Perturbative and semiclassical explorations of dualities

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Abstract

In this work we consider Abelian gauge fields defined on a $(3+1)$-dimensional bulk spacetime, which is allowed to interact with a matter CFT living on a plane boundary. This gives a family of boundary conformal field theories parameterized by the gauge coupling τ in the upper-half plane. Combining this with the recently discovered web of dualities connecting $(2+1)$-dimensional quantum field theories and $\mathrm{SL}(2,\mathbb{Z})$ duality in the bulk, one can perform powerful resummations in perturbation theory. We put this to work by considering a free scalar field theory on the boundary, which has a self-duality under $\tau \to -\frac{1}{\tau}$ in this setup. Leveraging on this we compute the two-loop anomalous dimension of the mass-squared operator, which is then resummed by imposing the self-duality. Our result can then be extrapolated to $\tau = 1$ which, after a bulk $\mathrm{SL}(2,\mathbb{Z})$ transformation, is then related to the 3d Gross-Neveu model by 3d bosonization. In particular this allows us to make a prediction for the anomalous dimension of the mass operator in the 3d Gross-Neveu model.

Declarations

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

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This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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The University's ethical procedures have been followed and, where appropriate, that ethical approval has been granted.

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1 Introduction

By now it is understood that the framework of quantum field theory (QFT) describes a wealth of physical phenomena. Examples in high energy theory are the standard model which has so far successfully described all collider experiments at CERN [1–5] (although see [6]), beyond the standard model theories used in primordial cosmology (see e.g. [7] and references therein), and the low-energy limits of the various string theories [8, 9] and D-branes [10, 11]. Other sources come from condensed matter. Here the degrees of freedom reside on the points of a lattice (and/or the links), and at sufficiently long distance scales where the lattice becomes irrelevant, the physics is described by a continuum QFT [12, 13]. In a similar way QFTs can also be used to understand purely classical theories, such as the Ising model in statistical mechanics [14]. All of this is perhaps not so surprising in view of Wilson's incredibly powerful picture of effective field theory [1, 3, 4, 15–18]. Here the input are the fields, symmetries, and parameters in an effective effective Lagrangian defined at some energy scale. The physics at a different energy scales is then encoded in the running of the parameters under the renormalization group flow.

This already seems enough motivation to try and improve our understanding of quantum field theory in general. Even though we are being quite abstract, we still want to discard those QFTs which are not physically sensible. For instance it should have a potential bounded from below [19], and if it is a gauge theory it should be absent of gauge anomalies [20]. Assuming we have a sensible QFT, one quickly discovers this is a rather difficult problem to analyse in any generality. Our best understanding comes from the following special cases:

(i) The theory may be weakly coupled so that perturbation theory works; at least in some energy window. Take for example quantum electrodynamics (QED) in $D = 3+1$ spacetime dimensions. Here perturbation theory works well at low energies where the coupling constant is small, although sadly perturbation theory breaks down at high-energies since the coupling constant grows as we flow toward the UV[1].

[1] In a physically motivated theory like the standard model, QED is only an effective theory which works up to the weak energy scale. Indeed the photon survives as the massless gauge boson due to electroweak symmetry breaking $SU(2) \to U(1)$.

(ii) The theory may have enough symmetries to be solvable, or at least be constraining enough to make interesting statements on the dynamics. This is the case, for example, in many supersymmetric models. Here the extra power of holomorphicity and R-symmetry have been incredibly powerful tools in obtaining many exact results including non-renormalization theorems, exact beta functions, low energy effective theory, vacuum structure, phases and more[2] [23–35].

(iii) The theory may be scale-invariant. In the case that the theory has a Lagrangian description, this means the action is invariant under coordinate rescalings $x \to \mathrm{e}^{\lambda} x$. If our QFT is relativistic and unitary then it generally believed that scale invariance + Poincaré invariance is enhanced to conformal symmetry[3], the extra generator is often denoted K_μ and generates special conformal transformations. Such a theory is thus referred to as a conformal field theory (CFT), and there now exist powerful methods to solve, or at least learn a lot about these theories, from the conformal bootstrap[4]. This was originally introduced for fields theories in two dimensions [41, 42]. More recently it has been understood how to extend this to higher dimensions in the breakthrough papers [43, 44]. There is already an enormous amount of literature on this subject, for a review see [45].

(iv) It could be a topological field theory [46, 47]. Perhaps the most familiar example is Chern-Simons theory which was famously solved in $D = 3$ spacetime dimensions by Witten in [48].

(v) In the case where we have gauge groups like $\mathrm{U}(N)$ or $\mathrm{O}(N)$, we can consider a large number of colours[5] N as was first famously done by 't Hooft [49] [50].

(vi) Finally we can consider theories in low spacetime dimensions. The case $D = 2$, which naturally arises in condensed matter models, was first considered in the high-energy context by Thirring who introduced a completely soluble interacting theory of fermions [51]. Subsequently Schwinger provided an exact solution to QED_2 [52] [53]. The subject

[2] Some reviews are [21, 22].
[3] A proof exists in $D = 2$ [36]. Attempts of proofs for $D \geq 3$ have been given in e.g. [37, 38].
[4] Good introductions are [39, 40].
[5] There are also variants where we consider theories with a large number of matter flavours N_f, although they will not be of interest to us.

of $2D$ CFT is under a lot of analytical control since the conformal group is so constraining in this case.

It has emerged from studies of supersymmetric theories [32–34, 54–60], field theories in $2D$ [61–63], and condensed matter models [64, 65], that working in terms of elementary degrees of freedom is not always the best way to understand strongly coupled theories[6]. In these examples it turns out that composite degrees of freedom known as solitons can become the fundamental excitation. This has led to a number of remarkable dualities between seemingly unrelated field theories where fundamental and solitonic excitations are exchanged. From these examples we see that there are (at least) two types of duality that need to be distinguished. One type is where two theories are exactly dual to one another, and another type where they flow to the same theory in the IR.

Our focus in this thesis will be on understanding dualities in $3D$ which turns out to be a particularly interesting case to consider. It is simple enough that we don't need to be so reliant on supersymmetry to make interesting statements, but it displays a vast array of rich physics which is interesting enough in its own right, and could help us in understanding theories in $D \geq 4$. The class of dualities we will consider fall into the second class mentioned above - they are emergent dualities in the IR.

The first duality established in 3D was between the XY model and the gauged version of the theory [66–68]. The dynamics of gauge fields in 3D is particularly interesting, one reason is the existence of a Chern-Simons interaction. This interaction ties magnetic flux to charged particles allowing their statistics to be altered [69, 70], in particular allowing bosons to be turned into fermions and vice versa. This statistical transmutation of flux led to early ideas of 3D bosonization [71–83].

More recently studies of $\mathrm{U}(N)$ Chern-Simons-matter theories in the large N limit [84–87], their supersymmetric versions [88–92] and RG flows [93, 94], has led to more powerful 3D bosonization dualities [95]. Studies in condensed matter proposed fermionic version of particle vortex duality [96–105], where a free fermion is conjectured to be dual its gauged version.

[6] In string theory the discovery of solitonic excitations called D-branes was responsible for the second superstring revolution.

In [106, 107] it was found that all these dualities can be derived from a single seed duality, forming a beautiful self-consistent duality web.

[107] took this a step further and showed that each of these dualities follows from a conjectured four-dimensional duality, where the CFT lives on the boundary and interacts with a $U(1)$ gauge field. In this four-dimensional setup, given a 3d CFT, call it $\mathfrak{T}$, we get a whole family of boundary conformal field theories (BCFTs) $\mathfrak{T}(\tau)$ parameterized by the complexified gauge coupling $\tau = \frac{\theta}{2\pi} + \frac{2\pi\,\mathrm{i}}{e^2}$ of the bulk Maxwell theory which lives in the upper half-plane. Moreover this setup enjoys an $SL(2,\mathbb{Z})$ electric-magnetic duality, which maps the coupling to $\tau \to \tau' = \frac{a\tau+b}{c\tau+d}$, whilst simultaneously adding topological degrees of freedom to the boundary.

Setup and results.

In this work we shall consider the case where the boundary theory $\mathfrak{T}$ is a free scalar interacting with a bulk Abelian gauge field in $(3+1)$-dimensions. This setup is conjectured to enjoy a self-duality under $\tau \to -\frac{1}{\tau}$ [107], which reduces to the bosonic particle-vortex duality in the appropriate decoupling limits. We can then consider the strongly coupled points $\tau = \pm 1$, which can be mapped by $SL(2,\mathbb{Z})$ to a dual theory which is decoupled from the bulk gauge field. By 3d bosonization this theory is itself dual to the Gross-Neveu model - strictly speaking, its UV completion in 3d which is the Gross-Neveu-Yukawa model. Now, for τ sufficiently close to $+\,\mathrm{i}\,\infty$ where the theory is weakly coupled we can reliably use perturbation theory for most computations[7], but of course this breaks down as we approach the strongly coupled region. However, as was first described in [108], we can leverage on the self duality $\tau \to -\frac{1}{\tau}$ to try to make sensible extrapolations of our perturbative results to the strongly coupled region. In particular, if we can make sensible predictions at $\tau = \pm 1$ then we can learn things about the Gross-Neveu model.

We put this to work to make an estimate for the anomalous dimension η_σ of the scalar in the Gross-Neveu model, at the critical point. We start by computing the anomalous dimension $\eta_{\overline{\phi}\phi}$ of the mass operator in the scalar theory $\mathfrak{T}(\tau)$ at weak coupling to two-loop order. We then extrapolate this to a non-perturbative result by using a duality-improved Padé approximant, demanding they be invariant under $\tau \to -\frac{1}{\tau}$. Similar ressumations have been performed

[7] Of course, perturbation theory cannot see instantons and other non-perturbative phenomena.

$\mathcal{N} = 4$ super Yang-Mills in [109]. There are two of their Padé functions we can use, both of which depend on two arbitrary parameters. These two parameters are fixed by our two loop perturbative result. Denoting the Padé approximants by F_1, F_2, extrapolating to strong coupling we get the estimate

$$\eta_\sigma \stackrel{?}{=} F_1(\tau = \pm 1) = \frac{37632}{97411 + 1764\pi^2} \approx 0.327745, \tag{1.1}$$

or

$$\eta_\sigma \stackrel{?}{=} F_2(\tau = \pm 1) = \frac{37632}{97411 + 14112\pi + 1764\pi^2} \approx 0.236449. \tag{1.2}$$

This critical exponent has been computed in a variety of ways in the literature. Our results, along with these, are summarized in Table 1 There is also a purely technical point in our

Method	η_σ
F_1	0.327745
F_2	0.236449
ϵ-expansion [110]	0.2934
Bootstrap [111]	0.320
Monte-Carlo [112]	0.31 ± 0.01
fRG [113]	0.372

Table 1: This table displays the method and corresponding prediction for the critical exponent η_ϕ in the Gross-Neveu-Yukawa model.

work that may be of interest to those who want to perform similar calculations to two loop order. A common regulator to use in practise is dimensional regularization, or its close relative dimensional reduction. However, the current technology cannot deal with the divergences that occur in perturbation theory. Specifically, there is one integral which appears ill-defined in these schemes. Thus we abandon these in favour of a momentum cutoff. The results of the two-loop integrals which occur in perturbation theory are then summarized in Appendix D.3 and D.4.

Layout of this thesis.

In Chapter 2 we discuss some well known examples to build intuition and introduce the necessary background material and ideas that permeate the subject. In particular, in Section 2.3 we introduce particle-vortex duality. This is an old duality boson-boson dating back to the 1980s which has been of tremendous use to condensed matter physicists and is now enjoying a renaissance in particle physics in light of the more recently discovered web of dualities between different 3d CFTs.

In Chapter 3 we review some of the ideas which has led to the more modern idea of 3d bosonization along with the duality web that it implies. In Chapter 4 we shall embed this in a four-dimensional setup, where we view these CFTs as living on a boundary which is allowed to interact with a bulk gauge field. We review electric-magnetic duality in this setup. We then consider the cases where we have free scalars on the boundary or free fermions, the various decoupling limits, their interplay with the 3d duality web, and the ressumation strategy to obtain non-perturbative results.

In chapter 5 we consider free scalars on the boundary. Here we begin by computing the anomalous dimension $\gamma_{\overline{\phi}\phi}$ of the mass operator in perturbation theory to two loop order. We then employ our resummation scheme to extrapolate this to strong coupling, and ultimately to the points $\tau = \pm 1$. 3d bosonization maps this point to the Gross-Neveu model, and in particular it maps the mass operator $\overline{\phi}\phi$ of the bosonic theory to the mass operator $\overline{\psi}\psi$ of the Gross-Neveu fermion. Thus our result makes a prediction for the anomalous dimension $\gamma_{\overline{\psi}\psi}$ of the mass operator $\overline{\psi}\psi$ in the Gross-Neveu model.

As mentioned above, with our current technology we cannot make sense of the scalar theory using dimensional regularization, so we choose to regulate our theory with a momentum cutoff instead. This breaks gauge invariance and so we should check that our result isn't nonsensical. Some preliminary checks are performed in Chapter 6, where we consider the case with fermions on the boundary. The benefit of this setup is that dimensional regularization works, and so we can explicitly check this more favourable dimensional-regularization results against the those obtained using a momentum cutoff.

The results of Section 5 onwards are my own unless stated otherwise, or accompanied with a reference.

2 Background

The first examples of (quantum) dualites were discovered in lattice Models. An illustrative example is Kramers-Wannier duality of the 1d quantum Ising model [64, 65]. Here we start with the standard Ising Hamiltonian written in terms of the Pauli matrices

$$\hat{H} = -J \sum_{i=1}^{n} \sigma_i^z \sigma_{i+1}^z - h \sum_{i=1}^{n} \sigma_i^x. \tag{2.1}$$

A natural basis to use are the tensor product states $|\vec{s}\rangle \equiv |s_1 s_2 \cdots s_n\rangle$, where $\sigma_i^z |\vec{s}\rangle = s_i |\vec{s}\rangle$ and each $s_i = \pm 1$. If we were to give this a name, we might call this the spin representation. One can rewrite the Hamiltonian in terms of the composite operators

$$\tau^z_{i+\frac{1}{2}} := \prod_{j \le i} \sigma_j^x, \quad \& \quad \tau^x_{i+\frac{1}{2}} := \sigma_i^z \sigma_{i+1}^z, \tag{2.2}$$

which live on the links of the lattice (labelled by a half-integer). Observe that the τ operators are highly non-local when written in terms of the Pauli matrices. It is not too hard to see that

- Physically the τ^z operator flips all of the spins to the left of the site i, for this reason it is sometimes called a domain wall operator (or a kink operator).
- The τ's satisfy the same $\mathfrak{su}(2)$ algebra as the Pauli matrices.
- One finds that the Hamiltonian takes the same form as in (2.1) when written in terms of the τ's, except the couplings h and J are interchanged.

With the Hamiltonian expressed in terms of the τ's, a natural basis to use would be eigenstates of τ^z, so we could call this the domain wall representation. Note that in this basis the $\tau's$ are local. Without much more effort one can also re-write the Hamiltonian in terms of Majorana fermions - this is the celebrated Jordan-Wigner transformation.

This introductory chapter shall proceed as follows. We shall start in Section 2.1 by gaining some intuition through studying some simple examples of boson-boson and boson-fermion dualities in $1+1$ dimensions. This introduces a lot of ideas which feed into the more complicated examples introduced in Chapter 3, but in a simple, solvable scenario, where all the details can be understood explicitly.

Section 2.2 discusses some of the particularly interesting dynamics of (Abelian) gauge fields in $2+1$ dimensions which will be relevant for us. This includes the Chern-Simons interaction and a magnetic symmetry, both of which are special to this number of dimensions.

Section 2.3 introduces our first interacting duality in $2+1$ dimensions - the bosonic particle-vortex duality [66–68] between the XY model and its gauged version. This is an old story dating back to the 1980s where the duality was shown to hold on the lattice near their respective critical points.

Moreover, we have overwhelming numerical evidence that there is a second order phase transition [114, 115], and thus on general principles we expect the theories near criticality to be described by a quantum field theory [116, 117], which flows to a conformal field theory at the critical point. In this case we thus obtain a duality between the two continuum QFTs, which becomes better and better as we approach the critical point. The continuum description of the XY model is provided by a Landau-Ginzburg type theory of a complex scalar field ϕ:

$$\mathscr{L} = |\partial\phi|^2 + \mu|\phi|^2 + \lambda|\phi|^4. \tag{2.3}$$

The critical CFT, reached in the infrared (IR), is the Wilson-Fisher fixed point [16, 118]. The continuum description of the gauged XY model is provided by minimally coupling the above theory to a dynamical gauge field

$$\mathscr{L} = |\,\mathrm{D}\phi|^2 + \mu|\phi|^2 + \lambda|\phi|^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{2.4}$$

The claim is that this can also be tuned to a CFT, which is exactly dual to the Wilson-Fisher CFT. This is sometimes referred to as the strong form of bosonic particle-vortex duality [119], to distinguish it from the weak form which holds on the lattice.

Specifically we will study the phase diagram in some detail, introduce the interesting vortex degrees of freedom, and study the mapping of operators under the duality. Again, all of these ideas and tools feed into our discussion of the more recent dualities starting in Chapter 3.

Moving on from Section 2.3 we want to start to describe the more recently discovered 3d fermion-fermion and fermion-boson dualities. Of course, to do this we must couple a fermion to a gauge field. However, as we discuss in 2.4 there is a slight subtlety that is important to be aware of, this is the so-called parity anomaly. This will be important in Section 3.3 when we discuss the duality web, and Section 4.4.3 when we couple boundary fermions to a bulk gauge field.

Finally in Section 2.5 we will introduce non-Abelian Chern-Simons theory, which is useful background for the discussion of large N Chern-Simons-matter theories in Chapter 3.3.

2.1 Toy models in $1+1$ dimensions.

In this section we discuss our first field-theoretic dualities in the simpler context of $1+1$ dimensions. Section 2.1.1 begins with a short discussion on the free Dirac fermion. Section 2.1.2 introduces the compact boson and a simple boson-boson duality known as T-duality. Section 2.1.3 combines this to derive a fermion-boson duality. We then discuss an application of the fermion-boson duality in Section 2.1.4.

General references for this section are the original papers [52, 53, 61–63], the reviews [120–122], Coleman's book [5] (in particular the chapter on classical lumps and their quantum descendants) and the book [12].

2.1.1 A Dirac fermion.

A Dirac fermion in two dimensions is a two component spinor

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \tag{2.5}$$

where the components of $\psi_\pm$ of ψ are Weyl spinors. We refer ψ_+ as right-movers and ψ_- as left-movers. The Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ can be represented by $\gamma^0 = \sigma^1$ and $\gamma^1 = \mathrm{i}\,\sigma^2$, where σ^1, σ^2 are the Pauli matrices. We then define the chirality matrix $\gamma^3 := \gamma^0\gamma^1 = \sigma^3$. The action for a free massless Dirac fermion is then

$$S_{\mathrm{f}} = \int \mathrm{d}^2x \; \mathrm{i}\,\overline{\psi}\partial\!\!\!/\psi. \tag{2.6}$$

This theory has two global symmetries. One is a vector symmetry which rotates both components $\psi_\pm$ of ψ with the same phase, and the other is an axial symmetry which rotates them by the opposite phase:

$$\mathrm{U}(1)_{\mathrm{v}}: \quad \psi_\pm \to \mathrm{e}^{\mathrm{i}\,\alpha}\,\psi_\pm, \quad \& \quad \mathrm{U}(1)_{\mathrm{a}}: \quad \psi_\pm \to \mathrm{e}^{\pm\,\mathrm{i}\,\alpha}\,\psi_\pm. \tag{2.7}$$

The corresponding conserved currents are

$$j^\mu_{\mathrm{v}} = \overline{\psi}\gamma^\mu\psi, \quad \& \quad j^\mu_{\mathrm{a}} = \overline{\psi}\gamma^\mu\gamma^3\psi. \tag{2.8}$$

From the action (2.6), we obtain the Hamiltonian $H = \int \mathrm{d}x\, \psi^\dagger(x)\gamma^3\partial_x\psi$. At the moment ψ is a classical (anticommuting) field and $\psi^\dagger$ just means the row vector $\psi^\dagger(x) = (\psi^*_+, \psi^*_-)$, where "$*$" means complex conjugate. Plugging in for the components $\psi_\pm$ of ψ the Hamiltonian reads

$$H = \int \mathrm{d}x \left[-\,\mathrm{i}\,\psi^*_+(x)\partial_x\psi_+(x) + \mathrm{i}\,\psi^*_-(x)\partial_x\psi_-(x) \right]. \tag{2.9}$$

We quantize this theory in the usual way by promoting ψ and ψ^* to operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ satisfying the canonical anticommutation relations

$$\{\hat{\psi}^\dagger_\pm(x), \hat{\psi}_\pm(y)\} = \delta(x-y), \tag{2.10}$$

where $\hat{\psi}^\dagger_\pm$ now means the Hermitian adjoint of the operator $\hat{\psi}_\pm$. This theory is easily solved by working in the momentum basis

$$\hat{\psi}_\pm(p) = \int_{-\infty}^{\infty} \mathrm{d}x\; \mathrm{e}^{-\,\mathrm{i}\,px}\,\hat{\psi}_\pm(x), \tag{2.11}$$

in terms of which the Hamiltonian reads

$$\hat{H} = \int \frac{\mathrm{d}p}{2\pi} \left[p\hat{\psi}_+^\dagger(p)\hat{\psi}_+(p) - p\hat{\psi}_-(p)^\dagger\hat{\psi}_-(p) \right]. \tag{2.12}$$

In this basis

$$\{\hat{\psi}_\pm^\dagger(p), \hat{\psi}_\pm(q)\} = 2\pi\delta(p-q). \tag{2.13}$$

The irreducible representation of the canonical commutation relations are built on top of the vacuum $|\mathrm{vac}\rangle$, which is the unique state of lowest energy. Looking at the expression for $\hat{H}$ in (2.12), we see that $|\mathrm{vac}\rangle$ should be filled with right-moving particles of negative momentum, and left-moving particles of positive momentum:

$$\hat{\psi}_+(p)\,|\mathrm{vac}\rangle = \hat{\psi}_-(-p)\,|\mathrm{vac}\rangle = 0, \quad (p > 0). \tag{2.14}$$

Since this is a free theory, it is completely determined from its two point functions by Wick's theorem. It is instructive to perform such computations once:

$$\langle\hat{\psi}_+(x)\hat{\psi}_+^\dagger(0)\rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{2\pi}\, \mathrm{e}^{\mathrm{i}\,px} \underbrace{\langle\hat{\psi}_+(p)\hat{\psi}_+^\dagger(q)\rangle}_{2\pi\delta(p-q)\Theta(q)} = \int_0^\infty \frac{\mathrm{d}p}{2\pi}\, \mathrm{e}^{\mathrm{i}\,px}. \tag{2.15}$$

As it stands, the integral on the far right is ill-defined due to contributions from large momenta. To regulate this divergence, we introduce small distance UV cutoff $\varepsilon > 0$ and include a factor $\mathrm{e}^{-\varepsilon p}$ in the integrand. This leaves the small momenta region unchanged and smoothly cuts off large momenta. The resulting integral is finite and gives

$$\langle\hat{\psi}_+(x)\hat{\psi}_+^\dagger(0)\rangle = \int_0^\infty \frac{\mathrm{d}p}{2\pi}\, \mathrm{e}^{-\varepsilon p + \mathrm{i}\,px} = \frac{\mathrm{i}}{2\pi} \cdot \frac{1}{x + \mathrm{i}\,\varepsilon}. \tag{2.16}$$

Repeating this exercise for ψ_-

$$\langle\hat{\psi}_\pm(x)\hat{\psi}_\pm^\dagger(y)\rangle = \pm\frac{\mathrm{i}}{2\pi} \cdot \frac{1}{(x-y) \pm \mathrm{i}\,\varepsilon}. \tag{2.17}$$

2.1.2 The compact boson and T-duality.

We now move on to discuss the compact boson in $1+1$ dimensions, whose action reads

$$S_{\mathrm{b}} = \int_M \mathrm{d}^2x \, \frac{\beta^2}{2} \partial_\mu \phi \partial^\mu \phi. \tag{2.18}$$

Here we take the spacetime M to be $M = \mathbb{R} \times \mathrm{S}^1$, with $t \in \mathbb{R}$ parameterizing time and $x \in \mathrm{S}^1$ parameterizing space. "compact" means that ϕ takes values in the target space $\phi \in [0, 2\pi)$, or equivalently ϕ is defined on $\mathbb{R}$ mod 2π:

$$\phi \cong \phi + 2\pi. \tag{2.19}$$

Ordinarily we canonically normalize the kinetic term of ϕ so that it is $\frac{1}{2}(\partial\phi)^2$, absorbing the overall factor of β^2 into ϕ. Doing this here would change the target space of ϕ, and so we expect to get a different theory for different values of β, a suspicion which is almost correct. Note that a mass term $\sim |\phi|^2$ or an interaction term like $\sim |\phi|^4$ are not allowed, since they do not respect the identification (2.19). More precisely, the well-defined variable is $\mathrm{e}^{\mathrm{i}\phi}$. The action is invariant under constant shifts of ϕ, with corresponding Noether current

$$j_{\mathrm{s}}^\mu = \beta^2 \partial^\mu \phi. \tag{2.20}$$

In the context of string theory this is referred to as the momentum current. This theory has another conserved quantity which is harder to spot. It is an example of a topological conservation law. To explain this, let us begin by working at a fixed time. Then, suppressing the time argument for now, $\phi(x)$ defines a map from the spatial S^1 to the target space S^1. Given such a map, as we traverse the coordinate circle going from $x = 0$ to $x = 2\pi$, we can count how many times $\phi(x)$ winds around the target space circle. Counting $+1$ every time we go around the target space clockwise, and -1 for every time we go around counterclockwise.

This defines a number called the winding number. In fact, we can write an integral expression for this number:

$$Q_{\mathrm{w}} := \frac{1}{2\pi} \int \mathrm{d}x \, \partial_x \phi. \tag{2.21}$$

Now it is straightforward to show that Q_{w} is conserved under time-evolution: Reinstating the time-dependence, suppose $\phi(t_0, x)$ has winding number n at time t_0, and consider the time-evolved solution $\phi(t, x)$ evaluated at a later time t_1. Then the field $\phi(t, x)$ can be interpreted as a homotopy between $\phi(t_0, x)$ and $\phi(t_1, x)$, and it is well-known that two continuous maps are homotopic if and only if they have the same winding number. Thus $\phi(t, x)$ has winding number n for all t. □ In fact, a moments thought reveals that this charge can be written as the integral of a local current

$$j_{\mathrm{w}}^{\mu} = \frac{1}{2\pi} \epsilon^{\mu\nu} \partial_\nu \phi, \tag{2.22}$$

from which the conservation of Q_{w} also follows.

T-duality, or particle-vortex duality $1+1$ dimensions.

We now show that the compact boson has another description in terms of a different scalar field. We start by looking at the partition function for the compact boson (2.18)

$$Z = \int \mathrm{D}\phi \, \exp\left[\int_M \mathrm{d}^2 x \, \frac{\beta^2}{2} \partial_\mu \phi \partial^\mu \phi \right]. \tag{2.23}$$

Observe that it is $\partial_\mu \phi$ which enters the action, so we might be tempted to treat $\partial_\mu \phi$ as the independent variable of integration. However this is a little fast, since

(i) $\partial_\mu \phi$ satisfies a Bianchi identity $\partial_\mu (\epsilon^{\mu\nu} \partial_\nu \phi) = 0$, and

(ii) $\frac{1}{2\pi} \int \mathrm{d}^2 x \, \partial_\mu (\epsilon^{\mu\nu} \partial_\nu \phi) = \frac{1}{2\pi} \int \mathrm{d}x^\mu \, \partial_\mu \phi \in \mathbb{Z}$.

Any old vector-valued function $v_\mu(x)$ satisfies neither of these conditions. Therefore we impose such a condition on v_μ with a Lagrange multiplier field $\tilde{\phi}$ as

$$Z = \int \mathrm{D}v\,\mathrm{D}\tilde{\phi}\,\exp\left[\int_M \mathrm{d}^2x\left(\frac{\beta^2}{2}v_\mu v^\mu + \frac{1}{2\pi}\epsilon^{\mu\nu}v_\mu\partial_\nu\tilde{\phi}\right)\right]. \tag{2.24}$$

As a check, we need to do the integral over $\tilde{\phi}$ and check that we end up back at (2.23). Since $\tilde{\phi}$ enters linearly, we can just replace it by its classical equation of motion. Varying $\tilde{\phi} \to \tilde{\phi} + \delta\tilde{\phi}$ and performing an integration-by-parts one finds a bulk term which sets $\partial_\mu v^\mu = 0$, taking care of (i). There is also a boundary term which enforces the quantization condition (ii) if and only if $\tilde{\phi}$ is compact:

$$\tilde{\phi} \cong \tilde{\phi} + 2\pi. \tag{2.25}$$

To get something different, we do the integral over v_μ instead. Since it enters quadratically we can just replace it by its equation of motion

$$v_\mu = \frac{1}{2\pi\beta^2}\epsilon_{\mu\nu}\partial^\nu\tilde{\phi}. \tag{2.26}$$

which gives

$$Z = \int \mathrm{D}\tilde{\phi}\,\exp\left[\int_M \mathrm{d}^2x\,\frac{\tilde{\beta}^2}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi}\right], \quad \tilde{\beta}^2 = \frac{1}{4\pi^2\beta^2}. \tag{2.27}$$

This is of the same form as our original theory (2.23) but with an inverted radius! The best way to get some intuition as to what is going on is to look at how the conserved currents on either side are mapped. We find that the particle density for ϕ is mapped to the winding density for $\tilde{\phi}$

$$\beta^2\partial^\mu\phi \quad \longleftrightarrow \quad \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\tilde{\phi}, \tag{2.28}$$

and vice versa, the winding of ϕ is mapped to the particle density for $\tilde{\phi}$

$$\frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi \quad \longleftrightarrow \quad \tilde{\beta}^2\partial^\mu\tilde{\phi}. \tag{2.29}$$

2.1.3 Bosonization in $1+1$ dimensions.

[62]. The bosonization map is naturally described in terms of left- and right-moving bosons

$$\hat{\phi}_+(x) := \frac{1}{\beta}\int_0^\infty \frac{\mathrm{d}p}{2\pi}\frac{1}{\sqrt{2|p|}}\left(\hat{\phi}(p)\,\mathrm{e}^{\mathrm{i}\,px} + \hat{\phi}^\dagger(p)\,\mathrm{e}^{-\mathrm{i}\,px}\right), \tag{2.30}$$

$$\hat{\phi}_-(x) := \frac{1}{\beta}\int_{-\infty}^0 \frac{\mathrm{d}p}{2\pi}\frac{1}{\sqrt{2|p|}}\left(\hat{\phi}(p)\,\mathrm{e}^{\mathrm{i}\,px} + \hat{\phi}^\dagger(p)\,\mathrm{e}^{-\mathrm{i}\,px}\right), \tag{2.31}$$

where $\hat{\phi}, \hat{\phi}^\dagger$ obey the familiar creation-annihilation algebra

$$[\hat{\phi}(p), \hat{\phi}^\dagger(p)] = 2\pi\delta(p-q), \tag{2.32}$$

and we have taken the liberty to add a convenient normalization. As we mentioned before, the well-defined operators in this model are derivatives of $\hat{\phi}$ and exponentials like[8] $\mathrm{e}^{\mathrm{i}\,\hat{\phi}}$. So the natural objects to compute are the two point functions $\langle \mathrm{e}^{\mathrm{i}\,\phi_\pm(x)}\,\mathrm{e}^{-\mathrm{i}\,\phi_\pm(y)}\rangle$. In computing these correlators one encounters ill-defined integrals like those occurring in (2.15). Regulating them in the same way as in the fermionic case, we find that

$$\langle \mathrm{e}^{\mathrm{i}\,\phi_\pm(x)}\,\mathrm{e}^{-\mathrm{i}\,\phi_\pm(y)}\rangle = \left(\pm\frac{\mathrm{i}\,\varepsilon}{(x-y)\pm\mathrm{i}\,\varepsilon}\right)^{1/4\pi\beta^2}. \tag{2.33}$$

In particular, this makes it clear that the theory indeed depends on the radius β.

[8] $\mathrm{e}^{\mathrm{i}\,\hat{\phi}}$ is defined by its Taylor series expansion and normal ordering each term so that the creation operators $\hat{\phi}^\dagger(p)$ stand to the left of the annihilation operators $\hat{\phi}(p)$.

The bosonization dictionary.

The bosonic correlators (2.33) are strikingly similar to the fermionic ones (2.17) at the radius $\beta^2 = \frac{1}{4\pi}$. Indeed, it looks like we could identify

$$\hat{\psi}_\pm(x) \quad \longleftrightarrow \quad \frac{1}{\sqrt{2\pi\varepsilon}}\, \mathrm{e}^{\pm\mathrm{i}\hat{\phi}_\pm(x)}\,. \tag{2.34}$$

Under this identification the fermion mass-operator $\overline{\psi}\psi$ then maps to

$$\overline{\psi}\psi \quad \longleftrightarrow \quad -\frac{1}{\pi\varepsilon}\cos\phi. \tag{2.35}$$

In all the dualities presented in this work, the most important thing to understand is the mapping between the conserved currents on both sides. Under the duality (2.34) we find that the vector fermionic current maps to the winding current for ϕ, and the axial current maps to the momentum current:

$$j_{\mathrm{v}}^\mu \quad \longleftrightarrow \quad j_{\mathrm{w}}^\mu \tag{2.36}$$

$$j_{\mathrm{a}}^\mu \quad \longleftrightarrow \quad j_{\mathrm{s}}^\mu\,. \tag{2.37}$$

The first identification (2.36) suggests that the ψ quanta are mapped to the winding of ϕ. Indeed Q_{v} measures the number of ψ quanta, whilst its proposed dual Q_{w} measures the winding of ϕ.

2.1.4 Application: Massive Thirring = Sine Gordon.

In this section we put our bosonization duality to work. Consider the massive Thirring model

$$S_{\mathrm{f}} = \int \mathrm{d}^2x \left[\mathrm{i}\,\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi - g(\overline{\psi}\gamma^\mu\psi)(\overline{\psi}\gamma_\mu\psi)\right]. \tag{2.38}$$

Consulting our bosonization dictionary, this is dual to the Sine-Gordon model

$$S_{\mathrm{b}} = \int \mathrm{d}^2x \left[\frac{\beta^2}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m}{\pi\varepsilon}\cos\phi\right], \tag{2.39}$$

where the radius of the boson is

$$\beta^2 = \frac{1}{4\pi} + \frac{g}{2\pi^2}. \tag{2.40}$$

In fact, much like the 3d dualities we describe in Chapter 3, these dualities fit into a 2d duality web [123].

2.2 Abelian gauge fields in $2+1$ dimensions

It may seem that bosonization is special to $1+1$ dimensions. Indeed bosons and fermions are distinguished by their behaviour when we interchange them, one cannot interchange particles in $1+1$d without bringing them together, and so presumably this argument breaks down. Moreover there is no notion of spin in this number of spacetime dimensions, so there is no conflict with the spin-statistics theorem.

Going up to $2+1$ dimensions we can understand particle statistics by orbiting one particle around another, so we would expect a more sophisticated mechanism to allow bose-fermi dualities to exist. This mechanism is found in the interesting behaviour of gauge fields in $2+1$ dimensions; in fact it is not an understatement to say that all of the dualities discussed in this work is down entirely to their rich dynamics in this special number of dimensions. In particular the existence of Chern-Simons interactions will allow us to turn bosons into fermions and vice versa. Most of our discussion is by now rather old and more details can be found in Polyakov's book [13], or more modern references [122, 124].

2.2.1 Free Maxwell fields.

Abelian gauge fields in 3d behave very different to in 4d. One aspect of this is seen by writing out Maxwell's equations with source, one finds that oppositely charged particles separated by a distance r experience a logarithmic potential $V(r) \sim \log r$ - we say that charges are logarithmically confined. In 3d the magnetic field is a (pseudo-) scalar B, whilst the electric field continues to be a vector E_i:

$$B := F_{12}, \quad E_i := F_{0i}. \tag{2.41}$$

Let us start by considering the free Maxwell action. It turns out that this theory is equivalent to the theory of a free massless scalar field $\sigma(x)$. Unsurprisingly σ is referred to as the dual photon. To see how this comes about, start with the partition function for free Maxwell fields

$$Z = \int \mathrm{D}A \, \exp\left[\frac{\mathrm{i}}{2e^2}\int F \wedge \star F\right]. \tag{2.42}$$

Like in our discussion of T-duality in Section 2.1.2, the trick is to promote the field strength $F_{\mu\nu}$ to an independent variable of integration. To do this we must introduce a Lagrange multiplier field σ to enforce the Bianchi identity $\mathrm{d}F = 0$

$$Z = \int \mathrm{D}\sigma \, \mathrm{D}F \, \exp\left[\mathrm{i}\int\left(\frac{1}{2e^2}F \wedge \star F + \frac{1}{4\pi}\sigma \, \mathrm{d}F\right)\right]. \tag{2.43}$$

Moreover F should satisfy the Dirac quantization condition $\frac{1}{2\pi}\int F \in \mathbb{Z}$, using the same arguments as before this means that σ should be compact:

$$\sigma \cong \sigma + 2\pi. \tag{2.44}$$

To get something interesting we now integrate over F. Since it only enters quadratically, this means we just replace it using its equation of motion

$$F_{\mu\nu} = -\frac{e^2}{2\pi}\epsilon^{\mu\nu\rho}\partial_\rho\sigma. \tag{2.45}$$

Plugging this back in

$$Z = \int \mathrm{D}\sigma \, \exp\left[\frac{\mathrm{i}\, e^2}{8\pi^2}\int \mathrm{d}^3x \, (\partial\sigma)^2\right]. \tag{2.46}$$

Remarkably, in the dual photon formulation it is plainly obvious that the photon has a global symmetry given by constant shift in $\sigma \to \sigma + c$, which is generated by the current

$$J^\mu = -\frac{e^2}{4\pi^2}\partial^\mu \sigma. \tag{2.47}$$

Importantly this symmetry group is $\mathrm{U}(1)$ rather than $\mathbb{R}$ because σ is compact. It is interesting to rewrite the current in terms of our original variable A_μ using the equation of motion (2.45) for σ

$$J^\mu = \frac{1}{4\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho}. \tag{2.48}$$

The global symmetry of this theory is rather unusual, and is similar the the winding symmetry of the compact boson discussed in Section 2.1.2. It is an example of a what is called a topological symmetry. Different to usual Noether currents, this is conserved off-shell; i.e. without the use of the classical equations of motion. Rather its conservation is a consequence of the Bianchi identity $\mathrm{d}F = 0$. The associated conserved quantity is the magnetic charge

$$Q := \int \mathrm{d}^2x\, J^0 = \frac{1}{2\pi}\int \mathrm{d}^2x\, B. \tag{2.49}$$

Operators which are charged under Q are called monopole operators[9]. The definition of the monopole creation operator $\mathcal{M}^\dagger(x)$ is slightly indirect, and is most naturally described in terms of the path integral. We define the action of the monopole operator $\mathcal{M}^\dagger(x)$ by first removing the point x from the path integral and integrating over all gauge fields subject to the monopole boundary condition

$$\int_x F = 2\pi. \tag{2.50}$$

[9] More generally Q is an example of a topological charge. Operators which carry such a topological charge are called disorder or defect operators.

