FIBRANT RESOLUTIONS FOR MOTIVIC THOM SPECTRA

GRIGORY GARKUSHA AND ALEXANDER NESHITOV

ABSTRACT. Using the theory of framed correspondences developed by Voevodsky [32] and the machinery of framed motives introduced and developed in [13], various explicit fibrant resolutions for a motivic Thom spectrum E are constructed in this paper. It is shown that the bispectrum

$$M_E^{\mathbb{G}}(X) = (M_E(X), M_E(X)(1), M_E(X)(2), \ldots),$$

each term of which is a twisted E-framed motive of X, introduced in the paper, represents $X_+ \wedge E$ in the category of bispectra. As a topological application, it is proved that the E-framed motive with finite coefficients $M_E(pt)(pt)/N$, $N > 0$, of the point $pt = \operatorname{Spec} k$ evaluated at pt is a quasi-fibrant model of the topological S^2-spectrum $Re^{\varepsilon}(E)/N$ whenever the base field k is algebraically closed of characteristic zero with an embedding $\varepsilon : k \hookrightarrow \mathbb{C}$. Furthermore, the algebraic cobordism spectrum MGL is computed in terms of Ω-correspondences in the sense of [21]. It is also proved that MGL is represented by a bispectrum each term of which is a sequential colimit of simplicial smooth quasi-projective varieties.

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1. INTRODUCTION

Voevodsky [32] introduced framed correspondences in order to suggest a new approach to stable motivic homotopy theory which will be more amenable to explicit computations. In [13] the machinery of (big) framed motives is developed converting the classical Morel–Voevodsky stable motivic homotopy theory into a local theory of framed bispectra and yielding a new model for $SH(k)$ in [14]. A key computation of [13] is to give explicit fibrant resolutions of the suspension spectra/bispectra of smooth algebraic varieties.

The main results of this paper are concentrated around explicit computations of motivic Thom spectra, decribed below, which play a central role in stable motivic homotopy theory. We use computational miracles of Voevodsky's framed correspondences to extend the machinery of framed motives that are of crucial importance in [13] to "E-framed motives", where E is a motivic Thom T-spectrum.

By definition, E is called a *Thom spectrum* if every space E_n has the form

$$E_n = \operatorname{colim}_i E_{n,i},\ E_{n,i} = V_{n,i}/(V_{n,i} - Z_{n,i}),$$

where $V_{n,i} \to V_{n,i+1}$ is a directed sequence of smooth varieties, $Z_{n,i} \to Z_{n,i+1}$ is a directed system of smooth closed subschemes in $V_{n,i}$. We say that a Thom spectrum E *has the bounding constant* d if d is the minimal integer such that codimension of $Z_{n,i}$ in $V_{n,i}$ is strictly greater than $n-d$ for all i,n. If E is also symmetric then it is said to be a *spectrum with contractible alternating group action*, if for any n and any even permutation $\tau \in \Sigma_n$ there is an $\mathbb{A}^1$-homotopy $E_n \to \underline{\mathrm{Hom}}(\mathbb{A}^1, E_n)$ between the action of τ and the identity map. In other words, E neglects the action of even permutations up to $\mathbb{A}^1$-homotopy. The most interesting examples of such symmetric Thom spectra, all of which have the bounding constant $d = 1$, are given by the spectra MGL, MSL or MSp (the latter two are regarded as T^2-symmetric spectra for which the above definitions remain the same). These Thom spectra are of fundamental importance. If we regard E as a $\mathbb{P}^1$-spectrum, denote by $\Theta^\infty(E)$ the standard stabilization $\operatorname{colim}_n \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, E[n])$ of E. Taking the Suslin complex at each level, we get a $\mathbb{P}^1$-spectrum $C_*\Theta^\infty(E)$.

Our first computation (see Theorem 5.4) is as follows.

1.1. **Theorem.** *Let E be a Thom spectrum with the bounding constant d. Let $C_*\Theta^\infty(E)^f$ be a spectrum obtained from $C_*\Theta^\infty(E)$ by taking a level Nisnevich local fibrant replacement. Then the spectrum $C_*\Theta^\infty(E)^f$ is motivically fibrant starting from level* $\max(0,d)$ *and is stably equivalent to E.*

If E is a symmetric T-spectrum, then there is another natural stabilization functor $\Theta^\infty_{sym}(E)$ (see Definition 6.2). It is different from $\Theta^\infty(E)$ and involves actions of certain permutations on E. As above, we can take the Suslin complex at each level and form a $\mathbb{P}^1$-spectrum $C_*\Theta^\infty_{sym}(E)$.

Given a Thom T-spectrum E, denote by $\mathrm{Fr}^E_n(X)$ the space $\mathrm{Fr}^E_n(X) = \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, X_+ \wedge E_n)$ and $\mathrm{Fr}^E(X) := \operatorname{colim}_n \mathrm{Fr}^E_n(X) = \Theta^\infty(X_+ \wedge E)_0$. By the Voevodsky lemma 2.12 $\mathrm{Fr}^E_n(X)$ and $\mathrm{Fr}^E(X)$ have an explicit geometric description. We can similarly define the sheaves $\mathrm{Fr}^E(T^i)$, $i \geqslant 0$. Altogether they form a $\mathbb{P}^1$-spectrum $\mathrm{Fr}^E(S_T) := (\mathrm{Fr}^E(S^0), \mathrm{Fr}^E(T), \mathrm{Fr}^E(T^2), \ldots)$. As usual, denote by $C_*\mathrm{Fr}^E(S_T)$ the $\mathbb{P}^1$-spectrum obtained from $\mathrm{Fr}^E(S_T)$ by taking the Suslin complex levelwise.

The next computation (see Theorem 8.1) gives the following fibrant resolutions of E (starting from some level).

1.2. **Theorem.** *For a symmetric Thom T-spectrum E with the bounding constant d and contractible alternating group action the following $\mathbb{P}^1$-spectra are isomorphic to E in $SH(k)$ and motivically fibrant starting from level* $\max(0,d)$*:*

- $C_*\operatorname{Fr}^E(S_T)^f$
- $C_*\Theta^\infty(E)^f$
- $C_*\Theta^\infty_{sym}(E)^f$,

where "f" refers to levelwise Nisnevich local fibrant replacements of the corresponding spectra.

Our next goal is to represent a Thom spectrum E in the category of $(S^1,\mathbb{G}_m^{\wedge 1})$-bispectra and construct an explicit fibrant resolution for it. To this end, we introduce and study in Section 9 E-framed motives of smooth algebraic varieties $M_E(X)$, $X \in \mathbf{Sm}_k$. They are defined similarly to framed motives introduced in [13] and are explicit sheaves of S^1-spectra.

The main result here (see Theorem 9.13) is as follows.

1.3. **Theorem.** *Suppose $X \in \mathbf{Sm}_k$ and E is a symmetric Thom T-spectrum with the bounding constant d and contractible alternating group action.*

(1) *If $d = 1$ then the $(S^1,\mathbb{G}_m^{\wedge 1})$-bispectrum*

$$M_E^{\mathbb{G}}(X)_f := (M_E(X)_f, M_E(X_+ \wedge \mathbb{G}_m^{\wedge 1})_f, M_E(X_+ \wedge \mathbb{G}_m^{\wedge 2})_f, \dots)$$

is motivically fibrant and represents the T-spectrum $X_+ \wedge E$ in the category of bispectra, where "f" refers to stable local fibrant replacements of S^1-spectra.

(2) *If $d < 1$ then the $(S^1,\mathbb{G}_m^{\wedge 1})$-bispectrum*

$$M_E^{\mathbb{G}}(X)_f := (M_E(X)_f, M_E(X_+ \wedge \mathbb{G}_m^{\wedge 1})_f, M_E(X_+ \wedge \mathbb{G}_m^{\wedge 2})_f, \dots)$$

is motivically fibrant and represents the T-spectrum $X_+ \wedge E$ in the category of bispectra, where "f" refers to level local fibrant replacements of S^1-spectra.

(3) *If $d > 1$ then the $(S^1,\mathbb{G}_m^{\wedge 1})$-bispectrum*

$$\Omega^{d-1}_{S^1\wedge\mathbb{G}_m^{\wedge 1}}((M_{E[d-1]}(X)_f, M_{E[d-1]}(X_+ \wedge \mathbb{G}_m^{\wedge 1})_f, M_{E[d-1]}(X_+ \wedge \mathbb{G}_m^{\wedge 2})_f, \dots))$$

is motivically fibrant and represents the T-spectrum $X_+ \wedge E$ in the category of bispectra, where "f" refers to stable local fibrant replacements of S^1-spectra. Here $E[d-1]$ stands for the $(d-1)$-th shift of E in the sense of Definition 2.6. Another equivalent model for the T-spectrum $X_+ \wedge E$ in the category of bispectra is given by

$$\Omega^{d-1}_{S^1\wedge\mathbb{G}_m^{\wedge 1}}((M_{T^{d-1}\wedge E}(X)_f, M_{T^{d-1}\wedge E}(X_+ \wedge \mathbb{G}_m^{\wedge 1})_f, M_{T^{d-1}\wedge E}(X_+ \wedge \mathbb{G}_m^{\wedge 2})_f, \dots)).$$

This bispectrum is motivically fibrant and "f" refers to stable local fibrant replacements of S^1-spectra.

One of the most impressive applications of the theory of framed correspondences and the machinery of framed motives is that they lead to computing explicit fibrant resolutions of classical topological objects in terms of algebraic varieties. These computations are far relatives for the celebrated constructions of Pontrjagin [25] who interpreted homotopy groups of spheres in terms of smooth manifolds. For example, the classical topological sphere spectrum is computed in [13] as the framed motive $M_{fr}(pt)(pt)$ of the point $pt = \operatorname{Spec} k$ evaluated at the point whenever the base field k is algebraically closed of characteristic zero. We use the preceding theorem to get a similar topological application in Theorem 10.3. The main example here concerns the motivic cobordism spectrum MGL whose realization is isomorphic to the topological complex cobordism spectrum MU.

1.4. **Theorem.** *Let k be an algebraically closed field of characteristic zero with an embedding $\varepsilon : k \hookrightarrow \mathbb{C}$. Suppose E is a symmetric Thom T-spectrum with the bounding constant $d \leqslant 1$ and contractible alternating group action. Then for all integers $N > 1$ and $n \in \mathbb{Z}$, the natural realisation functor $Re^{\varepsilon} : SH(k) \to SH$ in the sense of* [23] *induces an isomorphism*

$$\pi_n(M_E(pt)(pt); \mathbb{Z}/N) \cong \pi_n(Re^{\varepsilon}(E); \mathbb{Z}/N)$$

between stable homotopy groups with mod N coefficients.

Given a motivic Thom spectrum E and $X \in Sm/k$, we fix any group completion $Fr^E(\Delta^\bullet_k, X)^{\mathrm{gp}}$ of the space $Fr^E(\Delta^\bullet_k, X)$, which is functorial in X. For instance, one can take $Fr^E(\Delta^\bullet_k, X)^{\mathrm{gp}} = \Omega_{S^1} Fr^E(\Delta^\bullet_k, X \otimes S^1)$. Put

$$\pi^E_n(X) := \pi_n(Fr^E(\Delta^\bullet_k, X)^{\mathrm{gp}})$$

and call $\pi^E_n(X)$ the n-th singular algebraic E-homotopy group of X.

The following result on the singular algebraic E-homotopy is an analogue of the celebrated theorem of Suslin and Voevodsky [30] on the singular algebraic homology (see Theorem 10.5):

1.5. **Theorem.** *Suppose E is a symmetric Thom T-spectrum with the bounding constant $d \leqslant 1$ and contractible alternating group action. The assignment $X \mapsto \pi^E_*(X)$ is a generalized homology theory on $Sm/\mathbb{C}$. Moreover, passing to homotopy groups with finite coefficients, we get equalities*

$$\pi^E_n(X; \mathbb{Z}/m) = \pi_n(X(\mathbb{C})_+ \wedge Re^{\varepsilon}(E); \mathbb{Z}/m)$$

for all integers $n \geqslant 0$ and $m \neq 0$.

Also, the first part of this theorem is true over any perfect field k. Namely, the assignment $X \mapsto \pi^E_*(X)$ is a generalized homology theory on the category Sm/k.

We can simplify E-framed motives further by removing a bit of information in the definition of E-framed correspondences. In this way we arrive at "normally framed motives $\widetilde{M}_E(X)$" (see Definition 11.24). They play a pivotal role in our analysis and – most importantly – lead to explicit computations of the algebraic cobordism spectrum MGL (see below).

We prove the following result (see Theorem 11.26) computing E in terms of normally framed motives.

1.6. **Theorem.** *Suppose $X \in \mathbf{Sm}_k$ and E is a symmetric Thom T-spectrum with the bounding constant $d = 1$ and contractible alternating group action. Then we have a $(S^1, \mathbb{G}_m^{\wedge 1})$-bispectrum*

$$\widetilde{M}^{\mathbb{G}}_E(X)_f := (\widetilde{M}_E(X)_f, \widetilde{M}_E(X_+ \wedge \mathbb{G}_m^{\wedge 1})_f, \widetilde{M}_E(X_+ \wedge \mathbb{G}_m^{\wedge 2})_f, \ldots),$$

which is motivically fibrant and represents the T-spectrum $X_+ \wedge E$ in the category of bispectra, where "f" refers to stable local fibrant replacements of S^1-spectra.

The last section is dedicated to further explicit models representing the algebraic cobordism spectrum MGL in the category of bispectra. We first introduce Nisnevich sheaves $\mathrm{Emb}(-, X) = \mathrm{colim}_n \mathrm{Emb}_n(-, X)$, $X \in \mathbf{Sm}_k$, where $\mathrm{Emb}_n(U, X)$ is the set of couples (Z, f) such that Z is a closed l.c.i. subscheme in $\mathbb{A}^n_U$, finite and flat over U, and f is a regular map $f\colon Z\to X$. For $U,X\in\mathbf{Sm}_k$ we also denote by $Cor^\Omega_n(U,X)$ the groupoid with objects given by the set $\mathrm{Emb}_n(X,Y)$ whose morphisms between (Z_1,f_1) and (Z_2,f_2) are isomorphisms $\alpha\colon Z_1\to Z_2$ such that $\pi_{Z_2}\alpha=\pi_{Z_1}$ and $f_2\alpha=f_1$, where π_{Z_i} denotes the projection $\pi_{Z_i}\colon Z_i\to\mathbb{A}^n_X\to X$. There are natural stabilization maps $Cor^\Omega_n(-,X)\to Cor^\Omega_{n+1}(-,X)$ induced by the natural inclusions $\mathbb{A}^n_U\to\mathbb{A}^{n+1}_U$. Denote by $Cor^\Omega(-,X)$ the colimit $\mathrm{colim}_n Cor^\Omega_n(-,X)$.

In Theorem 12.15 we compute $M^{\mathbb{G}}_{MGL}(X)$ as follows.

1.7. **Theorem.** *For $X\in\mathbf{Sm}_k$ there is a natural levelwise stable local equivalence between $(S^1,\mathbb{G}_m^{\wedge 1})$-bispectra $M^{\mathbb{G}}_{MGL}(X)$ and*

$$(C_*\,\mathrm{Emb}(X_+\wedge\mathbb{S}),C_*\,\mathrm{Emb}(X_+\wedge\mathbb{G}_m^{\wedge 1}\wedge\mathbb{S}),\ldots)$$

or

$$(C_*NCor^\Omega(X_+\wedge\mathbb{S}),C_*NCor^\Omega(X_+\wedge\mathbb{G}_m^{\wedge 1}\wedge\mathbb{S}),\ldots).$$

Here "N" refers to the nerve of isomorphisms. In particular, the $(S^1,\mathbb{G}_m^{\wedge 1})$-bispectra

$$(C_*\,\mathrm{Emb}(X_+\wedge\mathbb{S})_f,C_*\,\mathrm{Emb}(X_+\wedge\mathbb{G}_m^{\wedge 1}\wedge\mathbb{S})_f,\ldots)$$

and

$$(C_*NCor^\Omega(X_+\wedge\mathbb{S})_f,C_*NCor^\Omega(X_+\wedge\mathbb{G}_m^{\wedge 1}\wedge\mathbb{S})_f,\ldots)$$

are motivically fibrant and represent the T-spectrum $X_+\wedge MGL$ in the category of bispectra, where "f" refers to stable local fibrant replacements of S^1-spectra.

We finish the paper by the following important computation (see Theorem 12.16) of the algebraic cobordism in terms of smooth quasi-projective varieties. This computation is an application of the preceding theorem.

1.8. **Theorem.** *The $(S^1,\mathbb{G}_m^{\wedge 1})$-bispectrum $M^{\mathbb{G}}_{MGL}(X)$ is isomorphic in $SH(k)$ to a bispectrum*

$$(E^{X_+\wedge\mathbb{S}},E^{X_+\wedge\mathbb{G}_m^{\wedge 1}\wedge\mathbb{S}},\ldots),$$

each term of which is given by a sequential colimit of simplicial smooth quasi-projective varieties $E^{X_+\wedge\mathbb{G}_m^{\wedge i}\wedge S^j}$, $i,j\geqslant 0$.

Throughout the paper we denote by $\mathbf{Sm}_k$ the category of smooth separated schemes of finite type over the base field k. We shall assume that k is perfect for the reason that the main result of [12] (complemented by [6] in characteristic 2 and by [5, A.27] for finite fields) says that over such fields for any $\mathbb{A}^1$-invariant quasi-stable radditive framed presheaf of Abelian groups $\mathscr{F}$, the associated Nisnevich sheaf $\mathscr{F}_{nis}$ is strictly $\mathbb{A}^1$-invariant. By a motivic space we shall mean a pointed simplicial Nisnevich sheaf on $\mathbf{Sm}_k$. If $\mathscr{C}$ is a category cotensored over the category of pointed motivic spaces $\mathscr{M}$, we shall write $\underline{\mathrm{Hom}}(A,C)\in\mathscr{C}$ for the cotensor object associated with $A\in\mathscr{M}$ and $C\in\mathscr{C}$ unless it is specified otherwise. We choose the flasque local/motivic model structures on motivic spaces (respectively S^1- or $\mathbb{P}^1$-spectra of motivic spaces) in the sense of [17].

Relations to other works. This paper (first appeared in the archive in April 2018) depends on a series of papers on framed motives [3, 10, 12, 13]. Computations of motivic Thom spectra like those of Theorem 9.13 in terms of tangentially framed correspondences as defined in [7] were later obtained in [8]. Our approach is based on Voevodsky's framed correspondences [32]. Technique developed in Sections 6 and 7 is crucial for the theory of motivic Γ-spaces [15], an extension of the celebrated Segal machine of Γ-spaces [28] to the world of motivic homotopy theory. A systematic study of normally framed correspondences associated with Thom spectra is given in Section 11. This type of correspondences associated with the motivic sphere spectrum is of great utility in [2, 7]. Normally framed correspondences lead to representability of some important motivic Thom spectra like MGL by schemes (see Theorem 12.16). The representability theorem is also proven in [8].

2. Preliminaries

In this section we collect basic facts about spectra and motivic spaces with framed correspondences.

Spectra of Thom type.

2.1. **Definition.** For every space X denote by C_*X its Suslin complex. It is the diagonal of the bisimplicial sheaf $(n,m) \mapsto \underline{\mathrm{Hom}}(\Delta^n_k, X_m)$ where X_m is the sheaf of m-simplices of X.

2.2. **Definition.** Given two spaces X, Y and maps $f, g\colon X \to Y$,

- a *simplicial homotopy* between f and g is a map $H\colon X \wedge \Delta[1]_+ \to Y$ such that the composition $Hi_0 = f$ and $Hi_1 = g$, where $i_0, i_1\colon X \to X \wedge \Delta[1]_+$ are the face maps;
- an $\mathbb{A}^1$*-homotopy* between f and g is a map $H\colon X \to \underline{\mathrm{Hom}}(\mathbb{A}^1, Y)$ such that $i_0^*H = f$ and $i_1^*H = g$, where $i_0, i_1\colon \underline{\mathrm{Hom}}(\mathbb{A}^1, Y) \to Y$ are maps induced by zero and unit embeddings of pt into $\mathbb{A}^1$.

2.3. **Remark.** Every $\mathbb{A}^1$-homotopy $H\colon X \to \underline{\mathrm{Hom}}(\mathbb{A}^1, Y)$ between f and g gives rise to a simplicial homotopy $H'\colon C_*X \wedge \Delta[1]_+ \to C_*Y$ between f and g.

2.4. **Convention.** We shall use the following notation:

- Given two motivic spaces B and C, we denote by tw the twist isomorphism $C \wedge B \xrightarrow{\cong} B \wedge C$.
- For brevity, we shall sometimes write (A, B) to denote $\underline{\mathrm{Hom}}(A, B)$, where A and B are motivic spaces. We shall use the canonical map

$$(A,B) \wedge C \to (A, C \wedge B)$$

which is adjoint to

$$(A,B) \xrightarrow{C \wedge id} (C \wedge A, C \wedge B) = (C, (A, C \wedge B).$$

When C and B are distinct spaces we shall often compose the previous map with the twist isomorphism $tw\colon C \wedge B \to B \wedge C$ to get the map

$$(A,B) \wedge C \to (A, B \wedge C).$$

- If there is no likelihood of confusion, we shall use the equality sign $\mathbb{P}^{\wedge m}\wedge\mathbb{P}^{\wedge n}=\mathbb{P}^{\wedge n}\wedge\mathbb{P}^{\wedge m}$ for the associativity isomorphism
$$\mathbb{P}^{\wedge m}\wedge\mathbb{P}^{\wedge n}\cong\mathbb{P}^{\wedge m+n}=\mathbb{P}^{\wedge n+m}\cong\mathbb{P}^{\wedge n}\wedge\mathbb{P}^{\wedge m}.$$
- For any m,n we shall identify the spaces (via associativity isomorphisms)
$$(\mathbb{P}^{\wedge m},(\mathbb{P}^{\wedge n},X))=(\mathbb{P}^{\wedge m}\wedge\mathbb{P}^{\wedge n},X)=(\mathbb{P}^{\wedge n}\wedge\mathbb{P}^{\wedge m},X)=(\mathbb{P}^{\wedge n},(\mathbb{P}^{\wedge m},X)).$$

Let T be the pointed Nisnevich sheaf $\mathbb{A}^1/(\mathbb{A}^1-0)$. A *$T$-spectrum* is a sequence of spaces E_n together with bonding maps, denoted by u. In what follows we work with right spectra, and so each bonding map is a map $u\colon E_n\wedge T\to E_{n+1}$. Denote by Σ_n the nth symmetric group. A *symmetric T-spectrum* is a spectrum E together with a left action of Σ_n on E_n such that the bonding maps satisfy the relevant equivariance properties.

2.5. **Definition.** Given $\tau\in\Sigma_n$ we shall write $\tau=(\tau(1),\dots,\tau(n))$. The reader should not confuse this notation with cyclic permutations. For any n,m, denote by $\chi_{n,m}\in\Sigma_{n+m}$ the obvious shuffle permutation $\chi_{n,m}=(n+1,\dots,n+m,1,\dots,n)$.

If we denote by S_T the symmetric motivic sphere T-spectrum $(S^0,T,T^2,\dots)$, then any symmetric T-spectrum is a right module over the monoid S_T in the category of symmetric sequences [16, 7.2].

2.6. **Definition.** Given a symmetric T-spectrum E and $n\geqslant 0$, denote by $u_l\colon T\wedge E_n\to E_{n+1}$ the composition
$$T\wedge E_n\xrightarrow{tw}E_n\wedge T\xrightarrow{u}E_{n+1}\xrightarrow{\chi_{n,1}}E_{1+n}.$$
Observe that the maps give a map of symmetric T-spectra $u_l : T\wedge E\to E[1]$. Here $E[1]$ is the *shift symmetric spectrum* whose spaces are given by $E[1]_n=E_{1+n}$ with action of Σ_n by restriction of the Σ_{1+n}-action on E_{1+n} along the obvious embedding $\Sigma_n\hookrightarrow\Sigma_{1+n}$ taking $\tau\in\Sigma_n$ to $1\oplus\tau\in\Sigma_{1+n}$. The structure maps of $E[1]$ are the reindexed structure maps for E. In turn, $T\wedge E$ is the *suspension spectrum* of E whose spaces are defined as $(T\wedge E)_n=T\wedge E_n$. The symmetric group Σ_n acts on $T\wedge E_n$ through the given action on E_n and trivially on T. Each structure map is the composite
$$(T\wedge E)_n\wedge T\cong T\wedge(E_n\wedge T)\xrightarrow{id\wedge u_n}(T\wedge E)_{n+1}.$$

2.7. **Definition.** A symmetric spectrum E is said to be a *spectrum with contractible alternating group action*, if for any n and any even permutation $\tau\in\Sigma_n$ there is an $\mathbb{A}^1$-homotopy $E_n\to\underline{\mathrm{Hom}}(\mathbb{A}^1,E_n)$ between the action of τ and the identity map. In other words, E neglects the action of even permutations up to $\mathbb{A}^1$-homotopy.

2.8. **Definition.** A T-spectrum E is called a *Thom spectrum* if every space E_n has the form
$$E_n=\mathrm{colim}_i E_{n,i},\ E_{n,i}=V_{n,i}/(V_{n,i}-Z_{n,i}),$$
where $V_{n,i}\to V_{n,i+1}$ is a directed sequence of smooth varieties, $Z_{n,i}\to Z_{n,i+1}$ is a directed system of smooth closed subschemes in $V_{n,i}$.

We shall say that a Thom spectrum E *has the bounding constant d* if d is the minimal integer such that codimension of $Z_{n,i}$ in $V_{n,i}$ is strictly greater than $n-d$ for all i,n.

2.9. **Example.** The suspension spectrum $\Sigma^\infty X_+ = X_+ \wedge S_T$ and the algebraic cobordism spectrum MGL of [31] (see also [22, 24]) are examples of symmetric Thom spectra with the bounding constant 1 and contractible alternating group action.

If E is a Thom T-spectrum with the bounding constant $d \geqslant 0$, then its nth shift $E[n]$ (see Definition 2.6) as well as the spectrum $T^n \wedge E$, $n \geqslant 0$, is a Thom spectrum with the bounding constant $d-n$. In turn, its negative shift $E[-n] = (*,\dots,*,E_0,E_1,\dots)$ having E_0 in the nth entry is a Thom T-spectrum with the bounding constant $n+d$. By definition, the trivial Thom spectrum $* = (*,*,\dots)$ has the bounding constant $+\infty$.

In practice we also deal with symmetric Thom T^2-spectra like MSL or MSp (see [24] for definitions). We also say that a Thom T^2-spectrum E *has the bounding constant d* if d is the minimal integer such that codimension of $Z_{n,i}$ in $V_{n,i}$ is strictly greater than $2n-d$ for all i,n. The Thom T^2-spectra MSL and MSp have the bounding constant $d=1$.

By construction (see [24]), the action of the symmetric group Σ_n on the spaces MSL_{2n} and MSp_{2n} factors through the action of SL_{2n} and Sp_{2n} respectively. Since SL_{2n} and Sp_{2n} are semisimple simply-connected groups, the sets of k-points $SL_{2n}(k)$ and $Sp_{2n}(k)$ are generated by the root subgroups $U_\alpha(k)$ (see [26]). Since every root subgroup U_α is isomorphic to the affine line $\mathbb{A}^1_k$, we have that for every element A of $G = SL_{2n}$ or $G = Sp_{2n}$ there exists a map $h\colon \mathbb{A}^1_k \to G$ such that $h(0) = I, h(1) = A$. It follows that MSL and MSp are T^2-spectra with contractible alternating group action as well.

2.10. **Lemma.** *Let $\Sigma_n \to \mathbf{GL}_n(k)$ be the standard inclusion and let τ be an even permutation. Then there is an $\mathbb{A}^1$-curve $L\colon \mathbb{A}^1_k \to \mathbf{GL}_n$ such that $L(0) = I$ is the identity matrix and $L(1) = \tau$.*

Proof. Since τ is even, its image belongs to $SL_n(k)$. Thus it can be written as a product of elementary matrices:

$$\tau = \prod_{l=1}^{m} e_{i_l,j_l}(\lambda_l), \qquad \text{where } 1 \leqslant i_l, j_l \leqslant n,\, \lambda_l \in k,\, i_l \neq j_l.$$

Here an elementary matrix $e_{i,j}(\lambda)$ is a matrix with all its diagonal elements equal to 1, λ being placed in the (i,j)-th entry and zero elsewhere. Then $L(t) = \prod_{l=1}^m e_{i_l,j_l}(t\lambda_l)$ defines a regular map $\mathbb{A}^1_k \to \mathbf{GL}_n(k)$ with $L(0) = I$ and $L(1) = \tau$. □

In order to avoid a heavy presentation, from now on we shall deal with Thom T-spectra only. The interested reader will be able to prove the relevant results for Thom T^2-spectra as well.

