



Particle-Hole Transformation in the Continuum and Determinantal Point Processes

Maryam Gharamah Ali Alshehri¹, Eugene Lytvynov² 

¹ Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk, Kingdom of Saudi Arabia.
E-mail: mgalshehri@ut.edu.sa

² Department of Mathematics, Swansea University, Swansea, UK.
E-mail: e.lytvynov@swansea.ac.uk

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Abstract: Let X be an underlying space with a reference measure σ . Let K be an integral operator in $L^2(X, \sigma)$ with integral kernel $K(x, y)$. A point process μ on X is called determinantal with the correlation operator K if the correlation functions of μ are given by $k^{(n)}(x_1, \dots, x_n) = \det[K(x_i, x_j)]_{i,j=1,\dots,n}$. It is known that each determinantal point process with a self-adjoint correlation operator K is the joint spectral measure of the particle density $\rho(x) = \mathcal{A}^+(x)\mathcal{A}^-(x)$ ($x \in X$), where the operator-valued distributions $\mathcal{A}^+(x)$, $\mathcal{A}^-(x)$ come from a gauge-invariant quasi-free representation of the canonical anticommutation relations (CAR). If the space X is discrete and divided into two disjoint parts, X_1 and X_2 , by exchanging particles and holes on the X_2 part of the space, one obtains from a determinantal point process with a self-adjoint correlation operator K the determinantal point process with the J -self-adjoint correlation operator $\widehat{K} = KP_1 + (1 - K)P_2$. Here P_i is the orthogonal projection of $L^2(X, \sigma)$ onto $L^2(X_i, \sigma)$. In the case where the space X is continuous, the exchange of particles and holes makes no sense. Instead, we apply a Bogoliubov transformation to a gauge-invariant quasi-free representation of the CAR. This transformation acts identically on the X_1 part of the space and exchanges the creation operators $\mathcal{A}^+(x)$ and the annihilation operators $\mathcal{A}^-(x)$ for $x \in X_2$. This leads to a quasi-free representation of the CAR, which is not anymore gauge-invariant. We prove that the joint spectral measure of the corresponding particle density is the determinantal point process with the correlation operator \widehat{K} .

1. Introduction

Let X be an underlying space with a reference measure σ . (Typically $X = \mathbb{R}^d$ and σ is the Lebesgue measure.) Let Γ_X denote the space of configurations in X , i.e., locally finite subsets of X . A point process in X is a probability measure on Γ_X , see e.g. [13]. A point process μ is called *determinantal* if the correlation functions of μ are given by

$$k_\mu^{(n)}(x_1, \dots, x_n) = \det [K(x_i, x_j)]_{i,j=1}^n, \quad n \in \mathbb{N}, \quad (1)$$

see e.g. [6, 14, 23, 26]. The function $K(x, y)$ is called the *correlation kernel* of μ . To study μ , one usually considers a (bounded) integral operator K in the (complex) space $\mathcal{H} := L^2(X, \sigma)$ with integral kernel $K(x, y)$. One calls K the *correlation operator* of μ .

Assume that the correlation kernel $K(x, y)$ is Hermitian, i.e., $K(x, y) = \overline{K(y, x)}$, equivalently the operator K is self-adjoint. Then, the Macchi–Soshnikov theorem [23, 26] gives a necessary and sufficient condition of existence of a determinantal point process μ with the correlation kernel $K(x, y)$.

It was shown in [18, 20] (see also [6, 27]) that this point process μ is the joint spectral measure of the particle density of a gauge-invariant quasi-free representation of the canonical anticommutation relations (CAR). More precisely, assume that operators $\mathcal{A}^+(\varphi)$ and $\mathcal{A}^-(\varphi)$ ($\varphi \in \mathcal{H}$) satisfy the CAR:

$$\mathcal{A}^-(\varphi) = (\mathcal{A}^+(\varphi))^*, \quad (2)$$

$$\{\mathcal{A}^+(\varphi), \mathcal{A}^+(\psi)\} = \{\mathcal{A}^-(\varphi), \mathcal{A}^-(\psi)\} = 0, \quad \{\mathcal{A}^-(\varphi), \mathcal{A}^+(\psi)\} = (\psi, \varphi)_{\mathcal{H}}. \quad (3)$$

Here $\{A, B\} = AB + BA$ is the anticommutator. Assume that the operators $\mathcal{A}^+(\varphi)$ and $\mathcal{A}^-(\varphi)$ act in $\mathcal{AF}(\mathcal{H} \oplus \mathcal{H})$, the antisymmetric Fock space over $\mathcal{H} \oplus \mathcal{H}$. Let \mathbf{A} be the CAR $*$ -algebra generated by these operators, and let τ be the vacuum state on \mathbf{A} . Furthermore, assume that the operators $\mathcal{A}^+(\varphi)$ and $\mathcal{A}^-(\varphi)$ are such that the state τ is gauge-invariant quasi-free, i.e.,

$$\tau(\mathcal{A}^+(\varphi_m) \cdots \mathcal{A}^+(\varphi_1) \mathcal{A}^-(\psi_1) \cdots \mathcal{A}^-(\psi_n)) = \delta_{m,n} \det [(K\varphi_i, \psi_j)_{\mathcal{H}}]_{i,j=1}^n, \quad (4)$$

where $\delta_{m,n}$ is the Kronecker symbol and K is the self-adjoint bounded linear operator in \mathcal{H} satisfying

$$(K\varphi, \psi)_{\mathcal{H}} = \tau(\mathcal{A}^+(\varphi)\mathcal{A}^-(\psi)), \quad \varphi, \psi \in \mathcal{H}. \quad (5)$$

See [2] or [12, Subsection 5.2.3].

Define operator-valued distributions $\mathcal{A}^+(x)$ and $\mathcal{A}^-(x)$ on X by

$$\mathcal{A}^+(\varphi) = \int_X \varphi(x) \mathcal{A}^+(x) \sigma(dx), \quad \mathcal{A}^-(\varphi) = \int_X \overline{\varphi(x)} \mathcal{A}^-(x) \sigma(dx), \quad \varphi \in \mathcal{H}. \quad (6)$$

The corresponding particle density is formally defined as the operator-valued distribution $\rho(x) := \mathcal{A}^+(x)\mathcal{A}^-(x)$, and in the smeared form,

$$\rho(\Delta) = \int_{\Delta} \rho(x) \sigma(dx) = \int_{\Delta} \mathcal{A}^+(x) \mathcal{A}^-(x) \sigma(dx),$$

where $\Delta \subset X$ is measurable and pre-compact. Note that, at least formally, each operator $\rho(\Delta)$ is Hermitian and for any sets Δ_1 and Δ_2 , the operators $\rho(\Delta_1)$ and $\rho(\Delta_2)$ commute. The main result of [20] was that, if K is an integral operator and its integral (Hermitian) kernel $K(x, y)$ is such that the corresponding determinantal point process μ exists, then the operators $\rho(\Delta)$ are well-defined, essentially self-adjoint, commuting, and furthermore

$$\tau(\rho(\Delta_1) \cdots \rho(\Delta_n)) = \int_{\Gamma_X} \gamma(\Delta_1) \cdots \gamma(\Delta_n) \mu(d\gamma), \quad (7)$$

where $\gamma(\Delta) := |\gamma \cap \Delta|$, the number of points of the configuration γ that belong to Δ . Formula (7) states that the moments of the operators $\rho(\Delta)$ under the gauge-invariant

quasi-free state τ are equal to the moments of the determinantal point process μ . It should be stressed that the set of all monomials $\gamma(\Delta_1) \cdots \gamma(\Delta_n)$ is total in $L^2(\Gamma_X, \mu)$.

For each measurable, pre-compact set $\Delta \subset X$, we denote by $\tilde{\rho}(\Delta)$ the closure of the operator $\rho(\Delta)$. (Note that the operators $\tilde{\rho}(\Delta)$ are self-adjoint.) According to [4, Chapter 3], formula (7) means that the determinantal point process μ is the joint spectral measure of the operators $\tilde{\rho}(\Delta)$.

Let us now assume that the underlying space X is divided into two disjoint parts, X_1 and X_2 . For $i = 1, 2$, let P_i denote the orthogonal projection of \mathcal{H} onto $\mathcal{H}_i := L^2(X_i, \sigma)$, and let $J := P_1 - P_2$. An (*indefinite*) J -scalar product in \mathcal{H} is defined by

$$[f, g] := (Jf, g)_{\mathcal{H}} = (P_1 f, P_1 g)_{\mathcal{H}} - (P_2 f, P_2 g)_{\mathcal{H}}, \quad f, g \in \mathcal{H},$$

see e.g. [3]. A bounded linear operator $\mathbb{K} \in \mathcal{L}(\mathcal{H})$ is called J -self-adjoint if $[\mathbb{K}f, g] = [f, \mathbb{K}g]$ for all $f, g \in \mathcal{H}$. If an integral operator $\mathbb{K} \in \mathcal{L}(\mathcal{H})$ is J -self-adjoint, then its integral kernel $\mathbb{K}(x, y)$ is called J -Hermitian. The integral kernel $\mathbb{K}(x, y)$ of an integral operator $\mathbb{K} \in \mathcal{L}(\mathcal{H})$ is J -Hermitian if and only if $\mathbb{K}(x, y) = \overline{\mathbb{K}(y, x)}$ for x and y that belong to the same part X_i and $\mathbb{K}(x, y) = -\overline{\mathbb{K}(y, x)}$ if $x \in X_i, y \in X_j$ with $i \neq j$.

For an arbitrary bounded linear operator $K \in \mathcal{L}(\mathcal{H})$, we define

$$\widehat{K} := K P_1 + (1 - K) P_2.$$

Note that the transformation $K \mapsto \widehat{K}$ is an involution. If an operator K is self-adjoint, then \widehat{K} is J -self-adjoint, and if an operator $\mathbb{K} \in \mathcal{L}(\mathcal{H})$ is J -self-adjoint, then $\widehat{\mathbb{K}}$ is self-adjoint.

A necessary and sufficient condition of existence of a determinantal point process with a J -Hermitian correlation kernel $\mathbb{K}(x, y)$ was given in [19]. We note that J -Hermitian correlation kernels naturally arise in the context of asymptotic representation theory of classical groups, such as symmetric or unitary groups of growing rank, see [8–11, 22].

Assume, for a moment, that the underlying space X is discrete and σ is the counting measure. Then any determinantal point process with a J -Hermitian correlation kernel can be obtained from a determinantal point process with a Hermitian correlation kernel through a particle-hole transformation. More precisely, let γ be a configuration in X , i.e., $\gamma \subset X$. We define a new configuration

$$I\gamma := (\gamma \cap X_1) \cup (X_2 \setminus \gamma), \quad (8)$$

i.e., $I\gamma$ coincides with γ in X_1 , and with the holes of γ (the points unoccupied by γ) in X_2 . Note that $I : \Gamma_X \rightarrow \Gamma_X$ is an involution. It was shown in [7] that, if μ is a determinantal point process with a correlation operator K , then $I_*\mu$, the pushforward of μ induced by I , is the determinantal point process with the correlation operator \widehat{K} . Therefore, μ is a determinantal point process with a J -Hermitian correlation kernel $\mathbb{K}(x, y)$ if and only if $\mu = I_*\nu$, where ν is the determinantal point process with the Hermitian correlation kernel $\widehat{\mathbb{K}}(x, y)$. (Note that $\nu = I_*\mu$.)

If the underlying space X is continuous and σ is a non-atomic measure, then a direct generalization of the above result is impossible. Indeed, for a configuration γ in X , the set $I\gamma$ defined by (8) is uncountable, hence it is not anymore a configuration in X . Furthermore, the Macchi–Soshnikov theorem and [19, Theorem 3] imply that, if $K(x, y)$, the correlation kernel of a determinantal point process, is either Hermitian or J -Hermitian, then the operator \widehat{K} is not even an integral operator, i.e., $\widehat{K}(x, y)$ does not exist.

The aim of this paper is to prove the following main result, which involves a Bogoliubov transformation of the CAR $*$ -algebra \mathbf{A} . We refer the reader to e.g. [12, Section 5.2.2] for the definition of a Bogoliubov transformation and a discussion of its properties.

Main result. *Let $\mathbb{K} \in \mathcal{L}(\mathcal{H})$ be a J -self-adjoint integral operator, and assume that the J -Hermitian integral kernel $\mathbb{K}(x, y)$ of the operator \mathbb{K} is the correlation kernel of a determinantal point process μ . For the self-adjoint operator $K := \widehat{\mathbb{K}}$, consider the corresponding gauge-invariant quasi-free representation of the CAR, i.e., the $*$ -algebra \mathbf{A} is generated by operators $\mathcal{A}^+(\varphi), \mathcal{A}^-(\varphi)$ in $\mathcal{AF}(\mathcal{H} \oplus \mathcal{H})$ that satisfy (2), (3) and, for the vacuum state τ on \mathbf{A} , formulas (4), (5) hold. Define a Bogoliubov transformation of the CAR $*$ -algebra \mathbf{A} by*

$$(\mathcal{A}^+(\varphi), \mathcal{A}^-(\varphi)) \mapsto (A^+(\varphi), A^-(\varphi)), \quad (9)$$

where for each $\varphi \in \mathcal{H}$,

$$A^+(\varphi) := \mathcal{A}^+(P_1\varphi) + \mathcal{A}^-(P_2\mathcal{C}\varphi), \quad A^-(\varphi) := \mathcal{A}^-(P_1\varphi) + \mathcal{A}^+(P_2\mathcal{C}\varphi). \quad (10)$$

Here $(\mathcal{C}\varphi)(x) := \overline{\varphi(x)}$ is the complex conjugation. (Note that the vacuum state τ on the CAR $*$ -algebra \mathbf{A} generated by the operators $A^+(\varphi), A^-(\varphi)$ ($\varphi \in \mathcal{H}$) is still quasi-free but not anymore gauge-invariant.) Let operator-valued distributions $A^+(x)$ and $A^-(x)$ be determined by $A^+(\varphi), A^-(\varphi)$ similarly to (6). Then the corresponding particle density

$$\rho(\Delta) = \int_{\Delta} A^+(x)A^-(x) \sigma(dx) \quad (\Delta \subset X \text{ measurable and pre-compact}) \quad (11)$$

is a family of well-defined, essentially self-adjoint, commuting operators in $\mathcal{AF}(\mathcal{H} \oplus \mathcal{H})$. Furthermore, for these operators $\rho(\Delta)$ and the determinantal point process μ with the correlation kernel $\mathbb{K}(x, y)$, formula (7) holds. In other words, the determinantal point process μ is the joint spectral measure of the family of self-adjoint operators $\tilde{\rho}(\Delta)$.

Note that formula (10) implies that

$$A^+(x) = \begin{cases} \mathcal{A}^+(x), & \text{if } x \in X_1, \\ \mathcal{A}^-(x), & \text{if } x \in X_2 \end{cases}, \quad A^-(x) = \begin{cases} \mathcal{A}^-(x), & \text{if } x \in X_1, \\ \mathcal{A}^+(x), & \text{if } x \in X_2. \end{cases} \quad (12)$$

In words, on the X_1 part of X we use the creation and annihilation operators of the original representation of the CAR, while on the X_2 part of X we exchange the creation and annihilation operators of the original representation. Hence, the Bogoliubov transformation (9), (10) can be thought of as a counterpart of the involution I defined by (8) in the case where the space X is discrete.

In fact, in our derivation of the main result, the existence of the determinantal point process μ follows from a general theorem regarding the joint spectral measure of a family of self-adjoint commuting operators, see [20, Theorem 1]. Hence, as a by-product of our considerations, we obtain a new proof of existence of a determinantal point process with a J -Hermitian correlation kernel.

In the case of a discrete space X , Koshida [16] proved that each pfaffian point process appears (in the terminology of the present paper) as the joint spectral measure of the particle density of a quasi-free representation of the CAR. Koshida notes: ‘it seems highly nontrivial if our construction can be extended to the case of continuous systems.’ While the present paper does not provide a solution to this problem, it still solves it for a particular class of (non-gauge-invariant) quasi-free states.

The paper is organized as follows. In Sect. 2, we discuss necessary preliminaries regarding determinantal point processes, J -Hermitian correlation kernels, the correlation measures and the joint spectral measure of a family of commuting self-adjoint operators, and quasi-free states on the CAR algebra.

In Sect. 3, we employ heuristic considerations, involving formula (11), in order to give a rigorous definition of Hermitian operators $\rho(\Delta)$. We also prove that these operators (algebraically) commute.

In Sect. 4, we derive rigorous formulas for the Wick (normal) product

$$:\rho(\Delta_1) \cdots \rho(\Delta_n): = \int_{\Delta_1 \times \cdots \times \Delta_n} A^+(x_n) \cdots A^+(x_1) A^-(x_1) \cdots A^-(x_n) \sigma(dx_1) \cdots \sigma(dx_n). \quad (13)$$

In Sect. 5, we formulate the main theorem of the paper (Theorem 5.2), which states that the the determinantal point process μ with the J -Hermitian correlation kernel $\mathbb{K}(x, y)$ is the joint spectral measure of the family of the commuting self-adjoint operators $\tilde{\rho}(\Delta)$. We start proving this result in Sect. 5.

Finally, in Sect. 6, we prove that the operators $\rho(\Delta)$ possess correlation functions and these are given by the right-hand side of formula (1) in which $K(x, y)$ is replaced by $\mathbb{K}(x, y)$. This concludes the proof of our main result.

2. Preliminaries

2.1. Determinantal point processes. Let X be a locally compact Polish space, let $\mathcal{B}(X)$ be the Borel σ -algebra on X , and let $\mathcal{B}_0(X)$ denote the collection of all sets from $\mathcal{B}(X)$ which are pre-compact. The *configuration space* over X is defined as the set of all locally finite subsets of X :

$$\Gamma_X := \{\gamma \subset X \mid \text{for all } \Delta \in \mathcal{B}_0(X) \mid \gamma \cap \Delta \mid < \infty\}.$$

Here, for a set Λ , $|\Lambda|$ denotes its capacity. Elements $\gamma \in \Gamma_X$ are called *configurations*. One identifies each configuration $\gamma = \{x_i\}_{i \geq 1}$ with the measure $\gamma = \sum_i \delta_{x_i}$ on X . Here, for $x \in X$, δ_x denotes the Dirac measure with mass at x . Through this identification, one gets the embedding of Γ_X into the space of all Radon (i.e., locally finite) measures on X .

The space Γ_X is endowed with the vague topology, i.e., the weakest topology on Γ_X with respect to which all maps $\Gamma_X \ni \gamma \mapsto \langle \gamma, f \rangle = \sum_{x \in \gamma} f(x)$, $f \in C_0(X)$, are continuous. Here $C_0(X)$ is the space of all continuous real-valued functions on X with compact support. We will denote by $\mathcal{B}(\Gamma_X)$ the Borel σ -algebra on Γ_X . A probability measure μ on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ is called a *point process* on X . For more detail, see e.g. [13, 15].