To check that $\mathcal{M}^\dagger(x)$ does the job, one can compute the commutator $[Q, \mathcal{M}^\dagger(x)]$, and show that it equals $+1$ (this is done in e.g. [122]). The finite form of which is

$$\mathcal{M}(x) \to e^{i\alpha}\, \mathcal{M}(x). \tag{2.51}$$

In fact, in the special case of free Maxwell fields that we are considering, there is a very explicit formula in terms of the dual photon. We claim that it is given by $\mathcal{M}^\dagger(x) = e^{i\sigma(x)}$. The easiest way to see this is to insert $e^{i\sigma(x)}$ into the path integral expression (2.43). Doing the integral over σ then just constrains F to obey the monopole boundary condition (2.50) - which is precisely our definition of $\mathcal{M}^\dagger(x)$.

Finally we note that one can view the photon as the massless Nambu-Goldstone mode of the broken $U(1)$ magnetic symmetry [125] - this way of thinking will tie in beautifully with our analysis of some phase diagrams in Section 2.3. The diagnostic of this spontaneous symmetry breaking is nothing but the monopole operator, which has a non-zero expectation value in the vacuum. In fact, this interpretation becomes immediate when we think in terms of the the dual photon $\sigma(x)$. Here the magnetic symmetry is just the shift $\sigma \to \sigma + \text{constant}$, but the action (2.46) has a degeneracy of ground states described by $\sigma = \text{constant}$.

Finally, one can consider more general theories with vacua for which the $U(1)$ magnetic symmetry is unbroken. We will encounter such an example in Section 2.3.1 when we couple the photon to a charged scalar field. We will find a phase whose vacua consists of magnetically charged vortices, which restore the magnetic symmetry. So what happens to the photon in this phase? For our interpretation to be of any substance, we should expect the photon to be massive. This is indeed the case! We will see that, in this vortex phase the gauge symmetry is broken, and the photon - whilst still present - becomes massive by the Higgs mechanism.

2.2.2 $U(1)$ **Chern-Simons theory.**

As we have already mentioned several times, gauge theories in $2+1$ dimensions have very different behaviour from those in one dimension higher. Another example of this is the existence of Chern-Simons interactions. The study of such theories has had numerous applications over the years; a non-exhaustive list is: Witten's work [48] connecting the observables in Chern-Simons theory - namely products of Wilson lines - with the Jones polynomial, which appears in the mathematical study of knots invariants, for which he won a Fields medal. There is also an intimate connection between Chern-Simons theory and rational conformal field theories in two dimensions [126, 127], Quantum gravity in three dimensions [128], and more down to earth physics such as describing quantum Hall states [129].

Our main focus will be on Abelian Chern-Simons theory, so we shall focus on that here, saving a short discussion on the non-Abelian theory for Section 2.5. Given a $\mathrm{U}(1)$ gauge field A_μ, the Chern-Simons action is defined to be

$$S_{\mathrm{CS}}[A;k] := \frac{k}{4\pi}\int \mathrm{d}^3x\,\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho. \tag{2.52}$$

Here k is a coupling constant called the **level**. This theory is sometimes denoted $\mathrm{U}(1)_k$. The action doesn't look manifestly gauge invariant, so let's check it. Under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\alpha$, the change in S_{CS} is a total derivative

$$\delta S_{\mathrm{CS}}[A;k] = \frac{k}{4\pi}\int \mathrm{d}^3x\,\epsilon^{\mu\nu\rho}\partial_\mu(\alpha\partial_\nu A_\rho).$$

This is gauge invariant if we can neglect boundary conditions, such as on spacetimes like $\mathbb{R}^3$. However, quantizing the theory on arbitrary surfaces we discover that k should be quantized:

$$k \in \mathbb{Z}. \tag{2.53}$$

We can write the Chern-Simons action in a geometric formulation as

$$S_{\mathrm{CS}}[A;k] = \frac{k}{4\pi}\int_X A\wedge \mathrm{d}A. \tag{2.54}$$

This makes it clear that the theory is topological, i.e. it is independent of the spacetime metric on X, at least classically. However it turns out that we must pick a metric when defining the quantum theory, this is the framing anomaly [48].

Flux attachment.

The crucial element in 3d bosonization is flux attachment [69, 72, 73]. It is interesting to consider coupling this to a conserved current

$$S = \int \mathrm{d}^3x \left(\frac{k}{4\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + A_\mu J^\mu \right).$$

The classical equation of motion is easily found to be $\frac{k}{4\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho} = J^\mu$. Decomposing the current as $J^\mu = (\rho, \vec{J})$, where ρ charge density and $\vec{J}$ current density, the only independent equation of motion is[10]

$$B = \frac{2\pi}{k} \cdot \rho, \tag{2.55}$$

where B is the magnetic field defined in (2.41). We see that the Chern-Simons fields have no dynamics of their own, their only effect is to attach a magnetic flux to each charged particle. This has an interesting effect when we orbit one particle around another. The state vector $|\psi\rangle$ describing two particles picks up an Aharonov-Bohm phase $|\psi\rangle \to \mathrm{e}^{2\pi \mathrm{i}/k} |\psi\rangle$. The exchange phase is then half of this

$$|\psi\rangle \to \pm \mathrm{e}^{\pi \mathrm{i}/k} |\psi\rangle \, . \tag{2.56}$$

Where we take the $-$ sign if both the underlying particles (i.e. with the CS interactions switched off) are fermions. Thus we can get a variety of statistics depending on the value of k, leading to the theory of anyons.

2.3 Particle-vortex duality.

In this section we discuss a rather old duality. This is the bosonic particle-vortex duality between the critical point of the O(2) model and its gauged version. The O(2) model (also

[10] The equation of motion involving $\vec{J}$ just tells this equation of motion is preserved under time evolution.

called the XY-model or Landau-Ginzburg model in condensed matter) is described by the Lagrangian

$$\mathscr{L}_{\mathrm{XY}} = \partial_\mu \overline{\phi} \partial^\mu \phi - U(|\phi|), \tag{2.57}$$

where ϕ is a complex scalar field and the potential $U(|\phi|)$ is given by

$$U(|\phi|) = \mu|\phi|^2 + \lambda|\phi|^4. \tag{2.58}$$

This theory arises, for example, in the continuum description of thin-film superconductors [130, 131], whilst the Euclidean version describes the superfluid transition in Helium [104, 132]. This theory has a global $\mathrm{U}(1)$ symmetry $\phi \to \mathrm{e}^{\mathrm{i}\,\alpha}\,\phi$. We can gauge the global $\mathrm{U}(1)$ symmetry of the $\mathrm{O}(2)$ model in the standard way. We promote the symmetry to a local $\mathrm{U}(1)$ gauge invariance $\phi \to \mathrm{e}^{\mathrm{i}\,\alpha(x)}\,\phi$ by introducing a gauge field A_μ and arrive at the Abelian Higgs model

$$\mathscr{L}_{\mathrm{AH}} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \mathrm{D}_\mu \overline{\phi} \mathrm{D}^\mu \phi - U(|\phi|). \tag{2.59}$$

Here the potential $U(|\phi|)$ is of the same form as in (2.58). One interesting physical application of this theory is the continuum description of superconductors. It is convenient to write the potential as

$$U(|\phi|) = \lambda(|\phi|^2 + v)^2 + \text{constant}, \tag{2.60}$$

where

$$v = \frac{\mu}{2\lambda}. \tag{2.61}$$

We shall then use the freedom to add a constant so that the minimum of $U(|\phi|)$ is zero.

Before going on to discuss these theories in detail and state the duality, let us discuss the philosophy that we will employ to conjecture the duality. This has provided a very powerful tool in the more modern dualities discussed in Chapter 3.

Philosophy.

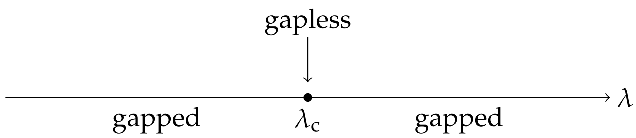
Consider a general relativistic QFT in D spacetime dimensions described by a Hamiltonian $\hat{H}$. As is usual in QFT, we assume the state of lowest energy - the vacuum - has zero energy $\hat{H}\,|\text{vac}\rangle = 0$. First of all we need to introduce some simple terminology:

- We say that the QFT is gapped if the energy difference ΔE between the vacuum state and the next lowest-energy state is finite.
- Conversely, we say that a QFT is gapless if there are states whose energy is arbitrarily close to zero.

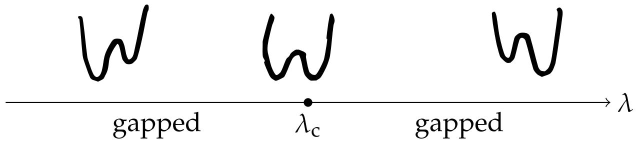
Our Hamiltonian will depend on a number of parameters $\hat{H}(\lambda, \dots)$. Generically it will be gapped, by which we mean that if the parameters are arbitrary, it is overwhelmingly likely that it will be gapped. On the other hand a gapless Hamiltonian is rather hard to come by. There are two well known mechanisms:

- The parameters can sometimes be very carefully chosen to lie on a point, or more generally a submanifold of the coupling space. If the Hamiltonian describes a unitary relativistic QFT, then it is believed that such points correspond to a conformal field theory.
- The spontaneous breaking of a continuous symmetry via the Nambu-Goldstone mechanism.

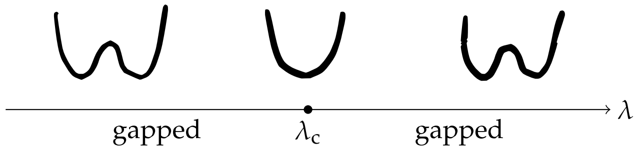
With this in mind, we now go on to consider phase transitions. Consider varying a parameter λ on which the Hamiltonian $\hat{H} = \hat{H}(\lambda, \dots)$ depends. As mentioned above we will generically lie in a gapped phase, but we could find a point describing a gapless phase



One possibility is that stable vacua are exchanged and become degenerate in energy at the critical point



Such a transition is called 1^{st} order. A famous example of this is the boiling point of water, where $\lambda = T$ is the temperature. One vacua describes liquid water, and the other vacua describes vapour[11]. Another possibility is that the vacua merge



Such a transition is called 2^{nd} order. This is the case of interest to us. Assuming our QFT is unitary, it is believed that such a point is described by a conformal field theory. By trying to understand the phase diagrams of various field theories, we may conjecture that they flow to the same CFT.

To conclude this discussion, let us note that, the characterization of gapped and gapless is of course rather crude. When one speaks of a gapped phase, it might be tempting to think that it is trivial at energies below the gap. But as we saw in our discussion of Chern-Simons theory in Section 2.2.2 we know that this is not all that can happen. The UV particle content still survives as classical, heavy probe particles. We can braid these particles and find non-trivial Aharanov-Bohm phases.

Now we move on to understanding the theories introduced in (2.57) and (2.59). We will begin by studying the gauged $O(2)$ model. At first glance it looks to be the richer theory, and we can take our time to study it in detail. The remarkable thing is that a lot of its behaviour is reproduced in the $O(2)$ model.

The duality we are going to describe was originally understood on lattice models [66], which are described by the continuum theories (2.57) and (2.59) in the long wavelength limit.

[11] There is, of course, another parameter which is the pressure. The water-vapour phase exists on a submanifold parameterized by the pressure and temperature.

For some useful background on this perspective see Polyakov's book [13] or the review articles [133–136].

2.3.1 The Gauged $\mathrm{O}(2)$ model.

A natural place to start is by determining what global symmetries the theory admits. We've gauged the $\mathrm{U}(1)$ phase rotation. However, recall from our study in Section 2.2.1, we saw that Abelian gauge theories are special in that they enjoy an ordinary[12] global magnetic symmetry generated by the current

$$J^{\mu}_{\text{mag}} = \frac{1}{4\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho}. \tag{2.62}$$

This is conserved by virtue of the Bianchi identity, so its conservation is not spoiled by the addition of matter. The next natural step is to determine the objects which carry this charge. It will turn out that the answer will fall out on our laps if we ask the far weaker question: What do the finite-energy solutions look like [5]? Our analysis will be simplest in temporal gauge $A_0 = 0$. It will be important to note that once we have picked this gauge, we still have the freedom to make time-independent gauge transformations. We can further simplify this question by considering a field configuration at a fixed time t_0, since time evolution takes finite energy solutions to finite energy solutions. The analysis is then most naturally done using polar coordinates (r, θ). Since we are working at a fixed time, we shall suppress the time arguments in the fields. It is convenient to split the Hamiltonian up into a kinetic energy piece and a potential energy piece, $H = T + V$ where

$$T := \int r\,\mathrm{d}r\,\mathrm{d}\theta\left(\frac{1}{2}\partial_t A_i \partial_t A_i + |\partial_t\phi|^2\right), \tag{2.63}$$

$$V := \int r\,\mathrm{d}r\,\mathrm{d}\theta\left(\frac{1}{4e^2}F_{ij}F_{ij} + \mathrm{D}_i\overline{\phi}\mathrm{D}_i\phi + U(|\phi|)\right). \tag{2.64}$$

Since each term is separately positive, it is necessary for the integral of each individual term to be finite, so we can focus on each term separately. The important part is the contribution

[12] [137]

from the potential energy V, and the most contribution to this is from the integral over $U(|\phi|)$. For this to be finite, it is necessary that $\phi(r,\theta)$ approach a zero of $U(|\phi)$ as $r \to \infty$:

$$\phi_\infty(\theta) := \lim_{r\to\infty} \phi(r,\theta) \text{ must be a zero of } U(|\phi|). \tag{2.65}$$

Now we look at the contribution from the covariant derivatives. This investigation is simplified further by using freedom to make a further time-independent gauge transformation to set[13] $A_r = 0$ on the time slice $t = t_0$ (of course A_r will generically be non-zero for $t \neq t_0$). In this gauge the covariant derivatives appearing in (2.64) simplify to

$$\mathrm{D}_r\phi = \partial_r\phi, \quad \& \quad \mathrm{D}_\theta\phi = \frac{1}{r}\Big(\partial_\theta + \mathrm{i}\,A_\theta\Big)\phi. \tag{2.66}$$

Thus for the contribution from this term to converge, it is necessary that

$$\lim_{r\to\infty} A_\theta(r,\theta) = \mathrm{i}\,\frac{1}{\phi_\infty(\theta)}\partial_\theta\phi_\infty(\theta). \tag{2.67}$$

Thus the asymptotic field data is determined entirely in terms of the behaviour of ϕ at $r = \infty$. I.e. it is determined entirely in terms of $\phi_\infty(\theta)$ which defines a mapping

$$\mathbb{S}^1 \to G/H, \quad \theta \mapsto \phi_\infty(\theta), \tag{2.68}$$

from the asymptotic boundary of space, parameterized by $\theta \in \mathbb{S}^1$, into the set of zeroes of $U(|\phi|)$, which we have mysteriously denoted G/H. Clearly then, everything hinges on the form of the potential, or more precisely its set of zeroes, which we refer to as the vacuum manifold. As we dial μ the behaviour of the potential qualitatively changes near $\mu = 0$. For $\mu > 0$ there is one unique minima of $U(|\phi|)$ given by $|\phi| = 0$, whilst for $\mu < 0$ any field configuration ϕ with $|\phi|^2 = |v|$ is a minimum. Thus the analysis now splits into these two cases.

[13] Such a gauge transformation is necessarily singular near $r = 0$. However we are only discussing finite enegy solutions, so only the behaviour near infinity matters.

The $\mu \gg 0$ phase.

In this case $v \geq 0$ and the gauge symmetry is unbroken, and G/H is trivial and there no vortices. At energies below μ this describes a gapless phase consisting of just free Maxwell theory which we studied in Section 2.2.1. There we saw that the $\mathrm{U}(1)_{\mathrm{mag}}$ symmetry was broken, and the photon could be viewed as the massless Nambu-Goldstone boson associated to this breaking. The ϕ field is now just a massive charged probe particle (i.e. it has no dynamics, we view it as a classical field). If we insert two oppositely charged probe particles, they experience Coulomb potential, hence we refer to this phase as the **Coulomb phase** of the gauged O(2) model. As we mentioned earlier, the Coulomb potential goes like $\log R$, where R is the particle separation - the particles are logarithmically confined.

The $\mu \ll 0$ phase.

In this phase the vacuum manifold G/H of $U(|\phi|)$ is a circle parameterized by $|\phi|^2 = |v|$. The scalar acquires a non-zero vacuum expectation value, breaking the U(1) gauge symmetry and giving a mass to the photon by the Higgs mechanism. Hence we refer to this phase as the **Higgs phase**. This phase is more interesting because it can support vortices. The asymptotic field data now defines a map into U(1)

$$\phi_\infty : \mathbb{S}^1 \to \mathrm{U}(1), \quad \theta \mapsto v\, \mathrm{e}^{-\mathrm{i}\sigma(\theta)} \tag{2.69}$$

An identical argument to the one given in Section 2.1.2 shows that the winding number

$$n = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \frac{d\sigma}{d\theta}. \tag{2.70}$$

is conserved under time evolution. Such solutions are known as Abrikosov-Nielsen-Olesen (ANO) vortices. We will now show that a vortex of winding number n carries precisely n units of magnetic charge. First of all, recall from (2.49) that the magnetic charge is defined as

$$Q := \frac{1}{2\pi} \int \mathrm{d}^2x\, B, \tag{2.71}$$

where $B = F_{12}$. We can write this as the integral of the 2-form $\mathrm{d}A = F_{12}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$ and then use Stokes' theorem to get

$$Q = \frac{1}{2\pi}\int \mathrm{d}A = \frac{1}{2\pi}\oint A. \tag{2.72}$$

Working in polar coordinates the remaining integral is

$$Q = \frac{1}{2\pi}\lim_{r\to\infty}\int_0^{2\pi}\mathrm{d}\theta\, A_\theta \overset{(2.67)}{=} \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\,\frac{\mathrm{i}}{\phi_\infty(\theta)}\partial_\theta\phi_\infty(\theta) \overset{(2.69)}{=} \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\,\frac{d\sigma}{d\theta}. \tag{2.73}$$

The remaining integral is precisely the definition of the winding number in (2.70), which proves our claim. In fact (running our earlier discussion on finite-energy solutions in reverse) the monopole operator $\mathcal{M}^{(n)\dagger}(x)$ necessarily creates a vortex carrying n units of magnetic charge at x.

Before going on to discuss the $\mathrm{O}(2)$ model, let us note that this is mathematically identical (at least for time-independent solutions) to Landau-Ginzburg theory of type II superconductors. The solutions we have discovered are called flux lines.

2.3.2 $\mathrm{O}(2)$ model and the Wilson-Fisher fixed point.

The **$\mathrm{O}(2)$ model** has a global $\mathrm{U}(1)$ symmetry[14] $\phi \to e^{\mathrm{i}\,\alpha}\,\phi$ with conserved current

$$J^\mu = \mathrm{i}(\overline{\phi}\partial^\mu\phi - \phi\partial^\mu\overline{\phi}), \tag{2.74}$$

which measures the particle density. As before we now consider the phases as we vary the mass parameter μ.

[14] The name $\mathrm{O}(2)$ model comes because we can decompose the complex scalar field ϕ into real and imaginary parts as $\phi = \phi^1 + \mathrm{i}\,\phi^2$, and form the 2-component vector

$$\vec{\phi} = \begin{pmatrix}\phi^1 \\ \phi^2\end{pmatrix}.$$

Writing the action in terms of $\vec{\phi}$, the symmetry group of the theory becomes $\mathrm{O}(2)$ acting on $\vec{\phi}$ as $\vec{\phi} \to O\vec{\phi}$, $O \in \mathrm{O}(2)$.

$\mu \gg 0$ **phase.**

This part of the parameter space is pretty boring. It describes a gapped phase in which the global $U(1)$ symmetry is unbroken. It consists of massive ϕ probe particles carrying unit $U(1)$ charge.

$\mu \ll 0$ **phase.**

In this phase ϕ acquires a vacuum expectation value, breaking the global $U(1)$ symmetry. Expanding ϕ around this vacuum as $\phi = \rho\, e^{i\sigma}$, we see that the theory contains a massless Nambu-Goldstone mode σ and a heavy ρ particle. We can write down an effective Lagrangian for the massless mode

$$\mathscr{L}_{\text{eff}} = \frac{1}{2}\Delta^2 \partial_\mu \sigma \partial^\mu \sigma + O((\partial\sigma)^4). \tag{2.75}$$

This form of the effective action follows from the fact that σ describes the phase of ϕ, and thus σ must be compact $\sigma \cong \sigma + 2\pi$.

Using a similar analysis to what we used in the Abelian Higgs model (essentially just turning off the gauge field), one can show that this phase admits vortex-antivortex solutions where σ winds asymptotically. In particular, solutions describing a single vortex or a single antivortex are not allowed as they cost infinite energy, but a vortex-antivortex pair is a perfectly well-defined finite energy solution - essentially because there is no asymptotic winding and the solution dies down to a constant at infinity. Performing this computation one finds that the energy of a vortex-antivortex solution grows logarithmically with their separation - i.e. they are logarithmically confined.

2.3.3 One more striking correspondence.

For fun we just mention one more striking similarity between these two theories. Consider adding a small symmetry breaking term to the Lagrangian of the $O(2)$ model

$$\mathscr{L}_{O(2)} \to \mathscr{L}_{O(2)} + \varepsilon \operatorname{Re} \phi.$$

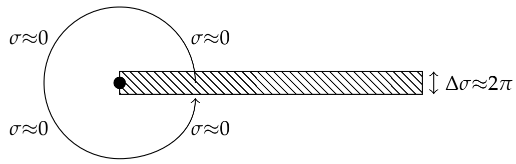


Figure 1: This is a schematic picture of the phase of the Goldstone mode σ. As we go around the origin it remains close to zero, and as we are about to come back to our starting point it suddenly jumps to 2π over the short width depicted by the hashed lined.

In the symmetry breaking phase $\mu \ll 0$ this adds $\sim \varepsilon \cos \sigma$ to the effective action for the Nambu-Goldstone mode σ. Now if we consider vortices in this phase, the potential favours $\sigma \approx 0$. Thus as we go around traverse the asymptotic boundary, σ stays zero almost everywhere and then makes a sudden jump from $\sigma \approx 0$ to $\sigma \approx 2\pi$. This looks like a confining flux tube! Indeed this behaviour is mirrored in the Abelian Higgs model by adding the monopole operator to the Lagrangian, explicitly breaking the $U(1)_{\text{mag}}$ symmetry:

$$\mathscr{L}_{\text{Abelian Higgs}} \to \mathscr{L}_{\text{Abelian Higgs}} + \varepsilon \mathcal{M}^{(1)},$$

This is Polyakov's confinement via monopole condensation [138], [139].

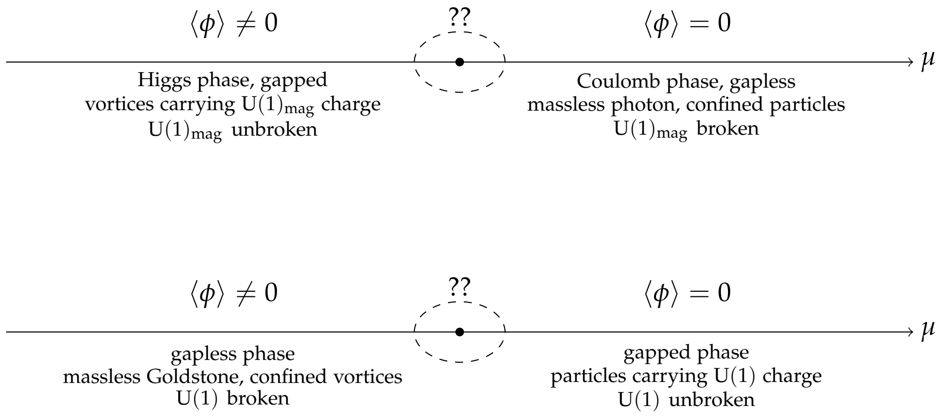


Figure 2: This depicts the phase diagram of the Abelian Higgs model (top) and the $O(2)$ model (bottom) as we vary the mass parameter μ. The region enclosed by the dashed circle is where the classical analysis breaks down and a quantum analysis is required.

2.3.4 The duality.

The phase diagrams are summarized in Figure 2. We see that there is a striking similarity between the phase diagrams! Indeed the unbroken Coulomb phase of the Abelian Higgs looks the same as the broken phase of the $O(2)$ model. The confined charges in the gauge theory mapping to the confined vortices in the $O(2)$ model. Similarly the symmetry-breaking Higgs phase looks identical to the unbroken phase of the $O(2)$ model.

Clearly both theories exhibit wildly different behaviour depending on whether $\mu \gg 0$ or $\mu \ll 0$, and so we expect a phase transition to occur at some point between the two phases. Moreover, being optimists, we might hope that this phase transition is second order and thus described by a conformal field theory. We might be even more optimistic and hope that both critical points are described by the same CFT. Do we have any analytic understanding of this possible critical point? In the case of the Abelian Higgs model the answer is negative, however we do have some understanding of the $O(2)$ model. The critical point of the $O(2)$ model can be understood in the ϵ-expansion [118, 140], which is reviewed in [1, 16]. Starting from the UV Lagrangian (2.57) and flowing the IR while tuning the physical mass to zero, we reach a critical fixed point where the $|\phi|^6$ interaction becomes irrelevant:

$$\mathscr{L} = \partial_\mu \bar{\phi} \partial^\mu \phi - \lambda_* |\phi|^4. \tag{2.76}$$

This is known as the Wilson-Fisher fixed point, and is of enormous importance in physics[15]. For instance, in $d = 3$ Euclidean dimensions it can be used to describe the superfluid transition in ^{4}He [132]. It is a strongly interacting CFT making analytic study hard. Recent advances in the conformal bootstrap has taught us a lot about this theory and its more general $\mathrm{O}(N)$ brethren [141–143]. is called the Wilson-Fisher fixed point. Since it has its own name, it will be of no surprise that it is of enormous importance in physics

The duality.

The statement of particle-vortex duality is that the critical point of the Abelian Higgs model lies in the same universality class as the $\mathrm{O}(2)$ model. Commonly written as [107]

$$|\partial\phi|^2 - \lambda_*|\phi|^4 \qquad \longleftrightarrow \qquad |\mathrm{D}_{\widehat{a}}\widehat{\phi}|^2 - \widehat{\lambda}_*|\widehat{\phi}|^4. \tag{2.77}$$

In particular we have dropped the gauge field kinetic term because it is irrelevant in the IR. This was proven to hold on a lattice theory in [66], and the approach to the continuum limit was studied numerically in [67] where they found strong evidence for the existence of a second order phase transition. There is now overwhelming numerical evidence that this is correct [114, 115].

On the face of it, it seems remarkable that these two theories could be dual. The $\mathrm{O}(2)$ model has less degrees of freedom and less parameters than the Abelian-Higgs model! The point is that it is an IR duality. The number of degrees of freedom is really a UV thing, as we approach the IR we can effectively lose degrees of freedom. Similarly for the parameters, these can become irrelevant and their effects die away in the IR. The duality holds at the CFT and in a small neighborhood of it. If we go too far from this point, the theories describe completely different physics again - there is no contradiction.

With the hard work now behind us, we can quite easily determine how the operators should map. Like any duality in physics, the conserved currents should map into each other:

$$J^\mu_{\mathrm{mag}} \quad \longleftrightarrow \quad J^\mu. \tag{2.78}$$

[15] As are the fixed points of $\mathrm{O}(3)$ or more generally $\mathrm{O}(N)$ models. See [135] for example.

This ties in nicely with the phase diagram in Figure 2, where we saw that the magnetically charged vortices in the Higgs phase map to the charged particles in the O(2) model. From our understanding of the phase diagram we expect

$$|\phi|^2 \quad \longleftrightarrow \quad -|\widehat{\phi}|^2, \tag{2.79}$$

although these are strongly coupled theories so there could be operator mixing. Again, from our understanding of the phase diagram we might write something like

$$\phi \text{ particles} \qquad \longleftrightarrow \qquad \widehat{\phi} \text{ vortices} \tag{2.80}$$

$$\phi \text{ vortices} \qquad \longleftrightarrow \qquad \widehat{\phi} \text{ particles.} \tag{2.81}$$

The first line can be interpreted more precisely as

$$\phi(x) \quad \longleftrightarrow \quad \mathcal{M}(x). \tag{2.82}$$

The second line is not really well-defined since $\widehat{\phi}$ is not gauge invariant; it is qualitatively true at best.

2.4 Fermions and the parity anomaly.

We now start progressing towards the more recent bosonization dualities. The first thing to discuss is the humble theory of a massless Dirac fermion coupled to a background gauge field. The action reads

$$S[\psi, A] = \int \mathrm{d}^3x\, \overline{\psi}\, \mathrm{i}\, \not{D}\psi. \tag{2.83}$$

However to define the quantum theory we must also specify a regularization scheme. As is well known, sometimes it is impossible to regulate a theory which preserves all of the classical symmetries (without introducing additional degrees of freedom). Classical symmetries which are not preserved at the quantum level are called anomalous. As is often the case in the study of anomalies, it is easiest to work in Euclidean signature, where the Dirac operator $\mathrm{i}\,\not{D}$ is Hermitian[16].

The action is both gauge-invariant, and T-invariant (as well as other symmetries like Poincaré symmetry, but these take a side role). It will turn out that we cannot define a quantum theory which preserves both of these symmetries - one of them is necessarily anomalous [144, 145] (c.f. also [146]). However it is crucial that we maintain gauge symmetries at the quantum level. Indeed gauge symmetry is not a symmetry at all, it is a redundancy in the description of our theory introduced for our convenience and nothing should depend on this redundancy (at the classical or quantum level). The place to look for these issues is the partition function [144, 145, 147]

$$Z[A] := \int \mathrm{D}\psi\, \mathrm{D}\overline{\psi}\ \mathrm{e}^{-S[\psi,A]}\,. \tag{2.84}$$

It turns out that this is not gauge-invariant! Formally this is equal to $\det(\mathrm{i}\,\not{D})$, and since $\mathrm{i}\,\not{D}$ is hermitian its eigenvalues λ_a are real. Thus

$$Z[A] = \det(\mathrm{i}\,\not{D}) = \prod_a \lambda_a. \tag{2.85}$$

This infinite product doesn't make much sense. For instance, how do we define its sign? Let us try method where we arbitrarily fix the sign of the partition function for some fiducial vector potential A_0. As we continuously change A the eigenvalues λ_a will of course change. We might hope to define the sign of $Z[A]$ for arbitrary A by continuously changing it from A_0 and each time an eigenvalue λ_a passes through zero we could declare that the sign of $Z[A]$ changes. With this definition, it had better be the case that the sign does not change under a gauge transformation $A^{(0)} \to A^{(1)} = A^{(0)} + \mathrm{d}\alpha$. The trick is to define interpolating gauge field

$$A(s;x) := (1-s)A^{(0)}(x) + sA^{(1)}(x), \tag{2.86}$$

[16] acting on the space of square integrable functions, equipped with the standard L^2 inner product $\langle f, g\rangle := \int \mathrm{d}^3x \overline{f}(x)g(x)$.

such that $A(0;x) = A^{(0)}(x)$ and $A(1;x) = A^{(1)}(x)$. We now view $A(s;x)$ as a four-dimensional gauge field living on $I \times \mathbb{R}^3$, where I is the closed interval $I := [0,1]$. Then number of eigenvalues which pass through 0 is given by the index of *four*-dimensional Dirac operator $\mathrm{i}\,\not{\mathcal{D}}$ on the space $I \times \mathbb{R}^3$. There is no reason for this to vanish, and indeed it does not in general.

However we should be dubious of the above analysis since we know that when we define a quantum theory, we should really have a regulator in mind. A particularly lucid regulator for the problem at hand, which manifestly preserves gauge-invariance (at the classical level at least), is a Pauli-Villars regulator. This adds a fictitious fermion Ψ with a large mass M, obeying the wrong spin-statistics relation - this ensures that the propagator from the Pauli-Villars fermion Ψ enters with the opposite sign to the physical fermion ψ. The modified action reads

$$S[\psi, A, \Psi; M] = \int \mathrm{d}^3x \left(\overline{\psi}\,\mathrm{i}\,\not{D}\psi + \overline{\Psi}\,\mathrm{i}\,\not{D}\Psi + \mathrm{i}\,M\overline{\Psi}\Psi \right). \tag{2.87}$$

As will be demonstrated below, it is now quite straightforward to show that the theory regulated in this way is gauge-invariant. In particular, note that the spectral flow argument given above makes no sense - the partition function is in general a complex number, and how do we define the sign of a complex number?

To understand the effect of this regulator, it is interesting to look at the effective action $S_{\mathrm{eff}}[\psi, A]$ for ψ and A obtained by integrating out the Pauli-Villars fermions. This is defined via

$$Z_{\pm}[A] := \lim_{M \to \pm\infty} \int \mathrm{D}\Psi\,\mathrm{D}\overline{\Psi}\,\mathrm{D}\psi\,\mathrm{D}\overline{\psi}\; \mathrm{e}^{-S[\psi, A, \Psi; M]}\,. \tag{2.88}$$

Since ψ is fermionic, its integral gives a factor of $\det(\mathrm{i}\,\not{D})$, whilst the integral over the bosonic field Ψ gives $[\det(\mathrm{i}\,\not{D} + \mathrm{i}\,M)]^{-1}$. Taking the limit $M \to \pm\infty$ gives

$$Z_{\pm}[A] = \prod_k \frac{\lambda_k}{\lambda_k + \mathrm{i}\,M} \overset{M\to\pm\infty}{\longrightarrow} |Z| \exp\left(\mp \frac{\mathrm{i}\,\pi}{2} \sum_k \mathrm{sgn}(\lambda_k) \right). \tag{2.89}$$

We are now almost there, only the sum over eigenvalues needs regulating. Atiyah, Patodi and Singer used the definition[17][148–150]

$$\eta[A] := \lim_{s\to 0}\sum_k \mathrm{sgn}(\lambda_k)|\lambda_k|^{-s}. \tag{2.90}$$

So finally

$$Z_\pm[A] = |Z|\, \mathrm{e}^{\mp\frac{\mathrm{i}\pi}{2}\eta[A]}, \tag{2.91}$$

gives us a well-defined, gauge-invariant, quantum theory. The minus (resp. plus) sign in the exponent corresponds to a Paulli-Villars fermion with mass $M \to +\infty$ (resp. $M \to -\infty$). Acting with T on the Pauli-Villars action before taking the limit $|M| \to \infty$ has the effect of replacing $M \to -M$, and thus the partition function is not T-invariant.

For reasons we shall now explain, the theory of a Dirac fermion coupled to a background gauge field, regulated using a Pauli-Villars regulator field of mass $M \to +\infty$ is commonly referred to as a fermion coupled to $\mathrm{U}(1)_{-1/2}$.

On an arbitrary Riemannian manifold X with boundary, the Atiyah-Patodi-Index theorem tells us that [147]

$$\widehat{A} - I_\theta - \frac{\eta}{2} \in \mathbb{Z}, \tag{2.92}$$

where I_θ is the instanton number of A (also called a theta term) and $\widehat{A}$ is a gravitational term:

$$I_\theta[F] := \frac{1}{2}\int_X \frac{F\wedge F}{(2\pi)^2}, \quad \widehat{A}[R] := -\frac{1}{48}\int_X \frac{\mathrm{Tr}\, R\wedge R}{(2\pi)^2}. \tag{2.93}$$

Here R is the Riemann tensor built out of the spin connection ω, viewed as a matrix-valued 2-form (the details will not concern us). In fact, the theorem tells us that the combination in (2.92) is equal to the Dirac index, however all that matters for us is that the combination above is integral. Now, the important point is that, up to numerical factors, $I_\theta[F]$ is equal to the Chern-Simons action $\mathrm{CS}[A]$ of A on the boundary ∂X, and similarly $\widehat{A}[R]$ is equal to the

[17] The expression $\sum_k \mathrm{sgn}(\lambda_k)|\lambda_k|^{-s}$ is well-defined and analytic for large $\mathrm{Re}\, s$. One then analytically continues to $s \to 0$.

gravitational Chern-Simons action $\mathrm{CS}_{\mathrm{grav}}[\omega]$ of the spin connection ω on ∂X. Plugging this back into (2.92) and inserting the correct numerical factors

$$\frac{\pi\eta}{2} = \frac{1}{2}\Big(\mathrm{CS}[A] - 2\mathrm{CS}_{\mathrm{grav}}[\omega]\Big) \mod \mathbb{Z}. \tag{2.94}$$

This formula will be important for us in the next couple of chapters. Presently it motivates our notation above for the theory regulated using a positive mass $M \to +\infty$ Pauli-Villars field, which we referred to as a fermion coupled to $\mathrm{U}(1)_{-1/2}$. Indeed the theory in question has partition function

$$Z[A] = |Z|\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}\eta[A]}, \tag{2.95}$$

and, up to a sign, we can use (2.94) to replace the phase factor by level- half Chern-Simons term.

2.5 Non-Abelian Chern-Simons theory.

One can easily generalize our discussion on Chern-Simons theory with Abelian gauge groups in Section 2.2.2 to more general gauge groups. The case of most interest to us will be for $\mathrm{U}(N)$ groups, but the discussion goes through with minor changes for orthogonal or symplectic gauge groups. This will be of use to us in the next chapter, where strong evidence of 3d bosonization has come from the study of large N Chern-Simons-matter theories.

The gauge fields A are thus valued in the Lie algebra $\mathfrak{u}(N)$ of traceless Hermitian matrices, and the Chern-Simons action reads[18]

$$S_{\mathrm{CS}}[A;k] := \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A\wedge \mathrm{d}A - \frac{2\,\mathrm{i}}{3}A\wedge A\wedge A\right). \tag{2.96}$$

[18] As is common in physics we define the Lie algebra $\mathfrak{u}(N)$ so that $\mathrm{e}^{\mathrm{i}A} \in \mathrm{U}(N)$ for $A \in \mathfrak{u}(N)$. Sometimes the Lie algebra is instead defined to be $\tilde{\mathfrak{u}}(N)$, defined so that $\mathrm{e}^{B} \in \mathrm{U}(N)$ for $B \in \tilde{\mathfrak{u}}(N)$. With this definition $\tilde{\mathfrak{u}}(N)$ consists of traceless *antihermitian* matrices. The two are simply related by $B = \mathrm{i}\,A$. The factor of i in the $A\wedge A\wedge A$ ensures that the Hamiltonian is hermitian in our conventions. If A were taken anti-hermitian, the factor of i would not be there. The wedge product of matrices is most easily defined by introducing generators T^a of $\mathfrak{u}(N)$ and expanding A as $A = A^a T^a$. The coefficients A^a are then normal differential forms and the wedge product reduces to the standard wedge product between forms.