2.11. **Definition.** There is a functorial fibrant replacement of motivic spaces $X \to X^f$ in the flasque Nisnevich local model structure (e.g. given by controlled fibrant models in the sense of [18, Section 1.2]) such that for any $\mathbb{P}^1$-, T- or S^1-spectrum $E = (E_0, E_1, \dots)$ the sequence $E^f = (E_0^f, E_1^f, \dots)$ can be canonically equipped with a structure of a spectrum and $E \to E^f$ is a map of spectra.

The Voevodsky Lemma. One of the key facts in the theory of framed correspondences is the following lemma of Voevodsky that computes Hom-sets between certain Nisnevich sheaves. Its proof can be found in [13, Section 3].

2.12. **Lemma** (Voevodsky's Lemma). *For $X,Y \in \mathbf{Sm}_k$ and a closed subset X' of X and open subset V of Y the set*

$$\mathrm{Hom}_{\mathbf{Shv}_\bullet}(X/X', Y/V)$$

is in a natural bijection with the set of equivalence classes of triples (U,Z,ϕ), where Z is a closed subset of X disjoint with X', U is an étale neighborhood of Z in X and $\phi\colon U \to Y$ is a regular map such that $\phi^{-1}(Y-V) = Z$. By definition, two triples (U,Z,ϕ) and (U',Z',ϕ') are equivalent if $Z = Z'$ and ϕ,ϕ' coincide on some common etale neighbourhood of Z in X.

2.13. **Corollary.** *For any Thom spectrum E there is a natural isomorphism of motivic spaces*

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge m}, E_n) \cong \underline{\mathrm{Hom}}(\mathbb{P}^m/\mathbb{P}^{m-1}, E_n).$$

As a consequence, the Σ_m-action on $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge m}, E_n)$ permuting factors of $\mathbb{P}^{\wedge m}$ can be extended to an action of $\mathbf{GL}_m(k)$ (it naturally acts on $\mathbb{P}^m/\mathbb{P}^{m-1}$), and thus for any even permutation $\tau \in \Sigma_m$ there is an $\mathbb{A}^1$-homotopy between the action of τ and the identity map of $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge m}, E_n)$ by Lemma 2.10.

Proof. By Definition 2.8 $E_n = \mathrm{colim}_i E_{n,i}$, where $E_{n,i} = V_{n,i}/(V_{n,i} - Z_{n,i})$. For any $X \in \mathbf{Sm}_k$ Lemma 2.12 implies both sets $\mathrm{Hom}(X_+ \wedge \mathbb{P}^{\wedge m}, E_{n,i})$ and $\mathrm{Hom}(X_+ \wedge \mathbb{P}^m/\mathbb{P}^{m-1}, E_{n,i})$ are naturally isomorphic. They are described up to isomorphism of sets (see [13, Section 3]) as the equivalences classes of triples (U,Z,ϕ), where Z is a closed subset of $\mathbb{A}^m_X$, finite over X, U is its étale neighborhood and $\phi\colon U \to V_{n,i}$ is such that $\phi^{-1}(Z_{n,i}) = Z$. □

2.14. **Definition.** Following [13] for $X,Y \in \mathbf{Sm}_k$ and an open subscheme U of Y, we set

$$\mathrm{Fr}_n(X, Y/U) := \mathrm{Hom}_{\mathbf{Shv}_\bullet}(X_+ \wedge \mathbb{P}^{\wedge n}, (Y/U) \wedge T^n).$$

$\mathrm{Fr}_n(X,Y/U)$ is pointed at the empty correspondence or, equivalently, at the zero map. By smashing the elements of $\mathrm{Fr}_n(X,Y/U)$ with the canonical motivic equivalence $\sigma : \mathbb{P}^{\wedge 1} \to T$, we get a map of pointed sets $\mathrm{Fr}_n(X,Y/U) \to \mathrm{Fr}_{n+1}(X,Y/U)$. Denote by

$$\mathrm{Fr}(X,Y/U) := \mathrm{colim}_n(\cdots \to \mathrm{Fr}_n(X,Y/U) \to \mathrm{Fr}_{n+1}(X,Y/U) \to \cdots).$$

We shall also write $C_* \mathrm{Fr}(Y/U)$ to denote the Suslin complex associated to the Nisnevich sheaf $X \mapsto \mathrm{Fr}(X,Y/U)$ (see Definition 2.1).

More generally, we can define the sets $\mathrm{Fr}_n(X,\mathscr{G}) := \mathrm{Hom}_{\mathbf{Shv}_\bullet}(X_+ \wedge \mathbb{P}^{\wedge n}, \mathscr{G} \wedge T^n)$, $\mathrm{Fr}(X,\mathscr{G})$ for every pointed Nisnevich sheaf $\mathscr{G}$ as well as the Suslin complex $C_* \mathrm{Fr}(\mathscr{G})$ associated to the Nisnevich sheaf $X \mapsto \mathrm{Fr}(X,\mathscr{G})$.

Below we shall often deal with sheaves of the form

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, \mathrm{Fr}_n(Y/U)), \quad i,n \geqslant 0.$$

By Voevodsky's Lemma its value at $X \in \mathbf{Sm}_k$ consists of the triples (W,Z,ϕ), where Z is a closed subset of $\mathbb{A}^{i+n}_X$, finite over X, W is its étale neighborhood and $\phi\colon W \to Y$ is such that $\phi^{-1}(Y-U) = Z$.

2.15. **Proposition** (Additivity Theorem). *Suppose $X,X' \in \mathbf{Sm}_k$ and Z,Z' are closed subsets of X and X' respectively. Denote $Y = X/(X-Z), Y' = X'/(X'-Z')$. Then for every $i \geqslant 0$ the canonical map*

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, C_* \mathrm{Fr}(Y \vee Y')) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, C_* \mathrm{Fr}(Y)) \times \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, C_* \mathrm{Fr}(Y'))$$

is a schemewise weak equivalence.

Proof. The proof is like that of the Additivity Theorem of [13]. □

2.16. **Corollary.** *Let Γ^{op} be the category of finite pointed sets and pointed maps. Under the notation of Proposition 2.15 the association*

$$K \in \Gamma^{\mathrm{op}} \mapsto \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, C_* \mathrm{Fr}(Y \wedge K))$$

is a special Γ-space in the sense of Segal [28]. As a result, the Segal S^1-spectrum

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y)) = \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, C_* \mathrm{Fr}(Y \wedge \mathbb{S}))$$

is sectionwise positively fibrant. Here $\mathbb{S} = (S^0, S^1, S^2, \dots)$ is the sphere spectrum and $M_{fr}(Y) := C_* \mathrm{Fr}(Y \wedge \mathbb{S})$ is the framed motive of Y in the sense of [13].

3. The functor Θ^∞ and the layer filtration

If there is no likelihood of confusion, we shall often regard T-spectra as $\mathbb{P}^1$-spectra by means of the canonical motivic equivalence $\sigma : \mathbb{P}^{\wedge 1} \to T$. Given a T-spectrum E, denote by $u_* \colon E_i \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, E_{i+1})$ the adjoint to the bonding map. Following Jardine [18, §2], we give the following definition.

3.1. **Definition.** Denote by $E \wedge T$ the *fake T-suspension spectrum* with terms $(E \wedge T)_i = E_i \wedge T$ and bonding maps given by

$$(E_i \wedge T) \wedge T \xrightarrow{u \wedge T} E_{i+1} \wedge T.$$

It is important to note that the bonding maps do not permute two copies of T on the left. Denote by $\Omega^\ell(E)$ the *fake loop $\mathbb{P}^1$-spectrum* with terms $\Omega^\ell(E)_i = \underline{\mathrm{Hom}}(\mathbb{P}^1, E_i)$ and bonding maps adjoint to

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, E_i) \xrightarrow{u_*} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, E_{i+1})).$$

We notice again that two copies of $\mathbb{P}^{\wedge 1}$ on the right are not permuted.

3.2. **Definition.** We denote by $E[1]$ the *shifted T-spectrum* $E[1]_i = E_{i+1}$. Its bonding maps $u \colon E_i \wedge T \to E_{i+1}$ induce a map of T-spectra $u \colon E \wedge T \to E[1]$.

3.3. **Definition.** Denote by $\Theta^1(E) = \Omega^\ell(E[1])$. By adjointness there is a canonical map of $\mathbb{P}^1$-spectra

$$E \to \Omega^\ell(E \wedge T) \to \Omega^\ell(E[1]) = \Theta^1(E).$$

Denote by $\Theta^n(E)$ the n-fold composition $\Theta^1(\Theta^1 \dots (E))$. There are natural stabilization maps $\Theta^n(E) \to \Theta^{n+1}(E)$ and $\Theta^\infty(E)$ denotes the colimit

$$\Theta^\infty(E) = \mathrm{colim}_n \Theta^n(E).$$

3.4. **Remark.** We shall need the following explicit description of spaces of the $\mathbb{P}^1$-spectrum $\Theta^n E$ and its bonding maps. The jth space equals

$$\Theta^n(E)_j = \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, E_{j+n}).$$

The bonding maps of $\Theta^n(E)$ are adjoint to

$$(\mathbb{P}^{\wedge n}, E_{n+j}) \xrightarrow{u_*} (\mathbb{P}^{\wedge n}, (\mathbb{P}^{\wedge 1}, E_{n+j+1})) = (\mathbb{P}^{\wedge n} \wedge \mathbb{P}^{\wedge 1}, E_{n+j+1}) = (\mathbb{P}^{\wedge 1}, (\mathbb{P}^{\wedge n}, E_{n+j+1})).$$

One should note that we do not permute copies of $\mathbb{P}^{\wedge 1}$ here. The stabilization map $\Theta^n(E)_j \to \Theta^{n+1}(E)_j$ can be described as the composite map

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, E_{j+n}) \xrightarrow{-\wedge\sigma} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}\wedge\mathbb{P}^{\wedge 1}, E_{j+n}\wedge T) \xrightarrow{u} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n+1}, E_{j+n+1}).$$

Here the left arrow smashes the simplices of the left space with $\sigma : \mathbb{P}^{\wedge 1} \to T$.

3.5. **Lemma.** *For any spectrum E the adjoint of each bonding map*

$$\Theta^\infty(E)_i \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, \Theta^\infty(E)_{i+1})$$

in the spectrum $\Theta^\infty E$ is an isomorphism.

Proof. For every n and i the adjoint of the bonding map

$$\Theta^n(E)_i \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, \Theta^n(E)_{i+1}) = \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n+1}, E_{i+n+1}) = \Theta^{n+1}(E)_i$$

coincides with the stabilization map $\Theta^n(E)_i \to \Theta^{n+1}(E)_i$. Thus we get an isomorphism of sequences $\Theta^{n+1}(E)_i$ and $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, \Theta^n(E)_{i+1})$. □

3.6. **Definition.** Given a T-spectrum E, define its *n-th layer* $L_n(E)$ as the T-spectrum

$$L_nE = (E_0, E_1, \dots, E_n, E_n\wedge T, E_n\wedge T^2, \dots).$$

The maps of spaces $E_n \wedge T^i = E_n\wedge T\wedge T^{i-1} \xrightarrow{u\wedge T^{i-1}} E_{n+1}\wedge T^{i-1}$ induce maps of spectra $L_nE \to L_{n+1}E$ and an obvious isomorphism of spectra $E \cong \mathrm{colim}_n L_nE$.

We recall the following lemma from [13, Section 13]. It says that Θ^∞ converts T-spectra into framed $\mathbb{P}^1$-spectra, i.e. spectra whose spaces are spaces with framed correspondences.

3.7. **Lemma.** *For any T-spectrum E there is a canonical isomorphism of $\mathbb{P}^1$-spectra*

$$\Theta^\infty E = \mathrm{colim}_n \Theta^\infty(L_nE).$$

Moreover, there is a canonical isomorphism of spaces $\Theta^\infty(L_nE)_i = \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, \mathrm{Fr}(E_n\wedge T^i))$.

Proof. Note that $\Theta^m(E) = \mathrm{colim}_n \Theta^m(L_nE)$ for every m. Thus passing to the colimit over m, we get the first isomorphism. For the second statement note that the left hand side is the colimit over m of the sequence $(\mathbb{P}^{\wedge m+n}, E_n\wedge T^{m+i}) \to (\mathbb{P}^{\wedge m+n}\wedge\mathbb{P}^{\wedge 1}, E_n\wedge T^{m+i}\wedge T)$. Thus the identification $(\mathbb{P}^{\wedge m+n}, E_n\wedge T^{m+i}) = (\mathbb{P}^{\wedge n}, (\mathbb{P}^{\wedge m}, E_n\wedge T^i\wedge T^m))$ (we do not permute copies of T here) provides its isomorphism with the sequence $(\mathbb{P}^{\wedge n}, \mathrm{Fr}_m(E_n\wedge T^i))$. □

4. THE MAYER–VIETORIS SEQUENCE

4.1. **Definition.** Suppose $X \in \mathbf{Sm}_k$ and Z is a smooth closed subvariety of codimension d. We say that the embedding $Z \to X$ is *trivial* if there is an étale map $\alpha\colon X \to \mathbb{A}^{n+d}$ such that $Z = X\times_{\mathbb{A}^{n+d}}\mathbb{A}^n$, where $\mathbb{A}^n \to \mathbb{A}^{n+d}$ is the standard linear embedding.

4.2. **Lemma.** *For every closed embedding of smooth varieties $Z \to X$ of codimension d there is an open cover of X by X_i such that the inclusion $X_i\cap Z \to X_i$ is trivial.*

Proof. Let n be the dimension of Z and let $x \in Z$ be a closed point. Since the embedding $Z \to X$ is regular, there is an open affine neighborhood X' of x in X such that $Z' = X' \cap Z$ is the zero locus of d regular functions $f_1, \ldots, f_d$. These regular functions define a map $f\colon X' \to \mathbb{A}^d$ that is flat by [29, Tag 00R4]. Since Z' is smooth, [29, Tag 01V9] implies that f is smooth at x, and by [29, Tag 054L] there is an affine open neighborhood U of x in X' such that $f\colon U \to \mathbb{A}^d$ can be presented as a composition of an étale map followed by a projection: $f\colon U \xrightarrow{\alpha} \mathbb{A}^n_{\mathbb{A}^d} \to \mathbb{A}^d$. Then $U \cap Z = U \times_{\mathbb{A}^d} \{0\} = U \times_{\mathbb{A}^{n+d}} \mathbb{A}^n$, and so the embedding $U \cap Z \to U$ is trivial. □

4.3. **Lemma.** *If $Z \to X$ is a trivial embedding of codimension d, then there is an isomorphism of Nisnevich sheaves $X/(X-Z) \xrightarrow{\cong} Z_+ \wedge T^d$.*

Proof. Fix an étale map $\alpha\colon X \to \mathbb{A}^{n+d}$ such that $Z = X \times_{\mathbb{A}^{n+d}} \mathbb{A}^n$. Consider the projection $pr_n\colon \mathbb{A}^{n+d} \to \mathbb{A}^n$ and the composition $pr_n \circ \alpha\colon X \to \mathbb{A}^n$. Then $X' = Z \times_{\mathbb{A}^n} X$ is an etale neighborhood of Z in X. Moreover, X' and $X - Z$ is an elementary Nisnevich cover of X, hence $X'/(X'-Z) \to X/(X-Z)$ is an isomorphism of Nisnevich sheaves. Then the composition

$$X' = Z \times_{\mathbb{A}^n} X \to Z \times_{\mathbb{A}^n} \mathbb{A}^{n+d} = Z \times_k \mathbb{A}^d$$

is an étale neighborhood of $Z = Z \times 0$ in $Z \times_k \mathbb{A}^d$, and thus it induces an isomorphism $X'/(X' - Z) \to Z \times \mathbb{A}^d/(Z \times \mathbb{A}^d - Z \times 0) = Z_+ \wedge T^d$. □

4.4. **Lemma.** *Suppose G is a strictly homotopy invariant Nisnevich sheaf, then for any bounded chain complex of presheaves X there is an isomorphism*

$$\mathrm{Hom}_{D^-_{Nis}}(Tot(C_*(X))_{Nis}, G[n]) \cong \mathrm{Hom}_{D^-_{Nis}}(X_{Nis}, G[n])$$

Proof. Consider the stupid truncation $\sigma_{\geqslant i} X$. Then there is a short exact sequence of complexes of presheaves

$$0 \to C_* X_i[-i] \to Tot(C_*(\sigma_{\geqslant i} X)) \to Tot(C_*(\sigma_{\geqslant i+1} X)) \to 0$$

Note that $\mathrm{Hom}_{D^-_{Nis}}((C_* X_i)_{Nis}, G[n]) \cong \mathrm{Hom}_{D^-_{Nis}}((X_i)_{Nis}, G[n])$ by [20, Prop. 12.19], and $\sigma_{\geqslant N_0} X = 0$ and $\sigma_{\geqslant N_1} X = X$ for some N_0, N_1. Then the statement follows by induction. □

4.5. **Lemma.** *Suppose F is a bounded complex of $\mathbb{Z}F_*$-presheaves such that F_{Nis} is quasi-isomorphic to zero and the homology presheaves $H_i(Tot(C_* F))$ are quasi-stable. Then the complex of sheaves $(Tot(C_* F))_{Nis}$ is locally quasi-isomorphic to zero.*

Proof. The presheaves $H_i(Tot(C_* F))$ are quasi-stable and homotopy invariant. By [12] the associated sheaves $H_i = H_i(Tot(C_* F))_{Nis}$ are strictly homotopy invariant, and hence by Lemma 4.4 there is an isomorphism

$$\mathrm{Hom}_{D^-_{Nis}}((Tot(C_* F))_{Nis}, H_i[n]) \cong \mathrm{Hom}_{D^-_{Nis}}(F_{Nis}, H_i[n]) = 0.$$

The inductive argument as in the proof of [20, 13.12] gives a map $(Tot(C_* F))_{Nis} \to H_i[i]$ inducing an isomorphism on homology sheaves. It is zero by the above arguments, hence $H_i = 0$. □

Suppose $Z \subseteq X$ is a closed subset of X, and $X = X_1 \cup X_2$ is a Zariski cover of X. Denote by $X_{12} = X_1 \cap X_2, Z_1 = X_1 \cap Z, Z_2 = X_2 \cap Z, Z_{12} = X_{12} \cap Z$ and $Y = X/(X-Z), Y_1 = X_1/(X_1 - Z_1), Y_2 = X_2/(X_2 - Z_2), Y_{12} = X_{12}/(X_{12} - Z_{12})$.

4.6. **Lemma.** *The maps $Y_{12} \to Y_1$, $Y_{12} \to Y_2$ are injective and the sheaf Y is the pushout of the diagram $Y_1 \hookleftarrow Y_{12} \hookrightarrow Y_2$.*

Proof. For any Henselian local scheme U the map

$$Y_{12}(U) = X_{12}(U)/(X_{12} - Z_{12})(U) \to X_1(U)/(X_1 - Z_1)(U) = Y_1(U)$$

is injective, because $(X_{12} - Z_{12})(U) = X_{12}(U) \cap (X_1 - Z_1)(U)$. Similarly, the map $Y_{12}(U) \to Y_2(U)$ is injective. Note that $Y(U) = X(U)/(X - Z)(U)$, $X(U) = X_1(U) \cup_{X_{12}(U)} X_2(U)$ and $(X - Z)(U) = (X_1 - Z_1)(U) \cup_{(X_{12}-Z_{12})(U)} (X_2 - Z_2)(U)$. Hence $Y(U) = Y_1(U) \cup_{Y_{12}(U)} Y_2(U)$. $\square$

4.7. **Definition.** Let $F^{\mathbb{P}^{\wedge i}}(U,Y)$, $U \in \mathbf{Sm}_k$, be the set of $x \in \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, \mathrm{Fr}(Y))(U)$ such that the support of x is connected. The free abelian group generated by $F^{\mathbb{P}^{\wedge i}}(U,Y)$ is denoted by $\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(U,Y)$. Then $\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(U,Y)$ is functorial in U. Moreover, $\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(-,Y)$ is a Nisnevich sheaf.

The following result gives an explicit computation of homology of the motivic S^1-spectrum $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y))$.

4.8. **Lemma.** *There are isomorphisms of graded presheaves*

$$\pi_*(\mathbb{Z}\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y))) = H_*(C_*\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y))$$

for all $i \geqslant 0$.

Proof. The proof repeats the proof in [10, 1.2] word for word. $\square$

4.9. **Lemma.** *For any $i \geqslant 0$, the natural maps $Y_{12} \to Y_2, Y_{12} \to Y_1, Y_1 \to Y, Y_2 \to Y$ give rise to a short exact sequence of Nisnevich sheaves*

$$0 \to \mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y_{12}) \to \mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y_1) \oplus \mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y_2)) \to \mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y) \to 0.$$

Proof. Let U be a local Henselian scheme. There is a coequalizer diagram of pointed sets

$$F^{\mathbb{P}^{\wedge i}}(U,Y_{12}) \rightrightarrows F^{\mathbb{P}^{\wedge i}}(U,Y_2) \vee F^{\mathbb{P}^{\wedge i}}(U,Y_1) \to F^{\mathbb{P}^{\wedge i}}(U,Y).$$

Thus it gives rise to a right exact sequence

$$\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y_{12}) \to \mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y_1) \oplus \mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y_2) \to \mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y) \to 0.$$

It remains to note that the latter sequence is also exact on the left. $\square$

4.10. **Corollary.** *The cone of the morphism of complexes*

$$C_*\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y_{12}) \to C_*\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y_1) \oplus C_*\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y_2)$$

is locally quasi-isomorphic to the complex $C_*\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y))$. In particular, we have a triangle in the derived category of complexes of sheaves

$$C_*\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y_{12}) \to C_*\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y_1) \oplus C_*\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y_2) \to C_*\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y).$$

Proof. Note that homology presheaves of $C_*\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y)$ are quasi-stable. Then by Lemmas 4.9 and 4.5 the totalization of the bicomplex

$$0 \to C_*\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y_{12}) \to C_*\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y_1) \oplus C_*\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y_2) \to C_*\mathbb{Z}F^{\mathbb{P}^{\wedge i}}(Y) \to 0$$

is locally quasi-isomorphic to zero. $\square$

4.11. **Proposition** (The Mayer–Vietoris sequence)**.** *For every $i \geqslant 0$ the square of S^1-spectra*

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y_{12})) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y_1)) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y_2)) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y)) \end{array}$$

is a homotopy pushout square in the local stable model structure of S^1-spectra.

Proof. The natural map from the cone of the morphism

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y_{12})) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y_1) \vee M_{fr}(Y_2))$$

to $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y))$ induces locally an equivalence on homology between connective spectra by Corollary 4.10. Then it is a local stable equivalence. □

For any space A, there is an obvious map $\mathrm{Fr}_n(A) \to \underline{\mathrm{Hom}}(B, \mathrm{Fr}_n(A \wedge B))$ defined by $(\mathbb{P}^{\wedge n}, A \wedge T^n) \to (B \wedge \mathbb{P}^{\wedge n}, A \wedge B \wedge T^n)$. It gives rise to a map of spectra $M_{fr}(A) \to \underline{\mathrm{Hom}}(B, M_{fr}(A \wedge B))$.

4.12. **Lemma.** *For any $X \in \mathbf{Sm}_k$ for $j \geqslant 1$ the map*

$$C_* \mathrm{Fr}(X_+ \wedge T^j) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, C_* \mathrm{Fr}(X \wedge T^j \wedge T^i))$$

is a local weak equivalence for any $i \geqslant 0$.

Proof. The map in question is obtained as the colimit of the maps

$$C_* \mathrm{Fr}_n(X_+ \wedge T^j) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, C_* \mathrm{Fr}_n(X_+ \wedge T^j \wedge T^i)). \tag{4.13}$$

Consider the triangle

$$\begin{array}{ccc} C_*\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i} \wedge \mathbb{P}^{\wedge n}, X_+ \wedge T^j \wedge T^{i+n}) & \longleftarrow & C_*\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i} \wedge \mathbb{P}^{\wedge n-i}, X_+ \wedge T^j \wedge T^i \wedge T^{n-i}) \\ \uparrow & \nearrow_{\cong} & \\ C_*\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, X_+ \wedge T^j \wedge T^n) & & \end{array}$$

where the vertical map is the map (4.13), the skew map is the isomorphism given by identification $\mathbb{P}^{\wedge n} = \mathbb{P}^{\wedge i} \wedge \mathbb{P}^{\wedge n-i}$ and $T^n = T^i \wedge T^{n-i}$, and the horizontal map is induced by the stabilization map $(\mathbb{P}^{\wedge i}, C_* \mathrm{Fr}_{n-i}(X_+ \wedge T^j)) \to (\mathbb{P}^{\wedge i}, C_* \mathrm{Fr}_n(X_+ \wedge T^j))$. The composite map of the triangle differs from the left vertical map by the shuffle permutation action $\chi_{n,i}$ on $\mathbb{P}^{\wedge i+n}$ and on T^{i+n} respectively. Thus if n is even then the triangle is commutative up to a simplicial homotopy by Corollary 2.13 and Remark 2.3. Note that the horizontal map induces an isomorphism on the colimit over n. Thus the vertical map induces a bijection on the colimits of sheaves π_*^{Nis}. For $j \geqslant 1$ the space $C_*(\mathrm{Fr}(X_+ \wedge T^j))$ is locally connected by [10, 8.1]. The space $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, C_*(\mathrm{Fr}(X_+ \wedge T^j \wedge T^i)))$ is isomorphic to $C_*(\mathrm{Fr}(X_+ \wedge T^j))$ by means of the horizontal map, and hence it is locally connected as well. We see that the vertical map induces a local weak equivalence. □

4.14. **Lemma.** *Suppose $Z \to X$ is a closed embedding of smooth varieties of codimension d. Then for $i < d$ the space $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, (C_* \mathrm{Fr}(X/X - Z)))$ is locally connected.*

Proof. If U is a local Henselian scheme, then every correspondence c in $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, \mathrm{Fr}(X/X - Z)) = \mathrm{colim}_n \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i+n}, (X/X-Z)\wedge T^n)$ can be described by triples $c = (S, U, \phi)$, where the support S is a closed subset of $\mathbb{A}_U^{i+n}$, finite over U, and $\phi : U \to X \times \mathbb{A}^n$ is a regular map from an étale neighborhood of S such that $S = \phi^{-1}(Z \times 0)$ (see Voevodsky's Lemma 2.12 and [13, Section 3] for details). Since S is finite over Henselian U, it is a disjoint union of local schemes S_j, finite over U, for $j = 1 \dots l$. Each map $S_j \to Z$ factors through $S_j \to Z_j$, where $Z_j = X_j \cap Z$ for some open X_j in X and such that $Z_j \to X_j$ is a trivial embedding. Thus the correspondence c lies in the image of $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, \mathrm{Fr}(\vee_j(X_j/X_j - Z_j))) = \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, \mathrm{Fr}(\vee(Z_j \wedge T^d))) = \mathrm{Fr}(\vee(Z_j \wedge T^{d-i})) = \mathrm{Fr}((\sqcup Z_j)_+ \wedge T^{d-i})$, and $\pi_0^{Nis}(C_* \mathrm{Fr}((\sqcup Z_j)_+ \wedge T^{d-i}) = *$ for $i < d$ by [10, A.1]. Since the class of $c \in \pi_0^{Nis}(\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, (C_* \mathrm{Fr}(X/X-Z))))$ belongs to the image of $\pi_0^{Nis}(C_* \mathrm{Fr}((\sqcup Z_j)_+ \wedge T^{d-i})) = *$, then c equals the class of the basepoint of $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, (C_* \mathrm{Fr}(X/X-Z)))$. We conclude that $\pi_0^{Nis}(\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, (C_* \mathrm{Fr}(X/X-Z)))) = *$. □

4.15. **Lemma.** *Suppose $Z \to X$ is a closed embedding of smooth varieties of codimension d. Then for $i < d$ the S^1-spectrum $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X/X-Z))$ is locally an Ω-spectrum and the S^1-spectrum $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X/X-Z))_f$, obtained from $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X/X-Z))$ by taking a local fibrant replacement levelwise, is motivically fibrant. In particular, the motivic space $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, C_* \mathrm{Fr}(X/X-Z))_f$ is motivically fibrant.*

Proof. It follows from Additivity Theorem 2.15, Corollary 2.16 and Lemma 4.14 that the Γ-space taking a finite pointed set K to $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, C_* \mathrm{Fr}((X/X-Z)\wedge K))$ is locally very special. By the Segal machine [28] $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X/X-Z)) = \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, C_* \mathrm{Fr}((X/X-Z)\wedge \mathbb{S}))$ is locally an Ω-spectrum, and hence so is $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X/X-Z))_f$. Since all spaces of $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X/X-Z))_f$ are locally fibrant, we see that $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X/X-Z))_f$ is sectionwise an Ω-spectrum. Since the sheaves of homotopy groups of $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X/X-Z))_f$ are strictly homotopy invariant by [12, 1.1], $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X/X-Z))_f$ is motivically fibrant by [13, 7.1]. □

4.16. **Corollary.** *For a Thom spectrum E with the bounding constant d, $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, C_* \mathrm{Fr}(E_n \wedge T^i))_f$ is a motivically fibrant space for $i \geqslant \max(0,d)$.*

4.17. **Lemma.** *Given $i, n \geqslant 0$, the natural map of S^1-spectra*

$$M_{fr}(X_+ \wedge T^n)_f \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X_+ \wedge T^n \wedge T^i)_f)$$

is a levelwise local weak equivalence in positive degrees, where "f" refers to a levelwise local fibrant replacement. In particular, the map is a stable local weak equivalence. If $n > 0$ then this map is a levelwise local weak equivalence of spectra in all degrees.

