^n \rightarrow E_{n+k} \wedge S^n \xrightarrow{\iota_{(j_1, \dots, j_k)}} \bigvee E_{n+k} \wedge S^n.$$

Here $\iota_{(j_1, \dots, j_k)}$ is the inclusion into the (j_1, \dots, j_k) th summand. The middle arrow comes from $E_n \wedge T^k \rightarrow E_{n+k}$. It is induced by structure maps of the symmetric T -spectrum E and is $(\Sigma_n \times \Sigma_k)$ -equivariant. Each natural morphism above coincides termwise with the composite map

$$u_{n+k-1} \circ \dots \circ (u_n \wedge \text{id}) : \text{Hom}_C(P^{\wedge n}, E_n \wedge S^n) \wedge S^k \rightarrow \text{Hom}_C(P^{\wedge(n+k)}, E_{n+k} \wedge S^{n+k}). \quad (4)$$

The fact that (4) is $(\Sigma_n \times \Sigma_k)$ -equivariant follows from commutativity of the diagram

$$\begin{array}{ccccccc} P^{\wedge n+k} & \xrightarrow{f \wedge \sigma^{\wedge k}} & (E_n \wedge S^n) \wedge T^k & \xrightarrow{\cong} & (E_n \wedge T^k) \wedge S^n & \longrightarrow & E_{n+k} \wedge S^n \xrightarrow{\iota_{(j_1, \dots, j_k)}} \bigvee E_{n+k} \wedge S^n \\ (\tau, \mu) \downarrow & & \tau \wedge \tau \wedge \mu \downarrow & & \tau \wedge \mu \wedge \tau \downarrow & & (\tau, \mu) \wedge \tau \wedge \mu^{\otimes k} \downarrow \\ P^{\wedge n+k} & & (E_n \wedge S^n) \wedge T^k & \xrightarrow{\cong} & (E_n \wedge T^k) \wedge S^n & \longrightarrow & E_{n+k} \wedge S^n \xrightarrow{\iota_{(j_{\mu(1)}, \dots, j_{\mu(k)})}} \bigvee E_{n+k} \wedge S^n \end{array}$$

in which $(\tau, \mu) \in \Sigma_n \times \Sigma_k$, $\mu^{\otimes k}$ permutes summands of the right upper corner by the rule $(j_1, \dots, j_k) \mapsto (j_{\mu(1)}, \dots, j_{\mu(k)})$. We see that $\mathbf{1}^E$ is a symmetric S^1 -spectrum.

(2). Suppose E is a symmetric ring T -spectrum. Define multiplication maps

$$\begin{aligned} \nu_{n,m} &: \text{Hom}_C(P^{\wedge n}, E_n \wedge S^n) \wedge \text{Hom}_C(P^{\wedge m}, E_m \wedge S^m) \\ &\rightarrow \text{Hom}_C(P^{\wedge(n+m)}, E_{n+m} \wedge S^{n+m}), \quad n, m \geq 0, \end{aligned}$$

by

$$(f, g) \mapsto (P^{\wedge(n+m)} \xrightarrow{f \wedge g} E_n \wedge S^n \wedge E_m \wedge S^m \cong E_n \wedge E_m \wedge S^{n+m} \xrightarrow{\mu_{n,m} \wedge \text{id}} E_{n+m} \wedge S^{n+m}),$$

where $\mu_{n,m}$ is the multiplication map of E (it is $\Sigma_n \times \Sigma_m$ -equivariant).

The map $\nu_{n,m}$ is $\Sigma_n \times \Sigma_m$ -equivariant. Indeed, this follows from commutativity of the diagram

$$\begin{array}{ccccc} P^{\wedge n+m} & \xrightarrow{f \wedge g} & E_n \wedge S^n \wedge E_m \wedge S^m & \xrightarrow{\cong} & E_n \wedge E_m \wedge S^{n+m} & \xrightarrow{\mu_{n,m} \wedge \text{id}} & E_{n+m} \wedge S^{n+m} \\ (\tau, \pi) \downarrow & & \tau \wedge \tau \wedge \pi \wedge \pi \downarrow & & \tau \wedge \pi \wedge (\tau, \pi) \downarrow & & \downarrow (\tau, \pi) \wedge (\tau, \pi) \\ P^{\wedge n+k} & & E_n \wedge S^n \wedge E_m \wedge S^m & \xrightarrow{\cong} & E_n \wedge E_m \wedge S^{n+m} & \xrightarrow{\mu_{n,m} \wedge \text{id}} & E_{n+m} \wedge S^{n+m} \end{array}$$

in which $(\tau, \pi) \in \Sigma_n \times \Sigma_m$. Clearly, the ‘‘associativity square’’

$$\begin{array}{ccc} (P^{\wedge n}, E_n \wedge S^n) \wedge (P^{\wedge m}, E_m \wedge S^m) \wedge (P^{\wedge p}, E_p \wedge S^p) & \xrightarrow{\text{id} \wedge \nu_{m,p}} & (P^{\wedge n}, E_n \wedge S^n) \wedge (P^{\wedge(m+p)}, E_{m+p} \wedge S^{m+p}) \\ \nu_{n,m} \wedge \text{id} \downarrow & & \downarrow \nu_{n,m+p} \\ (P^{\wedge(n+m)}, E_{n+m} \wedge S^{n+m}) \wedge (P^{\wedge p}, E_p \wedge S^p) & \xrightarrow{\nu_{n+m,p}} & (P^{\wedge(n+m+p)}, E_{n+m+p} \wedge S^{n+m+p}) \end{array}$$

is commutative for all $n, m, p \geq 0$ due to associativity of the multiplication maps $\mu_{*,*}$ for E .

Next, two unit maps

$$i_0 : S^0 \rightarrow \text{Hom}_C(S, E_0) \quad \text{and} \quad i_1 : S^1 \rightarrow \text{Hom}_C(P, E_1 \wedge S^1) \tag{5}$$

are defined as follows. Let $t_0 : S \rightarrow E_0$ and $t_1 : T \rightarrow E_1$ be the unit maps for the symmetric ring T -spectrum E . Then i_0 takes the unbased point of S^0 to t_0 and i_1 coincides termwise with the natural morphisms taking an unbased simplex j of $S^1_\ell = \{0, 1, \dots, \ell\}$ to the composite map

$$P \xrightarrow{\sigma} T \xrightarrow{t_1} E_1 \hookrightarrow \bigvee E_1,$$

where the right arrow is the inclusion into the j th summand.

The two ‘‘unit composites’’

$$(P^{\wedge n}, E_n \wedge S^n) \cong (P^{\wedge n}, E_n \wedge S^n) \wedge S^0 \xrightarrow{\text{id} \wedge i_0} (P^{\wedge n}, E_n \wedge S^n) \wedge (S, E_0) \xrightarrow{\nu_{n,0}} (P^{\wedge n}, E_n \wedge S^n)$$

$$(P^{\wedge n}, E_n \wedge S^n) \cong S^0 \wedge (P^{\wedge n}, E_n \wedge S^n) \xrightarrow{i_0 \wedge \text{id}} (S, E_0) \wedge (P^{\wedge n}, E_n \wedge S^n) \xrightarrow{\nu_{0,n}} (P^{\wedge n}, E_n \wedge S^n)$$

are the identity for all $n \geq 0$ due to the same properties for E .

Furthermore, the “centrality diagram”

$$\begin{array}{ccccc} (P^{\wedge n}, E_n \wedge S^n) \wedge S^1 & \xrightarrow{\text{id} \wedge i_1} & (P^{\wedge n}, E_n \wedge S^n) \wedge (P^{\wedge 1}, E_1 \wedge S^1) & \xrightarrow{\nu_{n,1}} & (P^{\wedge n+1}, E_{n+1} \wedge S^{n+1}) \\ \text{twist} \downarrow & & & & \downarrow \chi_{n,1} \\ S^1 \wedge (P^{\wedge n}, E_n \wedge S^n) & \xrightarrow{i_1 \wedge \text{id}} & (P^{\wedge 1}, E_1 \wedge S^1) \wedge (P^{\wedge n}, E_n \wedge S^n) & \xrightarrow{\nu_{1,n}} & (P^{\wedge 1+n}, E_{1+n} \wedge S^{1+n}) \end{array}$$

commutes for all $n \geq 0$ due to the same properties for E . Here $\chi_{n,m} \in \Sigma_{n+m}$ denotes the shuffle permutation that moves the first n elements past the last m elements, keeping each of the two blocks in order.

It follows that $\mathbf{1}^E$ is a symmetric ring S^1 -spectrum. Suppose E is commutative. Then the square

$$\begin{array}{ccc} (P^{\wedge n}, E_n \wedge S^n) \wedge (P^{\wedge m}, E_m \wedge S^m) & \xrightarrow{\text{twist}} & (P^{\wedge m}, E_m \wedge S^m) \wedge (P^{\wedge n}, E_n \wedge S^n) \\ \nu_{n,m} \downarrow & & \downarrow \nu_{m,n} \\ (P^{\wedge n+m}, E_{n+m} \wedge S^{n+m}) & \xrightarrow{\chi_{n,m}} & (P^{\wedge m+n}, E_{m+n} \wedge S^{m+n}) \end{array}$$

commutes for all $n, m \geq 0$. We also use here commutativity of the diagram

$$\begin{array}{ccccccc} P^{\wedge n+m} & \xrightarrow{f \wedge g} & E_n \wedge S^n \wedge E_m \wedge S^m & \xrightarrow{\cong} & E_n \wedge E_m \wedge S^n \wedge S^m & \xrightarrow{\mu_{n,m} \wedge \text{id}} & E_{n+m} \wedge S^{n+m} \\ \uparrow \chi_{m,n} & & \uparrow \text{twist} & & \uparrow \text{twist} \wedge \text{twist} & & \uparrow \chi_{m,n} \wedge \chi_{n,m} \\ P^{\wedge m+n} & \xrightarrow{g \wedge f} & E_m \wedge S^m \wedge E_n \wedge S^n & \xrightarrow{\cong} & E_m \wedge E_n \wedge S^m \wedge S^n & \xrightarrow{\mu_{m,n} \wedge \text{id}} & E_{m+n} \wedge S^{m+n} \end{array}$$

for all $f : P^{\wedge n} \rightarrow E_n \wedge S^n$ and $g : P^{\wedge m} \rightarrow E_m \wedge S^m$. It follows that $\mathbf{1}^E$ is commutative. \square

Corollary 3.3. *Suppose E is a commutative symmetric ring T -spectrum. The category of right $\mathbf{1}^E$ -modules $\text{Mod } \mathbf{1}^E$ is closed symmetric monoidal, where $\mathbf{1}^E$ is the commutative symmetric ring spectrum of Theorem 3.2.*

Lemma 3.4. *Under the assumptions of Theorem 3.2 two unit maps (5) can be extended to a ring morphism between symmetric ring spectra $\delta : S \rightarrow \mathbf{1}^E$, where S is the sphere spectrum.*

Proof. This is straightforward. \square

4 | RECONSTRUCTING STABLE HOMOTOPY THEORY SH

We refer the reader to [28] for basic facts on compactly generated triangulated categories. Below we will often use the following lemma.

Lemma 4.1 (see [13]). *Let S and \mathcal{T} be compactly generated triangulated categories. Suppose there exists a set of compact generators Σ in S and a triangulated functor $F : S \rightarrow \mathcal{T}$ that preserves direct sums such that*

- (1) *the collection $\{F(X) \mid X \in \Sigma\}$ is a set of compact generators in \mathcal{T} ,*

(2) for any X, Y in Σ , the induced map

$$F_{X,Y[n]} : \text{Hom}_{\mathcal{S}}(X, Y[n]) \rightarrow \text{Hom}_{\mathcal{T}}(FX, FY[n])$$

is an isomorphism for all $n \in \mathbb{Z}$.