This theory is often denoted $\mathrm{U}(N)_k$. Like in the Abelian case, gauge invariance of the action requires the level k to be integral. The proof of this fact is much more direct in the non-Abelian case. Under a gauge transformation $g(x) \in \mathrm{U}(N)$

$$A \to g^{-1} A g + \mathrm{i}\, g^{-1}\, \mathrm{d}g \tag{2.97}$$

one finds find

$$\delta S_{\mathrm{CS}} = \frac{k}{4\pi} \int \mathrm{d}^3x\, \epsilon^{\mu\nu\rho} \partial_\nu \operatorname{Tr}[\partial_\mu g g^{-1} A_\rho] + \frac{k}{12\pi} \int \mathrm{d}^3x\, \epsilon^{\mu\nu\rho} \operatorname{Tr}\left[(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)(g^{-1}\partial_\rho g)\right]. \tag{2.98}$$

Since the first term in (2.98) is a total derivative, it can be made to vanish by imposing suitable boundary conditions on A. This term was present in the Abelian case. The second term looks concerning as it is independent of the gauge field A. Remarkably, up to a numerical factor, it computes interesting topological quantity associated to the mapping $g(x)$. Writing this term as $2\pi k W(g)$, it is integer valued it is the so-called winding number of $g(x)$

$$W(g) := \frac{1}{24\pi^2} \int \mathrm{d}^3x\, \epsilon^{\mu\nu\rho} \operatorname{Tr}\left[(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)(g^{-1}\partial_\rho g)\right]. \tag{2.99}$$

Therefore $\delta S_{\mathrm{CS}} = 2\pi n k$, and so gauge invariance requires

$$k \in \mathbb{Z}. \tag{2.100}$$

2.5.1 Chern-Simons-matter theories.

Of particular interest to us are matter theories coupled to Chern-Simons gauge fields. Theories of this general type were studied in exquisite detail by [151] (c.f. also [152]. Abelian Chern-Simons-matter theories have been studied previously in [153, 154]). Since the matter actions depend on the spacetime metric, the total action is no longer topological. However gauge invariance demands that the level be an integer, and thus it does not run:

$$\beta_k := \frac{dk}{d\log\mu} = 0. \tag{2.101}$$

There are two common ways to regulate these theories. In one scheme we add an explicit Yang-Mills term to the Lagrangian

$$-\frac{1}{2e^2}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu}. \tag{2.102}$$

As we have mentioned before the coupling constant e^2 has dimensions of energy and can act as a UV cutoff. This scheme is manifestly gauge invariant, but it complicates the Feynman rules. For $\mathrm{U}(N)$ gauge groups the bare level is shifted by an integer amount:

$$k_{\mathrm{YM}} \to k_{\mathrm{YM}} + \operatorname{sgn}(k_{\mathrm{YM}})N. \tag{2.103}$$

Dimensional reduction.

Naive dimensional regularization is ambiguous for theories involving a Chern-Simons term because of the $\epsilon_{\mu\nu\rho}$ tensor. In the dimensional reduction scheme [155] tensor algebra is performed in 3 dimensions to obtain scalar integrands, and then the spacetime dimension is analytically continued in the standard way. This is not manifestly gauge invariant, but it does lead to simpler Feynman rules. We denote the bare level in the Chern-Simons-matter theories regulated in this way by k. There is no renormalization of the Chern-Simons level in this scheme. Thus the theory regulated with a Yang-Mills term and define the same physical theory if we identify the levels by

$$k = k_{\mathrm{YM}} + \operatorname{sgn}(k_{\mathrm{YM}})N. \tag{2.104}$$

Or equivalently

$$|k| = |k_{\mathrm{YM}}| + N. \tag{2.105}$$

2.5.2 Level-rank duality

Pure Chern-Simons theory enjoys level-rank duality [48, 156–160]. In contrast to the majority of the dualities presented in this work, this duality can be rigorously proven. There are two dualities which will be of use to us, for $k_{\mathrm{YM}} > 0$ they read

$$\mathrm{SU}(N)_{k_{\mathrm{YM}}} \quad \longleftrightarrow \quad \mathrm{U}(k_{\mathrm{YM}})_{-N}, \tag{2.106}$$

$$\mathrm{U}(N)_{k_{\mathrm{YM}},k_{\mathrm{YM}}+N} \quad \longleftrightarrow \quad \mathrm{U}(k_{\mathrm{YM}})_{-N,-k_{\mathrm{YM}}-N}. \tag{2.107}$$

Here

$$\mathrm{U}(N)_{k,k'} = (\mathrm{SU}(N)_k \times \mathrm{U}(1)_{Nk'})/\mathbb{Z}_N. \tag{2.108}$$

This is normalized so that $\mathrm{U}(N)_k \equiv \mathrm{U}(N)_{k,k}$ is the theory one gets by a Chern-Simons term involving a trace in the fundamental representation of $\mathrm{U}(N)$. Note that the gauge invariant theories are $\mathrm{U}(N)_{k_{\mathrm{YM}},k_{\mathrm{YM}}+nN}$, with $n \in \mathbb{Z}$. The level-rank dualities for $k_{\mathrm{YM}} < 0$ follow from the above by acting on both sides of the duality with a parity transformation, which just flips the sign of the levels on both sides.

This duality already gives a hint of the bosonization dualities described in the next chapter. The observables in pure Chern-Simons theories are built out of products of the Wilson lines

$$\mathrm{Tr}_{\mathcal{R}} \exp\left(\mathrm{i} \oint A\right), \tag{2.109}$$

where the trace is taken in the representation $\mathcal{R}$ of the gauge group. Along with the mapping on the level and rank in (2.106) and (2.107), the duality also tells us how the Wilson lines are exchanged. The first hint of 3d bosonization can be seen how these lines are exchanged.

3 The duality web in 2+1 dimensions

The study of large N Chern-Simons-matter theories was initiated to understand the gravity duals of free fields theories [84, 85]. Being free, they admit an infinite tower of conserved higher-spin currents[19]. The AdS/CFT dictionary tells us that a conserved current in the field theory corresponds to a massless gauge field in the bulk dual [163]. Thus the gravity theory should contain massless higher-spin gauge fields. Generically the correlation functions of these currents do not vanish. In the bulk this corresponds to a Witten diagram describing the scattering of the dual gauge fields. So this must be an interacting theory of higher spin currents.

In discussions of field theory in flat space, it is not known how to construct interacting theories of gauge fields of spins > 2, or even if it is possible. However, remarkably in AdS, such theories are known to exist. It was originally Fronsdal who noticed that perhaps an interacting higher-spin theory is possible in AdS [164]. Following this Fradkin and Vasiliev formulated an interacting theory of infinitely many such fields in AdS$_4$ [165–171] (for a reviews see [172–175]) - what is even more remarkable is that predated the discovery of AdS/CFT. These Vasiliev-type gravity theories are only understood at the classical level, it is not yet known how to define a consistent quantum theory.

3.1 Large N vector models

The canonical example of AdS/CFT is the duality between $\mathcal{N} = 4$ super Yang-Mills (SYM) theory with $\mathrm{SU}(N)$ gauge group and type IIB superstring theory on AdS$_5 \times \mathbb{S}^5$ [176–178]. The coupling constant g_{YM} in SYM is mapped to the string coupling constant g_s under the duality

$$g_{\mathrm{YM}}^2 \sim g_s. \tag{3.1}$$

Most examples of AdS/CFT are of this form, where the weakly coupled limit on one side maps to a strongly coupled limit on the other side. This is of course incredibly useful, but it makes explicit checks of the correspondence difficult. Taking the free limit on the field theory

[19] By the Coleman-Mandula theorem such a theory is necessarily non-interacting [161]. For a modern review see [162].

side, it is expected that the bulk dual is described by a massless higher-spin gauge theory [179–183]. In fact it seems to be that in general free CFTs of matrix-valued fields are dual to massless higher-spin fields [179–186]. However it is difficult to understand the bulk duals when we turn on even small interactions in these theories. The issue comes from operators of the form

$$\mathrm{Tr}[\Phi\partial^{i_1}\Phi\partial^{i_2}\cdots\partial^{i_k}\Phi]. \tag{3.2}$$

Such single-trace operators correspond to single-particle states in the bulk, but there are not enough fields in Vasiliev-type gravities to account for this proliferation of particles.

A key insight by Klebanov and Polyakov in [187] was that these issues can be avoided if one took vector matter instead.

- There is just one scalar primary, and thus one massive particle in the bulk. These theories have a chance to be dual to Vasiliev-type theories.
- Both sides are weakly coupled in the large N limit, so perhaps we could get a better understanding of how holography works in this example.

They demonstrated this with one of the simplest models one can write down, namely the theory of N free massless real scalars in d Euclidean dimensions[20]

$$S = \int \mathrm{d}^d x\, \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i. \tag{3.3}$$

Since this is a free theory, the dimension of ϕ^i is simply its classical dimension

$$\Delta_\phi = \frac{d}{2} - 1. \tag{3.4}$$

[20] As we proceed we will it become clear that $d = 3$ is particularly nice.

This defines (a rather trivial example of) a conformal field theory. In fact, it admits a far larger symmetry group, of which the conformal group is a tiny subgroup. This group is generated by the tower of conserved currents

$$J^{ij}_{\mu_1\cdots\mu_s} = \partial_{\mu_1}\cdots\partial_{\mu_k}\phi^i\partial_{\mu_{k+1}}\cdots\partial_{\mu_s}\phi^j + \cdots , \tag{3.5}$$

which generate so-called higher-spin symmetries. The extra terms represented by the dots are fixed uniquely by requiring that $J^{ij}_{\mu_1\cdots\mu_s}$ be symmetric, traceless, and conserved. It turns out that these conditions are equivalent to demanding that $J^{ij}_{\mu_1\cdots\mu_s}$ be annihilated by the generator K_μ of special conformal transformations - i.e. that $J^{ij}_{\mu_1\cdots\mu_s}$ be a conformal primary. Note that s is an even integer - if one tries to write down such objects with an odd number of indices it will vanish identically[21].

We now truncate to the $\mathrm{O}(N)$-singlet sector. Any $\mathrm{O}(N)$ singlet can be written as a product of "single-trace" operators[22]. These are operators built out of two fields ϕ^i with their $\mathrm{O}(N)$ indices contracted. In particular, the single-trace *primary* operators $J^{(s)}_{\mu_1\cdots\mu_s}$ are obtained from (3.5) by contracting the $\mathrm{O}(N)$ indices i and j:

$$\begin{aligned} J^{(0)} &:= \phi^i\cdot\phi^i, \\ J^{(s)}_{\mu_1\cdots\mu_s} &:= \partial_{\mu_1}\cdots\partial_{\mu_k}\phi^i\partial_{\mu_{k+1}}\cdots\partial_{\mu_s}\phi^i + \cdots , \quad s = 2,4,\ldots . \end{aligned} \tag{3.6}$$

This spectrum precisely matches the spectrum of the so-called minimal bosonic Vasiliev theory in AdS_{d+1} [165–167, 171]. Since the theory is non-interacting, the dimensions Δ_s of the operators $J^{(s)}$ are equal to their classical dimension

$$\Delta_s = d - 2 + s. \tag{3.7}$$

[21] in the complex case we have currents for each positive integer (c.f. below).
[22] We borrow the terminology used in theories with matrix degrees of freedom, where the gauge invariant operators are traces of products of fields.

We are now ready to try to understand the corresponding gravity theory. We start with the scalar operator $J^{(0)} = \vec{\phi} \cdot \vec{\phi}$. This should be dual to a bulk scalar field with mass $(m\ell_{\text{AdS}})^2 = \Delta_{\vec{\phi}^2}(\Delta_{\vec{\phi}^2} - d)$, which for $d = 3$ gives[23]

$$(m\ell_{\text{AdS}})^2 = -2. \tag{3.8}$$

The is precisely the mass of the scalar particle in Vasiliev theory! In three dimensions $\Delta_{\vec{\phi}^2} = 1$, which lies below $\frac{d}{2} = \frac{3}{2}$, and this means that there is an interesting twist in building an AdS/CFT correspondence [188]. Recall that the bulk dual of a scalar operator $\mathcal{O}$ with scaling dimension Δ is a scalar field $\varphi(z, \vec{x})$ with boundary behaviour

$$\varphi(z, \vec{x}) \sim z^{\Delta_-} \varphi_{(-)}(\vec{x}) + z^{\Delta_+} \varphi_{(+)} \quad (z \to 0), \tag{3.9}$$

where $\Delta_\pm$ are the roots of $(m\ell_{\text{AdS}})^2 = \Delta(\Delta - d)$:

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + (m\ell_{\text{AdS}})^2}. \tag{3.10}$$

In a unitary CFT the scaling dimensions satisfy $\Delta \geq \frac{d}{2} - 1$, so usually the positive branch Δ_+ is the only possible solution[24]. However, if the term in the square root is sufficiently small, the negative branch Δ_- is above the unitarity bound too. The case of relevance to us is $d = 3$ and scalar mass given by (3.8). Plugging in these values gives

$$\Delta_+ = 2, \quad \text{or} \quad \Delta_- = 1, \tag{3.11}$$

both of which are above the unitarity bound in $d = 3$! Picking the boundary condition $\varphi(z, \vec{x}) \sim z\varphi_0(\vec{x})$ thus corresponds to the free scalar theory which has $\Delta_{\vec{\phi}^2} = 1$. It now behooves us to ask: What does the boundary condition $\Delta_+ = 2$ correspond to on the CFT side? The results of [188, 189] (c.f. also [190, 191]) tell us this choice corresponds to the theory obtained by adding double-trace deformation $\sim (\vec{\phi}^2)^2$ to the Lagrangian and flowing from the

[23] It is a general fact that scalar fields in AdS have a range of negative mass for which they are stable. This is the Breitenlohner-Freedman bound $(m\ell_{\text{AdS}})^2 \geq -\frac{d^2}{4}$.

[24] In the bulk this is equivalent to the fact that the supergravity action evaluated on this solution is finite.

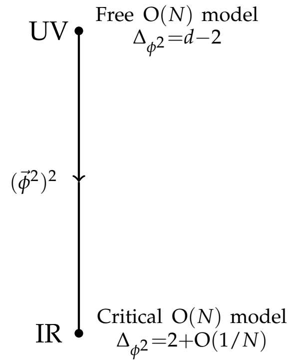


Figure 3: This is a schematic picture of the RG flow when we deform the free scalar theory by the relevant deformation $(\vec{\phi}^2)^2$. Flowing to the IR we reach another fixed point described by the interacting Wilson-Fisher CFT.

UV (the free theory) where $\vec{\phi}^2$ has scaling dimension Δ_-, to the IR fixed point where $\vec{\phi}^2$ has dimension $\Delta_+ = 2$. This is precisely the the critical $\mathbf{O(N)}$ model [192, 193]

$$S = \int d^3x \left(\frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i + \frac{\lambda}{2N} (\phi^i \phi^i)^2 \right). \tag{3.12}$$

This is a remarkable result! The same Vasiliev theory can describe the free scalar theory or the interacting fixed point, depending on the boundary conditions on the scalar field.

In fact, it was known way before these calculations were first done that, at the IR critical point the operator $\vec{\phi}^2$ has dimension $\Delta_{\vec{\phi}^2} = 2 + O(1/N)$ [16, 194, 195], but it is nice to see that everything fits together consistently.

The holographic dual of the $(\vec{\phi})^3$ deformation of the free vector model can be understood in a similar way [196].

We can generalise this in several ways. One can consider the U(N)-singlet sector of N complex scalars

$$S = \int d^d x \, \partial_\mu \overline{\phi}_i \partial^\mu \phi^i. \tag{3.13}$$

The spectrum of single-trace primary operators similarly consists of a tower of higher spin conserved currents

$$J^{(s)}_{\mu_1\cdots\mu_s} = \partial_{\mu_1}\cdots\partial_{\mu_k}\overline{\phi}_i\partial_{\mu_{k+1}}\cdots\partial_{\mu_s}\phi^j + \cdots . \tag{3.14}$$

Different from the O(N) case, there are now a currents of odd spin too. This is precisely the spectrum appearing in the bosonic AdS$_{d+1}$ Vasiliev theory [171]. Another case to consider is N free massless Dirac fermions [197, 198]

$$S = \int \mathrm{d}^d x\, \overline{\psi}_i \partial\!\!\!/ \psi^i . \tag{3.15}$$

This theory has a global U(N) symmetry. From the Lagrangian we read off the dimension $\Delta_\psi = \frac{d-1}{2}$. The U$(N)$ theory admits an infinite tower of higher-spin conserved currents. The single-trace currents are particularly simple for $d = 3$:

$$J^{(s)}_{\mu_1\cdots\mu_s} = \overline{\psi}_i\gamma_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_s}\psi^i + \cdots , \quad s = 1,2,3,\ldots , \tag{3.16}$$

and these have dimension

$$\Delta_s = 2\Delta_\psi + (s-1) = s+1. \tag{3.17}$$

On top of this is the parity odd scalar primary $J^{(0)} = \overline{\psi}_i\psi^i$ of dimension

$$\Delta_{\overline{\psi}\psi} = 2. \tag{3.18}$$

The reason for the (slight) complication in higher dimensions is that one can construct fermion bilinears involving more γ matrices. In three dimensions one can also impose a Majorana condition, so that the resulting theory has only O(N) invariance, and the O(N) singlet sector has only even spins.

The single trace spectrum U(N) theory is very similar to the one for free U(N) scalars, with the exception that the scalar operator here is parity odd and has $\Delta_0 = 2$ (compared to $\Delta_0^{\text{free scalars}} = 1$). Therefore we expect the AdS dual to be a higher-spin theory which includes a *pseudo*-scalar of mass $(m\ell_{\text{AdS}})^2 = -2$. Despite having the same spectrum, the bulk interactions should be different, since the three-point functions in the scalar and fermion theories are different. So we learn that there should be (a least) two inequivalent higher-spin theories in AdS_4, with identical spectrum (except for the parity of the scalar), but with different interactions. Such theories are known.

- ”type A” theory contains a parity even scalar,
- ”type B” theory contains a parity odd scalar [165, 170–172].

As we have already discussed, the A-type theory is conjectured to be dual to either the free or critical $\mathrm{U}(N)$ scalar vector model in the singlet sector, depending on the boundary condition imposed on the bulk scalar. Similarly the B-type theory has been conjectured to be dual to the free/critical $\mathrm{U}(N)$ fermion vector model [197]. Repeating out earlier arguments, assigning the bulk scalar the $\Delta = 2$ boundary condition gives us the free fermion theory on the boundary, whilst the $\Delta = 1$ condition should correspond to an interacting CFT obtained by deforming free theory by the double-trace operator $(\overline{\psi}\psi)^2$. This is just the familiar Gross-Neveu model

$$S = \int \mathrm{d}^3x \left[\overline{\psi}\partial\!\!\!/\psi^i + \frac{g}{2}(\overline{\psi}_i\psi^i)^2\right]. \tag{3.19}$$

This is the generalization of the original Gross-Neveu model which was defined in two spacetime dimensions [199], although this model is famously asymptotically free. This picture, summarized in Figure 4, is in agreement with field-theoretic analysis [200, 201]. By now many checks of the holographic dualities introduced in this section have been performed; see for example [193, 202, 203].

3.2 Large N Chern-Simons-matter theories.

3.2.1 Holography.

As has been mentioned before, the rationale for studying large N Chern-Simons-matter theories was to understand mysterious Vasiliev higher spin gravity theories [84, 85]. With the

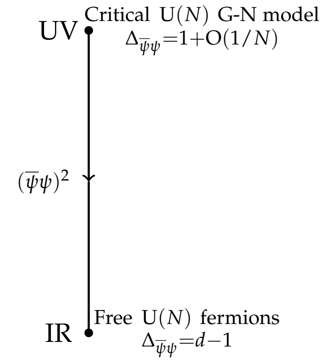


Figure 4: This is a schematic picture of the RG flow where we start with the Gross-Neveu CFT in the UV and flow to the IR where we eventually reach another fixed point described by a free fermion.

addition of these gauge fields, there is an extra parameter in the game, the Chern-Simons level k. The natural large N limit of these theories is a 't Hooft-like limit where we take N and k large such that the 't Hooft coupling

$$\lambda := \frac{N}{k} \tag{3.20}$$

is held fixed. In this limit λ effectively becomes a continuous parameter, and thus we get a one-parameter family of parity-breaking theories which preserve the higher spin symmetry in the bulk at the classical level. It was hoped that looking at how this mild deformation is mapped to the gravity side might help us understand the quantum Vasiliev theory. Vasiliev had already introduced a one-parameter family of such higher-spin theories which continuously interpolate between the parity-preserving type A and type B theories [165, 170–172, 186]. In fact, understanding their gravity duals gave the first hint that there could be a duality between fermions coupled to Chern-Simons and scalars coupled to Chern-Simons [85]. Since then the focus has shifted away from the gravity side, and fleshing out the bosonization duality.

3.2.2 Field theoretic analysis.

We now deviate away from holography and discuss the Chern-Simons-matter theories purely from a field theoretic point of view. Many exact results at leading order in N are known in these theories, by which we mean, valid to leading order in N but exact as a function of the 't Hooft coupling λ. A large part these results are due to a judicious choice of gauge, called light-cone gauge [85]. This gauge enormously simplifies the analysis since there are no gauge-boson self-interactions, and thus we don't have to worry about Fadeev-Popov ghosts either.

An important point to note is that the free energy of these large N Chern-Simons-matter theories vanishes as $|\lambda| \to 1$. The free energy is a measure of the number of available states of the system, and so this suggests the theory ceases to exist. In fact, the bound

$$|\lambda| \leq 1 \tag{3.21}$$

can also be seen as follows. Recall that k denotes the Chern-Simons level in the dimensional-reduction scheme, and is not renormalized. On the other hand, if we used a YM-regulator, we would naturally define a 't Hooft coupling $\lambda_{\mathrm{YM}} := \frac{N}{k_{\mathrm{YM}}}$. The two are then related as

$$|\lambda_{\mathrm{YM}}| = \frac{|\lambda|}{1-|\lambda|}, \quad \Leftrightarrow \quad |\lambda| = \frac{|\lambda_{\mathrm{YM}}|}{1+|\lambda_{\mathrm{YM}}|}. \tag{3.22}$$

Thus we see that, as $|\lambda_{\mathrm{YM}}|$ ranges between 0 and ∞, $|\lambda|$ ranges between 0 and 1.

[84] studied $\mathrm{U}(N)_k$ and $\mathrm{O}(N)_k$ Chern-Simons theory coupled to scalars in the fundamental representation, in the 't Hooft large N limit. This theory is often referred to as the regular boson (RB) theory in the literature. Coupling to gauge field naturally restricts to $\mathrm{U}(N)$-singlet sector, since these are just the gauge-invariant operators. The Euclidean action reads

$$I_{\mathrm{RB}} := \int \mathrm{d}^3x \left[\mathrm{D}_\mu \bar{\phi}_i \mathrm{D}^\mu \phi^i + \frac{g_6}{3!} (\bar{\phi}_i \phi^i)^3 \right] + I_{\mathrm{CS}}[A;k]. \tag{3.23}$$

As mentioned above, the natural large N limit is the 't Hooft-like limit $N, k, g_6 \to \infty$ with

$$\lambda := \frac{N}{k}, \quad \& \quad \lambda_6 := g_6 N^2 \text{ fixed.} \tag{3.24}$$

As mentioned in Section 2.5, it is very easy to engineer lines of fixed points in large N Chern-Simons-matter theories. The Chern-Simons level k is quantized and does not run, although it can be renormalized by an integer shift at one loop (2.103). The corresponding one-loop shift

λ_6

Figure 5: This is a schematic picture of the coupling λ_6 as we flow towards the IR. Note that since $\beta_\lambda = 0$, the RG flow is always in the λ_6 direction.

in λ subleading in N and thus can be ignored in the planar limit. In fact λ_6 is exactly marginal at leading order in N. Thus working to this order we get a whole plane of CFTs parameterized by λ and λ_6. At next-to-leading order a beta function for λ_6 is generated. Switching off the Chern-Simons coupling ($\lambda = 0$), the beta function is positive and so the theory with $\lambda_6 > 0$ is trivial in the IR (i.e. free). However, for $\lambda \neq 0$, there are two lines of non-trivial fixed points of the two-loop beta function, one is IR stable whilst the other is UV stable.

The spectrum of single-trace primary operators of the large N interacting fixed points discovered above, is found from the spectrum at $\lambda = 0$ by promoting the derivatives to gauge-covariant derivatives and then taking the symmetric traceless part

$$J^{(s)}_{\mu_1 \cdots \mu_s} = \mathrm{D}_{\mu_1} \cdots \mathrm{D}_{\mu_k} \overline{\phi}_i \mathrm{D}_{\mu_{k+1}} \cdots \mathrm{D}_{\mu_s} \phi^j + \cdots \; . \tag{3.25}$$

In addition, there is a scalar singlet operator $J^{(0)} = \overline{\phi}_i \phi^i$, also a primary. Due to interactions, the higher-spin symmetry is now weakly broken and has a non-trivial conservation equation

$$\partial \cdot J^{(s)} = \frac{1}{N} \sum_{s', s''} [J^{(s)}][J^{(s')}] + \frac{1}{N^2} \sum_{s', s'', s'''} [J^{(s)}][J^{(s')}][J^{(s'')}], \tag{3.26}$$

where $[J^{(s)}]$ refers to the conformal family. Counting dimensions on both sides we can restrict the sums to $s' + s'' \leq s$ and $s' + s'' + s''' \leq s$. The important point to take away is that the right-hand side contains no single-trace operators, only double- and triple-trace operators[25]. At finite N they are generally not conserved, and they also mix with multi-trace operators. However using the conformal algebra one can show that they are still conserved in the planar limit. It follows from the weakly broken higher spin symmetry (3.26) that the anomalous

[25] In particular there are no m-trace operators for $m \geq 4$ appearing on the right-hand side. This follows from the fact that m-trace operators satisfy $\Delta - s \geq m$. Indeed the left-hand side is spin $s - 1$ and scaling dimension $\Delta = (s + 1) + 1 = s + 2$. Thus their difference is $\Delta - (s - 1) = 3$.

dimensions of the higher-spin "currents" are $\mathrm{O}(1/N)$; i.e. the scaling dimensions Δ_s of the operators $J^{(s)}$ are equal to their classical value plus $\mathrm{O}(1/N)$ corrections:

$$\Delta_s = s + 1 + \mathrm{O}\left(\frac{1}{N}\right), \quad \text{(RB)}. \tag{3.27}$$

We could also study the theory with a non-vanishing quartic term $\sim (\overline{\phi}_i \phi^i)^2$ (at least in the large N limit), thus giving us the critical bosonic (CB) theory

$$I_{\mathrm{CB}} := \int \mathrm{d}^3x \left[\mathrm{D}_\mu \overline{\phi}_i \mathrm{D}^\mu \phi^i + \frac{g_4}{2} (\overline{\phi}_i \phi^i)^2 \right] + I_{\mathrm{CS}}[A; k]. \tag{3.28}$$

In this case the theory flows to another fixed point. At this fixed point it has the same spectrum of operators as near the free point, except the operator $\overline{\phi}_i \phi^i$ obtains a large anomalous dimension:

$$\Delta_{\overline{\phi}\phi} = 2 + \mathrm{O}\left(\frac{1}{N}\right). \qquad \text{(CB)} \tag{3.29}$$

This is to be compared with $\Delta_{\overline{\phi}\phi} = 1 + \mathrm{O}(\frac{1}{N})$ at the free fixed point.

We can repeat this analysis with fermions in the fundamental representation of $\mathrm{U}(N)$ [85]. Coupling to free fermions we get the regular fermion (RF) theory

$$I_{\mathrm{RF}} := \int \mathrm{d}^3x \, \overline{\psi} \, \mathrm{i} \, \not{D} \psi + I_{\mathrm{CS}}[A; k]. \tag{3.30}$$

The spectrum of primary operators in the interacting theory is unchanged

$$J^{(0)} = \overline{\psi}_i \psi^i, \quad \& \quad J^{(s)}_{\mu_1 \cdots \mu_s} = \overline{\psi}_i \gamma_{\mu_1} \mathrm{D}_{\mu_2} \cdots \mathrm{D}_{\mu_s} \psi^i + \cdots . \tag{3.31}$$

Like in the bosonic case the dimensions of the "currents" ($s \geq 1$) are equal to their classical value up to corrections of order $\frac{1}{N}$:

$$\Delta_s = s + 1 + \mathrm{O}(1/N), \quad (s \geq 1). \tag{3.32}$$

Similarly the anomalous dimension of the scalar primary $\overline{\psi}\psi$ is $O(1/N)$:

$$\Delta_{\overline{\psi}\psi} = 2 + O(1/N). \qquad \text{(RF)}. \tag{3.33}$$

3.2.3 Bosonization in the planar limit.

In this section we review the results of Maldacena and Zhiboedov in [86], which ultimately led to the bosonization dualities we will describe. They found that the three point functions $\langle J^{(s_1)} J^{(s_2)} J^{(s_3)} \rangle$ of the higher spin currents are highly constrained by the weakly broken higher spin symmetry (3.26).

To start with, conformal invariance restricts three-point functions to have the form [204] (c.f. also [205, 206] for explicit forms of these structures)

$$\langle J^{(s_1)} J^{(s_2)} J^{(s_3)} \rangle = \alpha_{s_1 s_2 s_3} \langle J^{(s_1)} J^{(s_2)} J^{(s_3)} \rangle_{\mathrm{b}} + \beta_{s_1 s_2 s_3} \langle J^{(s_1)} J^{(s_2)} J^{(s_3)} \rangle_{\mathrm{f}} + \gamma_{s_1 s_2 s_3} \langle J^{(s_1)} J^{(s_2)} J^{(s_3)} \rangle_{\mathrm{odd}}. \tag{3.34}$$

Here the subscript "b" and "f" refer to the result in the theory of a single free boson and a single Majorana fermion respectively. The subscript "odd" refers to structure which is only present in interacting theories. In [86] it was shown that the weakly broken higher spin symmetry (3.26) highly constrains the three-point functions in the planar limit. Their analysis showed that, in the planar limit, these correlators depend on 2 or 3 parameters, depending on whether the dimension of the scalar operator is 2 or 1, respectively[26]:

- If $\Delta_0 = 2 + O(\frac{1}{N})$ we call the theory **quasi-fermion**. This name was inspired by the regular fermion theory (3.30), which has $\Delta_{\overline{\psi}\psi} = 2 + O(\frac{1}{N})$. Here we start with free fermions and switching on a Chern-Simons interaction turns them into non-Abelian anyons, which for large k (the decoupling limit) become very close to ordinary fermions.
- If $\Delta_0 = 1 + O(\frac{1}{N})$, then we call the theory **quasi-boson** for now obvious reasons.

Two of the parameters were denoted $\widetilde{N}$ and $\tilde{\lambda}$ in [86]. The parameter $\widetilde{N}$ is fixed by the two point functions $\langle J^{(s)} J^{(s)} \rangle$ with $s \neq 0$. In particular this can be fixed by the energy-momentum

[26] Part of the analysis in [86] showed that only $\Delta_0 = 1$ or 2 are compatible with the weakly broken higher spin symmetry.

two point function. Since this is the same for both the quasi-fermion or quasi-boson theories, $\widetilde{N}$ is the same in either case. The parameter $\tilde{\lambda}$ is fixed by the two-point function of the scalar operator $\langle J^{(0)} J^{(0)} \rangle$ of the currents. This is necessarily different in the quasi-fermion and quasi-boson cases. For the case of large N Chern-Simons-matter theories, one can argue that $\widetilde{N}$ and $\tilde{\lambda}$ are related to the rank N and the 't Hooft coupling λ as

$$\widetilde{N} = N f(\lambda), \quad \tilde{\lambda} = h(\lambda), \tag{3.35}$$

for some functions f, h of the 't Hooft coupling λ. Considering the effect of a parity transformation, f should be an even function of λ and h an odd function. For quasi-fermionic theories, [86] found that the parameters α, β, γ appearing in (3.34) are fixed entirely in terms of $\widetilde{N}, \tilde{\lambda}$:

$$\alpha_{s_1 s_2 s_3} = \frac{\widetilde{N}\tilde{\lambda}_{\text{qf}}^2}{1+\tilde{\lambda}_{\text{qf}}^2}, \qquad (s_1, s_2, s_3 \neq 0), \tag{3.36}$$

$$\beta_{s_1 s_2 s_3} = \frac{\widetilde{N}}{1+\tilde{\lambda}_{\text{qf}}^2}, \qquad (s_1 \neq 0), \tag{3.37}$$

$$\gamma_{s_1 s_2 s_3} = \frac{\widetilde{N}\tilde{\lambda}_{\text{qf}}}{1+\tilde{\lambda}_{\text{qf}}^2}, \qquad (s_1, s_2 \neq 0), \tag{3.38}$$

where the remaining coefficients not listed vanish. In the quasi-boson case

$$\alpha_{s_1 s_2 s_3} = \frac{\widetilde{N}}{1+\tilde{\lambda}^2}, \qquad (s_1, s_2, s_3 \text{ not all } 0), \tag{3.39}$$

$$\beta_{s_1 s_2 s_3} = \frac{\widetilde{N}\tilde{\lambda}^2}{1+\tilde{\lambda}^2}, \qquad (s_1, s_2, s_3 \neq 0), \tag{3.40}$$

$$\gamma_{s_1 s_2 s_3} = \frac{\widetilde{N}\tilde{\lambda}}{1+\tilde{\lambda}^2}, \qquad (s_1, s_2 \neq 0), \tag{3.41}$$

and the extra parameter appears in the three-point function of the scalar operators, which was parameterized in [87] as

$$\alpha_{000} = \frac{\widetilde{N}}{(1+\tilde{\lambda}^2)^2} + z\left(\frac{\widetilde{N}}{1+\tilde{\lambda}^2}\right) a_3. \tag{3.42}$$

The extra parameter corresponding to the exactly marginal $\sim (\bar{\phi}\phi)^3$ interaction in the planar limit. The first hint of a bosonization duality was noted by Maldacena and Zhiboedov, where they observed that in the large coupling limit $\tilde{\lambda}_{\text{qf}} \to \infty$ of the quasi-fermion theory goes over to the correlators of the critical $\mathrm{O}(N)$ model. Similarly in the $\tilde{\lambda}_{\text{qb}} \to \infty$ limit, the free-boson correlators go over into those of the Gross-Neveu model. [87, 207] worked out the precise relation between abstract parameters $\widetilde{N}$ and $\tilde{\lambda}$ by explicitly computing them in the regular boson and regular fermion theories. They found

$$\widetilde{N} = 2N\frac{\sin(\pi\lambda)}{\pi\lambda}, \quad \tilde{\lambda}_{\text{qb}} = \tan\left(\frac{\pi\lambda}{2}\right), \quad \tilde{\lambda}_{\text{qf}} = \cot\left(\frac{\pi\lambda}{2}\right). \tag{3.43}$$

Taking strong coupling limit $\lambda \to 1$ we find that a critical scalar coupled to $\mathrm{U}(N)_k$ Chern-Simons theory is dual to a regular fermion coupled to a $\mathrm{U}(|k| - N)_{-k}$ Chern-Simons theory in the planar limit, and the same duality holds if we instead take regular bosons and critical fermions. We summarize either of these dualities as

$$\text{Scalar coupled to } \mathrm{U}(N)_k \quad \longleftrightarrow \quad \text{Fermion coupled to } \mathrm{U}(|k| - N)_{-k}. \tag{3.44}$$

The bosonization duality is more naturally expressed in terms of the level k_{YM}, obtained by using a Yang-Mills regulator (c.f. Section 2.5.1). The duality then reads

$$\text{Scalar coupled to } \mathrm{U}(N)_{k_{\text{YM}}} \quad \longleftrightarrow \quad \text{Fermion coupled to } \mathrm{U}(|k_{\text{YM}}|)_{-N}. \tag{3.45}$$

This is consistent with level-rank duality in the planar limit (c.f. Section 2.5.2). Note that this duality trivially generalises if we take N_f flavours of matter fields

$$N_f \text{ Scalars coupled to } \mathrm{U}(N)_{k_{\text{YM}}} \quad \longleftrightarrow \quad N_f \text{ Fermions coupled to } \mathrm{U}(|k_{\text{YM}}|)_{-N}. \tag{3.46}$$

So far we have only discussed the planar limit, now we want to see whether the duality holds at finite N. Unfortunately it is hard to test this, since at least one side of he duality is strongly coupled, and currently it not known how to perform exact computations at finite N. But we can perform a weak test at finite N, by comparing mass deformations on both

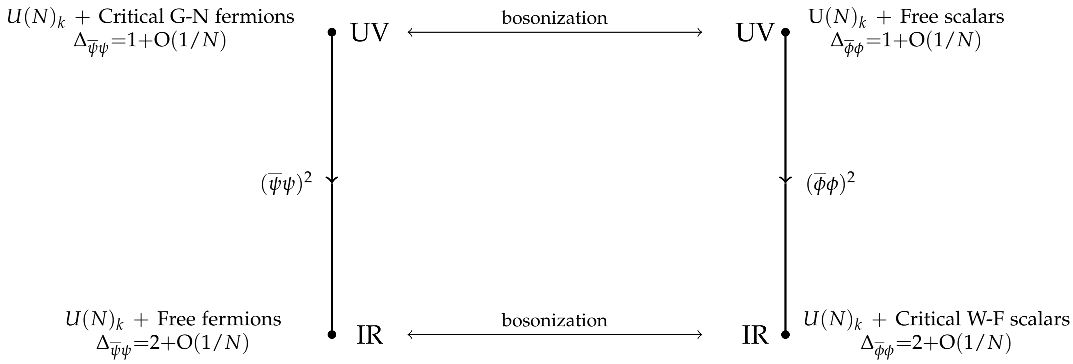


Figure 6: Here we summarize the RG flows of the various Chern-Simons-matter theories and their corresponding bosonization dualities. Note that we are not being precise about the level and ranks under the bosonization duality.

sides and flowing to the IR. In this limit both sides flow to a pure Chern-Simons theory. For the bosonization duality to be valid, it must be that these Chern-Simons theories should be level-rank dual. Aharony found that the following dualities were all consistent with these flows [95] (the notation $\mathrm{U}(N)_{k,k'}$ was introduced in Section 2.5.2)

$$\begin{aligned}
N_f \text{ bosons coupled to } \mathrm{SU}(N)_{k_{\mathrm{YM}}} &\quad\longleftrightarrow\quad N_f \text{ fermions coupled to } \mathrm{U}(k_{\mathrm{YM}})_{-N+N_f/2} \\
N_f \text{ bosons coupled to } \mathrm{U}(N)_{k_{\mathrm{YM}}} &\quad\longleftrightarrow\quad N_f \text{ fermions coupled to } \mathrm{SU}(k_{\mathrm{YM}})_{-N+N_f/2} \\
N_f \text{ bosons coupled to } \mathrm{U}(N)_{k_{\mathrm{YM}},k_{\mathrm{YM}}+N} &\quad\longleftrightarrow\quad \begin{matrix} N_f \text{ fermions coupled to} \\ \mathrm{U}(k_{\mathrm{YM}})_{-N+N_f/2,-N-k_{\mathrm{YM}}+N_f/2} \end{matrix} \quad .
\end{aligned} \tag{3.47}$$

In particular, all of the above dualities are indistinguishable in the planar limit; in fact they are all equivalent to (3.46) in this limit. The remarkable conjecture of Aharony is that all of the above dualities are correct.

At the time this mapping contradicted computations of the thermal free energy in [85], which is why this duality was not already established in [85, 86]. It turns out the computations of the free energy done previously were wrong, this was corrected in [208] where they found exact agreement with this duality.