Proof. The statement of the lemma can be reformulated as follows for $n \geqslant 0$: the map of S^1-spectra

$$M_{fr}(X_+ \wedge S^1 \wedge T^n)_f \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X_+ \wedge S^1 \wedge T^n \wedge T^i)_f)$$

is a levelwise local weak equivalence. The spectra $M_{fr}(X_+ \wedge S^1 \wedge T^n)_f, M_{fr}(X_+ \wedge S^1 \wedge T^n \wedge T^i)_f$ are both motivically fibrant by [13, 7.5].

The proof of [13, 4.1(2)] shows that the map in question is a levelwise local weak equivalence if so is the map

$$M_{fr}(X_+ \wedge S^1 \wedge T^n)_f \to \underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge i} \wedge S^i, M_{fr}(X_+ \wedge S^1 \wedge T^n \wedge \mathbb{G}_m^{\wedge i} \wedge S^i)_f),$$

where $\mathbb{G}_m^{\wedge 1}$ is the mapping cone of $pt_+ \to (\mathbb{G}_m)_+$ sending pt to $1 \in \mathbb{G}_m$ and $\mathbb{G}_m^{\wedge i}$ is the ith smash product of $\mathbb{G}_m^{\wedge 1}$. Our assertion now follows from the Cancellation Theorem for framed motives [3]. The same arguments apply to show that the map

$$M_{fr}(X_+ \wedge T^n)_f \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X_+ \wedge T^n \wedge T^i)_f)$$

is a levelwise local weak equivalence in all degrees for $n > 0$. □

4.18. **Proposition.** *Suppose $Z \to X$ is a closed embedding of smooth varieties of codimension d and $M_{fr}(X/(X-Z))_f$ is obtained from $M_{fr}(X/(X-Z))$ by taking a level local fibrant replacement. Then*

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X/(X-Z))) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X/(X-Z))_f)$$

is a levelwise local weak equivalence of S^1-spectra for $i < d$. In particular, the right spectrum is a fibrant replacement of the left spectrum in the stable motivic model structure of S^1-spectra whenever $i < d$. If $i = d$ then the map is a levelwise local weak equivalence in positive degrees. In particular, the map is a stable local weak equivalence for $i = d$.

Proof. Suppose $i < d$. By Lemma 4.2 there is a cover of X by open subsets X_j such that $X_j \cap Z \to X_j$ is a trivial embedding. We proceed by induction on n, the number of elements in the cover. For $n = 1$ we have $X/X - Z \cong Z_+ \wedge T^d$ by Lemma 4.3. Then the map in question fits into a commutative square

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Z_+ \wedge T^d)) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Z_+ \wedge T^d)_f) \\ \simeq \uparrow & & \simeq \uparrow \\ M_{fr}(Z_+ \wedge T^{d-i}) & \xrightarrow{\simeq} & M_{fr}(Z_+ \wedge T^{d-i})_f \end{array}$$

The left arrow is a levelwise local weak equivalence by Lemma 4.12, and the right arrow is a levelwise local weak equivalence by Lemma 4.17. Thus the upper map is a levelwise local weak equivalence.

For the induction step present X as the union of X_1 and X_2 such that X_1 can be covered by $n-1$ trivial open pieces, and $Z \cap X_2 \to X_2$ is a trivial embedding. Then for $X_{12} = X_1 \cap X_2$ the embedding $Z \cap X_{12} \to X_{12}$ is trivial. Denote by Y the sheaf $X/X - Z$ and by Y_i the sheaf $X_i/(X_i - (X_i \cap Z))$. Consider a commutative diagram of S^1-spectra

$$\begin{array}{ccccc} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y_{12})) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y_1)) \vee \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y_2)) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y)) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y_{12})_f) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y_1)_f) \vee \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y_2)_f) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y)_f) \end{array}$$

The upper row is a homotopy cofiber sequence in the local stable model structure of S^1-spectra by Proposition 4.11. By Lemma 4.15 $M_{fr}(Y_{12})_f, M_{fr}(Y_1)_f, M_{fr}(Y_2)_f, M_{fr}(Y)_f$ are motivically fibrant. It follows from Proposition 4.11 that the sequence

$$M_{fr}(Y_{12})_f \to M_{fr}(Y_1)_f \vee M_{fr}(Y_2)_f \to M_{fr}(Y)_f$$

is a homotopy cofiber sequence of motivically fibrant spectra in the local stable model structure, and hence so is the lower sequence of the commutative diagram above, because $\mathbb{P}^{\wedge i}$ is a flasque cofibrant motivic space. Two left vertical arrows are levelwise local weak equivalences by induction hypothesis. Hence the right arrow is a stable local weak equivalence factoring as

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y)) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y))_f \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(Y)_f).$$

Since the left arrow is a levelwise local equivalence by definition, then the right arrow is a stable local weak equivalence. But the middle spectrum is motivically fibrant by Lemma 4.15 as well as so is the right spectrum. It remains to observe that a stable local equivalence between motivically fibrant spectra must be a levelwise local weak equivalence.

If $i = d$ then we replace all framed motives and their levelwise local fibrant replacements by framed motives smashed with the unit circle S^1. Then all spaces of $M_{fr}(Y \wedge S^1)$ become connected and $M_{fr}(Y \wedge S^1)_f$ is a motivically fibrant S^1-spectrum. It is now enough to repeat the above arguments word for word (Lemma 4.12 is also satisfied for spaces of the form $C_* \mathrm{Fr}(X_+ \wedge S^1)$ which are automatically sectionwise connected) to show that

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge d}, M_{fr}(Y \wedge S^1)) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge d}, M_{fr}(Y \wedge S^1)_f)$$

is a stable local weak equivalence of spectra. By Corollary 2.16 the left spectrum is sectionwise an Ω-spectrum. Since a stable equivalence between Ω-spectra is a levelwise weak equivalence, it follows that the map of spectra is a levelwise local weak equivalence. Therefore, the map

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge d}, M_{fr}(Y)) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge d}, M_{fr}(Y)_f)$$

is a levelwise local weak equivalence in positive degrees. □

4.19. **Theorem.** *Suppose $Z \to X$ is a closed embedding of smooth varieties of codimension d. Then the space $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, C_* \mathrm{Fr}(X/X-Z)_f)$ is motivically fibrant and*

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, C_* \mathrm{Fr}(X/X-Z)) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, C_* \mathrm{Fr}(X/X-Z)_f)$$

is a local weak equivalence for $i < d$.

Proof. The statement follows from Proposition 4.18. □

4.20. **Corollary.** *If E is a Thom spectrum with the bounding constant d, then the motivic space $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge m}, C_* \mathrm{Fr}(E_n \wedge T^i)_f)$ is motivically fibrant and*

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge m}, C_* \mathrm{Fr}(E_n \wedge T^i)) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge m}, C_* \mathrm{Fr}(E_n \wedge T^i)_f)$$

is a local weak equivalence for $m \leqslant n + i - d$.

Proof. We have $E_n \wedge T^i = \mathrm{colim}_j V_{n,j} \times \mathbb{A}^i / (V_{n,j} \times \mathbb{A}^i - Z_{n,j} \times 0)$, where codimension of $Z_{n,j}$ in $V_{n,j}$ is strictly greater than $n - d$. Then codimension of $Z_{n,j} \times 0$ in $V_{n,j} \times \mathbb{A}^i$ is strictly greater than $n + i - d$. Then for $E_{n,j} = V_{n,j}/(V_{n,j} - Z_{n,j})$ we get that $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge m}, C_* \mathrm{Fr}(E_{n,j} \wedge T^i)_f)$ is motivically fibrant and the map $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge m}, C_* \mathrm{Fr}(E_{n,j} \wedge T^i)) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge m}, C_* \mathrm{Fr}(E_{n,j} \wedge T^i)_f)$ is a local weak equivalence for every j by Theorem 4.19. By passing to the colimit and using the fact that a directed colimit of flasque motivically fibrant spaces (respectively a directed colimit of local weak equivalences) is flasque motivically fibrant, we get the statement of the lemma. □

5. Fibrant replacements of Thom spectra

In this section we give a model for a fibrant replacement of a Thom spectrum E. First we need the following

5.1. **Lemma.** *Suppose E is a Thom spectrum with the bounding constant d. Then for $i \geqslant \max(0,d)$ and $n \geqslant 0$ the map of spaces $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, C_*\Theta^\infty(L_nE)_{i+1}) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, C_*\Theta^\infty(L_nE)^f_{i+1})$ is a local weak equivalence, where L_nE is the n-th layer of E and $(L_nE)^f_{i+1}$ is a local fibrant replacement of the space $(L_nE)_{i+1}$.*

Proof. By Corollary 4.20 the space $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, C_*\mathrm{Fr}(E_n \wedge T^{i+1})^f)$ is motivically fibrant. By Lemma 3.7 the map in question coincides with the horizontal map of the diagram

$$\begin{array}{ccc}
\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, C_*\mathrm{Fr}(E_n \wedge T^{i+1}))) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, C_*\mathrm{Fr}(E_n \wedge T^{i+1}))^f) \\
\downarrow & \swarrow & \\
\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, C_*\mathrm{Fr}(E_n \wedge T^{i+1})^f)) & &
\end{array} \tag{5.2}$$

The diagram (5.2) is obtained by applying $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, -)$ to the diagram

$$\begin{array}{ccc}
\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, C_*\mathrm{Fr}(E_n \wedge T^{i+1})) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, C_*\mathrm{Fr}(E_n \wedge T^{i+1}))^f \\
\downarrow & \swarrow & \\
\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, C_*\mathrm{Fr}(E_n \wedge T^{i+1})^f) & &
\end{array} \tag{5.3}$$

The slanted arrow exists by the right lifting property for fibrant spaces. The horizontal arrow of (5.3) is a local weak equivalence, and the vertical arrow of (5.3) is a local weak equivalence by Corollary 4.20. It follows that the slanted arrow of (5.3) is a local weak equivalence between fibrant spaces, and hence so is the slanted arrow of (5.2) since $\mathbb{P}^{\wedge 1}$ is a flasque cofibrant space. The vertical arrow of (5.2) is a local weak equivalence by Corollary 4.20. We see that the horizontal map of (5.2) is a local weak equivalence. □

The following theorem says that a fibrant replacement of a Thom spectrum E can be computed (starting at some level depending on its bounding constant) by first applying the Θ^∞-functor to E, then by taking the Suslin complex of each space of $\Theta^\infty(E)$ and finally by taking local fibrant replacements for $C_*\Theta^\infty(E)$.

5.4. **Theorem.** *Let E be a Thom spectrum with the bounding constant d. Let $C_*\Theta^\infty(E)^f$ be a spectrum obtained from $C_*\Theta^\infty(E)$ by taking a level local fibrant replacement. Then the spectrum $C_*\Theta^\infty(E)^f$ is motivically fibrant starting from level $\max(0,d)$ and is stably equivalent to E.*

Proof. Since a directed colimit of flasque locally fibrant spaces is flasque locally fibrant, it follows that $C_*\Theta^\infty(E)^f = \mathrm{colim}_n C_*\Theta^\infty(L_nE)^f$. Hence it is sufficient to prove that for every n the spectrum $C_*\Theta^\infty(L_nE)^f$ is motivically fibrant starting from level d. For $i \geqslant d$ the space $C_*\Theta^\infty(L_nE)^f_i$ equals $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, C_*\mathrm{Fr}(E_n \wedge T^i))^f$ by Lemma 3.7. Moreover, it is motivically fibrant by Corollary 4.16. Thus it remains to prove that each bonding map

$$C_*\Theta^\infty(L_nE)^f_i \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, C_*\Theta^\infty(L_nE)^f_{i+1})$$

is a local weak equivalence. It fits into the following commutative diagram:

$$\begin{array}{ccc} C_*\Theta^\infty(L_nE)^f_i & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, C_*\Theta^\infty(L_nE)^f_{i+1}) \\ \uparrow & & \uparrow \\ C_*\Theta^\infty(L_nE)_i & \xrightarrow{\cong} & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, C_*\Theta^\infty(L_nE)_{i+1}) \end{array}$$

where the right vertical arrow is a local weak equivalences by Lemma 5.1 and the lower arrow is an isomorphism by Lemma 3.5. Since the left vertical arrow is a local weak equivalence, then so is the upper arrow, as required. □

6. The functor Θ^∞_{sym}

Whenever a Thom T-spectrum E is symmetric, we can also construct further fibrant replacements for it. To this end, we introduce another stabilization functor Θ^∞_{sym} on the level of symmetric T-spectra, which is slightly different from Θ^∞. The spaces of $\Theta^\infty_{sym}(E)$ and $\Theta^\infty(E)$ are in fact isomorphic, but the bonding maps are different: the bonding maps of $\Theta^\infty_{sym}(E)$ require the structure of a symmetric spectrum on E, whereas the bonding maps of $\Theta^\infty(E)$ do not.

Given a T-spectrum E, let $T\wedge E$ be the suspension spectrum of E (see Definition 2.6). The functor $E\mapsto T\wedge E$ has a right adjoint loop functor $E\to\Omega_TE$, where Ω_TE has the spaces $(\Omega_TE)_i=\underline{\mathrm{Hom}}(T,E_i)$. If there is no likelihood of confusion, we denote by ΩE the $\mathbb{P}^1$-spectrum with $(\Omega E)_i=\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1},E_i)$ and the bonding maps are given by

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1},E_i)\wedge\mathbb{P}^{\wedge 1}\to\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1},E_i\wedge\mathbb{P}^{\wedge 1})\xrightarrow{\sigma}\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1},E_i\wedge T)\xrightarrow{u}(\mathbb{P}^{\wedge 1},E_{i+1}),$$

where $\mathbb{P}^{\wedge 1}\to T$ is a canonical motivic equivalence.

6.1. **Definition.** Define the functor $\Theta^1_{sym}(E)=\Omega(E[1])$, where $E[1]$ is the shift spectrum (see Definition 2.6), and

$$\Theta^n_{sym}(E):=\Theta^1_{sym}(\Theta^1_{sym}(\dots(E)))\quad (n\text{ times}).$$

6.2. **Definition.** If E is a symmetric T-spectrum, then there is a canonical map of T-spectra $T\wedge E\to E[1]$ (see Definition 2.6). Notice that this map requires the symmetric spectrum structure of E. By adjointness we have a map $E\to\Omega(T\wedge E)\to\Omega(E[1])=\Theta^1_{sym}(E)$. Iterating the latter map, we get a sequence of maps of spectra

$$E\to\Theta^1_{sym}(E)\to\Theta^2_{sym}(E)\to\cdots$$

Denote by $\Theta^\infty_{sym}(E)$ the colimit of this sequence. Then for every symmetric T-spectrum E there is a natural map of $\mathbb{P}^1$-spectra

$$\varepsilon:E\to\Theta^\infty_{sym}(E).$$

6.3. **Remark.** We need to describe bonding maps of $\Theta^n_{sym}(E)$ and stabilization maps $\Theta^n_{sym}(E)\to\Theta^{n+1}_{sym}(E)$ explicitly. One has,

$$\Theta^n_{sym}(E)_i=\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},E_{n+i}).$$

Each bonding map equals the composition

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},E_{n+i})\wedge\mathbb{P}^{\wedge 1}\to\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},E_{n+i}\wedge\mathbb{P}^{\wedge 1})\xrightarrow{\sigma}\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},E_{n+i}\wedge T)\xrightarrow{u}\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},E_{n+i+1})$$

and the stabilization map $\Theta^n_{sym}(E)_i \to \Theta^{n+1}_{sym}(E)_i$ equals the composition

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, E_{n+i}) \to (\mathbb{P}^{\wedge n}, \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, E_{n+i} \wedge T)) \xrightarrow{u} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n+1}, E_{n+i+1}) \xrightarrow{\chi_{i,1}} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n+1}, E_{n+1+i}),$$

where the left arrow is induced by the external smash product with $\sigma : \mathbb{P}^{\wedge 1} \to T$ and $\chi_{i,1}$ is the shuffle permutation in Σ_{n+i+1} permuting the last element with preceding i elements and preserves the first n elements.

6.4. **Lemma.** *For any symmetric T-spectrum E for any i there is an isomorphism of motivic spaces $\Theta^\infty(E)_i \xrightarrow{\cong} \Theta^\infty_{sym}(E)_i$.*

Proof. Define a map $f_n \colon \Theta^n(E)_i \to \Theta^n_{sym}(E)_i$ by the formula

$$f_n \colon \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, E_{i+n}) \xrightarrow{\chi_{i,n}} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, E_{n+i}),$$

where $\chi_{i,n}$ is the shuffle permutation that permutes the last n elements with the first i elements. Then the following diagram is commutative:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, E_{i+n}) & \xrightarrow{\chi_{i,n}} & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, E_{n+i}) \\ \downarrow u & & \downarrow \chi_{i,1}\circ u \\ \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n+1}, E_{i+n+1}) & \xrightarrow{\chi_{i,n+1}} & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n+1}, E_{n+1+i}). \end{array}$$

Here the left vertical arrow is the stabilization map $\Theta^n(E)_i \to \Theta^{n+1}(E)_i$ and the right vertical map is the stabilization map $\Theta^n_{sym}(E)_i \to \Theta^{n+1}_{sym}(E)_i$ of Remark 6.3. So the maps f_n induce a morphism of sequences. Then the maps f_n induce the desired isomorphism on colimits $f \colon \Theta^\infty(E)_i \xrightarrow{\cong} \Theta^\infty_{sym}(E)_i$. □

6.5. **Lemma.** *For any symmetric T-spectrum E there are isomorphisms of spaces*

$$\Theta^\infty(\Theta^m_{sym}(E))_i \cong \Theta^\infty_{sym}(E)_i, \quad \Theta^n(\Theta^\infty_{sym}(E))_i \cong \Theta^\infty_{sym}(E)_i.$$

Proof. Applying Lemma 6.4 to the symmetric spectrum $\Theta^m_{sym}(E)$, we have

$$\Theta^\infty(\Theta^m_{sym}(E))_i \cong \Theta^\infty_{sym}(\Theta^m_{sym}(E))_i = \Theta^\infty_{sym}(E)_i.$$

Also,

$$\Theta^n(\Theta^\infty_{sym}(E))_i = \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, \Theta^\infty_{sym}(E)_{i+n}) \cong \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, \Theta^\infty(E)_{i+n}) = \Theta^n(\Theta^\infty(E))_i = \Theta^\infty(E)_i,$$

as required. □

6.6. **Lemma.** *Suppose E is a Thom T-spectrum with the bounding constant d. Then the space $C_*\Theta^\infty(E)_i$ is locally connected for $i \geqslant \max(0,d)$.*

Proof. By Lemma 4.14 the space $C_*\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, \mathrm{Fr}(E_n \wedge T^i)) = C_*\Theta^\infty(L_n(E))_i$ is locally connected for every n. Then $C_*\Theta^\infty(E)_i = \mathrm{colim}_n C_*\Theta^\infty(L_n(E))_i$ is a locally connected space. □

6.7. **Proposition.** *Suppose E is a symmetric Thom T-spectrum with the bounding constant d and contractible alternating group action. Then the natural maps of spectra*

$$\xi : C_*\Theta^\infty_{sym}(E) \to C_*\Theta^\infty(\Theta^\infty_{sym}(E))$$

and

$$C_*\Theta^\infty(\varepsilon): C_*\Theta^\infty(E) \to C_*\Theta^\infty(\Theta^\infty_{sym}(E)),$$

obtained from the map $\varepsilon: E \to \Theta^\infty_{sym}(E)$ *by applying* $C_*\Theta^\infty$ *to it, induce local weak equivalences of spaces starting from level* $\max(0,d)$*. In particular, there is a commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{\eta} & C_*\Theta^\infty(E) \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle C_*\Theta^\infty(\varepsilon)} \\ C_*\Theta^\infty_{sym}(E) & \xrightarrow{\xi} & C_*\Theta^\infty(\Theta^\infty_{sym}(E)) \end{array}$$

of $\mathbb{P}^1$*-spectra, in which all arrows are stable motivic equivalences.*

Proof. Fix a number $i \geqslant d$. Consider a two-dimensional sequence

$$A_{n,m} = C_*\Theta^n(\Theta^m_{sym}(E))_i$$

with horizontal maps $A_{n,m} \to A_{n+1,m}$ induced by $\Theta^n \to \Theta^{n+1}$ and vertical maps $A_{n,m} \to A_{n,m+1}$ induced by $\Theta^m_{sym} \to \Theta^{m+1}_{sym}$. To prove the statement, we need to show that the maps $\mathrm{colim}_m A_{0,m} \to \mathrm{colim}_{n,m} A_{n,m}$ and $\mathrm{colim}_n A_{n,0} \to \mathrm{colim}_{n,m} A_{n,m}$ are local weak equivalences.

Without loss of generality it is sufficient to prove that for every m,n the maps

$$\mathrm{colim}_n A_{2n,2m} \to \mathrm{colim}_n A_{2n,2m+2} \tag{6.8}$$

and

$$\mathrm{colim}_m A_{2n,2m} \to \mathrm{colim}_m A_{2n+2,2m} \tag{6.9}$$

are local weak equivalences.

Note that the spaces $C_*\Theta^\infty(\Theta^m_{sym}(E))_i$ and $C_*\Theta^n(\Theta^\infty_{sym}(E))_i$ are isomorphic to $C_*\Theta^\infty(E)_i$ by Lemmas 6.5 and 6.4. Hence they are locally connected by Lemma 6.6.

To prove that (6.8) is a local weak equivalence, we apply Lemma 6.10 below for the case $A_n = A_{2n,2m}, B_n = A_{2n,2m+2}$ and the maps $i^A_n: A_{2n,2m} \to A_{2n+2,2m}$, $i^B_n: A_{2n,2m+2} \to A_{2n+2,2m+2}$, $f_n: A_{2n,2m} \to A_{2n,2m+2}$ are given by maps of the two dimensional sequences above. Define a map $g_n: A_{2n,2m+2} \to A_{2n+2,2m}$ as an identification via associativity isomorphism

$$A_{2n,2m+2} = C_*\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 2n}, \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 2m+2}, E_{2n+2m+2+i})) =$$
$$= C_*\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 2n+2}, \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 2m}, E_{2n+2m+2+i})) = A_{2n+2,2m}.$$

Then $g_n f_n$ differs from i^A_n by the action of an even permutation on $\mathbb{P}^{\wedge 2n+2m+2}$ and an even permutation on $E_{2n+2m+2+i}$. Thus $g_n f_n$ and i^A_n are simplicially homotopic by Corollary 2.13 and our assumption that E is a spectrum with contractible alternating group action as well as the fact that $\mathbb{A}^1$-homotopies become the usual ones after applying Suslins complex C_*. Also, $f_{n+1}g_n$ differs from i^B_n by the action of an even permutation on $\mathbb{P}^{\wedge 2n+2m+4}$ and an even permutation on $E_{2n+2m+4}$. Therefore $f_{n+1}g_n$ is simplicially homotopic to i^B_n for the same reasons as above. Thus the map on the colimits is a local weak equivalence by Lemma 6.10. The proof for the map (6.9) is analogous.

Finally, the map η of the commutative square of the proposition is a stable motivic equivalence by [16, 4.11], because the flasque motivic model structure on spaces is almost finitely generated in the sense of [16]. By the first part of the proof $\xi, C_*\Theta^\infty(\varepsilon)$ are stable motivic equivalences, and hence so is ε by the two-out-of-three property for weak equivalences. □

6.10. **Lemma.** *Suppose $i_n^A\colon A_n \to A_{n+1}$, $i_n^B\colon B_n \to B_{n+1}$ are directed systems of spaces, and $f_n\colon A_n \to B_n$ is a map of directed sequences. Suppose that there are maps $g_n\colon B_n \to A_{n+1}$ such that $g_n f_n$ is simplicially homotopic to i_n^A and $f_{n+1}g_n$ is simplicially homotopic to i_n^B. Also, suppose that the spaces $A = \operatorname{colim} A_n$ and $B = \operatorname{colim} B_n$ are locally connected. Then the map $f = \operatorname{colim} f_n\colon A \to B$ is a local weak equivalence.*

Proof. Given a local Henselian scheme U, the map $\pi_i(f)(U)\colon \pi_i(A(U)) \to \pi_i(B(U))$ equals the colimit of the system $\pi_i(f_n)(U)$. Note that the maps $\pi_i(g_n)(U)$ form a map of sequences $\pi_i(B_n(U)) \to \pi_i(A_{n+1}(U))$, which are inverse to $\pi_i(f_n)(U)$. Therefore the colimit $\pi_i(f)(U) = \operatorname{colim} \pi_i(f_n)(U)$ is bijective, and hence the map $f(U)$ induces a weak equivalence of connected simplicial sets $A(U) \to B(U)$. □

7. The spectrum $C_* \mathrm{Fr}^E(S_T)$

The purpose of this section is to introduce another spectrum $C_* \mathrm{Fr}^E(S_T)$ associated with a symmetric T-spectrum E. We show that it is stably equivalent to the spectrum $C_*\Theta^\infty_{sym}(E)$ whenever E is a Thom spectrum with the bounding constant d and contractible alternating group action (see Proposition 7.7).

7.1. **Definition.** Given a T-spectrum E and $X \in \mathbf{Sm}_k$, denote by $\mathrm{Fr}_n^E(X)$ the space $\Theta^n(X_+ \wedge E)_0$:

$$\mathrm{Fr}_n^E(X) = \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, X_+ \wedge E_n)$$

and $\mathrm{Fr}^E(X) := \operatorname{colim}_n \mathrm{Fr}_n^E(X) = \Theta^\infty(X_+ \wedge E)_0$.

7.2. **Definition.** For any symmetric T-spectrum E and any $m, n \geqslant 0$, define a pairing

$$\mathrm{Fr}_n(X,Y) \times \mathrm{Fr}_m^E(Y,Z) \to \mathrm{Fr}_{n+m}^E(X,Z)$$

as follows. Let $a \in \mathrm{Fr}_n(X,Y)$ be given by a map $a\colon X_+ \wedge \mathbb{P}^{\wedge n} \to Y_+ \wedge T^n$ and let $b \in \mathrm{Fr}_m^E(Y,Z)$ be given by $b\colon Y_+ \wedge \mathbb{P}^{\wedge m} \to Z_+ \wedge E_m$. Define $b \circ a$ as the composition

$$X_+ \wedge \mathbb{P}^{\wedge n} \wedge \mathbb{P}^{\wedge m} \xrightarrow{a \wedge \mathbb{P}^{\wedge m}} Y_+ \wedge T^n \wedge \mathbb{P}^{\wedge m} \xrightarrow{tw} Y_+ \wedge \mathbb{P}^{\wedge m} \wedge T^n \xrightarrow{b \wedge T^n} Z_+ \wedge E_m \wedge T^n \xrightarrow{tw}$$
$$\to Z_+ \wedge T^n \wedge E_m \xrightarrow{u_l} Z_+ \wedge E_{n+m},$$

where u_l is the map of Definition 2.6.

Note that if $E = S_T$, this definition coincides with the definition of the composition of framed correspondences defined in [32, 13].