A point process μ can be described with the help of its correlation measures. Denote $X^{(n)} := \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}$. The n -th *correlation measure* of μ is the symmetric measure $\theta^{(n)}$ on $X^{(n)}$ that satisfies

$$\int_{\Gamma_X} \sum_{\{x_1, \dots, x_n\} \subset \gamma} f^{(n)}(x_1, \dots, x_n) \mu(d\gamma) = \int_{X^{(n)}} f^{(n)}(x_1, \dots, x_n) \theta^{(n)}(dx_1 \cdots dx_n)$$

for all measurable symmetric functions $f^{(n)} : X^{(n)} \rightarrow [0, \infty)$. Let σ be a reference Radon measure on $(X, \mathcal{B}(X))$. If the correlation measure $\theta^{(n)}$ has density $k^{(n)} : X^{(n)} \rightarrow$

$[0, \infty)$ with respect to $\frac{1}{n!} \sigma^{\otimes n}$, then $k^{(n)}$ is called the n -th correlation function of μ . Under a mild condition on the growth of correlation measures as $n \rightarrow \infty$, they determine a point process uniquely [17].

Recall that a point process μ is called *determinantal* if there exists a complex-valued function $K(x, y)$ on X^2 , called the *correlation kernel*, such that (1) holds, see e.g. [6, 26]. The integral operator K in the complex L^2 -space $\mathcal{H} = L^2(X, \sigma)$ which has integral kernel $K(x, y)$ is called the *correlation operator* of μ .

Note that, for a given integral operator K in \mathcal{H} , the integral kernel $K(x, y)$ is defined up to a set of zero measure $\sigma^{\otimes 2}$. When calculating the value of $\det [K(x_i, x_j)]_{i,j=1}^n$, one has to use the values of $K(\cdot, \cdot)$ on the diagonal $\{(x, x) \in X^2 \mid x \in X\}$. However, the latter set is of zero measure $\sigma^{\otimes 2}$ if the measure σ is non-atomic, i.e., $\sigma(\{x\}) = 0$ for all $x \in X$. Hence, when speaking about the correlation operator K of a determinantal point process, one has to properly choose the values of the integral kernel of K on the diagonal in X^2 .

2.2. J -Hermitian correlation kernels. Assume that the underlying space X is split into two disjoint measurable parts, X_1 and X_2 , of positive measure σ . Just as in Introduction, we denote by P_i the orthogonal projection of \mathcal{H} onto $\mathcal{H}_i = L^2(X_i, \sigma)$, and we let $J = P_1 - P_2$.

According to the orthogonal sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, each operator $A \in \mathcal{L}(\mathcal{H})$ can be represented in the block form,

$$A = \begin{bmatrix} A^{11} & A^{21} \\ A^{12} & A^{22} \end{bmatrix}, \quad (14)$$

where $A^{ij} : \mathcal{H}_j \rightarrow \mathcal{H}_i$, $i, j = 1, 2$. Here $A^{ij} := P_i A P_j$. Then the operator A being J -self-adjoint means that $(A^{ii})^* = A^{ii}$ ($i = 1, 2$) and $(A^{21})^* = -A^{12}$.

We denote by $\mathcal{S}_1(\mathcal{H})$ the set of all trace-class operators in \mathcal{H} , and by $\mathcal{S}_2(\mathcal{H})$ the set of all Hilbert–Schmidt operators in \mathcal{H} . For $\Delta \in \mathcal{B}_0(X)$, we denote by P_Δ the orthogonal projection of \mathcal{H} onto $L^2(\Delta, \sigma)$. For $i = 1, 2$, we denote $\mathcal{B}_0(X_i) := \{\Delta \in \mathcal{B}_0(X) \mid \Delta \subset X_i\}$.

We say that an operator $K \in \mathcal{L}(\mathcal{H})$ is *locally trace-class* on X_i ($i = 1, 2$) if, for each $\Delta_i \in \mathcal{B}_0(X_i)$, we have $K_{\Delta_i} := P_{\Delta_i} K P_{\Delta_i} \in \mathcal{S}_1(\mathcal{H})$.

The following lemma can be easily checked by using basic properties of trace-class and Hilbert–Schmidt operators, see e.g. [25].

Lemma 2.1. *Let $K \in \mathcal{L}(\mathcal{H})$ satisfy $\mathbf{0} \leq K \leq \mathbf{1}$. Define $K_1 := \sqrt{K}$, $K_2 := \sqrt{\mathbf{1} - K}$. Then the following statements are equivalent.*

- (i) *The operator K is locally trace-class on X_1 and the operator $\mathbf{1} - K$ is locally trace-class on X_2 .*
- (ii) *For any $\Delta_i \in \mathcal{B}_0(X_i)$ ($i = 1, 2$), we have $K_i P_{\Delta_i} \in \mathcal{S}_2(\mathcal{H})$, or equivalently $P_{\Delta_i} K_i \in \mathcal{S}_2(\mathcal{H})$.*

Note that, in Lemma 2.1, $K_i P_i$ and $P_i K_i$ ($i = 1, 2$) are integral operators and their respective integral kernels $K_i(x, y)$ with $(x, y) \in (X \times X_i) \cup (X_i \times X)$ satisfy

$$\int_{(X \times \Delta_i) \cup (\Delta_i \times X)} |K_i(x, y)|^2 \sigma(dx) \sigma(dy) < \infty, \quad (15)$$

for any $\Delta_i \in \mathcal{B}_0(X_i)$. Without loss of generality, we may assume that $K_i(x, y) = \overline{K_i(y, x)}$ for all $(x, y) \in (X \times X_i) \cup (X_i \times X)$, and $\int_X |K(x, y)|^2 \sigma(dy) < \infty$ for all $x \in X_i$.

Now consider a J -self-adjoint operator $\mathbb{K} \in \mathcal{L}(\mathcal{H})$ and denote $K := \widehat{\mathbb{K}} = \mathbb{K}P_1 + (1 - \mathbb{K})P_2$. Note that $\mathbb{K}^{11} = K^{11}$, $\mathbb{K}^{22} = (1 - K)^{22}$, $\mathbb{K}^{21} = K^{21}$, and $\mathbb{K}^{12} = -K^{12} = -(K^{21})^*$. Let us assume that the operator K satisfies the assumptions of Lemma 2.1, equivalently the operator \mathbb{K} is locally trace-class on both X_1 and X_2 and $\mathbf{0} \leq \widehat{\mathbb{K}} \leq \mathbf{1}$.

Let us show that \mathbb{K} is an integral operator, and let us present an integral kernel of \mathbb{K} . For $i = 1, 2$, we set

$$\mathbb{K}(x, y) = \int_X K_i(x, z)K_i(z, y) \sigma(dz), \quad (x, y) \in X_i^2, \quad (16)$$

which is an integral kernel of \mathbb{K}^{ii} . Note that, for all $(x, y) \in X_i^2$, we have $\mathbb{K}(y, x) = \overline{\mathbb{K}(x, y)}$. Next, for any $\Delta_i \in \mathcal{B}_0(X_i)$ ($i = 1, 2$), we have $P_{\Delta_2}\mathbb{K}P_{\Delta_1} = P_{\Delta_2}K P_{\Delta_1} \in \mathcal{S}_2(\mathcal{H})$. Hence, \mathbb{K}^{21} is an integral operator. We choose an arbitrary integral kernel of \mathbb{K}^{21} , denoted by $\mathbb{K}(x, y)$ with $x \in X_2$ and $y \in X_1$. Finally, we set $\mathbb{K}(x, y) = -\overline{\mathbb{K}(y, x)}$ for $x \in X_1$ and $y \in X_2$. Thus, we have constructed an integral kernel of the operator \mathbb{K} .

The following theorem is shown in [19, Theorem 2].

Theorem 2.2. *Let a J -self-adjoint operator $\mathbb{K} \in \mathcal{L}(\mathcal{H})$ be locally trace-class on both X_1 and X_2 and such that $\mathbf{0} \leq \widehat{\mathbb{K}} \leq \mathbf{1}$. Let the integral kernel $\mathbb{K}(x, y)$ of the integral operator \mathbb{K} be chosen as above. Then there exists a unique determinantal point process with the correlation kernel $\mathbb{K}(x, y)$.*

Remark 2.3. In fact, the conditions of Theorem 2.2 are necessary for the existence of a determinantal point process with a J -Hermitian correlation kernel, see [19, Theorem 3].

Remark 2.4. While Theorem 2.2 will serve as a motivation for our studies, we will not actually use it and will derive the existence of a determinantal point process as in Theorem 2.2 by methods different to [19]. Nevertheless, we will use the choice of the integral kernel $\mathbb{K}(x, y)$ as described above.

2.3. Joint spectral measure of a family of commuting self-adjoint operators. Let us now present a result from [20] on the joint spectral measure of a family of commuting self-adjoint operators. Our brief presentation essentially follows [1, Section 4].

Let \mathcal{F} be a separable Hilbert space and let \mathcal{D} be a dense subspace of \mathcal{F} . For each $\Delta \in \mathcal{B}_0(X)$, let $\rho(\Delta) : \mathcal{D} \rightarrow \mathcal{D}$ be a linear Hermitian operator in \mathcal{F} . We further assume:

- for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$ with $\Delta_1 \cap \Delta_2 = \emptyset$, we have $\rho(\Delta_1 \cup \Delta_2) = \rho(\Delta_1) + \rho(\Delta_2)$;
- the operators $\rho(\Delta)$ commute, i.e., $[\rho(\Delta_1), \rho(\Delta_2)] = 0$ for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$.

Let \mathcal{A} denote the (commutative) $*$ -algebra generated by $(\rho(\Delta))_{\Delta \in \mathcal{B}_0(X)}$. Let Ω be a fixed vector in \mathcal{D} with $\|\Omega\|_{\mathcal{F}} = 1$, and let a state $\tau : \mathcal{A} \rightarrow \mathbb{C}$ be defined by $\tau(a) := (a\Omega, \Omega)_{\mathcal{F}}$ for $a \in \mathcal{A}$.

We define *Wick polynomials* in \mathcal{A} by the following recurrence formula:

$$\begin{aligned} &:\rho(\Delta): = \rho(\Delta), \\ :\rho(\Delta_1) \cdots \rho(\Delta_{n+1}): &= \rho(\Delta_{n+1}) : \rho(\Delta_1) \cdots \rho(\Delta_n): \end{aligned}$$

$$- \sum_{i=1}^n : \rho(\Delta_1) \cdots \rho(\Delta_{i-1}) \rho(\Delta_i \cap \Delta_{n+1}) \rho(\Delta_{i+1}) \cdots \rho(\Delta_n) :, \quad (17)$$

where $\Delta, \Delta_1, \dots, \Delta_{n+1} \in \mathcal{B}_0(X)$ and $n \in \mathbb{N}$. It is easy to see that, for each permutation $\pi \in S_n$,

$$: \rho(\Delta_1) \cdots \rho(\Delta_n) : = : \rho(\Delta_{\pi(1)}) \cdots \rho(\Delta_{\pi(n)}) :.$$

We assume that, for each $n \in \mathbb{N}$, there exists a symmetric measure $\theta^{(n)}$ on X^n that is concentrated on $X^{(n)}$ (i.e., $\theta^{(n)}(X^n \setminus X^{(n)}) = 0$) and such that

$$\theta^{(n)}(\Delta_1 \times \cdots \times \Delta_n) = \frac{1}{n!} \tau(: \rho(\Delta_1) \cdots \rho(\Delta_n) :), \quad \Delta_1, \dots, \Delta_n \in \mathcal{B}_0(X). \quad (18)$$

Note that, if the measure $\theta^{(n)}$ exists, then it is unique. The $\theta^{(n)}$ is called the *n-th correlation measure of the operators* $\rho(\Delta)$. If $\theta^{(n)}$ has a density $k^{(n)}$ with respect to $\frac{1}{n!} \sigma^{\otimes n}$, then $k^{(n)}$ is called the *n-th correlation function of the operators* $\rho(\Delta)$.

Theorem 2.5. [20] *Let $(\rho(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ be a family of Hermitian operators in \mathcal{F} as above. In particular, these operators have correlation measures $(\theta^{(n)})_{n=1}^{\infty}$ respective the state τ . Furthermore, we assume that the following two conditions are satisfied.*

(LB1) *For each $\Delta \in \mathcal{B}_0(X)$, there exists a constant $C_{\Delta} > 0$ such that*

$$\theta^{(n)}(\Delta^n) \leq C_{\Delta}^n, \quad n \in \mathbb{N}. \quad (19)$$

(LB2) *For any sequence $\{\Delta_l\}_{l \in \mathbb{N}} \subset \mathcal{B}_0(X)$ such that $\Delta_l \downarrow \emptyset$ (i.e., $\Delta_1 \supset \Delta_2 \supset \Delta_3 \supset \cdots$ and $\bigcap_{l=1}^{\infty} \Delta_l = \emptyset$), we have $C_{\Delta_l} \rightarrow 0$ as $l \rightarrow \infty$.*

Then the following statements hold.

- (i) *Let $\mathfrak{D} := \{a\Omega \mid a \in \mathcal{A}\}$ and let \mathfrak{F} denote the closure of \mathfrak{D} in \mathcal{F} . Each operator $(\rho(\Delta), \mathfrak{D})$ is essentially self-adjoint in \mathfrak{F} , i.e., the closure of $\rho(\Delta)$, denoted by $\tilde{\rho}(\Delta)$, is a self-adjoint operator in \mathfrak{F} .*
- (ii) *For any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, the projection-valued measures (resolutions of the identity) of the operators $\tilde{\rho}(\Delta_1)$ and $\tilde{\rho}(\Delta_2)$ commute.*
- (iii) *There exist a unique point process μ on X and a unique unitary operator $U : \mathfrak{F} \rightarrow L^2(\Gamma_X, \mu)$ satisfying $U\Omega = 1$ and*

$$U(\rho(\Delta_1) \cdots \rho(\Delta_n)\Omega) = \gamma(\Delta_1) \cdots \gamma(\Delta_n) \quad (20)$$

for any $\Delta_1, \dots, \Delta_n \in \mathcal{B}_0(X)$ ($n \in \mathbb{N}$). In particular,

$$\tau(\rho(\Delta_1) \cdots \rho(\Delta_n)) = \int_{\Gamma_X} \gamma(\Delta_1) \cdots \gamma(\Delta_n) \mu(d\gamma). \quad (21)$$

- (iv) *The correlations measures of the point process μ are $(\theta^{(n)})_{n=1}^{\infty}$.*

According to [4, Chapter 3], the point process μ from Theorem 2.5 is the *joint spectral measure of the family of commuting self-adjoint operators* $(\tilde{\rho}(\Delta))_{\Delta \in \mathcal{B}_0(X)}$.

2.4. *Quasi-free states on the CAR algebra.* Let \mathcal{F} be a separable Hilbert space, and let $a^+(\varphi)$ and $a^-(\varphi)$ ($\varphi \in \mathcal{H}$) be bounded linear operators in \mathcal{F} such that $a^+(\varphi)$ linearly depends on φ and $a^-(\varphi) = (a^+(\varphi))^*$. Let $a^+(\varphi)$ and $a^-(\varphi)$ satisfy the CAR, i.e., formula (3) holds in which the operators $\mathcal{A}^+(\varphi)$, $\mathcal{A}^-(\varphi)$ are replaced by $a^+(\varphi)$, $a^-(\varphi)$. Let \mathbb{A} be the $*$ -algebra generated by these operators. We define *field operators* $b(\varphi) := a^+(\varphi) + a^-(\varphi)$ ($\varphi \in \mathcal{H}$). As easily seen, these operators also generate \mathbb{A} .

Let $\tau : \mathbb{A} \rightarrow \mathbb{C}$ be a state on \mathbb{A} . The state τ is completely determined by the functionals $T^{(n)} : \mathcal{H}^n \rightarrow \mathbb{C}$ ($n \in \mathbb{N}$) defined by

$$T^{(n)}(\varphi_1, \dots, \varphi_n) := \tau(b(\varphi_1) \cdots b(\varphi_n)). \quad (22)$$

The state τ is called *quasi-free* if

$$T^{(2n-1)} = 0, \quad (23)$$

$$T^{(2n)}(\varphi_1, \dots, \varphi_{2n}) = \sum (-1)^{\text{Cross}(\nu)} T^{(2)}(\varphi_{i_1}, \varphi_{j_1}) \cdots T^{(2)}(\varphi_{i_n}, \varphi_{j_n}), \quad n \in \mathbb{N}, \quad (24)$$

where the summation is over all partitions $\nu = \{\{i_1, j_1\}, \dots, \{i_n, j_n\}\}$ of $\{1, \dots, 2n\}$ with $i_k < j_k$ ($k = 1, \dots, n$) and $\text{Cross}(\nu)$ denotes the number of all crossings in ν , i.e., the number of all choices of $\{i_k, j_k\}, \{i_l, j_l\} \in \nu$ such that $i_k < i_l < j_k < j_l$, see e.g. [12, Section 5.2.3].

The state τ is called *gauge-invariant* if, for each $q \in \mathbb{C}$ with $|q| = 1$, we have $T^{(n)}(q\varphi_1, \dots, q\varphi_n) = T^{(n)}(\varphi_1, \dots, \varphi_n)$ for all $\varphi_1, \dots, \varphi_n \in \mathcal{H}$, $n \in \mathbb{N}$. The state τ can also be uniquely characterized by the n -point functions $S^{(m,n)} : \mathcal{H}^{m+n} \rightarrow \mathbb{C}$ ($m+n \geq 1$) defined by

$$S^{(m,n)}(\varphi_1, \dots, \varphi_m, \psi_1, \dots, \psi_n) := \tau(a^+(\varphi_1) \cdots a^+(\varphi_m) a^-(\psi_1) \cdots a^-(\psi_n)). \quad (25)$$

The state τ is gauge-invariant quasi-free if and only if

$$S^{(m,n)}(\varphi_m, \dots, \varphi_1, \psi_1, \dots, \psi_n) = \delta_{m,n} \det \left[S^{(1,1)}(\varphi_i, \psi_j) \right]_{i,j=1, \dots, n}. \quad (26)$$

Let us briefly recall the Araki–Wyss [2] construction of the gauge-invariant quasi-free states. Let \mathcal{G} denote a separable complex Hilbert space. Let $\mathcal{AF}(\mathcal{G}) := \bigoplus_{n=0}^{\infty} \mathcal{G}^{\wedge n} n!$ denote the *antisymmetric Fock space of \mathcal{G}* . Here \wedge denotes the antisymmetric tensor product and elements of the Hilbert space $\mathcal{AF}(\mathcal{G})$ are sequences $g = (g^{(n)})_{n=0}^{\infty}$ with $g^{(n)} \in \mathcal{G}^{\wedge n}$ ($\mathcal{G}^{\wedge 0} := \mathbb{C}$) and $\|g\|_{\mathcal{AF}(\mathcal{G})}^2 = \sum_{n=0}^{\infty} \|g^{(n)}\|_{\mathcal{G}^{\wedge n}}^2 n! < \infty$. The vector $\Omega = (1, 0, 0, \dots)$ is called the *vacuum*.