Finally let us comment on the appearance of half-integer Chern-Simons levels in the dualities (3.47) - this is a slight generalization of our discussion at the end of Section 2.4 for a single Dirac fermion. To be concrete, let us consider the second duality. This actually stands for

$$\begin{gathered} |D_a\phi|^2 - |\phi|^4 + \frac{k_{\rm YM}}{4\pi}\,\mathrm{Tr}\left(a\,\mathrm{d}a - \frac{2\,\mathrm{i}}{3}a^3\right) \\ \updownarrow \\ \mathrm{i}\,\bar{\psi}\not{D}_b\psi + \frac{-N+N_f}{4\pi}\,\mathrm{Tr}\left(b\,\mathrm{d}b - \frac{2\,\mathrm{i}}{3}b^3\right) \end{gathered} \tag{3.48}$$

where a is a $\mathrm{U}(N)$ gauge field and b an $\mathrm{SU}(k_{\rm YM})$ gauge field. In particular, observe that the Chern-Simons level is actually an integer. However this is not the full story, since we need to specify a regularization scheme to deal with the divergences which appear in the quantum theory. One way to regulate the fermionic theory is to introduce N_f Pauli-Villars fields, one for each flavour of fermion. The integral over a single flavour of Pauli-Villars fermion generates a level $-\frac{1}{2}$ $\mathrm{SU}(k_{\rm YM})$ Chern-Simons interaction, so integrating over all of them gives us an effective level

$$-N + N_f - \frac{N_f}{2} = -N + \frac{N_f}{2}, \tag{3.49}$$

which is what appears in the duality (3.47).

3.3 The duality web.

Starting from Aharony's dualities (3.47), we now make the leap to $N = k = N_f = 1$ and reduces to

$$\begin{array}{lcl} \text{A fermion coupled to } \mathrm{U}(1)_{-\frac{1}{2}} & \longleftrightarrow & \text{A scalar} \\ \text{A fermion} & \longleftrightarrow & \text{A scalar coupled to } \mathrm{U}(1)_1 \\ \text{A fermion coupled to } \mathrm{U}(1)_{-\frac{3}{2}} & \longleftrightarrow & \text{A scalar coupled to } \mathrm{U}(1)_2. \end{array} \tag{3.50}$$

Here we have interpreted $\mathrm{SU}(1)$ as trivial; in particular $\mathrm{U}(1)_{k,k'} \equiv \mathrm{U}(1)_{k'}$. Here we are using the common notation explained at the end of Section 2.4 where $\mathrm{U}(1)_{-1/2}$ refers to a fermion coupled to a background gauge field, regulated using a positive mass Pauli-Villars field. The authors of [106] and [107] (c.f. also [209]) found that these dualities are but a small part of an enormous duality web. They proved this by first assuming the second bosonization duality[27] in (3.50), which we write as

$$\mathrm{i}\,\overline{\psi}\not{D}_B\psi - U_\mathrm{f}(\psi) \qquad \longleftrightarrow \qquad |D_a\phi|^2 - U_\mathrm{b}(\phi) + \frac{1}{4\pi}a \wedge \mathrm{d}a + \frac{1}{2\pi}B \wedge \mathrm{d}a. \tag{3.51}$$

The meaning of this is as follows. Recall that the bosonization dualities written as in, e.g. (3.50) describe two different dualities. One is between a free fermion and a critical Wilson-Fisher scalar, and the other is between a critical Gross-Neveu fermion and a free scalar. So picking the potential terms $U_\mathrm{f}(\psi), U_\mathrm{b}(\phi)$ appropriately we get either of these dualities. Recall also that the Maxwell kinetic term for the dynamical gauge field a is also implicitly there. Here we have coupled background gauge field B to the conserved currents on both sides. The current on the fermion side is $\mathrm{U}(1)$ particle density and the current on the scalar side is the $\mathrm{U}(1)_\mathrm{mag}$ magnetic symmetry of Section 2.2.1.

A simple check is to understand what happens when we deform both sides of the proposed duality by a mass term. On the scalar side this means adding a term $+\mu|\phi|^2$ to the potential $U_\mathrm{b}(\phi)$. For $\mu \gg 0$, the ϕ field is very heavy and we can drop it at low energies, leaving us with the low energy effective theory

$$Z_\mathrm{eff}[B] = \int \mathrm{D}a \,\exp\left[\mathrm{i}\int \mathrm{d}^3x \left(\frac{1}{4\pi}a \wedge \mathrm{d}a + \frac{1}{2\pi}B \wedge \mathrm{d}a\right)\right]. \tag{3.52}$$

By completing the square in a using $\frac{1}{4\pi}(a+B) \wedge \mathrm{d}(a+B) - \frac{1}{4\pi}B \wedge \mathrm{d}B$, we can integrate it out leaving behind a level $\mathrm{U}(1)_{-1}$ Chern-Simons coupling for the background field B

$$Z_\mathrm{eff}[B] = \exp\left(-\frac{\mathrm{i}}{4\pi}B \wedge \mathrm{d}B\right). \tag{3.53}$$

For $\mu \ll 0$ the scalar field condenses Higgsing the gauge field. All that is left is heavy charged probe particles $\phi, \overline{\phi}$.

[27] In particular, their analysis showed

$$\text{Bosonization} \;\Rightarrow\; \text{Particle-vortex duality}$$

but not the reverse implication.

Recall that, with a Pauli-Villars regulator, the partition function for the fermion reads

$$Z_{\mathrm{f}}[B] = |Z_{\mathrm{f}}[B]| \cdot \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}\eta(B)} \,. \tag{3.54}$$

Adding a mass term $+\mu\overline{\psi}\psi$ to the potential $U_{\mathrm{f}}(\phi)$, for $\mu \gg 0$ we can integrate out the fermion which gives a factor $\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}\eta(B)}$. This combines with the factor in (3.54) to give the low energy theory

$$Z_{\mathrm{eff}}[B] = \exp\Big(-\mathrm{i}\,\pi\eta(B)\Big) = \exp\Big(-\frac{\mathrm{i}}{4\pi} B \wedge \mathrm{d}B\Big). \tag{3.55}$$

Here we have used the Atiyah-Patodi-Singer index theorem (2.94). For $\mu \ll 0$ we can again integrate out the fermion which gives a factor of $\mathrm{e}^{+\frac{\mathrm{i}\pi}{2}\eta(B)}$, this cancels the phase from the Pauli-Villars fermion and nothing is left at low energies.

To derive further dualities from (3.51), the authors of [106, 107], used two operations.

(i) The first operation is to apply time-reversal T to both sides of the duality.

(ii) The second operation consists of two steps: First promote the background gauge field B to a dynamical gauge field. Next, to end up back with a theory of the class we started with, we need to find a conserved current in this new theory and couple it to a new background gauge field C. The sought after current is $\frac{1}{2\pi} \star \mathrm{d}b$ corresponding to the magnetic symmetry and thus the second step involves adding $+\frac{1}{2\pi} C \wedge \mathrm{d}b$ to the Lagrangian.

The technique (ii) was first introduced in [59] and heavily used in [210]. In fact (as described in [107]), this operation arises very naturally if we view these $2+1$ dimensional theories as living on the boundary of a $3+1$ dimensional bulk spacetime in which the background gauge field B propagates. This interpretation will be discussed at length in Section 4.

Let's start by deriving a new duality using time-reversal. On the fermionic side this changes the sign of the mass of the Pauli-Villars regulator, and the resulting partition function is $|Z_{\mathrm{f}}|\,\mathrm{e}^{+\frac{\mathrm{i}\pi}{2}\eta(B)}$. The phase differs from the one in (3.54) by $\pi\eta[B]$, which we can replace by $S_{\mathrm{CS}}[B]$ by the index theorem (2.94). On the scalar side this just flips the signs of the

Chern-Simons term and the BF term $\sim B \wedge \mathrm{d}a$. Thus applying T to both sides of (3.51) we obtain a new duality

$$\mathrm{i}\,\overline{\psi}\not{D}_B\psi - U_\mathrm{f}(\psi) + \frac{1}{4\pi}B\wedge \mathrm{d}B \quad\longleftrightarrow\quad |\mathrm{D}_a\phi|^2 - U_\mathrm{b}(\phi) - \frac{1}{4\pi}a\wedge\mathrm{d}a - \frac{1}{2\pi}B\wedge\mathrm{d}a. \tag{3.56}$$

The next thing to try is applying the operation (ii) to both sides of (3.51). This results in the duality

$$\mathrm{i}\,\overline{\chi}\not{D}_a\chi - \frac{1}{2}\cdot\frac{1}{4\pi}a\wedge\mathrm{d}a - \frac{1}{2\pi}a\wedge\mathrm{d}B - \frac{1}{4\pi}B\wedge\mathrm{d}B \quad\longleftrightarrow\quad |\mathrm{D}_B\phi|^2 - |\phi|^4 \tag{3.57}$$

Taking the time-reversal of both sides

$$\mathrm{i}\,\overline{\chi}\not{D}_a\chi + \frac{1}{2}\cdot\frac{1}{4\pi}a\wedge\mathrm{d}a + \frac{1}{2\pi}a\wedge\mathrm{d}B + \frac{1}{4\pi}B\wedge\mathrm{d}B \quad\longleftrightarrow\quad |\mathrm{D}_B\phi|^2 - |\phi|^4 \tag{3.58}$$

Playing this game a bit longer one eventually arrives at the purely bosonic particle-vortex dualities [106]

$$|\mathrm{D}_B\phi|^2 - U_\mathrm{b}(\phi) \quad\longleftrightarrow\quad |\mathrm{D}_{\widehat{a}}\widehat{\phi}|^2 - \widehat{U}_\mathrm{b}(\widehat{\phi}) + \frac{1}{2\pi}B\wedge\mathrm{d}\widehat{a}. \tag{3.59}$$

This includes the duality we discussed in Section 2.3 where the scalars are tuned to the Wilson-Fisher fixed point on both sides, but it now also includes the case where the scalars are near the free fixed point. We also get a fermionic version of particle-vortex duality

$$\mathrm{i}\,\overline{\chi}\not{D}_a\chi - \frac{\mathrm{i}}{2\pi}a\wedge\mathrm{d}b + \frac{\mathrm{i}}{2\pi}b\wedge\mathrm{d}b - \frac{\mathrm{i}}{2\pi}b\wedge\mathrm{d}A + \frac{\mathrm{i}}{4\pi}A\wedge\mathrm{d}A \quad\longleftrightarrow\quad \mathrm{i}\,\overline{\psi}\not{D}_A\psi. \tag{3.60}$$

4 Bulk gauge fields and boundary CFTs

4.1 Motivation and setup

We now embed our duality web in a $3+1$ dimensional setup. Specifically, we shall consider a $3+1$ dimensional bulk spacetime which has a boundary on which some matter theory lives. This setup exhibits rich behaviour. Some theoretical motivations to study this are:

- The bulk gauge coupling e^2 is dimensionless, and so it describes a marginal deformation of the 3d boundary conformal field theory (BCFT). On physical grounds the bulk gauge coupling cannot get renormalized[28], in particular it must vanishing beta function $\beta_{e^2} = 0$. Hence it describes an *exactly* marginal deformation of the 3d BCFT - such deformations are extremely hard to come by and are of great interest. In this way we obtain a whole family of BCFTs parameterized by the bulk gauge coupling.

- Our theories are examples of boundary conformal field theories (BCFTs) [212, 213], or more generally defect CFTs [214] and are amenable to the powerful methods of the boundary conformal bootstrap [108, 215–217].

- Maxwell theory on $\mathbb{R}^4$ is well known to enjoy an exact $SL(2,\mathbb{Z})$ duality [218–221], and one might expect this continues to hold in the presence of a boundary. Understanding this duality transformation ties in beautifully with the methods we used in uncovering the duality web in Section - specifically the method of introducing background gauge fields and making them dynamical has a very natural interpretation in this context. It will emerge from this study that every conjectured duality in $2+1$ dimensions can be derived from a conjectured duality in the $3+1$ dimensional setup. In particular, the duality web can be uncovered in the $3+1$ setup.

- Leveraging on the conjectured dualities in $3+1$ and the exact $SL(2,\mathbb{Z})$ duality, a powerful computation method can be developed allowing one to perform accurate ressumations of

[28] Physically, e controls the force between two probe particles in the bulk. Going infinitely far from the boundary, clearly this value cannot be effected by the boundary degrees of freedom. This does not constitute a proof that e is exactly marginal, only that this should be the case in a physically sensible model. A proof in the case of fermions on the boundary was supplied by [211].

results obtained by doing perturbation theory in the gauge coupling, and make powerful predictions on various pieces of CFT data.

There are also motivations to studying such a setup coming from condensed matter:

- In the case where the matter on the boundary is taken to be free fermions, this setup is closely related to models of graphene [222–227].
- This has applications to topological insulators [98, 99, 103, 105], which can be described in terms of a massless Dirac fermion on the boundary. Similarly it has applications to topological superconductors, which can be described as a massless Majorana fermion on the boundary [147]. Reviews include [228, 229]. More generally these are examples of symmetry protected topological (SPT) phases of matter [230].

Specifically, in this work we study $\mathrm{U}(1)$ gauge theory in $3+1$ dimensions interacting with a CFT living in $2+1$ dimensions. As we have mentioned above, the electromagnetic field in Minkowski space enjoys an exact $\mathrm{SL}(2,\mathbb{Z})$ duality at the quantum level, which acts on the complexified gauge coupling τ as $\tau \to \tau' = \frac{a\tau+b}{c\tau+d}$. With the addition of a CFT on the boundary, one finds that after performing an $\mathrm{SL}(2,\mathbb{Z})$ transformation in the bulk and going to the decoupling limit in the new frame ($\tau' \to \mathrm{i}\,\infty$) one finds a new boundary field theory related to the original one by Witten's $\mathrm{SL}(2,\mathbb{Z})$ transformation [210].

4.2 Coupling the bulk to the boundary.

We shall work on the half-space $X = \mathbb{R}^3 \times \mathbb{R}_+$, whose boundary is $\partial X = \mathbb{R}^3$. Our coordinates are packaged as $x^\mu = (x^i, y)$, where $\mu = 1, \ldots, 4$ labels the bulk components and $i = 1, 2, 3$ labels the boundary ones. We shall work exclusively in Euclidean space. The bulk gauge field is taken to have the standard Maxwell form including a θ-term

$$I_{\mathrm{EM}}[A] := \int_{y\geq 0} \mathrm{d}^4 x \left(\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \frac{\mathrm{i}\,\theta}{16\pi^2} F_{\mu\nu} \widetilde{F}^{\mu\nu} \right). \tag{4.1}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the standard field strength of a U(1) gauge field A_μ, and $\widetilde{F}_{\mu\nu}$ is the dual field strength

$$\widetilde{F}^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{4.2}$$

Note that, if we were working on all of $\mathbb{R}^4$ the θ-term would be integer-valued, in particular the theory would be invariant under $\theta \to \theta + 2\pi$. However the inclusion of a boundary spoils this and the theories with θ-angles θ and $\theta + 2\pi$ are different. Electric-magnetic duality is most conveniently described in terms of the complexified gauge coupling

$$\tau := \frac{\theta}{2\pi} + \frac{2\pi\,\mathrm{i}}{e^2}. \tag{4.3}$$

Since $e^2 \geq 0$, τ lives in the upper half-plane, and the weak-coupling limit corresponds to large $\operatorname{Im}\tau$. In particular the decoupling limit is $\tau \to +\mathrm{i}\,\infty$. One can write the Maxwell action in terms of τ as

$$I_{\mathrm{EM}}[A;\tau] = -\frac{\mathrm{i}}{8\pi}\int_{y\geq 0} \mathrm{d}^4x \left(\tau F^+_{\mu\nu}F^{+\mu\nu} - \overline{\tau}F^-_{\mu\nu}F^{-\mu\nu}\right) \tag{4.4}$$

where $F^+_{\mu\nu}$ (resp. $F^-_{\mu\nu}$) is the self-dual (resp. anti-self-dual) field strengths

$$F^\pm_{\mu\nu} := \frac{1}{2}(F_{\mu\nu} \pm \widetilde{F}_{\mu\nu}). \tag{4.5}$$

This ends our discussion of the bulk theory for now. The next thing to do is pick a 3d CFT whose fields we collectively denote as Φ, which we assume to be described by an action

$$I_{\mathrm{bdry}}[\Phi] = \int \mathrm{d}^3x\,\mathscr{L}_{\mathrm{bdry}}(\Phi). \tag{4.6}$$

However not any old CFT will do, we have to be able to define a gauge-invariant coupling to the bulk $\mathrm{U}(1)$ gauge field. There is a standard way to do this - we must require the boundary theory has a global $\mathrm{U}(1)$ symmetry and a gauge-invariant extension

$$I_{\text{bdry}}[\Phi, A] = \int \mathrm{d}^3x \, \mathscr{L}_{\text{bdry}}(\Phi, A). \tag{4.7}$$

We shall denote such a theory as $\mathfrak{T}(\tau)$. Of course, this suppresses the dependence of the theory on the various parameters of the original boundary theory, but the gauge coupling τ really takes the centre role so we suppress them. Now we should worry whether the combined bulk + boundary system is still conformal. In the simpler situation where the boundary theory has no marginal operators, then, at least for sufficiently weak gauge coupling (where classical dimension counting works) as the bulk is decoupling from the BCFT, the combined theory can certainly be tuned to a conformal fixed point. If the boundary theory has marginal operators then we cannot say much, even at weak coupling. Indeed, if λ is the corresponding marginal coupling, corrections due to the gauge field can be such that no real solutions to $\beta_\lambda = 0$ exist.

As one would expect, the situation at strong gauge coupling is even murkier. Even if we are in the simpler case above where we begin with no marginal operators, there is nothing to say that they couldn't obtain a large anomalous dimension and become marginal in this regime.

Despite all of this, we shall be optimists and gloss over this. In fact in Section we shall consider a theory which has a marginal operator, where we find that a BCFT exists, at least for weak coupling. Thus we are lead to a continuous family of so-called boundary conformal field theories (BCFTs) parameterized by the gauge coupling τ in the upper half-plane. In fact we don't need to be so abstract, since there are only two cases we shall be interested in. One is the setup (considered in Section) where we have a free Dirac fermion on the boundary

$$I_{\mathrm{f}}[\psi] = \int \mathrm{d}^3x \ \mathrm{i}\,\overline{\psi}\partial\!\!\!/\psi. \tag{4.8}$$

Of course, this theory is invariant under the global U(1) symmetry $\psi \to e^{i\alpha}\psi$ with corresponding conserved current

$$J_{\mathrm{f}}^a = \mathrm{i}\,\overline{\psi}\gamma^a\psi. \tag{4.9}$$

the gauge-invariant extension is obtained via the familiar minimal-coupling procedure by adding the gauge-invariant term $\mathrm{i}\,J_{\mathrm{f}}^a A_a$ to the Lagrangian:

$$I_{\mathrm{f}}[\psi, A] := \int \mathrm{d}^3x\ \mathrm{i}\,\overline{\psi}\not{D}\psi. \tag{4.10}$$

Going back to our discussion on finding fixed points, we see that this is straightforward for fermions. There is just one relevant operator (a mass term) and no marginal operators. If need be one can fine tune fermion mass to zero[29]. The free fermion theory realises the simpler situation above - it has no marginal operators and thus we expect the combined system to define a family of BCFTs. The other theory we shall consider is a free complex scalar on the boundary

$$I_{\mathrm{b}}[\phi] := \int \mathrm{d}^3x\, \partial_a\overline{\phi}\partial^a\phi. \tag{4.11}$$

Again this theory has global U(1) symmetry with conserved current

$$J_{\mathrm{b}}^a = \mathrm{i}(\phi\partial^a\overline{\phi} - \overline{\phi}\partial^a\phi), \tag{4.12}$$

and familiar gauge-invariant extension

$$S_{\mathrm{b}}[\phi, A] := \int \mathrm{d}^3x\, \mathrm{D}_a\overline{\phi}\mathrm{D}^a\phi, \tag{4.13}$$

This theory is more subtle. It has relevant operators $|\phi|^2$ and $|\phi|^4$ which are not a problem, but it also has a marginal operator $|\phi|^6$, and we shall have to check that the combined system can be tuned to a well-defined BCFT.

[29] In some regularization schemes (e.g. dimensional regularization) it remains zero once it is set to zero

4.3 Electric-magnetic duality in the presence of a boundary.

As we noted at the beginning of this chapter, pure Maxwell theory on $\mathbb{R}^4$ enjoys an exact $\mathrm{SL}(2,\mathbb{Z})$ duality which acts on the gauge coupling τ as

$$\tau \to \tau' := \frac{a\tau + b}{c\tau + d}, \quad \text{where } ad - bc = 1. \tag{4.14}$$

This action of $\mathrm{SL}(2,\mathbb{Z})$ is generated by the transformations

$$S(\tau) := -\frac{1}{\tau}, \quad T(\tau) := \tau + 1. \tag{4.15}$$

The statement is then that pure Maxwell with gauge coupling τ is equivalent to the same theory with gauge coupling τ', or more precisely, that their partition functions are the same [221]:

$$\int \mathrm{D}A \ \mathrm{e}^{-I_{\mathrm{EM}}[A;\tau]} = \int \mathrm{D}A' \ \mathrm{e}^{-I_{\mathrm{EM}}[A';\tau']} . \tag{4.16}$$

It is easy to see that the T operation is a symmetry of the theory. This is equivalent to shifting $\theta \to \theta + 2\pi$. But as we discussed in the previous section, on $\mathbb{R}^4$ the instanton number is equal to $I_\theta = \mathrm{i}\, n$ for some integer n, and thus $\mathrm{e}^{-\theta I_\theta}$ is unchanged. The more non-trivial statement is that the S transformation is a symmetry of the theory. This can be proven by the path-integral methods developed in this section[30]. We might expect that this duality extends to a duality of the combined bulk + boundary theory $\mathfrak{T}(\tau)$. The purpose of this section is to show that this is indeed the case, although the duality transformation is richer.

Given a given BCFT $\mathfrak{T}(\tau)$ coupled to a bulk gauge field with coupling τ, the dumbest guess we could make it that it is equivalent to $\mathfrak{T}(\tau')$ for any τ' related to τ by an $\mathrm{SL}(2,\mathbb{Z})$ transformation. This is wrong, but it is not far off from the right answer. Rather it is equivalent to $\mathfrak{T}(\tau')$ plus a topological field theory living on the boundary. Of course, the precise TQFT depends on the on the $\mathrm{SL}(2,\mathbb{Z})$ transformation connecting τ and τ'.

[30] We are only working on very simple manifolds, namely $\mathbb{R}^4$ or $\mathbb{R}^3 \times \mathbb{R}_+$. On an arbitrary Riemannian four-manifold there can be a c-number gravitational anomaly [221].

We begin with the partition function Z of the theory $\mathfrak{T}(\tau)$:

$$Z = \int \mathrm{D}\Phi\, \mathrm{D}A \ \exp\left[- \int_{\partial X} \mathscr{L}_{\mathrm{bdry}}[\Phi, A] - \int_X \mathscr{L}_{\mathrm{EM}}[F;\tau] \right]. \tag{4.17}$$

As alluded to above, we shall prove this duality using path integral methods, for which it is convenient to work in a more geometric notation. Here we package the components A_μ of the gauge field into a 1-form $A = A_\mu \, \mathrm{d}x^\mu$, and then the components of the field strength $F_{\mu\nu}$ are precisely the components of the two-form $F = \mathrm{d}A$. In terms of these variables the Maxwell action reads

$$I_{\mathrm{EM}}[A;\tau] = \int_X \left(\frac{1}{2e^2} F \wedge \star F + \frac{\mathrm{i}\,\theta}{8\pi^2} F \wedge F \right). \tag{4.18}$$

The analysis follows exactly the same logic as our study of T-duality for the compact boson in Section 2.1.2, or the study of the dual photon in $2+1$ dimensions in Section 2.2.1. We promote F to an independent variable by introducing a Lagrange multiplier field F' to enforce the constraint $F = \mathrm{d}A$. Thus the partition function for our combined bulk + boundary theory reads

$$Z = \int \mathrm{D}\Phi\, \mathrm{D}A\, \mathrm{D}F\, \mathrm{D}F' \ \exp\left[- \int_{\partial X} \mathscr{L}_{\mathrm{bdry}}[\Phi, A] - \int_X \mathscr{L}_{\mathrm{EM}}[F;\tau] - \frac{\mathrm{i}}{2\pi} \int_X F' \wedge (F - \mathrm{d}A) \right]. \tag{4.19}$$

Indeed, integrating over F' just sets $F = \mathrm{d}A$ and we get back to (4.17). Note that the action still has an explicit dependence on A from the coupling to the boundary degrees of freedom. To get something new we integrate over F. Since it enters quadratically we can just replace it by its equation of motion giving

$$Z = \int \mathrm{D}\Phi\, \mathrm{D}A\, \mathrm{D}F' \ \exp\left[- \int_{\partial X} \mathscr{L}_{\mathrm{bdry}}[\Phi, A] - \int_X \mathscr{L}_{\mathrm{EM}}[F';\tau'] + \frac{\mathrm{i}}{2\pi} \int_X F' \wedge \mathrm{d}A \right], \tag{4.20}$$

where $\tau' = -\frac{1}{\tau}$. Next we want to integrate over A. We cannot quite integrate over "all" of A due to its coupling to the boundary system, but we can integrate over its bulk part. To do this, we first integrate-by-parts using $F' \wedge \mathrm{d}A = \mathrm{d}(F' \wedge A) - \mathrm{d}F' \wedge A$, so that the partition function reads

$$Z = \int \mathrm{D}\Phi\, \mathrm{D}A\, \mathrm{D}F' \exp\left[-\int_{\partial X} \mathscr{L}_{\mathrm{bdry}}[\Phi, A] + \frac{\mathrm{i}}{2\pi} \int_{\partial X} F' \wedge A - \int_X \mathscr{L}_{\mathrm{EM}}[F'; \tau'] - \frac{\mathrm{i}}{2\pi} \int_X \mathrm{d}F' \wedge A \right]. \tag{4.21}$$

Now we can integrate over the bulk part of A, which tells us that F' is closed ($\mathrm{d}F' = 0$). Since every closed form on $X = \mathbb{R}^3 \times \mathbb{R}_+$ is exact, we must have $F' = \mathrm{d}A'$ for some 1-form A'. Thus all that remains in the boundary value of A, which we now denote a:

$$Z = \int \mathrm{D}\Phi\, \mathrm{D}a\, \mathrm{D}A' \exp\left[-\int_{\partial X} \mathscr{L}_{\mathrm{bdry}}[\Phi, a] + \frac{\mathrm{i}}{2\pi} \int_{\partial X} a \wedge \mathrm{d}A' - \int_X \mathscr{L}_{\mathrm{EM}}[A'; \tau'] \right]. \tag{4.22}$$

To summarize, we have found that our BCFT $\mathfrak{T}(\tau)$ with bulk coupling τ is equivalent to another BCFT $\mathfrak{T}'(\tau')$ interacting with a bulk gauge field A' and gauge coupling $\tau' = -\frac{1}{\tau}$. and BF coupling to a purely 3-dimensional gauge field a. this exchanges electric and magnetic charges, so we will refer to this operation as the S-operation.

What about the T-operation? In some sense this is simpler than in the case without a boundary. in the case without boundary, we invoked theorem ensuring that this was a symmetry. We do not need to invoke any theorems in this case because the transformation $\theta \to \theta + 2\pi$ is simply not a symmetry! The easiest way to understand what is going on, is to back to the geometric formulation of the action (4.18). This makes it obvious that the θ-term is a total derivative $F \wedge F = \mathrm{d}(A \wedge \mathrm{d}A)$, thus the contribution of this term to the action depends entirely on the boundary values of A. In fact, we recognize $A \wedge \mathrm{d}A$ as the Chern-Simons interaction:

$$I_\theta[A] = \frac{\mathrm{i}\,\theta}{8\pi^2} \int_{\partial X} A \wedge \mathrm{d}A \equiv \frac{\theta}{2\pi} I_{\mathrm{CS}}[A]. \tag{4.23}$$

Thus the transformation $\theta \to \theta + 2\pi$ shifts

$$I \to I + \frac{\mathrm{i}}{4\pi} \int_{\partial X} A \wedge \mathrm{d}A.$$

So the combination of changing $\theta \to \theta + 2\pi$ and adding $-\frac{i}{4\pi}\int_{\partial X} A \wedge dA$ to the action is a symmetry. This is our definition of the T-operation.

To summarize, it is convenient to refer to the bulk gauge field A as a background field:

- The S-operation sends $\tau \to -\frac{1}{\tau}$, promotes A to a dynamical field, and introduces a new background field B coupled to the current $\frac{1}{2\pi}\, da$ by adding to the action $-\frac{i}{2\pi}\int_{\partial X} B \wedge da$ to the action:

$$\mathrm{S}: \quad \tau \to -\frac{1}{\tau} \quad \& \quad \mathscr{L}_{\text{bdry}}[\Phi, A] \to \mathscr{L}_{\text{bdry}}[\Phi, a] - \frac{i}{2\pi} B \wedge da. \tag{4.24}$$

- The T-operation shifts $\tau \to \tau + 1$ and adds a level $k = 1$ Chern-Simons interaction term for the gauge field:

$$T: \quad \tau \to \tau + 1 \quad \& \quad \mathscr{L}_{\text{bdry}}[\Phi, A] \to \mathscr{L}_{\text{bdry}}[\Phi, A] - \frac{i}{4\pi} A \wedge dA. \tag{4.25}$$

Finally, note that when we introduced the Lagrange multiplier term in (4.19) there was an ambiguity in the overall sign that it entered with. If we had chosen the other sign we would've added $\frac{i}{2\pi} B \wedge da$ in (4.24) rather than subtracting it. Rather suggestively we will denote this operation as S^{-1}:

$$S^{-1}: \quad \tau \to -\frac{1}{\tau} \quad \& \quad \mathscr{L}_{\text{bdry}}[\Phi, A] \to \mathscr{L}_{\text{bdry}}[\Phi, a] + \frac{i}{2\pi} B \wedge da. \tag{4.26}$$

As we will see in the next section, this operation is precisely the inverse of S.

4.3.1 $\mathrm{SL}(2,\mathbb{Z})$ action

Following Witten[31] [210], it is instructive to compute S^2.

$$\mathscr{L}_{\text{bdry}}[\Phi, A] \xrightarrow{S} \mathscr{L}_{\text{bdry}}[\Phi, a] - \frac{i}{2\pi} B \wedge da \xrightarrow{S} \mathscr{L}_{\text{bdry}}[\Phi, a] - \frac{i}{2\pi} b \wedge da - \frac{i}{2\pi} C \wedge db. \tag{4.27}$$

[31] Continuing back to Minkowski space, our definitions of S and T agree with Witten.

Now observe that the action involving the boundary gauge field b is very simple.

$$S = -\frac{\mathrm{i}}{2\pi}\int_{\partial X}(b\wedge \mathrm{d}a + C\wedge \mathrm{d}b) = -\frac{\mathrm{i}}{2\pi}\int_{\partial X} b\wedge \mathrm{d}(a+C),$$

where we have integrated-by-parts. The path-integral over b sets $a+C$ to zero, up to a gauge transformation. Thus we can do the integral over a too, which sets $a = -C + \mathrm{d}\alpha$ for some function α, which is integrated over. So at this point the partition function reads

$$Z[C] = \int \mathrm{D}\Phi\, \mathrm{D}\alpha\ \mathrm{e}^{-\mathscr{L}_{\mathrm{bdry}}[\Phi,-C+\mathrm{d}\alpha]}\,. \tag{4.28}$$

By hypothesis, $\mathscr{L}_{\mathrm{bdry}}$ is gauge-invariant, so $\mathscr{L}_{\mathrm{bdry}}[\Phi, -C+\mathrm{d}\alpha] = \mathscr{L}_{\mathrm{bdry}}[\Phi, -C]$ and thus the integral over α factors out leaving $\mathscr{L}_{\mathrm{bdry}}[\Phi, -C]$. To summarize, (relabelling $C \to A$) the operation of S^2 simply reverses the sign of the background gauge field A:

$$\mathscr{L}_{\mathrm{bdry}}[\Phi, A] \xrightarrow{S^2} \mathscr{L}_{\mathrm{bdry}}[\Phi, -A]. \tag{4.29}$$

This is simply the original theory with the sign of the current reversed, which is equivalent to acting with charge conjugation. Repeating the above derivation with $S^{-1}S$ and SS^{-1}, changes $-A$ to $+A$ on the right-hand side of (4.29). I.e. $S^{-1}S = SS^{-1} = 1$.

4.4 Bulk $\mathrm{SL}(2,\mathbb{Z})$ action and the duality web.

In this section we will demonstrate how the purely 3d dualities follow from a conjectured 4d duality concerning a bulk gauge field coupled to a boundary CFT and study interesting limits of τ which describe a decoupled theory in an alternative duality frame.

4.4.1 Scalars on the boundary.

We begin with scalars on the boundary coupled to a bulk gauge field A. The system is thus described by the partition function

$$Z_{\mathrm{b}}(\tau) := \int \mathrm{D}\phi\, \mathrm{D}\overline{\phi}\, \mathrm{D}A\ \exp\left[-\int_{\partial X}\left(|\mathrm{D}_A\phi|^2 + U_{\mathrm{b}}(\phi)\right) - \int_X \mathscr{L}_{\mathrm{EM}}[A;\tau]\right]. \tag{4.30}$$

We consider, in turn, the cusps[32] at $\tau = 0$ and $\tau = \pm 1$. To get a dual weakly coupled theory describing the cusp at $\tau = 0$, we apply the S operation to get

$$Z_{\mathrm{b}}(\tau) = \int \mathrm{D}B\,\mathrm{D}\phi\,\mathrm{D}\overline{\phi}\,\mathrm{D}a \exp\left[-\int_{\partial X} \left(|\mathrm{D}_a\phi|^2 + U_{\mathrm{b}}(\phi) - \frac{\mathrm{i}}{2\pi} B\wedge \mathrm{d}a \right) - \int_X \mathscr{L}_{\mathrm{EM}}[B; -1/\tau] \right]. \tag{4.31}$$

Now let's consider the limit $\tau \to 0$ in the duality frame (4.31).

$$Z_{\mathrm{b}}(\tau = 0) = \int \mathrm{D}\phi\,\mathrm{D}\overline{\phi}\,\mathrm{D}a \exp\left[-\int_{\partial X} \left(|\mathrm{D}_a\phi|^2 + U_{\mathrm{b}}(\phi) - \frac{\mathrm{i}}{2\pi} B\wedge \mathrm{d}a \right) \right]. \tag{4.32}$$

By bosonic particle-vortex duality continued to Euclidean signature[33] this is equivalent to the original boundary theory! In the next section we will discuss a conjectured generalization of this which turns out to be incredibly powerful. Next consider the cusp at $\tau \to +1$. To get a dual weakly coupled theory in this limit, we apply ST^{-1} to get

$$\begin{aligned} Z_{\mathrm{b}}(\tau) = \int & \mathrm{D}B\,\mathrm{D}\phi\,\mathrm{D}\overline{\phi}\,\mathrm{D}a \\ & \exp\left\{ -\int_{\partial X} \left(|\mathrm{D}_a\phi|^2 + U_{\mathrm{b}}(\phi) + \frac{\mathrm{i}}{4\pi} a\wedge \mathrm{d}a - \frac{\mathrm{i}}{2\pi} B\wedge \mathrm{d}a \right) - \int_X \mathscr{L}_{\mathrm{EM}}\left[B; -\frac{1}{\tau-1}\right] \right\}. \end{aligned} \tag{4.33}$$

Taking the limit $\tau \to +1$ in this duality frame

$$Z_{\mathrm{b}}(\tau = 1) = \int \mathrm{D}\phi\,\mathrm{D}\overline{\phi}\,\mathrm{D}a \exp\left\{ -\int_{\partial X} \left(|\mathrm{D}_a\phi|^2 + U_{\mathrm{b}}(\phi) + \frac{\mathrm{i}}{4\pi} a\wedge \mathrm{d}a - \frac{\mathrm{i}}{2\pi} B\wedge \mathrm{d}a \right) \right\}. \tag{4.34}$$

---

[32] More generally (in this context) a **cusp** is any point τ in the upper-half plane which can be mapped to $+\mathrm{i}\infty$ by an $SL(2,\mathbb{Z})$ transformation (4.14).

[33] As described in Appendix , under Wick rotation $t = -\mathrm{i}\tau$ to Euclidean space the Chern-Simons action S_{CS} is unchanged. However we conventionally define the Euclidean action I_{CS} so that it enters as $\mathrm{e}^{-I_{\mathrm{CS}}}$, thus I_{CS} and S_{CS} are related by $\mathrm{e}^{\mathrm{i}S_{\mathrm{CS}}} = \mathrm{e}^{-I_{\mathrm{CS}}}$. I.e. so $-I_{\mathrm{CS}} = \mathrm{i}\,S_{\mathrm{CS}}$. The same rules apply to the BF action.

By the bosonization duality (3.51) this is equivalent to a Dirac fermion on the boundary. Finally, to get a weakly coupled dual theory describing the cusp $\tau \to -1$, we instead apply ST to get

$$Z_{\mathrm{b}}(\tau) = \int \mathrm{D}B\,\mathrm{D}\phi\,\mathrm{D}\overline{\phi}\,\mathrm{D}a$$
$$\exp\left\{-\int_{\partial X}\left(|\mathrm{D}_a\phi|^2 + U_{\mathrm{b}}(\phi) - \frac{\mathrm{i}}{4\pi}a\wedge \mathrm{d}a - \frac{\mathrm{i}}{2\pi}B\wedge \mathrm{d}a\right) - \int_X \mathscr{L}_{\mathrm{EM}}\left[B; -\frac{1}{\tau+1}\right]\right\}. \tag{4.35}$$

Using the same reasoning applied to the cusp at $\tau = -1$ and the duality (3.56), the theory is dual to a Dirac fermion coupled to a the background gauge field B with $\mathrm{U}(1)_1$ Chern-Simons interaction using the duality. Equivalently we can view this as a Dirac fermion regulated using a Pauli-Villars regulator fermion of the opposite sign mass. The resulting picture is summarized in Figure 7.