7.3. **Lemma.** *The pairing above endows $\mathrm{Fr}^E(X)$ with a structure of a presheaf with framed correspondences.*

Proof. This is straightforward. □

7.4. **Definition.** Given a T-spectrum E and $X \in \mathbf{Sm}_k$, denote by $\mathrm{Fr}_n^E(X_+ \wedge S_T)$ the T-spectrum with the spaces

$$\mathrm{Fr}_n^E(X_+ \wedge S_T)_i := \mathrm{Fr}_n^E(X_+ \wedge T^i) = \mathrm{Fr}_n^{T^i \wedge E}(X).$$

The bonding maps $\mathrm{Fr}_n^E(X_+\wedge T^i)\wedge T\to \mathrm{Fr}_n^E(X_+\wedge T^{i+1})$ are defined as the composite maps

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},X_+\wedge T^i\wedge E_n)\wedge T\to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},X_+\wedge T^i\wedge E_n\wedge T)\xrightarrow{tw}\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},X_+\wedge T^i\wedge T\wedge E_n).$$

In what follows we normally regard $\mathrm{Fr}_n^E(X_+\wedge S_T)$ as a $\mathbb{P}^1$-spectrum. The stabilization maps $\mathrm{Fr}_n^E(X_+\wedge T^i)\to \mathrm{Fr}_{n+1}^E(X_+\wedge T^i)$, given by the compositions

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},X_+\wedge T^i\wedge E_n)\xrightarrow{-\wedge\sigma}\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}\wedge\mathbb{P}^{\wedge 1},X_+\wedge T^i\wedge E_n\wedge T)\xrightarrow{u_n}\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n+1},X_+\wedge T^i\wedge E_{n+1}),$$

define a map of $\mathbb{P}^1$-spectra $\mathrm{Fr}_n^E(X_+\wedge S_T)\to \mathrm{Fr}_{n+1}^E(X_+\wedge S_T)$. Denote by

$$\mathrm{Fr}^E(X_+\wedge S_T):=\mathrm{colim}_n\,\mathrm{Fr}_n^E(X_+\wedge S_T).$$

If $E=S_T$ the spectrum $\mathrm{Fr}^E(X_+\wedge S_T)$ coincides with the spectrum $\mathrm{Fr}_{\mathbb{P}^1,T}(X)$ defined in [13].

Note that for $X\in\mathbf{Sm}_k$ the spectrum $\mathrm{Fr}^E(X_+\wedge S_T)$ is isomorphic to the spectrum $\mathrm{Fr}^{X_+\wedge E}(S_T)$. If E is a symmetric Thom spectrum with the bounding constant d, then so is $X_+\wedge E$. Thus we shall consider spectra of the form $\mathrm{Fr}^E(S_T)$ in what follows.

For any $n\geqslant 0$ and any symmetric T-spectrum E, construct a map of $\mathbb{P}^1$-spectra $f_n\colon \mathrm{Fr}_n^E(S_T)\to \Theta^n_{sym}(E)$ as the composition at each level $i\geqslant 0$

$$f_{n,i}\colon \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},T^i\wedge E_n)\xrightarrow{tw}\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},E_n\wedge T^i)\xrightarrow{u}\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},E_{n+i}),$$

where the first map is induced by twist $T^i\wedge E_n\to E_n\wedge T^i$.

7.5. **Lemma.** *Each map f_n, $n\geqslant 0$, is a morphism of spectra commuting with stabilization maps $\mathrm{Fr}_n^E(S_T)\to \mathrm{Fr}_{n+1}^E(S_T)$ and $\Theta^n_{sym}(E)\to\Theta^{n+1}_{sym}(E)$. In particular, they induce a map of spectra*

$$f\colon\ \mathrm{Fr}^E(S_T)\to\Theta^\infty_{sym}(E).$$

Proof. The following diagram commutes:

$$\begin{array}{ccccc}
\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},T^i\wedge E_n)\wedge T & \xrightarrow{tw} & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},E_n\wedge T^i)\wedge T & \xrightarrow{u} & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},E_{n+i})\wedge T\\
\downarrow & & \downarrow & & \downarrow\\
\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},T^{i+1}\wedge E_n) & \xrightarrow{tw} & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},E_n\wedge T^{i+1}) & \xrightarrow{u} & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},E_{n+i+1}),
\end{array}$$

where the left vertical arrow is the ith bonding map of the spectrum $\mathrm{Fr}_n^E(S_T)$, and the right vertical map is the ith bonding map of $\Theta_n^{sym}(E)$. We see that each map f_n is a morphism of spectra. Consider a commutative diagram

$$\begin{array}{ccccc}
\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},T^i\wedge E_n) & \xrightarrow{tw} & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},E_n\wedge T^i) & \xrightarrow{u} & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},E_{n+i})\\
\downarrow & & \downarrow & & \downarrow\\
\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n+1},T^i\wedge E_{n+1}) & \xrightarrow{tw} & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n+1},E_{n+1}\wedge T^i) & \xrightarrow{u} & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n+1},E_{n+1+i}),
\end{array}$$

in which the left vertical map is the stabilization $\mathrm{Fr}_n^E(S_T)_i\to\mathrm{Fr}_{n+1}^E(S_T)_i$ from Definition 7.4, and the right vertical map is the stabilization map $\Theta^n_{sym}(E)_i\to\Theta^{n+1}_{sym}(E)_i$ (see Remark 6.3). The middle vertical arrow equals the composite map

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},E_n\wedge T^i)\xrightarrow{-\wedge\sigma}\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n+1},E_n\wedge T^{i+1})\xrightarrow{(\chi_{i,1})_*}\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n+1},E_n\wedge T^{1+i})\xrightarrow{u_n}\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n+1},E_{n+1}\wedge T^i),$$

where $(\chi_{i,1})_*$ is induced by the shuffle map $\chi_{i,1} : T^{i+1} \to T^{1+i}$. For commutativity of the right square we also use here the fact that the diagram

$$\begin{array}{ccccc} E_n \wedge T^{i+1} & \xrightarrow{u} & E_{n+i} \wedge T & \xrightarrow{u} & E_{n+i+1} \\ \downarrow{\scriptstyle id\wedge\chi_{i,1}} & & & & \downarrow{\scriptstyle 1\oplus\chi_{i,1}} \\ E_n \wedge T^{1+i} & \xrightarrow{u} & E_{n+1} \wedge T^i & \xrightarrow{u} & E_{n+1+i}, \end{array}$$

is commutative because the compositions of horizontal maps are $\Sigma_n \times \Sigma_{i+1}$-equivariant maps. Thus the maps $f_{n,i}$ are compatible with stabilization. □

7.6. **Corollary.** *If $E = X_+ \wedge S_T$ then the map f of Lemma 7.5 gives an isomorphism of spectra $\mathrm{Fr}_{\mathbb{P}^1,T}(X) = \mathrm{Fr}^{X_+\wedge E}(S_T) \xrightarrow{\cong} \Theta^\infty_{sym}(X_+ \wedge S_T)$.*

Proof. It suffices to note that the bonding maps of $X_+ \wedge S_T$ are isomorphisms. □

7.7. **Proposition.** *For a symmetric Thom T-spectrum E with the bounding constant d and contractible alternating group action, the map f induces a local weak equivalence for any $i \geqslant \max(0,d)$:*

$$f_i \colon C_* \mathrm{Fr}^E(S_T)_i \to C_* \Theta^\infty_{sym}(E)_i.$$

Proof. The map $f_{n,i} \colon \mathrm{Fr}^E_n(T^i) \to \Theta^n_{sym}(E)_i$ fits into the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Fr}^E_n(T^i) & & \xrightarrow{f_{n,i}} & & \Theta^n_{sym}(E)_i \\ \Vert & & & & \uparrow{\scriptstyle \chi_{i,n}} \\ \Theta^n(T^i \wedge E)_0 & \xrightarrow{u_l} & \Theta^n(E[i])_0 & = & \Theta^n(E)_i \end{array}$$

where $\chi_{i,n}$ is the map of Lemma 6.4 and u_l is the left bonding map from Definition 2.6. Note that the maps of the diagram are compatible with stabilization maps, and hence we can pass to the colimit over n. Thus our assertion follows from Lemma 7.9 below. □

7.8. **Lemma.** *The natural map of T-spectra $u \colon E \wedge T^i \to E[i]$ induces a levelwise isomorphism of spaces $\Theta^\infty(E \wedge T^i) \xrightarrow{\cong} \Theta^\infty(E[i])$ for any T-spectrum E.*

Proof. For any m the map $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, E_{n+m} \wedge T^i) \xrightarrow{u} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, E_{n+m+i})$ commutes with stabilization by n and induces an isomorphism on colimits $\Theta^\infty(E \wedge T^i)_m \to \Theta^\infty(E[i])_m$. □

7.9. **Lemma.** *For a symmetric Thom T-spectrum E with the bounding constant d and contractible alternating group action, the map of spectra $u_l \colon T^i \wedge E \to E[i]$ induces a local weak equivalence of spaces*

$$C_*\Theta^\infty(T^i \wedge E)_0 \to C_*\Theta^\infty(E[i])_0$$

for any $i \geqslant \max(0,d)$.

Proof. For $i \geqslant \max(0,d)$ the space $C_*\Theta^\infty(E)_i$ is locally connected by Lemma 6.6. We apply Lemma 7.10 below to spaces

$$A_n = C_*\Theta^{2n}(T^i \wedge E)_0 = C_*\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 2n}, T^i \wedge E_{2n}),$$

$$B_n = C_*\Theta^{2n}(E\wedge T^i)_0 = C_*\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 2n}, E_{2n}\wedge T^i),$$

where the maps i_n^A and i_n^B from that lemma are induced by stabilization maps $\Theta^{2n}\to\Theta^{2n+2}$ and $C = C_*\Theta^\infty(E[i])_0$. Define $f_n\colon A_n\to B_n$ to be the map induced by the twist $tw\colon T^i\wedge E_{2n}\to E_{2n}\wedge T^i$. Then the composition $f_{n+1}\circ i_n^A$ coincides with the composition

$$f_{n+1}\circ i_n^A\colon C_*\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 2n}, T^i\wedge E_{2n})\to C_*\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 2n+2}, E_{2n}\wedge T^2\wedge T^i)\to C_*\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 2n+2}, E_{2n+2}\wedge T^i).$$

It differs from the composition $i_n^B\circ f_n$ by the permutation $E_{2n}\wedge T^2\wedge T^i\xrightarrow{tw} E_{2n}\wedge T^i\wedge T^2 \overset{assoc}{=} E_{2n}\wedge T^2\wedge T^i$. Since it is an even permutation and E is a spectrum with contractible alternating group action, we have that $f_{n+1}\circ i_n^A$ and $i_n^B\circ f_n$ are simplicially homotopic. Similarly, in the triangle

$$\begin{array}{ccccc} C_*(\mathbb{P}^{\wedge 2n}, T^i\wedge E_{2n}) & \xrightarrow{u_l} & C_*(\mathbb{P}^{\wedge 2n}, E_{2n+i}) & \longrightarrow & C_*\Theta^\infty(E[i])_0 \\ \downarrow{\scriptstyle tw} & \nearrow{\scriptstyle u} & & & \\ C_*(\mathbb{P}^{\wedge 2n}, E_{2n}\wedge T^i) & & & & \end{array}$$

the composition $u\circ tw$ differs from u_l by the action of the shuffle permutation $\chi_{2n,i}$ on E_{2n+i}, which is an even permutation. Thus the triangle commutes up to simplicial homotopy, because E is a spectrum with contractible alternating group action. Then for any $i\geqslant d$ the space $C_*\Theta^\infty(E[i])_0$ is connected and our statement follows from Lemmas 7.10 and 7.8. □

7.10. **Lemma.** *Suppose $i_n^A\colon A_n\to A_{n+1}, i_n^B\colon B_n\to B_{n+1}$ are directed sequences of spaces, C is a locally connected space and there are maps of sequences $A_n\to C$ and $B_n\to C$. Suppose that for any n there is a local weak equivalence $f_n\colon A_n\to B_n$ such that the diagrams*

$$\begin{array}{ccc} A_n & \longrightarrow & A_{n+1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ B_n & \longrightarrow & B_{n+1} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A_n & \longrightarrow & C \\ \downarrow{\scriptstyle f_n} & \nearrow & \\ B_n & & \end{array}$$

commute up to a simplicial homotopy. Let $A = \operatorname{colim} A_n, B = \operatorname{colim} B_n$. Then the map $B\to C$ is a local weak equivalence if and only if so is the map $A\to C$.

Proof. Given local Henselian scheme U and $i\geqslant 0$, the maps $\pi_i(f_n(U))$ form a map of sequences $\pi_i(f_n(U))\colon \pi_i(A_n(U))\to\pi_i(B_n(U))$, and the map $\operatorname{colim}_n\pi_i(f_n(U))$ fits into the commutative diagram

$$\begin{array}{ccc} \pi_i(A(U)) & \longrightarrow & \pi_i(C(U)) \\ {\scriptstyle \operatorname{colim}\pi_i(f_n)}\downarrow & \nearrow & \\ \pi_i(B(U)) & & \end{array}$$

Every $\pi_i(f_n(U))$ is a bijection, and hence so is $\operatorname{colim}\pi_i(f_n(U))$. If the map $B\to C$ is a local weak equivalence, then $B(U)$ is connected and all maps $\pi_i(B(U))\to\pi_i(C(U))$ are bijective. Then $A(U)$ is connected, and all the maps $\pi_i(A(U))\to\pi_i(C(U))$ are bijective. Therefore the map $A\to C$ is a local weak equivalence. Similarly, if $A\to C$ is a local weak equivalence, then so is $B\to C$. □

8. Fibrant resolutions of symmetric Thom spectra

We have discussed three types of spectra associated with a symmetric Thom T-spectrum E each of which is obtained from E by a certain stabilization and taking the Suslin complex at each level: $C_*\operatorname{Fr}^E(S_T)$, $C_*\Theta^\infty(E)$ and $C_*\Theta^\infty_{sym}(E)$. Moreover, by Propositions 6.7 and 7.7 they are isomorphic to each other in $SH(k)$ under certain reasonable assumptions on E. The next theorem says that if we take local fibrant replacements at each level in these spectra, they become motivically fibrant starting from some level d onwards. More precisely, the following result is true:

8.1. **Theorem.** *For a symmetric Thom T-spectrum E with the bounding constant d and contractible alternating group action the following $\mathbb{P}^1$-spectra are isomorphic to E in $SH(k)$ and motivically fibrant starting from level* $\max(0,d)$*:*

- $C_*\operatorname{Fr}^E(S_T)^f$
- $C_*\Theta^\infty(E)^f$
- $C_*\Theta^\infty_{sym}(E)^f$,

where "f" refers to levelwise local fibrant replacements of the corresponding spectra.

Proof. By Propositions 6.7 and 7.7 we have the following levelwise local weak equivalences of $\mathbb{P}^1$-spectra starting from level d:

$$C_*\operatorname{Fr}^E(S_T)\to C_*\Theta^\infty_{sym}(E)\to C_*\Theta^\infty(\Theta^\infty_{sym}(E))\leftarrow C_*\Theta^\infty(E).$$

Since the canonical map $E\to C_*\Theta^\infty(E)$ is a stable motivic equivalence by Theorem 5.4, we see that E is isomorphic in $SH(k)$ to each of the spectrum of the theorem.

Sublemma. *Suppose a map of $\mathbb{P}^1$-spectra $f\colon E\to E'$ is a levelwise local weak equivalence and all spaces E_i,E_i', $i\geqslant 0$, are fibrant in the flasque local model structure. Then E is motivically fibrant if and only if so is E'.*

Proof. Since each map $f_i\colon E_i\to E_i'$, $i\geqslant 0$, is a local weak equivalence between locally fibrant spaces, it is a sectionwise weak equivalence. Therefore if E_i is motivically fibrant, then E_i' is $\mathbb{A}^1$-invariant, and hence motivically fibrant as well. Since E_i,E_i' are flasque fibrant by assumption, then the map $\underline{\operatorname{Hom}}(\mathbb{P}^{\wedge 1},E_{i+1})\to\underline{\operatorname{Hom}}(\mathbb{P}^{\wedge 1},E_{i+1}')$ is a sectionwise weak equivalence, because $\mathbb{P}^{\wedge 1}$ is flasque cofibrant. Hence the adjoint to the bonding map $E_i\to\underline{\operatorname{Hom}}(\mathbb{P}^{\wedge 1},E_{i+1})$ is a sectionwise weak equivalence if and only if so is the map $E_i'\to\underline{\operatorname{Hom}}(\mathbb{P}^{\wedge 1},E_{i+1}')$. □

Since the spectrum $C_*\Theta^\infty(E)^f$ is motivically fibrant starting from level d by Theorem 5.4, then so are $C_*\Theta^\infty_{sym}(E)^f$ and $C_*\Theta^\infty(\Theta^\infty_{sym}(E))^f$ by Proposition 6.7 and the sublemma above. Likewise, $C_*\operatorname{Fr}^E(S_T)^f$ is motivically fibrant starting from level d by Proposition 7.7 and the sublemma above. This completes the proof of the theorem. □

The spectrum $C_*\operatorname{Fr}^E(S_T)^f$, which is isomorphic to E in $SH(k)$ by Theorem 8.1, is of particular interest, because it will lead to an equivalent model of E in the category of $(S^1,\mathbb{G}_m^{\wedge 1})$-bispectra (see Theorem 9.13).

9. E-FRAMED MOTIVES AND BISPECTRA

Following Definition 7.1, for any space $\mathscr{X}$ and any T-spectrum E denote by $\mathrm{Fr}^E_n(\mathscr{X}) = \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, \mathscr{X} \wedge E_n)$, $n \geqslant 0$. Also, set $\mathrm{Fr}^E(\mathscr{X}) = \mathrm{colim}_n(\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, \mathscr{X} \wedge E_n)) = \Theta^\infty(\mathscr{X} \wedge E)_0$. If $\mathscr{X} = X_+$, $X \in \mathbf{Sm}_k$, then we shall write $\mathrm{Fr}^E(X)$ dropping $+$ from notation.

9.1. **Lemma.** *$\mathrm{Fr}^E(\mathscr{X})$ is functorial in $\mathscr{X}$ and E. If E is a directed colimit of T-spectra $\mathrm{colim}_k E_k$, then $\mathrm{Fr}^E(\mathscr{X}) = \mathrm{colim}_k \mathrm{Fr}^{E_k}(\mathscr{X})$. In particular, $\mathrm{Fr}^E(\mathscr{X}) = \mathrm{colim}_k \mathrm{Fr}^{L_k E}(\mathscr{X})$, where $L_k E$ is the k-th layer of E.*

9.2. **Definition.** Given a T-spectrum E, the assignment $K \mapsto C_* \mathrm{Fr}^E(\mathscr{X} \wedge K)$ is plainly a Γ-space. The *E-framed motive $M_E(\mathscr{X})$ of $\mathscr{X}$* is the Segal symmetric S^1-spectrum $C_* \mathrm{Fr}^E(\mathscr{X} \wedge \mathbb{S})$. If $E = S_T$ then $M_E(\mathscr{X})$ is the framed motive $M_{fr}(\mathscr{X})$ of $\mathscr{X}$ in the sense of [13].

Lemma 9.1 implies the following

9.3. **Corollary.** *$M_E(\mathscr{X})$ is functorial in $\mathscr{X}$ and E. If E is a directed colimit of T-spectra $\mathrm{colim}_k E_k$, then $M_E(\mathscr{X}) = \mathrm{colim}_k M_{E_k}(\mathscr{X})$. In particular, $M_E(\mathscr{X}) = \mathrm{colim}_k M_{L_k E}(\mathscr{X})$, where $L_k E$ is the k-th layer of E.*

The next statement is straightforward.

9.4. **Lemma.** *$M_{L_k E}(\mathscr{X}) = \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge k}, M_{fr}(\mathscr{X} \wedge E_k))$ for any $k \geqslant 0$.*

9.5. **Definition.** Given a Thom T-spectrum E, $U \in \mathbf{Sm}_k$ and $Y = X/(X - Z)$, where $X \in \mathbf{Sm}_k$ and Z is a closed subset in X, denote by $\mathbb{Z}F^E_n(U,Y)$ the free Abelian group generated by the elements of $\mathrm{Fr}^E_n(U,Y) = \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, Y \wedge E_n)$ with connected support (recall that the elements of $\mathrm{Fr}^E_n(U,Y)$ have an explicit geometric description using Voevodsky's Lemma 2.12). We also set $\mathbb{Z}F^E(U,Y) := \mathrm{colim}_n \mathbb{Z}F^E_n(U,Y)$, where the colimit maps are defined in the same fashion with those of $\mathrm{Fr}^E(U,Y)$.

The assignment $K \mapsto C_* \mathbb{Z}F^E(U, Y \wedge K)$ is plainly a Γ-space. The *linear E-framed motive $LM_E(Y)$ of Y* is the Segal symmetric S^1-spectrum $C_* \mathbb{Z}F^E(Y \wedge \mathbb{S})$. If $E = S_T$ then $LM_E(Y)$ is the linear framed motive $LM_{fr}(Y)$ of Y in the sense of [13]. Note that the presheaves of stable homotopy groups $\pi_*(LM_E(Y))$ are computed as the presheaves of homology groups of the complex $C_* \mathbb{Z}F^E(Y)$ (we freely use the Dold–Kan correspondence here).

As above we have the following

9.6. **Lemma.** *$LM_E(Y) = \mathrm{colim}_k LM_{L_k E}(Y)$ and $LM_{L_k E}(Y) = LM^{\mathbb{P}^{\wedge k}}_{fr}(Y \wedge E_k)$, where $LM^{\mathbb{P}^{\wedge k}}_{fr}(Y \wedge E_k)$ is Segal's spectrum associated with the Γ-space $K \mapsto C_* \mathbb{Z}F^{\mathbb{P}^{\wedge k}}(U, Y \wedge K)$ (see Definition 4.7).*

The following lemma says that $LM_E(Y)$ computes homology of the E-framed motive of Y.

9.7. **Lemma.** *Given a Thom T-spectrum E and Y as above, there is an isomorphism of graded presheaves $\pi_*(\mathbb{Z}M_E(Y)) = \pi_*(LM_E(Y))$.*

Proof. We have that $\pi_*(\mathbb{Z}M_E(Y)) = \mathrm{colim}_k \pi_*(\mathbb{Z}M_{L_k E}(Y)) = \mathrm{colim}_k \pi_*(\mathbb{Z}(\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge k}, M_{fr}(Y \wedge E_k)))) = \mathrm{colim}_k \pi_*(LM^{\mathbb{P}^{\wedge k}}_{fr}(Y \wedge E_k)) = \pi_*(LM_E(Y))$. We have used here Corollary 9.3, Lemmas 4.8, 9.4 and 9.6. □

Following notation of [10], denote by $\mathbb{A}^1//\mathbb{G}_m$ the mapping cone of the natural embedding $(\mathbb{G}_m)_+ \hookrightarrow \mathbb{A}^1_+$. It is represented by a simplicial scheme from $\mathrm{Fr}_0(k)$.

9.8. **Lemma.** *For a Thom T-spectrum E with the bounding constant d, the natural map*

$$M_E(T^\ell \wedge (\mathbb{A}^1//\mathbb{G}_m)^{\wedge i}) \to M_E(T^\ell \wedge T^i), \quad \ell := \max(0, d-1),$$

is a local stable weak equivalence for any $i > 0$.

Proof. By Corollary 9.3 and Lemma 9.4 $M_E(T^\ell \wedge (\mathbb{A}^1//\mathbb{G}_m)^{\wedge i}) = \operatorname{colim}_n \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge T^\ell \wedge (\mathbb{A}^1//\mathbb{G}_m)^{\wedge i}))$ and $M_E(T^\ell \wedge T^i) = \operatorname{colim}_n \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge T^\ell \wedge T^i))$. Since a directed colimit of stable local weak equivalences is a stable local weak equivalence, it is sufficient to check that the natural map

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge T^\ell \wedge (\mathbb{A}^1//\mathbb{G}_m)^{\wedge i})) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge T^\ell \wedge T^i))$$

is a local stable weak equivalence of spectra. By definition of the bounding constant d, the space $E_n \wedge T^\ell$ is a colimit of spaces of the form $X/X-Z$ where Z has codimension greater than or equal to n. Consider a commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge T^\ell \wedge (\mathbb{A}^1//\mathbb{G}_m)^{\wedge i})) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge T^\ell \wedge T^i)) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge T^\ell \wedge (\mathbb{A}^1//\mathbb{G}_m)^{\wedge i})_f) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge T^\ell \wedge T^i)_f), \end{array}$$

where "f" refers to a level local fibrant replacement. Then by Proposition 4.18 the vertical arrows are local stable weak equivalences, and the bottom arrow is a stable weak equivalence between motivically fibrant S^1-spectra by [10, 1.1; A.1]. By the two-out-of-three-property the upper arrow is a local stable weak equivalence. $\square$

9.9. **Proposition.** *Let E be a Thom T-spectrum with the bounding constant d and $\ell = \max(0, d-1)$. Then for every $i > 0$ the S^1-spectra $M_E(T^\ell \wedge (\mathbb{A}^1//\mathbb{G}_m)^{\wedge i})_f$, $M_E(T^\ell \wedge T^i)_f$ and $M_E(T^\ell \wedge (S^1 \wedge \mathbb{G}_m^{\wedge 1})^{\wedge i})_f$, where "$f$" refers to a level local fibrant replacement, are motivically fibrant.*

Proof. Without loss of generality we assume $d \leqslant 1$. Indeed, if $d > 1$ we replace E by $T^{d-1} \wedge E$, which has the bounding constant 1, observing that $M_E(T^{d-1} \wedge -) \cong M_{T^{d-1} \wedge E}(-)$. It suffices to prove the statement for the spectrum $M_E((\mathbb{A}^1//\mathbb{G}_m)^{\wedge i})_f$, because our arguments will be the same for the other two spectra.

By Corollary 9.3 and Lemma 9.4 one has $M_E((\mathbb{A}^1//\mathbb{G}_m)^{\wedge i}) = \operatorname{colim}_n \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge (\mathbb{A}^1//\mathbb{G}_m)^{\wedge i}))$. Since flasque motivically fibrant spectra are closed under filtered colimits, it is enough to show that each spectrum $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge (\mathbb{A}^1//\mathbb{G}_m)^{\wedge i}))_f$ is motivically fibrant. But this can shown similarly to Lemma 4.15 if we note that the space $C_* \mathrm{Fr}(X_+ \wedge (\mathbb{A}^1//\mathbb{G}_m)^{\wedge i})$, $X \in \mathbf{Sm}_k$, is locally connected by [10, A.1] and if we apply the proof of Lemma 4.14 to show that the space $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, C_* \mathrm{Fr}(E_n \wedge (\mathbb{A}^1//\mathbb{G}_m)^{\wedge i}))$ is locally connected. $\square$

9.10. **Lemma.** *Under the assumptions of Proposition 9.9 the map $M_E(T^\ell \wedge (\mathbb{A}^1//\mathbb{G}_m)^{\wedge i})_f \to M_E(T^\ell \wedge (S^1 \wedge \mathbb{G}_m^{\wedge 1})^{\wedge i})_f$ is a sectionwise level equivalence for $\ell = \max(0, d-1)$.*

Proof. The proof of Proposition 9.9 shows that it suffices to prove the assertion for $d \leqslant 1$ and that the map of $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge (\mathbb{A}^1//\mathbb{G}_m)^{\wedge i})) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge (S^1 \wedge \mathbb{G}_m^{\wedge 1})^{\wedge i})$ is a local stable weak equivalence for any $n \geqslant 0$. Since a map of locally connected spectra is a stable local equivalence if and only if so is the map on homology, then using Lemma 4.8 our assertion reduces to showing that the map of complexes $C_*\mathbb{Z}F^{\mathbb{P}^{\wedge n}}(E_n \wedge (\mathbb{A}^1//\mathbb{G}_m)^{\wedge i}) \to C_*\mathbb{Z}F^{\mathbb{P}^{\wedge n}}(E_n \wedge (S^1 \wedge \mathbb{G}_m^{\wedge 1})^{\wedge i})$ is locally a quasi-isomorphism. The latter fact is proved similar to [13, 8.2]. $\square$

We shall need the following fact which was proven in [13, Section 12].

9.11. **Lemma.** *Let $\mathscr{X}$ be an $\mathbb{A}^1$-local motivic S^1-spectrum whose presheaves of stable homotopy groups are homotopy invariant quasi-stable radditive presheaves with framed correspondences. Suppose $\mathscr{X}^f$ is a local stable fibrant replacement of $\mathscr{X}$. Then the map of spectra $\underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, \mathscr{X}) \to \underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, \mathscr{X}^f)$ is a local stable equivalence.*

9.12. **Proposition.** *The following statements are true:*

(1) *Suppose $Z \to X$ is a closed embedding of smooth varieties of codimension d. Then the natural map*

$$\alpha : \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X/X-Z)) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i} \wedge \mathbb{G}_m^{\wedge 1}, M_{fr}((X/X-Z) \wedge \mathbb{G}_m^{\wedge 1}))$$

is a stable local weak equivalence of S^1-spectra for all $i \leqslant d$.