Then F is an equivalence of triangulated categories.

Let $Sp_{S^1, \mathbb{G}}(k)$ denote the category of symmetric (S^1, \mathbb{G}) -bispectra associated with the closed symmetric monoidal category of pointed motivic spaces \mathcal{M} (see the notation). Here the \mathbb{G} -direction corresponds to the pointed motivic space \mathbb{G} , which is the mapping cone in \mathcal{M} of the map $1 : \text{pt}_+ \rightarrow (\mathbb{G}_m)_+$. The category $Sp_{S^1, \mathbb{G}}(k)$ is equipped with a stable motivic model category structure [24]. Denote by $\text{SH}(k)$ its homotopy category. If \mathcal{E} is a bispectrum and p, q are integers, recall that $\pi_{p,q}(\mathcal{E})$ is the Nisnevich sheaf of bigraded stable homotopy groups associated to the presheaf

$$U \in \text{Sm}/k \mapsto \text{SH}(k)(\Sigma_{S^1}^{\infty} \Sigma_{\mathbb{G}^{\wedge 1}}^{\infty} U_+, S^{p-q} \wedge \mathbb{G}^{\wedge q} \wedge \mathcal{E}).$$

A map of bispectra $f : \mathcal{E} \rightarrow \mathcal{E}'$ is a stable motivic equivalence if and only if $\pi_{*,*}(f)$ is an isomorphism. The category $\text{SH}(k)$ has a closed symmetric monoidal structure with monoidal unit being the motivic sphere bispectrum $S_k := \Sigma_{S^1}^{\infty} \Sigma_{\mathbb{G}}^{\infty} S^0$ (see [24] for details).

The following theorem was proven by Levine [25] for algebraically closed fields of characteristic zero, with embedding $\bar{k} \hookrightarrow \mathbb{C}$ and extended by Wilson-Østvær [37, Corollaries 1.2 and 6.5] to arbitrary algebraically closed fields.

Theorem 4.2. *Let \bar{k} be an algebraically closed field of exponential characteristic e and S be the sphere spectrum $\Sigma_{S^1}^{\infty} S^0$. For all $n \geq 0$ the homomorphism $\mathbb{L}c : \pi_n(S)[e^{-1}] \rightarrow \pi_{n,0}(S_{\bar{k}})[e^{-1}]$ is an isomorphism, where $c : SH \rightarrow SH(\bar{k})$ is the functor induced by the functor $c : \mathbf{S}_+ \rightarrow \mathcal{M}$ sending pointed simplicial sets to constant motivic spaces.*

The following statement was proven by Zargar [38, Theorem 1.1] by using the stable étale realization functor.

Corollary 4.3. *Let \bar{k} be an algebraically closed field of exponential characteristic e . The triangulated functor*

$$\mathbb{L}c : SH[1/e] \rightarrow SH(\bar{k})[1/e]$$

is full and faithful.

Proof. Using Lemma 4.1, our statement follows from Theorem 4.2 if we note that $SH[1/e]$ (respectively, the image of $SH[1/e]$) is compactly generated by $S[1/e]$ (respectively, by $S_{\bar{k}}[1/e]$). \square

Recall from [19] that one of the equivalent ways to define Voevodsky’s framed correspondences of level $n \geq 0$ between smooth k -schemes $X, Y \in \text{Sm}/k$ is as follows:

$$\text{Fr}_n(X, Y) := \text{Hom}_{\mathcal{M}}(X_+ \wedge \mathbb{P}^{\wedge n}, Y_+ \wedge T^n),$$

where $\mathbb{P}^{\wedge n}$ (respectively, T^n) is the smash product of n copies of $(\mathbb{P}^1, \infty) \in \mathcal{M}$ (respectively, $T = \mathbb{A}^1/(\mathbb{A}^1 - \{0\}) \in \mathcal{M}$).

Remark 4.4. Due to Voevodsky's lemma [19, section 3] there is an equivalent description of the sets $\text{Fr}_n(X, Y)$ in terms of explicit geometric data. In detail, each element in that description is the equivalence class of a quadruple (Z, U, φ, g) , where Z is a closed subset of \mathbb{A}_X^n that is finite over X , U is an étale neighborhood of Z in \mathbb{A}_X^n , $\varphi = (\varphi_1, \dots, \varphi_n)$ is a collection of regular functions on U such that $\bigcap_{i=1}^n \{\varphi_i = 0\} = Z$, and g is a morphism from U to Y . The equivalence relation on such quadruples depends on the choice of the neighborhood U . If the base field is \mathbb{C} and $X = Y = \text{pt}$, the Hom-sets $\text{Hom}_{\mathcal{M}}(\mathbb{P}^{\wedge m}, T^n)$, $m, n \geq 0$, can also be described in terms of *holomorphic framed correspondences* – see [12]. These are equivalence classes of triples (Z, U, f) , where Z consists of finitely many points in \mathbb{C}^m , U is an open neighborhood of Z , $f = (f_1, \dots, f_n) : U \rightarrow \mathbb{C}^n$ is a holomorphic map such that $Z = f^{-1}(0)$. The latter description essentially follows from the Implicit Function Theorem in Complex Analysis.

Following notation of [19] one sets for any finite pointed set K ,

$$\text{Fr}_n(X, Y \otimes K) := \text{Hom}_{\mathcal{M}}(X_+ \wedge \mathbb{P}^{\wedge n}, Y_+ \wedge T^n \wedge K), \quad X, Y \in \text{Sm}/k.$$

There is a distinguished framed correspondence $\sigma : (\mathbb{P}^1, \infty) \rightarrow T$ in $\text{Fr}_1(\text{pt}, \text{pt})$ associated with the triple $(\{0\}, \mathbb{A}^1, t)$. The external smash product by σ gives rise to a map $\text{Fr}_n(X, Y \otimes K) \rightarrow \text{Fr}_{n+1}(X, Y \otimes K)$. One sets,

$$\text{Fr}(X, Y \otimes K) := \text{colim}(\text{Fr}_0(X, Y \otimes K) \xrightarrow{-\wedge \sigma} \text{Fr}_1(X, Y \otimes K) \xrightarrow{-\wedge \sigma} \dots).$$

Let Δ_k^* denote the standard cosimplicial affine scheme $n \mapsto \text{Spec}(k[x_0, \dots, x_n]/(x_0 + \dots + x_n - 1))$ and $C_* \text{Fr}(X, Y \otimes K) := \text{Fr}(X \times \Delta_k^*, Y \otimes K)$. By the Additivity Theorem of [19], the assignment

$$K \in \Gamma^{\text{op}} \mapsto C_* \text{Fr}(X, Y \otimes K) \in \mathbf{S}, \quad (6)$$

gives rise to a special Γ -space of pointed simplicial sets. Its Segal S^1 -spectrum $C_* \text{Fr}(X, Y \otimes (\Sigma_{S^1}^\infty S^0))$ is denoted by $M_{fr}(Y)(X)$ and is called the *framed motive of Y* evaluated at X .

The following theorem was proven by Garkusha and Panin [19] for algebraically closed fields of characteristic zero, with embedding $\bar{k} \hookrightarrow \mathbb{C}$.

Theorem 4.5 (Garkusha–Panin [19]). *Let \bar{k} be an algebraically closed field of characteristic 0. Then the framed motive $M_{fr}(\text{pt})(\text{pt})$ of the point $\text{pt} = \text{Spec}(\bar{k})$ evaluated at pt has the stable homotopy type of the classical sphere spectrum $S = \Sigma_{S^1}^\infty S^0$. If \bar{k} is an algebraically closed field of positive characteristic $e > 0$ then $M_{fr}(\text{pt})(\text{pt})[1/e]$, $\text{pt} = \text{Spec}(\bar{k})$, has the stable homotopy type of $S[1/e]$.*

Proof. The proof literally repeats that of [19, Theorem 11.9] if we use Corollary 4.3. \square

Definition 4.6. Given a field k , denote by

$$\mathbf{1}_k := (\text{Fr}_0(\text{pt}, \text{pt}), \text{Fr}_1(\text{pt}, S^1), \text{Fr}_2(\text{pt}, S^2), \dots), \quad \text{pt} = \text{Spec}(k),$$

the right S^1 -spectrum of pointed simplicial sets with structure maps defined as

$$\text{Fr}_n(\text{pt}, S^n) \xrightarrow{-\wedge \sigma} \text{Fr}_{n+1}(\text{pt}, S^n) \rightarrow \underline{\text{Hom}}_{\mathbf{S}}(S^1, \text{Fr}_{n+1}(\text{pt}, S^{n+1})).$$

The *sphere spectrum over a field k* is the S^1 -spectrum of pointed simplicial sets

$$S_k := (\text{Fr}_0(\Delta_k^\bullet, \text{pt}), \text{Fr}_1(\Delta_k^\bullet, S^1), \text{Fr}_2(\Delta_k^\bullet, S^2), \dots).$$

The simplicial set $\text{Fr}_m(\text{pt}, S^m)$ can be regarded as the constant simplicial pointed set S^m together with the “coefficients set” $\text{Fr}_m(\text{pt}, \text{pt})$ attached to it. Hence, the S^1 -spectrum $\mathbf{1}_k$ can be regarded as a graded “tensor algebra” associated to S^1 with “ $\text{Fr}_*(\text{pt}, \text{pt})$ -coefficients”. In other words, we add $\text{Fr}_*(\text{pt}, \text{pt})$ -coefficients to the classical sphere spectrum $S = (S^0, S^1, \dots)$.

Proposition 4.7. *The spectra $\mathbf{1}_k, S_k$ are commutative symmetric ring spectra in $Sp_{S^1}^\Sigma$.*

Proof. We apply Theorem 3.2: C is replaced by the category of pointed motivic spaces \mathcal{M} , P (respectively, T) is replaced by $\mathbb{P}^{\wedge 1} := (\mathbb{P}^1, \infty)$ (respectively, by $T = \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$), E is replaced by the commutative symmetric sphere T -spectrum $S_k = (S^0, T, T^2, \dots)$. With this notation $\mathbf{1}_k = \mathbf{1}^E$ of Theorem 3.2, and hence a commutative symmetric ring spectrum.

The commutative symmetric ring structure on S_k is defined in a similar fashion if we use the cosimplicial diagonal morphism $\text{diag} : \Delta_k^\bullet \rightarrow \Delta_k^\bullet \times \Delta_k^\bullet$. □

Denote by $\text{Mod } S_k$ the category of right S_k -modules in $Sp_{S^1}^\Sigma$. The natural map of ring objects $S \rightarrow S_k$ induces a pair of adjoint functors

$$L : Sp_{S^1}^\Sigma \rightleftarrows \text{Mod } S_k : U, \tag{7}$$

where U is the forgetful functor and its left adjoint L is the “extension of framed scalars” functor. Following [32, section 4], we define the stable model structure on $\text{Mod } S_k$ by calling a map f of S_k -spectra a stable equivalence or fibration if so is $U(f)$. By [32, Theorem 4.1] this model structure is also cofibrantly generated monoidal satisfying the monoid axiom. By construction, (L, U) is a Quillen pair.

Definition 4.8. The category $\text{Mod } S_k$ is called the *category of framed symmetric S^1 -spectra over a field k* . The *stable homotopy category over a field k* , denoted by SH_k , is defined as the homotopy category of $\text{Mod } S_k$ with respect to the stable model structure. SH_k is a closed symmetric monoidal category.