For $\varphi \in \mathcal{G}$, we define a *creation operator* $a^+(\varphi) \in \mathcal{L}(\mathcal{AF}(\mathcal{G}))$ by $a^+(\varphi)g^{(n)} := \varphi \wedge g^{(n)}$ for $g^{(n)} \in \mathcal{G}^{\wedge n}$. For each $\varphi \in \mathcal{G}$, we define an *annihilation operator* $a^-(\varphi) := a^+(\varphi)^*$. Then,

$$a^-(\varphi)g_1 \wedge \cdots \wedge g_n = \sum_{i=1}^n (-1)^{i+1} (g_i, \varphi)_{\mathcal{G}} g_1 \wedge \cdots \wedge g_{i-1} \wedge g_{i+1} \wedge \cdots \wedge g_n$$

for all $g_1, \dots, g_n \in \mathcal{G}$. Note that

$$\|a^+(\varphi)\|_{\mathcal{L}(\mathcal{AF}(\mathcal{G}))} = \|a^-(\varphi)\|_{\mathcal{L}(\mathcal{AF}(\mathcal{G}))} = \|\varphi\|_{\mathcal{G}}. \quad (27)$$

The operators $a^+(\varphi)$, $a^-(\varphi)$ satisfy the CAR (over \mathcal{G}).

Let now $\mathcal{G} = \mathcal{H} \oplus \mathcal{H}$, and for $\varphi \in \mathcal{H}$ and $\diamond \in \{+, -\}$, we denote $a_1^\diamond(\varphi) := a^\diamond(\varphi, 0)$, $a_2^\diamond(\varphi) := a^\diamond(0, \varphi)$.

We fix any $K \in \mathcal{L}(\mathcal{H})$ such that $\mathbf{0} \leq K \leq \mathbf{1}$, and define the operators K_1 and K_2 as in Lemma 2.1. We define operators

$$\mathcal{A}^+(\varphi) := a_2^+(K_2\varphi) + a_1^-(CK_1\varphi), \quad \mathcal{A}^-(\varphi) := a_2^-(K_2\varphi) + a_1^+(CK_1\varphi), \quad (28)$$

where \mathcal{C} is the complex conjugation in \mathcal{H} . The operators $\mathcal{A}^+(\varphi)$, $\mathcal{A}^-(\varphi)$ satisfy the CAR (2), (3). Let \mathbf{A} denote the corresponding CAR $*$ -algebra. The vacuum state on \mathbf{A} is defined by $\tau(a) := (a\Omega, \Omega)_{\mathcal{AF}(\mathcal{G})}$ ($a \in \mathbf{A}$). This state is gauge-invariant quasi-free. More exactly, setting $\mathcal{F} = \mathcal{AF}(\mathcal{G})$ and $a^\pm(\varphi) = \mathcal{A}^\pm(\varphi)$, one shows that formulas (25), (26) hold, with $S^{(1,1)}(\varphi, \psi) = (K\varphi, \psi)_{\mathcal{H}}$ ($\varphi, \psi \in \mathcal{H}$). In fact, each gauge-invariant quasi-free state on the CAR algebra over \mathcal{H} can be constructed in such a way [2].

Next, just as in Sect. 2.2, we assume that the space X is divided into two disjoint parts, X_1 and X_2 . We define operators $A^+(\varphi)$, $A^-(\varphi)$ ($\varphi \in \mathcal{H}$) by formula (10). Hence, by (28),

$$\begin{aligned} A^+(\varphi) &= a^+(CK_1\mathcal{C}P_2\varphi, K_2P_1\varphi) + a^-(CK_1P_1\varphi, \mathcal{C}K_2P_2\varphi), \\ A^-(\varphi) &= a^-(\mathcal{C}K_1\mathcal{C}P_2\varphi, K_2P_1\varphi) + a^+(CK_1P_1\varphi, \mathcal{C}K_2P_2\varphi). \end{aligned} \quad (29)$$

These operators also satisfy the CAR and denote by \mathbb{A} the corresponding CAR $*$ -algebra. (Note that we have the equality of \mathbf{A} and \mathbb{A} as sets.) The vacuum state τ on \mathbb{A} is not anymore gauge-invariant but it is still quasi-free. More exactly, setting $\mathcal{F} = \mathcal{AF}(\mathcal{G})$ and $a^\pm(\varphi) = A^\pm(\varphi)$, one easily shows that formulas (22)–(24) hold, with

$$T^{(2)}(\varphi, \psi) = 2i\Im(K\mathbb{J}\varphi, \mathbb{J}\psi)_{\mathcal{H}} + (\mathbb{J}\psi, \mathbb{J}\varphi)_{\mathcal{H}}.$$

Here $\mathbb{J}\varphi := P_1\varphi + P_2\mathcal{C}\varphi$.

Remark 2.6. The n -point functions $S^{(m,n)}$ for the state τ on \mathbb{A} can be calculated as follows. First, we note that

$$\begin{aligned} S^{(1,1)}(\varphi, \psi) &= ((P_1\mathbb{K}P_1 + P_2\mathbb{K}P_2)\varphi, \psi)_{\mathcal{H}}, \\ S^{(2,0)}(\varphi, \psi) &= (\psi, (\mathcal{C}P_2K P_1 + P_1K_2\mathcal{C}K_2P_2)\varphi)_{\mathcal{H}}, \\ S^{(0,2)}(\varphi, \psi) &= \overline{S^{(2,0)}(\psi, \varphi)}. \end{aligned}$$

Next, if $m+n$ is odd, then $S^{(m,n)} = 0$, and if $m+n = 2k$ ($k \geq 2$), then, by using Lemma 6.1 below, we get

$$S^{(m,n)}(\varphi_1, \dots, \varphi_{m+n}) = \sum (-1)^{\text{Cross}(v)} S_{i_1, j_1}^{(2)}(\varphi_{i_1}, \varphi_{j_1}) \cdots S_{i_k, j_k}^{(2)}(\varphi_{i_k}, \varphi_{j_k}),$$

where the summation is over all partitions $v = \{\{i_1, j_1\}, \dots, \{i_k, j_k\}\}$ of $\{1, \dots, 2k\}$ with $i_l < j_l$ and $S_{i_l, j_l}^{(2)} := S^{(2,0)}$ if $j_l \leq m$, $S_{i_l, j_l}^{(2)} := S^{(1,1)}$ if $i_l \leq m < j_l$, $S_{i_l, j_l}^{(2)} := S^{(0,2)}$ if $i_l \geq m+1$ ($l = 1, \dots, k$).

3. Rigorous Construction of the Particle Density

Let $\mathbb{K} \in \mathcal{L}(\mathcal{H})$ be a J -self-adjoint operator satisfying the assumptions of Theorem 2.2, and let $K := \overline{\mathbb{K}}$. Let $\mathcal{G} = \mathcal{H} \oplus \mathcal{H}$, and let the operators $A^+(\varphi), A^-(\varphi) \in \mathcal{L}(\mathcal{AF}(\mathcal{G}))$ ($\varphi \in \mathcal{H}$) be defined by (29). Let the operator-valued distributions $A^+(x), A^-(x)$ ($x \in X$) be determined by $A^+(\varphi), A^-(\varphi)$ similarly to (6).

Recall that $\mathcal{AF}(\mathcal{G})$ consists of all sequences $g = (g^{(n)})_{n=0}^\infty$ with $g^{(n)} \in \mathcal{G}^{\wedge n}$ satisfying $\sum_{n=0}^\infty \|g^{(n)}\|_{\mathcal{G}^{\wedge n}}^2 n! < \infty$. We denote by $\mathcal{AF}_{\text{fin}}(\mathcal{G})$ the dense subspace of $\mathcal{AF}(\mathcal{G})$ that consists of all finite sequences $g = (g^{(n)})_{n=0}^\infty$ from $\mathcal{AF}(\mathcal{G})$, i.e., for some $N \in \mathbb{N}$ (depending on g), we have $g^{(n)} = 0$ for all $n \geq N$. We endow $\mathcal{AF}_{\text{fin}}(\mathcal{G})$ with the topology of the locally convex direct sum of the Hilbert spaces $\mathcal{G}^{\wedge n}$.

We denote by $\mathcal{L}(\mathcal{AF}_{\text{fin}}(\mathcal{G}))$ the space of continuous linear operators in $\mathcal{AF}_{\text{fin}}(\mathcal{G})$. Note that a linear operator A acting in $\mathcal{AF}_{\text{fin}}(\mathcal{G})$ is continuous if and only if, for each k there exists N such that $AG^{\wedge k} \subset \bigoplus_{n=0}^N \mathcal{G}^{\wedge n}$ and A acts continuously from $\mathcal{G}^{\wedge k}$ into $\bigoplus_{n=0}^N \mathcal{G}^{\wedge n}$, see e.g. [24, Chapter II, Section 6].

Recall the heuristic definition (11) of $\rho(\Delta)$. Our aim in this section is to rigorously define $\rho(\Delta)$ for each $\Delta \in \mathcal{B}_0(X)$ as an operator from $\mathcal{L}(\mathcal{AF}_{\text{fin}}(\mathcal{G}))$ which is Hermitian in $\mathcal{AF}(\mathcal{G})$. To this end, we start with heuristic considerations.

Think of K_1 and K_2 as self-adjoint integral operators with integral Hermitian kernels $K_1(x, y)$ and $K_2(x, y)$, respectively. Let $\Delta \in \mathcal{B}_0(X_1)$, and denote by χ_Δ the indicator function of the set Δ . We have

$$\begin{aligned} \int_{\Delta} A^+(x) \sigma(dx) &= A^+(\chi_\Delta) = a_2^+(K_2 \chi_\Delta) + a_1^-(\mathcal{C}K_1 \chi_\Delta) \\ &= \int_X (K_2 \chi_\Delta)(x) a_2^+(x) \sigma(dx) + \int_X (K_1 \chi_\Delta)(x) a_1^-(x) \sigma(dx) \\ &= \int_X \int_{\Delta} K_2(x, y) a_2^+(x) \sigma(dy) \sigma(dx) + \int_X \int_{\Delta} K_1(x, y) a_1^-(x) \sigma(dy) \sigma(dx) \\ &= \int_{\Delta} \int_X K_2(y, x) a_2^+(y) \sigma(dy) \sigma(dx) + \int_{\Delta} \int_X \overline{K_1(x, y)} a_1^-(y) \sigma(dy) \sigma(dx) \\ &= \int_{\Delta} (a_2^+(K_2(\cdot, x)) + a_1^-(K_1(x, \cdot))) \sigma(dx). \end{aligned}$$

From here, and using similar calculations when $\Delta \in \mathcal{B}_0(X_i)$ ($i = 1, 2$), we formally conclude that

$$A^+(x) = \begin{cases} a_2^+(K_2(\cdot, x)) + a_1^-(K_1(x, \cdot)) & \text{if } x \in X_1, \\ a_1^+(K_1(x, \cdot)) + a_2^-(K_2(\cdot, x)) & \text{if } x \in X_2, \end{cases}$$

$$A^-(x) = \begin{cases} a_2^-(K_2(\cdot, x)) + a_1^+(K_1(x, \cdot)) & \text{if } x \in X_1, \\ a_1^-(K_1(x, \cdot)) + a_2^+(K_2(\cdot, x)) & \text{if } x \in X_2. \end{cases}$$

Denoting $\rho(x) := A^+(x)A^-(x)$, we get, for $x \in X_1$,

$$\begin{aligned} \rho(x) &= a_2^+(K_2(\cdot, x))a_1^+(K_1(x, \cdot)) + a_2^+(K_2(\cdot, x))a_2^-(K_2(\cdot, x)) \\ &\quad + a_1^-(K_1(x, \cdot))a_1^+(K_1(x, \cdot)) + a_1^-(K_1(x, \cdot))a_2^-(K_2(\cdot, x)), \end{aligned} \quad (30)$$

and for $x \in X_2$,

$$\begin{aligned} \rho(x) &= a_1^+(K_1(x, \cdot))a_2^+(K_2(\cdot, x)) + a_1^+(K_1(x, \cdot))a_1^-(K_1(x, \cdot)) \\ &\quad + a_2^-(K_2(\cdot, x))a_2^+(K_2(\cdot, x)) + a_2^-(K_2(\cdot, x))a_1^-(K_1(x, \cdot)). \end{aligned} \quad (31)$$

Remark 3.1. Comparing formulas (30) and (31), we see that the right-hand side of (31) can be obtained from the right-hand side of (30) by swapping places of $a_1^\diamond(K_1(x, \cdot))$ and $a_2^\diamond(K_2(\cdot, x))$ ($\diamond \in \{+, -\}$).

For $\Delta \in \mathcal{B}_0(X_1)$, let us now rigorously define $\int_X \rho(x)\sigma(dx)$.

The creation and annihilation operators. We will now define

$$\int_{\Delta} a_2^+(K_2(\cdot, x))a_1^+(K_1(x, \cdot))\sigma(dx), \quad \int_{\Delta} a_1^-(K_1(x, \cdot))a_2^-(K_2(\cdot, x))\sigma(dx).$$

First, we note that, for $\varphi, \psi \in \mathcal{G}$ and $g^{(n)} \in \mathcal{G}^{\wedge n}$,

$$a^+(\varphi)a^+(\psi)g^{(n)} = \varphi \wedge \psi \wedge g^{(n)} = \mathcal{A}_{n+2}(\varphi \otimes \psi \otimes g^{(n)}).$$

Here, for each $n \in \mathbb{N}$, we denote by \mathcal{A}_n the antisymmetrization operator in $\mathcal{G}^{\otimes n}$, i.e., the orthogonal projection of $\mathcal{G}^{\otimes n}$ onto $\mathcal{G}^{\wedge n}$.

For $\varphi^{(2)} \in \mathcal{G}^{\otimes 2}$, we define a creation operator $a^+(\varphi^{(2)})$ in $\mathcal{AF}_{\text{fin}}(\mathcal{G})$ by

$$a^+(\varphi^{(2)})g^{(n)} := \mathcal{A}_{n+2}(\varphi^{(2)} \otimes g^{(n)}), \quad g^{(n)} \in \mathcal{G}^{\wedge n}.$$

Obviously $a^+(\varphi^{(2)}) \in \mathcal{L}(\mathcal{AF}_{\text{fin}}(\mathcal{G}))$. We also denote

$$a^-(\varphi^{(2)}) := a^+(\varphi^{(2)})^* \upharpoonright \mathcal{AF}_{\text{fin}}(\mathcal{G}),$$

which also belongs to $\mathcal{L}(\mathcal{AF}_{\text{fin}}(\mathcal{G}))$. One can easily derive explicit formulas for the action of this operator.

Thus, we formally have

$$\int_{\Delta} a_2^+(K_2(\cdot, x))a_1^+(K_1(x, \cdot))\sigma(dx) = a^+\left(\int_{\Delta} (0, K_2(\cdot, x)) \otimes (K_1(x, \cdot), 0) \sigma(dx)\right). \quad (32)$$

To give the above operator a rigorous meaning, we will first identify $\int_{\Delta} K_2(\cdot, x) \otimes K_1(x, \cdot) \sigma(dx)$ as an element of $\mathcal{H}^{\otimes 2}$.

For a linear operator $B \in \mathcal{L}(\mathcal{H})$, denote $\overline{B} := CBC$, i.e., \overline{B} is the complex conjugate of B . If B is an integral operator with integral kernel $B(x, y)$, then \overline{B} is the integral operator with integral kernel $\overline{B}(x, y)$. In particular, if B is self-adjoint, then the integral kernel of \overline{B} is $B(y, x)$.

Now, for any $\varphi, \psi \in \mathcal{H}$, we formally calculate

$$\begin{aligned} &\left(\int_{\Delta} K_2(\cdot, x) \otimes K_1(x, \cdot) \sigma(dx), \varphi \otimes \psi\right)_{\mathcal{H}^{\otimes 2}} \\ &= \int_{\Delta} (K_2(\cdot, x) \otimes K_1(x, \cdot), \varphi \otimes \psi)_{\mathcal{H}^{\otimes 2}} \sigma(dx) \\ &= \int_{\Delta} \int_X K_2(y, x) \overline{\varphi(y)} \sigma(dy) \int_X K_1(x, z) \overline{\psi(z)} \sigma(dz) \sigma(dx) \end{aligned}$$

$$\begin{aligned}
 &= \int_X \chi_\Delta(x) (K_1 \bar{\psi})(x) (\overline{K_2 \bar{\varphi}})(x) \sigma(dx) = \int_X \chi_\Delta(x) (K_1 \bar{\psi})(x) \overline{(K_2 \varphi)(x)} \sigma(dx) \\
 &= (P_\Delta K_1 \bar{\psi}, K_2 \varphi)_{\mathcal{H}} = (K_2 P_\Delta K_1 \bar{\psi}, \varphi)_{\mathcal{H}}.
 \end{aligned} \tag{33}$$

Since $P_\Delta K_1 \in \mathcal{S}_2(H)$ (see Lemma 2.1 (ii)), we get $K_2 P_\Delta K_1 \in \mathcal{S}_2(H)$. Therefore, $K_2 P_\Delta K_1$ is an integral operator and we denote its integral kernel by $(K_2 P_\Delta K_1)(x, y)$. Note that $K_2 P_\Delta K_1(\cdot, \cdot) \in \mathcal{H}^{\otimes 2}$. Thus, we continue (33) as follows:

$$= \int_X \int_X (K_2 P_\Delta K_1)(x, y) \overline{\psi(y)} \sigma(dy) \overline{\varphi(x)} \sigma(dx) = (K_2 P_\Delta K_1(\cdot, \cdot), \varphi \otimes \psi)_{\mathcal{H}^{\otimes 2}}.$$

Hence, we rigorously define

$$\int_\Delta K_2(\cdot, x) \otimes K_1(x, \cdot) \sigma(dx) := (K_2 P_\Delta K_1)(\cdot, \cdot) \in \mathcal{H}^{\otimes 2}. \tag{34}$$

We define an isometry

$$\begin{aligned}
 \mathcal{I}_{21} : \mathcal{H}^{\otimes 2} &\rightarrow \mathcal{G}^{\otimes 2} = (\mathcal{H} \oplus \mathcal{H}) \otimes (\mathcal{H} \oplus \mathcal{H}), \\
 \mathcal{I}_{21} \varphi \otimes \psi &= (0, \varphi) \otimes (\psi, 0), \quad \varphi, \psi \in \mathcal{H}.
 \end{aligned}$$

We denote $(K_2 P_\Delta K_1)_{2,1} := \mathcal{I}_{21}(K_2 P_\Delta K_1)(\cdot, \cdot)$. Then, in view of (32) and (34), we define

$$\int_\Delta a_2^+(K_2(\cdot, x)) a_1^+(K_1(x, \cdot)) \sigma(dx) := a^+((K_2 P_\Delta K_1)_{2,1}), \tag{35}$$

and so

$$\int_\Delta a_1^-(K_1(x, \cdot)) a_2^-(K_2(\cdot, x)) \sigma(dx) := a^-((K_2 P_\Delta K_1)_{2,1}). \tag{36}$$

The neutral operator. Our next aim is to rigorously define operators

$$\int_\Delta a_2^+(K_2(\cdot, x)) a_2^-(K_2(\cdot, x)) \sigma(dx), \quad \int_\Delta a_1^-(K_1(x, \cdot)) a_1^+(K_1(x, \cdot)) \sigma(dx).$$

For a linear operator $B \in \mathcal{G}$, the differential second quantization of B is defined as a linear operator $d\Gamma(B) \in \mathcal{L}(\mathcal{AF}_{\text{fin}}(\mathcal{G}))$ satisfying $d\Gamma(B)\Omega := 0$ (recall that Ω is the vacuum), and for any $g_1, \dots, g_n \in \mathcal{G}$,

$$d\Gamma(B)g_1 \wedge \dots \wedge g_n = \sum_{i=1}^n g_1 \wedge \dots \wedge g_{i-1} \wedge (B g_i) \wedge g_{i+1} \wedge \dots \wedge g_n.$$

As easily seen, for any $\varphi, \psi \in \mathcal{G}$, we have $a^+(\varphi)a^-(\psi) = d\Gamma((\cdot, \psi)_{\mathcal{G}} \varphi)$, i.e., the differential second quantization of the operator $\mathcal{G} \ni g \mapsto (g, \psi)_{\mathcal{G}} \varphi$. Hence, we may formally write

$$\int_\Delta a_2^+(K_2(\cdot, x)) a_2^-(K_2(\cdot, x)) \sigma(dx) = d\Gamma\left(\int_\Delta (\cdot, \mathcal{I}_2 K_2(\cdot, x))_{\mathcal{G}} \mathcal{I}_2 K_2(\cdot, x) \sigma(dx)\right), \tag{37}$$

where the isometry $\mathcal{I}_2 : \mathcal{H} \rightarrow \mathcal{G} = \mathcal{H} \oplus \mathcal{H}$ is defined by $\mathcal{I}_2 \varphi := (0, \varphi)$.