(2) *If E is a Thom T-spectrum with the bounding constant $d \leqslant 1$, then the natural map*

$$\beta : M_E(X) \to \underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, M_E(X_+ \wedge \mathbb{G}_m^{\wedge 1})), \quad X \in \mathbf{Sm}_k,$$

is a stable local weak equivalence of S^1-spectra.

(3) *If E is a Thom T-spectrum with the bounding constant $d \leqslant 1$ and $M_E(X)_f$ is a stable local fibrant replacement of $M_E(X)$, $X \in \mathbf{Sm}_k$, then $M_E(X)_f$ is a motivically fibrant S^1-spectrum.*

Proof. (1). First suppose $i = 0$. Without loss of generality we assume that $X/X-Z = Z_+ \wedge T^d$, because we can apply the Mayer–Vietoris sequence of Proposition 4.11 to reduce the general case to this particular one. Indeed, $M_{fr}(X/X-Z)$ and $M_{fr}((X/X-Z) \wedge \mathbb{G}_m^{\wedge 1})$ are homotopy pushouts (=homotopy pullbacks) of framed motives of the form $M_{fr}(Z_+ \wedge T^d)$ and $M_{fr}(Z_+ \wedge T^d \wedge \mathbb{G}_m^{\wedge 1})$ respectively. Using Lemma 9.11 the functor $\underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, -)$ respects homotopy pullbacks (=pushouts) for framed motives in question. Therefore our assertion reduces to showing that the natural map of S^1-spectra

$$M_{fr}(Z_+ \wedge T^d) \to \underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, M_{fr}(Z_+ \wedge T^d \wedge \mathbb{G}_m^{\wedge 1}))$$

is a stable local weak equivalence. It fits into a commutative diagram

$$\begin{array}{ccc} M_{fr}(Z_+ \wedge T^d) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, M_{fr}(Z_+ \wedge T^d \wedge \mathbb{G}_m^{\wedge 1})) \\ \downarrow & & \downarrow \\ M_{fr}(Z_+ \wedge (\mathbb{A}^1//\mathbb{G}_m)^{\wedge d}) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, M_{fr}(Z_+ \wedge (\mathbb{A}^1//\mathbb{G}_m)^{\wedge d} \wedge \mathbb{G}_m^{\wedge 1})) \end{array}$$

in which the lower arrow is a stable local weak equivalence by the Cancellation Theorem for framed motives of [3]. The vertical arrows are stable local weak equivalences by [10, 1.1] and Lemma 9.11. We see that the upper arrow is a stable local weak equivalence, as required.

Suppose now $i \leqslant d$. Consider a commutative diagram of S^1-spectra

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X/X-Z)) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i} \wedge \mathbb{G}_m^{\wedge 1}, M_{fr}((X/X-Z) \wedge \mathbb{G}_m^{\wedge 1})) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i}, M_{fr}(X/X-Z)_f) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge i} \wedge \mathbb{G}_m^{\wedge 1}, M_{fr}((X/X-Z) \wedge \mathbb{G}_m^{\wedge 1})_f) \end{array}$$

where "f" refers to a level local fibrant replacement. By Proposition 4.18 the left vertical map is a levelwise local weak equivalence in positive degrees, and hence a stable local weak equivalence. Since $M_{fr}(X/X-Z)_f, M_{fr}((X/X-Z) \wedge \mathbb{G}_m^{\wedge 1})_f$ are motivically fibrant spectra in positive degrees, the lower arrow is a sectionwise weak equivalence in positive degrees by the first assertion and Proposition 4.18, and hence a sectionwise stable weak equivalence. The right vertical map is a levelwise local weak equivalence in positive degrees by Lemma 9.11 and Proposition 4.18, hence it is a stable local weak equivalence. We see that the upper arrow is a stable local weak equivalence.

(2). If E is a Thom T-spectrum with the bounding constant $d \leqslant 1$, then the natural map

$$\beta : M_E(X) \to \underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, M_E(X_+ \wedge \mathbb{G}_m^{\wedge 1}))$$

is isomorphic to the sequential colimit of maps

$$\beta_k : M_{L_k E}(X) \to \underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, M_{L_k E}(X_+ \wedge \mathbb{G}_m^{\wedge 1})).$$

Every such map is isomorphic to

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge k}, M_{fr}(X_+ \wedge E_k)) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge k} \wedge \mathbb{G}_m^{\wedge 1}, M_{fr}(X_+ \wedge E_k \wedge \mathbb{G}_m^{\wedge 1})).$$

It follows from assertion (1) that each β_k is a stable local weak equivalence of S^1-spectra, and hence so is β as a sequential colimit of stable local weak equivalences of S^1-spectra.

(3). We can compute a stable local fibrant replacement $M_E(X)_f$ of $M_E(X)$ as the spectrum $\mathrm{colim}_k M_{L_k E}(X)_f = \mathrm{colim}_k \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge k}, M_{fr}(X_+ \wedge E_k))_f$ (we use Lemma 9.4 here), because a sequential colimit of fibrant spectra is fibrant in the flasque local stable model model structure of S^1-spectra. So it suffices to show that each spectrum $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge k}, M_{fr}(X_+ \wedge E_k))_f$ is $\mathbb{A}^1$-invariant. By the Mayer–Vietoris sequence of Proposition 4.11 this reduces to showing that every spectrum of the form $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge k}, M_{fr}(X_+ \wedge T^k))_f$ is $\mathbb{A}^1$-invariant. But the latter is obvious because $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge k}, M_{fr}(X_+ \wedge T^k))_f \cong M_{fr}(X)_f$ and $M_{fr}(X)_f$ is $\mathbb{A}^1$-invariant by [13, 7.1]. □

In what follows by bispectra we shall mean $(S^1, \mathbb{G}_m^{\wedge 1})$-bispectra in the category of motivic spaces. We are now in a position to prove the main result of the section. It gives an explicit fibrant resolution of a symmetric Thom spectrum in the category of bispectra.

9.13. **Theorem.** *Suppose $X \in \mathbf{Sm}_k$ and E is a symmetric Thom T-spectrum with the bounding constant d and contractible alternating group action.*

(1) *If $d = 1$ then the $(S^1, \mathbb{G}_m^{\wedge 1})$-bispectrum*

$$M_E^{\mathbb{G}}(X)_f := (M_E(X)_f, M_E(X_+ \wedge \mathbb{G}_m^{\wedge 1})_f, M_E(X_+ \wedge \mathbb{G}_m^{\wedge 2})_f, \ldots)$$

is motivically fibrant and represents the T-spectrum $X_+ \wedge E$ in the category of bispectra, where "f" refers to stable local fibrant replacements of S^1-spectra.

(2) *If $d<1$ then the $(S^1,\mathbb{G}_m^{\wedge 1})$-bispectrum*

$$M_E^{\mathbb{G}}(X)_f := (M_E(X)_f, M_E(X_+\wedge\mathbb{G}_m^{\wedge 1})_f, M_E(X_+\wedge\mathbb{G}_m^{\wedge 2})_f,\dots)$$

is motivically fibrant and represents the T-spectrum $X_+\wedge E$ in the category of bispectra, where "f" refers to level local fibrant replacements of S^1-spectra.

(3) *If $d>1$ then the $(S^1,\mathbb{G}_m^{\wedge 1})$-bispectrum*

$$\Omega^{d-1}_{S^1\wedge\mathbb{G}_m^{\wedge 1}}((M_{E[d-1]}(X)_f, M_{E[d-1]}(X_+\wedge\mathbb{G}_m^{\wedge 1})_f, M_{E[d-1]}(X_+\wedge\mathbb{G}_m^{\wedge 2})_f,\dots))$$

is motivically fibrant and represents the T-spectrum $X_+\wedge E$ in the category of bispectra, where "f" refers to stable local fibrant replacements of S^1-spectra. Here $E[d-1]$ stands for the $(d-1)$-th shift of E in the sense of Definition 2.6. Another equivalent model for the T-spectrum $X_+\wedge E$ in the category of bispectra is given by

$$\Omega^{d-1}_{S^1\wedge\mathbb{G}_m^{\wedge 1}}((M_{T^{d-1}\wedge E}(X)_f, M_{T^{d-1}\wedge E}(X_+\wedge\mathbb{G}_m^{\wedge 1})_f, M_{T^{d-1}\wedge E}(X_+\wedge\mathbb{G}_m^{\wedge 2})_f,\dots)).$$

This bispectrum is motivically fibrant and "f" refers to stable local fibrant replacements of S^1-spectra.

Proof. (1). The fact that the bispectrum $M_E^{\mathbb{G}}(X)_f$ is motivically fibrant follows from Proposition 9.12 and Lemma 9.11. It remains to show that it represent E in the category of bispectra.

Without loss of generality we may assume $X=pt$, because we can replace E with $X_+\wedge E$. It follows from Theorem 8.1 that E is isomorphic in $SH_T(k)$ to $C_*\operatorname{Fr}^E(S_T)^f$. The latter is a T-spectrum (see Definition 7.4). It is positively fibrant by Theorem 8.1. The motivic equivalence $\widetilde{T}:=\mathbb{A}^1//\mathbb{G}_m\to T$ induces an equivalence of categories $SH_T(k)\xrightarrow{\simeq}SH_{\widetilde{T}}(k)$. It takes E to a $\widetilde{T}$-spectrum isomorphic to $C_*\operatorname{Fr}^E(S_T)^f$. By Lemma 9.8 and Proposition 9.9 the natural map $C_*\operatorname{Fr}^E(S_{\widetilde{T}})^f\to C_*\operatorname{Fr}^E(S_T)^f$ is a sectionwise weak equivalence in positive degrees, where $S_{\widetilde{T}}=(S^0,\widetilde{T},\widetilde{T}^{\wedge 2},\dots)$. By the sublemma on p. 26 $C_*\operatorname{Fr}^E(S_{\widetilde{T}})^f$ is a motivically fibrant $\widetilde{T}$-spectrum in positive degrees (notice that each space $C_*\operatorname{Fr}^E(S_{\widetilde{T}})^f_{i>0}$ is motivically fibrant by Proposition 9.9). We see that $C_*\operatorname{Fr}^E(S_{\widetilde{T}})^f$ is a positively fibrant $\widetilde{T}$-spectrum representing E in $SH_{\widetilde{T}}(k)$. Consider now the canonical motivic weak equivalence $\widetilde{T}\to S^1\wedge\mathbb{G}_m^{\wedge 1}$. For the same reasons $C_*\operatorname{Fr}^E(S_{S^1\wedge\mathbb{G}_m^{\wedge 1}})^f$ is a positively fibrant $\widetilde{T}$-spectrum which is sectionwise weakly equivalent to $C_*\operatorname{Fr}^E(S_{\widetilde{T}})^f$. We use here Lemma 9.10 as well. It follows that $C_*\operatorname{Fr}^E(S_{S^1\wedge\mathbb{G}_m^{\wedge 1}})^f$ is a positively fibrant $S^1\wedge\mathbb{G}_m^{\wedge 1}$-spectrum representing E in $SH_{S^1\wedge\mathbb{G}_m^{\wedge 1}}(k)$. It remains to observe that this spectrum is equivalent to the diagonal spectrum for the bispectrum

$$M_E^{\mathbb{G}}(X) := (M_E(X), M_E(X_+\wedge\mathbb{G}_m^{\wedge 1}), M_E(X_+\wedge\mathbb{G}_m^{\wedge 2}),\dots).$$

(2). This immediately follows from (1) if we observe that a levelwise local fibrant replacement of each weighted E-framed motive $M_E(X_+\wedge\mathbb{G}_m^{\wedge n})_f$, $n\geqslant 0$, is a motivically fibrant S^1-spectrum. To see the latter, we repeat the proof of Proposition 9.12(3) and apply Lemma 4.15.

(3). This follows from (1) if we observe that $E[d-1]$ and $T^{d-1}\wedge E$ are symmetric Thom T-spectra with the bounding constant $d=1$. □

We finish the section by proving the following useful result.

9.14. **Theorem.** *Suppose E is a symmetric Thom T-spectrum with the bounding constant $d \leqslant 1$ and contractible alternating group action.*

(1) *For every elementary Nisnevich square*

$$\begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

the square of S^1-spectra

$$\begin{array}{ccc} M_E(U') & \longrightarrow & M_E(X') \\ \downarrow & & \downarrow \\ M_E(U) & \longrightarrow & M_E(X) \end{array}$$

is homotopy cartesian locally in the Nisnevich topology.

(2) *The natural map $M_E(X \times \mathbb{A}^1) \to M_E(X)$ is a stable local weak equivalence of S^1-spectra. The same is also true for linear E-framed motives.*

Proof. (1). The square of motivic T-spectra

$$\begin{array}{ccc} U'_+ \wedge E & \longrightarrow & X'_+ \wedge E \\ \downarrow & & \downarrow \\ U_+ \wedge E & \longrightarrow & X_+ \wedge E \end{array}$$

is homotopy cartesian in the stable motivic model structure. By Theorem 9.13 it induces a homotopy cartesian square of motivically fibrant bispectra

$$\begin{array}{ccc} M_E^{\mathbb{G}}(U')_f & \longrightarrow & M_E^{\mathbb{G}}(X')_f \\ \downarrow & & \downarrow \\ M_E^{\mathbb{G}}(U)_f & \longrightarrow & M_E^{\mathbb{G}}(X)_f \end{array} .$$

Here 'f' refers to local replacements in each weight (see Theorem 9.13). Passing to weight zero motivic S^1-spectra, one gets a homotopy cartesian square of motivically fibrant S^1-spectra

$$\begin{array}{ccc} M_E(U')_f & \longrightarrow & M_E(X')_f \\ \downarrow & & \downarrow \\ M_E(U)_f & \longrightarrow & M_E(X)_f \end{array}$$

Our statement now follows.

(2). It is proven similarly to (1) if we start with the stable motivic equivalence of T-spectra $(X \times \mathbb{A}^1)_+ \wedge E \to X_+ \wedge E$.

The same statements for linear E-framed motives follow from (1), (2) and Lemma 9.7. $\square$

10. TOPOLOGICAL THOM SPECTRA WITH FINITE COEFFICIENTS

In this section we give a topological application of Theorem 9.13. Namely, many important topological Thom spectra like MU can be obtained as the the realization of their motivic counterparts if the base field is $\mathbb{C}$. We shall prove below that the stable homotopy groups of such topological Thom spectra with finite coefficients can be computed by means of the stable homotopy groups with finite coefficients of weight zero of the associated E-framed motive, which is an *explicit* positively fibrant S^1-spectrum by the very construction. We first need a couple of useful lemmas.

By $f_0(SH(k))$ we shall mean the full triangulated subcategory of effective T-spectra, i.e., the subcategory which is compactly generated by the suspension T-spectra of the smooth algebraic varieties. We shall also write $f_\ell(SH(k))$, $\ell \in \mathbb{Z}$, to denote $f_0(SH(k)) \wedge T^\ell$.

10.1. **Lemma.** *Suppose $Z \to X$ is a closed embedding of smooth varieties of codimension ℓ. Then the suspension T-spectrum $\Sigma_T^\infty(X/X-Z)$ of the sheaf $X/X-Z$ belongs to $f_\ell(SH(k))$.*

Proof. If $X/X-Z = Z_+ \wedge T^\ell$ then our assertion is trivial. By using induction, we can cover X by open subsets X_1, X_2 such that $Z_2 = Z \cap X_2 \to X_2$ is a trivial embedding and X_1 is covered by $n-1$ open trivial pieces. Then for $X_{12} = X_1 \cap X_2$ the embedding $Z_{12} := Z \cap X_{12} \to X_{12}$ is trivial. Denote by $Y := X/X-Z$ and by $Y_i := X_i/X_i - Z_i$. Then $\Sigma_T^\infty Y_{12}, \Sigma_T^\infty Y_2 \in f_\ell(SH(k))$ and $\Sigma_T^\infty Y_1 \in f_\ell(SH(k))$ by induction hypothesis. By Lemma 4.6 Y is a pushout of sheaves embeddings $Y_1 \hookleftarrow Y_{12} \hookrightarrow Y_2$. Therefore we have a triangle in $SH(k)$

$$\Sigma_T^\infty Y_{12} \to \Sigma_T^\infty Y_1 \oplus \Sigma_T^\infty Y_2 \to \Sigma_T^\infty Y \xrightarrow{+}$$

in which the left two entries belong to $f_\ell(SH(k))$. It follows that $\Sigma_T^\infty Y \in f_\ell(SH(k))$. $\square$

10.2. **Lemma.** *Let E be a Thom T-spectrum with the bounding constant d. Then E belongs to $f_{1-d}(SH(k))$. In particular, E is an effective T-spectrum if $d \leqslant 1$.*

Proof. Since $f_d(SH(k)) = f_0(SH(k)) \wedge T^d$ for any integer d, we may assume $d = 1$ and show that E is an effective T-spectrum in this case.

We have $E = \operatorname{colim}_k L_k E$, where each layer has stable homotopy type of $\Omega_T^k((\Sigma_T^\infty E_k)^f)$. Here $(\Sigma_T^\infty E_k)^f$ stands for a stable motivic fibrant replacement of $\Sigma_T^\infty E_k$. Then E is isomorphic in $SH(k)$ to $\operatorname{hocolim}_k \Omega_T^k((\Sigma_T^\infty E_k)^f)$.

Each $E_k = \operatorname{colim}_j(V_{k,j}/V_{k,j} - Z_{k,j})$, where codimension of $Z_{k,j}$ in $V_{k,j}$ is larger than or equal to k. By Lemma 10.1 the flasque cofibrant T-spectrum $\Sigma_T^\infty(V_{k,j}/V_{k,j} - Z_{k,j})$ is in $f_k(SH(k))$. Since $\Sigma_T^\infty E_k$ is isomorphic in $SH(k)$ to $\operatorname{hocolim}_j(\Sigma_T^\infty(V_{k,j}/V_{k,j} - Z_{k,j}))$ and $f_k(SH(k))$ is closed under homotopy colimits, it follows that $\Sigma_T^\infty E_k \in f_k(SH(k))$, and hence $\Omega_T^k((\Sigma_T^\infty E_k)^f) \in f_0(SH(k))$. The isomorphism $E \cong \operatorname{hocolim}_k \Omega_T^k((\Sigma_T^\infty E_k)^f)$ in $SH(k)$ now implies $E \in f_0(SH(k))$. $\square$

Suppose that the base field k has an embedding $\varepsilon : k \hookrightarrow \mathbb{C}$. Following Panin–Pimenov–Röndigs [23, §A4] there is a natural realization functor

$$Re^\varepsilon : SH(k) \to SH,$$

where SH is the homotopy category of the stable model category of classical S^2-spectra of topological spaces (it is canonically equivalent to the homotopy category of the stable model category of classical S^1-spectra as well). Re^ε is an extension of the functor

$$An : \mathbf{Sm}_k \to \mathbf{Top}$$

sending a k-smooth variety X to $X^{an} := X(\mathbb{C})$ with the classical topology.

Following Levine's indexing [19], denote by $\pi^{\mathbb{A}^1}_{a,b}(E)$, where $E \in SH(k)$, the Nisnevich sheaf on $\mathbf{Sm}_k$ associated to the presheaf $U \mapsto \mathrm{Hom}_{SH(k)}(\Sigma^a_{S^1}\Sigma^b_{\mathbb{G}_m^{\wedge 1}}U_+, E)$. For $E \in SH(k)$ (respectively $E \in SH$) and a positive integer N, we let E/N denote an object of $SH(k)$ (respectively $E/N \in SH$) that fits into a triangle $E \xrightarrow{N\cdot id} E \to E/N \to E[1]$. By definition, $\pi^{\mathbb{A}^1}_{a,b}(E;\mathbb{Z}/N) := \pi^{\mathbb{A}^1}_{a,b}(E/N)$ (respectively $\pi_n(E;\mathbb{Z}/N) := \pi_n(E/N)$).

We are now in a position to prove the main result of the section.

10.3. **Theorem.** *Let k be an algebraically closed field of characteristic zero with an embedding $\varepsilon : k \hookrightarrow \mathbb{C}$. Suppose E is a symmetric Thom T-spectrum with the bounding constant $d \leqslant 1$ and contractible alternating group action. Then for all integers $N > 1$ and $n \in \mathbb{Z}$, the realization functor Re^ε induces an isomorphism*

$$\pi_n(M_E(pt)(pt);\mathbb{Z}/N) \cong \pi_n(Re^\varepsilon(E);\mathbb{Z}/N)$$

between stable homotopy groups with $\operatorname{mod} N$ coefficients.

Proof. By Lemma 10.2 E is an effective T-spectrum. It follows from [19, 7.1] that the map

$$\pi^{\mathbb{A}^1}_{n,0}(E;\mathbb{Z}/N)(pt) \to \pi_n(Re^\varepsilon(E);\mathbb{Z}/N)$$

is an isomorphism for all $n \in \mathbb{Z}$. Theorem 9.13 implies that $\pi^{\mathbb{A}^1}_{n,0}(E;\mathbb{Z}/N)$ is computed as the sheaf $\pi^{Nis}_n(M_E(pt)_f;\mathbb{Z}/N)$. It remains to observe that

$$\pi^{Nis}_n(M_E(pt)_f;\mathbb{Z}/N)(pt) = \pi_n(M_E(pt);\mathbb{Z}/N)(pt) = \pi_n(M_E(pt)(pt);\mathbb{Z}/N),$$

what completes the proof. □

As the realization of MGL is isomorphic to MU in SH, the complex cobordism S^2-spectrum, and, by Quillen's Theorem [27], $\pi_*(MU)$ is isomorphic to the Lazard ring $Laz = \mathbb{Z}[x_1, x_2, \ldots]$, $\deg(x_i) = 2i$, the preceding theorem implies the following

10.4. **Corollary.** *Let k be an algebraically closed field of characteristic zero with an embedding $\varepsilon : k \hookrightarrow \mathbb{C}$. For all $n > 1$ and $i \in \mathbb{Z}$, there is an isomorphism $\pi_i(M_{MGL}(pt)(pt);\mathbb{Z}/n) \cong Laz_i/nLaz_i$, where $M_{MGL}(pt)$ is the MGL-motive of the point $pt = \mathrm{Spec}(k)$.*

We finish the section by the following result about the singular algebraic E-homotopy defined in the introduction. It is an analogue of the celebrated theorem of Suslin and Voevodsky [30] on singular algebraic homology.

10.5. **Theorem.** *Let k be an algebraically closed field of characteristic zero with an embedding $\varepsilon : k \hookrightarrow \mathbb{C}$. Suppose E is a symmetric Thom T-spectrum with the bounding constant $d \leqslant 1$ and contractible alternating group action and $X \in Sm/k$. There are canonical isomorphisms of Abelian groups*

$$\pi^E_n(X;\mathbb{Z}/m) = \pi_n(X(\mathbb{C})_+ \wedge Re^\varepsilon(E);\mathbb{Z}/m)$$

for all integers $n \geqslant 0$ and $m \neq 0$.

Moreover, if k is any perfect field, then the assignment

$$X \mapsto \pi_*^E(X) = \pi_*(Fr^E(\Delta_k^\bullet, X)^{\mathrm{gp}})$$

is a generalized homology theory on Sm/k.

Proof. Since $M_E(X)$ can be identified with $M_{X_+\wedge E}(pt)$ and $X(\mathbb{C})_+ \wedge Re^\varepsilon(E) \cong Re^\varepsilon(X_+ \wedge E)$ by [23, A.23], Theorem 10.3 implies

$$\pi_n(M_E(X)(pt);\mathbb{Z}/m) \cong \pi_n(X(\mathbb{C})_+ \wedge Re^\varepsilon(E);\mathbb{Z}/m), \quad n \geqslant 0.$$

We have that

$$\pi_n(M_E(X)(pt);\mathbb{Z}/m) \cong \pi_n(\Omega_{S^1} Fr^E(\Delta_k^\bullet, X \otimes S^1);\mathbb{Z}/m) \cong \pi_n(Fr^E(\Delta_k^\bullet, X)^{\mathrm{gp}};\mathbb{Z}/m) = \pi_n^E(X;\mathbb{Z}/m).$$

Now the fact that the assignment

$$X \mapsto \pi_*^E(X) = \pi_*(Fr^E(\Delta_k^\bullet, X)^{\mathrm{gp}})$$

is a generalized homology theory on Sm/k with k perfect immediately follows from Theorem 9.14 (verifying the excision property and the homotopy invariance property for homology theories). □

11. Normally E-framed motives

Suppose E is a symmetric Thom T-spectrum with contractible alternating group action and the bounding constant $d = 1$. In Theorem 9.13 we have constructed an explicit fibrant bispectrum representing E in terms of E-framed motives. We can simplify E-framed motives further by forgetting a bit of information and construct, up to a local equivalence of S^1-spectra, an equivalent model for them, called normally E-framed motives. Then we construct in Theorem 11.26 an explicit fibrant bispectrum representing E whose entries are expressed in terms of weighted normally E-framed motives. Another advantage of normally E-framed motives is that they lead to representability of important Thom spectra like MGL by schemes (this material is treated in the next section in details).

11.1. **Convention.** From now on we shall assume that a symmetric Thom spectrum E with the bounding constant $d = 1$ and contractible alternating group action is of the form:

- for any $n \geqslant 0$, $E_n = Th(V_n)$ with $V_n \to Z_n$ a Σ_n-equivariant vector bundle of rank n over $Z_n \in \mathbf{Sm}_k$;
- The bonding maps $E_n \wedge T^m \to E_{n+m}$ are induced by closed embeddings $i_{n,m}\colon Z_n \to Z_{n+m}$ such that we have a Cartesian square

$$\begin{array}{ccc} V_n \times \mathbb{A}^m & \xrightarrow{I_{n,m}} & V_{n+m} \\ \downarrow & & \downarrow \\ Z_n & \xrightarrow{i_{n,m}} & Z_{n+m} \end{array}$$

and $i_{n+m,r} \circ i_{n,m} = i_{n,m+r}$. Applying the shuffle permutation $\chi_{n,m} \in \Sigma_{n+m}$, define the *left inclusion maps* $i^l_{n,m} := \chi_{n,m} \circ i_{n,m}$. We require the left inclusion maps $i^l_{n,m}$ to fit into

Cartesian squares

$$\begin{array}{ccc} \mathbb{A}^m \times V_n & \xrightarrow{I^l_{n,m}} & V_{m+n} \\ \downarrow & & \downarrow \\ Z_n & \xrightarrow{i^l_{n,m}} & Z_{m+n} \end{array}$$

where $I^l_{n,m}$ is the composition $\mathbb{A}^m \times V_n \xrightarrow{tw} V_n \times \mathbb{A}^m \xrightarrow{I_{n,m}} V_{n+m} \xrightarrow{\chi_{n,m}} V_{m+n}$. Observe that the maps $I^l_{n,m}$ induce the left bonding maps $u_l\colon T^m \wedge E_n \to E_{m+n}$ in the sense of Definition 2.6.

11.2. **Remark.** The spectrum $\Sigma^\infty_T X_+$ satisfies conditions of 11.1 with $Z_n = X, V_n = X \times \mathbb{A}^n$. The spectrum MGL is a directed colimit of spectra of the form 11.1. Indeed, for any $i \geqslant 0$ there is a spectrum $E^{(i)}$, where $E^{(i)}_n = Th(V^{(i)}_n)$ and $V^{(i)}_n = \mathscr{T}GL_{n,ni}$ is the tautological vector bundle over the Grassmannian $Z^{(i)}_n = Gr(n,ni)$. Then the spectra $E^{(i)}$ satisfy conditions of 11.1 and $MGL = \operatorname{colim}_i E^{(i)}$ [22, §2.1].