If $N \in Sp_{S^1}^\Sigma$ is a symmetric right S^1 -spectrum, define an S^1 -spectrum

$$\text{Fr}_*^\Sigma(N) := (N_0, \text{Hom}_{\mathcal{M}}(\mathbb{P}^{\wedge 1}, T \wedge N_1), \text{Hom}_{\mathcal{M}}(\mathbb{P}^{\wedge 2}, T^2 \wedge N_2), \dots).$$

Each structure map

$$v_n : \text{Hom}_{\mathcal{M}}(\mathbb{P}^{\wedge n}, T^n \wedge N_n) \wedge S^1 \rightarrow \text{Hom}_{\mathcal{M}}(\mathbb{P}^{\wedge n+1}, T^{n+1} \wedge N_{n+1})$$

coincides termwise with the natural morphisms

$$\bigvee \text{Hom}_{\mathcal{M}}(\mathbb{P}^{\wedge n}, T^n \wedge N_n) \rightarrow \text{Hom}_{\mathcal{M}}(\mathbb{P}^{\wedge n+1}, T^{n+1} \wedge N_{n+1}),$$

where coproducts are indexed by nonbasepoint elements of $S^1_\ell = \ell_+ = \{0, 1, \dots, \ell\}$. They take an element $f : \mathbb{P}^{\wedge n} \rightarrow T^n \wedge (N_n)_\ell$ of the k th summand to the composition

$$\begin{aligned} \mathbb{P}^{\wedge n+1} &\xrightarrow{f \wedge \sigma} (T^n \wedge (N_n)_\ell) \wedge T \cong T^{n+1} \wedge (N_n)_\ell \xrightarrow{\iota_k} \bigvee T^{n+1} \wedge (N_n)_\ell \\ &= (T^{n+1} \wedge (N_n)_\ell) \wedge S^1_\ell \cong T^{n+1} \wedge (N_n \wedge S^1)_\ell \xrightarrow{\text{id} \wedge u_n} T^{n+1} \wedge (N_{n+1})_\ell, \end{aligned}$$

where u_n is the n th structure map of N . We can equivalently define v_n by the composition

$$\begin{aligned} \text{Hom}_{\mathcal{M}}(\mathbb{P}^{\wedge n}, T^n \wedge N_n) \wedge S^1 &\rightarrow \text{Hom}_{\mathcal{M}}(\mathbb{P}^{\wedge n}, T^n \wedge N_n \wedge S^1) \xrightarrow{-\wedge \sigma} \text{Hom}_{\mathcal{M}}(\mathbb{P}^{\wedge n+1}, T^n \wedge N_n \wedge S^1 \wedge T) \\ &\cong \text{Hom}_{\mathcal{M}}(\mathbb{P}^{\wedge n+1}, T^{n+1} \wedge N_n \wedge S^1) \xrightarrow{(u_n)_*} \text{Hom}_{\mathcal{M}}(\mathbb{P}^{\wedge n}, T^n \wedge N_{n+1}). \end{aligned}$$

We also define an S^1 -spectrum

$$C_* \text{Fr}_*^\Sigma(N) := (N_0, \text{Hom}_{\mathcal{M}}((\Delta_k^*)_+ \wedge \mathbb{P}^{\wedge 1}, T \wedge N_1), \text{Hom}_{\mathcal{M}}((\Delta_k^*)_+ \wedge \mathbb{P}^{\wedge 2}, T^2 \wedge N_2), \dots)$$

with the structure maps defined as above.

$\text{Fr}_*^\Sigma(N)$ and $C_* \text{Fr}_*^\Sigma(N)$ are symmetric S^1 -spectra for the same reason as $\mathbf{1}^E$ and S_k are.

Following notation of [15, 19], one sets for any finite pointed set K and any integer $k \geq 0$,

$$\text{Fr}_n(X, (Y \times T^k) \otimes K) := \text{Hom}_{\mathcal{M}}(X_+ \wedge \mathbb{P}^{\wedge n}, Y_+ \wedge T^{k+n} \wedge K), \quad X, Y \in \text{Sm}/k.$$

The external smash product by σ gives rise to a map $\text{Fr}_n(X, (Y \times T^k) \otimes K) \rightarrow \text{Fr}_{n+1}(X, (Y \times T^k) \otimes K)$. One sets,

$$\text{Fr}(X, (Y \times T^k) \otimes K) := \text{colim}(\text{Fr}_0(X, (Y \times T^k) \otimes K) \xrightarrow{-\wedge \sigma} \text{Fr}_1(X, (Y \times T^k) \otimes K) \xrightarrow{-\wedge \sigma} \dots)$$

and

$$C_* \text{Fr}(X, (Y \times T^k) \otimes K) := \text{Fr}(X \times \Delta_k^*, (Y \times T^k) \otimes K).$$

By the Additivity Theorem of [19], the assignment

$$K \in \Gamma^{\text{op}} \mapsto C_* \text{Fr}(X, (Y \times T^k) \otimes K) \in \mathbf{S},$$

gives rise to a special Γ -space of pointed simplicial sets. The pointed motivic space $X \in \text{Sm}/k \mapsto C_* \text{Fr}(X, (Y \times T^k) \otimes K) \in \mathbf{S}$ is denoted by $C_* \text{Fr}((Y \times T^k) \otimes K)$ in [15, 19]. If $Y = \text{pt}$ the latter motivic space is denoted by $C_* \text{Fr}(T^k \otimes K)$.

Definition 4.9. The *symmetric framed motive* of a symmetric S^1 -spectrum $N \in Sp_{S^1}^\Sigma$ is the symmetric S^1 -spectrum

$$M_{fr}^\Sigma(N) := (C_* \text{Fr}(\text{pt}, \text{pt} \otimes N_0), \text{Hom}_{\mathcal{M}}(\mathbb{P}^{\wedge 1}, C_* \text{Fr}(T \otimes N_1)), \text{Hom}_{\mathcal{M}}(\mathbb{P}^{\wedge 2}, C_* \text{Fr}(T^2 \otimes N_2)), \dots)$$

with structure maps and actions of the symmetric groups defined similarly to $C_* \text{Fr}_*^\Sigma(N)$. The framed motive of the suspension spectrum $\Sigma_{S^1}^\infty X$ of a pointed simplicial set X will be denoted by $M_{fr}^\Sigma(X)$.

Remark 4.10. If N is the suspension symmetric spectrum $\Sigma_{S^1}^\infty X$ of a pointed simplicial set X , the framed motive in the sense of the preceding definition is a bit different from the framed motive of X evaluated at pt defined as

$$M_{fr}(X) := C_* \text{Fr}(\text{pt}, \text{pt} \otimes (\Sigma_{S^1}^\infty X)),$$

where $K \in \Gamma^{\text{op}} \mapsto C_* \text{Fr}(\text{pt}, \text{pt} \otimes K) \in \mathbf{S}_*$ is the special Γ -space (6).

Lemma 4.11. *If k is a perfect field, then the canonical map of ordinary S^1 -spectra $M_{fr}(X) \rightarrow M_{fr}^\Sigma(X)$ is a level equivalence in positive degrees for any pointed simplicial set X .*

Proof. Repeating the proof of [14, Lemma 4.12] word for word, we have that the canonical map of connected spaces

$$C_* \text{Fr}(\text{pt}, \text{pt} \otimes (X \wedge S^\ell)) \rightarrow \text{Hom}_{\mathcal{M}}(\mathbb{P}^{\wedge k}, C_* \text{Fr}(T^k \otimes (X \wedge S^\ell))), \quad k, \ell > 0,$$

is a weak equivalence. It follows that the map $M_{fr}(X) \rightarrow M_{fr}^\Sigma(X)$ is a level equivalence of spectra in positive degrees. \square

The reader should not confuse π_* -isomorphisms (i.e., maps inducing isomorphisms of stable homotopy groups) and stable equivalences of symmetric spectra. The first class is a proper subclass of the second.

Though $M_{fr}(X)$ is canonically a symmetric S^1 -spectrum, where Σ_n acts on each space by permuting S^n , the point is that it is not a S_k -module in contrast with $M_{fr}^\Sigma(X)$.

Proposition 4.12. *Given a field k and $N \in Sp_{S^1}^\Sigma$, the following statements are true.*

- (1) $C_* \text{Fr}_*(N)$ and $M_{fr}^\Sigma(N)$ are right S_k -modules.
- (2) Every map of symmetric S^1 -spectra $f : N \rightarrow N'$ induces morphisms of right S_k -modules $C_* \text{Fr}_*(f) : C_* \text{Fr}_*(N) \rightarrow C_* \text{Fr}_*(N')$ and $M_{fr}^\Sigma(f) : M_{fr}^\Sigma(N) \rightarrow M_{fr}^\Sigma(N')$.
- (3) The canonical map $\alpha : S_k \rightarrow M_{fr}^\Sigma(S^0)$ in $\text{Mod } S_k$ is a π_* -isomorphism (i.e., a stable equivalence of ordinary spectra) whenever the base field k is perfect.
- (4) For every $N \in Sp_{S^1}^\Sigma$ the canonical map $\beta : C_* \text{Fr}_*(N) \rightarrow M_{fr}^\Sigma(N)$ is a π_* -isomorphism whenever the base field k is perfect.

Proof.

- (1). The desired pairing

$$(C_* \text{Fr}_*(N) \wedge S_k)_m = \bigvee_{q+p=m} (\Sigma_{q+p})_+ \wedge_{\Sigma_q \times \Sigma_p} C_* \text{Fr}_q(N_q) \wedge \text{Fr}_p(\Delta^*, S^p) \rightarrow C_* \text{Fr}_m(N_m)$$

is defined as follows. Given two morphisms $(\beta : \Delta_+^\ell \wedge \mathbb{P}^{\wedge q} \rightarrow T^q \wedge N_q) \in \text{Fr}_q(\Delta^\ell, N_q)$ and $(\alpha : \Delta_+^\ell \wedge \mathbb{P}^{\wedge p} \rightarrow T^p \wedge S^p) \in \text{Fr}_p(\Delta^\ell, S^p)$, define a morphism $\beta \star \alpha \in \text{Fr}(\Delta^\ell, N_{q+p})$ as the composite

$$\begin{aligned} (\Delta^\ell)_+ \wedge \mathbb{P}^{\wedge(q+p)} &\xrightarrow{\text{diag} \wedge \text{id}} (\Delta^\ell \times \Delta^\ell)_+ \wedge \mathbb{P}^{\wedge(q+p)} \cong (\Delta_+^\ell \wedge \mathbb{P}^{\wedge q}) \wedge (\Delta_+^\ell \wedge \mathbb{P}^{\wedge p}) \\ &\xrightarrow{\beta \wedge \alpha} (T^q \wedge N_q) \wedge (T^p \wedge S^p) \cong T^{q+p} \wedge N_q \wedge S^p \rightarrow T^{q+p} \wedge N_{q+p}. \end{aligned}$$

It is straightforward to see that this pairing is $\Sigma_q \times \Sigma_p$ -equivariant, satisfies associativity and unit conditions, hence it defines the structure of a right S_k -module on $C_* \text{Fr}_*^\Sigma(N)$. For the same reasons, $M_{f_r}^\Sigma(N)$ is a right S_k -module.

- (2). This is straightforward.
- (3). Let $\Theta_{S^1}^\infty$ be the naive stabilization functor of S^1 -spectra. It has the property that $X \rightarrow \Theta_{S^1}^\infty(X)$ is a stable equivalence for every S^1 -spectrum X [22, Proposition 4.7].