4.4.2 Bosonic particle-vortex duality.

Consider the bosonic particle-vortex duality, which in Euclidean space reads

$$|\mathrm{D}_B\phi|^2 + U_{\mathrm{b}}(\phi) \qquad \longleftrightarrow \qquad |\mathrm{D}_{\widehat{a}}\widehat{\phi}|^2 + U_{\mathrm{b}}(\widehat{\phi}) - \frac{\mathrm{i}}{2\pi}B\wedge\widehat{a}. \tag{4.36}$$

Following [107] we will show that this duality follows from the conjectured $(3+1)$-dimensional duality

$$Z_{\mathrm{b}}(\tau) \overset{!?}{=} Z_{\mathrm{b}}(-1/\tau). \tag{4.37}$$

Essentially by construction, this reduces to the particle-vortex dualities (4.36) in the decoupling limit $\tau \to +\mathrm{i}\,\infty$, but it is instructive to run through the logic. Taking the limit $\tau \to +\mathrm{i}\,\infty$ on the

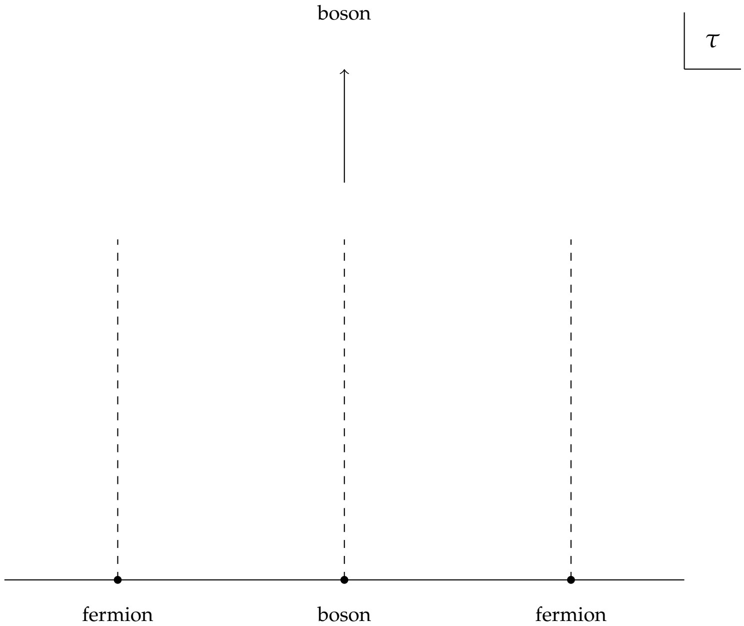


Figure 7: In this figure we plot the upper-half plane parameterized by $\tau := \frac{\theta}{2\pi} + \frac{2\pi \mathrm{i}}{e^2}$. The dashed line in the centre is the line $\mathrm{Re}\,\tau = 0$ (equivalently $\theta = 0$), whereas the left (resp. right) dashed line is the line $\mathrm{Re}\,\tau = -1$ (resp. $\mathrm{Re}\,\tau = +1$). To explain the boson and fermion labels, consider the setup where we have either a free or Wilson-Fisher boson ϕ on the boundary coupled to bulk gauge field with gauge coupling τ, thus giving the theory $Z_{\mathrm{b}}(\tau)$ in (4.30). Then, of course, taking the decoupling limit $\tau \to +\mathrm{i}\,\infty$ we are left with a Wilson-Fisher boson on the boundary - this is represented by the vertical line going off to $\tau = +\mathrm{i}\,\infty$ in the figure. As we approach the cusp at the origin $\tau \to 0$, the theory $Z_{\mathrm{b}}(\tau)$ is strongly coupled. But by the conjectured duality (4.37), this is equivalent to the vortex-theory described by $\widehat{\phi}$ with dual coupling $\widehat{\tau} = -\frac{1}{\tau}$. In the limit $\tau \to 0$ the dual coupling $\widehat{\tau} \to +\mathrm{i}\,\infty$. i.e. the gauge field decouples in this limit leaving behind decoupled theory $|\mathrm{D}_{\widehat{a}}\widehat{\phi}|^2 + U_{\mathrm{b}}(\widehat{\phi}) - \frac{\mathrm{i}}{2\pi} D \wedge \mathrm{d}\widehat{a}$, where $\widehat{a}$ is a purely three-dimensional dynamical gauge field. Similarly as we approach the cusp at $\tau \to 1$, the theory $Z_{\mathrm{b}}(\tau)$ is strongly coupled. But in the duality frame (4.34) the theory decouples in this limit and is equivalent to a Dirac fermion by 3d bosonization. Finally we apply the same logic to the cusp at $\tau = -1$, which in the duality frame (4.35) is described by a fermionic theory.

left-hand side of (4.37) we simply get back the purely bosonic theory coupled to background gauge field A:

$$Z_{\mathrm{b}}(\tau) \xrightarrow{\tau \to +\mathrm{i}\infty} \int \mathrm{D}\phi \, \exp\left[- \int_{\partial X} \left(|\mathrm{D}_A \phi|^2 + |\phi|^4 \right) \right]. \tag{4.38}$$

If we take the limit $\tau \to +\mathrm{i}\infty$ on the right-hand side of (4.36) we end up with a strongly interacting theory. However, the S-dual description gives a weakly coupled theory in this limit. Applying the S-operation to $Z_{\mathrm{b}}(-1/\tau)$ we get

$$Z_{\mathrm{b}}(-1/\tau) = \int \mathrm{D}B \, \mathrm{D}\phi \, \mathrm{D}\overline{\phi} \, \mathrm{D}a \, \exp\left[- \int_{\partial X} \left(|\mathrm{D}_a \phi|^2 + |\phi|^4 - \frac{\mathrm{i}}{2\pi} B \wedge \mathrm{d}a \right) - \int_X \mathscr{L}_{\mathrm{EM}}[B;\tau] \right].$$

The bulk gauge fields now decouple in the limit $\tau \to +\mathrm{i}\infty$, reproducing the right-hand side of (4.36).

As we now describe, [108] leveraged on this idea to develop a powerful computational scheme. We shall be interested in the case where we have free scalars on the boundary, but similar arguments can be used for other cases. In this scenario the cusps at $\tau = \pm 1$ are described by the strongly coupled critical Gross-Neveu fermion, about which not much is known. However this setup allows us to make non-trivial quantitative predictions about this point. We start by doing perturbation theory for weak coupling in the free scalar theory. One might try to extrapolate the perturbative results to an all-orders result using the self-duality $\tau \leftrightarrow -\frac{1}{\tau}$. However, recall that the operator map is, in general, highly non-local. But there are certain operators which map into each other under the duality and thus we can use this ressumation technique. In particular the mass operators and conserved currents. Since these are conformal field theories, such operators are characterized by their anomalous dimensions. Using the ressumed result, we then try to extrapolate this to $\tau = \pm 1$ to get predictions for the mass operator and conserved currents in the Gross-Neveu model.

4.4.3 Fermionic particle-vortex duality

Finally let's us try to play a similar game with the fermionic particle-vortex duality of (3.60). In Euclidean signature this reads

$$\mathrm{i}\,\overline{\chi}\not{D}_a\chi - \frac{\mathrm{i}}{2\pi}a\wedge \mathrm{d}b + \frac{\mathrm{i}}{2\pi}b\wedge \mathrm{d}b - \frac{\mathrm{i}}{2\pi}b\wedge \mathrm{d}A + \frac{\mathrm{i}}{4\pi}A\wedge \mathrm{d}A \qquad \longleftrightarrow \qquad \mathrm{i}\,\overline{\psi}\not{D}_A\psi. \tag{4.39}$$

The theory on the left-hand side looks like it can be obtained from the free theory $\mathrm{i}\,\overline{\chi}\not{D}_A\chi$ for some $\mathrm{SL}(2,\mathbb{Z})$ transformation. Indeed

$$\begin{aligned}\mathrm{i}\,\overline{\chi}\not{D}_A\chi &\xrightarrow{S} \mathrm{i}\,\overline{\chi}\not{D}_a\chi - \frac{\mathrm{i}}{2\pi}B\wedge \mathrm{d}a \xrightarrow{T^{-2}} \mathrm{i}\,\overline{\chi}\not{D}_a\chi - \frac{\mathrm{i}}{2\pi}B\wedge \mathrm{d}a + \frac{\mathrm{i}}{2\pi}B\wedge \mathrm{d}B \\ &\xrightarrow{S} \mathrm{i}\,\overline{\chi}\not{D}_a\chi - \frac{\mathrm{i}}{2\pi}b\wedge \mathrm{d}a + \frac{\mathrm{i}}{2\pi}b\wedge \mathrm{d}b - \frac{\mathrm{i}}{2\pi}C\wedge \mathrm{d}b,\end{aligned} \tag{4.40}$$

then applying T^{-1} we get the left-hand side of (4.39) (after some integration-by-parts and relabelling the background field $C\to A$). The inverse of $T^{-1}ST^{-2}S$ is $S^{-1}T^2S^{-1}T$, which maps

$$\tau \to \tau' = -\frac{1}{2-\frac{1}{\tau+1}} \equiv -\frac{\tau+1}{2\tau+1}. \tag{4.41}$$

So, like in the previous section, we postulate that the theory

$$S_{\mathrm{f}}[\psi, A;\tau] := \int_{\partial X}\mathrm{i}\,\overline{\psi}\not{D}_A\psi + \int_X \mathscr{L}_{\mathrm{EM}}[A;\tau] \tag{4.42}$$

is equivalent to

$$S_{\mathrm{f}}[\chi, A;\tau'] := \int_{\partial X}\mathrm{i}\,\overline{\chi}\not{D}_A\chi + \int_X \mathscr{L}_{\mathrm{EM}}[A;\tau'], \tag{4.43}$$

with gauge coupling τ' given in (4.41). That is, we postulate that the corresponding partition function satisfies the self-duality

$$Z_{\mathrm{f}}(\tau) \overset{!?}{=} Z_{\mathrm{f}}\Big(-\frac{\tau+1}{2\tau+1}\Big). \tag{4.44}$$

By construction, this equivalence reduces to (4.39) in the (appropriate) decoupling limits. Near the cusp $\tau = 1$ we get a dual weakly coupled description by applying ST^{-1} to $Z_{\mathrm{f}}(\tau)$

$$Z_{\mathrm{f}}(\tau) = \int \exp\left\{ -\int_{\partial X} \left(\mathrm{i}\,\overline{\psi} \not{D}_a \psi + \frac{\mathrm{i}}{4\pi} a \wedge \mathrm{d}a - \frac{\mathrm{i}}{2\pi} B \wedge \mathrm{d}a \right) - \int_X \mathscr{L}_{\mathrm{EM}}[B; -1/(\tau-1)] \right\}. \tag{4.45}$$

Similarly near the cusp $\tau = -1$ we get a dual weakly coupled description by applying ST to $Z_{\mathrm{f}}(\tau)$

$$Z_{\mathrm{f}}(\tau) = \int \exp\left\{ -\int_{\partial X} \left(\mathrm{i}\,\overline{\psi} \not{D}_a \psi - \frac{\mathrm{i}}{4\pi} a \wedge \mathrm{d}a - \frac{\mathrm{i}}{2\pi} B \wedge \mathrm{d}a \right) - \int_X \mathscr{L}_{\mathrm{EM}}[B; -1/(\tau+1)] \right\}. \tag{4.46}$$

By bosonization dualities in Section 3.3 these are dual to corresponding bosonic theories. The picture is summarized in Figure 8. This is the setup that was originally considered in [108]. For more details on this construction see [107], and for some discussion on the self-dualities (4.37) and (4.44) see [119, 231, 232].

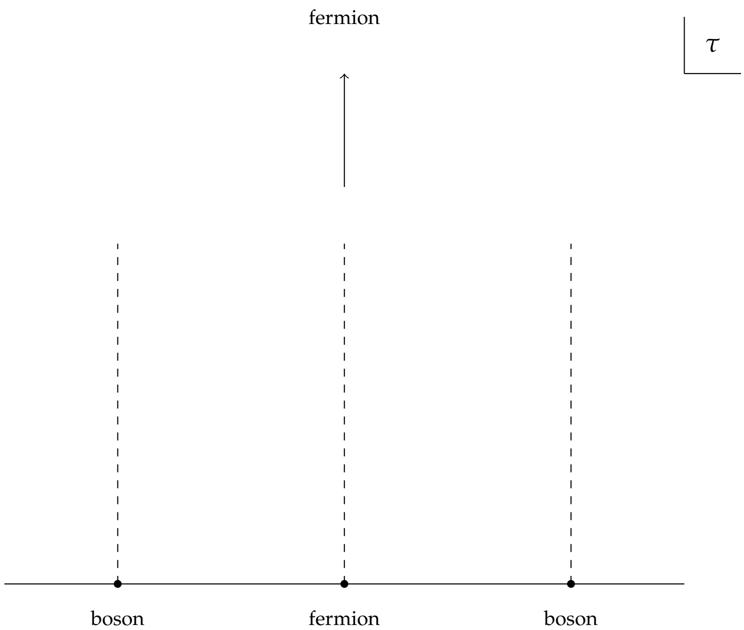


Figure 8: This is upper-half plane parameterized by τ in the duality frame which describes a fermion in the decoupling limit $\tau \to +\mathrm{i}\infty$.

5 Scalars on the boundary

In this section we begin the program laid out in the previous section by taking free scalars on the boundary (in particular see the discussion at the end of Section 4.4.2). The action we start with is thus

$$I_{\mathrm{bdry}}[\phi] = \int \mathrm{d}^3x\, \partial_a\phi\partial^a\overline{\phi} \equiv -\int \mathrm{d}^3x\, \overline{\phi}\partial^2\phi. \tag{5.1}$$

The conserved current corresponding to the global $\mathrm{U}(1)$ phase rotation symmetry

$$j_a = \mathrm{i}\,\overline{\phi}\overleftrightarrow{\partial_a}\phi, \qquad \overline{\phi}\overleftrightarrow{\partial_a}\phi := \partial_a\overline{\phi}\phi - \overline{\phi}\partial_a\phi. \tag{5.2}$$

The fundamental object in perturbation theory is the propagator

$$\Delta(x-y) := \langle\phi(x)\overline{\phi}(y)\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\Delta(p)\,\mathrm{e}^{\mathrm{i}\,p(x-y)}, \tag{5.3}$$

where we can Fourier transform on the boundary since we have translation invariance there. A simple way to determine the propagator [10] is to expand out the right-hand side of

$$0 = \frac{1}{Z}\int \mathrm{D}\phi\,\mathrm{D}\overline{\phi}\,\frac{\delta}{\delta\overline{\phi}(x)}(\overline{\phi}(y)\,\mathrm{e}^{-S}), \tag{5.4}$$

from which we derive the well-known fact that the propagator is a Green's function for the Laplacian

$$-\partial_x^2\langle\phi(x)\overline{\phi}(y)\rangle \equiv -\partial_x^2\Delta(x-y) = \delta^3(x-y). \tag{5.5}$$

Expanding both sides in momentum space and plugging in the Fourier expansion of the delta function $\delta(x-y) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\,p(x-y)}$, we determine

$$\Delta(p) = \frac{1}{p^2}, \quad \Leftrightarrow \quad \Delta(x-y) = \frac{1}{4\pi|x-y|}. \tag{5.6}$$

To derive the position-space propagator $\Delta(x-y)$ we have used the integral formula (E.1). Coupling the scalars to the bulk gauge field we obtain the boundary action

$$I[\phi, A] = \int \mathrm{d}^3x \left(|\mathrm{D}\phi|^2 + \frac{\lambda}{(3!)^2}|\phi|^6 \right). \tag{5.7}$$

Here we are obliged to include the marginal $|\phi|^6$ term - it is generated at the quantum level whether we like it or not. We also need the photon propagator $\langle A_\mu(x)A_\nu(x')\rangle$. Since we are doing perturbation theory on the boundary, we can integrate out the bulk components and consider the propagator $\langle A_a(\boldsymbol{x})A_b(\boldsymbol{x}')\rangle$ between two points $\boldsymbol{x}, \boldsymbol{x}'$ on the boundary. This was worked out in [108] by starting from the free propagator on $\mathbb{R}^4$, using the method of images to get it on the half-space $X = \mathbb{R}^3 \times \mathbb{R}_+$ and then taking the limit as x and x' approach the boundary. The result is

$$\langle A_a(\boldsymbol{x})A_b(\boldsymbol{x}')\rangle = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\, \Delta_{ab}(\boldsymbol{p})\, \mathrm{e}^{\mathrm{i}\,\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}, \tag{5.8}$$

where

$$\Delta_{ab}(\boldsymbol{p}) = \frac{\alpha^2}{|\boldsymbol{p}|}\left[\delta_{ab} - (1-\xi)\frac{p_a p_b}{|\boldsymbol{p}|^2} + \vartheta\epsilon_{abc}\frac{p^c}{|\boldsymbol{p}|}\right]. \tag{5.9}$$

Here ξ parameterizes a choice of gauge from using covariant gauge fixing term $\sim \frac{1}{\xi}(\partial_\mu A^\mu)^2$ in the bulk action, and α is the effective boundary coupling given by

$$\alpha^2 := \frac{e^2}{1+\vartheta^2}, \quad \vartheta := \frac{\theta e^2}{4\pi^2}. \tag{5.10}$$

In terms of the complexified gauge coupling $\tau = \frac{\theta}{2\pi} + \frac{2\pi\,\mathrm{i}}{e^2}$

$$\vartheta = \frac{\mathrm{Re}\,\tau}{\mathrm{Im}\,\tau}, \quad \alpha^2 = \frac{2\pi\,\mathrm{Im}\,\tau}{|\tau|^2}. \tag{5.11}$$

5.1 Renormalization schemes

As is familiar in quantum field theory we often encounter divergent integrals due to high momentum modes. The way to make sense of these is renormalization. This involves regulating the theory in some way to make it well-defined, compute whatever we need to compute, and only at the end remove the regulator. In this work we shall consider both dimensional regularization (DR) and a momentum cutoff. In practise dimensional regularization is a far more convenient regulator to work with, however our current technology is unable to deal with one of the divergent integrals that occur in this theory. A brief introduction to the state of the art technology for computing dimensionally regulated loop integrals, along with the hurdle in our scalar theory, is presented in Appendix D.2. Using a momentum cutoff is rather more cumbersome, but it has the benefit that we can compute all the divergent integrals which occur in our calculations. We will begin by laying out the dimension-regularization scheme in some detail, and then go on to discuss the minor differences in defining the theory with a momentum cutoff. Then we discuss the renormalization of the mass operator.

5.1.1 Dimensional regularization.

In this method we continue the boundary dimension

$$d = 3 - 2\epsilon, \tag{5.12}$$

whilst keeping the codimension fixed, i.e. [211, 222, 226, 233–235]

$$D - d = 1. \tag{5.13}$$

Putative divergences now appear in the guise of poles as $\epsilon \to 0$, which can be absorbed into the normalization of the fields and parameters (couplings and masses) which enter into the Lagrangian. The fields and parameters appearing in (5.7) are referred to as the bare fields; we indicate this by adding a subscript "0" to them so that the action reads

$$I = \int \mathrm{d}^d x \left(\partial_a \phi_0 \partial^a \overline{\phi}_0 + \mathrm{i}\, \alpha_0 A_{0a} \overline{\phi}_0 \overleftrightarrow{\partial_a} \phi_0 + \alpha_0^2 A_{0a} A_{0a} |\phi_0|^2 + \frac{\lambda_0}{(3!)^2} |\phi_0|^6 \right). \tag{5.14}$$

We now introduce renormalized fields and couplings

$$\phi_0 = Z_\phi^{1/2}\phi, \quad A_{0a} = Z_A^{1/2} A_a, \quad \alpha_0 = Z_\phi^{-1} Z_A^{-1/2} Z_\alpha \mu^\epsilon \alpha, \quad \lambda_0 = Z_\phi^{-3} Z_\lambda \mu^{4\epsilon}\lambda, \tag{5.15}$$

in terms of which the action reads

$$I = \int \mathrm{d}^d x \left(Z_\phi \partial_a \phi \partial^a \overline{\phi} + \mathrm{i}\, Z_\alpha \mu^\epsilon \alpha A_a \overline{\phi} \overleftrightarrow{\partial_a} \phi + Z_\phi^{-1} Z_\alpha^2 (\mu^\epsilon \alpha)^2 A_a A_a |\phi|^2 + \frac{Z_\lambda \mu^{4\epsilon}\lambda}{(3!)^2} |\phi|^6 \right). \tag{5.16}$$

The extra factors of Z_ϕ and Z_A have been inserted into the definition of the renormalized couplings α and λ so that the action takes the relatively simple form above. We have also introduced an auxiliary parameter

$$\mu \equiv \text{Renormalization scale} \tag{5.17}$$

with dimensions of mass so that the renormalized couplings are dimensionless for all d. Indeed, doing standard dimensional analysis with the boundary action (5.16) in $d = 3 - 2\epsilon$ dimensions and bulk action in $D = 4 - 2\epsilon$ dimensions (c.f. (5.13)), it is straightforward to verify

$$[\phi_0] = \frac{1}{2} - \epsilon, \quad [A_0] = 1 - \epsilon, \quad [\alpha_0] = \epsilon, \quad [\lambda_0] = 4\epsilon. \tag{5.18}$$

Renormalization is a quantum effect, meaning that the renormalization constants are (at least in perturbation theory) of the form $Z = 1 + \mathrm{O}(\alpha, \lambda)$, and so it is convenient to define the $\mathrm{O}(\alpha, \lambda)$ quantities

$$\delta Z_\phi := Z_\phi - 1, \quad \delta Z_\alpha := Z_\alpha - 1, \quad \delta Z_2 := Z_\phi^{-1} Z_\alpha^2 - 1, \quad \delta Z_\lambda := Z_\lambda - 1. \tag{5.19}$$

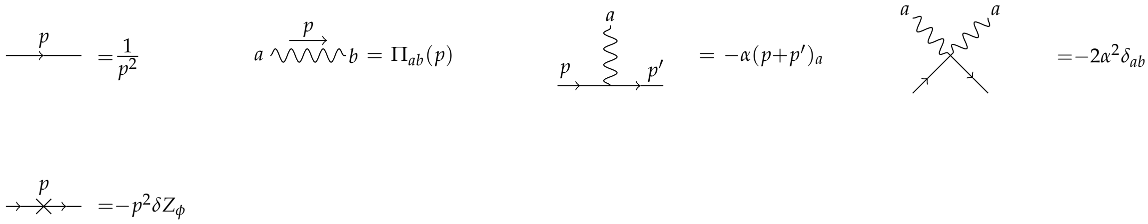


Figure 9: This summarizes the Feynman rules for the scalar BCFT in momentum space.

Thus we finally arrive at the action expressed in terms of the renormalized fields and parameters

$$
\begin{aligned}
I = \int \mathrm{d}^d x \Bigg(& \partial_a \phi \partial^a \overline{\phi} + \mathrm{i}\, \mu^\epsilon \alpha A_a \overline{\phi} \overleftrightarrow{\partial_a} \phi + (\mu^\epsilon \alpha)^2 A_a A_a |\phi|^2 + \frac{\mu^{4\epsilon}\lambda}{(3!)^2} |\phi|^6 \\
& + \delta Z_\phi \partial_a \phi \partial^a \overline{\phi} + \mathrm{i}\, \delta Z_\alpha \mu^\epsilon \alpha A_a \overline{\phi} \overleftrightarrow{\partial_a} \phi + \delta Z_2 (\mu^\epsilon \alpha)^2 A_a A_a |\phi|^2 + \frac{\mu^{4\epsilon}\lambda \delta Z_\lambda}{(3!)^2} |\phi|^6 \Bigg). \qquad (5.20)
\end{aligned}
$$

The momentum-space Feynman rules derived from this action are recorded in Figure 9, and deserve a small comment. First of all, we find it convenient to take the photon propagator to be

$$
\Pi_{ab}(p) := \frac{1}{|p|} \left(\delta_{ab} - (1-\xi) \frac{p_a p_b}{p^2} + \vartheta \epsilon_{abc} \frac{p^c}{|p|} \right), \qquad (5.21)
$$

which is to be compared with $\Delta_{ab}(p)$ in (5.9). The two differ by a factor of α^2, which we have absorbed into a simple rescaling of the gauge field, after which the factors of the boundary gauge coupling α now appear in the vertices. One could just as well use the original Δ_{ab}, but we find it easier to keep track of the renormalization constants this way. The last step to have a bona fide renormalization scheme is to specify exactly how the renormalization constants Z_ϕ etc. are defined. The scheme that stands out for multiloop calculations is the minimal subtraction (MS) scheme, where we subtract *only* the divergent part.

5.1.2 Momentum cutoff.

In this method we introduce a large momentum scale Λ and regulate the theory by only integrating over momenta p with $|p| < \Lambda$:

$$\int \frac{d^3p}{(2\pi)^3} \to \int_{|p|<\Lambda} \frac{d^3p}{(2\pi)^3}. \tag{5.22}$$

Putative divergences as $\Lambda \to \infty$ are now absorbed into a set of renormalized fields and parameters

$$\phi_0 = Z_\phi^{1/2}\phi, \quad A_{0a} = Z_A^{1/2}A_a, \quad \alpha_0 = Z_\phi^{-1}Z_A^{-1/2}Z_\alpha\alpha, \quad \lambda_0 = Z_\phi^{-3}Z_\lambda\lambda. \tag{5.23}$$

The Feynman diagrams now depend on the cutoff Λ and an external momentum scale p^2. The renormalization constants are defined by a minimal-subtraction scheme where we subtract the divergent part of the diagram at the momentum scale

$$|p| = \mu. \tag{5.24}$$

The action in terms of renormalized fields and couplings is just as in (5.20), but now with the boundary dimension d strictly equal to 3:

$$\begin{aligned} I = \int d^3x \Bigg(&\partial_a\phi\partial^a\overline{\phi} + i\,\alpha A_a\overline{\phi}\overleftrightarrow{\partial_a}\phi + \alpha^2 A_aA_a|\phi|^2 + \frac{\lambda}{(3!)^2}|\phi|^6 \\ &+ \delta Z_\phi\partial_a\phi\partial^a\overline{\phi} + i\,\delta Z_\alpha\alpha A_a\overline{\phi}\overleftrightarrow{\partial_a}\phi + \delta Z_2\alpha^2 A_aA_a|\phi|^2 + \frac{\lambda\delta Z_\lambda}{(3!)^2}|\phi|^6\Bigg). \end{aligned} \tag{5.25}$$

5.2 Renormalization group equations.

Both renormalization schemes depend on an arbitrary mass scale μ. For physical quantities to be independent of μ, the parameters entering the theory also depend on this scale. Of fundamental importance are how the couplings run, which is the subject of this subsection.

As discussed at the beginning of Section 4, a crucial aspect of this setup is the fact that the bulk gauge coupling e, and thus the effective boundary coupling α (defined in (5.10)), is not renormalized. That the bare gauge coupling α_0 is not renormalized means, in either scheme (c.f. (5.15) or (5.23))

$$Z_\phi^{-1} Z_A^{-1/2} Z_\alpha = 1. \tag{5.26}$$

This has been proven in the case of fermions on the boundary [211], but not (to our knowledge) for the case of scalars to our knowledge. We do not attempt to prove it here - suffice to say that this is true to order α^2, which is good enough for our purposes.

5.2.1 Dimensional regularization

In dimensional regularization we define the beta functions

$$\beta_\alpha := \left.\frac{d\alpha}{d\log\mu}\right|_{\epsilon=0}, \quad \& \quad \beta_\lambda := \left.\frac{d\lambda}{d\log\mu}\right|_{\epsilon=0}. \tag{5.27}$$

Plugging (5.26) into the definition of the renormalized gauge coupling (5.15), it is related to the bare coupling by

$$\alpha_0 = \mu^\epsilon \alpha. \tag{5.28}$$

Taking the logarithm of both sides and differentiating with respect to $\log\mu$, we obtain the trivial RG equation

$$\frac{d\alpha}{d\log\mu} = -\epsilon\alpha. \tag{5.29}$$

The beta function β_α, obtained by taking the limit $\epsilon \to 0$ in the above, thus vanishes

$$\beta_\alpha = 0. \tag{5.30}$$

Repeating the logic used to obtain (5.29) for the bare coupling λ_0, we obtain the renormalization group equation for λ (c.f. (5.15))

$$\frac{d\lambda}{d\log\mu} + 4\epsilon\lambda + \lambda\frac{\partial\log(Z_\phi^{-3}Z_\lambda)}{\partial\log\mu} = 0. \tag{5.31}$$

Taking the limit $\epsilon \to 0$ we obtain an equation for β_λ. However, as it stands, this expression is not so useful. The reason is because the dependence of the renormalization constants Z_ϕ, Z_λ on the renormalization scale μ is through the couplings $\alpha(\mu)$ and $\lambda(\mu)$. It does not take too much work to get a more explicit formula. For notational simplicity, define

$$G(\alpha,\lambda) := \log(Z_\phi^{-3}Z_\lambda), \tag{5.32}$$

which appears in the RGE (5.31). Note that this completely determines the beta function for λ. To get a more explicit formula, we start by using the chain rule

$$\begin{aligned}\frac{\partial G}{\partial\log\mu} &= \frac{d\alpha}{d\log\mu}\frac{\partial G}{\partial\alpha} + \frac{d\lambda}{d\log\mu}\frac{\partial G}{\partial\lambda}\\ &\overset{(5.29)}{=} -\epsilon\alpha\frac{\partial G}{\partial\alpha} + \frac{d\lambda}{d\log\mu}\frac{\partial G}{\partial\lambda},\end{aligned} \tag{5.33}$$

so that the renormalization group equation reads

$$4\epsilon\lambda - \epsilon\lambda\alpha\frac{\partial G}{\partial\alpha} + \left(1 + \lambda\frac{\partial G}{\partial\lambda}\right)\frac{d\lambda}{d\log\mu} = 0. \tag{5.34}$$

Just like the renormalization constants, G is determined in perturbation theory as a Laurent series in ϵ

$$G(\alpha,\lambda) = \sum_{\ell=1}^{\infty}\frac{G^{(\ell)}(\alpha,\lambda)}{\epsilon^\ell}, \tag{5.35}$$

where the coefficients $G^{(\ell)}(\alpha,\lambda)$ are themselves power series in the couplings α and λ. Plugging this expansion for G back into (5.34), and a similar expansion for $\frac{d\lambda}{d\log\mu}$, we get a Laurent series whose coefficients must separately vanish. Looking at the $O(\epsilon)$ coefficient is enough to establish

$$\frac{d\lambda}{d\log\mu} = -4\epsilon\lambda + \beta_\lambda. \tag{5.36}$$

Looking at the $O(\epsilon^0)$ coefficient determines

$$\beta_\lambda = \lambda\alpha\frac{\partial G^{(1)}}{\partial\alpha} + 4\lambda^2\frac{\partial G^{(1)}}{\partial\lambda}. \tag{5.37}$$

This is our final formula for β_λ in the dimensional regularization scheme. $G^{(1)}$ are the coefficients of $\frac{1}{\epsilon}$ appearing in the Laurent series of $\log(Z_\phi^{-3}Z_\lambda)$ (c.f. equations (5.32) and (5.35)). It is a general, but rather non-trivial fact, that the higher order terms automatically vanish [2].

5.2.2 Momentum cutoff

With a momentum cutoff we define the beta functions

$$\beta_\alpha := \frac{d\alpha}{d\log\mu}, \quad \& \quad \beta_\lambda := \frac{d\lambda}{d\log\mu}. \tag{5.38}$$

Similar arguments to the beginning of the previous subsection yield the renormalization group equation

$$\beta_\lambda = -\lambda\frac{\partial\log(Z_\phi^{-3}Z_\lambda)}{\partial\log\mu}. \tag{5.39}$$

5.3 Renormalization of composite operators.

As discussed at the end of Section 4.4.2, an interesting quantity to compute in this theory is the anomalous dimension of the mass squared operator. More details on the renormalization of composite operators can be found in [1]. The bare mass-squared operator is

$$\mathcal{O}_0(x) := \overline{\phi}_0(x)\phi_0(x). \tag{5.40}$$

Figure 10: The momentum space Feynman rules for the mass-squared operator carrying zero momentum in the scalar BCFT.

The renormalized operator is

$$\mathcal{O}(x) := Z_{\mathcal{O}}\overline{\phi}_0(x)\phi_0(x) \equiv Z_{\mathcal{O}}Z_{\phi}\overline{\phi}(x)\phi(x). \tag{5.41}$$

Although the renormalization constant $Z_{\mathcal{O}}$ is what we are after, when doing perturbation theory it is convenient to package the combination $Z_{\mathcal{O}}Z_{\phi}$ into a single renormalization constant

$$Z_{m^2} := Z_{\mathcal{O}}Z_{\phi}. \tag{5.42}$$

In terms of which the renormalized operator reads

$$\mathcal{O}(x) = Z_{m^2}\overline{\phi}(x)\phi(x). \tag{5.43}$$

The Feynman rules for an insertion of $\mathcal{O}$ are recorded in Figure 10. Once we have computed Z_{ϕ} and Z_{m^2} in perturbation theory, it is a simple matter of algebra to re-write this in terms of $Z_{\mathcal{O}}$:

$$Z_{\mathcal{O}} = Z_{m^2}Z_{\phi}^{-1}. \tag{5.44}$$

The renormalization constant $Z_{\mathcal{O}}$ will depend on the renormalization scale μ. In dimensional-regularization this dependence enters through the definition of the renormalized couplings in (5.15). With a cutoff the dependence enters through (5.24). In either scheme, the anomalous dimension of $\mathcal{O}$ is then given by the formula

$$\gamma_{\mathcal{O}} := \frac{d\log Z_{\mathcal{O}}}{d\log\mu}. \tag{5.45}$$

It is important to note that, on physical grounds, $\gamma_{\mathcal{O}}$ should be gauge invariant. Indeed the value of $\gamma_{\mathcal{O}}$ at the critical point is an example of a critical exponent, and a priori measurable.

5.4 One loop analysis.

In this section we summarise the 1-loop structure of the theory. More details are given in Appendix B. The first thing we need to do is check we have a bona fide CFT on the boundary. As we have discussed, the gauge coupling - or better yet, the effective boundary coupling α - does not run. But we have a marginal coupling λ and we need to check for zeroes of the beta function β_λ. Before doing this we will need the wavefunction renormalization Z_ϕ of the scalar, so we begin there.

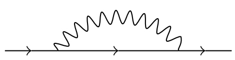
Since we are looking for fixed points of λ, we can assume that $\lambda = \mathrm{O}(\alpha^2)$, which simplifies the organization of the perturbative series. Of course, we still need to check that $\lambda \geq 0$ at the fixed point, leading to a well-defined scalar potential bounded from below.

The scalar wavefunction renormalization is obtained by renormalizing the 1-particle-irreducible (1PI) graphs contributing to the exact ϕ propagator. We denote the sum of such graphs with the external propagators removed as $\Sigma_\phi(p)$, where p is the external momentum. Then, working in perturbation theory, $\Sigma_\phi(p)$ can be expressed as a power series in α^2. We denote the coefficient of α^n as $\Sigma_\phi(p|\alpha^n)$. Therefore

$$\Sigma_\phi(p) = \sum_{\ell=0}^{\infty} (\alpha^2)^\ell \cdot \Sigma_\phi(p|\alpha^{2\ell}). \tag{5.46}$$

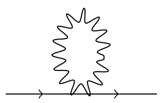
There are three graphs which contribute at order α^2:

$$\Sigma^{(1)}_\phi(p|\alpha^2) =$$



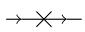
(5.47)

$$\Sigma^{(2)}_\phi(p|\alpha^2) =$$



(5.48)

$$\Sigma_{\phi,\mathrm{ct}}|\alpha^2) =$$



(5.49)

Even though the divergent part suffices to determine the renormalization parameters at one-loop, it will turn out that we need the finite part of the scalar wavefunction renormalization at two-loops. Using a momentum cutoff we find

$$\Sigma_\phi^{(1)}(p|\alpha^2) = \frac{\alpha^2}{(2\pi)^2}\left\{\xi\Lambda^2 + p^2\left(\frac{8}{3}-\xi\right)\log\frac{\Lambda^2}{p^2} + \left(\frac{16}{9}-2\xi\right)p^2\right\}, \tag{5.50}$$

and

$$\Sigma_\phi^{(2)}(p|\alpha^2) = -\frac{\alpha^2}{\pi^2}\left(1+\frac{\xi}{2}\right)\Lambda^2. \tag{5.51}$$

The Λ^2 divergences can be cancelled by a mass counterterm for ϕ. Importantly these independent of the external momentum p, so we can assume the mass counterterm is arranged such that the physical mass vanishes and this will remain unchanged under RG flow. The logarithmic divergence is more interesting. Adding to this the counterterm contribution and demanding that the result be finite as $\Lambda \to \infty$ for external momenta $|p| = \mu$ determines the wavefunction renormalization to be

$$\delta Z_\phi = \frac{(8-3\xi)\alpha^2}{12\pi^2}\log\frac{\Lambda^2}{\mu^2} + \mathrm{O}(\alpha^4). \tag{5.52}$$

It will be enough for our purposes only to determine the divergent parts of the remaining graphs. The next piece of information we need is the renormalization constant Z_λ for the 6-point vertex. This can be obtained by renormalizing the 1PI graphs contributing to the six point function $\langle\bar\phi\phi\bar\phi\phi\bar\phi\phi\rangle$. The contributions at order α^4 fall into two classes, which we have rather schematically displayed in Figure 11. There are two topologically inequivalent arrangements for the external momenta in the first graph, and three in the second. Extracting the divergent parts and inserting appropriate combinatorial factors, requiring that these graphs cancel with the counterterm $-\lambda\delta Z_\lambda$ thus determines

$$\delta Z_\lambda = \left(\frac{7\lambda}{24\pi^2} - \frac{3\xi\alpha^2}{4\pi^2}\right)\log\frac{\Lambda^2}{\mu^2} + \cdots, \tag{5.53}$$

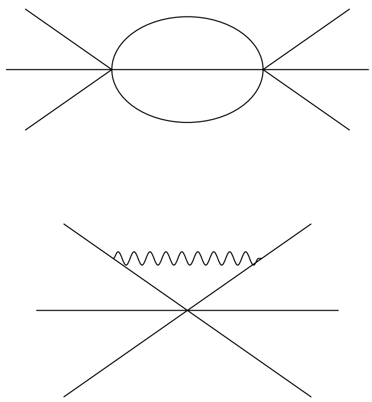


Figure 11: The leading order corrections to the scalar six-point vertex.

where the dots stand for higher order terms. We have everything we need to compute the beta function for λ (c.f. (5.39) and (5.52)), we find

$$\beta_\lambda = \frac{4\lambda}{\pi^2}\left(\frac{7\lambda}{48} - \alpha^2\right). \tag{5.54}$$

This admits a non-trivial fixed point at

$$\lambda_* = \frac{48\alpha^2}{7} + O(\alpha^4). \tag{5.55}$$

Importantly $\lambda_* > 0$, so this gives a well-defined scalar potential bounded from below. One can ask what happens to the trivial fixed point $\lambda = 0$ if we go to higher order. The next corrections are the $O(\alpha^6)$ diagrams pictured in Figure 12. The divergent parts of the first three graphs are proportional to ξ. Including the fourth graph, the ξ dependence cancels out and their sum is of the form $\sim -\#\alpha^6 \log\frac{\Lambda^2}{p^2}$, where # is independent of ξ, but more importantly it is *positive* (and non-zero!). The contribution to the beta function is then

$$\beta_\lambda = \frac{4\lambda}{\pi^2}\left(\frac{7\lambda}{48} - \alpha^2\right) - \#'\alpha^6, \tag{5.56}$$

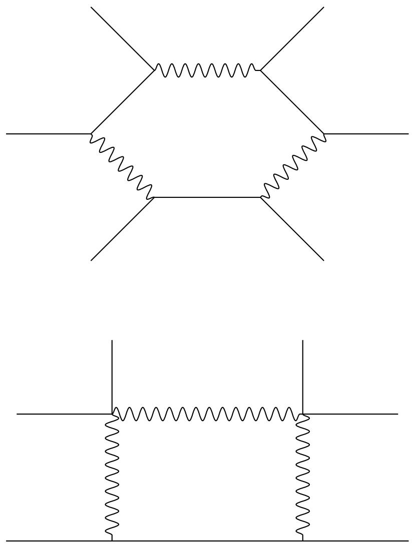


Figure 12: Order α^6 corrections to the scalar six-point vertex. The ordering of the diagrams (as referred to in the main text) is from right-to-left top-to-bottom.

where $\#'$ is some other constant, but importantly it is still positive (and non-zero!). This is pictured schematically in Figure 13, along with the earlier result (5.54) for the beta function that is valid to order α^4. Since the correction is negative, this shifts the $\lambda = 0$ fixed point to a *negative* number of order α^4, and so the quantum theory does not make sense.