When E is a symmetric Thom T^2-spectrum with the bounding constant $d=1$ and contractible alternating group action, we impose analogous conditions:

- $E_n = Th(V_n)$, where $V_n \to Z_n$ is a Σ_n-equivariant vector bundle of rank $2n$ over $Z_n \in \mathbf{Sm}_k$ for any $n \geqslant 0$;
- The bonding maps $E_n \wedge T^{2m} \to E_{n+m}$ are induced by closed embeddings $i_{n,m}\colon Z_n \to Z_{n+m}$ such that we have a Cartesian square

$$\begin{array}{ccc} V_n \times \mathbb{A}^{2m} & \xrightarrow{I_{n,m}} & V_{n+m} \\ \downarrow & & \downarrow \\ Z_n & \xrightarrow{i_{n,m}} & Z_{n+m} \end{array}$$

and $i_{n+m,r} \circ i_{n,m} = i_{n,m+r}$. Applying the shuffle permutation $\chi_{n,m} \in \Sigma_{n+m}$, one sets $i^l_{n,m} = \chi_{n,m} \circ i_{n,m}$. The maps $i^l_{n,m}$ are required to fit into Cartesian squares

$$\begin{array}{ccc} \mathbb{A}^{2m} \times V_n & \xrightarrow{I^l_{n,m}} & V_{m+n} \\ \downarrow & & \downarrow \\ Z_n & \xrightarrow{i^l_{n,m}} & Z_{m+n} \end{array}$$

where $I^l_{n,m}$ is the composition $\mathbb{A}^m \times V_n \xrightarrow{tw} V_n \times \mathbb{A}^m \xrightarrow{I_{n,m}} V_{n+m} \xrightarrow{\chi_{n,m}} V_{m+n}$. Observe that the maps $I^l_{n,m}$ induce the left bonding maps $u_l\colon T^{2m} \wedge E_n \to E_{m+n}$ in the sense of Definition 2.6.

11.3. **Remark.** The T^2-spectra MSL and MSp are directed colimits of spectra that satisfy the above assumptions. Namely, $MSL = \operatorname{colim} E^{(i)}$ where $E^{(i)}_n = Th(V^{(i)}_n)$, $V^{(i)}_n = \mathscr{T}SL_{n,ni}$ is the tautological special bundle over the special Grassmannian $Z^{(i)}_n = SGr(n,ni)$ [24, §4]. The spectrum

$MSp = \operatorname{colim} E^{(i)}$ where $E_n^{(i)} = Th(V_n^{(i)})$, $V_n^{(i)} = \mathscr{T}Sp_{n,ni}$ is the tautological symplectic bundle over the symplectic Grassmannian $Z_n^{(i)} = HGr(n,ni)$ [24, §6].

As in the previous sections we shall only consider the case of a T-spectrum E. The interested reader will easily do the same constructions for T^2-spectra in a similar fashion.

11.4. **Definition.** (Cf. [9, B.7.1]) Suppose $X \to Y$ is a closed embedding. We call X a *locally complete intersection (l.c.i.) subscheme of* Y if for every point of X there is an affine neighborhood in Y such that ideal of definition of X is generated by a regular sequence.

11.5. **Remark.** ([1, Corollary 4.5]) If Y is regular and X is a closed subscheme of codimension d, then X is an l.c.i. subscheme if and only if the ideal of definition of X is locally generated by d elements.

11.6. **Lemma.** *For $X,Y \in \mathbf{Sm}_k$, there is a natural bijection between the set $\mathrm{Fr}_n^E(X,Y)$ and the set of equivalence classes of quadruples (U,Z,ϕ,f), where*

- *Z is a closed l.c.i. subscheme of $\mathbb{A}^n_X$, finite and flat over X;*
- *U is an étale neighborhood of Z in $\mathbb{A}^n_X$;*
- *$\phi\colon U \to V_n$ is a regular map, called a framing, such that $Z = U \times_{V_n} Z_n$;*
- *$f\colon U \to Y$ is a regular map.*

Two quadruples (U,Z,ϕ,f) and (U',Z',ϕ',f') are equivalent if $Z = Z'$ and there is an open neighborhood U_0 of Z in $U \times_{\mathbb{A}^n_X} U'$ such that the framings ϕ,ϕ' as well as regular maps f,f' coincide on U_0.

Proof. By Voevodsky's Lemma 2.12, the elements of $\mathrm{Fr}_n^E(X,Y)$ can be described as the sets of equivalence classes of quadruples (U,Z,ϕ,f), where Z is a closed subset of $\mathbb{A}^n_X$, finite over X, U is its étale neighborhood, and $\phi\colon U \to V_n$ is a regular map such that $Z = \phi^{-1}(Z_n)$, and $f\colon U \to Y$ is a regular map. Two quadruples (U,Z,ϕ,f) and (U,Z',ϕ',f') are equivalent if $Z = Z'$ and there is an open neighbourhood of Z in $U \times_{\mathbb{A}^n_X} U'$, where ϕ coincides with ϕ', and f coincides with f'.

For any such quadruple the framing $\phi\colon U \to E_n$ defines a closed subscheme $Z' = U \times_{V_n} Z_n$. Then $(Z')_{red} = Z$, hence Z has codimension n in U, and is locally defined by n equations. Then it is an l.c.i. subscheme of U by Remark 11.5. Since $(Z')_{red} = Z$ is finite over X, Z is finite over X as well, and the composition $Z' \to U \to \mathbb{A}^n_X$ is a closed embedding by [29, Tag 04XV]. Since $U \to \mathbb{A}^n_X$ is étale, it induces an isomorphism between conormal sheaves of Z' in U and Z' in $\mathbb{A}^n_X$ by [29, Tag 0635]. Then by Nakayama's lemma Z' locally is defined in $\mathbb{A}^n_X$ by n equations. Thus Z' is an l.c.i. subscheme in $\mathbb{A}^n_X$, finite over X. It is flat over X by [29, Tag 00R3]. Then the assignment $(U,Z,\phi,f) \mapsto (U,Z',\phi,f)$ defines the desired bijection between $\mathrm{Fr}_n^E(X,Y)$ and the set of the statement of the lemma. □

11.7. **Definition.** For $X,Y \in \mathbf{Sm}_k$ the *set of normally framed correspondences* $\widetilde{\mathrm{Fr}}_n^E(X,Y)$ is the set of equivalence classes of quintuples (U,Z,ϕ,ψ,f), where

- Z is an l.c.i. subscheme of $\mathbb{A}^n_X$, finite and flat over X;
- U is an étale neighborhood of Z in $\mathbb{A}^n_X$;
- $\psi\colon U \to Z_n$ is a regular map and $\phi\colon N_{Z/\mathbb{A}^n_X} \cong (\psi i)^* V_n$ is an isomorphism of vector bundles, where i is the inclusion $i\colon Z \to U$;

- $f\colon Z \to Y$ is a regular map.

Two quintuples (U,Z,ϕ,ψ,f) and (U',Z',ϕ',ψ',f') are equivalent if $Z = Z'$ as subschemes of $\mathbb{A}^n_X$ and there is an open neighborhood U'' of Z in $U \times_{\mathbb{A}^n_X} U'$ such that $\psi = \psi'$ on U'', $\phi = \phi'$, and $f = f'$.

11.8. **Definition.** For an affine scheme $X = \operatorname{Spec} A$ and its closed subscheme $Z = \operatorname{Spec} A/I$ of X denote by $X^h := \operatorname{Spec} A^h$, where (A^h, I^h) is the Henselian pair associated to (A,I)[29, Tag 09XD]. We call X^h the *Henselization of X in Z*. If (A,I) is a Henselian pair, we will call $(X,Z) = (\operatorname{Spec} A, \operatorname{Spec} A/I)$ a *Henselian pair of schemes*.

11.9. **Remark.** Suppose $i\colon Z \to U$ is a closed l.c.i. subscheme and $\phi\colon U \to V_n$ is a regular map such that $\phi^* J \subseteq I$, where I is the sheaf of ideals defining Z in U and J is the sheaf of ideals defining Z_n in V_n. Then it defines a morphism of vector bundles

$$N(\phi)\colon N_{Z/U} \to (\pi\phi i)^* V_n, \tag{11.10}$$

where π is the projection $\pi\colon V_n \to Z_n$, which is dual to the morphism of sheaves

$$J/J^2 \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_Z \to I/I^2.$$

11.11. **Lemma.** *In the notation of Remark 11.9 one has:*

(1) *if $\phi\colon U \to V_n$ is a framing of Z, then $N(\phi)$ is an isomorphism;*
(2) *if U is affine, (Z,U) is a Henselian pair, and the morphism $N(\phi)$ is an isomorphism, then ϕ is a framing of Z in U.*

Proof. (1). Note that when ϕ is a framing, I is generated by the image of J. Hence the map $J/J^2 \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_Z \to I/I^2$ induced by ϕ is is a surjection of locally free sheaves of rank n, and so it is an isomorphism. Thus $N(\phi)$ is an isomorphism.

(2). Let $U = \operatorname{Spec} R$ and let $I' \subseteq I$ denote the ideal generated by the image of J. Since $N(\phi)$ is an isomorphism, the dual map $J/J^2 \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_Z \to I/I^2$ is surjective, hence $I = I' + I^2$. Since $I \subseteq Jac(R)$, then $I = I'$ by Nakayama's lemma. We see that I is generated by the image of J, and hence ϕ is a framing. □

There is a forgetful map

$$fog\colon \operatorname{Fr}^E_n(X,Y) \to \widetilde{\operatorname{Fr}}^E_n(X,Y), \ \ (U,Z,\phi,f) \mapsto (U,Z,N(\phi),\pi\phi,f).$$

There is also a stabilization map

$$\widetilde{\operatorname{Fr}}^E_n(X,Y) \to \widetilde{\operatorname{Fr}}^E_{n+1}(X,Y),\ (U,Z,\psi,\phi,f) \mapsto (U \times \mathbb{A}^1, Z \times 0, \psi', \phi', f),$$

where ψ' is the composition $\psi'\colon U \times \mathbb{A}^1 \to U \xrightarrow{\psi} Z_n \to Z_{n+1}$ and ϕ' is the composition

$$\phi'\colon N_{Z\times 0/\mathbb{A}^{n+1}_X} = N_{Z/\mathbb{A}^n_X} \oplus 1 \xrightarrow{\phi\oplus 1} (\psi i)^*(V_n \oplus 1) = (\psi' i)^*(V_{n+1}).$$

Denote by $\widetilde{\operatorname{Fr}}^E(-,Y)$ the colimit of the presheaves $\widetilde{\operatorname{Fr}}^E_n(-,Y)$ with respect to the stabilization maps

$$\widetilde{\operatorname{Fr}}^E(-,Y) := \operatorname{colim}_n \widetilde{\operatorname{Fr}}^E_n(-,Y).$$

11.12. **Lemma.** *The presheaf* $\widetilde{\mathrm{Fr}}^E(-,Y)$ *admits framed transfers and the forgetful map induces a map* $fog\colon \mathrm{Fr}^E(-,Y) \to \widetilde{\mathrm{Fr}}^E(-,Y)$ *of presheaves with framed transfers.*

Proof. We shall construct a pairing

$$\mathrm{Fr}_n(X,Y) \times \widetilde{\mathrm{Fr}}^E_m(Y,W) \to \widetilde{\mathrm{Fr}}^E_{n+m}(X,W), (b,a) \mapsto b^*(a)$$

as follows. If $a = (U,Z,\phi,\psi,f) \in \widetilde{\mathrm{Fr}}^E_m(Y,W)$ and $b = (U',Z',\phi',f') \in \mathrm{Fr}_n(X,Y)$, define $b^*(a) = (U'',Z'',\phi'',\psi'',f'')$, where $U'' = U' \times_Y U, Z'' = Z' \times_Y Z$, ψ'' is the composition

$$\psi''\colon U' \times_Y U \to U \to Z_m \xrightarrow{i^l_{m,n}} Z_{n+m}.$$

Since the canonical map $\tau : (N_{Z'/U'})|_{Z''} \oplus (N_{Z/U})|_{Z''} \to N_{Z''/U''}$ is a surjection of vector bundles of the same rank, it is an isomorphism. Define the isomorphism ϕ'' as the composition

$$\phi''\colon N_{Z''/U''} \xrightarrow{\tau^{-1}} (N_{Z'/U'})|_{Z''} \oplus (N_{Z/U})|_{Z''} \xrightarrow{N(\phi')\oplus\phi} 1^n \oplus (i\psi)^*V_m \to (i\psi'')^*V_{n+m},$$

where $N(\phi)$ stands for the isomorphism of formula (11.10) for $E = S_T$. The function f'' is, by definition, the composition $U'' \to U \to W$. This pairing is plainly compatible with stabilization by m and endows $\widetilde{\mathrm{Fr}}^E(-,Y)$ with the structure of a framed presheaf such that the forgetful map $\mathrm{Fr}^E(-,Y) \to \widetilde{\mathrm{Fr}}^E(-,Y)$ is a morphism of framed presheaves. □

11.13. **Remark.** If X is an affine smooth variety, the set $\mathrm{Fr}^E_n(X,Y)$ (respectively $\widetilde{\mathrm{Fr}}^E_n(X,Y)$) is in bijective correspondence with the set of triples (Z,ϕ,f), where Z is an l.c.i. closed subscheme in $\mathbb{A}^n_X$, finite and flat over X, $\phi\colon (\mathbb{A}^n_X)^h \to V_n$ such that $Z = (\mathbb{A}^n_X)^h \times_{V_n} Z_n$, and $f\colon (\mathbb{A}^n_X)^h \to Y$ (respectively with the set of quadruples (Z,ϕ,ψ,f), where Z is an l.c.i. closed subscheme in $\mathbb{A}^n_X$, finite and flat over X, $\psi\colon (\mathbb{A}^n_X)^h \to Z_n$, and $\phi : N_{Z/\mathbb{A}^n_X} \xrightarrow{\cong} (\psi i)^*V_n$, $f\colon (\mathbb{A}^n_X)^h \to Y$).

11.14. **Lemma.** *Suppose there is an étale map* $Y \to \mathbb{A}^d$. *Then the forgetful map of presheaves* $fog\colon \mathrm{Fr}^E_n(-,Y) \to \widetilde{\mathrm{Fr}}^E_n(-,Y)$ *is locally surjective in the Nisnevich topology.*

Proof. Suppose X is a local Henselian scheme and $(Z,\phi,\psi,f) \in \widetilde{\mathrm{Fr}}^E_n(X,Y)$. Then Z is semi-local Henselian and the map $\psi\colon U \to Z_n$, where U is the Henselization of Z in $\mathbb{A}^n_X$, factors as $U \xrightarrow{\widetilde{\psi}} Z^0_n \subseteq Z_n$, where Z^0_n is an open subset of Z_n such that the fiber V_n over Z^0_n is a trivial vector bundle. Let i denote the inclusion $Z \hookrightarrow U$. Fix a trivialization $V_n|_{Z^0_n} \cong Z^0_n \times \mathbb{A}^n$. It gives a trivialization of $(\psi i)^*V_n$. Composing the latter trivialization with ϕ, one gets a trivialization $\widetilde{\phi}$ of the bundle $N_{Z/\mathbb{A}^n_X} = N_{Z/U}$. The trivialization $\widetilde{\phi}$ provides a basis of the $k[Z]$-module I/I^2, where I is the ideal of definition of Z in U. The basis of I/I^2 lifts to a set of generators $\gamma = (\gamma_1,\ldots,\gamma_n)$ of I. They define a map $\phi' = (\widetilde{\psi},\gamma)\colon U \to Z^0_n \times \mathbb{A}^n \to V_n$ such that $\phi = N(\phi')$ in the sense of formula (11.10).

Now let us extend the regular map $f\colon Z \to Y$ to $f'\colon U \to Y$. By assumption, there is an étale map $g\colon Y \to \mathbb{A}^d$. There is also a map $h\colon U \to \mathbb{A}^d$ that extends the composition $gf\colon Z \to \mathbb{A}^d$. Then $W = U \times_{\mathbb{A}^d} Y$ will give an étale neighbourhood of Z in U, hence there is a section $U \to W$. Then the composition $f'\colon U \to W \to Y$ extends f. We see that the triple $(Z,\phi',f') \in \mathrm{Fr}^E_n(X,Y)$ is a preimage of $(Z,\phi,\psi,f) \in \widetilde{\mathrm{Fr}}^E_n(X,Y)$. □

11.15. **Definition.** For $Y \in \mathbf{Sm}_k$ define a presheaf of S^1-spectra $\widetilde{\mathrm{Fr}}^E(Y \otimes \mathbb{S})$ associated to the presheaf of Γ-spaces $K \mapsto \widetilde{\mathrm{Fr}}^E(Y \otimes K)$ (cf. [13, Section 5])

$$\widetilde{\mathrm{Fr}}^E(Y \otimes \mathbb{S}) = (\widetilde{\mathrm{Fr}}^E(Y), \widetilde{\mathrm{Fr}}^E(Y \otimes S^1), \widetilde{\mathrm{Fr}}^E(Y \otimes S^2), \ldots).$$

The *normally E-framed motive of Y* is the presheaf of S^1-spectra

$$\widetilde{M}_E(Y) = C_*\widetilde{\mathrm{Fr}}^E(Y \otimes \mathbb{S}) = (C_*\widetilde{\mathrm{Fr}}^E(Y), C_*\widetilde{\mathrm{Fr}}^E(Y \otimes S^1), C_*\widetilde{\mathrm{Fr}}^E(Y \otimes S^2), \ldots).$$

It follows from Lemma 11.12 that both $\widetilde{\mathrm{Fr}}^E(Y \otimes \mathbb{S})$ and $\widetilde{M}_E(Y)$ are presheaves of S^1-spectra with framed transfers.

11.16. **Lemma.** *The presheaves of stable homotopy groups $\pi_i(\widetilde{\mathrm{Fr}}^E(Y \otimes \mathbb{S}))$ have $\mathbb{Z}F_*$-transfers and the presheaves of stable homotopy groups $\pi_i(\widetilde{M}_E(Y))$ are $\mathbb{A}^1$-invariant stable $\mathbb{Z}F_*$-presheaves.*

Proof. For X_1, X_2 there is a natural bijection $\widetilde{\mathrm{Fr}}_n^E(X_1 \sqcup X_2, Y) \to \widetilde{\mathrm{Fr}}_n^E(X_1, Y) \times \widetilde{\mathrm{Fr}}_n^E(X_2, Y)$, hence there is an isomorphism $\widetilde{\mathrm{Fr}}^E(X_1 \sqcup X_2, Y \otimes \mathbb{S}) \to \widetilde{\mathrm{Fr}}_n^E(X_1, Y \otimes \mathbb{S}) \times \widetilde{\mathrm{Fr}}_n^E(X_2, Y \otimes \mathbb{S})$ of S^1-spectra. Then the presheaves $\pi_i(\widetilde{\mathrm{Fr}}^E(Y \otimes \mathbb{S}))$ are radditive with framed transfers, and hence these are $\mathbb{Z}F_*$-presheaves.

Recall that $\sigma_X \in \mathrm{Fr}_1(X, X)$ uniquely corresponds to the canonical motivic equivalence $X_+ \wedge \mathbb{P}^{\wedge 1} \to X_+ \wedge T$ and is given by the quadruple $(X \times 0, X \times \mathbb{A}^1, pr_{\mathbb{A}^1}, pr_X)$. Then $\sigma_X^* \colon \widetilde{\mathrm{Fr}}_n^E(X, Y) \to \widetilde{\mathrm{Fr}}_{n+1}^E(X, Y)$ differs from the stabilization map $\widetilde{\mathrm{Fr}}_n^E(X, Y) \to \widetilde{\mathrm{Fr}}_{n+1}^E(X, Y)$ by the action of the shuffle permutation $\chi_{1,n}$ on $\mathbb{A}_X^{n+1}$ and on the vector bundle V_{n+1}. As usual, when n is even, they differ by an $\mathbb{A}^1$-homotopy, hence induce homotopic maps $C_*\widetilde{\mathrm{Fr}}_n^E(X, Y) \to C_*\widetilde{\mathrm{Fr}}_{n+1}^E(X, Y)$. Thus σ_X^* induces the identity map on presheaves of homotopy groups. □

11.17. **Definition.** For a map of simplicial presheaves $f \colon X \to Y$ denote by $\check{C}(f)$ the diagonal of the Čech bisimplicial presheaf with n-simplices given by simplicial presheaf

$$\check{C}(f)_n = X \times_Y \times \ldots \times_Y X \qquad (n+1 \text{ times})$$

with the usual face and degeneracy maps. Then f factors as a composition

$$X \xrightarrow{d(f)} \check{C}(f) \xrightarrow{p(f)} Y$$

where $d(f)$ is the diagonal map

$$d(f)_n \colon X_n \to X_n \times_{Y_n} \times \ldots \times_{Y_n} X_n$$

and $p(f)$ is the projection

$$p(f)_n \colon X_n \times_{Y_n} \times \ldots \times_{Y_n} X_n \to Y_n.$$

Note that if $X(U) \to Y(U)$ is surjective for $U \in \mathbf{Sm}_k$, then $p(f) \colon \check{C}(f)(U) \to Y(U)$ is a weak equivalence of simplicial sets.

11.18. **Definition.** For every simplicial presheaf X denote by C_*X the diagonal of the bisimplicial presheaf $n \mapsto X(\Delta^n_k)$. Then there is a canonical inclusion map $c_0\colon X \to C_*(X)$, and for every $U \in \mathbf{Sm}_k$ the map

$$C_*(c_0)\colon C_*X(U) \to C_*C_*X(U)$$

is a weak equivalence of simplicial sets.

11.19. **Lemma.** *Suppose there is an étale map $g\colon Y \to \mathbb{A}^d$ and $fog_Y\colon \mathrm{Fr}^E_n(Y) \to \widetilde{\mathrm{Fr}}^E_n(Y)$ is the forgetful map. Then there exists a map of simplicial presheaves $H_Y\colon \check{C}(fog_Y) \to C_*\mathrm{Fr}^E_n(Y)$ on the category of smooth affine varieties, compatible with stabilization by n, and that fits into the commutative diagram*

$$\begin{array}{ccc} \mathrm{Fr}^E_n(Y) & \xrightarrow{\quad c_0 \quad} & C_*\mathrm{Fr}^E_n(Y) \\ \downarrow{\scriptstyle d(fog)} & \nearrow{\scriptstyle H_Y} & \downarrow{\scriptstyle C^*(fog_Y)} \\ \check{C}(fog_Y) & \xrightarrow{c_0\circ p(fog)} & C_*(\widetilde{\mathrm{Fr}}^E_n(Y)). \end{array} \tag{11.20}$$

Moreover, the map H_Y is functorial in Y in the following sense: if $g\colon Y \to \mathbb{A}^d$ is étale, $g'\colon Y' \to \mathbb{A}^d$ is étale, and $q\colon Y \to Y'$ is a map such that $g'q = g$, then the diagram

$$\begin{array}{ccc} \check{C}(fog_Y) & \xrightarrow{H_Y} & C_*\mathrm{Fr}^E_n(Y) \\ \downarrow & & \downarrow \\ \check{C}(fog_{Y'}) & \xrightarrow{H_{Y'}} & C_*\mathrm{Fr}^E_n(Y') \end{array} \tag{11.21}$$

is commutative. Here the vertical arrows are induced by q.

Proof. For brevity we sometimes write fog instead of fog_Y. For an affine smooth X the set of m simplices $\check{C}(fog)_m(X)$ consists of $(m+1)$-triples of correspondences $(Z,\phi_0,f_0),\dots,(Z,\phi_m,f_m)$ in $\mathrm{Fr}^E_n(X,Y)$ such that the maps $\pi\phi_0,\dots,\pi\phi_m\colon U \to Z_n$ are equal, isomorphisms on normal bundles $N(\phi_i)\colon N_{Z/A^n_X} \to (\pi\phi_i i)^*V_n$ are equal for $i = 0,\dots,m$, and the regular maps $f_i\colon U \to Y$ coincide on Z. Here U denotes the Henselization of Z in $\mathbb{A}^n_X$ and we use Remark 11.13 here.

The addition map $V_n \times_{Z_n} V_n \to V_n$ and scalar multiplication map $\mathbb{A}^1 \times V_n \to V_n$ give rise to the linear combination map

$$V_n \times_{Z_n} V_n \times \ldots \times_{Z_n} V_n \times \mathbb{A}^{m+1} \to V_n,$$
$$((v_0,\dots,v_m),(t_0,\dots,t_m)) \mapsto t_0v_0 + \ldots + t_mv_m.$$

For a $(m+1)$-tuple $(Z,\phi_0,f_0),\dots(Z,\phi_m,f_m)$ in $\check{C}(fog)_m(X)$ the maps $\phi_0,\dots,\phi_m$ coincide after composing them with $\pi\colon V_n \to Z_n$, hence they define a map $\phi\colon U \to V_n \times_{Z_n} \times \ldots \times_{Z_n} V_n$. Taking composition with the linear combination map, we get a map

$$\Phi = t_0\phi_0 + \ldots + t_m\phi_m\colon U \times \Delta^m_k \to V_n,$$

where $t_0,\dots,t_m$ denote the barycentric coordinates on Δ^m_k. Let J denote the sheaf of ideals defining Z_n in V_n. For every ϕ_i we have that $\phi_i^*(J)$ lies inside the ideal I defining Z in U. Then $\Phi^*(J)$ lies inside the ideal $I \otimes_k k[\Delta^m] \subseteq k[U] \otimes_k k[\Delta^m]$ which defines $Z \times \Delta^m_k$ inside $U \times \Delta^m_k$.

Let $\Phi^h : (U \times \Delta^m_k)^h \to V_n$ denote the map on Henselization induced by Φ. Since $\Phi^*(J) \subseteq I \otimes k[\Delta^m]$, then $(\Phi^h)^*(J) \subseteq (I \otimes k[\Delta^m])^h$, where the latter denotes the corresponding ideal in the Henselization ring $k(U \times \Delta^m)^h$. The normal bundles $N_{Z\times\Delta^m_k/U\times\Delta^m_k}$ and $N_{Z\times\Delta^m_k/(U\times\Delta^m_k)^h}$ are canonically isomorphic. We denote them by $N_{Z\times\Delta^m_k}$ for brevity. By 11.9 Φ^h defines a morphism of vector bundles

$$N(\Phi^h)\colon N_{Z\times\Delta^m_k} \to (\pi\Phi^h i_\Delta)^* V_n,$$

where $i_\Delta\colon Z\times\Delta^m_k \to (U\times\Delta^m_k)^h$ denotes the inclusion. Let $i\colon Z \to U$ denote the inclusion, and $p\colon Z\times\Delta^m_k \to Z$ denote the projection. Then $\pi\Phi^h i_\Delta = \pi\phi_j ip$ for every $j = 0,\dots,m$. In particular, $\pi\Phi^h i_\Delta = \pi\phi_0 ip$. The normal bundle $N_{Z\times\Delta^m_k}$ is canonically isomorphic to the pullback $p^*N_{Z/U}$. By construction, the morphism of bundles $N(\Phi^h)$ equals the sum

$$N(\Phi^h) = t_0 p^* N(\phi_0) + \ldots + t_m p^* N(\phi_m)\colon\ p^*N_{Z/U} \to p^*(\pi\phi_0 i)^* V_n.$$

Since $N(\phi_i) = N(\phi_0)$ for all $i = 0,\dots,m$, then $N(\Phi^h) = p^*N(\phi_0)$ is an isomorphism, because so is $N(\phi_0)$. Then Φ^h is a framing of $Z\times\Delta^m_k$ in $(U\times\Delta^m_k)^h$ by Lemma 11.11(2).

The maps $f_0,\dots,f_m\colon U \to Y$ coincide on Z. Consider the map

$$t_0 g f_0 + \ldots + t_m g f_m\colon U\times\Delta^m_k \to \mathbb{A}^d.$$

Then the fiber product $U' = (U\times\Delta^m_k)\times_{\mathbb{A}^d} Y$ is an étale neighborhood of $Z\times\Delta^m_k$ in $U\times\Delta^m_k$, hence there is a unique section $s\colon (U\times\Delta^m_k)^h \to U'$, where $(U\times\Delta^m_k)^h$ is the Henselization of $Z\times\Delta^m_k$ in $U\times\Delta^m_k$. Denote by $t_0 f_0 + \ldots + t_m f_m$ the composition

$$(U\times\Delta^m_k)^h \xrightarrow{s} U' \to Y.$$

Then for every $m \geqslant 0$ one gets a map $H_m\colon \check{C}(fog)_m \to C_m \mathrm{Fr}^E_n(Y)$ defined as

$$H_m\colon (Z,\phi_0,f_0),\dots(Z,\phi_m,f_m) \mapsto (Z\times\Delta^m_k, \Phi^h, t_0 f_0 + \ldots + t_m f_m)$$

in the notation of Remark 11.13. Clearly, the maps H_m are compatible with the face and degeneracy maps and yield the desired morphism of simplicial presheaves $H_Y\colon \check{C}(fog_Y) \to C_* \mathrm{Fr}^E_n(Y)$ on the category of smooth affine varieties.

If a m-tuple $((Z,\phi_0,f_0),\dots(Z,\phi_m,f_m))$ in $\check{C}(fog)_m$ is in the image of $d(fog)$, then $(Z,\phi_i,f_i) = (Z,\phi_0,f_0)$ for $i = 0,\dots m$. Thus $(Z\times\Delta^m_k, \Phi^h, t_0 f_0 + \ldots + t_m f_m) = (Z\times\Delta^m_k, \phi_0\circ pr, f_0\circ pr)$, where $pr\colon U\times\Delta^m_k \to U$ is the projection. Then the left triangle in the diagram (11.20) is commutative.