Let $A_{n,r} := C_*(\mathbb{P}^{\wedge n}, T^n \wedge S^r)$, $B_{m,n,r} := C_*(\mathbb{P}^{\wedge(m+n)}, T^{m+n} \wedge S^r)$, $m, n, r \geq 0$. We have maps of spaces

$$A_{n,r} \xrightarrow{\sigma} A_{n+1,r} \quad \text{and} \quad B_{m,n,r} \xrightarrow{\sigma} B_{m+1,n,r} \xrightarrow{\sigma} B_{m+1,n+1,r}.$$

Let $A_r := \text{colim}_{n \geq r} A_{n,r}$, $B_{m,r} := \text{colim}_n B_{m,n,r}$, $B_r := \text{colim}_{m,n} B_{m,n,r}$. The spaces A_r, B_r constitute right S^1 -spectra A and B . Each structure map $A_r \rightarrow \underline{\text{Hom}}(S^1, A_{r+1})$ is the composite map determined by the following commutative diagram

$$\begin{array}{ccccc} A_r : & (\mathbb{P}^{\wedge r}, T^r \wedge S^r) & \xrightarrow{\sigma} & (\mathbb{P}^{\wedge(r+1)}, T^{r+1} \wedge S^r) & \xrightarrow{\sigma} \dots \\ & \downarrow \sigma & & \downarrow \sigma & \\ & (\mathbb{P}^{\wedge(r+1)}, T^{r+1} \wedge S^r) & \xrightarrow{\sigma} & (\mathbb{P}^{\wedge(r+2)}, T^{r+2} \wedge S^r) & \xrightarrow{\sigma} \dots \\ & \downarrow -\wedge S^1 & & \downarrow -\wedge S^1 & \\ \underline{\text{Hom}}(S^1, A_{r+1}) : & (S^1, (\mathbb{P}^{\wedge(r+1)}, T^{r+1} \wedge S^{r+1})) & \xrightarrow{(\sigma, S^1)} & (S^1, (\mathbb{P}^{\wedge(r+2)}, T^{r+2} \wedge S^{r+1})) & \xrightarrow{(\sigma, S^1)} \dots \end{array}$$

(we omit C_* here and below for brevity). Each structure map $B_r \rightarrow \underline{\text{Hom}}(S^1, B_{r+1})$ is defined in a similar fashion.

There is a commutative diagram of spectra

$$\begin{array}{ccccc} & & \xrightarrow{j_{S_k}} & & \\ S_k & \xrightarrow{a} & A & \xrightarrow{j_A} & \Theta_{S^1}^\infty(A) \\ \downarrow \alpha & & \downarrow c & & \downarrow \Theta_{S^1}^\infty(c) \\ M_{f_r}^\Sigma(S^0) & \xrightarrow{b} & B & \xrightarrow{j_B} & \Theta_{S^1}^\infty(B) \\ & & \xrightarrow{j_{M_{f_r}^\Sigma(S^0)}} & & \end{array}$$

in which $\Theta_{S^1}^\infty(A) = \Theta_{S^1}^\infty(S_k)$, $\Theta_{S^1}^\infty(B) = \Theta_{S^1}^\infty(M_{f_r}^\Sigma(S^0))$. The maps a, b, c are defined in a canonical way. By the two-out-of-three property a, b are stable equivalences. Therefore, α is a stable equivalence if and only if c is.

But c is the infinite composition

$$A_r \cong B_{0,r} \rightarrow B_{1,r} \rightarrow B_{2,r} \rightarrow \dots B_r.$$

Each composition $A_r \rightarrow B_{m,r}$ is isomorphic to the canonical map of spaces

$$C_* \text{Fr}(S^r) \rightarrow \text{Hom}_{\mathcal{M}}(\mathbb{P}^{\wedge m}, C_* \text{Fr}(T^m \otimes S^r)).$$

Repeating the proof of [14, Lemma 4.12] word for word, the map is a weak equivalence for positive r , and hence c is a weak equivalence in positive degrees.

- (4). The proof literally repeats that of (3) if we replace S^r by N_r . □

Recall that we distinguish classical (also called naive) stable homotopy groups $\pi_*(N)$ of $N \in Sp_{S^1}^\Sigma$ and “true” stable homotopy groups (denote them by $\underline{\pi}_*(N)$, see, e.g., [31] for details). They coincide for semistable symmetric spectra.

Corollary 4.13. *Given a perfect field k , the symmetric spectrum S_k is semistable and $\pi_0(S_k) = \underline{\pi}_0(S_k) = K_0^{MW}(k)$, where $K_0^{MW}(k)$ is the Milnor–Witt group of k .*

Proof. By [19, Corollary 11.2] $\pi_{0,0}(\Sigma_{S^1}^\infty \Sigma_{G_m}^\infty S^0) = \pi_0(M_{fr}(\text{pt}))$. By a theorem of Morel [26], one has $\pi_{0,0}(\Sigma_{S^1}^\infty \Sigma_{G_m}^\infty S^0) = K_0^{MW}(k)$. By definition [19], $M_{fr}(\text{pt})$ is the Segal symmetric spectrum associated with a special Γ -space. Therefore, $M_{fr}(\text{pt})$ is semistable by [30, Example 4.2]. Our statement now follows from Lemma 4.11 and Proposition 4.12(3). \square

By e^{-1} -stable equivalences (respectively $\pi_*[e^{-1}]$ -isomorphisms), we mean maps of symmetric spectra (respectively, ordinary spectra) inducing isomorphisms in $SH[e^{-1}]$ (respectively, isomorphisms of stable homotopy groups with e^{-1} -coefficients).

Theorem 4.14. *If \bar{k} is an algebraically closed field of exponential characteristic e , then the natural maps of symmetric S^1 -spectra*

$$\nu : N \rightarrow C_* \text{Fr}_*^\Sigma(N), \quad \beta \circ \nu : N \rightarrow M_{fr}^\Sigma(N)$$

are $\pi_*[e^{-1}]$ -isomorphisms, where $\beta : C_* \text{Fr}_*^\Sigma(N) \rightarrow M_{fr}^\Sigma(N)$ is a canonical map. In particular, a map $\gamma : N \rightarrow N'$ of symmetric S^1 -spectra is an e^{-1} -stable equivalence or a $\pi_*[e^{-1}]$ -isomorphism if and only if $M_{fr}^\Sigma(\gamma)$ is.

Proof. The map β is a π_* -isomorphism by Proposition 4.12(4). Consider a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\nu} & C_* \text{Fr}_*^\Sigma(N) \\ & \searrow \chi_N & \downarrow a \\ & & M_{fr}(N) \end{array} \quad \begin{array}{ccc} & & \xrightarrow{j_{C_* \text{Fr}_*(N)}} \\ & & \Theta_{S^1}^\infty M_{fr}(N) \\ & & \downarrow j_{M_{fr}(N)} \\ & & M_{fr}(N) \end{array}$$

We see that a is a π_* -isomorphism. Therefore, ν is a $\pi_*[e^{-1}]$ -isomorphism if and only if χ_N is.

We claim that the map of S^1 -spectra $\chi_X : \Sigma_{S^1}^\infty X \rightarrow M_{fr}(X)$ is a $\pi_*[e^{-1}]$ -isomorphism. Indeed, if X is a finite pointed set regarded as a constant simplicial pointed set, this follows from Theorem 4.5. If X is any pointed set, then χ_X is a directed colimit of maps χ_W , where W runs over finite pointed subsets of X . Hence, χ_X is a $\pi_*[e^{-1}]$ -isomorphism as directed colimits preserve $\pi_*[e^{-1}]$ -isomorphisms. Finally, as the geometric realization of a simplicial $\pi_*[e^{-1}]$ -isomorphism is an $\pi_*[e^{-1}]$ -isomorphism, then so is χ_X for an arbitrary pointed simplicial set X as claimed.

Next, every (ordinary) spectrum $N \in Sp_{S^1}$ equals $\text{colim}_i L_i(N)$, where each spectrum $L_i(N) = (N_0, \dots, N_{i-1}, N_i, N_i \wedge S^1, N_i \wedge S^2, \dots)$. Then $\chi_N : N \rightarrow M_{fr}(N)$ equals $\chi_N = \text{colim}_i \chi_{L_i(N)}$. It is a $\pi_*[e^{-1}]$ -isomorphism as each $\chi_{L_i(N)}$ is (this follows from the previous claim about χ_X). Therefore, the natural map of S^1 -spectra $\chi_N : N \rightarrow M_{fr}(N)$ is a $\pi_*[e^{-1}]$ -isomorphism for all $N \in Sp_{S^1}$. \square

We are now in a position to prove the main result of this section saying that the stable homotopy category of classical symmetric spectra can be recovered from the stable homotopy category of framed spectra over an algebraically closed field (after inverting the exponential characteristic).

Theorem 4.15. *Suppose $k = \bar{k}$ is an algebraically closed field of exponential characteristic e . The Quillen pair (L, U) (7) is a Quillen equivalence. In particular, it induces an equivalence of compactly generated triangulated categories*

$$L : SH[e^{-1}] \rightleftarrows SH_k[e^{-1}] : U.$$

Moreover, the equivalence L is isomorphic to the functor

$$M_{fr}^\Sigma : SH[e^{-1}] \xrightarrow{\sim} SH_k[e^{-1}]$$

that takes a symmetric S^1 -spectrum N to its symmetric framed motive $M_{fr}^\Sigma(N)$.

Proof. $SH = \text{Ho}(Sp_{S^1}^\Sigma)$ is compactly generated by the sphere spectrum S . SH_k is compactly generated by the framed sphere spectrum S_k . By construction, $L(S) = S_k$. Therefore, our statement that (L, U) is a Quillen equivalence reduces to showing that the composite map of S^1 -spectra is a e^{-1} -stable equivalence

$$\varphi : S \rightarrow UL(S) = U(S_k) \rightarrow U(S_k^f),$$

where S_k^f is a fibrant replacement of S_k in $\text{Mod } S_k$ (we also use Lemma 4.1 here).

We claim that $S_k^f = \Omega_{S^1} M_{fr}^\Sigma(S^1)$. Indeed, the canonical map $\alpha : S_k \rightarrow M_{fr}^\Sigma(S^0)$ in $\text{Mod } S_k$ is a π_* -isomorphism by Proposition 4.12(3). It follows from [19, Theorem 4.1] that $M_{fr}^\Sigma(S^0)$ is a positively fibrant symmetric Ω -spectrum, and hence $\Omega_{S^1} M_{fr}^\Sigma(S^1)$ is an Ω -spectrum. It follows that $S_k^f = \Omega_{S^1} M_{fr}^\Sigma(S^1)$. Theorem 4.14 implies that φ is a $\pi_*[e^{-1}]$ -isomorphism.

Next, Theorem 4.14 implies that $N \mapsto M_{fr}^\Sigma(N)$ induces a functor

$$M_{fr}^\Sigma : SH[e^{-1}] \rightarrow SH_k[e^{-1}].$$

We have that $M_{fr}^\Sigma(S) \cong S_k$ is a compact generator. The first part of the proof implies that

$$SH[e^{-1}](S[*], S) \rightarrow SH_k[e^{-1}](M_{fr}^\Sigma(S)[*], M_{fr}^\Sigma(S))$$

is an isomorphism of graded Abelian groups, hence M_{fr}^Σ is an equivalence of compactly generated triangulated categories by Lemma 4.1. By Theorem 4.14, the canonical map of S^1 -spectra $N \rightarrow M_{fr}^\Sigma(N)$ is a $\pi_*[e^{-1}]$ -isomorphism for any $N \in Sp_{S^1}^\Sigma$. Therefore, $\text{id} \rightarrow U \circ M_{fr}^\Sigma$ is an isomorphism of functors. Composing it with $(M_{fr}^\Sigma)^{-1}$, where $(M_{fr}^\Sigma)^{-1}$ is a quasi-inverse functor to M_{fr}^Σ , we get an isomorphism of functors $(M_{fr}^\Sigma)^{-1} \simeq U$. By the first part of the proof L is a quasi-inverse functor to U , and hence L is isomorphic to the functor M_{fr}^Σ , as was to be shown. \square

5 | SPECTRAL CATEGORIES ASSOCIATED TO SYMMETRIC RING SPECTRA

In what follows by a *spectral category*, we mean a category enriched over the closed symmetric monoidal category of symmetric S^1 -spectra $Sp_{S^1}^\Sigma$.