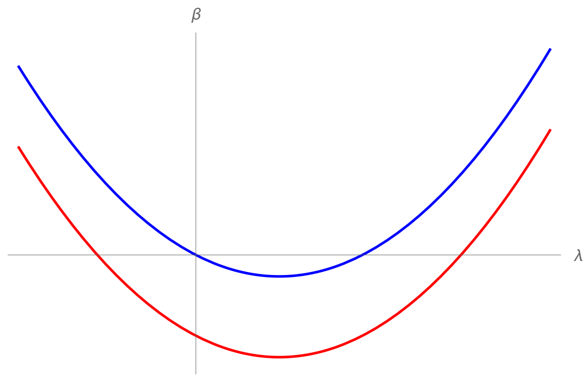


Figure 13: This is a schematic plot of the beta function $\beta_\lambda(\lambda, \alpha)$ as a function of λ for fixed α. The blue line is (5.54), which has a trivial fixed point at $\lambda = 0$. The red line displays the beta function (5.56) which includes the α^6 correction. Here the trivial fixed point is shifted away from zero to an unstable negative value.

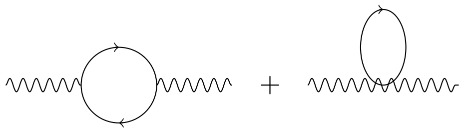


Figure 14: One-loop photon self-energy graphs.

We now go on to computing the remaining renormalization constants Z_A and Z_α, as these will appear in our two loop computations. The diagrams which contribute to the 1PI photon propagator are given in Figure 14. These are easily seen to be finite, thus in either scheme

$$\delta Z_A = 0 + \mathrm{O}(\alpha^4). \qquad \text{(DR or Cutoff)}. \tag{5.57}$$

Next we consider the 1PI 3-point vertex. The graphs at one-loop order are displayed in Figure 15. To determine δZ_α it suffices to consider the kinematic arrangement $p' = p$ where the diagrams significantly simplify. To cancel with counterterm $-2p^a \alpha \delta Z_\alpha$ determines

$$\delta Z_\alpha = \frac{\alpha^2}{12\pi^2}(8 - 3\xi) \log \frac{\Lambda^2}{\mu^2} + \mathrm{O}(\alpha^4). \tag{5.58}$$

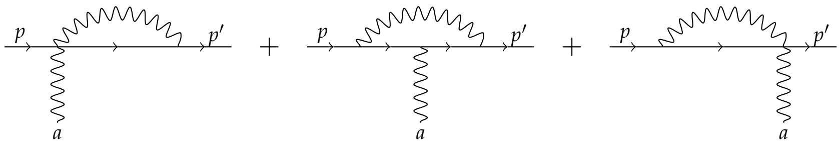


Figure 15: One loop corrections to the three-point vertex.

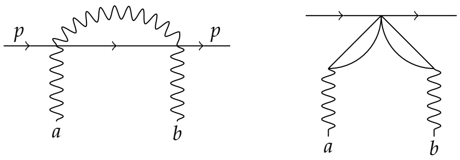


Figure 16: One loop corrections to the four-point vertex.

Finally we have the 1PI graphs for the 4-point vertex pictured in Figure 16. The second diagram is finite, but the first is divergent. One can easily check that the 4-point vertex is rendered finite with the above choice of δZ_α too. Another check is to verify that the gauge coupling is not renormalized. As discussed at the beginning of Section 4, this is expected on physical grounds (c.f. also the discussion at the beginning of Section 5.2). First of all, comparing (5.52) and (5.58), we see that the wavefunction and vertex renormalizations are equal up to corrections of order α^4:

$$\delta Z_\alpha = \delta Z_\phi + O(\alpha^4), \quad \Leftrightarrow \quad Z_\alpha = Z_\phi + O(\alpha^4). \tag{5.59}$$

Thus plugging this into (5.15)

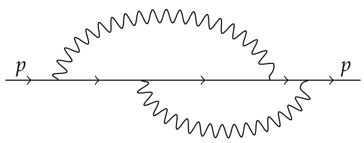
$$\alpha_0 = \alpha + O(\alpha^4). \tag{5.60}$$

I.e. the gauge-coupling is not renormalized, as we had hoped.

5.5 Two-loop scalar wavefunction renormalization

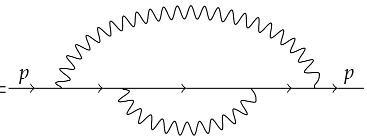
The graphs which contribute to the scalar self energy $\Sigma_\phi(p)$ at order α^4 are are

$$\Sigma_\phi^{(1)}(p|\alpha^4) =$$



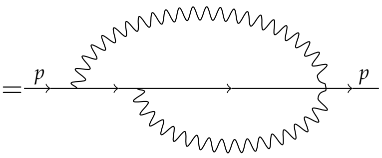
(5.61)

$$\Sigma_\phi^{(2)}(p|\alpha^4) =$$



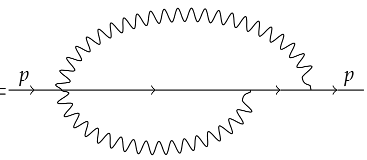
(5.62)

$$\Sigma_\phi^{(3)}(p|\alpha^4) =$$



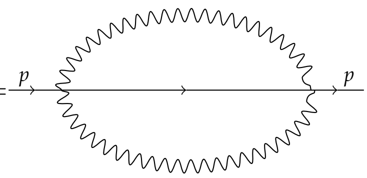
(5.63)

$$\Sigma_\phi^{(4)}(p|\alpha^4) =$$



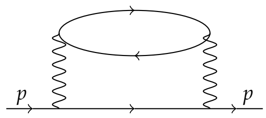
(5.64)

$$\Sigma_\phi^{(5)}(p|\alpha^4) =$$



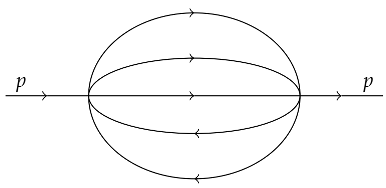
(5.65)

$$\Sigma_\phi^{(6)}(p|\alpha^4) =$$

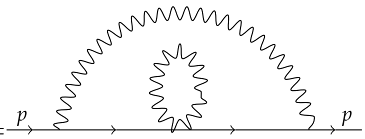


(5.66)

$$\Sigma_\phi^{(7)}(p|\alpha^4) =$$



(5.67)

$$\Sigma_\phi^{(8)}(p|\alpha^4) =$$ 

(5.68)

Before analysing the diagrams, we note the immediate fact that the dependence of Σ_ϕ on θ is an $\mathrm{O}(e^6)$ effect (recall that α is defined in terms of the charge e in (5.10)). To see this, we look at the expression (5.21) for the photon propagator, which is repeated here for convenience:

$$\Pi_{ab}(p) = \frac{1}{|p|}\left(\delta_{ab} - (1-\xi)\frac{p_a p_b}{p^2} + \vartheta\epsilon_{abc}\frac{p^c}{|p|}\right). \tag{5.69}$$

The θ dependence is through ϑ, which was defined in (5.10). Since ϑ is of order e^2, we can lump the θ dependence of the graphs (5.61)-(5.68) into $\mathrm{O}(e^6)$ corrections. So any θ dependence at the order we are working to (i.e. order e^4) would come from our one-loop analysis in the previous section - but we found no θ dependence in any of our results in the previous section!

Now we move on to analysing the divergences of the two loop diagrams. Here we just quote our results, relegating details of the computation to Appendix B. Working in the gauge $\xi = 1$ we find

$$\Sigma_\phi^{(1)}(p|\alpha^4) \sim \alpha^4 p^2\left\{\frac{977}{48\pi^4}\log\frac{\Lambda^2}{p^2} + \frac{9}{4\pi^4}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}, \tag{5.70}$$

$$\Sigma_\phi^{(2)}(p|\alpha^4) \sim \alpha^4 p^2\left\{\frac{265}{432\pi^4}\log\frac{\Lambda^2}{p^2} + \frac{25}{288\pi^4}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}, \tag{5.71}$$

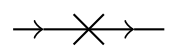
$$\Sigma_\phi^{(3)}(p|\alpha^4) \sim -\alpha^4 p^2\left\{\frac{487}{48\pi^4}\log\frac{\Lambda^2}{p^2} + \frac{323}{288\pi^4}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}, \tag{5.72}$$

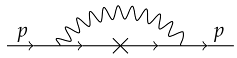
$$\Sigma_\phi^{(5)}(p|\alpha^4) \sim -\frac{\alpha^4 p^2}{8\pi^4}\log\frac{\Lambda^2}{p^2}, \tag{5.73}$$

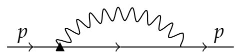
$$\Sigma_\phi^{(6)}(p|\alpha^4) \sim -\frac{\alpha^4 p^2}{24\pi^4}\log\frac{\Lambda^2}{p^2}, \tag{5.74}$$

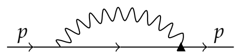
$$\Sigma_\phi^{(7)}(p|\alpha^4) \sim \frac{8\alpha^4 p^2}{147\pi^4}\log\frac{\Lambda^2}{p^2}, \tag{5.75}$$

and $\Sigma_\phi^{(4)}(p|\alpha^4) = \Sigma_\phi^{(3)}(p|\alpha^4)$. Also $\Sigma_\phi^{(8)}$ does not contribute to the wavefunction renormalization since the photon loop gives a Λ^2 divergence, so and we drop it. Note also that we are excluding power divergences $\sim \Lambda^2$. On top of this we also have diagrams which include counterterms

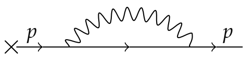
$$\Sigma^{(1)}_{\phi,\mathrm{ct}}|\alpha^4) = $$  $$= -p^2 \delta Z_\phi|\alpha^4) \tag{5.76}$$

$$\Sigma^{(2)}_{\phi,\mathrm{ct}}|\alpha^4) = $$  (5.77)

$$\Sigma^{(3)}_{\phi,\mathrm{ct}}|\alpha^4) = $$  (5.78)

$$\Sigma^{(4)}_{\phi,\mathrm{ct}}|\alpha^4) = $$  (5.79)

The last three are all equal by (5.52) and (5.58)

$$\Sigma^{(2)}_{\phi,\mathrm{ct}}(p|\alpha^4) + \Sigma^{(3)}_{\phi,\mathrm{ct}}(p|\alpha^4) + \Sigma^{(4)}_{\phi,\mathrm{ct}}(p|\alpha^4) = -3\delta Z_\phi|\alpha^2) \times$$ 

Using (5.50) and (5.52) for the 1-loop graph and $\delta Z_\phi|\alpha^2)$ we find

$$\Sigma^{(2)}_{\phi,\mathrm{ct}}(p|\alpha^4) + \Sigma^{(3)}_{\phi,\mathrm{ct}}(p|\alpha^4) + \Sigma^{(4)}_{\phi,\mathrm{ct}}(p|\alpha^4) = \alpha^4 p^2 \cdot \left\{ \frac{5}{72\pi^4} \log \frac{\Lambda^2}{\mu^2} - \frac{25}{48\pi^4} \log \frac{\Lambda^2}{\mu^2} \log \frac{\Lambda^2}{p^2} \right\}. \tag{5.80}$$

Adding everything together and requiring the result be finite at $p^2 = \mu^2$ determines

$$\delta Z_\phi|\alpha^4) = \alpha^4 p^2 \left\{ \left(\frac{3571}{5292\pi^4} - \frac{1}{24\pi^2} \right) \log \frac{\Lambda^2}{\mu^2} - \frac{41}{96\pi^4} \left(\log \frac{\Lambda^2}{\mu^2} \right)^2 \right\}. \tag{5.81}$$

5.6 Mass-squared operator

To determine $Z_{\overline{\phi}\phi}$ (or equivalently Z_{m^2}) we renormalize the correlator

$$\langle \overline{\phi}\phi(x) \overline{\phi}(y) \phi(z) \rangle. \tag{5.82}$$

Fourier transforming to momentum space, this means renormalizing the 1PI graphs. It suffices to renormalize these graphs in the special case that the operator $\overline{\phi}\phi$ carries zero momentum, and we denote the sum of graphs as $\Sigma_{\overline{\phi}\phi}(p)$. Using perturbation theory we can write $\Sigma_{\overline{\phi}\phi}(p)$ as a power series in the coupling α; we denote the coefficient of α^ℓ as $\Sigma_{\overline{\phi}\phi}(p|\alpha^\ell)$, so that

$$\Sigma_{\overline{\phi}\phi}(p) = \sum_{\ell=0}^{\infty} (\alpha^2)^\ell \Sigma_{\overline{\phi}\phi}(p|\alpha^{2\ell}). \tag{5.83}$$

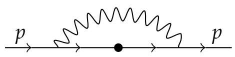
Of course, only even powers $(\alpha^2)^\ell$ of α occur. Absorbing divergences into δZ_{m^2} order-by-order in perturbation theory, we get a corresponding series

$$\delta Z_{m^2} = \sum_{\ell=1}^{\infty} \alpha^{2\ell} \delta Z_{m^2}|\alpha^{2\ell}). \tag{5.84}$$

5.6.1 One loop analysis.

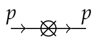
There is just one graph at one loop

$$\Sigma_{\overline{\phi}\phi}|\alpha^2) = \tag{5.85}$$



along with the counterterm

$$\Sigma_{\overline{\phi}\phi,\mathrm{ct}}|\alpha^2) = \tag{5.86}$$



We find

$$\Sigma_{\overline{\phi}\phi}(p|\alpha^2) = \frac{\alpha^2}{(2\pi)^2}\left(8 - 5\xi + \xi \log \frac{\Lambda^2}{p^2}\right). \tag{5.87}$$

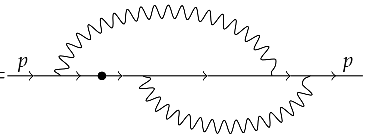
Requiring the divergence cancel at $p^2 = \mu^2$ determines

$$\delta Z_{m^2} = -\frac{\alpha^2 \xi}{4\pi^2} \log \frac{\Lambda^2}{\mu^2} + \mathrm{O}(\alpha^4). \tag{5.88}$$

5.6.2 Two-loop analysis.

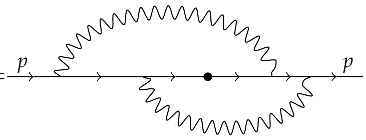
The two loop contributions $\Sigma_{\overline{\phi}\phi}(p|\alpha^4)$ to $\Sigma_{\overline{\phi}\phi}(p)$ are simple to enumerate. They are given by summing over insertions of the mass operator on the internal lines of the self-energy graphs (5.61)-(5.68). For instance, there are three insertions we can make on the graph (5.61). Summing over insertions from left-to-right these correspond to

$$\Sigma^{(1;1)}_{\overline{\phi}\phi}(p|\alpha^4) =$$



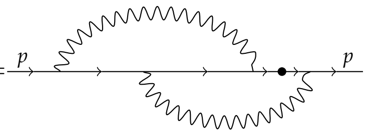
(5.89)

$$\Sigma^{(1;2)}_{\overline{\phi}\phi}(p|\alpha^4) =$$



(5.90)

$$\Sigma^{(1;3)}_{\overline{\phi}\phi}(p|\alpha^4) =$$



(5.91)

Here the superscript "$(1;1)$" means the first insertion on the first graph, the subscript "$(1;2)$" means the second insertion on the first graph, and similarly for "$(1;3)$". Working in the gauge $\xi = 1$ we find

$$\Sigma^{(1;1)}_{\overline{\phi}\phi}(p|\alpha^4) \sim \alpha^4 \left\{ \frac{19}{48\pi^4} \log\frac{\Lambda^2}{p^2} + \frac{143}{4512\pi^4}\left(\log\frac{\Lambda^2}{p^2}\right)^2 \right\}, \tag{5.92}$$

$$\Sigma^{(1;2)}_{\overline{\phi}\phi}(p|\alpha^4) \sim -\alpha^4 \left\{ \frac{127}{96\pi^4} \log\frac{\Lambda^2}{p^2} + \frac{1}{8\pi^4}\left(\log\frac{\Lambda^2}{p^2}\right)^2 \right\}, \tag{5.93}$$

and $\Sigma^{(1;3)}_{\bar{\phi}\phi} = \Sigma^{(1;1)}_{\bar{\phi}\phi}(p|\alpha^4)$. Summing over insertions from right-to-left on the second graph (5.62) we get

$$\Sigma^{(2;1)}_{\bar{\phi}\phi}(p|\alpha^4) \sim \alpha^4 \left\{ \frac{37}{72\pi^4} \log \frac{\Lambda^2}{p^2} + \frac{1}{24\pi^4} \left(\log \frac{\Lambda^2}{p^2} \right)^2 \right\}, \tag{5.94}$$

$$\Sigma^{(2;2)}_{\bar{\phi}\phi}(p|\alpha^4) \sim -\alpha^4 \left\{ \frac{19}{48\pi^4} \log \frac{\Lambda^2}{p^2} + \frac{1}{16\pi^4} \left(\log \frac{\Lambda^2}{p^2} \right)^2 \right\}, \tag{5.95}$$

and $\Sigma^{(2;3)}_{\bar{\phi}\phi} = \Sigma^{(2;1)}_{\bar{\phi}\phi}(p|\alpha^4)$. To belabour the point, the subscript "(2;1)" means first insertion on the second graph, the subscript "(2;2)" means second insertion on the second graph, etc. where the insertions go left-to-right. There are two insertions on the third self-energy graph leading to

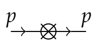
$$\Sigma^{(3;1)}_{\bar{\phi}\phi}(p|\alpha^4) \sim -\alpha^4 \left\{ \frac{31}{36\pi^4} \log \frac{\Lambda^2}{p^2} + \frac{1}{12\pi^4} \left(\log \frac{\Lambda^2}{p^2} \right)^2 \right\}, \tag{5.96}$$

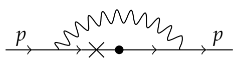
$$\Sigma^{(3;2)}_{\bar{\phi}\phi}(p|\alpha^4) \sim -\frac{7\alpha^4}{192\pi^4} \log \frac{\Lambda^2}{p^2}. \tag{5.97}$$

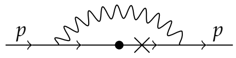
The two insertions $\Sigma^{(4;1)}_{\bar{\phi}\phi}(p|\alpha^4)$ and $\Sigma^{(4;2)}_{\bar{\phi}\phi}(p|\alpha^4)$ on the fourth self-energy graph give the same contribution as on the third. There is just one insertion on the fifth graph

$$\Sigma^{(5)}_{\bar{\phi}\phi}(p|\alpha^4) \sim' -\frac{\alpha^4}{32\pi^4} \log \frac{\Lambda^2}{p^2}. \tag{5.98}$$

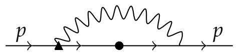
It turns out that the insertions on graphs 6, 7 and 8 do not contribute. Finally, we have the following graphs which include counterterms

$$\Sigma^{(1)}_{\bar{\phi}\phi,\mathrm{ct}}(p|\alpha^4) = $$  $$= \delta Z_{m^2}|\alpha^4) \tag{5.99}$$

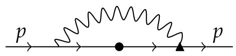
$$\Sigma^{(2)}_{\bar{\phi}\phi,\mathrm{ct}}(p|\alpha^4) = $$  $$\tag{5.100}$$

$$\Sigma^{(3)}_{\overline{\phi}\phi,\mathrm{ct}}(p|\alpha^4) = $$ 

(5.101)

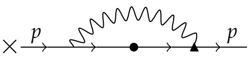
$$\Sigma^{(4)}_{\overline{\phi}\phi,\mathrm{ct}}(p|\alpha^4) = $$ 

(5.102)

$$\Sigma^{(5)}_{\overline{\phi}\phi,\mathrm{ct}}(p|\alpha^4) = $$ 

(5.103)

Graphs 2-5 are all equal

$$\Sigma^{(2)}_{\overline{\phi}\phi,\mathrm{ct}}(p|\alpha^4) + \cdots + \Sigma^{(5)}_{\overline{\phi}\phi,\mathrm{ct}}(p|\alpha^4) = -4\delta Z_\phi|\alpha^2) \times $$ 

(5.104)

Thus using (5.52) and (5.87) the sum of the graphs (5.99)-(5.103) is

$$\Sigma_{\overline{\phi}\phi,\mathrm{ct}}(p|\alpha^4) = -\alpha^4\left(\frac{5}{4\pi^4}\log\frac{\Lambda^2}{\mu^2} + \frac{5}{12\pi^4}\log\frac{\Lambda^2}{\mu^2}\log\frac{\Lambda^2}{p^2}\right) + \delta Z_{m^2}|\alpha^4). \tag{5.105}$$

Adding everything together and requiring that the result be finite at $p^2 = \mu^2$ determines

$$\delta Z_{m^2}|\alpha^4) = \frac{857}{288\pi^4}\log\frac{\Lambda^2}{\mu^2} + \frac{88}{141\pi^4}\left(\log\frac{\Lambda^2}{\mu^2}\right)^2. \tag{5.106}$$

Finally, as explained in Section 5.5, the dependence of our result on θ is invisible to the order we are working to.

5.6.3 Anomalous dimension.

Now, recall that the renormalization constant we are after is not Z_{m^2}, but rather $Z_{\overline{\phi}\phi}$. This is determined by inverting the series expansion for Z_ϕ and then plugging it into (5.44), which gives

$$Z_{\overline{\phi}\phi} = 1 + \alpha^2\Big(\delta Z_{m^2}|\alpha^2) - \delta Z_\phi|\alpha^2)\Big) + \alpha^4\Big(\delta Z_{m^2}|\alpha^4) + \delta Z_\phi|\alpha^2)^2 - \delta Z_\phi|\alpha^4)\Big) + \mathrm{O}(\alpha^6). \tag{5.107}$$

From which read off the coefficients $\delta Z_{\overline{\phi}\phi}|\alpha^2)$, and $\delta Z_{\overline{\phi}\phi}|\alpha^4)$. Plugging in our one loop results (5.88) and (5.52) into the above determines the one-loop coefficient to be

$$\delta Z_{\overline{\phi}\phi}|\alpha^2) = -\frac{2\alpha^2}{3\pi^2}\log\frac{\Lambda^2}{\mu^2}. \tag{5.108}$$

In particular, this one-loop result is gauge invariant as expected. Using also (5.106) and (5.81) determines the two-loop coefficient to be

$$\delta Z_{\overline{\phi}\phi}|\alpha^4) = \left(\frac{97411}{42336\pi^4} + \frac{1}{24\pi^2}\right)\log\frac{\Lambda^2}{\mu^2} + \frac{16579}{13536\pi^4}\left(\log\frac{\Lambda^2}{\mu^2}\right)^2. \tag{5.109}$$

With these coefficients in hand, we can plug them in to the logarithm

$$\log Z_{\overline{\phi}\phi} = \alpha^2\delta Z_{\overline{\phi}\phi}|\alpha^2) + \alpha^4\left(\delta Z_{\overline{\phi}\phi}|\alpha^4) - \frac{1}{2}\delta Z_{\overline{\phi}\phi}|\alpha^2)^2\right) + \mathrm{O}(\alpha^6), \tag{5.110}$$

to get

$$\log Z_{\overline{\phi}\phi} = -\frac{2\alpha^2}{3\pi^2}\log\frac{\Lambda^2}{\mu^2} + \alpha^4\left(\frac{97411}{42336\pi^4} + \frac{1}{24\pi^2}\right)\log\frac{\Lambda^2}{\mu^2} + \mathrm{O}(\alpha^6). \tag{5.111}$$

Differentiating with respect to $\log\mu$, the anomalous dimension is

$$\gamma_{\overline{\phi}\phi} = \frac{4\alpha^2}{3\pi^2} - \alpha^4\left(\frac{97411}{21168\pi^4} + \frac{1}{12\pi^2}\right) + \mathrm{O}(\alpha^6). \tag{5.112}$$

5.7 Extrapolating to the upper half plane

Now we shall attempt to extrapolate our result (5.112) using the proposed self-duality $\tau \to -\frac{1}{\tau}$ described in Section 4.4.1. To begin with, we need to translate our result which is written in terms of the effective boundary gauge coupling α defined in (5.10), back in terms of the bulk gauge coupling e and the theta angle θ. But this is easy since we can replace α^2 by e^2 up to corrections of order e^6

$$\alpha^2 = e^2 + \mathrm{O}(e^6). \tag{5.113}$$

Thus, in terms of e, the anomalous dimension reads

$$\gamma_{\bar{\phi}\phi} = \frac{4e^2}{3\pi^2} - \left(\frac{97411}{21168\pi^4} + \frac{1}{12\pi^2}\right)e^4 + \mathrm{O}(e^6). \tag{5.114}$$

In particular the θ dependence is an $\mathrm{O}(e^6)$ effect (c.f. also the discussion in Section 5.5). Our result (5.114) is valid in a neighborhood of $\tau = +\mathrm{i}\infty$, and by S-duality, a neighborhood of $\tau = 0$. We might hope that, if we can find a smooth interpolating function $F(\tau)$ which agrees with with our perturbative result (5.114) near $\tau = +\mathrm{i}\infty$ and the S-dual result near $\tau = 0$, it might be somewhat close to the true anomalous dimension for all τ. Presumably there will be many functions $F(\tau)$ which satisfy this criteria, so which one do we pick? A particularly useful place to start is by choosing F to be a Padé approximant

$$P_{[n/m]}(g) = \frac{a_0 + a_1 g + \cdots + a_n g^n}{b_0 + b_1 g + \cdots + b_m g^m}, \tag{5.115}$$

where g is an arbitrary coupling constant for now. Ultimately g will be equal to e^2. The a's and the b's are chosen so that the Taylor series around $g = 0$ agrees with the perturbative result to order g^{m+n+1}. We don't want just any $P_{[n/m]}$, rather we want one that is invariant under the S transformation. Following [236] just such a problem was considered in [109], where they constructed two families of such Padé approximants. The first is of the form

$$P^{\mathrm{S}}_{[n/m]}(g) := \frac{a_0 g_{\mathrm{S}}^{-n} + a_1 g_{\mathrm{S}}^{-n+1} + \cdots + a_{n-1} g_{\mathrm{S}}^{-1} + a_n}{b_0 g_{\mathrm{S}}^{-m} + b_1 g_{\mathrm{S}}^{-m+1} + \cdots + b_{m-1} g_{\mathrm{S}}^{-1} + b_m}. \tag{5.116}$$

where

$$g_{\mathrm{S}}^k := g^k + (\mathrm{S}\cdot g)^k, \quad k \in \mathbb{Z}. \tag{5.117}$$

Whilst the second is an odd-degree Padé approximant

$$\tilde{P}^{\mathrm{S}}_{[n/m]}(g) = \frac{a_0 g_{\mathrm{S}}^{-n-1/2} + a_1 g_{\mathrm{S}}^{-n+1/2} + \cdots + a_{n-1} g_{\mathrm{S}}^{-3/2} + a_n g_{\mathrm{S}}^{-1/2}}{b_0 g_{\mathrm{S}}^{-m-1/2} + b_1 g_{\mathrm{S}}^{-m+1/2} + \cdots + b_{m-1} g_{\mathrm{S}}^{-3/2} + b_m g_{\mathrm{S}}^{-1/2}}. \tag{5.118}$$

Since $P^{S}_{[n/m]}(g) \sim \frac{a_0 g^{m-n}}{b_0}$ for small g, we require $n = m-1$ to match weak coupling behaviour. The same applies to $\tilde{P}^{S}_{[n/m]}$. Working through the details, at the order we are working to, we arrive at the integral-power Padé

$$F_1(e,\theta) := \frac{c_1}{e^{-2} + e'^{-2} - c_1'}, \tag{5.119}$$

and the half-integral-power Padé

$$F_2(e,\theta) = \frac{c_2(e^{-1} + e'^{-1})}{e^{-3} + e'^{-3} + c_2'(e^{-1} + e'^{-1})}. \tag{5.120}$$

Here e' is the S-dual charge of e. More precisely, if we denote by $\tau' := -\frac{1}{\tau}$ the S-dual gauge coupling, then the dual theta angle θ' and dual charge e' are defined so that

$$\tau' \equiv \frac{\theta'}{2\pi} + \frac{2\pi \mathrm{i}}{e'^2}. \tag{5.121}$$

Explicitly

$$\theta' = -\frac{\theta}{\frac{\theta^2}{4\pi^2} + \frac{4\pi^2}{e^4}}, \quad \& \quad e'^2 = \frac{\theta^2 e^2}{4\pi^2} + \frac{4\pi^2}{e^2}. \tag{5.122}$$

It now remains to fixed the coefficients c_i, c_i' $(i = 1, 2)$. To fix them we expand the Padé functions as a series in e^2. Doing this for F_1 up to order e^4, and comparing with our result (5.114) we read off

$$c_1 = \frac{4}{3\pi^2}, \quad \& \quad c_1' = -\left(\frac{97411}{28224\pi^2} + \frac{1}{16}\right). \tag{5.123}$$

Playing the same game with F_2, we fix the remaining coefficients to be

$$c_2 = \frac{4}{3\pi^2}, \quad \& \quad c_2' = \frac{97411}{28224\pi^2} + \frac{1}{16} + \frac{1}{2\pi}. \tag{5.124}$$

Extrapolating the first to $\tau = \pm 1$

$$F_1(\infty, 2\pi) = \frac{37632}{97411 + 1764\pi^2} \approx 0.327745. \tag{5.125}$$

And the second

$$F_2(\infty, 2\pi) = \frac{37632}{97411 + 14112\pi + 1764\pi^2} \approx 0.236449. \tag{5.126}$$

5.8 Comparisons with the Gross-Neveu model

We expect to map to a critical exponent in the 3d Gross-Neveu model. We briefly review this model, then go on to make comparisons with the known data. The Gross-Neveu (GN) model was [199] originally formulated in $d = 2$ dimensions

$$\mathscr{L} = \overline{\psi}_j \partial\!\!\!/ \psi^j + \frac{g}{2}(\overline{\psi}_j \psi^j)^2, \tag{5.127}$$

where ψ^j is a four-component Dirac spinor and $j = 1, \ldots, N_{\mathrm{f}}$ is a flavour index. This theory has a discrete chiral symmetry

$$\psi \to \gamma_5 \psi, \quad \overline{\psi} \to -\overline{\psi}\gamma_5. \tag{5.128}$$

In $d = 2$ this theory is famously asymptotically free and displays dynamical breakdown of chiral symmetry. For $d > 2$ this theory is not perturbatively renormalizable, although it is renormalizable in a $\frac{1}{N}$ expansion [200]. The $\frac{1}{N}$ expansion is expedited by performing a Hubbard-Stratonovic transformation

$$\mathscr{L} = \overline{\psi}_i \partial\!\!\!/ \psi_i + \frac{1}{2}\sqrt{g}\sigma\overline{\psi}_i\psi_i - \frac{1}{2}\sigma^2, \tag{5.129}$$

where σ is a real scalar field. This suggests [237, 238] that the UV completion can be obtained by making the scalar field dynamical, and we thus arrive at the Gross-Neveu-Yukawa (GNY) model

$$\mathscr{L} := \overline{\psi}_i \partial\!\!\!/ \psi_i + \frac{1}{2}(\partial\sigma)^2 + \frac{1}{2}m^2\sigma^2 + g_1\sigma\overline{\psi}_i\psi_i + \frac{1}{4!}g_2\sigma^4, \tag{5.130}$$

also known as the chiral Ising model. It is perturbatively renormalizable in 3 and 4 dimensions. The discrete chiral symmetry of the Gross-Neveu model is preserved via

$$\psi \to \gamma_5\psi, \quad \overline{\psi} \to -\overline{\psi}\gamma_5, \quad \sigma \to -\sigma. \tag{5.131}$$

In particular, this symmetry prevents the appearance of a cubic scalar interaction. Since $\sigma \sim \overline{\psi}\psi$ in the large N limit, we expect our result to map to the anomalous dimension γ_σ at the critical point, commonly denoted $\eta_\sigma \equiv \gamma_\sigma(g_1^*, g_2^*)$ (here g_i^*, $i = 1, 2$, denotes the value of the coupling g_i at the critical point). The GNY model has been studied using a variety of methods. The prediction for η_ϕ and the corresponding method is displayed in Table 2 below We see that, whilst F_2 does not work so well, F_1 fits into the known data remarkably well.

Method	η_σ
F_1	0.327745
F_2	0.236449
ϵ-expansion [110]	0.2934
Bootstrap [111]	0.320
Monte-Carlo [112]	0.31 ± 0.01
fRG [113]	0.372

Table 2: This table displays the method and corresponding prediction for the critical exponent η_ϕ in the Gross-Neveu-Yukawa model.

6 Fermions on the boundary

At this point we would like to stop and try to check that our cutoff regularization scheme (c.f. Section 5.1.2), and the two-loop integrals computed in Sections D.3 and D.4 of the Appendix, make sense. A relatively simple check we can perform is to start with free fermions on the boundary instead. We essentially repeat the calculation we did in the previous section, but this time we can use dimensional-regularization too. Since the anomalous dimension $\gamma_{\overline{\psi}\psi}$ of the mass operator is an a priori measurable quantity, it should be independent of the regularization scheme used. A lot of the two-loop integrals tabulated in Appendix D that enter into the computation of $\gamma_{\overline{\phi}\phi}$ in the scalar case also feed into the computation of $\gamma_{\overline{\psi}\psi}$, so this provides a non-trivial check of these integrals.

We begin by setting up the fermionic BCFT. The free fermion action reads

$$I = \int \mathrm{d}^3x \; \mathrm{i}\, \overline{\psi} \partial\!\!\!/ \psi. \tag{6.1}$$

We define the propagator (including spinor indices)

$$S_{\alpha\beta}(x-y) := \langle \overline{\psi}_\beta(y) \psi_\alpha(x) \rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, S_{\alpha\beta}(p) \, \mathrm{e}^{\mathrm{i}\, p(x-y)}, \tag{6.2}$$

which is a Green's function for the Dirac operator $-\mathrm{i}\, \partial\!\!\!/_x$, and thus

$$S(p) = \frac{1}{p\!\!\!/}. \tag{6.3}$$

Coupling to the bulk gauge field we obtain the boundary action

$$I = \int \mathrm{d}^3x \; \mathrm{i}\, \overline{\psi} D\!\!\!\!/ \psi. \tag{6.4}$$

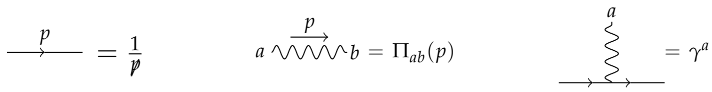


Figure 17: Feynman rules for the Dirac fermion BCFT in momentum space.