Let us define $\int_{\Delta} (\cdot, K_2(\cdot, x))_{\mathcal{H}} K_2(\cdot, x) d\sigma(x)$ as a bounded linear operator in \mathcal{H} . For $\varphi, \psi \in \mathcal{H}$, we formally calculate

$$\begin{aligned} & \left(\int_{\Delta} (\varphi, K_2(\cdot, x))_{\mathcal{H}} K_2(\cdot, x) \sigma(dx), \psi \right)_{\mathcal{H}} \\ &= \int_{\Delta} (\varphi, K_2(\cdot, x))_{\mathcal{H}} (K_2(\cdot, x), \psi)_{\mathcal{H}} \sigma(dx) \\ &= \int_{\Delta} \int_X \varphi(y) \overline{K_2(y, x)} d\sigma(y) \int_X K_2(z, x) \overline{\psi(z)} \sigma(dz) \sigma(dx) \\ &= \int_{\Delta} (K_2\varphi)(x) \overline{(K_2\psi)(x)} \sigma(dx) = (K_2 P_{\Delta} K_2 \varphi, \psi)_{\mathcal{H}}. \end{aligned}$$

Thus, we rigorously define

$$\int_{\Delta} (\cdot, K_2(\cdot, x))_{\mathcal{H}} K_2(\cdot, x) \sigma(dx) := K_2 P_{\Delta} K_2. \quad (38)$$

In view of (37) and (38), we define

$$\int_{\Delta} a_2^+(K_2(\cdot, x)) a_2^-(K_2(\cdot, x)) \sigma(dx) := d\Gamma(\mathbf{0} \oplus K_2 P_{\Delta} K_2). \quad (39)$$

Next, using the CAR, we formally have

$$\begin{aligned} & \int_{\Delta} a_1^-(K_1(x, \cdot)) a_1^+(K_1(x, \cdot)) \sigma(dx) \\ &= - \int_{\Delta} a_1^+(K_1(x, \cdot)) a_1^-(K_1(x, \cdot)) \sigma(dx) + \int_{\Delta} (K_1(x, \cdot), K_1(x, \cdot))_{\mathcal{H}} \sigma(dx) \\ &= - \int_{\Delta} a_1^+(K_1(x, \cdot)) a_1^-(K_1(x, \cdot)) \sigma(dx) + \int_{\Delta} \int_X |K_1(x, y)|^2 \sigma(dy) \sigma(dx). \end{aligned} \quad (40)$$

Since $P_{\Delta} K_1$ is a Hilbert–Schmidt operator (see Lemma 2.1), we have

$$\int_{\Delta} \int_X |K_1(x, y)|^2 \sigma(dy) \sigma(dx) = \|P_{\Delta} K_1\|_2^2, \quad (41)$$

where $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm in $\mathcal{S}_2(\mathcal{H})$. Noting that the operator $K_{\Delta} = P_{\Delta} K P_{\Delta}$ is self-adjoint, we easily see that

$$\|P_{\Delta} K_1\|_2^2 = \text{Tr}(K_{\Delta}) = \text{Tr}(\mathbb{K}_{\Delta}). \quad (42)$$

Hence, by (40)–(42) and similarly to (39), we rigorously define

$$\int_{\Delta} a_1^-(K_1(x, \cdot)) a_1^+(K_1(x, \cdot)) \sigma(dx) := d\Gamma(\overline{-K_1 P_{\Delta} K_1} \oplus \mathbf{0}) + \text{Tr}(\mathbb{K}_{\Delta}). \quad (43)$$

Thus, formulas (35), (36), (39), and (43) imply a rigorous definition of $\rho(\Delta)$ for $\Delta \in \mathcal{B}_0(X_1)$:

$$\begin{aligned} \rho(\Delta) &:= a^+((K_2 P_{\Delta} K_1)_{2,1}) + a^-((K_2 P_{\Delta} K_1)_{2,1}) \\ &\quad + d\Gamma(\overline{-K_1 P_{\Delta} K_1} \oplus K_2 P_{\Delta} K_2) + \text{Tr}(\mathbb{K}_{\Delta}), \end{aligned}$$

Next, we note, if $\varphi^{(2)} \in \mathcal{G}^{\otimes 2}$ and $\psi^{(2)} \in \mathcal{G}^{\otimes 2}$ is defined by $\psi^{(2)}(x, y) := \varphi^{(2)}(y, x)$, then $a^+(\psi^{(2)}) = -a^+(\varphi^{(2)})$. Using this observation and Remark 3.1, we similarly define $\rho(\Delta)$ for $\Delta \in \mathcal{B}_0(X_2)$:

$$\begin{aligned} \rho(\Delta) &:= -a^+((K_2 P_\Delta K_1)_{2,1}) - a^-((K_2 P_\Delta K_1)_{2,1}) \\ &\quad - d\Gamma(\overline{-K_1 P_\Delta K_1} \oplus K_2 P_\Delta K_2) + \text{Tr}(\mathbb{K}_\Delta). \end{aligned}$$

Finally, for each $\Delta \in \mathcal{B}_0(X)$, we define $\rho(\Delta) := \rho(\Delta \cap X_1) + \rho(\Delta \cap X_2)$.

We sum up our considerations in the following definition.

Definition 3.2. For each $\Delta \in \mathcal{B}_0(X)$, we define

$$\begin{aligned} \rho(\Delta) &:= a^+((K_2 J_\Delta K_1)_{2,1}) + a^-((K_2 J_\Delta K_1)_{2,1}) \\ &\quad + d\Gamma(\overline{-K_1 J_\Delta K_1} \oplus K_2 J_\Delta K_2) + \text{Tr}(\mathbb{K}_{\Delta \cap X_1}) + \text{Tr}(\mathbb{K}_{\Delta \cap X_2}), \end{aligned}$$

where $J_\Delta := P_{\Delta \cap X_1} - P_{\Delta \cap X_2}$. Each operator $\rho(\Delta)$ belongs to $\mathcal{L}(\mathcal{AF}_{\text{fin}}(\mathcal{G}))$.

We note that, for each $\Delta \in \mathcal{B}_0(X)$, $\rho(\Delta)$ is a densely defined, Hermitian operator in $\mathcal{AF}(\mathcal{G})$.

Let $(e_i)_{i=1}^\infty$ be an orthonormal basis for \mathcal{H} consisting of real-valued functions. The following proposition can be easily proved by using Definition 3.2.

Proposition 3.3. For $\Delta \in \mathcal{B}_0(X_1)$, we have

$$\begin{aligned} \rho(\Delta) &= \sum_{i,j=1}^\infty \left[(K_2 P_\Delta K_1 e_j, e_i)_{\mathcal{H}} a_2^+(e_i) a_1^+(e_j) + (K_1 P_\Delta K_2 e_j, e_i)_{\mathcal{H}} a_1^-(e_i) a_2^-(e_j) \right. \\ &\quad \left. + (K_2 P_\Delta K_2 e_j, e_i)_{\mathcal{H}} a_2^+(e_i) a_2^-(e_j) + (K_1 P_\Delta K_1 e_j, e_i)_{\mathcal{H}} a_1^-(e_i) a_1^+(e_j) \right], \quad (44) \end{aligned}$$

and for $\Delta \in \mathcal{B}_0(X_2)$, we have

$$\begin{aligned} \rho(\Delta) &= \sum_{i,j=1}^\infty \left[(K_2 P_\Delta K_1 e_i, e_j)_{\mathcal{H}} a_1^+(e_i) a_2^+(e_j) + (K_1 P_\Delta K_2 e_i, e_j)_{\mathcal{H}} a_2^-(e_i) a_1^-(e_j) \right. \\ &\quad \left. + (K_1 P_\Delta K_1 e_i, e_j)_{\mathcal{H}} a_1^+(e_i) a_1^-(e_j) + (K_2 P_\Delta K_2 e_i, e_j)_{\mathcal{H}} a_2^-(e_i) a_2^+(e_j) \right]. \quad (45) \end{aligned}$$

In formulas (44) and (45), the series converge strongly in $\mathcal{L}(\mathcal{F}_{\text{fin}}(\mathcal{G}))$, i.e., for each $f \in \mathcal{F}_{\text{fin}}(\mathcal{G})$, the series applied to the vector f converges in $\mathcal{F}_{\text{fin}}(\mathcal{G})$ (hence also in $\mathcal{F}(\mathcal{G})$).

Remark 3.4. Formulas (44) and (45) could serve as an alternative definition of $\rho(\Delta)$ for Δ from $\mathcal{B}_0(X_1)$ or $\mathcal{B}_0(X_2)$, respectively. However, if we initially accepted (44) and (45) as the definition of $\rho(\Delta)$, it would not be *a priori* clear if such a definition does not depend on the choice of a real orthonormal basis in \mathcal{H} .

Proposition 3.5. For any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, we have $\rho(\Delta_1)\rho(\Delta_2) = \rho(\Delta_2)\rho(\Delta_1)$.

To prove Proposition 3.5, let us recall a result on strong convergence of bounded linear operators. Let E_1 and E_2 be Hilbert (or even Banach) spaces. Let $(B_n)_{n=1}^\infty$ be a sequence from $\mathcal{L}(E_1, E_2)$ and assume that $(B_n)_{n=1}^\infty$ converges strongly to $B \in \mathcal{L}(E_1, E_2)$. Then, the uniform boundedness principle states that $\sup_{n \in \mathbb{N}} \|B_n\| < \infty$, see e.g. [5, Chapter 8, Section 2]. This immediately implies

Lemma 3.6. *Let E_1, E_2, E_3 be Hilbert spaces, let $\{B_n\}_{n=1}^\infty \in \mathcal{L}(E_1, E_2)$ and $\{C_n\}_{n=1}^\infty \in \mathcal{L}(E_2, E_3)$. Assume that the series $\sum_{n=1}^\infty B_n$ and $\sum_{n=1}^\infty C_n$ strongly converge in $\mathcal{L}(E_1, E_2)$ and $\mathcal{L}(E_2, E_3)$, respectively. Then the series $\sum_{m=1}^\infty \sum_{n=1}^\infty B_m C_n$ converges strongly in $\mathcal{L}(E_1, E_3)$.*

Proof of Proposition 3.5. The result follows from Proposition 3.3 and Lemma 3.6 by a tedious calculation that uses the CAR.

4. Wick Polynomials of the Particle Density

Recall that the Hermitian operators $\rho(\Delta)$ were heuristically defined by (11) and the corresponding Wick polynomials $:\rho(\Delta_1) \cdots \rho(\Delta_n):$ were defined by (17). Then the heuristic formula (13) holds, see e.g. [21, Section C]. For the reader's convenience, we will now present a heuristic proof of (13).

Define the operator-valued distribution $:\rho(x_1) \cdots \rho(x_n):$ so that the following formula holds:

$$:\rho(\Delta_1) \cdots \rho(\Delta_n): = \int_{\Delta_1 \times \cdots \times \Delta_n} :\rho(x_1) \cdots \rho(x_n): \sigma(dx_1) \cdots \sigma(dx_n)$$

for all $\Delta_1, \dots, \Delta_n \in \mathcal{B}_0(X)$. Then, by (17), we have $:\rho(x): = \rho(x)$ and

$$:\rho(x_1) \cdots \rho(x_{n+1}): = \rho(x_{n+1}) : \rho(x_1) \cdots \rho(x_n): - \sum_{i=1}^n \delta(x_i, x_{n+1}) : \rho(x_1) \cdots \rho(x_n):. \quad (46)$$

Here the generalized function $\delta(x_1, x_2)$ is defined so that

$$\int_{X^2} f^{(2)}(x_1, x_2) \delta(x_1, x_2) \sigma(dx_1) \sigma(dx_2) := \int_X f^{(2)}(x, x) \sigma(dx).$$

It follows from the CAR that the operator-valued distributions $A^+(x), A^-(x)$ satisfy the commutation relations:

$$\{A^+(x_1), A^+(x_2)\} = \{A^-(x_1), A^-(x_2)\} = 0, \quad \{A^-(x_1), A^+(x_2)\} = \delta(x_1, x_2). \quad (47)$$

Now (46) and (47) imply, by induction,

$$:\rho(x_1) \cdots \rho(x_n): = A^+(x_n) \cdots A^+(x_1) A^-(x_1) \cdots A^-(x_n). \quad (48)$$

Note that formula (48) does not depend on the representation of the CAR, and just states that $:\rho(x_1) \cdots \rho(x_n):$ is the Wick (normal) ordering of the product $\rho(x_1) \cdots \rho(x_n)$.

Formula (13) can be recurrently written as

$$\begin{aligned} :\rho(\Delta): &= \rho(\Delta), \\ :\rho(\Delta_1) \cdots \rho(\Delta_n): &= \int_{\Delta_n} A^+(x_n) : \rho(\Delta_1) \cdots \rho(\Delta_{n-1}): A^-(x_n) \sigma(dx_n), \quad n \geq 2. \end{aligned} \quad (49)$$

The aim of this section is to derive a rigorous form of formula (49).

We start with presenting a rigorous form of the heuristic operator

$$W(\Delta, R) = :\rho(\Delta)R: = \int_{\Delta} A^+(X) R A^-(x) \sigma(dx), \quad (50)$$

where $\Delta \in \mathcal{B}_0(X)$ and $R \in \mathcal{L}(\mathcal{AF}_{\text{fin}}(\mathcal{G}))$. The following result is inspired by Proposition 3.3 and formula (50).

Proposition 4.1. *Let $R \in \mathcal{L}(\mathcal{AF}_{\text{fin}}(\mathcal{G}))$. For $\Delta \in \mathcal{B}_0(X_1)$, we define*

$$W(\Delta, R) := \sum_{i,j=1}^{\infty} \left[(K_2 P_{\Delta} K_1 e_j, e_i)_{\mathcal{H}} a_2^+(e_i) R a_1^+(e_j) + (K_1 P_{\Delta} K_2 e_j, e_i)_{\mathcal{H}} a_1^-(e_i) R a_2^-(e_j) \right. \\ \left. + (K_2 P_{\Delta} K_2 e_j, e_i)_{\mathcal{H}} a_2^+(e_i) R a_2^-(e_j) + (K_1 P_{\Delta} K_1 e_j, e_i)_{\mathcal{H}} a_1^-(e_i) R a_1^+(e_j) \right], \quad (51)$$

and for $\Delta \in \mathcal{B}_0(X_2)$, we define

$$W(\Delta, R) := \sum_{i,j=1}^{\infty} \left[(K_2 P_{\Delta} K_1 e_i, e_j)_{\mathcal{H}} a_1^+(e_i) R a_2^+(e_j) + (K_1 P_{\Delta} K_2 e_i, e_j)_{\mathcal{H}} a_2^-(e_i) R a_1^-(e_j) \right. \\ \left. + (K_1 P_{\Delta} K_1 e_i, e_j)_{\mathcal{H}} a_1^+(e_i) R a_1^-(e_j) + (K_2 P_{\Delta} K_2 e_i, e_j)_{\mathcal{H}} a_2^-(e_i) R a_2^+(e_j) \right]. \quad (52)$$

For $\Delta \in \mathcal{B}_0(X)$, we define $W(\Delta, R) := W(\Delta \cap X_1, R) + W(\Delta \cap X_2, R)$. Then $W(\Delta, R) \in \mathcal{L}(\mathcal{AF}_{\text{fin}}(\mathcal{G}))$. In formulas (51) and (52), the series converge strongly in $\mathcal{L}(\mathcal{AF}_{\text{fin}}(\mathcal{G}))$.

Before proving Proposition 4.1, let us present the main result of this section, which gives a rigorous form of formula (49).

Proposition 4.2. *For any $\Delta_1, \dots, \Delta_n \in \mathcal{B}_0(X)$, $n \geq 2$, we have*

$$:\rho(\Delta_1) \cdots \rho(\Delta_n): = W(\Delta_n, :\rho(\Delta_1) \cdots \rho(\Delta_{n-1}):). \quad (53)$$

Let us now prove Propositions 4.1 and 4.2.

Proof of Proposition 4.1. We will present the proof only in the case $\Delta \in \mathcal{B}_0(X_1)$. Below, we will denote by $\mathbf{1}_k$ the identity operator in $\mathcal{G}^{\otimes k}$.

Step 1. Let $R_{m,n+1} \in \mathcal{L}(\mathcal{G}^{\wedge(n+1)}, \mathcal{G}^{\wedge m})$. For any $\varphi, \psi \in \mathcal{G}$ and $g^{(n)} \in \mathcal{G}^{\wedge n}$, we have

$$a^+(\varphi) R_{m,n+1} a^+(\psi) g^{(n)} = \mathcal{A}_{m+1}(\mathbf{1}_1 \otimes R_{m,n+1})(\mathbf{1}_1 \otimes \mathcal{A}_{n+1})(\varphi \otimes \psi \otimes g^{(n)}).$$

We set $e_n^{(1)} := (e_n, 0)$, $e_n^{(2)} := (0, e_n)$ ($n \in \mathbb{N}$), which is an orthonormal basis for \mathcal{G} . Then, for any $M, N \in \mathbb{N}$,

$$\sum_{i=1}^M \sum_{j=1}^N (K_2 P_{\Delta} K_1 e_j, e_i)_{\mathcal{H}} a_2^+(e_i) R_{m,n+1} a_1^+(e_j) g^{(n)} \\ = \mathcal{A}_{m+1}(\mathbf{1}_1 \otimes R_{m,n+1})(\mathbf{1}_1 \otimes \mathcal{A}_{n+1}) \left[\left(\sum_{i=1}^M \sum_{j=1}^N (K_2 P_{\Delta} K_1 e_j, e_i)_{\mathcal{H}} e_i^{(2)} \otimes e_j^{(1)} \right) \otimes g^{(n)} \right] \\ \rightarrow \mathcal{A}_{m+1}(\mathbf{1}_1 \otimes R_{m,n+1})(\mathbf{1}_1 \otimes \mathcal{A}_{n+1}) \left[(K_2 P_{\Delta} K_1)_{2,1} \otimes g^{(n)} \right]$$

in $\mathcal{G}^{\wedge(m+1)}$ as $M, N \rightarrow \infty$. This implies that the series

$$\sum_{i,j=1}^{\infty} (K_2 P_{\Delta} K_1 e_j, e_i)_{\mathcal{H}} a_2^+(e_i) R a_1^+(e_j)$$

converges strongly in $\mathcal{AF}_{\text{fin}}(\mathcal{G})$.