Recall from [31, Construction 5.6] that for every pair of symmetric spectra X, Y a morphism $X \wedge Y \rightarrow Z$ to a symmetric spectrum Z is the same as giving a bimorphism $b : (X, Y) \rightarrow Z$. We define a *bimorphism* $b : (X, Y) \rightarrow Z$ as a collection of $\Sigma_p \times \Sigma_q$ -equivariant maps of pointed simplicial sets $b_{p,q} : X_p \wedge Y_q \rightarrow Z_{p+q}$ for $p, q \geq 0$, such that the “bilinearity diagram” commutes for all $p, q \geq 0$:

$$\begin{array}{ccccc}
 & & X_p \wedge Y_q \wedge S^1 & \xrightarrow{X_p \wedge twist} & X^p \wedge S^1 \wedge Y_q \\
 & \swarrow^{X_p \wedge \sigma_q} & \downarrow b_{p,q} \wedge S^1 & & \downarrow \sigma_p \wedge Y_q \\
 X_p \wedge Y_{q+1} & & Z_{p+q} \wedge S^1 & & X_{p+1} \wedge Y_q \\
 & \searrow^{b_{p,q+1}} & \downarrow \sigma_{p+q} & & \downarrow b_{p+1,q} \\
 & & Z_{p+q+1} & \xleftarrow{1 \times \chi_{1,q}} & Z_{p+1+q}
 \end{array} \tag{8}$$

In this section, \mathcal{C} is the category from Section 2.

Definition 5.1. Suppose \mathcal{B} is a full subcategory of \mathcal{C} closed under \wedge and $\sigma : P \rightarrow T$ is a morphism in \mathcal{C} . Let E be a symmetric ring T -spectrum in $Sp^\Sigma(\mathcal{C}, T)$. We define the *symmetric S^1 -spectrum* of (E, σ) -correspondences $\text{Corr}_*^{E,\sigma}(X, Y)$ between two objects $X, Y \in \mathcal{B}$ as follows. First, let

$$\text{Corr}_n^{E,\sigma}(X, Y) := \text{Hom}_{\mathcal{C}}(X \wedge P^{\wedge n}, Y \wedge E_n \wedge S^n).$$

This simplicial set is pointed at the zeroth map. By definition, $\text{Corr}_0^{E,\sigma}(X, Y) := \text{Hom}_{\mathcal{C}}(X, Y \wedge E_0)$. Similarly to (3), each $\text{Corr}_n^{E,\sigma}(X, Y)$ is a Σ_n -simplicial set. The left action of Σ_n on $\text{Corr}_n^{E,\sigma}(X, Y)$ is given by conjugation: for each $f : X \wedge P^{\wedge n} \rightarrow Y \wedge E_n \wedge S^n$ and each $\tau \in \Sigma_n$ the morphism $\tau \cdot f$ is defined as the composition

$$X \wedge P^{\wedge n} \xrightarrow{X \wedge \tau^{-1}} X \wedge P^{\wedge n} \xrightarrow{f} Y \wedge E_n \wedge S^n \xrightarrow{Y \wedge \tau \wedge \tau} Y \wedge E_n \wedge S^n.$$

Second, repeating the proof of Theorem 3.2(1) word for word the morphism σ induces natural $(\Sigma_n \times \Sigma_k)$ -equivariant maps

$$\text{Corr}_n^{E,\sigma}(X, Y) \wedge S^k \rightarrow \text{Corr}_{n+k}^{E,\sigma}(X, Y),$$

so that

$$\text{Corr}_*^{E,\sigma}(X, Y) := (\text{Corr}_0^{E,\sigma}(X, Y), \text{Corr}_1^{E,\sigma}(X, Y), \text{Corr}_2^{E,\sigma}(X, Y), \dots)$$

becomes a symmetric S^1 -spectrum.

Define a pairing

$$\varphi_{X,Y,Z}^\sigma : \text{Corr}_n^{E,\sigma}(X, Y) \wedge \text{Corr}_m^{E,\sigma}(Y, Z) \rightarrow \text{Corr}_{n+m}^{E,\sigma}(X, Z)$$

by the rule: $\varphi_{X,Y,Z}^\sigma(f : X \wedge P^{\wedge n} \rightarrow Y \wedge E_n \wedge S^n, g : Y \wedge P^{\wedge m} \rightarrow Z \wedge E_m \wedge S^m)$ is given by the composition

$$\begin{aligned} X \wedge P^{\wedge n} \wedge P^{\wedge m} &\xrightarrow{f \wedge P^{\wedge m}} Y \wedge E_n \wedge S^n \wedge P^{\wedge m} \xrightarrow{tw} Y \wedge P^{\wedge m} \wedge E_n \wedge S^n \xrightarrow{g \wedge E_n} \\ Z \wedge E_m \wedge S^m \wedge E_n \wedge S^n &\xrightarrow{tw} Z \wedge E_n \wedge E_m \wedge S^n \wedge S^m \xrightarrow{Z \wedge \mu_{n,m} \wedge S^n \wedge S^m} Z \wedge E_{n+m} \wedge S^{n+m}. \end{aligned}$$

Theorem 5.2. *Let E be a symmetric ring T -spectrum in $Sp^\Sigma(\mathcal{C}, T)$ and \mathcal{B} is a full subcategory of \mathcal{C} closed under monoidal product. Then \mathcal{B} can be enriched over the closed symmetric monoidal category of symmetric S^1 -spectra $Sp_{S^1}^\Sigma$. Namely, $Sp_{S^1}^\Sigma$ -objects of morphisms are defined by the symmetric spectra $\text{Corr}_*^{E,\sigma}(X, Y)$ of (E, σ) -correspondences. Compositions are defined by pairings $\varphi_{X,Y,Z}$. The resulting $Sp_{S^1}^\Sigma$ -category is denoted by $\text{Corr}_*^{E,\sigma}(\mathcal{B})$. Moreover, the $Sp_{S^1}^\Sigma$ -category $\text{Corr}_*^{E,\sigma}(\mathcal{B})$ is symmetric monoidal with the same monoidal product on objects as in \mathcal{B} whenever E is a commutative ring T -spectrum.*

Proof. The identity morphism is defined by

$$X \cong X \wedge S \xrightarrow{\rho^{-1}} X \wedge S \xrightarrow{\text{id}_X \wedge \iota_0} X \wedge E_0 \in \text{Corr}_0^E(X, X),$$

where $\iota_0 : S \rightarrow E_0$ is the unit map. Our proof now literally repeats that of Theorem 2.4. The only thing one has to care about is that the pairings occurring here are bimorphisms of symmetric spectra. This is clearly the case if one chases over the diagram (8). It is also worth noting that $(E_0, E_1 \wedge S^1, E_2 \wedge S^2, \dots)$ is a ring object in the category of simplicial symmetric sequences in \mathcal{C} , which is commutative whenever E is. \square

The proof of the theorem implies the following result.

Corollary 5.3. *Under the notation of Theorem 5.2 any morphism of two symmetric ring T -spectra $\gamma : E \rightarrow E'$ in $Sp^\Sigma(\mathcal{C}, T)$ induces a morphism of spectral categories $\gamma_* : \text{Corr}_*^{E,\sigma}(\mathcal{B}) \rightarrow \text{Corr}_*^{E',\sigma}(\mathcal{B})$.*

Day's Theorem [2] together with Theorem 5.2 also imply the following result.

Corollary 5.4. *Under the notation of Theorem 5.2 if E is a commutative symmetric ring T -spectrum then the category of right $\text{Corr}_*^{E,\sigma}(\mathcal{B})$ -modules is a closed symmetric monoidal category.*

Theorem 5.5. *Suppose E is a commutative symmetric ring T -spectrum in $Sp^\Sigma(\mathcal{C}, T)$. Under the assumptions of Theorem 5.2 the spectral category $\text{Corr}_*^{E,\sigma}(\mathcal{B})$ is also a symmetric monoidal $\text{Mod } \mathbf{1}^E$ -category with the same monoidal product on objects as in \mathcal{B} , where $\text{Mod } \mathbf{1}^E$ is the closed symmetric monoidal category of Corollary 3.3.*

Proof. Each symmetric spectrum $\text{Corr}_*^{E,\sigma}(\mathcal{B})(U, V)$, $U, V \in \text{Ob } \mathcal{B}$, is canonically in $\text{Mod } \mathbf{1}^E$ if we define compositions $\theta_{p,r} : \text{Corr}_{p,r}^{E,\sigma}(\mathcal{B})(U, V) \wedge \mathbf{1}_r^E \rightarrow \text{Corr}_{p+r}^{E,\sigma}(\mathcal{B})(U, V)$ by

$$U \wedge P^{\wedge p} \wedge P^{\wedge r} \xrightarrow{f \wedge g} V \wedge E_p \wedge S^p \wedge E_r \wedge S^r \xrightarrow{tw} V \wedge E_n \wedge E_r \wedge S^{n+r} \xrightarrow{V \wedge \mu_{n,r} \wedge S^{n+r}} V \wedge E_{n+r} \wedge S^{n+r},$$

where $f \in \text{Corr}_p^{E,\sigma}(\mathcal{B})(U, V)$, $g \in \mathbf{1}_r^E$ and $\mu_{*,*}$ is the multiplication map of E . The proof is like that of Theorem 5.2 if we observe that the diagram

$$\begin{array}{ccccc}
 & X_p \wedge Y_q \wedge \mathbf{1}_r^E & \xrightarrow{X_p \wedge \text{twist}} & X^p \wedge \mathbf{1}_r^E \wedge Y_q & \\
 & \swarrow X_p \wedge \theta_{q,r} & \downarrow b_{p,q} \wedge \mathbf{1}_r^E & \downarrow \theta_{p,r} \wedge Y_q & \\
 X_p \wedge Y_{q+r} & & Z_{p+q} \wedge \mathbf{1}_r^E & & X_{p+r} \wedge Y_q \\
 & \searrow b_{p,q+r} & \downarrow \theta_{p+q,r} & & \downarrow b_{p+r,q} \\
 & & Z_{p+q+r} & \xleftarrow{1 \times \chi_{r,q}} & Z_{p+r+q}
 \end{array} \tag{9}$$

is commutative with $X = \text{Corr}_*^{E,\sigma}(\mathcal{B})(U, V)$, $Y = \text{Corr}_*^{E,\sigma}(\mathcal{B})(V, W)$, $Z = \text{Corr}_*^{E,\sigma}(\mathcal{B})(U, W)$ and $b = \varphi_{U,V,W}$. □

Corollary 5.6. *Under the notation of Theorem 5.5 any morphism of two symmetric ring T -spectra $\gamma : E \rightarrow E'$ in $Sp^\Sigma(\mathcal{C}, T)$ induces a morphism of $\text{Mod } \mathbf{1}^E$ -categories $\gamma_* : \text{Corr}_*^{E,\sigma}(\mathcal{B}) \rightarrow \text{Corr}_*^{E',\sigma}(\mathcal{B})$.*

Day’s Theorem [2] together with Theorem 5.2 also imply the following result.