6.1 Renormalization schemes.

6.1.1 Dimensional regularization.

Like before we continue the boundary dimension d to $d \to d = 3 - 2\epsilon$, where ϵ is a complex number, whilst keeping the codimension fixed $D - d = 1$, and introduce renormalized fields and couplings

$$\psi_0 = Z_\psi^{1/2}\psi, \quad A_0 = Z_A^{1/2}A, \quad \alpha_0 = Z_\psi^{-1}Z_A^{-1/2}Z_\alpha\alpha. \tag{6.5}$$

In terms of which the action reads

$$I = \int \mathrm{d}^3x \left(\mathrm{i}\,\overline{\psi}\partial\!\!\!/\psi + \alpha\overline{\psi}A\!\!\!/\psi + \mathrm{i}\,\delta Z_\psi\overline{\psi}\partial\!\!\!/\psi + \delta Z_\alpha\mu^\epsilon\alpha\overline{\psi}A\!\!\!/\psi \right), \tag{6.6}$$

where we've defined the $\mathrm{O}(\alpha)$ quantities

$$\delta Z_A := Z_A - 1, \quad \delta Z_\psi := Z_\psi - 1, \quad \delta Z_\alpha := Z_\alpha - 1. \tag{6.7}$$

Finally we introduce an auxiliary parameter μ with dimensions off mass. They Feynman rules are recorded in Figure 17. Again we shall be interested in computing the anomalous dimension of the mass-squared operator. The bare mass-squared operator is

$$\mathscr{O}_0(x) := \overline{\psi}_0\psi_0(x) \tag{6.8}$$

The renormalized operator is

$$\mathscr{O}(x) := Z_{\overline{\psi}\psi}\overline{\psi}_0\psi_0(x) \equiv Z_{\overline{\psi}\psi}Z_\psi\overline{\psi}\psi(x). \tag{6.9}$$

We package the combination $Z_{\overline{\psi}\psi}Z_\psi$ into a single renormalization constant

$$Z_m := Z_{\overline{\psi}\psi}Z_\psi. \tag{6.10}$$

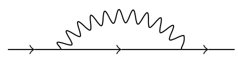
In terms of which the renormalized operator reads

$$\mathscr{O}(x) := Z_m\overline{\psi}\psi(x). \tag{6.11}$$

6.2 One-loop analysis.

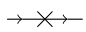
There are no marginal operators at the free fermion fixed point, so the combined bulk + boundary system defines a CFT, at least for weak gauge coupling. We start with the wavefunction renormalization of the fermion. Using the analogous notation as in the previous section (c.f. (5.46)), there is just one contribution (more details on our analysis of the Feynman diagrams can be found in Appendix C)

$$\Sigma_\psi(p|\alpha^2) = \tag{6.12}$$



along with the counterterm

$$\Sigma_{\phi,\mathrm{ct}}(p|\alpha^2) = \tag{6.13}$$



We find

$$\Sigma_\psi(p|\alpha^2) = \frac{\alpha^2 \not{p}}{4\pi^2}\left\{\left(\frac{2}{3}-\xi\right)\frac{1}{\epsilon}+\left(\frac{2}{3}-\xi\right)\log\frac{\overline{\mu}^2}{p^2}+\frac{14}{9}-2\xi\right\}. \qquad \text{(DR)} \tag{6.14}$$

Here we have introduced the rescaled parameter

$$\overline{\mu} := \sqrt{\pi\,\mathrm{e}^{-\gamma}}\mu, \tag{6.15}$$

where γ is the Euler-Mascheroni constant. Including counterterm $\Sigma_{\text{ct}}(p|\alpha^2) = -\not{p}\delta Z_\psi$ get that

$$\delta Z_\psi = \frac{\alpha^2}{4\pi^2}\left(\frac{2}{3} - \xi\right)\frac{1}{\epsilon}. \qquad \text{(DR)} \tag{6.16}$$

Using a momentum cutoff instead

$$\Sigma_\psi(p|\alpha^2) = \frac{\alpha^2 \not{p}}{4\pi^2}\left\{\left(\frac{2}{3} - \xi\right)\log\frac{\Lambda^2}{p^2} - \frac{17}{9} + \xi\right\}. \qquad \text{(Cutoff)} \tag{6.17}$$

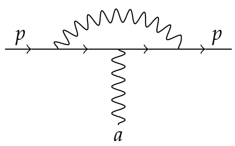
thus

$$\delta Z_\psi|\alpha^2) = \frac{\alpha^2}{4\pi^2}\left(\frac{2}{3} - \xi\right)\log\frac{\Lambda^2}{p^2}. \qquad \text{(Cutoff)} \tag{6.18}$$

Like before, there is no photon wavefunction renormalization in either scheme:

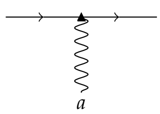
$$\delta Z_A = 0 + \mathrm{O}(\alpha^4). \tag{6.19}$$

There is just one diagram contributing to the 1PI 3-point vertex

$\Sigma^a_{3\text{pt}}|\alpha^3) =$ 

(6.20)

along with the counterterm

$\Sigma^a_{3\text{pt,ct}}|\alpha^3) =$ 

(6.21)

Since we only need the divergent bit, it suffices to consider the limit $p' = p$, for which we find

$$\Sigma^a_{3\text{pt}}(p) \sim \frac{\alpha^3}{4\pi^2}\left(-\frac{2}{3} + \xi\right)\frac{1}{\epsilon}\cdot\gamma^a. \qquad \text{(DR)} \tag{6.22}$$

Requiring the counterterm cancel the divergence determines

$$\delta Z_\alpha = \frac{\alpha^2}{4\pi^2}\left(\frac{2}{3} - \xi\right)\frac{1}{\epsilon}, \qquad \text{(DR)} \tag{6.23}$$

which is equal to δZ_ψ as required by charge non-renormalization. Using a cutoff instead

$$\Sigma^a_{3\text{pt}}(p) \sim \frac{\alpha^3}{4\pi^2}\left(-\frac{2}{3} + \xi\right)\gamma^a \log\frac{\Lambda^2}{p^2}, \qquad \text{(Cutoff)} \tag{6.24}$$

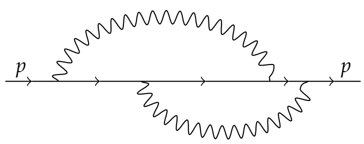
which determines

$$\delta Z_\alpha = \frac{\alpha^2}{12\pi^2}(2 - 3\xi)\log\frac{\Lambda^2}{p^2}. \qquad \text{(Cutoff)} \tag{6.25}$$

6.3 Two-loop fermion wavefunction renormalization

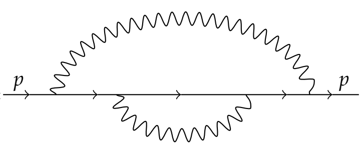
The graphs which contribute to the two-loop fermion self energy $\Sigma_\psi(p|\alpha^4)$ are

$$\Sigma^{(1)}_\psi(p|\alpha^4) =$$



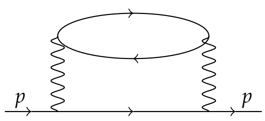
(6.26)

$$\Sigma^{(2)}_\psi(p|\alpha^4) =$$



(6.27)

$$\Sigma^{(3)}_\phi(p|\alpha^4) =$$



(6.28)

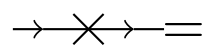
Working in the gauge $\xi = 1$ we find

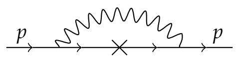
$$\Sigma_{\psi}^{(1)} \sim -\alpha^4 \left\{ \frac{13}{216\pi^4} \log \frac{\Lambda^2}{p^2} + \frac{1}{144\pi^4} \left(\log \frac{\Lambda^2}{p^2} \right)^2 \right\}, \tag{6.29}$$

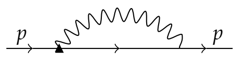
$$\Sigma_{\psi}^{(2)} \sim \alpha^4 \left\{ \frac{7}{216\pi^4} \log \frac{\Lambda^2}{p^2} + \frac{1}{288\pi^4} \left(\log \frac{\Lambda^2}{p^2} \right)^2 \right\}, \tag{6.30}$$

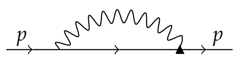
$$\Sigma_{\psi}^{(3)} \sim -\frac{\alpha^4}{96\pi^2} \log \frac{\Lambda^2}{p^2}. \tag{6.31}$$

We also have diagrams which include counterterms

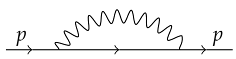
$\Sigma^{(1)}_{\psi,\mathrm{ct}}|\alpha^4) =$  $= -p^2 \delta Z_{\phi}|\alpha^4)$ (6.32)

$\Sigma^{(2)}_{\psi,\mathrm{ct}}|\alpha^4) =$  (6.33)

$\Sigma^{(3)}_{\psi,\mathrm{ct}}|\alpha^4) =$  (6.34)

$\Sigma^{(4)}_{\psi,\mathrm{ct}}|\alpha^4) =$  (6.35)

$\Sigma^{(3)}_{\psi,\mathrm{ct}}|\alpha^4)$ and $\Sigma^{(4)}_{\psi,\mathrm{ct}}|\alpha^4)$ are each equal and opposite to $\Sigma^{(2)}_{\psi,\mathrm{ct}}|\alpha^4)$, thus

$\Sigma^{(2)}_{\psi,\mathrm{ct}}(p|\alpha^4) + \Sigma^{(3)}_{\psi,\mathrm{ct}}(p|\alpha^4) + \Sigma^{(4)}_{\psi,\mathrm{ct}}(p|\alpha^4) = \delta Z_{\psi}|\alpha^2) \times$  (6.36)

Plugging in using (6.17) and (6.18) gives

$$\Sigma^{(2)}_{\phi,\mathrm{ct}}(p|\alpha^4) + \Sigma^{(3)}_{\phi,\mathrm{ct}}(p|\alpha^4) + \Sigma^{(4)}_{\phi,\mathrm{ct}}(p|\alpha^4) = \alpha^4 \left\{ \frac{1}{54\pi^4} \log \frac{\Lambda^2}{\mu^2} + \frac{1}{144\pi^4} \left(\log \frac{\Lambda^2}{\mu^2} \right)^2 \right\}. \tag{6.37}$$

Adding everything together

$$\delta Z_\psi|\alpha^4) = \alpha^4 \left\{ -\frac{8+9\pi^2}{864\pi^4} \log\frac{\Lambda^2}{\mu^2} + \frac{1}{288\pi^4}\left(\log\frac{\Lambda^2}{\mu^2}\right)^2 \right\}. \qquad \text{(Cutoff)} \qquad (6.38)$$

Compared with[34] [108]

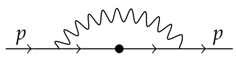
$$\delta Z_\psi|\alpha^4) = \alpha^4 \left\{ -\frac{16+9\pi^2}{1728\pi^4}\frac{1}{\epsilon} + \frac{1}{288\pi^4}\frac{1}{\epsilon^2} \right\} \qquad \text{(DR)} \qquad (6.39)$$

obtained in DR.

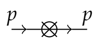
6.4 Mass-squared operator.

6.4.1 One-loop analysis.

There is just one graph contributing at one loop

$$\Sigma_{\overline{\psi}\psi}|\alpha^2) =$$  (6.40)

along with the counterterm

$$\Sigma_{\overline{\psi}\psi,\mathrm{ct}}|\alpha^2) =$$  (6.41)

We find

$$\Sigma_{\overline{\psi}\psi}(p|\alpha^2) = \frac{\alpha^2}{4\pi^2}\left\{ (2+\xi)\left(\frac{1}{\epsilon} + \log\frac{\overline{\mu}^2}{p^2}\right) + 2(3+2\xi) \right\}. \qquad \text{(DR)} \qquad (6.42)$$

Including the counterterm and requiring the result be finite determines

$$\delta Z_m = -\frac{(2+\xi)\alpha^2}{4\pi^2\epsilon}. \qquad \text{(DR)} \qquad (6.43)$$

[34] In fact the authors of [108] work with arbitrary ξ. We quote their result evaluated at $\xi = 1$.

Using a cutoff instead

$$\Sigma_{\overline{\psi}\psi}(p) = \frac{\alpha^2}{4\pi^2}\left\{4 + 2\xi + (2+\xi)\log\frac{\Lambda^2}{p^2}\right\}, \qquad \text{(Cutoff)} \qquad (6.44)$$

and thus

$$\delta Z_m = -\frac{(2+\xi)\alpha^2}{4\pi^2}\log\frac{\Lambda^2}{\mu^2}. \qquad \text{(Cutoff)} \qquad (6.45)$$

Using (5.107) this determines the renormalization constant $Z_{\overline{\psi}\psi}$ to order α^2. In dimensional regularization

$$\delta Z_{\overline{\psi}\psi} = -\frac{2\alpha^2}{3\pi^2\epsilon}. \qquad \text{(DR)} \qquad (6.46)$$

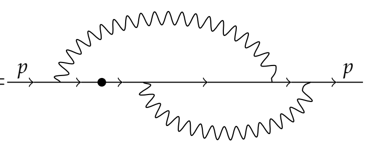
And with a momentum cutoff

$$\delta Z_{\overline{\psi}\psi} = -\frac{2\alpha^2}{3\pi^2}\log\frac{\Lambda^2}{\mu^2}. \qquad \text{(Cutoff)} \qquad (6.47)$$

6.4.2 Two-loop analysis.

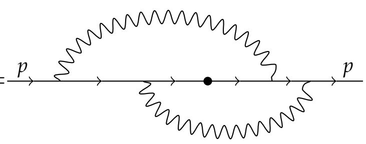
Like before the two loop graphs contributing to $\Sigma_{\overline{\psi}\psi}$ are given by summing over insertions of the mass operator on the internal lines of the self-energy graphs for ψ. Summing over insertions on the first graph (6.26)

$$\Sigma^{(1;1)}_{\overline{\psi}\psi}(p|\alpha^4) =$$

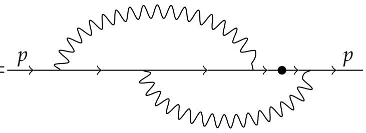


(6.48)

$$\Sigma^{(1;2)}_{\overline{\psi}\psi}(p|\alpha^4) =$$



(6.49)

$$\Sigma^{(1;3)}_{\overline{\psi}\psi}(p|\alpha^4) =$$


(6.50)

We find

$$\Sigma^{(1;1)}_{\overline{\psi}\psi}(p|\alpha^4) \sim \frac{\alpha^4}{\pi^4}\left\{\frac{1}{8}\log\frac{\Lambda^2}{p^2} + \frac{3}{64}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}, \tag{6.51}$$

$$\Sigma^{(1;2)}_{\overline{\psi}\psi}(p|\alpha^4) \sim -\frac{5\alpha^4}{512\pi^4}\log\frac{\Lambda^2}{p^2}, \tag{6.52}$$

and $\Sigma^{(1;3)}_{\overline{\psi}\psi}(p|\alpha^4) = \Sigma^{(1;1)}_{\overline{\psi}\psi}(p|\alpha^4)$. Summing over the three possible insertions on the second self-energy graph (6.27)

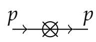
$$\Sigma^{(2;1)}_{\overline{\psi}\psi}(p|\alpha^4) \sim -\frac{\alpha^4}{\pi^4}\left\{\frac{7}{1536}\log\frac{\Lambda^2}{p^2} + \frac{1}{2048}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}, \tag{6.53}$$

$$\Sigma^{(2;2)}_{\overline{\psi}\psi}(p|\alpha^4) \sim \frac{\alpha^4}{\pi^4}\left\{\frac{9}{256}\log\frac{\Lambda^2}{p^2} + \frac{9}{2048}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}, \tag{6.54}$$

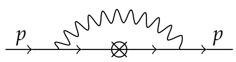
and $\Sigma^{(2;3)}_{\overline{\psi}\psi}(p|\alpha^4) = \Sigma^{(2;1)}_{\overline{\psi}\psi}(p|\alpha^4)$. Summing over insertions on the third self-energy graph (6.28) we find

$$\Sigma^{(3)}_{\overline{\psi}\psi}(p|\alpha^4) \sim -\frac{5\alpha^4}{32\pi^2}\log\frac{\Lambda^2}{p^2}. \tag{6.55}$$

Finally we also have graphs including counterterms

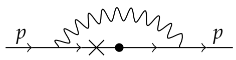
$$\Sigma^{(1)}_{\overline{\phi}\phi,\mathrm{ct}}(p|\alpha^4) =$$


$$= \delta Z_{m^2}|\alpha^4)$$
(6.56)

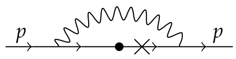
$$\Sigma^{(2)}_{\overline{\phi}\phi,\mathrm{ct}}(p|\alpha^4) =$$


(6.57)

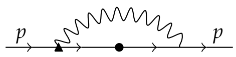
$$\Sigma^{(3)}_{\overline{\phi}\phi,\mathrm{ct}}(p|\alpha^4) = \tag{6.58}$$



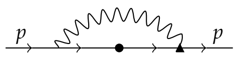
$$\Sigma^{(4)}_{\overline{\phi}\phi,\mathrm{ct}}(p|\alpha^4) = \tag{6.59}$$



$$\Sigma^{(5)}_{\overline{\phi}\phi,\mathrm{ct}}(p|\alpha^4) = \tag{6.60}$$



$$\Sigma^{(6)}_{\overline{\phi}\phi,\mathrm{ct}}(p|\alpha^4) = \tag{6.61}$$



The wavefunction renormalization insertions cancel against the vertex renormalizations, thus leaving

$$\begin{aligned}\Sigma^{(1)}_{\overline{\psi}\psi,\mathrm{ct}} + \Sigma^{(2)}_{\overline{\psi}\psi,\mathrm{ct}} &= \delta Z_m|\alpha^4) + \delta Z_m|\alpha^2) \cdot \Sigma_{\overline{\psi}\psi}|\alpha^2) \\ &\sim \delta Z_{\psi\overline{\psi}}|\alpha^4) - \alpha^4\left[\frac{980+9\pi^2}{864\pi^4}\log\frac{\Lambda^2}{\mu^2} + \frac{145}{288\pi^4}\left(\log\frac{\Lambda^2}{\mu^2}\right)^2\right].\end{aligned} \tag{6.62}$$

All together this determines the mass renormalization

$$\delta Z_{\psi\overline{\psi}}|\alpha^4) = \frac{\alpha^4}{\pi^4}\left[\frac{36\pi^2-32}{216}\log\frac{\Lambda^2}{\mu^2} + \frac{2}{9}\left(\log\frac{\Lambda^2}{\mu^2}\right)^2\right]. \qquad \text{(Cutoff)} \tag{6.63}$$

Compared to [108]

$$\delta Z_{\psi\overline{\psi}}|\alpha^4) = \frac{\alpha^4}{\pi^4}\left[\frac{9\pi^2-8}{108}\frac{1}{\epsilon} + \frac{2}{9}\frac{1}{\epsilon^2}\right], \qquad \text{(DR)} \tag{6.64}$$

obtained in dim-reg.

6.4.3 Anomalous dimension

Combining (6.46) and (6.64), our result for $\delta Z_{\overline{\psi}\psi}$ to order α^4 using dimensional regularization is

$$\delta Z_{\psi\overline{\psi}} = -\frac{2\alpha^2}{3\pi^2\epsilon} + \frac{\alpha^4}{\pi^4}\left[\frac{9\pi^2-8}{108}\frac{1}{\epsilon} + \frac{2}{9}\frac{1}{\epsilon^2}\right]. \qquad \text{(DR)} \qquad (6.65)$$

To compute the anomalous dimension $\gamma_{\overline{\psi}\psi}$ in dimensional regularization, we use the chain rule $\frac{d}{d\log\mu} = \frac{d\alpha}{d\log\mu}\frac{\partial}{\partial\alpha}$ and (5.29) to get the more useful equation

$$\gamma_{\overline{\psi}\psi} = -\epsilon\frac{\partial\log Z_{\overline{\psi}\psi}}{\partial\log\alpha}. \qquad (6.66)$$

From which we compute the anomalous dimension

$$\gamma_{\psi\overline{\psi}} = \frac{4\alpha^2}{3\pi^2} + \frac{(32-36\pi^2)\alpha^4}{108\pi^4}. \qquad \text{(DR)} \qquad (6.67)$$

Combining (6.47) and (6.63), our result for $\delta Z_{\overline{\psi}\psi}$ to order α^4 using a momentum cutoff is

$$\delta Z_{\overline{\psi}\psi} = -\frac{2\alpha^2}{3\pi^2}\log\frac{\Lambda^2}{\mu^2} + \frac{\alpha^4}{\pi^4}\left[\frac{36\pi^2-32}{216}\log\frac{\Lambda^2}{\mu^2} + \frac{2}{9}\left(\log\frac{\Lambda^2}{\mu^2}\right)^2\right]. \qquad \text{(Cutoff)} \qquad (6.68)$$

From which we compute the anomalous dimension

$$\gamma_{\overline{\psi}\psi} = \frac{4\alpha^2}{3\pi^2} + \frac{(32-36\pi^2)\alpha^4}{108\pi^4} + O(\alpha^6), \qquad \text{(Cutoff)} \qquad (6.69)$$

in precise agreement with the result (6.67) obtained using dimensional-regularization above.

7 Conclusion

In this work we have successfully overcome technical hurdles in considering a bulk U(1) gauge field interacting with a boundary scalar field allowing us to perform perturbative computations to two loop order. A regularization scheme of great practical use is dimensional regularization, but our current technology is not sufficient to deal with the divergences that occur in this theory. Our workaround in this thesis has been to use a momentum cutoff, which is harder to use in practise, but has the benefit that we can determine the two loop divergence structure of the theory allowing us to perform an RG analysis to this order.

As a result we have computed the anomalous dimension $\gamma_{\bar{\phi}\phi}$ of the mass squared operator $\bar{\phi}\phi$ to two loop order at weak coupling. By the conjectured S-duality of this setup, this determines the anomalous dimension at strong coupling. We have performed a "duality-improved" extrapolation to arbitrary values of the coupling τ. Near the point $\tau = +1$, we expect the theory to be dual to the critical Gross-Neveu model by 3d bosonization. Extrapolating our non-perturbative result to this region, we have made an estimate for the anomalous dimension γ_σ of the scalar field in this theory at the fixed point, often denoted η_σ. To the order we are working, we found that there are two consistent ressumations which imply

$$\eta_\sigma = \frac{19200}{41659 + 900\pi^2} = 0.379885, \tag{7.1}$$

or

$$\eta_\sigma = \frac{19200}{41659 + 7200\pi + 900\pi^2} = 0.262435. \tag{7.2}$$

Comparing with the known data we obtain Table 2, which is repeated below for convenience. As for future work, let us note that another way to study this theory is by using the powerful methods of the boundary conformal bootstrap, which we have have briefly mentioned a few times in this work. In fact a very general analysis was performed in [108]. They found the intriguing result that the current-current two point function (in our setup this is the current (5.2)) is related to the hemisphere partition function [239]. It would be a nice exercise to

Method	η_σ
F_1	0.327745
F_2	0.236449
ϵ-expansion [110]	0.2934
Bootstrap [111]	0.320
Monte-Carlo [112]	0.31 ± 0.01
fRG [113]	0.372

Table 3: This table displays the method and corresponding prediction for the critical exponent η_ϕ in the Gross-Neveu-Yukawa model.

compute this in perturbation theory, resum it, and extrapolate it to make a prediction for the hemisphere free energy in the Gross-Neveu model. Finally, purely from a practical point of view, it would be a nice exercise to computerize the perturbative approach using our momentum cutoff procedure.

A conventions

A.1 Geometry.

In Minkowski space we use the mostly minus convention $\eta_{\mu\nu} = (+1, -1, -1, \ldots, -1)$. When discussing forms we write components of an r-form $\omega \in \Omega^r(M)$ on an arbitrary manifold M as

$$\omega = \frac{1}{r!}\omega_{\mu_1\cdots\mu_r}\,\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}. \tag{A.1}$$

The exterior derivative is defined as

$$\mathrm{d}\omega := \frac{1}{r!}\partial_\nu\omega_{\mu_1\cdots\mu_r}\,\mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}. \tag{A.2}$$

If $\xi \in \Omega^q(M), \omega \in \Omega^r(M)$ then

$$\mathrm{d}(\xi \wedge \omega) = \mathrm{d}\xi \wedge \omega + (-1)^q \xi \wedge \mathrm{d}\omega. \tag{A.3}$$

The Hodge star is a map $\star : \Omega^r(M) \to \Omega^{d-r}(M)$ (where $d := \dim M$). Acting on the coordinate basis forms

$$\star(\mathrm{d}x^{\mu_1} \wedge \cdots \mathrm{d}x^{\mu_r}) := \frac{1}{(d-r)!}\epsilon^{\mu_1\cdots\mu_r}{}_{\nu_{r+1}\cdots\nu_d}\,\mathrm{d}x^{\nu_{r+1}} \wedge \cdots \wedge \mathrm{d}x^{\nu_d} \tag{A.4}$$

here $\epsilon^{\mu_1\cdots\mu_d}$ totally antisymmetric symbol with

$$\epsilon^{01\cdots(d-1)} := +1. \tag{A.5}$$

A.2 Minkowski space actions.

The Dirac algebra in Minkowski space reads $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, where we can take the gamma matrices to obey the hermicity property $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$. The QED Lagrangian reads

$$\mathscr{L}_{\text{QED}} := \overline{\psi}(\mathrm{i}\,\not{D} - m)\psi - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}, \tag{A.6}$$

where $\overline{\psi} := \psi^\dagger\gamma^0$ is the standard Dirac conjugate, $\not{D} := \gamma^\mu \mathrm{D}_\mu$ where the covariant derivative D_μ acts on the charge $+1$ field ψ and the charge -1 field $\overline{\psi}$ as

$$\mathrm{D}_\mu\psi(x) := (\partial_\mu + \mathrm{i}\,A_\mu(x))\psi(x), \quad \& \quad \mathrm{D}_\mu\overline{\psi} := (\partial_\mu - \mathrm{i}\,A_\mu(x))\overline{\psi}(x). \tag{A.7}$$

The Lagrangian for a massive complex scalar field in Minkowski space is

$$\mathscr{L}_{\text{SQED}} = \mathrm{D}_\mu\overline{\phi}(\mathrm{D}^\mu\phi) - m^2|\phi|^2 - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} \tag{A.8}$$

where

$$\mathrm{D}_\mu\phi(x) := (\partial_\mu + \mathrm{i}\,A_\mu(x))\phi(x), \quad \& \quad \mathrm{D}_\mu\overline{\phi} := (\partial_\mu - \mathrm{i}\,A_\mu(x))\overline{\phi}(x). \tag{A.9}$$

A.3 CPT in 2+1 dimensions

In three dimensions parity is really a reflection since the naive parity operation $\vec{x} \to -\vec{x}$ is continuously connected to the identity. Time-reversal T is defined in the standard way:

$$\mathsf{T}(t, x^1, x^2) := (-t, x^1, x^2), \quad \mathsf{P}(t, x^1, x^2) := (t, -x^1, x^2), \tag{A.10}$$

in particular it is antiunitary $\mathsf{T}\,\mathrm{i}\,\mathsf{T}^{-1} = -\,\mathrm{i}$. The action of P and T on the fields is defined so that the standard Maxwell action $\sim \int F_{\mu\nu}F^{\mu\nu}$ and Dirac action $\sim \int \overline{\psi}\,\mathrm{i}\,\not{\partial}\psi$ are invariant. We take parity to act as

$$\mathsf{P}A_1(t, x^1, x^2)\mathsf{P}^{-1} = -A_1(t, -x^1, x^2), \quad \mathsf{P}\psi(t, x^1, x^2)\mathsf{P}^{-1} = \mathrm{i}\,\gamma^2\psi(t, -x^1, x^2), \tag{A.11}$$

and time reversal to act as

$$\mathsf{T} A_i(t,\vec{x})\mathsf{T}^{-1} = -A_i(-t,\vec{x}), \quad \mathsf{T}\psi(t,\vec{x})\mathsf{T}^{-1} = -\gamma^1\psi(-t,\vec{x}). \tag{A.12}$$

Finally charge conjugation is defined so that it turns the electron Dirac equation $(\mathrm{i}\,\partial\!\!\!/ - A\!\!\!/ - m)\psi = 0$ into the positron Dirac equation $(\mathrm{i}\,\partial\!\!\!/ + A\!\!\!/ - m)\psi^{\mathsf{C}}$, where $\psi^{\mathsf{C}} \equiv \mathsf{C}\psi\mathsf{C}^{-1}$. Thus we take charge conjugation to act as

$$\mathsf{C}A_\mu\mathsf{C}^{-1} = -A_\mu, \quad \mathsf{C}\psi\mathsf{C}^{-1} = \mathrm{i}\,\gamma^2\psi^*. \tag{A.13}$$

What is more interesting is when we consider interacting theories, which can break these discrete symmetries. By the famous CPT theorem, any relativistic quantum field theory is invariant under CPT. With these definitions one can check a fermion mass term breaks parity

$$\mathsf{P}\overline{\psi}\psi\mathsf{P}^{-1} = -\overline{\psi}\psi. \tag{A.14}$$

Similarly the Chern-Simons term breaks parity

$$\mathsf{P}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho\mathsf{P}^{-1} = -\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho. \tag{A.15}$$

A.4 Euclidean continuation.

One way to motivate the Euclidean quantum field theory is by considering thermal correlators as follows [3, 4, 240]. When we study QFT in Minkowski signature we are often interested in time-ordered correlation functions

$$\langle\Psi'|\,\hat{\mathscr{O}}_1(t_1,\vec{x}_1)\cdots\hat{\mathscr{O}}_n(t_n,\vec{x}_n)\,|\Psi\rangle\,, \tag{A.16}$$

where $|\Psi\rangle\,, |\Psi'\rangle$ are some asymptotic states and $\hat{\mathscr{O}}(t,\vec{x})$ are Heisenberg-picture operators related to the Schrödinger-picture ones by

$$\hat{\mathscr{O}}(t,\vec{x}) := \mathrm{e}^{\mathrm{i}\,\hat{H}t}\,\hat{\mathscr{O}}(0,\vec{x})\,\mathrm{e}^{-\mathrm{i}\,\hat{H}t}\,. \tag{A.17}$$

Other useful observables include the correlation functions

$$\langle \Psi' | \, \hat{\mathscr{O}}_1(\tau_1, \vec{x}_1) \cdots \hat{\mathscr{O}}_n(\tau_n, \vec{x}_n) \, |\Psi\rangle \, , \tag{A.18}$$

where now the operators $\hat{\mathscr{O}}(\tau, \vec{x})$ have the time dependence

$$\hat{\mathscr{O}}(t, \vec{x}) := \mathrm{e}^{\hat{H}\tau} \, \hat{\mathscr{O}}(0, \vec{x}) \, \mathrm{e}^{-\hat{H}\tau} \, . \tag{A.19}$$

Repeating the Minkowski space derivation of the path integral where the time evolution into infinitesimal intervals, we find that these correlators can be represented as a path integral weighted by the Boltzmann factor[35] $\mathrm{e}^{-\widehat{S}}$:

$$\langle \mathrm{vac} | \, \hat{\mathscr{O}}_1(\tau_1, \vec{x}_1) \cdots \hat{\mathscr{O}}_n(\tau_n, \vec{x}_n) \, |\mathrm{vac}\rangle = \int \mathrm{D}\Phi \, \mathscr{O}_1(\tau_1, \vec{x}_1) \cdots \mathscr{O}_n(\tau_n, \vec{x}_n) \, \mathrm{e}^{-\widehat{S}} \, . \tag{A.20}$$

$\widehat{S}$ is called the Euclidean action and there is a simple prescription to obtain it from the corresponding Minkowski space action.

A.4.1 Matter theories.

The recipe for the Euclidean continuation of these theories is straightforward [3, 4, 240]. First we define Wick rotated variables y^μ

$$x^0 = -\,\mathrm{i}\, y^0, \tag{A.21}$$

and $y^i = x^i$. Then the Euclidean Lagrangian $\widehat{\mathscr{L}}$ is equal to minus $\mathscr{L}$, where we replace derivatives $\frac{\partial}{\partial t}$ with respect to t by derives with respect to τ by the chain rule $\frac{\partial}{\partial t} = \mathrm{i} \frac{\partial}{\partial \tau}$; schematically

$$\widehat{\mathscr{L}} = -\mathscr{L}(\partial_t = \mathrm{i}\, \partial_\tau). \tag{A.22}$$

[35] only in this section will we denote Euclidean variables with a hat.

Using this rule, Euclidean continuation of the free scalar is

$$\widehat{\mathscr{L}} = \left(\frac{\partial\phi}{\partial\tau}\right)^2 + \sum_i \left(\frac{\partial\phi}{\partial y^i}\right)^2 + m^2|\phi|^2. \tag{A.23}$$

Similarly for the free fermion

$$\widehat{\mathscr{L}} = \overline{\psi}\left(\gamma^0 \frac{\partial}{\partial\tau} - \mathrm{i}\,\gamma^i \frac{\partial}{\partial y^i} + m\right)\psi. \tag{A.24}$$

It is conventional and convenient define Euclidean gamma matrices

$$\widehat{\gamma}^0 := \gamma^0, \quad \widehat{\gamma}^i := -\mathrm{i}\,\gamma^i, \quad \{\widehat{\gamma}^\mu, \widehat{\gamma}^\nu\} = 2\delta^{\mu\nu}, \tag{A.25}$$

so that the Euclidean Dirac Lagrangian reads

$$\widehat{\mathscr{L}_{\mathrm{f}}} = \mathrm{i}\,\overline{\psi}(\widehat{\gamma}^a \partial_a + m)\psi. \tag{A.26}$$

Here the overall factor of i comes from changing variables $\psi \to \mathrm{i}\,\psi$. Note that in the path integral formulation ψ and $\overline{\psi}$ are to be treated as independent variables.

A.4.2 Gauge theories

There is one more ingredient for the continuation of gauge fields. Under the Wick rotation (A.21) the vector potential $A = A_\mu \,\mathrm{d}x^\mu = A_0 \,\mathrm{d}x^0 + A_i \,\mathrm{d}x^i$ transforms as

$$A = A_0 \,\mathrm{d}x^0 + A_i \,\mathrm{d}x^i \longrightarrow A_t \mathrm{d}(-\mathrm{i}\,y^0) + A_i \,\mathrm{d}x^i = -\mathrm{i}\,A_0 \,\mathrm{d}y^0 + A_i \,\mathrm{d}x^i. \tag{A.27}$$

Thus define Euclidean vector potential $\widehat{A} = \widehat{A}_\mu \,\mathrm{d}y^\mu$ with components $\widehat{A}_\mu$ given by

$$\widehat{A}_0 := -\mathrm{i}\,A_0, \quad \widehat{A}_i := A_i. \tag{A.28}$$

Applying the recipe (A.22) and writing the Maxwell action in terms of the Euclidean components (A.28)

$$\widehat{\mathscr{L}} = \frac{1}{4e^2}\widehat{F}_{\mu\nu}\widehat{F}^{\mu\nu} + \frac{\mathrm{i}\,\theta}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\widehat{F}_{\mu\nu}\widehat{F}_{\rho\sigma}. \tag{A.29}$$

Note that the epsilon symbol $\epsilon^{\mu_1\cdots\mu_D}$ defined (in either signature) so that

$$\epsilon^{01\cdots d} := 1. \tag{A.30}$$

One can combine this with the continuation rules in the previous section to derive the more general matter + radiation Lagrangians. The $\mathrm{U}(1)$ Chern-Simons action continues to

$$\widehat{\mathscr{L}} = -\frac{\mathrm{i}}{4\pi}\epsilon^{\mu\nu\rho}\widehat{A}_\mu\frac{\partial}{\partial y^\nu}\widehat{A}_\rho. \tag{A.31}$$

Similarly the non-Abelian $\mathrm{U}(N)$ Chern-Simons theory continues to

$$\widehat{\mathscr{L}} = -\frac{\mathrm{i}}{4\pi}\,\mathrm{Tr}\left(\widehat{A}\wedge \mathrm{d}\widehat{A} + \frac{2}{3}\widehat{A}\wedge\widehat{A}\wedge\widehat{A}\right). \tag{A.32}$$

A.5 Gamma matrix technology.

The Dirac algebra in Euclidean space reads

$$\{\gamma^a,\gamma^b\} = 2\delta^{ab}. \tag{A.33}$$

We record the useful algebraic identities

$$\gamma^a \not{a}\gamma^a = -\not{a}, \tag{A.34}$$

$$\gamma^a \not{a}\not{b}\gamma^a = \not{b}\not{a} + 2(a\cdot b), \tag{A.35}$$

$$\gamma^a \not{a}\not{b}\not{c}\gamma^a = -\not{a}\not{b}\not{c} - 2\not{b}\not{a}\not{c} + 2\not{c}\not{a}\not{b}, \tag{A.36}$$

and trace identities

$$\mathrm{Tr}\, \not{a} = 0, \tag{A.37}$$

$$\mathrm{Tr}(\not{a}\not{b}) = 2(a \cdot b), \tag{A.38}$$

$$\mathrm{Tr}(\not{a}\not{b}\not{c}) = 2\,\mathrm{i}(a \times b) \cdot c, \tag{A.39}$$

$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 2(a \cdot b)(c \cdot d) - 2(a \cdot c)(b \cdot d) + 2(a \cdot d)(b \cdot c). \tag{A.40}$$

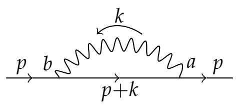
Here "$\times$" refers to the cross product $(a \times b)_\rho = a^\mu b^\nu \epsilon_{\mu\nu\rho}$ so that, in components, (A.39) reads

$$\mathrm{Tr}(\not{a}\not{b}\not{c}) = 2\,\mathrm{i}\, a^\mu b^\nu c^\rho \epsilon_{\mu\nu\rho}. \tag{A.41}$$

B Feynman diagram analysis of the scalar BCFT

B.1 One-loop analysis

To provide familiarity with our regularization schemes, we shall provide the details for computing the one-loop scalar self-energy graph (5.47) , which we repeat here for convenience

$$\Sigma_\phi^{(1)}(p|\alpha^2) =$$ 

(B.1)

Dimensional-regularization.

Applying the Feynman rules

$$\Sigma_\phi^{(1)}(p|\alpha^2) = (\mu^\epsilon\alpha)^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(2p+k)_a \Pi_{ab}(k)(2p+k)_b}{(p+k)^2}. \tag{B.2}$$

Plugging in for the photon propagator, and replacing scalar products $p \cdot k$ with propagators using

$$2p \cdot k = (p+k)^2 - p^2 - k^2, \tag{B.3}$$

the graph can be written in terms of the family of integrals

$$I(n,m) := \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{((p+k)^2)^n (k^2)^m}. \tag{B.4}$$

These are discussed in more detail in Section D.1.1 of the Appendix. In terms of these integrals, our diagram reads

$$\begin{aligned}\Sigma_\phi^{(1)}(p|\alpha^2) = (\mu^\epsilon\alpha)^2 \bigg\{ & 2p^2 I(1,1/2) + 2I(0,1/2) - I(1,-1/2) \\ & - (1-\xi)\Big[I(-1,3/2) - 2p^2 I(0,3/2) + p^4 I(1,3/2)\Big]\bigg\}.\end{aligned} \tag{B.5}$$

$I(n,m)$ can be expressed as a rational product $G(n,m)$ of Gamma functions (c.f. equations (D.4) and (D.5))

$$\Sigma_{\phi}^{(1)}(p|\alpha^2) = \frac{\alpha^2 p^2}{(4\pi)^{3/2}} \left(\frac{4\pi\mu^2}{p^2}\right)^{\epsilon} \left\{ 2G(1,1/2) - G(1,-1/2) - (1-\xi)G(1,3/2) \right\}. \tag{B.6}$$

Performing expansion around $\epsilon = 0$

$$\Sigma_{\phi}^{(1)}(p|\alpha^2) = \frac{\alpha^2 p^2}{4\pi^2} \left\{ \left(\frac{8}{3} - \xi\right)\frac{1}{\epsilon} + \left(\frac{8}{3} - \xi\right)\frac{5}{3}\log\frac{\pi\, \mathrm{e}^{-\gamma}\mu^2}{p^2} + \frac{56}{9} \right\}, \tag{B.7}$$

where γ is the Euler-Mascheroni constant. Requiring that the $\frac{1}{\epsilon}$ pole cancel against the counterterm $-p^2\delta Z_{\phi}$ determines

$$\delta Z_{\phi} = \frac{\alpha^2}{4\pi^2}\left(\frac{8}{3} - \xi\right)\frac{1}{\epsilon} + \mathrm{O}(\alpha^4). \tag{B.8}$$

Momentum cutoff.

Writing out the diagram (B.1) using the Feynman rules in our cutoff regulated theory and following analogous steps that led to (B.5), we get

$$\begin{aligned}\Sigma_{\phi}^{(1)}(p|\alpha^2) = \alpha^2 \Big\{ & 2p^2 I_{(\Lambda)}(1,1/2) + 2I_{(\Lambda)}(0,1/2) - I_{(\Lambda)}(1,-1/2) \\ & - (1-\xi)\left[I_{(\Lambda)}(-1,3/2) - 2p^2 I_{(\Lambda)}(0,3/2) + p^4 I_{(\Lambda)}(1,3/2) \right] \Big\},\end{aligned} \tag{B.9}$$

where

$$I_{(\Lambda)}(n,m) := \int\limits_{|k|<\Lambda} \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{((p+k)^2)^n (k^2)^m}. \tag{B.10}$$

The relevant integrals are recorded in equations (D.7)-(D.12). Some of them have infra-red divergences coming from regions of small loop momenta k, but using $\log\frac{p^2}{\delta^2} - \log\frac{\Lambda^2}{\delta^2} = -\log\frac{\Lambda^2}{p^2}$ these divergences cancel leaving

$$\Sigma_\phi^{(1)}(p|\alpha^2) = \frac{\alpha^2}{(2\pi)^2}\left\{\xi\Lambda^2 + p^2\left(\frac{8}{3} - \xi\right)\log\frac{\Lambda^2}{p^2} + \left(\frac{16}{9} - 2\xi\right)p^2\right\}. \tag{B.11}$$

B.2 Two-loop scalar self-energy.

B.2.1 First diagram.