Step 2. Similarly to $\mathcal{AF}_{\text{fin}}(\mathcal{G})$, we define the topological vector space $\mathcal{F}_{\text{fin}}(\mathcal{G})$ that consists of all finite sequences $g = (g^{(n)})_{n=0}^{\infty}$ with $g^{(n)} \in \mathcal{G}^{\otimes n}$ (here $\mathcal{G}^{\otimes 0} := \mathbb{C}$).

For $\varphi \in \mathcal{G}$, we denote by $l^+(\varphi)$ the left creation operator and by $l^-(\varphi)$ the left annihilation operator in $\mathcal{F}_{\text{fin}}(\mathcal{G})$:

$$\begin{aligned} l^+(\varphi)g_1 \otimes g_2 \otimes \cdots \otimes g_n &= \varphi \otimes g_1 \otimes \cdots \otimes g_n, \\ l^-(\varphi)g_1 \otimes g_2 \otimes \cdots \otimes g_n &= (g_1, \varphi)_{\mathcal{G}} g_2 \otimes g_3 \otimes \cdots \otimes g_n, \end{aligned}$$

for $g_1, \dots, g_n \in \mathcal{G}$. Obviously, $l^+(\varphi), l^-(\varphi) \in \mathcal{L}(\mathcal{F}_{\text{fin}}(\mathcal{G}))$. Then, for each $g^{(n)} \in \mathcal{G}^{\wedge n}$,

$$a^+(\varphi)g^{(n)} = \mathcal{A}_{n+1}l^+(\varphi)g^{(n)}, \quad a^-(\varphi)g^{(n)} = nl^-(\varphi)g^{(n)}.$$

Let $\varphi^{(2)} \in \mathcal{G}^{\otimes 2}$. We also define an operator $l^-(\varphi^{(2)}) \in \mathcal{L}(\mathcal{F}_{\text{fin}}(\mathcal{G}))$ by

$$l^-(\varphi^{(2)})g_1 \otimes g_2 \otimes \cdots \otimes g_n = (g_1 \otimes g_2, \varphi^{(2)})_{\mathcal{G}^{\otimes 2}} g_3 \otimes g_4 \otimes \cdots \otimes g_n.$$

In particular, if $\varphi^{(2)} = \varphi_1 \otimes \varphi_2$ with $\varphi_1, \varphi_2 \in \mathcal{G}$, then $l^-(\varphi_1 \otimes \varphi_2) = l^-(\varphi_2)l^-(\varphi_1)$.

Consider $R_{m,n-1} \in \mathcal{L}(\mathcal{G}^{\wedge(n-1)}, \mathcal{G}^{\wedge m})$. Then, for $\varphi, \psi \in \mathcal{G}$, $g^{(n)} \in \mathcal{G}^{\wedge n}$, we have

$$\begin{aligned} a^-(\varphi)R_{m,n-1}a^-(\psi)g^{(n)} &= ml^-(\varphi)R_{m,n-1}nl^-(\psi)g^{(n)} \\ &= mn l^-(\varphi)R_{m,n-1}(l^-(\psi) \upharpoonright_{\mathcal{G}} \otimes \mathbf{1}_{n-1})g^{(n)} = mn l^-(\varphi)(l^-(\psi) \upharpoonright_{\mathcal{G}} \otimes R_{m,n-1})g^{(n)} \\ &= mn l^-(\varphi)(l^-(\psi) \upharpoonright_{\mathcal{G}} \otimes \mathbf{1}_m)(\mathbf{1}_1 \otimes R_{m,n-1})g^{(n)} = mn l^-(\varphi)l^-(\psi)(\mathbf{1}_1 \otimes R_{m,n-1})g^{(n)} \\ &= mn l^-(\psi \otimes \varphi)(\mathbf{1}_1 \otimes R_{m,n-1})g^{(n)}. \end{aligned}$$

Hence, for $M, N \in \mathbb{N}$,

$$\begin{aligned} &\sum_{i=1}^M \sum_{j=1}^N (K_1 P_{\Delta} K_2 e_j, e_i)_{\mathcal{H}} a_1^-(e_i) R_{m,n-1} a_2^-(e_j) g^{(n)} \\ &= mn l^-\left(\sum_{i=1}^N \sum_{j=1}^M (K_2 P_{\Delta} K_1 e_j, e_i)_{\mathcal{H}} e_i^{(2)} \otimes e_j^{(1)} \right) (\mathbf{1}_1 \otimes R_{m,n-1}) g^{(n)} \\ &\rightarrow mn l^-\left((K_2 P_{\Delta} K_1)_{2,1} \right) (\mathbf{1}_1 \otimes R_{m,n-1}) g^{(n)} \end{aligned}$$

in $\mathcal{G}^{\wedge n}$ as $M, N \rightarrow \infty$. This implies that the series

$$\sum_{i,j=1}^{\infty} (K_1 P_{\Delta} K_2 e_j, e_i)_{\mathcal{H}} a_1^-(e_i) R_{m,n-1} a_2^-(e_j) g^{(n)}$$

converges strongly in $\mathcal{AF}_{\text{fin}}(\mathcal{G})$.

Step 3. Let $R_{m,n-1} \in \mathcal{L}(\mathcal{G}^{\wedge(n-1)}, \mathcal{G}^{\wedge m})$. For $\varphi, \psi \in \mathcal{G}$ and $g^{(n)} \in \mathcal{G}^{\wedge n}$, we have

$$a^+(\varphi)R_{m,n-1}a^-(\psi)g^{(n)} = n\mathcal{A}_{m+1}([\cdot, \psi]_{\mathcal{G}} \varphi] \otimes R_{m,n-1})g^{(n)}.$$

Hence, for any $M, N \in \mathbb{N}$,

$$\begin{aligned} &\sum_{i=1}^M \sum_{j=1}^N (K_2 P_{\Delta} K_2 e_j, e_i)_{\mathcal{H}} a_2^+(e_i) R_{m,n-1} a_2^-(e_j) g^{(n)} \\ &= n\mathcal{A}_{m+1} \left(\left(\sum_{i=1}^M \sum_{j=1}^N (K_2 P_{\Delta} K_2 e_j, e_i)_{\mathcal{H}} (\cdot, e_j^{(2)})_{\mathcal{G}} e_i^{(2)} \right) \otimes R_{m,n-1} \right) g^{(n)} \end{aligned}$$

$$\rightarrow n\mathcal{A}_{m+1}((\mathbf{0} \oplus K_2 P_\Delta K_2) \otimes R_{m,n-1})g^{(n)}$$

in $\mathcal{G}^{\wedge(m+1)}$ as $M, N \rightarrow \infty$. This implies that the series

$$\sum_{i,j=1}^{\infty} (K_2 P_\Delta K_2 e_j, e_i)_{\mathcal{H}} a_2^+(e_i) R a_2^-(e_j)$$

converges strongly in $\mathcal{AF}_{\text{fin}}(\mathcal{G})$.

Step 4. Recall that, for all $x \in X_1$, we have $K_1(x, \cdot) \in \mathcal{H}$. Therefore, for all $x \in X_1$, the operators $a_1^+(K_1(x, \cdot))$ and $a_1^-(K_1(x, \cdot))$ are bounded in $\mathcal{AF}(\mathcal{G})$ and have norm $\|K_1(x, \cdot)\|_{\mathcal{H}}$.

Let $R_{m,n+1} \in \mathcal{L}(\mathcal{G}^{\wedge(n+1)}, \mathcal{G}^{\wedge m})$. We will now show that

$$\int_{\Delta} a_1^-(K_1(x, \cdot)) R_{m,n+1} a_1^+(K_1(x, \cdot)) \sigma(dx) \quad (54)$$

exists as a Bochner integral with values in $\mathcal{L}(\mathcal{G}^{\wedge n}, \mathcal{G}^{\wedge(m-1)})$. For the definition and properties of a Bochner integral, see e.g. [5, Chapter 10, Section 3].

Lemma 4.3. *The mappings*

$$\Delta \ni x \mapsto a_1^+(K_1(x, \cdot)) \in \mathcal{L}(\mathcal{AF}(\mathcal{G})), \quad \Delta \ni x \mapsto a_1^-(K_1(x, \cdot)) \in \mathcal{L}(\mathcal{AF}(\mathcal{G}))$$

are strongly measurable.

Proof. Consider the product space $\Delta \times X$ equipped with the corresponding σ -algebra. We will say that a function $g : \Delta \times X \rightarrow \mathbb{C}$ is *rectangle-simple* if it is of the form

$$g(x, y) = \sum_{i=1}^n c_i \chi_{A_i \times B_i}(x, y),$$

where $c_i \in \mathbb{C}$, $A_i \in \mathcal{B}(X_1)$, $A_i \subset \Delta$, $B_i \in \mathcal{B}(X)$, $\sigma(B_i) < \infty$ ($i = 1, \dots, n$), and the sets $A_i \times B_i$ are mutually disjoint. Since $K_1(\cdot, \cdot) \in L^2(\Delta \times X_1, \sigma^{\otimes 2})$, there exists a sequence $(g_n)_{n=1}^{\infty}$ of rectangle-simple functions such that $g_n(x, y) \rightarrow K_1(x, y)$ and $|g_n(x, y)| \leq |K_1(x, y)|$ for $\sigma^{\otimes 2}$ -a.a. $(x, y) \in \Delta \times X$. In particular, $g_n(\cdot, \cdot) \rightarrow K_1(\cdot, \cdot)$ in $L^2(\Delta \times X, \sigma^{\otimes 2})$. By (27), this implies that $(a_1^+(g_n(x, \cdot)))_{n=1}^{\infty}$ and $(a_1^-(g_n(x, \cdot)))_{n=1}^{\infty}$ are sequences of simple functions on Δ with values in $\mathcal{L}(\mathcal{AF}(\mathcal{G}))$ such that, for σ -a.a. $x \in \Delta$,

$$a_1^+(g_n(x, \cdot)) \rightarrow a_1^+(K_1(x, \cdot)), \quad a_1^-(g_n(x, \cdot)) \rightarrow a_1^-(K_1(x, \cdot)),$$

where convergence is in $\mathcal{L}(\mathcal{AF}(\mathcal{G}))$. This implies that the statement of the lemma holds. \square

Lemma 4.3 easily implies that the mapping

$$\Delta \ni x \mapsto a_1^-(K_1(x, \cdot)) R_{m,n+1} a_1^+(K_1(x, \cdot)) \in \mathcal{L}(\mathcal{G}^{\wedge n}, \mathcal{G}^{\wedge(m-1)})$$

is strongly measurable. Furthermore, by (27),

$$\begin{aligned} & \int_{\Delta} \|a_1^-(K_1(x, \cdot))R_{m,n+1}a_1^+(K_1(x, \cdot))\|_{\mathcal{L}(\mathcal{G}^{\wedge n}, \mathcal{G}^{\wedge(m-1)})} \sigma(dx) \\ & \leq \|R_{m,n+1}\|_{\mathcal{L}(\mathcal{G}^{\wedge(n+1)}, \mathcal{G}^{\wedge m})} \int_{\Delta \times X} |K_1(x, y)|^2 \sigma(dx) \sigma(dy) < \infty. \end{aligned}$$

Hence, by [5, Chapter 10, Theorem 3.1], the Bochner integral (54) exists.

We have,

$$\begin{aligned} a_1^+(K_1(x, \cdot)) &= \sum_{i=1}^{\infty} \int_X K_1(x, y) e_i(y) d\sigma(y) a_1^+(e_i), \\ a_1^-(K_1(x, \cdot)) &= \sum_{i=1}^{\infty} \int_X \overline{K_1(x, y)} e_i(y) d\sigma(y) a_1^-(e_i), \end{aligned}$$

where the series converges in $\mathcal{L}(\mathcal{AF}(\mathcal{G}))$. Note also that, for each $N \in \mathbb{N}$,

$$\begin{aligned} & \left\| \sum_{i=1}^N \int_X K_1(x, y) e_i(y) \sigma(dy) a_1^+(e_i) \right\|_{\mathcal{L}(\mathcal{AF}(\mathcal{G}))} \leq \|K_1(x, \cdot)\|_{\mathcal{H}}, \\ & \left\| \sum_{i=1}^N \int_X \overline{K_1(x, y)} e_i(y) \sigma(dy) a_1^-(e_i) \right\|_{\mathcal{L}(\mathcal{AF}(\mathcal{G}))} \leq \|K_1(x, \cdot)\|_{\mathcal{H}}. \end{aligned}$$

Using the dominated convergence theorem for a Bochner integral (see e.g. [5, Chapter 10, Exercise 3.6]), we get

$$\begin{aligned} & \int_{\Delta} a_1^-(K_1(x, \cdot)) R_{m,n+1} a_1^+(K_1(x, \cdot)) \sigma(dx) \\ &= \sum_{i,j=1}^{\infty} \int_{\Delta} \sigma(dx) \int_X \sigma(dy) K_1(x, y) e_j(y) \int_X \sigma(dy') \overline{K_1(x, y')} e_i(y') a_1^-(e_i) R_{m,n+1} a_1^+(e_j) \\ &= \sum_{i,j=1}^{\infty} (K_1 P_{\Delta} K_1 e_j, e_i)_{\mathcal{H}} a_1^-(e_i) R_{m,n+1} a_1^+(e_j), \end{aligned}$$

where the series converges in $\mathcal{L}(\mathcal{G}^{\wedge n}, \mathcal{G}^{\wedge(m+1)})$.

To prove Proposition 4.2, we first need the following

Lemma 4.4. *Let $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$ and $R \in \mathcal{L}(\mathcal{AF}_{\text{fin}}(\mathcal{G}))$. Then*

$$\rho(\Delta_1)W(\Delta_2, R) = W(\Delta_2, \rho(\Delta_1)R) + W(\Delta_1 \cap \Delta_2, R). \quad (55)$$

Proof. By linearity, we can assume that $\Delta_1 \in \mathcal{B}_0(X_{i_1})$, $\Delta_2 \in \mathcal{B}_0(X_{i_2})$, where $i_1, i_2 \in \{1, 2\}$. Consider, for example, the case where $\Delta_1, \Delta_2 \in \mathcal{B}_0(X_1)$. By (44), (51) and Lemma 3.6, we then write down the left-hand side of (55) as

$$\begin{aligned} & \rho(\Delta_1)W(\Delta_2, R) \\ &= \sum_{i,j,k,l=1}^{\infty} \left[(K_2 P_{\Delta_1} K_1 e_j, e_i)_{\mathcal{H}} a_2^+(e_i) a_1^+(e_j) + (K_1 P_{\Delta_1} K_2 e_j, e_i)_{\mathcal{H}} a_1^-(e_i) a_2^-(e_j) \right. \\ & \quad \left. + (K_2 P_{\Delta_1} K_2 e_j, e_i)_{\mathcal{H}} a_2^+(e_i) a_2^-(e_j) + (K_1 P_{\Delta_1} K_1 e_j, e_i)_{\mathcal{H}} a_1^-(e_i) a_1^+(e_j) \right] \end{aligned}$$

$$\begin{aligned}
 & \times \left[(K_2 P_{\Delta_2} K_1 e_l, e_k)_{\mathcal{H}} a_2^+(e_k) R a_1^+(e_l) + (K_1 P_{\Delta_2} K_2 e_l, e_k)_{\mathcal{H}} a_1^-(e_k) R a_2^-(e_l) \right. \\
 & \left. + (K_2 P_{\Delta_2} K_2 e_l, e_k)_{\mathcal{H}} a_2^+(e_k) R a_2^-(e_l) + (K_1 P_{\Delta_2} K_1 e_l, e_k)_{\mathcal{H}} a_1^-(e_k) R a_1^+(e_l) \right], \quad (56)
 \end{aligned}$$

and similarly we write down the right-hand side of (55). (The appearing series converge strongly in $\mathcal{L}(\mathcal{AF}_{\text{fin}}(\mathcal{G}))$.) Through lengthy but rather straightforward calculations, one shows that both expressions are equal. To give the reader a feeling how these calculations are carried out, we consider the following term appearing in (56):

$$\begin{aligned}
 & \sum_{i,j,k,l=1}^{\infty} (K_1 P_{\Delta_1} K_2 e_j, e_i)_{\mathcal{H}} (K_2 P_{\Delta_2} K_1 e_l, e_k)_{\mathcal{H}} a_1^-(e_i) a_2^-(e_j) a_2^+(e_k) R a_1^+(e_l) \\
 & = \sum_{i,j,k,l=1}^{\infty} (K_1 P_{\Delta_1} K_2 e_j, e_i)_{\mathcal{H}} (K_2 P_{\Delta_2} K_1 e_l, e_k)_{\mathcal{H}} a_2^+(e_k) a_1^-(e_i) a_2^-(e_j) R a_1^+(e_l) \\
 & + \sum_{i,j,k,l=1}^{\infty} (K_1 P_{\Delta_1} K_2 e_j, e_i)_{\mathcal{H}} (K_2 P_{\Delta_2} K_1 e_l, e_k)_{\mathcal{H}} \delta_{j,k} a_1^-(e_i) R a_1^+(e_l), \quad (57)
 \end{aligned}$$

where we used the CAR. The second sum on the right-hand side of (57) is equal to

$$\begin{aligned}
 & \sum_{i,k,l=1}^{\infty} (K_1 P_{\Delta_1} K_2 e_i, e_k)_{\mathcal{H}} (K_2 P_{\Delta_2} K_1 e_l, e_i)_{\mathcal{H}} a_1^-(e_k) R a_1^+(e_l) \\
 & = \sum_{k,l=1}^{\infty} (K_2 P_{\Delta_2} K_1 e_l, K_2 P_{\Delta_1} K_1 e_k)_{\mathcal{H}} a_1^-(e_k) R a_1^+(e_l) \\
 & = \sum_{k,l=1}^{\infty} (K_1 P_{\Delta_1} (\mathbf{1} - K) P_{\Delta_2} K_1 e_l, e_k)_{\mathcal{H}} a_1^-(e_k) R a_1^+(e_l) \\
 & = \sum_{k,l=1}^{\infty} (K_1 P_{\Delta_1 \cap \Delta_2} K_1 e_l, e_k)_{\mathcal{H}} a_1^-(e_k) R a_1^+(e_l) \\
 & - \sum_{k,l=1}^{\infty} (K_1 P_{\Delta_1} K P_{\Delta_2} K_1 e_l, e_k)_{\mathcal{H}} a_1^-(e_k) R a_1^+(e_l). \quad (58)
 \end{aligned}$$

On the other hand, another term appearing in (56) is:

$$\begin{aligned}
 & \sum_{i,j,k,l=1}^{\infty} (K_1 P_{\Delta_1} K_1 e_j, e_i)_{\mathcal{H}} (K_1 P_{\Delta_2} K_1 e_l, e_k)_{\mathcal{H}} a_1^-(e_i) a_1^+(e_j) a_1^-(e_k) R a_1^+(e_l) \\
 & = \sum_{i,j,k,l=1}^{\infty} (K_1 P_{\Delta_1} K_1 e_j, e_i)_{\mathcal{H}} (K_1 P_{\Delta_2} K_1 e_l, e_k)_{\mathcal{H}} a_1^-(e_k) a_1^-(e_i) a_1^+(e_j) R a_1^+(e_l) \\
 & + \sum_{i,j,k,l=1}^{\infty} (K_1 P_{\Delta_1} K_1 e_j, e_i)_{\mathcal{H}} (K_1 P_{\Delta_2} K_1 e_l, e_k)_{\mathcal{H}} \delta_{j,k} a_1^-(e_i) R a_1^+(e_l). \quad (59)
 \end{aligned}$$

Similarly, the second sum on the right-hand side of (59) is equal to

$$\begin{aligned}
& \sum_{i,k,l=1}^{\infty} (K_1 P_{\Delta_1} K_1 e_i, e_k)_{\mathcal{H}} (K_1 P_{\Delta_2} K_1 e_l, e_i)_{\mathcal{H}} a_1^-(e_k) R a_1^+(e_l) \\
&= \sum_{k,l=1}^{\infty} (K_1 P_{\Delta_2} K_1 e_l, K_1 P_{\Delta_1} K_1 e_k)_{\mathcal{H}} a_1^-(e_k) R a_1^+(e_l) \\
&= \sum_{k,l=1}^{\infty} (K_1 P_{\Delta_1} K_1 P_{\Delta_2} K_1 e_l, e_k)_{\mathcal{H}} a_1^-(e_k) R a_1^+(e_l). \tag{60}
\end{aligned}$$

Thus, we see that, in formula (57), the ‘wrong’ term given by (60) cancels out, the first sum on the right-hand side of (57) and the first sum on the right-hand side of (59) come from $W(\Delta_2, \rho(\Delta_1)R)$, and the first sum on the right-hand side of (58) comes from $W(\Delta_1 \cap \Delta_2, R)$.