Corollary 5.7. *Under the notation of Theorem 5.5 if E is a commutative symmetric ring T -spectrum then the category of $\text{Corr}_*^{E,\sigma}(\mathcal{B})$ -modules in the category $\text{Mod } \mathbf{1}^E$ is a closed symmetric monoidal category.*

6 | ENRICHED MOTIVIC HOMOTOPY THEORY

One of the approaches to Morel–Voevodsky’s stable motivic homotopy theory $SH(k)$ over a field k is by means of symmetric T -spectra $Sp_T^\Sigma(k)$, where $T = \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$ (see, e.g., [24]). In detail, we start with motivic spaces \mathcal{M} equipped with the flasque motivic model structure in the sense of [23] and then pass to $Sp_T^\Sigma(k)$ equipped with the stable model structure. The homotopy category of the latter model category is denoted by $SH(k)$.

A genuinely local approach to $SH(k)$, envisioned by Voevodsky in 2001, is presented in [20]. It is based on Voevodsky’s framed correspondences and the machinery of framed motives [19].

In this section, we suggest yet another (genuinely local) approach to $SH(k)$ and, more generally, a local model for the category of E -modules in $SH(k)$, where E is a symmetric Thom ring spectrum. It is an application of enriched category theory of spectral categories and spectral modules of Section 5. The same approach was used in [16, 17] to construct the theory of K -motives.

Following [14], a symmetric T -spectrum E is called a *Thom spectrum* if each motivic space E_n has the form

$$E_n = \text{colim}_i E_{n,i}, \quad E_{n,i} = V_{n,i}/(V_{n,i} - Z_{n,i}),$$

where $V_{n,i} \rightarrow V_{n,i+1}$ is a directed sequence of smooth varieties, $Z_{n,i} \rightarrow Z_{n,i+1}$ is a directed system of smooth closed subschemes in $V_{n,i}$. We say that a Thom spectrum E has the *bounding constant* d if d is the minimal integer such that codimension of $Z_{n,i}$ in $V_{n,i}$ is strictly greater than $n - d$ for all i, n . The T -spectrum E is said to be a *spectrum with contractible alternating group action*, if for

any n and any even permutation $\tau \in \Sigma_n$ there is an \mathbb{A}^1 -homotopy $E_n \rightarrow \underline{\text{Hom}}(\mathbb{A}^1, E_n)$ between the action of τ and the identity map. In other words, E neglects the action of even permutations up to \mathbb{A}^1 -homotopy.

Unless it is specified otherwise E is a symmetric Thom ring T -spectrum with the bounding constant $d \leq 1$ and contractible alternating group action throughout this section. By [14, Lemma 10.2], $E \in SH^{\text{eff}}(k)$, where $SH^{\text{eff}}(k)$ is the full triangulated subcategory of $SH(k)$ of effective T -spectra. It is compactly generated by the suspension T -spectra $\Sigma_T^\infty X_+$, $X \in \text{Sm}/k$. For instance, E is the algebraic cobordism T -spectrum MGL or motivic sphere spectrum $S_k = (S^0, T, T^2, \dots)$. Other examples are commutative symmetric ring T^2 -spectra MSL and MSp [29]. The results that use T^2 -spectra are the same with those proven in this section and which use T -spectra. For brevity, we will deal with T -spectra only.

We can apply Theorem 5.5 to the following data:

- ◊ $C = \mathcal{M}$;
- ◊ the canonical map $\sigma : \mathbb{P}^{\wedge 1} \rightarrow T$, where $\mathbb{P}^{\wedge 1} := (\mathbb{P}^1, \infty) \in \mathcal{M}$, given by the framed correspondence $(\{0\}, \mathbb{A}^1, t) \in \text{Fr}_1(\text{pt}, \text{pt})$;
- ◊ $B = \{X_+ \mid X \in \text{Sm}/k\}$.

Within this notation, the symmetric monoidal spectral category $\text{Corr}_*^{E, \sigma}(B)$ of Theorem 5.5 will be denoted by \mathcal{O}^E for brevity and each $\text{Corr}_n^{E, \sigma}(B)(X, Y) = \text{Hom}_{\mathcal{M}}(X_+ \wedge \mathbb{P}^{\wedge n}, Y_+ \wedge E_n \wedge S^n)$ will be denoted by $\text{Fr}_n^E(X, Y \otimes S^n)$. Recall from [19, section 3] that each simplicial set $\text{Fr}_n^E(X, Y \otimes S^n) = \text{Hom}_{\mathcal{M}}(X_+ \wedge \mathbb{P}^{\wedge n}, Y_+ \wedge E_n \wedge S^n)$ has an explicit geometric description due to Voevodsky's lemma.

Similarly to Definition 4.6, we can consider a spectral category \mathcal{O}_Δ^E that is obtained from \mathcal{O}^E by applying the Suslin complex to symmetric spectra of morphisms:

$$\mathcal{O}_\Delta^E(X, Y) := (\text{Fr}_0^E(\Delta_k^* \times X, Y), \text{Fr}_1^E(\Delta_k^* \times X, Y \otimes S^1), \dots).$$

Let \mathcal{O} be a spectral category and let $\text{Mod } \mathcal{O}$ be the category of \mathcal{O} -modules. Recall that the projective stable model structure on $\text{Mod } \mathcal{O}$ is defined as follows (see [33]). The weak equivalences are the objectwise stable weak equivalences and fibrations are the objectwise stable projective fibrations. The stable projective cofibrations are defined by the left lifting property with respect to all stable projective acyclic fibrations.

Let \mathcal{Q} denote the set of elementary distinguished squares in Sm/k (see [27, 3.1.3])

$$\begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & \mathcal{Q} & \downarrow \varphi \\ U & \xrightarrow{\psi} & X \end{array}$$

and let \mathcal{O} be a spectral category over Sm/k . By $\mathcal{Q}_{\mathcal{O}}$ denote the set of squares

$$\begin{array}{ccc} \mathcal{O}(-, U') & \longrightarrow & \mathcal{O}(-, X') \\ \downarrow & \mathcal{O}\mathcal{Q} & \downarrow \varphi \\ \mathcal{O}(-, U) & \xrightarrow{\psi} & \mathcal{O}(-, X) \end{array} \quad (10)$$

which are obtained from the squares in \mathcal{Q} by taking $X \in \text{Sm}/k$ to $\mathcal{O}(-, X)$. The arrow $\mathcal{O}(-, U') \rightarrow \mathcal{O}(-, X')$ can be factored as a cofibration $\mathcal{O}(-, U') \rightarrow \text{Cyl}$ followed by a simplicial homotopy

equivalence $Cyl \rightarrow \mathcal{O}(-, X')$. There is a canonical morphism $A_{\mathcal{O}Q} := \mathcal{O}(-, U) \bigsqcup_{\mathcal{O}(-, U')} Cyl \rightarrow \mathcal{O}(-, X)$.

Definition 6.1 (see [16, 17]). We say that \mathcal{O} is *Nisnevich excisive* if for every elementary distinguished square Q the square $\mathcal{O}Q$ (10) is homotopy pushout in the Nisnevich local model structure on $Sp_{S^1}^\Sigma(k) := Sp^\Sigma(\mathcal{M}, S^1)$.

The *Nisnevich local model structure* on $\text{Mod } \mathcal{O}$ is the Bousfield localization of the stable projective model structure with respect to the family of projective cofibrations

$$\mathcal{N}_{\mathcal{O}} = \{cyl(A_{\mathcal{O}Q} \rightarrow \mathcal{O}(-, X))\}_{Q_{\mathcal{O}}}.$$

The homotopy category for the Nisnevich local model structure will be denoted by $SH_{S^1}^{nis} \mathcal{O}$.

Suppose \mathcal{O} is symmetric monoidal. By a theorem of Day [2], $\text{Mod } \mathcal{O}$ is a closed symmetric monoidal category with smash product \wedge and $\mathcal{O}(-, pt)$ being the monoidal unit. The smash product is defined as

$$M \wedge_{\mathcal{O}} N = \int^{\text{Ob } \mathcal{O} \otimes \mathcal{O}} M(X) \wedge N(Y) \wedge \mathcal{O}(-, X \times Y). \tag{11}$$

The internal Hom functor, right adjoint to $- \wedge_{\mathcal{O}} M$, is given by

$$\underline{\text{Mod}} \mathcal{O}(M, N)(X) := Sp^\Sigma(M, N(X \times -)) = \int_{Y \in \text{Ob } \mathcal{O}} \underline{Sp}^\Sigma(M(Y), N(X \times Y)).$$

By [6, Corollary 2.7] that there is a natural isomorphism

$$\mathcal{O}(-, X) \wedge_{\mathcal{O}} \mathcal{O}(-, Y) \cong \mathcal{O}(-, X \times Y).$$

Theorem 6.2 [16]. *Suppose \mathcal{O} is a Nisnevich excisive spectral category. Then the Nisnevich local model structure on $\text{Mod } \mathcal{O}$ is cellular, proper, spectral and weakly finitely generated. Moreover, a map of \mathcal{O} -modules is a weak equivalence in the Nisnevich local model structure if and only if it is a weak equivalence in the Nisnevich local model structure on $Sp_{S^1}^\Sigma(k)$. If \mathcal{O} is a symmetric monoidal spectral category then the model structure on $\text{Mod } \mathcal{O}$ is symmetric monoidal with respect to the smash product (11) of \mathcal{O} -modules.*

In our setting, we regard spectral categories $\mathcal{O}^E, \mathcal{O}_\Delta^E$ as symmetric monoidal $\text{Mod } \mathbf{1}^E$ -categories with the same monoidal product on objects as in Sm/k (see Theorem 5.5), where $\text{Mod } \mathbf{1}^E$ is the closed symmetric monoidal category of Corollary 3.3. Denote by $\text{Mod } \mathcal{O}^E$ and $\text{Mod } \mathcal{O}_\Delta^E$ the closed symmetric monoidal categories of \mathcal{O}^E - and \mathcal{O}_Δ^E -modules in the category $\text{Mod } \mathbf{1}^E$ (see Corollary 5.7).

The Nisnevich local model structure on $\text{Mod } \mathcal{O}^E$ and $\text{Mod } \mathcal{O}_\Delta^E$ as well as their homotopy categories $SH_{S^1}^{nis} \mathcal{O}^E$ and $SH_{S^1}^{nis} \mathcal{O}_\Delta^E$ are defined similarly to Definition 6.1.

Given $X \in \text{Sm}/k$ and a motivic space $G \in \mathcal{M}$, denote by $C_* \text{Fr}_n^E(X_+ \wedge G)$ the pointed motivic space $U \in \text{Sm}/k \mapsto \text{Hom}_{\mathcal{M}}((U \times \Delta_k^*)_+ \wedge \mathbb{P}^{\wedge n}, X_+ \wedge G \wedge E_n)$. One has a canonical map $C_* \text{Fr}_n^E(X_+ \wedge G) \rightarrow C_* \text{Fr}_{n+1}^E(X_+ \wedge G)$ defined by the composition

$$\begin{aligned} & \text{Hom}_{\mathcal{M}}((U \times \Delta_k^*)_+ \wedge \mathbb{P}^{\wedge n}, X_+ \wedge G \wedge E_n) \xrightarrow{-\wedge \sigma} \\ & \text{Hom}_{\mathcal{M}}((U \times \Delta_k^*)_+ \wedge \mathbb{P}^{\wedge n+1}, X_+ \wedge G \wedge E_n \wedge T) \rightarrow \underline{\text{Hom}}_{\mathcal{M}}((U \times \Delta_k^*)_+ \wedge \mathbb{P}^{\wedge n+1}, X_+ \wedge G \wedge E_{n+1}). \end{aligned}$$

We set,

$$C_* \operatorname{Fr}^E(X_+ \wedge G) := \operatorname{colim}(C_* \operatorname{Fr}_0^E(X_+ \wedge G) \rightarrow C_* \operatorname{Fr}_1^E(X_+ \wedge G) \rightarrow \dots).$$

If we drop Δ_k^* from the definition of $C_* \operatorname{Fr}^E(X_+ \wedge G)$, one gets motivic spaces $\operatorname{Fr}^E(X_+ \wedge G)$.