As we have already proved, if $\Phi = t_0\phi_0 + \ldots + t_m\phi_m$ then $N(\Phi^h) = p^*N(\phi_0)$. It follows that the right triangle in the diagram (11.20) is commutative as well.

To see that the diagram (11.21) commutes when $q\colon Y \to Y'$ is a map over $\mathbb{A}^d$, we note that the following maps coincide:

$$t_0 g f_0 + \ldots + t_m g f_m = t_0 g' q f_0 + \ldots + t_m g' q f_m\colon U\times\Delta^m_k \to \mathbb{A}^d.$$

Then the diagram

$$\begin{array}{ccccc} (U\times\Delta^m_k)^h & \longrightarrow & (U\times\Delta^m_k)\times_{\mathbb{A}^d} Y & \longrightarrow & Y \\ & \searrow & \downarrow & & \downarrow{\scriptstyle q} \\ & & (U\times\Delta^m_k)\times_{\mathbb{A}^d} Y' & \longrightarrow & Y' \end{array}$$

is commutative. Thus we get that

$$q(t_0 f_0 + \ldots + t_m f_m) = t_0 q f_0 + \ldots + t_m q f_m\colon (U\times\Delta^m_k)^h \to Y',$$

and hence (11.21) commutes. ☐

11.22. **Lemma.** *Suppose there is an étale map $g\colon Y \to \mathbb{A}^d$. Then the natural map $fog\colon \mathrm{Fr}^E(Y) \to \widetilde{\mathrm{Fr}}^E(Y)$ induces a local stable weak equivalence of S^1-spectra $M_E(Y) \xrightarrow{\simeq} \widetilde{M}_E(Y)$.*

Proof. For every $n \geqslant 0$ the map $fog\colon \mathrm{Fr}_n^E(Y) \to \widetilde{\mathrm{Fr}}_n^E(Y)$ is locally surjective by Lemma 11.14. It follows that the induced map $\check{C}(fog) \to \widetilde{\mathrm{Fr}}_n^E(Y)$ is a local weak equivalence. Let $\check{C}(fog \otimes \mathbb{S})$ denote the presheaf of Segal S^1-spectra associated to the presheaf of Γ-spaces $K \mapsto \check{C}(fog \otimes K)$, where $fog \otimes K$ is the forgetful map $fog\colon \mathrm{Fr}_n^E(Y \otimes K) \to \widetilde{\mathrm{Fr}}_n^E(Y \otimes K)$. Then the induced map

$$\check{C}(fog \otimes \mathbb{S}) \to \widetilde{\mathrm{Fr}}_n^E(Y \otimes \mathbb{S})$$

is a levelwise local weak equivalence of S^1-spectra.

For any finite pointed set K we have that $Y \otimes K$ is étale over $\mathbb{A}^d$ via the natural composition $Y \otimes K \to Y \xrightarrow{g} \mathbb{A}^d$ and for any map $K \to K'$ of pointed sets the induced map $Y \otimes K \to Y \otimes K'$ is a map of varieties over $\mathbb{A}^d$. Then the maps $H_{Y \otimes K}$ of Lemma 11.19 induces a map of presheaves of S^1-spectra $H\colon \check{C}(fog \otimes \mathbb{S}) \to C_* \mathrm{Fr}_n^E(Y \otimes \mathbb{S})$. Applying C_* we get a commutative diagram

$$\begin{array}{ccc} C_* \mathrm{Fr}_n^E(Y \otimes \mathbb{S}) & \longrightarrow & C_* C_* \mathrm{Fr}_n^E(Y \otimes \mathbb{S}) \\ \downarrow{\scriptstyle C_* d(fog)} & \overset{C_* H}{\nearrow} & \downarrow{\scriptstyle C_* C_*(fog)} \\ C_* \check{C}(fog \otimes \mathbb{S}) & \longrightarrow & C_* C_* \widetilde{\mathrm{Fr}}_n^E(Y \otimes \mathbb{S}) \end{array}$$

The horizontal arrows in the diagram are motivic stable weak equivalences. Then $C_* H$ has both a left and a right inverse in $SH_{S^1}(k)$. So $C_* H$ is a motivic stable weak equivalence as well, and hence so are the vertical arrows. Since C_* is an idempotent operation up to motivic equivalence and sequential colimits preserve stable motivic equivalences, it follows that $C_*(fog) : M_E(Y) \to \widetilde{M}_E(Y)$ is a motivic stable weak equivalence. It follows from Lemmas 9.12(3), 11.16, [13, 7.1] and [12, 1.1] that local stable fibrant replacements $M_E(Y)_f, M_E(Y)_f$ of $M_E(Y), M_E(Y)$ are motivically fibrant S^1-spectra. Therefore the induced map $C_*(fog)_f : M_E(Y)_f \to \widetilde{M}_E(Y)_f$ is a sectionwise level weak equivalence of spectra, and hence $C_*(fog) : M_E(Y) \to \widetilde{M}_E(Y)$ is a stable local weak equivalence, as required. ☐

11.23. **Lemma.** *Suppose $Y \in \mathbf{Sm}_k$ equals the union of two open subschemes Y_1 and Y_2. Let $Y_{12} = Y_1 \cap Y_2$. Then*

$$\begin{array}{ccc} \widetilde{M}_E(Y_{12}) & \longrightarrow & \widetilde{M}_E(Y_1) \\ \downarrow & & \downarrow \\ \widetilde{M}_E(Y_2) & \longrightarrow & \widetilde{M}_E(Y) \end{array} \qquad \begin{array}{ccc} M_E(Y_{12}) & \longrightarrow & M_E(Y_1) \\ \downarrow & & \downarrow \\ M_E(Y_2) & \longrightarrow & M_E(Y) \end{array}$$

are homotopy pushout squares in the local stable model structure of S^1-spectra.

Proof. Similarly to [13, Definition 8.3] one can introduce the presheaves of abelian groups $\widetilde{\mathbb{Z}F}^E(Y)$ imposing the additivity relation on supports in $\mathbb{Z}\widetilde{\mathrm{Fr}}^E(Y)$. The same reasons as in [10, Theorem 1.2] show that homology of the complex $C_*\widetilde{\mathbb{Z}F}(Y)(X)$ computes homology of the S^1-spectrum $\widetilde{M}_E(Y)(X)$ for any $X \in \mathbf{Sm}_k$. Repeating Lemma 4.9, Corollary 4.10 and Proposition 4.11 literally, one gets that the sequence

$$0 \to \widetilde{\mathbb{Z}F}(Y_{12}) \to \widetilde{\mathbb{Z}F}(Y_1) \oplus \widetilde{\mathbb{Z}F}(Y_2) \to \widetilde{\mathbb{Z}F}(Y) \to 0$$

is locally exact, hence there is a triangle in the derived category

$$C_*\widetilde{\mathbb{Z}F}(Y_{12}) \to C_*\widetilde{\mathbb{Z}F}(Y_1) \oplus C_*\widetilde{\mathbb{Z}F}(Y_2) \to C_*\widetilde{\mathbb{Z}F}(Y)$$

of complexes of sheaves. So the first square in the statement of the lemma is homotopy pushout. The same proof applies to showing that the second square is homotopy pushout. □

11.24. **Definition.** Suppose E is a directed colimit of spectra $E^{(i)}$ satisfying Condition 11.1. Define the presheaf $\widetilde{\mathrm{Fr}}^E(Y)$ as the directed colimit

$$\widetilde{\mathrm{Fr}}^E(Y) = \mathrm{colim}_i \widetilde{\mathrm{Fr}}^{E^{(i)}}(Y),$$

and the *normally E-framed motive* $\widetilde{M}_E(Y)$ as the directed colimit $\widetilde{M}_E(Y) = \mathrm{colim}_i \widetilde{M}_{E^{(i)}}(Y)$.

11.25. **Proposition.** *Suppose $Y \in \mathbf{Sm}_k$ and E is a directed colimit of spectra $E^{(i)}$ satisfying Condition 11.1. Then the natural forgetful map $fog\colon \mathrm{Fr}^E(Y) \to \widetilde{\mathrm{Fr}}^E(Y)$ induces a local stable weak equivalence of S^1-spectra $M_E(Y) \xrightarrow{\simeq} \widetilde{M}_E(Y)$.*

Proof. Every smooth variety Y of dimension d has a Zariski cover by varieties Y_i that admit étale maps $Y_i \to \mathbb{A}^d$. Then the statement follows by induction on the number of varieties in the cover of Y if we apply Lemmas 11.22 and 11.23 as well as the fact that $M_E(Y) = \mathrm{colim}_i M_{E^{(i)}}(Y)$. □

Similarly to framed correspondences there is a natural action of the category $\mathrm{Fr}_0(k)$ on Nisnevich sheaves $\widetilde{\mathrm{Fr}}^E(-,X)$ that takes $U \in \mathrm{Fr}_0(k)$ to the sheaf $\widetilde{\mathrm{Fr}}^E(- \times U, X \times U)$. The action gives rise to maps of S^1-spectra

$$a_n : \widetilde{M}_E(X_+ \wedge \mathbb{G}_m^{\wedge n}) \to \underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, \widetilde{M}_E(X_+ \wedge \mathbb{G}_m^{\wedge n+1})), \quad n \geqslant 0,$$

literally repeating the construction of the same maps for weighted K-motives in [11, Section 3].

We finish the section by the following computation.

11.26. **Theorem.** *Suppose $X \in \mathbf{Sm}_k$ and E is a symmetric Thom T-spectrum with the bounding constant $d = 1$ and contractible alternating group action. Then the $(S^1, \mathbb{G}_m^{\wedge 1})$-bispectrum*

$$\widetilde{M}_E^{\mathbb{G}}(X)_f := (\widetilde{M}_E(X)_f, \widetilde{M}_E(X_+ \wedge \mathbb{G}_m^{\wedge 1})_f, \widetilde{M}_E(X_+ \wedge \mathbb{G}_m^{\wedge 2})_f, \dots)$$

with bonding maps induced by a_n-s above is motivically fibrant and represents the T-spectrum $X_+ \wedge E$ in the category of bispectra, where "f" refers to stable local fibrant replacements of S^1-spectra.

Proof. By Lemma 11.16 the sheaves of stable homotopy groups of each S^1-spectrum in $\widetilde{M}_E^{\mathbb{G}}(X)_f$ are $\mathbb{A}^1$-invariant, stable with framed transfers. It follows from [12] that they are strictly $\mathbb{A}^1$-invariant. By [13, 7.1] all S^1-spectra of the bispectrum are motivically fibrant. Observe that the natural map of bispectra

$$M_E^{\mathbb{G}}(X)_f \to \widetilde{M}_E^{\mathbb{G}}(X)_f,$$

induced by the forgetful map, is a level equivalence by Proposition 11.25. It follows from Theorem 9.13 that $\widetilde{M}_E^{\mathbb{G}}(X)_f$ is motivically fibrant and represents the T-spectrum $X_+ \wedge E$ in the category of bispectra. □

12. Computing the algebraic cobordism spectrum MGL

In this section we give another description of the bispectrum $M_E^{\mathbb{G}}(X)$ for the case $E = MGL$ in terms of Hilbert schemes and Ω-correspondences.

12.1. **Definition.** Given a ring R, we call a submodule M of R^N *admissible* if the quotient R^N/M is projective. If M is admissible, then it is also projective. We say that a map $f\colon M \to R^N$ is an *admissible embedding* if f is injective and $f(M)$ is an admissible submodule of R^N.

12.2. **Definition.** Given a ring R, denote by $R[\Delta^n] = R[t_0,\dots t_n]/(t_0+\dots+t_n-1)$ the coordinate ring on Δ^n_R. Also, $R[\partial\Delta^n] := R[\Delta^n]/(t_0t_1\dots t_n)$ and for every $0 \leqslant i \leqslant n$, $R[\partial_i\Delta^n] := R[\Delta^n]/t_i$ denotes the ring of functions on the i-th face. We also set $R[\partial_{ij}\Delta^n] := R[\Delta^n]/(t_i,t_j)$.

For every $R[\Delta^n]$-module M denote by $\partial M = M \otimes_{R[\Delta^n]} R[\partial\Delta^n]$, $\partial_i M = M \otimes_{R[\Delta^n]} R[\partial_i\Delta^n]$, $\partial_{ij} M = M \otimes_{R[\Delta^n]} R[\partial_{ij}\Delta^n]$.

12.3. **Lemma.** *For any affine X there is a bijection between $\widetilde{\mathrm{Fr}}_n^{MGL}(X,Y)$ and the set of quadruples (Z,ϕ,ψ,f), where Z is a closed l.c.i. subscheme of $\mathbb{A}^n_X$, finite and flat over X, R is the Henselization ring of Z in $\mathbb{A}^n_X$, $i\colon Z \to \operatorname{Spec} R$ is the embedding, $\psi\colon \operatorname{Spec} R \to Gr(n)$, $\phi\colon N_{Z/\mathbb{A}^n_X} \to (\psi i)^*\tau_n$ is an isomorphism of $k[Z]$-modules, and $f\colon Z \to Y$ is a regular map.*

Proof. This follows from Remark 11.13, Definition 11.24 and the fact that $Gr(n)(R)$ equals $\operatorname{colim} Gr(n,N)(R)$ for any k-algebra R. □

12.4. **Definition.** For $X,Y \in \mathbf{Sm}_k$ denote by $\mathrm{Emb}_n(X,Y)$ the set of couples (Z,f), where Z is a closed l.c.i. subscheme in $\mathbb{A}^n_X$, finite and flat over X, and f is a regular map $f\colon Z \to Y$. Note that $\mathrm{Emb}_n(X,Y)$ is pointed at the couple $(\emptyset, \emptyset \to Y)$.

We need the following intermediate object:

12.5. **Definition.** For $X,Y \in \mathbf{Sm}_k$ denote by $B_n(X,Y)$ the set of quadruples (Z,ϕ,ψ,f), where Z is a closed l.c.i. subscheme of $\mathbb{A}^n_X$, finite and flat over X, $\psi\colon Z \to Gr(n)$, $\phi\colon N_{Z/\mathbb{A}^n_X} \to \psi^*\tau_n$ is an isomorphism of vector bundles over Z, and $f\colon Z \to Y$ is a regular map.

12.6. **Remark.** The motivic space $Gr(n) = \operatorname{colim}_N Gr(n,N)$ is a directed colimit of closed embeddings of smooth varieties. For a k-scheme Z by a regular map $\psi\colon Z \to Gr(n)$ we mean an element of $\operatorname{colim}_N \operatorname{Hom}(Z, Gr(n,N))$. Then every regular map $\psi\colon Z \to Gr(n)$ induces a vector bundle $\psi^*\tau(n)$ over Z. Note that for a k-algebra R the set $Gr(n,N)(R)$ is in bijective correspondence with the set of rank n admissible submodules of R^N (see [29, Tag 089R]).

Note that $B_n(-,Y), \mathrm{Emb}_n(-,Y)$ are presheaves on $\mathbf{Sm}_k$. There are natural forgetful maps $\widetilde{\mathrm{Fr}}_n^{MGL}(-,Y) \to B_n(-,Y) \to \mathrm{Emb}_n(-,Y)$. We shall prove that for any smooth affine X these maps induce weak equivalences of simplicial sets

$$C_*\widetilde{\mathrm{Fr}}_n^{MGL}(X,Y) \to C_*B_n(X,Y) \to C_*\mathrm{Emb}_n(X,Y).$$

12.7. **Lemma.** *For every affine smooth X the map $C_*\widetilde{\mathrm{Fr}}_n^{MGL}(X,Y) \to C_*B_n(X,Y)$ is a trivial Kan fibration of simplicial sets.*

Proof. The map on zero simplices $\widetilde{\mathrm{Fr}}_n^{MGL}(X,Y) \to B_n(X,Y)$ is surjective by Lemma A.5. Suppose $\sigma\colon \Delta[m] \to C_*(B_n(X,Y))$ is a m-simplex and there is a lift of the boundary $\gamma\colon \partial\Delta[m] \to C_*\widetilde{\mathrm{Fr}}_n^{MGL}(X,Y)$. Then γ is represented by a collection $\gamma_i\colon \partial_i\Delta[m] \to C_*\widetilde{\mathrm{Fr}}_n^{MGL}(X,Y)$, such that γ_i and γ_j agree on the intersection $\partial_i\Delta[m] \cap \partial_j\Delta[m]$.

Suppose σ is represented by a quadruple $(Z,\phi,\psi,f) \in B_n(\Delta^m_X,Y)$. Let $\partial\Delta^m_X$ be the variety $\operatorname{Spec} k[\Delta^m_X]/(t_0t_1\dots t_m)$, where $t_0,\dots,t_m$ are the barycentric coordinates of the algebraic simplex Δ^m_X. Let $\partial Z = Z \times_{\Delta^m_X} \partial\Delta^m_X$ denote the fiber of Z over $\partial\Delta^m_X$.

Let R denote the Henselization ring of Z inside $\mathbb{A}^n \times \Delta^m_X$. Note that by [29, Tag09XK] the ring $R' = R \otimes_{k[\Delta^m_X]} k[\partial\Delta^m_X]$, is the Henselization ring of ∂Z, and $R'_i = R \otimes_{k[\Delta^m_X]} k[\partial_i\Delta^m_X]$ is the Henselization of $\partial_i Z$, and $R'_i \otimes_R R'_j$ is the Henselization of $\partial_i Z \cap \partial_j Z$.

Then each γ_i is represented by a quadruple $(\partial_i Z, \phi|_{\partial_i Z}, \psi'_i, f|_{\partial_i Z})$ as in Lemma 12.3, where $\psi'_i\colon \operatorname{Spec} R'_i \to Gr(n)$ extends the map $\psi|_{\partial_i Z}\colon \partial_i Z \to Gr(n)$. For any i,j the maps ψ'_i and ψ'_j agree on $\operatorname{Spec} R'_i \otimes_R R'_j$. Then they descend to a map $\psi'\colon \operatorname{Spec} R' \to Gr(n)$.

Then by Lemma A.5 there exists a map $\psi''\colon \operatorname{Spec} R \to Gr(n)$ that extends ψ' and ψ. Clearly, the quadruple (Z,ψ'',ϕ,f) in $\widetilde{\mathrm{Fr}}_n^{MGL}(\Delta^m_X,Y)$ is the desired lift of σ that extends γ. □

12.8. **Lemma.** *For any n and any affine smooth X the forgetful map $f\colon B_n(-,Y) \to \mathrm{Emb}_n(-,Y)$ induces a trivial Kan fibration of simplicial sets $C_*f\colon C_*B_n(X,Y) \to C_*\,\mathrm{Emb}_n(X,Y)$.*

Proof. Suppose $m \geqslant 0$, $\sigma\colon \Delta[m] \to C_*\,\mathrm{Emb}_n(X,Y)$ is a m-simplex and $\gamma\colon \partial\Delta[m] \to C_*B_n(X,Y)$ is a lift of its boundary. Let us prove that there is a m-simplex $\sigma'\colon \Delta[m] \to C_*B_n(X,Y)$ making the diagram

$$\begin{array}{ccc} \partial\Delta[m] & \xrightarrow{\ \gamma\ } & C_*B_n(X,Y) \\ \downarrow & \overset{\sigma'}{\nearrow} & \downarrow{\scriptstyle C_*f} \\ \Delta[m] & \xrightarrow[\ \sigma\ ]{} & C_*\,\mathrm{Emb}_n(X,Y) \end{array}$$

commutative.

Suppose σ is given by a couple $(Z,f) \in \mathrm{Emb}_n(\Delta^m_X,Y)$. The map γ is given by a collection of quadruples $\gamma_i = (\partial_i Z, \phi_i, \psi_i, f|_{\partial_i Z}) \in B_n(\partial_i\Delta^m_X, Y)$ as in Definition 12.5, where $\partial_i\Delta^m_X$ denotes the i-th face of the algebraic simplex Δ^m_X and $\partial_i Z$ is the fiber of Z over $\partial_i\Delta^m_X$. The elements γ_i coincide on the intersections $\partial_i\Delta^m_X \cap \partial_j\Delta^m_X$, and hence the regular maps $\psi_i\colon \partial_i Z \to Gr(n)$ coincide on the intersections $\partial_{ij}Z$. So they descend to a regular map $\psi\colon \partial Z \to Gr(n,N)$ for some number N by Remark 12.6. The map ψ defines an admissible submodule $j\colon P = \psi^*\tau(n,N) \subseteq k[\partial Z]^N$. The isomorphisms $\phi_i\colon N_{\partial_i Z} \to \partial_i P$ coincide on intersections $N_{\partial_{ij}Z}$, and then by Lemma A.6 there is a unique isomorphism $\phi\colon N_{\partial Z} \to P$ that extends ϕ_i.

Then $j \circ \phi\colon N_{\partial Z} \to k[\partial Z]^N$ is an admissible embedding. By Lemma A.8 it can be extended to an admissible embedding $\Phi\colon N_Z \to k[Z]^N \oplus k[Z]^d$ such that $\partial\Phi$ equals the composition of $j \circ \phi$ and the standard embedding $k[\partial Z]^N \to k[\partial Z]^N \oplus k[\partial Z]^d$. It follows that the image $\Phi(N_Z) \subseteq k[Z]^{N+d}$ is a rank n admissible submodule, and so it corresponds to a regular map $\Psi\colon Z \to$

$Gr(n,N+d) \to Gr(n)$ by Remark 12.6 such that the composition $\partial_i Z \to Z \xrightarrow{\Psi} Gr(n)$ equals ψ_i for any $i = 0,\dots,n$. Abusing notation, denote by Φ the isomorphism $\Phi\colon N_{Z/\mathbb{A}^n_X} \to \Phi(N_{Z/\mathbb{A}^n_X})$. Then the quadruple $(Z,\Phi,\Psi,f) \in B_n(\Delta^m_X,Y)$ provides the desired m-simplex σ'. □

12.9. **Proposition.** *For $Y \in \mathbf{Sm}_k$ the sheaf $\mathrm{Emb}_n(-,Y)$ is representable by a countable disjoint union $E^Y_n := \bigsqcup_{d\geqslant 0} E^Y_{n,d}$ of smooth quasi-projective varieties.*

Proof. Denote by $\mathrm{Emb}_n(U,Y)_d$ the set of couples (Z,f) where Z is a closed l.c.i. subscheme of $\mathbb{A}^n_U$, finite of degree d and flat over U. Then $\mathrm{Emb}_n(-,Y)_d$ is a subpresheaf of $\mathrm{Emb}_n(-,Y)$, and $\mathrm{Emb}_n(U,Y)$ is the disjoint union of $\mathrm{Emb}_n(U,Y)_d$, $d \geqslant 0$, for any connected $U \in \mathbf{Sm}_k$.

By [7, Lemma 5.1.3] the presheaf $\mathrm{Emb}_n(-,k)_d$ is represented by a smooth quasi-projective scheme $Hilb^{lci}_d(\mathbb{A}^n)$. There is the universal finite flat map $W_d \to Hilb^{lci}_d(\mathbb{A}^n)$. Then the Weil restriction functor $R_{W_d/Hilb^{lci}_d(\mathbb{A}^n)}(W_d \times_k Y)$ coincides with $\mathrm{Emb}_n(-,Y)_d$ and is represented by a quasi-projective smooth scheme $E^Y_{n,d}$ over k by [4, 7.6.4-7.6.5]. □

The natural inclusions of affine spaces $\mathbb{A}^n \to \mathbb{A}^{n+1}$ induce stabilization maps of pointed sheaves $\mathrm{Emb}_n(-,Y) \to \mathrm{Emb}_{n+1}(-,Y)$. Denote by $\mathrm{Emb}(-,Y)$ the pointed sheaf $\mathrm{Emb}(-,Y) = \mathrm{colim}_n \mathrm{Emb}_n(-,Y)$. Note that forgetful maps $\widetilde{\mathrm{Fr}}^{MGL}_n(-,Y) \to \mathrm{Emb}_n(-,Y)$ are consistent with the stabilization maps.

12.10. **Corollary.** *The sheaf $\mathrm{Emb}(-,Y)$ is isomorphic to a sequential colimit E^Y of smooth quasi-projective varieties.*

Proof. This follows from Proposition 12.9 and the fact that $\bigsqcup_{d\geqslant 0} E^Y_{n,d}$ is $\mathrm{colim}_{k\geqslant 0}(E^Y_{n,d_1} \sqcup \dots \sqcup E^Y_{n,d_k})$. Hence $\mathrm{Emb}(-,Y)$ is isomorphic to $E^Y := \mathrm{colim}_{n,k\geqslant 0}(E^Y_{n,d_1} \sqcup \dots \sqcup E^Y_{n,d_k})$. □

We shall give an alternative description of the space $C_* \mathrm{Emb}(-,Y)$ in terms of Ω-correspondences studied in [21].

12.11. **Definition.** For $X,Y \in \mathbf{Sm}_k$ denote by $Cor^{\Omega}_n(X,Y)$ the groupoid with objects given by the set $\mathrm{Emb}_n(X,Y)$ whose morphisms between (Z_1,f_1) and (Z_2,f_2) are isomorphisms $\alpha\colon Z_1 \to Z_2$ such that $\pi_{Z_2}\alpha = \pi_{Z_1}$ and $f_2\alpha = f_1$, where π_{Z_i} denotes the projection $\pi_{Z_i}\colon Z_i \to \mathbb{A}^n_X \to X$. The assignment $X \mapsto Cor^{\Omega}_n(X,Y)$ defines a presheaf of groupoids on $\mathbf{Sm}_k$. There are natural stabilization maps $Cor^{\Omega}_n(-,Y) \to Cor^{\Omega}_{n+1}(-,Y)$ induced by the natural inclusions $\mathbb{A}^n_X \to \mathbb{A}^{n+1}_X$. Denote by $Cor^{\Omega}(X,Y)$ the colimit $Cor^{\Omega}(X,Y) = \mathrm{colim}_n Cor^{\Omega}_n(X,Y)$.

12.12. **Lemma.** *Suppose $f\colon X \to Y$ is a l.c.i. embedding, $g\colon X \to W$ is any regular map and W is regular. Then the map $(f,g)\colon X \to Y \times W$ is an l.c.i. embedding.*

Proof. The map (f,g) is the composition $X \xrightarrow{\Gamma_g} X \times W \xrightarrow{f\times id} Y \times W$. The map $f \times id$ is a l.c.i. embedding. The graph inclusion $\Gamma_g\colon X \to X \times W$ fits into the pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & W \\ \big\downarrow{\scriptstyle \Gamma_g} & & \big\downarrow \\ X \times W & \longrightarrow & W \times W \end{array}$$

where the right arrow is the diagonal embedding. Since W is regular, the diagonal map $W \to W \times W$ is a l.c.i. embedding, so the ideal defining X in $X \times W$ is locally generated by n elements, where $n = \dim W$. Then $\Gamma_g \colon X \to X \times W$ is a l.c.i. embedding by Remark 11.5. Then the composition of Γ_g and $f \times id$ is an l.c.i. embedding by [9, B.7.4]. □

12.13. **Lemma.** *Let $NCor^{\Omega}(X,Y)$ be the nerve of the groupoid $Cor^{\Omega}(X,Y)$. Then for a smooth affine X and $Y \in \mathbf{Sm}_k$ the natural map $f\colon \operatorname{Emb}(X,Y) \to NCor^{\Omega}(X,Y)$ induces a weak equivalence of simplicial sets*

$$C_* f\colon C_* \operatorname{Emb}(X,Y) \to C_* NCor^{\Omega}(X,Y).$$

Proof. Note that $C_*NCor^{\Omega}(X,Y)$ is a bisimplicial set with m-simplices given by $C_*N_mCor^{\Omega}(X,Y)$. Thus it is sufficient to prove that for any m the map

$$C_* f\colon C_* \operatorname{Emb}(X,Y) \to C_* N_m Cor^{\Omega}(X,Y)$$

is a weak equivalence of simplicial sets. Note that the map of presheaves $f\colon \operatorname{Emb}_n(-,Y) \to N_mCor^{\Omega}_n(-,Y)$ is an inclusion admitting a retraction

$$p\colon N_mCor^{\Omega}_n(-,Y) \to \operatorname{Emb}_n(-,Y)$$

that sends $((Z_0,f_0) \xrightarrow{\alpha_0} (Z_1,f_1) \to \cdots \xrightarrow{\alpha_{m-1}} (Z_m,f_m)) \in N_mCor^{\Omega}_n(X,Y)$ to $(Z_0,f_0) \in \operatorname{Emb}_n(X,Y)$.