We leave the rest of calculations to the interested reader. \square

Proof of Proposition 4.2. For $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, we have by Lemma 4.4,

$$\rho(\Delta_1)\rho(\Delta_2) = \rho(\Delta_1)W(\Delta_2, \mathbf{1}) = W(\Delta_2, \rho(\Delta_1)) + \rho(\Delta_1 \cap \Delta_2).$$

Hence, by (17),

$$:\rho(\Delta_1)\rho(\Delta_2): = W(\Delta_2, \rho(\Delta_1)),$$

i.e., formula (53) holds for $n = 2$. Assume formula (53) holds for n and let us prove it for $n + 1$. By Lemma 4.4 and (17),

$$\begin{aligned}
\rho(\Delta_1):\rho(\Delta_2)\cdots\rho(\Delta_{n+1}): &= \rho(\Delta_1)W(\Delta_{n+1}, :\rho(\Delta_2)\cdots\rho(\Delta_n):) \\
&= W(\Delta_{n+1}, \rho(\Delta_1):\rho(\Delta_2)\cdots\rho(\Delta_n):) + W(\Delta_1 \cap \Delta_{n+1}, :\rho(\Delta_2)\cdots\rho(\Delta_n):) \\
&= W(\Delta_{n+1}, :\rho(\Delta_1)\rho(\Delta_2)\cdots\rho(\Delta_n):) \\
&\quad + \sum_{i=2}^n W(\Delta_{n+1}, :\rho(\Delta_2)\cdots\rho(\Delta_1 \cap \Delta_i)\cdots\rho(\Delta_n):) + :\rho(\Delta_1 \cap \Delta_{n+1})\rho(\Delta_2)\cdots\rho(\Delta_n): \\
&= W(\Delta_{n+1}, :\rho(\Delta_1)\rho(\Delta_2)\cdots\rho(\Delta_n):) + \sum_{i=2}^{n+1} :\rho(\Delta_2)\cdots\rho(\Delta_1 \cap \Delta_i)\cdots\rho(\Delta_{n+1}):,
\end{aligned}$$

which, by (17), implies that formula (53) holds for $n + 1$.

5. The Joint Spectral Measure of the Particle Density

Our aim now is to apply the results of Sect. 2.3 by setting $\mathcal{F} = \mathcal{AF}(\mathcal{G})$, $\mathcal{D} = \mathcal{AF}_{\text{fin}}(\mathcal{G})$, \mathcal{A} to be the commutative $*$ -algebra generated by the particle density $(\rho(\Delta))_{\Delta \in \mathcal{B}_0(X)}$, and τ to be the vacuum state on \mathcal{A} , i.e., $\tau(a) = (a\Omega, \Omega)_{\mathcal{AF}(\mathcal{G})}$ ($a \in \mathcal{A}$), where Ω is the vacuum in $\mathcal{AF}(\mathcal{G})$.

Recall the construction of the integral kernel $\mathbb{K}(x, y)$ of the operator \mathbb{K} in Sect. 2.2.

Theorem 5.1. *The operators $(\rho(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ have correlation measures $(\theta^{(n)})_{n=1}^{\infty}$ respective the vacuum state state τ . Furthermore, the corresponding correlation functions are given by*

$$k^{(n)}(x_1, \dots, x_n) = \det [\mathbb{K}(x_i, x_j)]_{i,j=1,\dots,n}. \tag{61}$$

We will prove Theorem 5.1 below, in Sect. 6, but let us now formulate and prove the main theorem of the paper.

Theorem 5.2. *The operators $(\rho(\Delta))_{\Delta \in \mathcal{B}_0(X)}$, together with their correlation measures $(\theta^{(n)})_{n=1}^\infty$, satisfy the assumptions of Theorem 2.5. Thus, the following statements hold:*

- (i) *Let $\mathfrak{D} := \{a\Omega \mid a \in \mathcal{A}\}$ and let \mathfrak{F} denote the closure of \mathfrak{D} in $\mathcal{AF}(\mathcal{G})$. Each operator $(\rho(\Delta), \mathfrak{D})$ is essentially self-adjoint in \mathfrak{F} , i.e., the closure of $\rho(\Delta)$, denoted by $\tilde{\rho}(\Delta)$, is a self-adjoint operator in \mathfrak{F} .*
- (ii) *For any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, the projection-valued measures (resolutions of the identity) of the operators $\tilde{\rho}(\Delta_1)$ and $\tilde{\rho}(\Delta_2)$ commute.*
- (iii) *There exist a unique point process μ in X and a unique unitary operator $U : \mathfrak{F} \rightarrow L^2(\Gamma_X, \mu)$ satisfying $U\Omega = 1$ and (20). In particular, (21) holds.*
- (iv) *The correlations functions of the point process μ are given by (61).*

Proof. We first prove the following

Lemma 5.3. *For any $\Delta \in \mathcal{B}_0(X)$,*

$$\int_{\Delta} \mathbb{K}(x, x) \sigma(dx) = \text{Tr}(\mathbb{K}_{\Delta \cap X_1}) + \text{Tr}(\mathbb{K}_{\Delta \cap X_2}), \quad (62)$$

and for any $\Delta_1, \dots, \Delta_n \in \mathcal{B}_0(X)$ ($n \geq 2$), we have

$$\int_{\Delta_1 \times \dots \times \Delta_n} \det [\mathbb{K}(x_i, x_j)]_{i,j=1,\dots,n} \sigma(dx_1) \cdots \sigma(dx_n) = \sum_{\xi \in S_n} (-1)^{\text{sgn}(\xi)} \prod_{\psi \in \text{Cycles}(\xi)} \mathbb{T}_\psi, \quad (63)$$

where, for a cycle $\psi = (l_1 l_2 \cdots l_k)$ in a permutation $\xi \in S_n$, we have

$$\mathbb{T}_\psi = \int_{\Delta_{l_1}} \mathbb{K}(x, x) \sigma(dx) = \text{Tr}(\mathbb{K}_{\Delta_{l_1} \cap X_1}) + \text{Tr}(\mathbb{K}_{\Delta_{l_1} \cap X_2}), \quad \text{if } k = 1, \quad (64)$$

$$\mathbb{T}_\psi = \text{Tr}(P_{\Delta_{l_1}} \mathbb{K} P_{\Delta_{l_2}} \mathbb{K} P_{\Delta_{l_3}} \mathbb{K} \cdots \mathbb{K} P_{\Delta_{l_k}} \mathbb{K} P_{\Delta_{l_1}}), \quad \text{if } k \geq 2. \quad (65)$$

Proof. Recall that that, if $S, T \in \mathcal{S}_2(\mathcal{H})$ with integral kernels $S(x, y)$ and $T(x, y)$, respectively, then

$$\text{Tr}(ST) = \int_X (ST)(x, x) \sigma(dx) = \int_{X^2} S(x, y) T(y, x) \sigma(dx) \sigma(dy).$$

Hence, formula (62) holds by statement (ii) of Lemma 2.1 and the construction of the integral kernel $\mathbb{K}(x, y)$.

Next, we note that, for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, we have $P_{\Delta_1} \mathbb{K} P_{\Delta_2} \in \mathcal{S}_2(\mathcal{H})$, hence the operator appearing in (65) is indeed of trace class. Formula (63) obviously holds with \mathbb{T}_ψ given, for $k = 1$ by (64), and for $k \geq 2$,

$$\begin{aligned} \mathbb{T}_\psi &= \int_{\Delta_{l_1} \times \Delta_{l_2} \times \dots \times \Delta_{l_k}} \mathbb{K}(x_{l_1}, x_{l_2}) \mathbb{K}(x_{l_2}, x_{l_3}) \\ &\quad \times \cdots \times \mathbb{K}(x_{l_{k-1}}, x_{l_k}) \mathbb{K}(x_{l_k}, x_{l_1}) \sigma(dx_{l_1}) \cdots \sigma(dx_{l_k}) \\ &= \int_{X^k} (P_{\Delta_{l_1}} \mathbb{K} P_{\Delta_{l_2}})(x_1, x_2) (P_{\Delta_{l_2}} \mathbb{K} P_{\Delta_{l_3}})(x_2, x_3) \\ &\quad \times \cdots \times (P_{\Delta_{l_{k-1}}} \mathbb{K} P_{\Delta_{l_k}})(x_{k-1}, x_k) (P_{\Delta_{l_k}} \mathbb{K} P_{\Delta_{l_1}})(x_k, x_1) \sigma(dx_1) \cdots \sigma(dx_k) \end{aligned}$$

$$\begin{aligned}
&= \int_X (P_{\Delta_{l_1}} \mathbb{K} P_{\Delta_{l_2}} \mathbb{K} P_{\Delta_{l_3}} \mathbb{K} \cdots \mathbb{K} P_{\Delta_{l_k}} \mathbb{K} P_{\Delta_{l_1}})(x, x) \sigma(dx) \\
&= \text{Tr} (P_{\Delta_{l_1}} \mathbb{K} P_{\Delta_{l_2}} \mathbb{K} P_{\Delta_{l_3}} \mathbb{K} \cdots \mathbb{K} P_{\Delta_{l_k}} \mathbb{K} P_{\Delta_{l_1}}).
\end{aligned}$$

□

Let us prove that condition (LB1) of Theorem 2.5 is satisfied. For $l \geq 2$, we have

$$\begin{aligned}
|\text{Tr}(\mathbb{K}_{\Delta}^l)| &\leq \|\mathbb{K}_{\Delta}^l\|_1 \leq \|\mathbb{K}_{\Delta}\|_2 \|\mathbb{K}_{\Delta}^{l-1}\|_2 \leq \|\mathbb{K}_{\Delta}\|_2^2 \|\mathbb{K}_{\Delta}^{l-2}\| \\
&\leq \max \{ \|\mathbb{K}_{\Delta}\|_2, \|\mathbb{K}_{\Delta}\| \}^l,
\end{aligned} \tag{66}$$

where $\|\cdot\|_1$ denotes the norm in $\mathcal{S}_1(\mathcal{H})$. Theorem 5.1, Lemma 5.3 and formula (66) imply that condition (LB1) is satisfied with

$$C_{\Delta} = \max \{ \|\mathbb{K}_{\Delta \cap X_1}\|_1 + \|\mathbb{K}_{\Delta \cap X_2}\|_1, \|\mathbb{K}_{\Delta}\|_2, \|\mathbb{K}_{\Delta}\| \}.$$

In the case where Δ is a subset of either X_1 or X_2 , we can find a finer estimate of $\theta^{(n)}(\Delta^n)$. Indeed, assume, for example that $\Delta \subset X_1$. Then, for any $x, y \in \Delta$, we have, by (16),

$$\mathbb{K}(x, y) = (K_1(x, \cdot), K_1(y, \cdot))_{\mathcal{H}},$$

and so, for any $x_1, \dots, x_n \in \Delta$,

$$\det [\mathbb{K}(x_i, x_j)]_{i,j=1,\dots,n} = (K_1(x_1, \cdot) \wedge \cdots \wedge K_1(x_n, \cdot), K_1(x_1, \cdot) \wedge \cdots \wedge K_1(x_n, \cdot))_{\mathcal{H}^{\wedge n}} n!,$$

which implies

$$|\det [\mathbb{K}(x_i, x_j)]_{i,j=1,\dots,n}| \leq \|K_1(x_1, \cdot)\|_{\mathcal{H}}^2 \cdots \|K_1(x_n, \cdot)\|_{\mathcal{H}}^2 n!.$$

Hence, by Theorem 5.1, condition (LB1) is satisfied with

$$C_{\Delta} = \int_{\Delta \times X} |K_1(x, y)|^2 \sigma(dx) \sigma(dy). \tag{67}$$

Similarly, for $\Delta \in \mathcal{B}_0(X_2)$, (LB1) is satisfied with

$$C_{\Delta} = \int_{\Delta \times X} |K_2(x, y)|^2 \sigma(dx) \sigma(dy). \tag{68}$$

It follows from the proof of Theorem 2.5 in [20] that, when checking condition (LB2), it is sufficient to assume that all sets in the sequence $\{\Delta_l\}_{l \in \mathbb{N}}$ are subsets of either X_1 or X_2 . But then (LB2) is an immediate consequence of formulas (67) and (68). □

6. Proof of Theorem 5.1

We will now prove Theorem 5.1. Our strategy here is to prove the existence of correlation measures and that of correlation functions at the same time.

We first state, for any $x_1, \dots, x_n \in X$,

$$\det [\mathbb{K}(x_i, x_j)]_{i,j=1,\dots,n} \geq 0. \tag{69}$$

Indeed, if all points x_1, \dots, x_n belong to the same part, X_1 or X_2 , then the matrix $[\mathbb{K}(x_i, x_j)]_{i,j=1,\dots,n}$ is Hermitian, hence its determinant is ≥ 0 . Otherwise, without loss of generality, we may assume that for some m with $1 < m < n$, we have $x_1, \dots, x_m \in X_1$ and $x_{m+1}, \dots, x_n \in X_2$. But then formula (69) follows from [22, Proposition 1.4].

Next, it is easy to see that, if among points x_1, \dots, x_n , at least two points coincide, then $\det [\mathbb{K}(x_i, x_j)]_{i,j=1,\dots,n} = 0$. Therefore, the measure

$$\det [\mathbb{K}(x_i, x_j)]_{i,j=1,\dots,n} \frac{1}{n!} \sigma(dx_1) \cdots \sigma(dx_n)$$

is concentrated on $X^{(n)}$.

Hence, to prove Theorem 5.1, it suffices to show that, for any $\Delta_1, \dots, \Delta_m \in \mathcal{B}_0(X_1)$, $\Delta_{m+1}, \dots, \Delta_{m+n} \in \mathcal{B}_0(X_2)$, $m, n \in \mathbb{N}_0$, $m+n \geq 1$, we have

$$\begin{aligned} & \tau(\rho(\Delta_1) \cdots \rho(\Delta_{m+n})) \\ &= \int_{\Delta_1 \times \cdots \times \Delta_{m+n}} \det [\mathbb{K}(x_i, x_j)]_{i,j=1,\dots,m+n} \sigma(dx_1) \cdots \sigma(dx_{m+n}). \end{aligned} \quad (70)$$

We divide the proof of this formula into several steps.

Step 1. To shorten our notations, we denote, for $i, j \in \mathbb{N}$ and $\Delta \in \mathcal{B}_0(X_1)$,

$$\begin{aligned} c_{ij}^{++}(\Delta) &= (K_2 P_\Delta K_1 e_j, e_i)_{\mathcal{H}}, & c_{ij}^{--}(\Delta) &= (K_1 P_\Delta K_2 e_j, e_i)_{\mathcal{H}}, \\ c_{ij}^{+-}(\Delta) &= (K_2 P_\Delta K_2 e_j, e_i)_{\mathcal{H}}, & c_{ij}^{-+}(\Delta) &= (K_1 P_\Delta K_1 e_j, e_i)_{\mathcal{H}}, \end{aligned}$$

for $\Delta \in \mathcal{B}_0(X_2)$,

$$\begin{aligned} c_{ij}^{++}(\Delta) &= (K_2 P_\Delta K_1 e_i, e_j)_{\mathcal{H}}, & c_{ij}^{--}(\Delta) &= (K_1 P_\Delta K_2 e_i, e_j)_{\mathcal{H}}, \\ c_{ij}^{+-}(\Delta) &= (K_1 P_\Delta K_1 e_i, e_j)_{\mathcal{H}}, & c_{ij}^{-+}(\Delta) &= (K_2 P_\Delta K_2 e_i, e_j)_{\mathcal{H}}, \end{aligned}$$

and

$$A_i^+ := a_2^+(e_i), \quad A_i^- := a_1^-(e_i), \quad B_i^+ := a_1^+(e_i), \quad B_i^- := a_2^-(e_i).$$

Then, by Proposition 3.3, for $\Delta \in \mathcal{B}_0(X_1)$,

$$\rho(\Delta) = \sum_{i,j=1}^{\infty} \sum_{\diamond_1, \diamond_2 \in \{+, -\}} c_{ij}^{\diamond_1 \diamond_2}(\Delta) A_i^{\diamond_1} B_j^{\diamond_2},$$

and for $\Delta \in \mathcal{B}_0(X_2)$,

$$\rho(\Delta) = \sum_{i,j=1}^{\infty} \sum_{\diamond_1, \diamond_2 \in \{+, -\}} c_{ij}^{\diamond_1 \diamond_2}(\Delta) B_i^{\diamond_1} A_j^{\diamond_2}.$$

We define an ordered set

$$\mathfrak{E} := \{1, \dots, m, m+1, \dots, m+n, (m+n)', \dots, (m+1)', m', (m-1)', \dots, 1'\} \quad (71)$$

(the elements of \mathfrak{E} being listed in (71) in the increasing order). By Proposition 4.2,

$$\begin{aligned} & \tau(\rho(\Delta_1) \cdots \rho(\Delta_{m+n})) \\ &= \sum_{i_1, \dots, i_{m+n}, i_{(m+n)'}, \dots, i_{1'} \in \mathbb{N}} \sum_{\diamond_1, \dots, \diamond_{m+n}, \diamond_{(m+n)'}, \dots, \diamond_{1'} \in \{+, -\}} c_{i_1 i_{1'}}^{\diamond_1 \diamond_{1'}}(\Delta_1) \cdots c_{i_{m+n} i_{(m+n)'}}^{\diamond_{m+n} \diamond_{(m+n)'}}(\Delta_{m+n}) \\ & \quad \times \tau(A_{i_1}^{\diamond_1} \cdots A_{i_m}^{\diamond_m} B_{i_{m+1}}^{\diamond_{m+1}} \cdots B_{i_{m+n}}^{\diamond_{m+n}} A_{i_{(m+n)'}}^{\diamond_{(m+n)'}} \cdots A_{i_{(m+1)'}}^{\diamond_{(m+1)'}} B_{i_{m'}}^{\diamond_{m'}} \cdots B_{i_{1'}}^{\diamond_{1'}}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i_1, \dots, i_{m+n}, i_{(m+n)'}, \dots, i_{1'} \in \mathbb{N}} c_{i_1 i_1'}^{-+}(\Delta_1) \cdots c_{i_m i_m'}^{-+}(\Delta_m) \\
&\times \sum_{\diamond_{m+1}, \dots, \diamond_{m+n}, \diamond_{(m+n)'}, \dots, \diamond_{(m+1)'} \in \{+, -\}} c_{i_{m+1} i_{(m+1)'}}^{\diamond_{m+1} \diamond_{(m+1)'}}(\Delta_{m+1}) \cdots c_{i_{m+n} i_{(m+n)'}}^{\diamond_{m+n} \diamond_{(m+n)'}}(\Delta_{m+n}) \\
&\times \tau \left(A_{i_1}^- \cdots A_{i_m}^- B_{i_{m+1}}^{\diamond_{m+1}} \cdots B_{i_{m+n}}^{\diamond_{m+n}} A_{i_{(m+n)'}}^{\diamond_{(m+n)'}} \cdots A_{i_{(m+1)'}}^{\diamond_{(m+1)'}} B_{i_{m'}}^+ \cdots B_{i_{1'}}^+ \right). \tag{72}
\end{aligned}$$

Step 2. We remind the reader that τ is a quasi-free state. As easily seen, the following lemma holds.