Definition 6.3. The *symmetric E -framed motive* of a smooth algebraic variety $X \in \operatorname{Sm}/k$ is the symmetric S^1 -spectrum

$$M_E^\Sigma(X) := (C_* \operatorname{Fr}^E(X), \underline{\operatorname{Hom}}_{\mathcal{M}}(\mathbb{P}^{\wedge 1}, C_* \operatorname{Fr}^E(X_+ \wedge E_1 \wedge S^1)), \underline{\operatorname{Hom}}_{\mathcal{M}}(\mathbb{P}^{\wedge 2}, C_* \operatorname{Fr}^E(X_+ \wedge E_2 \wedge S^2)), \dots)$$

with structure maps defined similarly to $M_{fr}^\Sigma(N)$ of Definition 4.9.

Remark 6.4. The E -framed motive in the sense of the preceding definition is a bit different from the E -framed motive of X in the sense of [14] defined as

$$M_E(X) := (C_* \operatorname{Fr}^E(X), C_* \operatorname{Fr}^E(X \wedge S^1), C_* \operatorname{Fr}^E(X \wedge S^2), \dots).$$

Lemma 6.5. *If k is a perfect field, then the canonical map of ordinary S^1 -spectra $M_E(X) \rightarrow M_E^\Sigma(X)$ is a level local equivalence in positive degrees for any $X \in \operatorname{Sm}/k$.*

Proof. The proof is like that of Lemma 4.11. We also use [14, section 7] here. \square

Though $M_E(X)$ is canonically a symmetric S^1 -spectrum in $S\mathcal{P}_{S^1}^\Sigma(k)$, where Σ_n acts on each space by permuting S^n , the point is that it is not an \mathcal{O}_Δ^E -module in contrast with $M_E^\Sigma(X)$.

Proposition 6.6. *Given a field k and $X \in \operatorname{Sm}/k$, the following statements are true:*

- (1) $M_E^\Sigma(X)$ is an \mathcal{O}_Δ^E -module.
- (2) The canonical map $\alpha : \mathcal{O}_\Delta^E(-, X) \rightarrow M_E^\Sigma(X)$ in $\operatorname{Mod}\mathcal{O}_\Delta^E$ is a sectionwise π_* -isomorphism (i.e., a stable equivalence of ordinary spectra) whenever the base field k is perfect.

Proof.

- (1). $M_E^\Sigma(X)$ is an \mathcal{O}_Δ^E -module for the same reasons as the representable $\mathcal{O}_\Delta^E(-, X)$ is.
- (2). The proof is like that of Proposition 4.12. We also use [14, section 7] here. \square

Theorem 6.7. *Let k be a perfect field. The commutative spectral category \mathcal{O}_Δ^E is Nisnevich excisive and the Nisnevich local model structure on $\operatorname{Mod}\mathcal{O}_\Delta^E$ has all the properties of Theorem 6.2. The category $\operatorname{SH}_{S^1}^{\operatorname{nis}}\mathcal{O}_\Delta^E$ is closed symmetric monoidal compactly generated triangulated with compact generators being the symmetric E -framed motives $\{M_E^\Sigma(X) \mid X \in \operatorname{Sm}/k\}$. The monoidal product $M_E^\Sigma(X) \wedge^L M_E^\Sigma(Y)$ in $\operatorname{SH}_{S^1}^{\operatorname{nis}}\mathcal{O}_\Delta^E$ is isomorphic to $M_E^\Sigma(X \times Y)$.*

Proof. \mathcal{O}_Δ^E is Nisnevich excisive by [14, section 9], Lemma 6.5, and Proposition 6.6. The fact that the Nisnevich local model structure on $\operatorname{Mod}\mathcal{O}_\Delta^E$ has all the properties of Theorem 6.2 follows from the fact that \mathcal{O}_Δ^E is Nisnevich excisive symmetric monoidal.

$\operatorname{SH}_{S^1}^{\operatorname{nis}}\mathcal{O}_\Delta^E$ is closed symmetric monoidal compactly generated triangulated with compact generators being the representable \mathcal{O}_Δ^E -modules $\{\mathcal{O}_\Delta^E(-, X) \mid X \in \operatorname{Sm}/k\}$. The isomorphism $M_E^\Sigma(X) \wedge$

$M_E^\Sigma(Y) \cong M_E^\Sigma(X \times Y)$ in $\mathrm{SH}_{S^1}^{\mathrm{nis}} \mathcal{O}_\Delta^E$ follows from the isomorphism

$$\mathcal{O}_\Delta^E(-, X \times Y) \cong \mathcal{O}_\Delta^E(-, X) \wedge \mathcal{O}_\Delta^E(-, Y) \cong \mathcal{O}_\Delta^E(-, X) \wedge^L \mathcal{O}_\Delta^E(-, Y)$$

and Proposition 6.6(2). The previous proposition also shows that compact generators can be given by the symmetric E -framed motives $\{M_E^\Sigma(X) \mid X \in \mathrm{Sm}/k\}$. \square

Let $\mathbb{G}_m^{\wedge 1}$ be the mapping cone in $\Delta^{\mathrm{op}} \mathrm{Fr}_0(k)$ associated with $1 : \mathrm{pt} \rightarrow \mathbb{G}_m$. There is a suspension functor $\Sigma_{\mathbb{G}_m}^\infty$ from $\mathrm{Mod}_{\mathbb{G}_m^{\wedge 1}} \mathcal{O}_\Delta^E$ to (S^1, \mathbb{G}) -bispectra $Sp_{S^1, \mathbb{G}}(k)$:

$$\Sigma_{\mathbb{G}_m}^\infty(\mathcal{X}) := (\mathcal{X}(\mathrm{pt}), \mathcal{X}(\mathbb{G}_m^{\wedge 1}), \mathcal{X}(\mathbb{G}_m^{\wedge 2}), \dots).$$

Here $\mathcal{X}(\mathbb{G}_m^{\wedge n}) := \mathcal{X} \wedge_{\mathrm{Mod}_{\mathbb{G}_m^{\wedge 1}} \mathcal{O}_\Delta^E} \mathcal{O}_\Delta^E(-, \mathbb{G}_m^{\wedge n})$ is regarded as a presheaf of S^1 -spectra. Each structure map is induced by the adjunction unit morphism

$$\mathcal{X}(\mathbb{G}_m^{\wedge n}) \rightarrow \underline{\mathrm{Hom}}_{\mathrm{Mod}_{\mathbb{G}_m^{\wedge 1}} \mathcal{O}_\Delta^E}(\mathcal{O}_\Delta^E(-, \mathbb{G}_m^{\wedge 1}), \mathcal{X}(\mathbb{G}_m^{\wedge n+1})).$$

Corollary 6.8. *Let k be a perfect field. There is a triangulated equivalence of compactly generated triangulated categories*

$$\mathrm{SH}_{S^1}^{\mathrm{nis}} \mathcal{O}_\Delta^E \simeq \mathrm{Mod}_{SH(k)}^{\mathrm{eff}} E.$$

Proof. $SH(k) = \mathrm{Ho}(Sp_T^\Sigma(k))$ is naturally zigzag equivalent to the category of bispectra $\mathrm{SH}(k) = \mathrm{Ho}(Sp_{S^1, \mathbb{G}}(k))$. Let $\widetilde{\mathrm{Mod}}_{SH(k)}^{\mathrm{eff}} E$ be the essential image of $\mathrm{Mod}_{SH(k)}^{\mathrm{eff}} E$ under this zigzag equivalence. The category $\widetilde{\mathrm{Mod}}_{SH(k)}^{\mathrm{eff}} E$ is compactly generated by the images of $\{X_+ \wedge E \mid X \in \mathrm{Sm}/k\}$ in $\widetilde{\mathrm{Mod}}_{SH(k)}^{\mathrm{eff}} E$. By the proof of [14, Theorem 9.13] the latter are isomorphic in $\widetilde{\mathrm{Mod}}_{SH(k)}^{\mathrm{eff}} E$ to motivically fibrant bispectra

$$M_E^\mathbb{G}(X)_f := (M_E(X)_f, M_E(X_+ \wedge \mathbb{G}_m^{\wedge 1})_f, M_E(X_+ \wedge \mathbb{G}_m^{\wedge 2})_f, \dots),$$

where “ f ” refers to level local fibrant replacements of motivic S^1 -spectra. We have a triangulated functor of compactly generated triangulated categories

$$L\Sigma_{\mathbb{G}_m}^\infty : \mathrm{SH}_{S^1}^{\mathrm{nis}} \mathcal{O}_\Delta^E \rightarrow \widetilde{\mathrm{Mod}}_{SH(k)}^{\mathrm{eff}} E.$$

By Lemma 6.5, Proposition 6.6, and Theorem 6.7, $M_E^\mathbb{G}(X)_f \cong L\Sigma_{\mathbb{G}_m}^\infty(\mathcal{O}_\Delta^E(-, X))$. It follows that $L\Sigma_{\mathbb{G}_m}^\infty$ takes compact generators to compact generators with isomorphic Hom-sets. It remains to apply Lemma 4.1. \square

Next, we can stabilize our constructions in the $\mathbb{G}_m^{\wedge 1}$ -direction as follows. Denote by $- \boxtimes \mathbb{G}_m^{\wedge 1}$ the endofunctor $\mathcal{X} \in \mathrm{Mod}_{\mathbb{G}_m^{\wedge 1}} \mathcal{O}_\Delta^E \mapsto \mathcal{X}(\mathbb{G}_m^{\wedge 1})$. Following Hovey [22, section 8], we consider the stable model structure on $\mathbb{G}_m^{\wedge 1}$ -symmetric spectra $Sp^\Sigma(\mathrm{Mod}_{\mathbb{G}_m^{\wedge 1}} \mathcal{O}_\Delta^E, \mathbb{G}_m^{\wedge 1})$ (we start with the Nisnevich local stable model structure on $\mathrm{Mod}_{\mathbb{G}_m^{\wedge 1}} \mathcal{O}_\Delta^E$). Its homotopy category is denoted by $\mathrm{SH}_{S^1, \mathbb{G}_m} \mathcal{O}_\Delta^E$. Given $\mathcal{X} \in \mathrm{SH}_{S^1}^{\mathrm{nis}} \mathcal{O}_\Delta^E$ we write $\mathcal{X}(1)$ to denote $\mathcal{X} \boxtimes^L \mathbb{G}_m^{\wedge 1}$.

Corollary 6.9. *Let k be a perfect field. There is a triangulated equivalence of compactly generated triangulated categories*

$$\mathrm{SH}_{S^1, \mathbb{G}_m}^{\mathrm{nis}} \mathcal{O}_\Delta^E \simeq \mathrm{Mod}_{\mathrm{SH}(k)} E,$$

where $\mathrm{Mod}_{\mathrm{SH}(k)} E$ is the category of E -modules in $\mathrm{SH}(k)$. Moreover, the functor

$$\Sigma_{\mathbb{G}_m}^\infty : \mathrm{SH}_{S^1}^{\mathrm{nis}} \mathcal{O}_\Delta^E \rightarrow \mathrm{SH}_{S^1, \mathbb{G}_m}^{\mathrm{nis}} \mathcal{O}_\Delta^E$$

is fully faithful. In particular, $\mathrm{Hom}_{\mathrm{SH}_{S^1}^{\mathrm{nis}} \mathcal{O}_\Delta^E}(\mathcal{X}, \mathcal{X}') \rightarrow \mathrm{Hom}_{\mathrm{SH}_{S^1, \mathbb{G}_m}^{\mathrm{nis}} \mathcal{O}_\Delta^E}(\mathcal{X}(1), \mathcal{X}'(1))$ is an isomorphism for all $\mathcal{X}, \mathcal{X}' \in \mathrm{SH}_{S^1}^{\mathrm{nis}} \mathcal{O}_\Delta^E$.