For every smooth affine X and $((Z_0,f_0) \xrightarrow{\alpha_0} (Z_1,f_1) \xrightarrow{\alpha_1} \cdots \to (Z_m,f_m)) \in N_mCor^{\Omega}_n(X,Y)$ consider the map

$$r_i\colon Z_i \times \mathbb{A}^1 \to \mathbb{A}^n_X \times_X \mathbb{A}^n_X \times \mathbb{A}^1, \quad (z,t) \mapsto ((1-t)z + t\beta_i(z), t(t-1)z, t).$$

Here t denotes the coordinate on $\mathbb{A}^1$ and $\beta_i\colon Z_i \to Z_0$ is the isomorphism of X-schemes $\beta_i = \alpha_0^{-1} \circ \ldots \circ \alpha_{i-1}^{-1}$.

Note that r_i is a map of schemes over $X \times \mathbb{A}^1$. The map r_i, restricted to the fiber over $X \times (\mathbb{A}^1 - \{0,1\})$, fits into the diagram

$$\begin{array}{ccc} Z_i \times (\mathbb{A}^1 - \{0,1\}) & \xrightarrow{r_i} & \mathbb{A}^n_X \times_X \mathbb{A}^n_X \times (\mathbb{A}^1 - \{0,1\}) \\ & \searrow^{r_i'} & \downarrow{\scriptstyle (x,y,t) \mapsto (x, \frac{y}{t(t-1)}, t)} \\ & & \mathbb{A}^n_X \times_X \mathbb{A}^n_X \times (\mathbb{A}^1 - \{0,1\}), \end{array}$$

where $r_i'\colon (z,t) \mapsto ((1-t)z + t\beta_i(z), z, t)$ is a l.c.i. embedding by Lemma 12.12. The fiber of r_i over $X \times 0$ and $X \times 1$ is a l.c.i. embedding. Then the map r_i is a l.c.i. embedding by Lemma A.9. Let us denote by Z_i' the image of r_i. Note that the composition $Z_i \times \mathbb{A}^1 \xrightarrow{r_i} Z_i' \subset \mathbb{A}^n_X \times_X \mathbb{A}^n_X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$ coincides with $\pi \times id_{\mathbb{A}^1}$, where π is the projection $\pi : Z_i \subset \mathbb{A}^n_X \to X$. Therefore Z_i' is finite and flat over $X \times \mathbb{A}^1$.

We construct an $\mathbb{A}^1$-homotopy

$$H\colon N_mCor^{\Omega}_n(-,Y) \to N_mCor^{\Omega}_{2n}(\mathbb{A}^1 \times -, Y)$$

as follows. We set

$$H\colon ((Z_0,f_0) \xrightarrow{\alpha_0} (Z_1,f_1) \xrightarrow{\alpha_1} \cdots (Z_m,f_m)) \mapsto ((Z_0',f_0') \xrightarrow{\gamma_0} (Z_1',f_1') \xrightarrow{\gamma_1} \cdots (Z_m',f_m')),$$

where each isomorphism γ_i is given by the composition

$$\gamma_i\colon Z'_i \xrightarrow{r_i^{-1}} Z_i \times \mathbb{A}^1 \xrightarrow{\alpha_i \times id} Z_{i+1} \times \mathbb{A}^1 \xrightarrow{r_{i+1}} Z'_{i+1},$$

and each map $f'_i\colon Z'_i \to Y$ is given by the composition

$$f'_i\colon Z'_i \xrightarrow{r_i^{-1}} Z_i \times \mathbb{A}^1 \xrightarrow{\pi_{Z_i}} Z_i \xrightarrow{f_i} Y.$$

Then $H_0\colon N_mCor_n^{\Omega}(-,Y) \to N_mCor_{2n}^{\Omega}(-,Y)$ is the stabilization map and $H_1\colon N_mCor_n^{\Omega}(-,Y) \to N_mCor_{2n}^{\Omega}(-,Y)$ equals the composition

$$H_1\colon N_mCor_n^{\Omega}(-,Y) \xrightarrow{p} \mathrm{Emb}_n(-,Y) \xrightarrow{f} N_mCor_n^{\Omega}(-,Y) \xrightarrow{stab} N_mCor_{2n}^{\Omega}(-,Y),$$

where the last arrow is the stabilization map. The $\mathbb{A}^1$-homotopy H gives rise to a simplicial homotopy

$$H\colon C_*N_mCor_n^{\Omega}(-,Y) \times \Delta[1] \to C_*N_mCor_{2n}^{\Omega}(-,Y).$$

By construction, for any X we have $H(C_*\mathrm{Emb}_n(X,Y) \times \Delta[1]) \subseteq C_*\mathrm{Emb}_{2n}(X,Y)$. Then by Lemma 12.14 the map $f\colon C_*\mathrm{Emb}(X,Y) \to C_*N_mCor^{\Omega}(X,Y)$ is a weak equivalence. □

12.14. **Lemma.** *Suppose $X_n \subseteq X_{n+1}$ is a directed system of inclusions of simplicial sets, $Y_n \subseteq Y_{n+1}$ is a directed system of simplicial subsets $Y_n \subseteq X_n$, and $p_n\colon X_n \to Y_n$ is a sequence of retractions that agree with inclusions $X_n \subseteq X_{n+1}$ and $Y_n \subseteq Y_{n+1}$. Assume that for every n there is a homotopy $H(n)\colon X_n \times \Delta[1] \to X_{2n}$ such that $H(n)_0\colon X_n \to X_{2n}$ is the inclusion map, $H(n)(Y_n \times \Delta[1]) \subseteq Y_{2n}$, and the map $H(n)_1\colon X_n \to X_{2n}$ equals the composition*

$$X_n \xrightarrow{p_n} Y_n \subseteq Y_{2n} \subseteq X_{2n}.$$

Then the inclusion $Y \to X$ is a weak equivalence, where $Y = \mathrm{colim}_n Y_n, X = \mathrm{colim}_n X_n$.

Proof. Consider a point $y \in Y_n$. The inclusion map $j\colon X_n \to X_{2n}$ and the composition $f\colon X_n \xrightarrow{p_n} Y_n \subseteq X_{2n}$ are homotopic by means of the free homotopy $H(n)$. Then the two induced maps

$$\pi_i(j), \pi_i(f)\colon \pi_i(X_n,y) \to \pi_i(X_{2n},y)$$

differ by the action $[\gamma]_*$ of the class $[\gamma] \in \pi_1(X_{2n},y)$ on $\pi_i(X_{2n},y)$, where $\gamma\colon \Delta[1] \to Y_{2n}, \gamma(t) = H(n)(y,t)$ is the loop given by the image of the base point y under the homotopy $H(n)$. Since the loop γ lies inside Y_n, the action of $[\gamma]$ on $\pi_i(Y_{2n},y)$ preserves the image of $\pi_i(Y_n,y)$ under the inclusion map $Y_n \to X_{2n}$. Then the image

$$\pi_i(j)(\pi_i(X_n,y)) = [\gamma]_*\pi_i(f)(\pi_i(X_n,y))$$

lies inside the image of $\pi_i(Y_n,y)$. Then $\pi_i(Y,y) \to \pi_i(X,y)$ is surjective for any point $y \in Y$. The existence of retractions p_n implies that the map $\pi_i(Y,y) \to \pi_i(X,y)$ is also injective and for every point $x \in X_n$ the map $t \mapsto H(n)(x,t)$ gives a path between x and the point of Y_n. We see that $\pi_0(Y) \to \pi_0(X)$ is surjective, and hence $Y \to X$ is a weak equivalence. □

Note that for any pointed finite set K the assignment

$$K \mapsto \mathrm{Emb}(X_+ \wedge K)$$

defines a sheaf of Γ-spaces. Denote by $\mathrm{Emb}(X_+ \wedge \mathbb{S})$ the corresponding S^1-spectrum. For any $W \in \mathbf{Sm}_k$ there is a canonical map

$$\mathrm{Emb}(-,X) \to \mathrm{Emb}(- \times W, X \times W),$$

functorial in $W \in \mathrm{Fr}_0(k)$. These two constructions give rise to a $(S^1, \mathbb{G}_m^{\wedge 1})$-bispectrum

$$(C_* \mathrm{Emb}(X_+ \wedge \mathbb{S}), C_* \mathrm{Emb}(X_+ \wedge \mathbb{S} \wedge \mathbb{G}_m^{\wedge 1}), \ldots).$$

Its structure maps literally repeat the construction of the structure maps for K-motives [11, Section 3]. We also define a $(S^1, \mathbb{G}_m^{\wedge 1})$-bispectrum

$$(C_* NCor^{\Omega}(X_+ \wedge \mathbb{S}), C_* NCor^{\Omega}(X_+ \wedge \mathbb{G}_m^{\wedge 1} \wedge \mathbb{S}), \ldots)$$

in a similar fashion.

The following theorem computes $M_{MGL}^{\mathbb{G}}(X)$ as the above two bispectra.

12.15. **Theorem.** *For $X \in \mathbf{Sm}_k$ there is a natural levelwise stable local equivalence between $(S^1, \mathbb{G}_m^{\wedge 1})$-bispectra $M_{MGL}^{\mathbb{G}}(X)$ and*

$$(C_* \mathrm{Emb}(X_+ \wedge \mathbb{S}), C_* \mathrm{Emb}(X_+ \wedge \mathbb{G}_m^{\wedge 1} \wedge \mathbb{S}), \ldots)$$

or

$$(C_* NCor^{\Omega}(X_+ \wedge \mathbb{S}), C_* NCor^{\Omega}(X_+ \wedge \mathbb{G}_m^{\wedge 1} \wedge \mathbb{S}), \ldots).$$

In particular, the $(S^1, \mathbb{G}_m^{\wedge 1})$-bispectra

$$(C_* \mathrm{Emb}(X_+ \wedge \mathbb{S})_f, C_* \mathrm{Emb}(X_+ \wedge \mathbb{G}_m^{\wedge 1} \wedge \mathbb{S})_f, \ldots)$$

and

$$(C_* NCor^{\Omega}(X_+ \wedge \mathbb{S})_f, C_* NCor^{\Omega}(X_+ \wedge \mathbb{G}_m^{\wedge 1} \wedge \mathbb{S})_f, \ldots)$$

are motivically fibrant and represent the T-spectrum $X_+ \wedge MGL$ in the category of bispectra, where "f" refers to stable local fibrant replacements of S^1-spectra.

Proof. The first claim follows from Proposition 11.25 and Lemmas 12.3, 12.7, 12.8, 12.13. The proof of Theorem 11.26 shows that the bispectra

$$(C_* \mathrm{Emb}(X_+ \wedge \mathbb{S})_f, C_* \mathrm{Emb}(X_+ \wedge \mathbb{G}_m^{\wedge 1} \wedge \mathbb{S})_f, \ldots)$$

and

$$(C_* NCor^{\Omega}(X_+ \wedge \mathbb{S})_f, C_* NCor^{\Omega}(X_+ \wedge \mathbb{G}_m^{\wedge 1} \wedge \mathbb{S})_f, \ldots)$$

are motivically fibrant and represent the T-spectrum $X_+ \wedge MGL$ in the category of bispectra. □

We already know from Corollary 12.10 that the sheaf $\mathrm{Emb}(-,Y)$ is isomorphic to a sequential colimit E^Y of smooth quasi-projective varieties. Thus the $(S^1, \mathbb{G}_m^{\wedge 1})$-bispectrum

$$(\mathrm{Emb}(X_+ \wedge \mathbb{S}), \mathrm{Emb}(X_+ \wedge \mathbb{G}_m^{\wedge 1} \wedge \mathbb{S}), \ldots), \quad X \in \mathbf{Sm}_k,$$

can be presented as the $(S^1, \mathbb{G}_m^{\wedge 1})$-bispectrum $(E^{X_+ \wedge \mathbb{S}}, E^{X_+ \wedge \mathbb{G}_m^{\wedge 1} \wedge \mathbb{S}}, \ldots)$. By construction, the (i,j)-th term of the latter bispectrum is a sequential colimit of simplicial smooth quasi-projective varieties $E^{X_+ \wedge \mathbb{G}_m^{\wedge i} \wedge S^j}$.

By using the preceding theorem, we therefore get the following result:

12.16. **Theorem.** *The $(S^1, \mathbb{G}_m^{\wedge 1})$-bispectrum $M_{MGL}^{\mathbb{G}}(X)$ is isomorphic in $SH(k)$ to the bispectrum $(E^{X_+\wedge\mathbb{S}}, E^{X_+\wedge\mathbb{G}_m^{\wedge 1}\wedge\mathbb{S}}, \dots)$, each term of which is given by a sequential colimit of simplicial smooth quasi-projective varieties $E^{X_+\wedge\mathbb{G}_m^{\wedge i}\wedge S^j}$, $i, j \geqslant 0$.*

APPENDIX A. TECHNICAL LEMMAS

In this section we recall standard facts about projective modules over Henselian pairs. Throughout this section R denotes a Noetherian k-algebra. By 12.6 the set $Gr(n,N)(R)$ equals the set of rank n admissible submodules of R^N. If $R \to S$ is a map of k-algebras, by $P \otimes_R S$ we shall mean the image of P in S^N. It gives an element of $Gr(n,N)(S)$. It is important to recall from [29, Tag 089R] that $Gr(n,N)(R)$ is *functorial* in R.

A.1. **Lemma.** *Suppose (R,I) is a Henselian pair, and J is an ideal in R. Suppose B is an integral R-algebra, and $e' \in B/IB$, $e'' \in B/JB$ are two idempotents that coincide in $B/(I+J)B$. Then there is an idempotent $e \in B$ such that $e + IB = e'$ and $e + JB = e''$.*

Proof. By [29, Tag 09XI] there is a bijections between idempotents in B and B/IB as well as there is a bijection between idempotents in B/J and $B/(I+J)B$. If e is an idempotent in B such that $e + IB = e'$, then $e + JB = e''$. □

Denote by $Idemp_n(R)$ the set of idempotents of the matrix ring $M_n(R)$.

A.2. **Lemma.** *Suppose (R,I) is a Henselian pair, J is an ideal in R. Consider a diagram of sets:*

$$Idemp_n(R) \to Idemp_n(R/I) \times Idemp_n(R/J) \rightrightarrows Idemp_n(R/(I+J)).$$

Suppose $(x',x'') \in Idemp_n(R/I) \times Idemp_n(R/J)$ and the images of x', x'' coincide in $Idemp_n(R/(I+J))$. Then there is $x \in Idemp_n(R)$ such that the image of x in $Idemp_n(R/I)$ equals x', and the image of x in $Idemp(R/J)$ equals x''.

Proof. There is a right exact sequence of R-modules

$$M_n(R) \to M_n(R/I) \oplus M_n(R/J) \to M_n(R/(I+J)) \to 0.$$

Take a matrix $y \in M_n(R)$ to be a preimage of (x',x''). Let $f(t) \in R[t]$ be the characteristic polynomial of the matrix y. Denote by $B = R[t]/f(t)$. We follow the proof of [29, Tag07M5]. Note that B is integral over R and there is a ring map $g \colon B \to M_n(R)$ that sends t to y. For any prime ideal p containing J the image of $f(t)$ in $k(p)[t]$ is the characteristic polynomial of an idempotent matrix, hence it divides $t^n(t-1)^n$. Then $t^n(1-t)^n \in \sqrt{J}B$ and there exists a constant N_0 such that for any $N \geqslant N_0$ the element $t^N + (1-t)^N$ is invertible in B/JB. It follows that $e'' := \frac{t^N}{t^N+(1-t)^N}$ in B/JB is an idempotent and a preimage of x'' in $M_n(R/J)$. Likewise there is a constant N_1 such that for any $N > N_1$ the element $e' = \frac{t^N}{t^N+(1-t)^N}$ in B/IB is an idempotent and a preimage of x' in $M_n(R/I)$. Then for $N > max(N_0,N_1)$ the images of e' and e'' coincide in $B/(I+J)B$ and by the previous lemma there is an idempotent e in B lifting e' and e''. Then $x = g(e)$ is an idempotent matrix in $M_n(R)$ such that the image of x in $M_n(R/I)$ equals x' and the image of x in $M_n(R/J)$ equals x''. □

A.3. **Lemma.** *Suppose (R,I) is a Henselian pair, $i\colon P \subseteq R^n$ is an admissible submodule. Then $i'\colon P\otimes_R R/I \subseteq (R/I)^n$ is an admissible submodule of $(R/I)^n$. Assume that there is projection $\pi'\colon (R/I)^n \to P\otimes_R R/I$ such that $\pi' i' = id$. Then there is a projection $\pi\colon R^n \to P$ such that $\pi i = id$ and $\pi\otimes R/I = \pi'$.*

Proof. Let e_i, $i=1,\dots,n$, denote the standard basis of R^n and let $\bar{e}_i$ be the standard basis of $(R/I)^n$. Take a map $p\colon R^n \to P$ sending e_i to some preimage of $\pi'(\bar{e}_i)$. Then $p\circ i$ is an endomorphism of P such that $(p\circ i)\otimes_R R/I$ is the identity endomorphism of $P\otimes_R R/I$. Then $p\circ i$ is invertible by Nakayama's lemma. It follows that $\pi = (p\circ i)^{-1}p\colon R^n \to P$ is a projection onto P lifting π' and $\pi\circ i = id$. □

A.4. **Lemma.** *Suppose (R,I) is a Henselian pair, J is an ideal in R. Suppose $P_1 \in Gr(n,N)(R/I)$ and $P_2 \in Gr(n,N)(R/J)$ are such that $P_1\otimes_{R/I}(R/I+J) = P_2\otimes_{R/J}(R/I+J)$ in $Gr(n,N)(R/(I+J))$. Then there is $P\in Gr(n,N)(R)$ such that $P\otimes_R R/I = P_1$ in $Gr(n,N)(R/I)$ and $P\otimes_R R/J = P_2$ in $Gr(n,N)(R/J)$.*

Proof. Choose a projection $p_1\colon (R/I)^N \to P_1$. Then $p_1\otimes_{R/I} R/(I+J)$ is a projection onto $P_1\otimes_{R/I} R/(I+J) = P_2\otimes_{R/J} R/(I+J)$. The pair $(R/J, I/I\cap J)$ is Henselian by [29, Tag 09XK], then by Lemma A.3 there is a projection $p_2\colon (R/J)^N \to P_2$ that lifts $p_1\otimes_{R/I} R/(I+J)$. Then $A_1 = i_1p_1$ and $A_2 = i_2p_2$ are idempotents that coincide in $Idemp_N(R/(I+J))$. By Lemma A.2 there is an idempotent $A\in Idemp_N(R)$ such that $A\otimes_R R/I = A_1$ and $A\otimes R/J = A_2$. Then $P = A(R^N)$ is an element of $Gr(n,N)(R)$ such that $P\otimes R/I = P_1$ and $P\otimes R/J = P_2$. □

A.5. **Lemma.** *Suppose (R,I) is a Henselian pair and J is an ideal in R. Suppose $f_1\colon \operatorname{Spec} R/I \to Gr(n)$ and $f_2\colon \operatorname{Spec} R/J \to Gr(n)$ coincide on $\operatorname{Spec} R/(I+J)$. Then there is $f\colon \operatorname{Spec} R \to Gr(n)$ that extends f_1 and f_2.*

Proof. This follows from the previous lemma and the fact that $Gr(n)(R) = \operatorname{colim}_N Gr(n,N)(R)$. □

In the ring $R[\Delta^n]$ denote by t the product $t = t_0\dots t_n$ of barycentric coordinates in $R[\Delta^n]$. For any $R[\Delta^n]$-module we denote by M_t the localization $M_t = M\otimes_{R[\Delta^n]} R[\Delta^n][1/t]$.

A.6. **Lemma.** *Suppose ∂B is a finite flat $R[\partial\Delta^n]$-algebra, M is a finitely generated projective ∂B-module, $P\subseteq (\partial B)^N$ is an admissible submodule, and for every $i = 0,\dots,n$ there is an isomorphism $f_i\colon \partial_i M \to \partial_i P$, and for every i,j the maps $f_i\otimes_{\partial_i B}\partial_{ij}B$ and $f_j\otimes_{\partial_j B}\partial_{ij}B$ coincide on $\partial_{ij}M$ (see Definition 12.2). Then there is an isomorphism $f\colon M\to P$ such that $f\otimes\partial_i B = f_i$.*

Proof. There is a left exact sequence of $R[\partial\Delta^n]$-modules

$$0 \to R[\partial\Delta^n] \to \oplus_{i=0}^n R[\partial_i\Delta^n] \to \oplus_{i<j} R[\partial_{ij}\Delta^n].$$

Tensoring it with ∂B over $R[\partial\Delta^n]$, we get a left exact sequence for every projective ∂B-module. The maps f_i induce a commutative square in the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & \oplus_i\partial_i M & \longrightarrow & \oplus_{i<j}\partial_{ij}M \\ & & & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{ij}} \\ 0 & \longrightarrow & P & \longrightarrow & \oplus_i\partial_i P & \longrightarrow & \oplus_{i<j}\partial_{ij}P. \end{array}$$

Since ∂B is flat $R[\partial\Delta^n]$, both M and P are flat as $R[\partial\Delta^n]$-modules, then the rows in the diagram are exact, as they are obtained by tensoring with the left exact sequence above. Then there is a unique isomorphism $f\colon M \to P$ that makes the diagram commutative. □

A.7. **Lemma.** *Suppose M is a finitely generated projective $R[\Delta^n]$-module. A map*

$$f\colon M \to R[\Delta^n]^N$$

is an admissible embedding if and only if its restriction to the boundary

$$f \otimes R[\partial\Delta^n]\colon M \otimes_{R[\Delta^n]} R[\partial\Delta^n] \to R[\partial\Delta^n]^N$$

is an admissible embedding and the localized map

$$f_t\colon M_t \to R[\Delta^n]_t^N$$

is an admissible embedding.

Proof. Let K and C denote the kernel and cokernel of f respectively. We need to check that $K = 0$ and C is projective. Note that K is a submodule of a free finite rank $R[\Delta^n]$-module. Since t is not a zero divisor in $R[\Delta^n]$, then the localization map $K \to K_t$ is injective and $K_t = \ker(f_t) = 0$, hence $K = 0$. Let r denote the rank of M. For every maximal ideal m of $R[\Delta^n]$ if $t \notin m$ then C_m is a localization of C_t, hence it is a free module of rank $N-r$. If $t \in m$, then C/mC is a free module of rank $N-r$. By Nakayama's lemma there is a surjection $g\colon R[\Delta^n]_m^{N-r} \to C_m$ of modules over the local ring $R[\Delta^n]_m$. Then localization $(C_m)_t$ is a localization of the projective module C_t of rank $N-r$. Then $(C_m)_t$ is projective of rank $N-r$. Since g_t is a surjective map between projective modules of the same rank, then it is an isomorphism, and so $Ker(g)_t = 0$. Then $Ker(g) = 0$, since $Ker(g)$ is a submodule of the free module $R[\Delta^n]_m^{N-r}$, and t is not a zero divisor of $R[\Delta^n]$. □

A.8. **Lemma.** *Suppose M is a finitely generated projective $R[\Delta^n]$-module. Assume that there is an admissible embedding $f'\colon M \otimes_{R[\Delta^n]} R[\partial\Delta^n] \to R[\partial\Delta^n]^N$. Then there is a number d and an admissible embedding $f\colon M \to R[\Delta^n]^N \oplus R[\Delta^n]^d$ such that the map*

$$f \otimes R[\partial\Delta^n]\colon M \otimes_{R[\Delta^n]} R[\partial\Delta^n] \to R[\partial\Delta^n]^N \oplus R[\partial\Delta^n]^d$$

equals the composition of f' and the standard embedding $R[\partial\Delta^n]^N \to R[\partial\Delta^n]^N \oplus R[\partial\Delta^n]^d$.

Proof. Consider some admissible embedding $j\colon M \to R[\Delta^n]^d$ and some projection $p\colon R[\Delta^n]^d \to M$ such that $p \circ j = id_M$. Let $e_1, \ldots, e_d$ denote the standard basis of $R[\Delta^n]^d$ and let $\bar{e}_1, \ldots, \bar{e}_d$ be the standard basis of $R[\partial\Delta^n]^d$. Consider the composition

$$R[\partial\Delta^n]^d \xrightarrow{p\otimes id} M \otimes_{R[\Delta^n]} R[\partial\Delta^n] \xrightarrow{f'} R[\partial\Delta^n]^N.$$

For $i = 1, \ldots, d$ take $x_i \in R[\Delta^n]^N$ to be any preimage of $f'((p \otimes id)(\bar{e}_i)) \in R[\partial\Delta^n]^N$. Then there is a homomorphism

$$F\colon R[\Delta^n]^d \to R[\Delta^n]^N \oplus R[\Delta^n]^d, \quad e_i \mapsto (x_i, (t_0 t_1 \ldots t_n) e_i),$$

where $t_0, \ldots, t_n$ denote the coordinates in the ring $R[\Delta^n]$. Take $f\colon M \to R[\Delta^n]^N \oplus R[\Delta^n]^d$ to be the composition $f = F \circ j$. Let us check that f is an admissible embedding.

Note that $f \otimes_{R[\Delta^n]} R[\partial\Delta^n]$ is the composition of f' and the standard embedding $R[\partial\Delta^n]^N \to R[\partial\Delta^n]^N \oplus R[\partial\Delta^n]^d$. In particular, $f \otimes_{R[\Delta^n]} R[\partial\Delta^n]$ is an admissible embedding.

The localization $f_t\colon M_t \to R[\Delta^n]_t^{N+d}$ is the composition of $F_t \circ j_t$ and F_t fits into a commutative triangle

$$\begin{array}{ccc} R[\Delta^n]_t^d & \xrightarrow{g\oplus id} & R[\Delta^n]_t^N \oplus R[\Delta^n]_t^d \\ & \searrow^{F_t} & \uparrow{\scriptstyle id\oplus\frac{1}{t}id} \\ & & R[\Delta^n]_t^N \oplus R[\Delta^n]_t^d \end{array}$$

where $g\colon R[\Delta^n]_t^d \to R[\Delta^n]_t^N$ is the map that sends e_i to x_i. The right arrow of the triangle is an isomorphism and $g \oplus id$ is an admissible embedding. Then F_t is an admissible embedding, and hence so is f_t. By Lemma A.7 f is an admissible embedding. □

A.9. **Lemma.** *Suppose X is an affine variety over k, A and Y are equidimensional flat affine X-schemes, $A \to X$ is finite, Y is Cohen–Macaulay, and $f\colon A \to Y$ is a morphism over X. Suppose Z is a closed subset of X and the map on the fiber products $f_Z\colon A \times_X Z \to Y \times_X Z$ and $A \times_X (X - Z) \to Y \times_X (X - Z)$ are l.c.i. embeddings. Then f is an l.c.i. embedding.*

Proof. Denote by $n = \dim Y - \dim A$ and let A_Z (resp. Y_Z, A_{X-Z}, Y_{X-Z}) be the fiber product $A \times_X Z$ (respectively $Y \times_X Z$, $A \times_X (X - Z)$, $Y \times_X (X - Z)$). Let us check that $k[Y] \to k[A]$ is surjective. For every point $x \in X$ if $x \in X - Z$, then the localization map $k[Y]_x \to k[A]_x$ is surjective. If $x \in Z$, then the map $k[Y] \otimes_{k[X]} k(x) \to k[A] \otimes_{k[X]} k(x)$ is surjective. It follows from Nakayama's lemma that the map on localizations $k[Y]_x \to k[A]_x$ is surjective. Then $k[Y] \to k[A]$ is surjective, hence $A \to Y$ is a closed embedding. Let I denote the kernel of $k[Y] \to k[A]$. For every point $y \in Y$ if y is in Y_{X-Z}, then I_y is generated by a regular sequence of length n. If y is in Y_Z, the sequence

$$0 \to I_y \otimes_{k[X]} k[Z] \to k[Y_Z]_y \to k[A_Z]_y \to 0$$

is exact, because $k[A]$ is flat over $k[X]$. Then $I_y \otimes_{k[X]} k[Z]$ is generated by n elements over $k[Y_Z]_y$, hence $I_y \otimes_{k[Y]_y} k(y)$ is generated by n elements. By Nakayama's lemma I_y is generated by n elements. Since A has codimension n in Y, these elements form a regular sequence [1, III.4.5]. Then A is an l.c.i. subscheme in Y. □

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Department of Mathematics, Swansea University, Fabian Way, Swansea SA1 8EN, UK
E-mail address: g.garkusha@swansea.ac.uk

Department of Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada
E-mail address: alexander.neshitov@gmail.com