Lemma 6.1. *Let $n \in \mathbb{N}$ and let $g_1, \dots, g_{2n} \in \mathcal{G}$. Let $\diamond_1, \dots, \diamond_{2n} \in \{+, -\}$ and assume that the number of plusses among $\diamond_1, \dots, \diamond_{2n}$ is the same as the number of minuses. Then,*

$$\left(a^{\diamond_1}(g_1) \cdots a^{\diamond_n}(g_{2n}) \Omega, \Omega \right)_{\mathcal{AF}(\mathcal{G})} = \sum (-1)^{\text{Cross}(v)} \left(\prod_{\substack{\{i, j\} \in v \\ i < j}} (g_j, g_i)_{\mathcal{G}} \right),$$

where the summation is over all partitions $v = \{\{i_1, j_1\}, \dots, \{i_n, j_n\}\}$ of $\{1, \dots, 2n\}$ with $i_k < j_k$ and such that $\diamond_{i_k} = -, \diamond_{j_k} = +$ ($k = 1, \dots, n$).

We define four (ordered) subsets of \mathcal{E} as follows:

$$\begin{aligned}
\mathfrak{A} &:= \{1, \dots, m\}, \quad \mathfrak{B} := \{m+1, \dots, m+n\}, \\
\mathfrak{C} &:= \{(m+n)', \dots, (m+1)'\}, \quad \mathfrak{D} := \{m', \dots, 1'\}. \tag{73}
\end{aligned}$$

Denote by \mathfrak{R} the collection of all partitions v of \mathcal{E} into n two-point sets such that, if $\{i, j\} \in v$ with $i < j$, then one of the following four statements holds: (i) $i \in \mathfrak{A}$ and $j \in \mathfrak{B}$; (ii) $i \in \mathfrak{B}$ and $j \in \mathfrak{C}$; (iii) $i \in \mathfrak{C}$ and $j \in \mathfrak{D}$; (iv) $i \in \mathfrak{A}$ and $j \in \mathfrak{D}$.

By Lemma 6.1 and in view of the definition of A_i^\diamond and B_i^\diamond ($\diamond \in \{+, -\}$), we continue (72) as follows:

$$\begin{aligned}
&= \sum_{i_1, \dots, i_{m+n}, i_{(m+n)'}, \dots, i_{1'} \in \mathbb{N}} c_{i_1 i_1'}^{-+}(\Delta_1) \cdots c_{i_m i_m'}^{-+}(\Delta_m) \sum_{v \in \mathfrak{R}} (-1)^{\text{Cross}(v)} c_{i_{m+1} i_{(m+1)'}}^{\diamond_{m+1} \diamond_{(m+1)'}}(v) (\Delta_{m+1}) \\
&\times \cdots \times c_{i_{m+n} i_{(m+n)'}}^{\diamond_{m+n} \diamond_{(m+n)'}}(v) (\Delta_{m+n}) \prod_{\{u, v\} \in v} \delta_{i_u, i_v}. \tag{74}
\end{aligned}$$

Here, for $u \in \mathcal{E}$, we denote $\diamond_u(v) := -$ if $\{u, v\} \in v$ and $u < v$, and $\diamond_u(v) := +$ if $\{v, u\} \in v$ and $u < v$.

Step 3. Let $I : \mathcal{E} \rightarrow \mathcal{E}$ be the bijective map that is defined as follows: I acts as the identity on \mathfrak{A} and \mathfrak{D} and swaps the elements of \mathfrak{B} and \mathfrak{C} , i.e., $I(i) = i'$ for all $i \in \mathfrak{B}$ and $I(i') = i$ for all $i' \in \mathfrak{C}$.

Denote by \mathfrak{S} the collection of all partitions v of \mathcal{E} into n two-point sets such that, for each $\{i, j\} \in v$, we have $i \in \mathfrak{A} \cup \mathfrak{B}$ and $j \in \mathfrak{C} \cup \mathfrak{D}$. As easily seen, the map I induces a bijection (still denoted by I) of \mathfrak{R} onto \mathfrak{S} .

Lemma 6.2. (i) *For $v \in \mathfrak{R}$, denote by $k_1(v)$ the number of all $\{i, j\} \in v$ such that $i \in \mathfrak{A}$, $j \in \mathfrak{B}$, and denote $k_2(v)$ the number of all $\{i, j\} \in v$ such that $i \in \mathfrak{C}$, $j \in \mathfrak{D}$. Then $k_1(v) = k_2(v)$ and we denote $l(v) := k_1(v) = k_2(v)$.*

(ii) *We have, for each $v \in \mathfrak{R}$*

$$(-1)^{\text{Cross}(v)} = (-1)^{\text{Cross}(I(v))} (-1)^{l(v)}. \tag{75}$$

Proof of Proposition 4.2. Part (i) is obvious, so we only prove part (ii). For $\nu \in \mathfrak{A}$, under the map I , the change in the number of crossings happens in the following three cases:

- (a) Let $\{i_1, j_1\}, \{i_2, j_2\} \in \nu$, $i_1, i_2 \in \mathfrak{A}$, $j_1, j_2 \in \mathfrak{B}$. Then $\{I(i_1), I(j_1)\}, \{I(i_2), I(j_2)\}$ have a crossing if and only if $\{i_1, j_1\}, \{i_2, j_2\}$ do not have a crossing;
- (b) Let $\{i_1, j_1\}, \{i_2, j_2\} \in \nu$, $i_1, i_2 \in \mathfrak{C}$, $j_1, j_2 \in \mathfrak{D}$. Then $\{I(i_1), I(j_1)\}, \{I(i_2), I(j_2)\}$ have a crossing if and only if $\{i_1, j_1\}, \{i_2, j_2\}$ do not have a crossing;
- (c) Let $\{i_1, j_1\}, \{i_2, j_2\} \in \nu$, $i_1 \in \mathfrak{A}$, $j_1 \in \mathfrak{B}$, $i_2 \in \mathfrak{C}$, $j_2 \in \mathfrak{D}$. Then $\{i_1, j_1\}, \{i_2, j_2\}$ do not have a crossing while $\{I(i_1), I(j_1)\}$ and $\{I(i_2), I(j_2)\}$ have a crossing.

Let $N_a(\nu)$, $N_b(\nu)$, and $N_c(\nu)$ denote the number of $\{\{i_1, j_1\}, \{i_2, j_2\}\} \subset \nu$ as in (a), (b), and (c), respectively. Since for any $i, j \in \mathbb{N}_0$, $(-1)^{i-j} = (-1)^{i+j}$, we therefore have:

$$(-1)^{\text{Cross}(I(\nu))} = (-1)^{\text{Cross}(\nu) + N_a(\nu) + N_b(\nu) + N_c(\nu)}. \quad (76)$$

By part (i), $N_a(\nu) = N_b(\nu)$, while $N_c(\nu) = \ell(\nu)^2$. Hence, (76) implies (75). \square

Step 4. We will now identify the set of partitions \mathfrak{A} with the symmetric group S_{m+n} . We define $m : \mathfrak{C} \cup \mathfrak{D} \rightarrow \mathfrak{A} \cup \mathfrak{B}$ by $mi' = i$ for $i' \in \mathfrak{C} \cup \mathfrak{D}$.

We define $\mathcal{J} : \mathfrak{A} \rightarrow S_{m+n}$ as follows: for $\nu \in \mathfrak{A}$ let $\xi = \mathcal{J}(\nu)$ be given by:

- for each $\{u, v\} \in \nu$ such that $u \in \mathfrak{A}$ and $v \in \mathfrak{D}$, $\xi(u) = mv$,
- for each $\{u, v\} \in \nu$ such that $u \in \mathfrak{A}$ and $v \in \mathfrak{B}$, $\xi(u) = v$,
- for each $\{u, v\} \in \nu$ such that $u \in \mathfrak{B}$ and $v \in \mathfrak{C}$, $\xi(mv) = u$,
- for each $\{u, v\} \in \nu$ such that $u \in \mathfrak{C}$ and $v \in \mathfrak{D}$, $\xi(mu) = mv$.

As easily seen, for each $\nu \in \mathfrak{A}$,

$$\text{Cross}(I(\nu)) = \text{sgn}(\mathcal{J}(\nu)).$$

Then formula (75) implies

$$(-1)^{\text{Cross}(\mathcal{J}^{-1}(\xi))} = (-1)^{\text{sgn}(\xi)} (-1)^{\ell(\mathcal{J}^{-1}(\xi))}, \quad \xi \in S_{m+n}.$$

Hence, we continue (74) as follows:

$$\begin{aligned} &= \sum_{\xi \in S_{m+n}} (-1)^{\text{sgn}(\xi)} (-1)^{\ell(\mathcal{J}^{-1}(\xi))} \sum_{i_1, \dots, i_{m+n}, i_{(m+n)'}, \dots, i_{1'} \in \mathbb{N}} c_{i_1 i_{1'}}^{-+}(\Delta_1) \cdots c_{i_m i_{m'}}^{-+}(\Delta_n) \\ &\times c_{i_{m+1} i_{(m+1)'}}^{\diamond_{m+1}(\mathcal{J}^{-1}(\xi)) \diamond_{(m+1)' }(\mathcal{J}^{-1}(\xi))}(\Delta_{m+1}) \cdots c_{i_{m+n} i_{(m+n)'}}^{\diamond_{m+n}(\mathcal{J}^{-1}(\xi)) \diamond_{(m+n)' }(\mathcal{J}^{-1}(\xi))}(\Delta_{m+n}) \\ &\times \prod_{\{u, v\} \in \mathcal{J}^{-1}(\xi)} \delta_{i_u, i_v}. \end{aligned} \quad (77)$$

Step 5. We define mappings $\tau_1 : \mathfrak{A} \cup \mathfrak{B} \rightarrow \mathfrak{A} \cup \mathfrak{C}$ and $\tau_2 : \mathfrak{A} \cup \mathfrak{B} \rightarrow \mathfrak{B} \cup \mathfrak{D}$ as follows:

- for $u \in \mathfrak{A}$, $\tau_1(u) := u$ and $\tau_2(u) := u'$;
- for $u \in \mathfrak{B}$, $\tau_1(u) := u'$ and $\tau_2(u) := u$.

Then, for each $\xi \in S_{m+n}$, we have $\{u, v\} \in \mathcal{J}^{-1}(\xi)$ if and only if, for some $i \in \{1, \dots, m+n\}$, we have $\{u, v\} = \{\tau_1(u), \tau_2(\xi(u))\}$. Hence, by (72), (74), and (77),

$$\begin{aligned}
& \tau(\rho(\Delta_1) \cdots \rho(\Delta_{m+n})) \\
&= \sum_{\xi \in S_{m+n}} (-1)^{\text{sgn}(\xi)} (-1)^{l(\mathcal{J}^{-1}(\xi))} \sum_{i_1, \dots, i_{m+n}, i_{(m+n)'}, \dots, i_{1'} \in \mathbb{N}} c_{i_1 i_{1'}}^{-+}(\Delta_1) \cdots c_{i_m i_{m'}}^{-+}(\Delta_m) \\
&\quad \times c_{i_{m+1} i_{(m+1)'}}^{\diamond_{m+1}(\mathcal{J}^{-1}(\xi)) \diamond_{(m+1)'(\mathcal{J}^{-1}(\xi))}}(\Delta_{m+1}) \cdots c_{i_{m+n} i_{(m+n)'}}^{\diamond_{m+n}(\mathcal{J}^{-1}(\xi)) \diamond_{(m+n)'(\mathcal{J}^{-1}(\xi))}}(\Delta_{m+n}) \\
&\quad \times \prod_{i=1}^{m+n} \delta_{i_{\tau_1(u)}, i_{\tau_2(\xi(u))}}. \tag{78}
\end{aligned}$$

Step 6. Next, we prove

Lemma 6.3. *Let $\xi \in S_{m+n}$ and let $(l_1 l_2 \cdots l_k)$ be a cycle in ξ . Then*

$$\begin{aligned}
& \sum_{i_1, i_2, \dots, i_k, i_{1'}, i_{2'}, \dots, i_{k'} \in \mathbb{N}} c_{i_1 i_{1'}}^{\diamond_{l_1}(\mathcal{J}^{-1}(\xi)) \diamond_{l_1'(\mathcal{J}^{-1}(\xi))}}(\Delta_{l_1}) c_{i_2 i_{2'}}^{\diamond_{l_2}(\mathcal{J}^{-1}(\xi)) \diamond_{l_2'(\mathcal{J}^{-1}(\xi))}}(\Delta_{l_2}) \\
&\quad \times \cdots \times c_{i_k i_{k'}}^{\diamond_{l_k}(\mathcal{J}^{-1}(\xi)) \diamond_{l_k'(\mathcal{J}^{-1}(\xi))}}(\Delta_{l_k}) \delta_{i_{\tau_1(l_1)}, i_{\tau_2(l_2)}} \delta_{i_{\tau_1(l_2)}, i_{\tau_2(l_3)}} \cdots \delta_{i_{\tau_1(l_k)}, i_{\tau_2(l_1)}} \\
&= \text{Tr} \left(P_{\Delta_{l_k}} R(l_k, l_{k-1}) P_{\Delta_{l_{k-1}}} R(l_{k-1}, l_{k-2}) P_{\Delta_{l_{k-2}}} \cdots P_{\Delta_{l_1}} R(l_1, l_k) P_{\Delta_{l_k}} \right). \tag{79}
\end{aligned}$$

Here, for $u, v \in \{1, 2, \dots, m+n\}$,

$$R(u, v) := \begin{cases} K, & \text{if } \min\{u, v\} \leq m, \\ \mathbf{1} - K, & \text{if } \min\{u, v\} \geq m+1. \end{cases} \tag{80}$$

Proof. Let us first consider the case where $l_1, l_2, \dots, l_k \in \{1, \dots, m\}$. Then the left-hand side of (79) becomes

$$\begin{aligned}
& \sum_{i_1, i_2, \dots, i_k, i_{1'}, i_{2'}, \dots, i_{k'} \in \mathbb{N}} c_{i_1 i_{1'}}^{-+}(\Delta_{l_1}) c_{i_2 i_{2'}}^{-+}(\Delta_{l_2}) \cdots c_{i_k i_{k'}}^{-+}(\Delta_{l_k}) \delta_{i_{l_1}, i_{l_2}} \delta_{i_{l_2}, i_{l_3}} \cdots \delta_{i_k, i_{l_1}} \\
&= \sum_{i_1, i_2, \dots, i_k \in \mathbb{N}} c_{i_1 i_k}^{-+}(\Delta_{l_1}) c_{i_2 i_1}^{-+}(\Delta_{l_2}) \cdots c_{i_k i_{k-1}}^{-+}(\Delta_{l_k}).
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{i_1 \in \mathbb{N}} c_{i_1 i_k}^{-+}(\Delta_{l_1}) c_{i_2 i_1}^{-+}(\Delta_{l_2}) &= \sum_{i_1 \in \mathbb{N}} (K_1 P_{\Delta_{l_1}} K_1 e_{i_k}, e_{i_1})_{\mathcal{H}} (K_1 P_{\Delta_{l_2}} K_1 e_{i_1}, e_{i_2})_{\mathcal{H}} \\
&= (K_1 P_{\Delta_{l_2}} K P_{\Delta_{l_1}} K_1 e_{i_k}, e_{i_2})_{\mathcal{H}}.
\end{aligned}$$

Next,

$$\begin{aligned}
& \sum_{i_2 \in \mathbb{N}} (K_1 P_{\Delta_{l_2}} K P_{\Delta_{l_1}} K_1 e_{i_k}, e_{i_2})_{\mathcal{H}} c_{i_3 i_2}^{-+}(\Delta_{l_3}) \\
&= \sum_{i_2 \in \mathbb{N}} (K_1 P_{\Delta_{l_2}} K P_{\Delta_{l_1}} K_1 e_{i_k}, e_{i_2})_{\mathcal{H}} (K_1 P_{\Delta_{l_3}} K_1 e_{i_2}, e_{i_3})_{\mathcal{H}} \\
&= (K_1 P_{\Delta_{l_3}} K P_{\Delta_{l_2}} K P_{\Delta_{l_1}} K_1 e_{i_k}, e_{i_3})_{\mathcal{H}}.
\end{aligned}$$

Continuing by analogy, we conclude:

$$\begin{aligned} & \sum_{i_{l_1}, i_{l_2}, \dots, i_{l_{k-1}} \in \mathbb{N}} c_{i_{l_1} i_{l_k}}^{-+}(\Delta_{l_1}) c_{i_{l_2} i_{l_1}}^{-+}(\Delta_{l_2}) \cdots c_{i_{l_k} i_{l_{k-1}}}^{-+}(\Delta_{l_k}) \\ & = (K_1 P_{\Delta_{l_k}} K P_{\Delta_{l_{k-1}}} K \cdots K P_{\Delta_{l_1}} K_1 e_{i_{l_k}}, e_{i_{l_k}})_{\mathcal{H}}, \end{aligned}$$

which implies

$$\begin{aligned} & \sum_{i_{l_1}, i_{l_2}, \dots, i_{l_k} \in \mathbb{N}} c_{i_{l_1} i_{l_k}}^{-+}(\Delta_{l_1}) c_{i_{l_2} i_{l_1}}^{-+}(\Delta_{l_2}) \cdots c_{i_{l_k} i_{l_{k-1}}}^{-+}(\Delta_{l_k}) \\ & = \sum_{i_k \in \mathbb{N}} (K_1 P_{\Delta_{l_k}} K P_{\Delta_{l_{k-1}}} K \cdots K P_{\Delta_{l_1}} K_1 e_{i_k}, e_{i_k})_{\mathcal{H}} \\ & = \text{Tr} (K_1 P_{\Delta_{l_k}} K P_{\Delta_{l_{k-1}}} K \cdots K P_{\Delta_{l_1}} K_1) \\ & = \text{Tr} (P_{\Delta_{l_k}} K P_{\Delta_{l_{k-1}}} K \cdots K P_{\Delta_{l_1}} K) \\ & = \text{Tr} (P_{\Delta_{l_k}} K P_{\Delta_{l_{k-1}}} K \cdots K P_{\Delta_{l_1}} K P_{\Delta_{l_k}}). \end{aligned}$$

We similarly treat the case where $l_1, l_2, \dots, l_k \in \{m+1, \dots, m+n\}$. Finally we consider the case where

$$\{l_1, l_2, \dots, l_k\} \cap \{1, \dots, m\} \neq \emptyset, \quad \{l_1, l_2, \dots, l_k\} \cap \{m+1, \dots, m+n\} \neq \emptyset.$$

Without loss of generality, we may assume that $l_1 \in \{1, \dots, m\}$ and $l_k \in \{m+1, \dots, m+n\}$. To simplify the notation, we will additionally assume that, for some $\alpha \in \{1, \dots, k-1\}$,

$$l_1, \dots, l_\alpha \in \{1, \dots, m\}, \quad l_{\alpha+1}, \dots, l_k \in \{m+1, \dots, m+n\}.$$

The interested reader can easily extend our arguments to the more general case.