Proof. The proof is similar to that of Corollary 6.8. We compare compact generators and Hom-sets between them in both categories. \square

It is worth mentioning that we do not use any motivic equivalences or the \mathbb{A}^1 -relation in any of our definitions above (similarly to constructions of [20]). All constructions here are genuinely local. On the other hand, we can do the usual Voevodsky approach [34] to constructing the triangulated category of motives $DM^{\mathrm{eff}}(k)$. We start with the spectral category \mathcal{O}^E and Čech local model structure on the stable model category of \mathcal{O}^E -modules $\mathrm{Mod}_{\mathcal{O}^E}$.

For each finite Nisnevich cover $\{U_i \rightarrow X\}$ we let $\mathcal{O}^E(-, \check{U}_*)$ be the realization of the simplicial module which in dimension n is $\vee_{i_0, \dots, i_n} \mathcal{O}^E(-, U_{i_0 \dots i_n})$, with the obvious face and degeneracy maps. Here $U_{i_0 \dots i_n}$ stands for the smooth scheme $U_{i_0} \times_X \cdots \times_X U_{i_n}$. The reader should not confuse $\mathcal{O}^E(-, \check{U}_*)$ with the realization $\mathcal{O}^E(-, \check{C}(U_*))$ of the simplicial module which in dimension n is $\mathcal{O}^E(-, \sqcup_{i_0, \dots, i_n} U_{i_0 \dots i_n})$.

Lemma 6.10. *Each natural map $\vee_{i_0, \dots, i_n} \mathcal{O}^E(-, U_{i_0 \dots i_n}) \rightarrow \mathcal{O}^E(-, \check{C}(U_n))$ is a schemewise stable equivalence of ordinary S^1 -spectra.*

Proof. It is enough to show that the natural map

$$\begin{aligned} \beta : (\mathrm{Fr}_0^E(X, V), \mathrm{Fr}_1^E(X, V \otimes S^1), \dots) \vee (\mathrm{Fr}_0^E(X, W), \mathrm{Fr}_1^E(X, W \otimes S^1), \dots) \rightarrow \\ \rightarrow (\mathrm{Fr}_0^E(X, V \sqcup W), \mathrm{Fr}_1^E(X, (V \sqcup W) \otimes S^1), \dots) \end{aligned}$$

is a stable equivalence of ordinary S^1 -spectra for any $X, V, W \in \mathrm{Sm}/k$. This is a stable equivalence if and only if $\Theta_{S^1}^\infty(\beta)$ is. The latter map is a stable equivalence if and only if

$$\begin{aligned} \gamma : (\mathrm{Fr}^E(X, V), \mathrm{Fr}^E(X, V \otimes S^1), \dots) \vee (\mathrm{Fr}^E(X, W), \mathrm{Fr}^E(X, W \otimes S^1), \dots) \rightarrow \\ \rightarrow (\mathrm{Fr}^E(X, V \sqcup W), \mathrm{Fr}^E(X, (V \sqcup W) \otimes S^1), \dots) \end{aligned}$$

is a stable equivalence. This is a map of Segal S^1 -spectra associated to Segal spaces of the form $K \in \Gamma^{\mathrm{op}} \mapsto \mathrm{Fr}^E(X, V \otimes K)$, hence γ is a map of connective spectra. The Stable Whitehead Theorem [31, Proposition II.6.30] implies γ is a stable equivalence if and only if

$$\begin{aligned} \mathbb{Z}(\gamma) : (\mathbb{Z} \mathrm{Fr}^E(X, V), \mathbb{Z} \mathrm{Fr}^E(X, V \otimes S^1), \dots) \vee (\mathbb{Z} \mathrm{Fr}^E(X, W), \mathbb{Z} \mathrm{Fr}^E(X, W \otimes S^1), \dots) \rightarrow \\ \rightarrow (\mathbb{Z} \mathrm{Fr}^E(X, V \sqcup W), \mathbb{Z} \mathrm{Fr}^E(X, (V \sqcup W) \otimes S^1), \dots) \end{aligned}$$

is a stable equivalence, where $\mathbb{Z} \text{Fr}^E(X, V)$ is the reduced free Abelian group of the pointed set $\text{Fr}^E(X, V)$. Repeating the proof of [15, Theorem 1.2] word for word, $\mathbb{Z}(\gamma)$ is stably equivalent to the map

$$\begin{aligned} \delta : (\mathbb{Z}F^E(X, V), \mathbb{Z}F^E(X, V \otimes S^1), \dots) \vee (\mathbb{Z}F^E(X, W), \mathbb{Z}F^E(X, W \otimes S^1), \dots) \rightarrow \\ \rightarrow (\mathbb{Z}F^E(X, V \sqcup W), \mathbb{Z}F^E(X, (V \sqcup W) \otimes S^1), \dots), \end{aligned}$$

where $\mathbb{Z}F^E(X, V) = \text{colim}_n \mathbb{Z}F_n^E(X, V)$ with $\mathbb{Z}F_n^E(X, V)$ the free Abelian group freely generated by E -framed correspondences with connected support [14]. As $\mathbb{Z}F^E(X, V \sqcup W) = \mathbb{Z}F^E(X, V) \times \mathbb{Z}F^E(X, W)$, the map δ equals the stable equivalence of S^1 -spectra

$$\begin{aligned} (\mathbb{Z}F^E(X, V), \mathbb{Z}F^E(X, V \otimes S^1), \dots) \vee (\mathbb{Z}F^E(X, W), \mathbb{Z}F^E(X, W \otimes S^1), \dots) \rightarrow \\ \rightarrow (\mathbb{Z}F^E(X, V), \mathbb{Z}F^E(X, V \otimes S^1), \dots) \times (\mathbb{Z}F^E(X, W), \mathbb{Z}F^E(X, W \otimes S^1), \dots). \end{aligned}$$

This completes the proof of the lemma. \square

The *Cech model category* $\text{Mod}_{\text{Cech}}^E$ associated with Nisnevich topology is obtained from $\text{Mod}\mathcal{O}^E$ by Bousfield localization with respect to all maps $\eta : \mathcal{O}^E(-, \check{U}_*) \rightarrow \mathcal{O}^E(-, X)$ running over the set of finite Nisnevich covers. It follows from [36, Corollary 5.10] (see also [5]) that $\text{Mod}_{\text{Cech}}^E$ coincides with the Nisnevich local model category $\text{Mod}_{\text{nis}}^E$, with stable weak equivalences defined stalkwise.

We say that a spectral category \mathcal{O} is *Cech excisive* if for any finite Nisnevich cover $\{U_i \rightarrow X\}$ the induced map $\eta : \mathcal{O}(-, \check{U}_*) \rightarrow \mathcal{O}(-, X)$ is a local stable weak equivalence.

Theorem 6.11. *Let k be any field. The commutative spectral category \mathcal{O}^E is Cech excisive. The Cech model structure coincides with Nisnevich local model structure on $\text{Mod}\mathcal{O}^E$. This model structure has all the properties of Theorem 6.2. The homotopy category $D\mathcal{O}^{E, \text{eff}}(k)$ of $\text{Mod}_{\text{Cech}}^E$ is closed symmetric monoidal compactly generated triangulated with compact generators being the representables $\{\mathcal{O}^E(-, X) \mid X \in \text{Sm}/k\}$. The monoidal product $\mathcal{O}^E(-, X) \wedge \mathcal{O}^E(-, Y)$ in $D\mathcal{O}^{E, \text{eff}}(k)$ is isomorphic to $\mathcal{O}^E(-, X \times Y)$.*

Proof. By Lemma 6.10 each map $\vee_{i_0, \dots, i_n} \mathcal{O}^E(-, U_{i_0 \dots i_n}) \rightarrow \mathcal{O}^E(-, \check{C}(U_n))$ is a schemewise stable equivalence, and hence the realization is. The proof of [35, Theorem 4.4] shows that the map $\mathcal{O}^E(-, \check{C}(U_*)) \rightarrow \mathcal{O}^E(-, X)$ is a level local equivalence. We see that η is a local stable weak equivalence. The rest is now straightforward. \square

The homotopy category $D\mathcal{O}^{E, \text{eff}}(k)$ plays the same role as the derived category $D(\text{Shv}_{tr}^{\text{nis}}(\text{Sm}/k))$ of cochain complexes of Nisnevich sheaves with transfers. Recall from [34] that Voevodsky's category of motives $DM^{\text{eff}}(k)$ is the localization of $D(\text{Shv}_{tr}^{\text{nis}}(\text{Sm}/k))$ with respect to the family $\{\mathbb{Z}_{tr}(-, X \times \mathbb{A}^1) \rightarrow \mathbb{Z}_{tr}(-, X) \mid X \in \text{Sm}/k\}$. If k is perfect, $DM^{\text{eff}}(k)$ is equivalent to the full subcategory of $D(\text{Shv}_{tr}^{\text{nis}}(\text{Sm}/k))$ consisting of chain complexes with homotopy invariant cohomology sheaves [34]. Likewise, localize $\text{Mod}_{\text{Cech}}^E$ with respect to the maps $\{\mathcal{O}^E(-, X \times \mathbb{A}^1) \rightarrow \mathcal{O}^E(-, X) \mid X \in \text{Sm}/k\}$. Denote by $D\mathcal{O}_{\text{mot}}^{E, \text{eff}}(k)$ its homotopy category.

Theorem 6.12. *Let k be a perfect field. The homotopy category $D\mathcal{O}_{\text{mot}}^{E,\text{eff}}(k)$ is equivalent to the full triangulated subcategory $DE^{\text{eff}}(k)$ of $D\mathcal{O}^{E,\text{eff}}(k)$ consisting of modules with homotopy invariant sheaves of stable homotopy groups. The inclusion $DE^{\text{eff}}(k) \rightarrow D\mathcal{O}^{E,\text{eff}}(k)$ has a right adjoint C_* taking a module $M \in D\mathcal{O}^{E,\text{eff}}(k)$ to its Suslin complex $C_*(M)$.*

Moreover, there is a triangulated equivalence of compactly generated triangulated categories

$$DE^{\text{eff}}(k) \simeq \text{SH}_{S^1}^{\text{nis}} \mathcal{O}_{\Delta}^E.$$

Proof. The proof of the first part is like that of [17, Theorem 3.5]. One also uses here the fact that if k is perfect then by [18] (complemented by [4] in characteristic 2 and by [3, A.27] for finite fields) any \mathbb{A}^1 -invariant quasi-stable additive framed presheaf of Abelian groups \mathcal{F} , the associated Nisnevich sheaf \mathcal{F}_{nis} is strictly \mathbb{A}^1 -invariant.

The equivalence $DE^{\text{eff}}(k) \simeq \text{SH}_{S^1}^{\text{nis}} \mathcal{O}_{\Delta}^E$ follows from the fact that both categories are compactly generated by symmetric E -framed motives with the same Hom-sets (as usual we use Lemma 4.1 here). \square

We call the category $DE^{\text{eff}}(k)$ from the preceding theorem the *triangulated category of E -framed motives*. We finish the paper with the following result saying that $DE^{\text{eff}}(k)$ recovers the category of effective E -modules in $SH(k)$.

Corollary 6.13. *Let k be a perfect field. There is a triangulated equivalence of compactly generated triangulated categories*

$$DE^{\text{eff}}(k) \simeq \text{Mod}_{SH(k)}^{\text{eff}} E.$$

Proof. This follows from the preceding theorem and Corollary 6.8. \square

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