We consider separately three cases:

Case 1: $\alpha = k-1$. Then the left-hand side of (79) becomes

$$\sum_{i_{l_1}, \dots, i_{l_{k-1}}, i_{l'_k} \in \mathbb{N}} c_{i_{l_1} i_{l'_k}}^{-+}(\Delta_{l_1}) c_{i_{l_2} i_{l_1}}^{-+}(\Delta_{l_2}) \cdots c_{i_{l_{k-1}} i_{l_{k-2}}}^{-+}(\Delta_{l_{k-1}}) c_{i_{l_{k-1}} i_{l'_k}}^{+-}(\Delta_{l_k}).$$

Analogously to the above calculations, we get

$$\begin{aligned} & \sum_{i_{l_1}, \dots, i_{l_{k-2}} \in \mathbb{N}} c_{i_{l_1} i_{l'_k}}^{-+}(\Delta_{l_1}) c_{i_{l_2} i_{l_1}}^{-+}(\Delta_{l_2}) \cdots c_{i_{l_{k-1}} i_{l_{k-2}}}^{-+}(\Delta_{l_{k-1}}) \\ & = (K_1 P_{\Delta_{l_{k-1}}} K P_{\Delta_{l_{k-2}}} K \cdots K P_{\Delta_{l_1}} K_1 e_{i_{l'_k}}, e_{i_{l_{k-1}}})_{\mathcal{H}}. \end{aligned}$$

Then

$$\begin{aligned} & \sum_{i_{l_{k-1}} \in \mathbb{N}} (K_1 P_{\Delta_{l_{k-1}}} K P_{\Delta_{l_{k-2}}} K \cdots K P_{\Delta_{l_1}} K_1 e_{i_{l'_k}}, e_{i_{l_{k-1}}})_{\mathcal{H}} c_{i_{l_{k-1}} i_{l'_k}}^{+-}(\Delta_{l_k}) \\ & = \sum_{i_{l_{k-1}} \in \mathbb{N}} (K_1 P_{\Delta_{l_{k-1}}} K P_{\Delta_{l_{k-2}}} K \cdots K P_{\Delta_{l_1}} K_1 e_{i_{l'_k}}, e_{i_{l_{k-1}}})_{\mathcal{H}} (K_1 P_{\Delta_{l_k}} K_1 e_{i_{l_{k-1}}}, e_{i_{l'_k}})_{\mathcal{H}} \end{aligned}$$

$$= (K_1 P_{\Delta_{l_k}} K P_{\Delta_{l_{k-1}}} K P_{\Delta_{l_{k-2}}} K \cdots K P_{\Delta_{l_1}} K_1 e_{i'_{l'_k}}, e_{i'_{l'_k}})_{\mathcal{H}},$$

and

$$\begin{aligned} & \sum_{i'_{l'_k} \in \mathbb{N}} (K_1 P_{\Delta_{l_k}} K P_{\Delta_{l_{k-1}}} K P_{\Delta_{l_{k-2}}} K \cdots K P_{\Delta_{l_1}} K_1 e_{i'_{l'_k}}, e_{i'_{l'_k}})_{\mathcal{H}} \\ &= \text{Tr} (P_{\Delta_{l_k}} K P_{\Delta_{l_{k-1}}} K P_{\Delta_{l_{k-2}}} K \cdots K P_{\Delta_{l_1}} K P_{\Delta_{l_k}}). \end{aligned}$$

Case 2: $\alpha = k - 2$. Then the left-hand side of (79) becomes

$$\begin{aligned} & \sum_{i_{l_1}, \dots, i_{l_{k-2}}, i'_{l'_{k-1}}, i'_{l'_k} \in \mathbb{N}} c_{i_{l_1} i'_{l'_k}}^{-+} (\Delta_{l_1}) c_{i_{l_2} i'_{l'_1}}^{-+} (\Delta_{l_2}) \cdots c_{i_{l_{k-2}} i_{l_{k-3}}}^{-+} (\Delta_{l_{k-2}}) \\ & \times c_{i_{l_{k-2}} i'_{l'_{k-1}}}^{++} (\Delta_{l_{k-1}}) c_{i'_{l'_{k-1}} i'_{l'_k}}^{--} (\Delta_{l_k}) \\ &= \sum_{i_{l_{k-2}}, i'_{l'_{k-1}}, i'_{l'_k} \in \mathbb{N}} (K_1 P_{\Delta_{l_{k-2}}} K P_{\Delta_{l_{k-3}}} K \cdots K P_{\Delta_{l_1}} K_1 e_{i'_{l'_k}}, e_{i_{l_{k-2}}})_{\mathcal{H}} \\ & \times (K_2 P_{\Delta_{l_{k-1}}} K_1 e_{i_{l_{k-2}}}, e_{i'_{l'_{k-1}}})_{\mathcal{H}} (K_1 P_{\Delta_{l_k}} K_2 e_{i'_{l'_{k-1}}}, e_{i'_{l'_k}})_{\mathcal{H}} \\ &= \sum_{i'_{l'_{k-1}}, i'_{l'_k} \in \mathbb{N}} (K_2 P_{\Delta_{l_{k-1}}} P_{\Delta_{l_{k-2}}} K P_{\Delta_{l_{k-3}}} K \cdots K P_{\Delta_{l_1}} K_1 e_{i'_{l'_k}}, e_{i'_{l'_{k-1}}})_{\mathcal{H}} \\ & \times (K_1 P_{\Delta_{l_k}} K_2 e_{i'_{l'_{k-1}}}, e_{i'_{l'_k}})_{\mathcal{H}} \\ &= \sum_{i'_{l'_k} \in \mathbb{N}} (K_1 P_{\Delta_{l_k}} (\mathbf{1} - K) P_{\Delta_{l_{k-1}}} K P_{\Delta_{l_{k-2}}} \cdots K P_{\Delta_{l_1}} K_1 e_{i'_{l'_k}}, e_{i'_{l'_k}})_{\mathcal{H}} \\ &= \text{Tr} (P_{\Delta_{l_k}} (\mathbf{1} - K) P_{\Delta_{l_{k-1}}} K P_{\Delta_{l_{k-2}}} K \cdots K P_{\Delta_{l_1}} K P_{\Delta_{l_k}}). \end{aligned}$$

Case 3: $\alpha \leq k - 3$. Then the left-hand side of (79) becomes

$$\begin{aligned} &= \sum_{i_{l_1}, \dots, i_{l_\alpha}, i'_{l'_{\alpha+1}}, i'_{l'_{\alpha+2}}, \dots, i'_{l'_k} \in \mathbb{N}} c_{i_{l_1} i'_{l'_k}}^{-+} (\Delta_{l_1}) c_{i_{l_2} i_{l_1}}^{-+} (\Delta_{l_2}) \cdots c_{i_{l_\alpha} i_{l_{\alpha-1}}}^{-+} (\Delta_{l_\alpha}) \\ & \times c_{i_{l_\alpha} i'_{l'_{\alpha+1}}}^{++} (\Delta_{l_{\alpha+1}}) c_{i'_{l'_{\alpha+1}} i'_{l'_{\alpha+2}}}^{-+} (\Delta_{l_{\alpha+2}}) \cdots c_{i'_{l'_{k-2}} i'_{l'_{k-1}}}^{-+} (\Delta_{l_{k-1}}) c_{i'_{l'_{k-1}} i'_{l'_k}}^{--} (\Delta_{l_k}). \end{aligned}$$

Similarly to Case 2, we obtain:

$$\begin{aligned} & \sum_{i_{l_1}, \dots, i_{l_\alpha} \in \mathbb{N}} c_{i_{l_1} i'_{l'_k}}^{-+} (\Delta_{l_1}) c_{i_{l_2} i_{l_1}}^{-+} (\Delta_{l_2}) \cdots c_{i_{l_\alpha} i_{l_{\alpha-1}}}^{-+} (\Delta_{l_\alpha}) c_{i_{l_\alpha} i'_{l'_{\alpha+1}}}^{++} (\Delta_{l_{\alpha+1}}) \\ &= (K_2 P_{\Delta_{l_{\alpha+1}}} K P_{\Delta_{l_\alpha}} K \cdots K P_{\Delta_{l_1}} K_1 e_{i'_{l'_k}}, e_{i'_{l'_{\alpha+1}}})_{\mathcal{H}}. \end{aligned}$$

Then

$$\begin{aligned} & \sum_{i'_{l'_{\alpha+1}}, \dots, i'_{l'_{k-2}} \in \mathbb{N}} (K_2 P_{\Delta_{l_{\alpha+1}}} K P_{\Delta_{l_\alpha}} K \cdots K P_{\Delta_{l_1}} K_1 e_{i'_{l'_k}}, e_{i'_{l'_{\alpha+1}}})_{\mathcal{H}} \\ & \times c_{i'_{l'_{\alpha+1}} i'_{l'_{\alpha+2}}}^{-+} (\Delta_{l_{\alpha+2}}) \cdots c_{i'_{l'_{k-2}} i'_{l'_{k-1}}}^{-+} (\Delta_{l_{k-1}}) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i'_{\alpha+1}, \dots, i'_{k-2} \in \mathbb{N}} (K_2 P_{\Delta_{l_{\alpha+1}}} K P_{\Delta_{l_{\alpha}}} K \cdots K P_{\Delta_{l_1}} K_1 e_{i'_{l_k}}, e_{i'_{l_{\alpha+1}}}) \mathcal{H} \\
 &\quad \times (K_2 P_{\Delta_{l_{\alpha+2}}} K_2 e_{i'_{l_{\alpha+1}}}, e_{i'_{l_{\alpha+2}}}) \mathcal{H} \cdots (K_2 P_{\Delta_{l_{k-1}}} K_2 e_{i'_{l_{k-2}}}, e_{i'_{l_{k-1}}}) \mathcal{H} \\
 &= (K_2 P_{\Delta_{l_{k-1}}} (\mathbf{1} - K) P_{\Delta_{l_{k-2}}} (\mathbf{1} - K) \cdots (\mathbf{1} - K) P_{\Delta_{l_{\alpha+1}}} K P_{\Delta_{l_{\alpha}}} K \\
 &\quad \cdots K P_{\Delta_{l_1}} K_1 e_{i'_{l_k}}, e_{i'_{l_{k-1}}}) \mathcal{H}.
 \end{aligned}$$

Finally,

$$\begin{aligned}
 &\sum_{i'_{k-1}, i'_{l_k} \in \mathbb{N}} (K_2 P_{\Delta_{l_{k-1}}} (\mathbf{1} - K) P_{\Delta_{l_{k-2}}} (\mathbf{1} - K) \cdots (\mathbf{1} - K) P_{\Delta_{l_{\alpha+1}}} K P_{\Delta_{l_{\alpha}}} K \\
 &\quad \cdots K P_{\Delta_{l_1}} K_1 e_{i'_{l_k}}, e_{i'_{l_{k-1}}}) \mathcal{H} c_{i'_{k-1}, i'_{l_k}}^{-}(\Delta_{l_k}) \\
 &= \sum_{i'_{k-1}, i'_{l_k} \in \mathbb{N}} (K_2 P_{\Delta_{l_{k-1}}} (\mathbf{1} - K) P_{\Delta_{l_{k-2}}} (\mathbf{1} - K) \cdots (\mathbf{1} - K) P_{\Delta_{l_{\alpha+1}}} K P_{\Delta_{l_{\alpha}}} K \\
 &\quad \cdots K P_{\Delta_{l_1}} K_1 e_{i'_{l_k}}, e_{i'_{l_{k-1}}}) \mathcal{H} (K_1 P_{\Delta_{l_k}} K_2 e_{i'_{l_{k-1}}}, e_{i'_{l_k}}) \mathcal{H} \\
 &= \text{Tr} (P_{\Delta_{l_k}} (\mathbf{1} - K) P_{\Delta_{l_{k-1}}} (\mathbf{1} - K) \cdots (\mathbf{1} - K) P_{\Delta_{l_{\alpha+1}}} K P_{\Delta_{l_{\alpha}}} K \cdots K P_{\Delta_{l_1}} K P_{\Delta_{l_k}}) \mathcal{H}.
 \end{aligned}$$

Step 7. For a given cycle $\theta = (l_1 l_2 \cdots l_k)$ in a permutation $\xi \in S_{m+n}$, we denote by $\tilde{\mathbb{T}}_{\theta}$ the value given by (the right hand-side of) formula (79). Denote by $\mathfrak{t}(\theta)$ the number of $i \in \{1, \dots, k\}$ such that $l_i \in \{m+1, \dots, m+n\}$ but $l_{i+1} \in \{1, \dots, m\}$, where $l_{k+1} := l_1$. Then, by (64), (65), and (80),

$$(-1)^{\mathfrak{t}(\theta)} \tilde{\mathbb{T}}_{\theta} = \text{Tr} (P_{\Delta_{l_k}} \mathbb{K} P_{\Delta_{l_{k-1}}} \mathbb{K} \cdots \mathbb{K} P_{\Delta_{l_1}} \mathbb{K} P_{\Delta_{l_k}}) = \mathbb{T}_{\theta^{-1}}. \quad (81)$$

By Lemma 6.2 (i),

$$\sum_{\theta \in \text{Cycles}(\xi)} \mathfrak{t}(\theta) = l(\mathcal{J}^{-1}(\xi)). \quad (82)$$

Thus, by (78), (81), (82) and Lemma 6.3,

$$\begin{aligned}
 \tau(\rho(\Delta_1) \cdots \rho(\Delta_{m+n})) &= \sum_{\xi \in S_{m+n}} (-1)^{\text{sgn}(\xi)} \prod_{\theta \in \text{Cycles}(\xi)} \mathbb{T}_{\theta^{-1}} \\
 &= \sum_{\xi \in S_{m+n}} (-1)^{\text{sgn}(\xi)} \prod_{\theta \in \text{Cycles}(\xi)} \mathbb{T}_{\theta}.
 \end{aligned} \quad (83)$$

Formulas (63) and (83) imply (70).

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References

1. Alshehri, M.G.A., Lytvynov, E.: Hafnian point processes and quasi-free states on the CCR algebra. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **25**, no. 1, Paper No. 2250002, 25 pp (2022)
2. Araki, H., Wyss, W.: Representations of canonical anticommutation relation. *Helv. Phys. Acta* **37**, 136–159 (1964)
3. Azizov, T.Ya., Iokhvidov, I.S.: *Linear Operators in Spaces with an Indefinite Metric*. Wiley, Chichester (1989)
4. Berezansky, Y.M., Kondratiev, Y.G.: *Spectral Methods in Infinite-Dimensional Analysis*, vol. 2. Kluwer Academic Publishers, Dordrecht (1995)
5. Berezansky, Y.M., Sheftel, Z.G., Us, G.F.: *Functional Analysis*, vol. I. Birkhäuser Verlag, Basel (1996)
6. Borodin, A.: Determinantal point processes. In: *The Oxford Handbook of Random Matrix Theory*, pp. 231–249. Oxford University Press, Oxford (2011)
7. Borodin, A., Okounkov, A., Olshanski, G.: Asymptotics of Plancherel measures for symmetric groups. *J. Am. Math. Soc.* **13**, 481–515 (2000)
8. Borodin, A., Olshanski, G.: Point processes and the infinite symmetric group. *Math. Res. Lett.* **5**, 799–816 (1998)
9. Borodin, A., Olshanski, G.: Distributions on partitions, point processes, and the hypergeometric kernel. *Commun. Math. Phys.* **211**, 335–358 (2000)
10. Borodin, A., Olshanski, G.: Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes. *Ann. Math.* **2**(161), 1319–1422 (2005)
11. Borodin, A., Olshanski, G.: Random partitions and the gamma kernel. *Adv. Math.* **194**, 141–202 (2005)
12. Bratteli, O., Robinson, D.W.: *Operator Algebras and Quantum Statistical Mechanics*, vol. 2. Equilibrium states. *Models in Quantum Statistical Mechanics*, 2nd edn. Springer, Berlin (1997)
13. Daley, D.J., Vere-Jones, D.: *An Introduction to the Theory of Point Processes*. Springer, New York (1988)
14. Georgii, H.-O., Yoo, H.J.: Conditional intensity and Gibbsianness of determinantal point processes. *J. Stat. Phys.* **118**, 55–84 (2005)
15. Kallenberg, O.: *Random Measures*. Akad.-Verl, Berlin (1983)
16. Koshida, S.: Pfaffian point processes from free fermion algebras: perfectness and conditional measures. *SIGMA Symmetry Integrability Geom. Methods Appl.* **17**, Paper No. 008, 35 pp (2021)
17. Lenard, A.: Correlation functions and the uniqueness of the state in classical statistical mechanics. *Commun. Math. Phys.* **30**, 35–44 (1973)
18. Lytvynov, E.: Fermion and boson random point processes as particle distributions of infinite free Fermi and Bose gases of finite density. *Rev. Math. Phys.* **14**, 1073–1098 (2002)
19. Lytvynov, E.: Determinantal point processes with J -Hermitian correlation kernels. *Ann. Probab.* **41**, 2513–2543 (2013)
20. Lytvynov, E., Mei, L.: On the correlation measure of a family of commuting Hermitian operators with applications to particle densities of the quasi-free representations of the CAR and CCR. *J. Funct. Anal.* **245**, 62–88 (2007)
21. Menikoff, R., Sharp, D.H.: Representations of a local current algebra: their dynamical determination. *J. Math. Phys.* **16**, 2341–2352 (1975)
22. Olshanski, G.: Point processes and the infinite symmetric group. Part V: analysis of the matrix Whittaker kernel. arXiv preprint, <http://arxiv.org/abs/math/9810014> (1998)
23. Macchi, O.: The coincidence approach to stochastic point processes. *Adv. Appl. Probab.* **7**, 83–122 (1975)

24. Schaefer, H.H., Wolff, M.P.: *Topological Vector Spaces*, 2nd edn. Springer, New York (1999)
25. Simon, B.: *Trace Ideals and Their Applications*. Cambridge University Press, Cambridge (1979)
26. Soshnikov, A.: Determinantal random point fields. *Russ. Math. Surv.* **55**, 923–975 (2000)
27. Tamura, H., Ito, K.R.: A canonical ensemble approach to the fermion/boson random point processes and its applications. *Commun. Math. Phys.* **263**, 353–380 (2006)

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