Applying the Feynman rules to the graph in (5.61)

$$\begin{aligned}\Sigma_\phi^{(1)}(p|\alpha^4) &= \alpha^4 \int \frac{\mathrm{d}^3k\mathrm{d}^3q}{(2\pi)^6}\frac{[2(p+q)+k]_a\Pi_{(ab)}(k)(2p+k)_b(2p+q)_c\Pi_{(cd)}(q)[2(p+k)+q]_d}{(p+k)^2(p+k+q)^2(p+q)^2} \\ &\equiv \Sigma_\phi^{(1),1}(p|\alpha^4) - (1-\xi)\Sigma_\phi^{(1),2}(p|\alpha^4) + (1-\xi)^2\Sigma_\phi^{(1),3}(p|\alpha^4),\end{aligned} \tag{B.12}$$

where the round brackets mean the symmetric part. This just means that we drop the part of the photon propagator (5.21) proportional to ϑ, since this term is higher order in e (c.f. (5.10)). The notation "$(p|\alpha^4)$" was introduced in (5.46). Expanding out the photon propagator, we get a term independent of ξ, a term proportional to $(1-\xi)$, and a term proportional to $(1-\xi)^2$, whose coefficients we have denoted by $\Sigma_\phi^{(1),1}, \Sigma_\phi^{(1),2}$, and $\Sigma_\phi^{(1),3}$ respectively. Explicitly

$$\Sigma_\phi^{(1),1}(p|\alpha^4) = \alpha^4 \int \frac{\mathrm{d}^3k\mathrm{d}^3q}{(2\pi)^6}\frac{N_{1,1}}{|k||q|(p+k)^2(p+k+q)^2(p+q)^2}, \tag{B.13}$$

where

$$N_{1,1} = \left\{3p^2 + 2p\cdot k + (p+k)^2 + 4p\cdot q + 2k\cdot q\right\}\cdot\left\{3p^2 + 2pq + (p+q)^2 + 4p\cdot k + 2k\cdot q\right\}. \tag{B.14}$$

Using the reduction formula (B.3), as well as

$$2p\cdot q = (p+q)^2 - p^2 - q^2, \quad 2k\cdot q = (p+k+q)^2 - (p+k)^2 - (p+q)^2 + p^2, \tag{B.15}$$

the numerator $N_{1,1}$ reduces to

$$\begin{aligned} N_{1,1} &= 2k^4 - 3p^2k^2 + p^4 + 5k^2q^2 - 3p^2q^2 + 2q^4 + (p+k)^4 + (p+q)^4 + (p+k+q)^4 \\ &\quad + (p+q)^2(2p^2 - 3k^2 - 3q^2) + 2(p+k)^2(p+q)^2 + (p+k)^2(2p^2 - 3k^2 - 3q^2) \\ &\quad + 2(p+k+q)^2\Big\{(p+k)^2 + (p+q)^2\Big\} + (p+k+q)^2(2p^2 - 3k^2 - 3q^2). \end{aligned} \tag{B.16}$$

Plugging this back in

$$\begin{aligned} \Sigma_\phi^{(1),1}(p|\alpha^4) = \alpha^4 \Bigg\{ & 2\left[\frac{3,-1}{2,2,2}\right] - 3p^2\left[\frac{1,-1}{2,2,2}\right] + p^4\left[\frac{-1,-1}{2,2,2}\right] + 5\left[\frac{1,1}{2,2,2}\right] - 3p^2\left[\frac{-1,1}{2,2,2}\right] \\ & + 2\left[\frac{-1,3}{2,2,2}\right] + \left[\frac{-1,-1}{-2,2,2}\right] + \left[\frac{-1,-1}{2,2,-2}\right] + \left[\frac{-1,-1}{2,-2,2}\right] + 2p^2\left[\frac{-1,-1}{2,2,0}\right] - 3\left[\frac{1,-1}{2,2,0}\right] \\ & - 3\left[\frac{-1,1}{2,2,0}\right] + 2\left[\frac{-1,-1}{0,2,0}\right] + 2p^2\left[\frac{-1,-1}{0,2,2}\right] - 3\left[\frac{1,-1}{0,2,2}\right] - 3\left[\frac{-1,1}{0,2,2}\right] + 2\left[\frac{-1,-1}{0,0,2}\right] \\ & + 2\left[\frac{-1,-1}{2,0,0}\right] + 2p^2\left[\frac{-1,-1}{2,0,2}\right] - 3\left[\frac{1,-1}{2,0,2}\right] - 3\left[\frac{-1,1}{2,0,2}\right] \Bigg\}, \end{aligned} \tag{B.17}$$

where we have defined

$$\left[\frac{a,b}{c,d,e}\right] := \int \frac{\mathrm{d}^3k\mathrm{d}^3q}{(2\pi)^6} \frac{|k|^a|q|^b}{(p+k)^c(p+k+q)^d(p+q)^e}. \tag{B.18}$$

This has the obvious invariance

$$(a \leftrightarrow b, c \leftrightarrow e). \tag{B.19}$$

Using this invariance, and the fact that that $\left[\frac{-1,-1}{2,2,2}\right]$ is finite, the divergent part is

$$\begin{aligned} \Sigma_{1,1}(p|\alpha^4) \sim \alpha^4 \Bigg\{ & 4\left[\frac{-1,3}{2,2,2}\right] - 6p^2\left[\frac{-1,1}{2,2,2}\right] + 5\left[\frac{1,1}{2,2,2}\right] + 2\left[\frac{-1,-1}{2,2,-2}\right] + \left[\frac{-1,-1}{2,-2,2}\right] \\ & + 4p^2\left[\frac{-1,-1}{2,2,0}\right] - 6\left[\frac{1,-1}{2,2,0}\right] - 6\left[\frac{-1,1}{2,2,0}\right] + 2\left[\frac{-1,-1}{0,2,0}\right] - 6\left[\frac{1,-1}{2,0,2}\right] \\ & + 2p^2\left[\frac{-1,-1}{2,0,2}\right] \Bigg\}. \end{aligned} \tag{B.20}$$

The logarithmic divergences of all the necessary integrals are recorded in Section D.3. We find

$$\Sigma_{1,1}(p|\alpha^4) \sim \frac{\alpha^4 p^2}{\pi^4}\left\{\frac{977}{48}\log\frac{\Lambda^2}{p^2}+\frac{9}{4}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}. \tag{B.21}$$

We have only computed the diagrams in the gauge $\xi = 1$, which corresponds to dropping $\Sigma_{\phi}^{(1),2}(p|\alpha^4)$ and $\Sigma_{\phi}^{(1),3}(p|\alpha^4)$ in (B.12). A more thorough analysis would do this in an arbitrary gauge, and since the anomalous dimension of $\overline{\phi}\phi$ is a physical quantity, a good check would be that the gauge dependence drops out. However this adds to the zoo of integrals that we would have to compute. All of them are easily do-able using our methods. We report for completeness

$$\begin{aligned}\Sigma_{\phi}^{(1),2}(p|\alpha^4) = 2\alpha^4\Bigg\{&\left[\frac{-1,-3}{2,-2,0}\right]-p^2\left[\frac{-1,-3}{2,-2,2}\right]-\left[\frac{-1,-3}{-2,2,0}\right]+p^2\left[\frac{-1,-3}{-2,2,2}\right]+\left[\frac{-1,-3}{2,0,-2}\right]\\&-\left[\frac{1,-3}{2,0,0}\right]-2\left[\frac{-1,-1}{2,0,0}\right]+p^2\left[\frac{1,-3}{2,0,2}\right]-p^4\left[\frac{-1,-3}{2,0,2}\right]+2p^2\left[\frac{-1,-1}{2,0,2}\right]\\&-\left[\frac{-1,-3}{0,2,-2}\right]+\left[\frac{1,-3}{0,2,0}\right]+2\left[\frac{-1,-1}{0,2,0}\right]+p^4\left[\frac{-1,-3}{0,2,2}\right]-p^2\left[\frac{1,-3}{0,2,2}\right]\\&-2p^2\left[\frac{-1,-1}{0,2,2}\right]\Bigg\},\end{aligned} \tag{B.22}$$

and

$$\begin{aligned}\Sigma_{\phi}^{(1),3}(p|\alpha^4) = \alpha^4\Bigg\{&\left[\frac{-3,-3}{-2,2,-2}\right]-p^2\left[\frac{-3,-3}{-2,2,0}\right]+p^4\left[\frac{-3,-3}{0,2,0}\right]-p^2\left[\frac{-3,-3}{0,2,-2}\right]\\&+p^4\left[\frac{-3,-3}{2,-2,2}\right]-p^2\left[\frac{-3,-3}{2,-2,0}\right]+\left[\frac{-3,-3}{0,-2,0}\right]-p^2\left[\frac{-3,-3}{0,-2,2}\right]+p^2\left[\frac{-3,-3}{2,0,-2}\right]\\&-p^4\left[\frac{-3,-3}{2,0,0}\right]-\left[\frac{-3,-3}{-2,0,0}\right]+p^2\left[\frac{-3,-3}{-2,0,2}\right]-\left[\frac{-3,-3}{0,0,-2}\right]+2p^2\left[\frac{-3,-3}{0,0,0}\right]\\&-p^4\left[\frac{-3,-3}{0,0,2}\right]\Bigg\}.\end{aligned} \tag{B.23}$$

B.2.2 Second diagram

Applying the Feynman rules to the graph in (5.62)

$$\Sigma_\phi^{(2)}(p|\alpha^4) = \alpha^4 \int \frac{\mathrm{d}^3k\mathrm{d}^3q}{(2\pi)^6} \frac{(2p+k)_a\Pi_{(ab)}(k)(2p+k)_b(2(p+k)+q)_c\Pi_{(cd)}(q)(2(p+k)+q)_d}{(p+k)^4(p+k+q)^2}$$
$$\equiv \Sigma_\phi^{(2),1}(p|\alpha^4) - (1-\xi)\Sigma_\phi^{(2),2}(p|\alpha^4) + (1-\xi)^2\Sigma_\phi^{(2),3}(p|\alpha^4). \tag{B.24}$$

After reducing

$$\Sigma_\phi^{(2),1}(p|\alpha^4) = \alpha^4\Bigg\{4\begin{bmatrix}-1,-1\\ \hline 0,2,0\end{bmatrix} + 4\begin{bmatrix}-1,-1\\ \hline 2,0,0\end{bmatrix} - 2\begin{bmatrix}1,-1\\ \hline 4,0,0\end{bmatrix} + 4p^2\begin{bmatrix}-1,-1\\ \hline 4,0,0\end{bmatrix} + \begin{bmatrix}1,1\\ \hline 4,2,0\end{bmatrix}$$
$$- 2p^2\begin{bmatrix}-1,1\\ \hline 4,2,0\end{bmatrix} + 4p^2\begin{bmatrix}-1,-1\\ \hline 2,2,0\end{bmatrix} - 2\begin{bmatrix}1,-1\\ \hline 2,2,0\end{bmatrix} - 2\begin{bmatrix}-1,1\\ \hline 2,2,0\end{bmatrix}\Bigg\}. \tag{B.25}$$

Using the formulae in Section D.3

$$\Sigma_\phi^{(2),1}(p|\alpha^4) \sim \frac{\alpha^4p^2}{\pi^4}\left\{\frac{265}{432}\log\frac{\Lambda^2}{p^2} + \frac{25}{288}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}. \tag{B.26}$$

For completeness we include

$$\Sigma_\phi^{(2),2}(p|\alpha^4) = \alpha^4\Bigg\{2\begin{bmatrix}-1,-3\\ \hline -2,2,0\end{bmatrix} + 2p^2\begin{bmatrix}-1,-3\\ \hline 0,2,0\end{bmatrix} - \begin{bmatrix}1,-3\\ \hline 0,2,0\end{bmatrix} + 2\begin{bmatrix}-1,-3\\ \hline 2,-2,0\end{bmatrix} - \begin{bmatrix}1,-3\\ \hline 4,-2,0\end{bmatrix}$$
$$+ 2p^2\begin{bmatrix}-1,-3\\ \hline 4,-2,0\end{bmatrix} - 4\begin{bmatrix}-1,-3\\ \hline 0,0,0\end{bmatrix} + 2\begin{bmatrix}1,-3\\ \hline 2,0,0\end{bmatrix} - 4p^2\begin{bmatrix}-1,-3\\ \hline 2,0,0\end{bmatrix} + 2\begin{bmatrix}-3,-1\\ \hline -2,2,0\end{bmatrix}$$
$$+ 2\begin{bmatrix}-3,-1\\ \hline 0,0,0\end{bmatrix} - 4p^2\begin{bmatrix}-3,-1\\ \hline 2,0,0\end{bmatrix} + 2p^4\begin{bmatrix}-3,-1\\ \hline 4,0,0\end{bmatrix} - p^4\begin{bmatrix}-3,1\\ \hline 4,2,0\end{bmatrix} - 4p^2\begin{bmatrix}-3,-1\\ \hline 0,2,0\end{bmatrix}$$
$$+ +2p^2\begin{bmatrix}-3,1\\ \hline 2,2,0\end{bmatrix} - \begin{bmatrix}-3,1\\ \hline 0,2,0\end{bmatrix} + 2p^4\begin{bmatrix}-3,-1\\ \hline 2,2,0\end{bmatrix}\Bigg\}, \tag{B.27}$$

and

$$\Sigma_\phi^{(2),3}(p|\alpha^4) = \alpha^4 \Bigg\{ \left[\frac{-3,-3}{-4,2,0}\right] - 2p^2\left[\frac{-3,-3}{-2,2,0}\right] + p^4\left[\frac{-3,-3}{0,2,0}\right] + \left[\frac{-3,-3}{0,-2,0}\right] - 2p^2\left[\frac{-3,-3}{2,-2,0}\right] + p^4\left[\frac{-3,-3}{4,-2,0}\right] - 2\left[\frac{-3,-3}{-2,0,0}\right] + 4p^2\left[\frac{-3,-3}{0,0,0}\right] - 2p^4\left[\frac{-3,-3}{2,0,0}\right] \Bigg\}. \tag{B.28}$$

However, we have not computed the logarithmic divergences of all the integrals occurring in (B.27) and (B.28).

B.2.3 Third and fourth self-energy diagrams

Applying the Feynman rules to the graph in (5.63)

$$\begin{aligned} \Sigma_\phi^{(3)}(p|\alpha^4) &= \alpha^4 \int \frac{d^3k d^3q}{(2\pi)^6} \frac{(-2\delta_{ac})\Pi_{(cd)}(q)(2(p+k)+q)_d \Pi_{(ab)}(k)(2p+k)_b}{(p+k)^2(p+k+q)^2} \\ &\equiv \Sigma_\phi^{(3),1}(p|\alpha^4) - (1-\xi)\Sigma_\phi^{(3),2}(p|\alpha^4) + (1-\xi)^2\Sigma_\phi^{(3),3}(p|\alpha^4). \end{aligned}$$

After reducing

$$\Sigma_\phi^{(3),1}(p|\alpha^4) = \alpha^4 \Bigg\{ 2\left[\frac{1,-1}{2,2,0}\right] + 2\left[\frac{-1,1}{2,2,0}\right] - 5\left[\frac{-1,-1}{0,2,0}\right] - \left[\frac{-1,-1}{2,2,-2}\right] - \left[\frac{-1,-1}{2,0,0}\right] - p^2\left[\frac{-1,-1}{2,2,0}\right] \Bigg\}. \tag{B.29}$$

Using the formulae in Section D.3

$$\Sigma_\phi^{(3),1}(p|\alpha^4) \sim \frac{\alpha^4 p^2}{\pi^4} \Bigg\{ -\frac{487}{48}\log\frac{\Lambda^2}{p^2} - \frac{323}{288}\left(\log\frac{\Lambda^2}{p^2}\right)^2 \Bigg\}. \tag{B.30}$$

For completeness, the gauge dependent parts are

$$\begin{aligned}\Sigma_\phi^{(3),2}(p|\alpha^4) = -\alpha^4 \Bigg\{ &\left[\frac{-1,-3}{-2,2,0}\right] + \left[\frac{-1,-3}{2,-2,0}\right] + \left[\frac{-1,-3}{2,0,-2}\right] - p^2\left[\frac{-1,-3}{2,0,0}\right] - 2\left[\frac{-1,-3}{0,0,0}\right] \\ &- 2\left[\frac{-1,-1}{2,0,0}\right] + p^2\left[\frac{-1,-3}{0,2,0}\right] + 2\left[\frac{-1,-1}{0,2,0}\right] - \left[\frac{-1,-3}{0,2,-2}\right] + \left[\frac{-3,-1}{-2,2,0}\right] \\ &+ p^2\left[\frac{-3,-1}{2,2,-2}\right] - 2p^2\left[\frac{-1,-1}{2,2,0}\right] + p^4\left[\frac{-3,-1}{2,2,0}\right] + 2\left[\frac{-1,-1}{0,2,0}\right] - 2p^2\left[\frac{-3,-1}{0,2,0}\right] \\ &- \left[\frac{-3,-1}{0,2,-2}\right] + \left[\frac{-3,-1}{0,0,0}\right] - p^2\left[\frac{-3,-1}{2,0,0}\right] \Bigg\}, \end{aligned} \tag{B.31}$$

and

$$\begin{aligned}\Sigma_\phi^{(3),2}(p|\alpha^4) = -\alpha^4 \Bigg\{ &\left[\frac{-3,-3}{-4,2,0}\right] + \left[\frac{-3,-3}{-2,2,-2}\right] - 2p^2\left[\frac{-3,-3}{-2,2,0}\right] + \left[\frac{-3,-3}{0,-2,0}\right] - p^2\left[\frac{-3,-3}{2,-2,0}\right] \\ &+ p^4\left[\frac{-3,-3}{0,2,0}\right] - p^2\left[\frac{-3,-3}{0,2,-2}\right] + p^2\left[\frac{-3,-3}{2,0,-2}\right] - 2\left[\frac{-3,-3}{-2,0,0}\right] - p^4\left[\frac{-3,-3}{2,0,0}\right] \\ &+ 3p^2\left[\frac{-3,-3}{0,0,0}\right] - \left[\frac{-3,-3}{0,0,-2}\right] \Bigg\}. \end{aligned} \tag{B.32}$$

B.2.4 Fifth self-energy diagram

Applying the Feynman rules to the graph in (5.65)

$$\begin{aligned}\Sigma_\phi^{(5)}(p|\alpha^4) &= \frac{\alpha^4}{2}\int \frac{\mathrm{d}^3k\mathrm{d}^3q}{(2\pi)^6}\frac{4\delta_{ac}\delta_{bd}\Pi_{(ab)}(k)\Pi_{(cd)}(q)}{(p+k+q)^2} \\ &\equiv \Sigma^{(5),1}(p|\alpha^4) - (1-\xi)\Sigma^{(5),2}(p|\alpha^4) + (1-\xi)^2\Sigma^{(5),3}(p|\alpha^4), \end{aligned} \tag{B.33}$$

where the symmetry factor of 2 is from interchanging internal photon lines. The gauge independent bit is simply

$$\Sigma_\phi^{(5),1}(p|\alpha^4) = 6\alpha^4\left[\frac{-1,-1}{0,2,0}\right]. \tag{B.34}$$

Plucking out the relevant integral from Section D.3 we thus have

$$\Sigma_{\phi}^{(5),1}(p|\alpha^4) \sim -\frac{\alpha^4 p^2}{8\pi^4}\log\frac{\Lambda^2}{p^2}. \tag{B.35}$$

B.2.5 Sixth self-energy diagram

Applying the Feynman rules to the graph in (5.66)

$$\begin{aligned}\Sigma_{\phi}^{(6)}(p|\alpha^4) &= \alpha^4\int\frac{\mathrm{d}^3k\mathrm{d}^3q}{(2\pi)^6}\,\frac{(2q+k)_a\Pi_{(ab)}(k)(2p+k)_b(2p+k)_c\Pi_{(cd)}(k)(2q+k)_d}{(p+k)^2(k+q)^2q^2}\\ &\equiv \Sigma_{\phi}^{(6),1}(p|\alpha^4)-(1-\xi)\Sigma_{\phi}^{(6),2}(p|\alpha^4)+(1-\xi)^2\Sigma_{\phi}^{(6),3}(p|\alpha^4).\end{aligned}$$

We have

$$\Sigma_{\phi}^{(6),1}(p|\alpha^4) = \alpha^4\int\frac{\mathrm{d}^3k\mathrm{d}^3q}{(2\pi)^6}\,\frac{(4p\cdot q+2p\cdot k+2k\cdot q+k^2)^2}{k^2(p+k)^2(k+q)^2q^2} \tag{B.36}$$

Since the denominator structure here is different to the cases above, we now use the reduction formulae

$$2p\cdot k = (p+k)^2-p^2-k^2,\quad 2k\cdot q = (k+q)^2-k^2-q^2. \tag{B.37}$$

Reducing the numerator in this way leads to

$$\begin{aligned}\Sigma_{\phi}^{(6),1}(p|\alpha^4) = \alpha^4\Bigg\{&\left[\frac{2,-2}{2,2}\right]+2p^2\left[\frac{0,-2}{2,2}\right]+p^4\left[\frac{-2,-2}{2,2}\right]-8|p|\left[\frac{0,-1}{2,2}\middle|\cos\theta_{pq}\right]\\ &-8|p|^3\left[\frac{-2,-1}{2,2}\middle|\cos\theta_{pq}\right]+2\left[\frac{0,0}{2,2}\right]+2p^2\left[\frac{-2,0}{2,2}\right]+16p^2\left[\frac{-2,0}{2,2}\middle|\cos^2\theta_{pq}\right]\\ &-8|p|\left[\frac{-2,1}{2,2}\middle|\cos\theta_{pq}\right]+\left[\frac{-2,2}{2,2}\right]+\left[\frac{-2,-2}{-2,2}\right]+\left[\frac{-2,-2}{2,-2}\right]\\ &+8|p|\left[\frac{-2,-1}{0,2}\middle|\cos\theta_{pq}\right]-2p^2\left[\frac{-2,-2}{0,2}\right]-2\left[\frac{0,-2}{0,2}\right]-2\left[\frac{-2,0}{0,2}\right]\\ &+2\left[\frac{-2,-2}{0,0}\right]+8|p|\left[\frac{-2,-1}{2,0}\middle|\cos\theta_{pq}\right]-2p^2\left[\frac{-2,-2}{2,0}\right]-2\left[\frac{0,-2}{2,0}\right]-2\left[\frac{-2,0}{2,0}\right]\Bigg\},\end{aligned} \tag{B.38}$$

where we have defined

$$\left[\frac{a,b}{c,d}\right] := \int \frac{d^3k d^3q}{(2\pi)^6} \frac{|k|^a |q|^b}{(p+k)^c (k+q)^d}, \tag{B.39}$$

and

$$\left[\frac{a,b}{c,d}\middle| \cos\theta_{pq}\right] := \int \frac{d^3k d^3q}{(2\pi)^6} \frac{|k|^a |q|^b \cos\theta_{pq}}{(p+k)^c (k+q)^d}. \tag{B.40}$$

We immediately see that the terms on the bottom line do not contribute to the wavefunction renormalization - i.e. they have no logarithmic divergences. Also $\left[\frac{-2,-2}{2,2}\right]$ is finite, so can be dropped too. A more detailed analysis of the integrals recorded in Section D.4 yields

$$\Sigma_\phi^{(6),1}(p|\alpha^4) \sim -\frac{\alpha^4 p^2}{24\pi^2} \log \frac{\Lambda^2}{p^2}. \tag{B.41}$$

B.2.6 Seventh diagram.

In this subsection we shall extract the divergent part of $\Sigma_\phi^{(7)}$. It is easiest to isolate the divergence in position space and then Fourier transform back to momentum space. Similar methods were used in [241] and [242] for example. In position space the graph reads

$$\Gamma_\phi^{(7)}(x|\alpha^4) = \frac{\lambda_*^2}{12} \int d^3w d^3z\, \Delta(x-w)\Delta(w-z)^5 \Delta(z), \tag{B.42}$$

where the factor of 12 in the denominator is symmetry factor. The logarithmic divergence comes from w, z near 0

$$\begin{aligned} \Gamma_\phi^{(7)}(x|\alpha^4) &\sim \frac{\lambda_*^2}{12}\Delta(x) \int d^3w d^3z\, \Delta(w-z)^5 \Delta(z) \\ &= \frac{\lambda_*^2}{12\cdot(2\pi)^7} \frac{1}{|x|} \int d^3w d^3z \frac{1}{|w-z|^5} \frac{1}{|z|}. \end{aligned} \tag{B.43}$$

In going to the second line we have plugged in for the position-space propagator given in (5.6). The remaining integral can now be done in closed form. After a bit of work, we find the divergent part to be

$$\Gamma_{\phi}^{(7)}(x|\alpha^4) \sim \frac{\lambda_*^2}{1728\pi^5}\frac{1}{|x|}\log(|x|\Lambda). \tag{B.44}$$

Using (E.2) we Fourier transform to momentum space

$$\begin{aligned}\Gamma_{\phi}^{(7)}(p|\alpha^4) &= \int \mathrm{d}^3x\ \mathrm{e}^{-\mathrm{i}\,px}\,\Gamma_{\phi}^{(7)}(x|\alpha^4)\\ &\sim \frac{1}{p^2}\frac{\lambda_*^2}{864\pi^4}\log\frac{\Lambda^2}{p^2}.\end{aligned} \tag{B.45}$$

Chopping off the external propagators, we obtain the contribution to the self-energy

$$\Sigma_{\phi}^{(7)}(p|\alpha^4) \sim \frac{p^2\lambda_*^2}{864\pi^4}\log\frac{\Lambda^2}{p^2}. \tag{B.46}$$

Plugging in for λ_* in terms of α using (5.55)

$$\Sigma_{\phi}^{(7)}(p|\alpha^4) \sim \frac{8\alpha^4 p^2}{147\pi^4}\log\frac{\Lambda^2}{p^2}. \tag{B.47}$$

B.3 Mass-squared operator

B.3.1 First diagram

In this section we isolate the logarithmic divergences of the graphs enumerated in Section 5.6.2. Applying the Feynman rules to the graph in (5.89) we get

$$\begin{aligned}\Sigma_{\bar\phi\phi}^{(1;1)}(p|\alpha^4) &= \alpha^4\int\frac{\mathrm{d}^3k\mathrm{d}^3q}{(2\pi)^6}\,\frac{[2(p+q)+k]_a\Pi_{(ab)}(k)(2p+k)_b(2p+q)_c\Pi_{(cd)}(q)[2(p+k)+q]_d}{(p+k)^4(p+k+q)^2(p+q)^2}\\ &\equiv \Sigma_{\phi}^{(1;1),1}(p|\alpha^4) - (1-\xi)\Sigma_{\phi}^{(1;1),2}(p|\alpha^4) + (1-\xi)^2\Sigma_{\phi}^{(1;1),3}(p|\alpha^4),\end{aligned} \tag{B.48}$$

where the notation $\Sigma^{(1;1),i}$, $i = 1,2,3$, is analogous to that in Section B.2. Expanding out the numerator and reducing

$$\begin{aligned}\Sigma_{\phi}^{(1;1),1}(p|\alpha^4) = \alpha^4 \Bigg\{ & 2\left[\frac{3,-1}{4,2,2}\right] - 3p^2\left[\frac{1,-1}{4,2,2}\right] + p^4\left[\frac{-1,-1}{4,2,2}\right] + 5\left[\frac{1,1}{4,2,2}\right] - 3p^2\left[\frac{-1,1}{4,2,2}\right] \\ & + 2\left[\frac{-1,3}{4,2,2}\right] + \left[\frac{-1,-1}{0,2,2}\right] + \left[\frac{-1,-1}{4,2,-2}\right] + \left[\frac{-1,-1}{4,-2,2}\right] + 2p^2\left[\frac{-1,-1}{4,2,0}\right] \\ & - 3\left[\frac{1,-1}{4,2,0}\right] - 3\left[\frac{-1,1}{4,2,0}\right] + 2\left[\frac{-1,-1}{2,2,0}\right] + 2p^2\left[\frac{-1,-1}{2,2,2}\right] - 3\left[\frac{1,-1}{2,2,2}\right] \\ & - 3\left[\frac{-1,1}{2,2,2}\right] + 2\left[\frac{-1,-1}{2,0,2}\right] + 2\left[\frac{-1,-1}{4,0,0}\right] + 2p^2\left[\frac{-1,-1}{4,0,2}\right] - 3\left[\frac{1,-1}{4,0,2}\right] \\ & - 3\left[\frac{-1,1}{4,0,2}\right] \Bigg\}. \end{aligned} \tag{B.49}$$

Using the formulae in Section D.3

$$\Sigma_{\phi}^{(1;1),1}(p|\alpha^4) \sim \frac{\alpha^4}{\pi^4}\left\{\frac{19}{48}\log\frac{\Lambda^2}{p^2} + \frac{143}{4512}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\} + \frac{\alpha^4}{32\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}. \tag{B.50}$$

Here $\varepsilon > 0$ is an IR regulator, coming from collinear loop momentum. Applying the Feynman rules to the graph with the mass operator inserted in the middle

$$\begin{aligned}\Sigma_{\phi}^{(1;2)}(p|\alpha^4) &= \alpha^4 \int \frac{d^3k d^3q}{(2\pi)^6} \frac{[2(p+q)+k]_a \Pi_{(ab)}(k)(2p+k)_b(2p+q)_c\Pi_{(cd)}(q)[2(p+k)+q]_d}{(p+k)^2(p+k+q)^4(p+q)^2} \\ &\equiv \Sigma_{\phi}^{(1;2),1}(p|\alpha^4) - (1-\xi)\Sigma_{\phi}^{(1;2),2}(p|\alpha^4) + (1-\xi)^2\Sigma_{\phi}^{(1;2),3}(p|\alpha^4),\end{aligned} \tag{B.51}$$

where

$$\begin{aligned}\Sigma_{\phi}^{(1;2),1}(p|\alpha^4) = \alpha^4 \Bigg\{ & 4\left[\frac{-1,3}{2,4,2}\right] - 6p^2\left[\frac{-1,1}{2,4,2}\right] + 5\left[\frac{1,1}{2,4,2}\right] + 2\left[\frac{-1,-1}{2,4,-2}\right] + \left[\frac{-1,-1}{2,0,2}\right] \\ & + 4p^2\left[\frac{-1,-1}{2,4,0}\right] - 6\left[\frac{1,-1}{2,4,0}\right] - 6\left[\frac{-1,1}{2,4,0}\right] + 2\left[\frac{-1,-1}{0,4,0}\right] - 6\left[\frac{1,-1}{2,2,2}\right] \\ & + 2p^2\left[\frac{-1,-1}{2,2,2}\right] \Bigg\}. \end{aligned} \tag{B.52}$$

Using the formulae in Section D.3

$$\Sigma_{\phi}^{(1;2),1}(p|\alpha^4) \sim \frac{\alpha^4}{\pi^4}\left\{-\frac{127}{96}\log\frac{\Lambda^2}{p^2} - \frac{1}{8}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}$$
$$-\frac{1}{32\pi^4}\log\frac{2}{\varepsilon}\log\frac{\Lambda^2}{p^2} + \frac{13}{192\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}. \tag{B.53}$$

B.3.2 Second diagram.

Applying the Feynman rules

$$\Sigma_{\bar{\phi}\phi}^{(2;1)} = \alpha^4\int\frac{\mathrm{d}^3k\mathrm{d}^3q}{(2\pi)^6}\frac{(2p+k)_a\Pi_{(ab)}(k)(2p+k)_b(2(p+k)+q)_c\Pi_{(cd)}(q)(2(p+k)+q)_d}{(p+k)^6(p+k+q)^2}$$
$$\equiv \Sigma_{\bar{\phi}\phi}^{(2;1),1} - (1-\xi)\Sigma_{\bar{\phi}\phi}^{(2;1),2} + (1-\xi)^2\Sigma_{\bar{\phi}\phi}^{(2;1),3}, \tag{B.54}$$

where

$$\Sigma_{\bar{\phi}\phi}^{(2;1),1} = \alpha^4\Bigg\{4\left[\frac{-1,-1}{2,2,0}\right] + 4\left[\frac{-1,-1}{4,0,0}\right] - 2\left[\frac{1,-1}{6,0,0}\right] + 4p^2\left[\frac{-1,-1}{6,0,0}\right] + \left[\frac{1,1}{6,2,0}\right]$$
$$-2p^2\left[\frac{-1,1}{6,2,0}\right] + 4p^2\left[\frac{-1,-1}{4,2,0}\right] - 2\left[\frac{1,-1}{4,2,0}\right] - 2\left[\frac{-1,1}{4,2,0}\right]\Bigg\}. \tag{B.55}$$

Using the formulae in Section D.3

$$\Sigma_{\bar{\phi}\phi}^{(2;1),1} \sim \frac{\alpha^4}{\pi^4}\left\{\frac{37}{72}\log\frac{\Lambda^2}{p^2} + \frac{1}{24}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}$$
$$+\frac{\alpha^4}{\pi^3}\left(\frac{9}{128}\frac{1}{\varepsilon} - \frac{1}{192}\frac{1}{\varepsilon^3}\right)\log\frac{\Lambda^2}{p^2}. \tag{B.56}$$

Applying the Feynman rules to the next graph

$$\Sigma_{\bar{\phi}\phi}^{(2;2)} = \alpha^4\int\frac{\mathrm{d}^3k\mathrm{d}^3q}{(2\pi)^6}\frac{(2p+k)_a\Pi_{(ab)}(k)(2p+k)_b(2(p+k)+q)_c\Pi_{(cd)}(q)(2(p+k)+q)_d}{(p+k)^4(p+k+q)^4}$$
$$\equiv \Sigma_{\bar{\phi}\phi}^{(2;2),1} - (1-\xi)\Sigma_{\bar{\phi}\phi}^{(2;2),2} + (1-\xi)^2\Sigma_{\bar{\phi}\phi}^{(2;2),3}, \tag{B.57}$$

where

$$\Sigma^{(2;2),1}_{\bar{\phi}\phi} = \alpha^4 \Bigg\{ 4\left[\frac{-1,-1}{0,4,0}\right] + 4\left[\frac{-1,-1}{2,2,0}\right] - 2\left[\frac{1,-1}{4,2,0}\right] + 4p^2\left[\frac{-1,-1}{4,2,0}\right] + \left[\frac{1,1}{4,4,0}\right] - 2p^2\left[\frac{-1,1}{4,4,0}\right] + 4p^2\left[\frac{-1,-1}{2,4,0}\right] - 2\left[\frac{1,-1}{2,4,0}\right] - 2\left[\frac{-1,1}{2,4,0}\right] \Bigg\}. \tag{B.58}$$

Using the formulae in Section D.3

$$\Sigma^{(2;2),1}_{\bar{\phi}\phi} \sim \frac{\alpha^4}{\pi^4}\left\{ -\frac{19}{48}\log\frac{\Lambda^2}{p^2} - \frac{1}{16}\left(\log\frac{\Lambda^2}{p^2}\right)^2 \right\} + \left(-\frac{1}{16\pi^4}\log\frac{2}{\varepsilon} + \frac{13}{96\pi^3}\frac{1}{\varepsilon} \right)\log\frac{\Lambda^2}{p^2}. \tag{B.59}$$

B.3.3 Third and fourth diagrams

Applying the Feynman rules

$$\begin{aligned} \Sigma^{(3;1)}_{\bar{\phi}\phi}(p|\alpha^4) &= \alpha^4 \int \frac{d^3k d^3q}{(2\pi)^6} \frac{(-2\delta_{ac})\Pi_{(cd)}(q)(2(p+k)+q)_d \Pi_{(ab)}(k)(2p+k)_b}{(p+k)^4(p+k+q)^2} \\ &\equiv \Sigma^{(3;1),1}_{\bar{\phi}\phi}(p|\alpha^4) - (1-\xi)\Sigma^{(3;1),2}_{\bar{\phi}\phi} + (1-\xi)^2\Sigma^{(3;1),3}_{\bar{\phi}\phi}, \end{aligned}$$

where

$$\Sigma^{(3;1),1}_{\bar{\phi}\phi}(p|\alpha^4) = -\alpha^4 \Bigg\{ 5\left[\frac{-1,-1}{2,2,0}\right] + \left[\frac{-1,-1}{4,2,-2}\right] + \left[\frac{-1,-1}{4,0,0}\right] + p^2\left[\frac{-1,-1}{4,2,0}\right] - 2\left[\frac{1,-1}{4,2,0}\right] - 2\left[\frac{-1,1}{4,2,0}\right] \Bigg\}. \tag{B.60}$$

Using the formulae in Section D.3

$$\Sigma^{(3;1),1}_{\bar{\phi}\phi}(p|\alpha^4) \sim \frac{\alpha^4}{\pi^4}\left\{ -\frac{31}{36}\log\frac{\Lambda^2}{p^2} - \frac{1}{12}\left(\log\frac{\Lambda^2}{p^2}\right)^2 \right\}. \tag{B.61}$$

Similarly

$$\Sigma^{(3;2)}_{\bar{\phi}\phi}(p|\alpha^4) = \alpha^4 \int \frac{d^3k d^3q}{(2\pi)^6} \frac{(-2\delta_{ac})\Pi_{(cd)}(q)(2(p+k)+q)_d \Pi_{(ab)}(k)(2p+k)_b}{(p+k)^2(p+k+q)^2}$$
$$\equiv \Sigma^{(3;2),1}_{\bar{\phi}\phi}(p|\alpha^4) - (1-\xi)\Sigma^{(3;2),2}_{\bar{\phi}\phi}(p|\alpha^4) + (1-\xi)^2\Sigma^{(3;2),3}_{\bar{\phi}\phi}(p|\alpha^4). \tag{B.62}$$

where

$$\Sigma^{(3;2),1}_{\bar{\phi}\phi}(p|\alpha^4) = \alpha^4 \Bigg\{ 2\left[\frac{1,-1}{2,4,0}\right] + 2\left[\frac{-1,1}{2,4,0}\right] - 5\left[\frac{-1,-1}{0,4,0}\right] - \left[\frac{-1,-1}{2,4,-2}\right]$$
$$- \left[\frac{-1,-1}{2,2,0}\right] - p^2\left[\frac{-1,-1}{2,4,0}\right] \Bigg\}. \tag{B.63}$$

Using the formulae in Section D.3

$$\Sigma^{(3;2),1}_{\bar{\phi}\phi}(p|\alpha^4) \sim \alpha^4 \left(-\frac{1}{\pi^4}\frac{7}{192} + \frac{1}{16\pi^4}\log\frac{2}{\varepsilon} - \frac{41}{384\pi^3}\frac{1}{\varepsilon} \right) \log\frac{\Lambda^2}{p^2}. \tag{B.64}$$

B.3.4 Fifth diagram.

For the fifth graph

$$\Sigma^{(5)}_{\bar{\phi}\phi}(p|\alpha^4) = \frac{\alpha^4}{2} \int \frac{d^3k d^3q}{(2\pi)^6} \frac{4\delta_{ac}\delta_{bd}\Pi_{(ab)}(k)\Pi_{(cd)}(q)}{(p+k+q)^4}$$
$$\equiv \Sigma^{(5),1}_{\bar{\phi}\phi}(p|\alpha^4) - (1-\xi)\Sigma^{(5),2}_{\bar{\phi}\phi}(p|\alpha^4) + (1-\xi)^2\Sigma^{(5),3}_{\bar{\phi}\phi}(p|\alpha^4). \tag{B.65}$$

Including symmetry factor of 2 and using the formulae in Section D.3

$$\Sigma^{(5),1}_{\bar{\phi}\phi}(p|\alpha^4) = 6\alpha^4\left[\frac{-1,-1}{0,4,0}\right] \sim' \alpha^4 \left(-\frac{1}{32\pi^4} + \frac{1}{64\pi^3}\frac{1}{\varepsilon} \right) \log\frac{\Lambda^2}{p^2}. \tag{B.66}$$

B.3.5 Sixth diagram.

For the first insertion

$$\Sigma^{(6;1)}_{\overline{\phi}\phi}(p|\alpha^4) = \alpha^4 \int \frac{\mathrm{d}^3k\mathrm{d}^3q}{(2\pi)^6} \frac{(2q+k)_a\Pi_{(ab)}(k)(2p+k)_b(2p+k)_c\Pi_{(cd)}(k)(2q+k)_d}{(p+k)^2(k+q)^4q^2}$$
$$\equiv \Sigma^{(6;1),1}_{\overline{\phi}\phi}(p|\alpha^4) - (1-\xi)\Sigma^{(6;1),2}_{\overline{\phi}\phi}(p|\alpha^4) + (1-\xi)^2\Sigma^{(6;1),3}_{\overline{\phi}\phi}(p|\alpha^4), \tag{B.67}$$

where

$$\begin{aligned}\Sigma^{(6;1),1}_{\overline{\phi}\phi}(p|\alpha^4) = \alpha^4 \Bigg\{ & \left[\frac{2,-2}{2,4}\right] + 2p^2\left[\frac{0,-2}{2,4}\right] + p^4\left[\frac{-2,-2}{2,4}\right] - 8|p|\left[\frac{0,-1}{2,4}\middle|\cos\theta_{pq}\right] \\ & + 2\left[\frac{0,0}{2,4}\right] + 2p^2\left[\frac{-2,0}{2,4}\right] + 16p^2\left[\frac{-2,0}{2,4}\middle|\cos^2\theta_{pq}\right] - 8|p|\left[\frac{-2,1}{2,4}\middle|\cos\theta_{pq}\right] \\ & + \left[\frac{-2,-2}{-2,4}\right] + \left[\frac{-2,-2}{2,0}\right] + 8|p|\left[\frac{-2,-1}{0,4}\middle|\cos\theta_{pq}\right] - 2p^2\left[\frac{-2,-2}{0,4}\right] \\ & + 2\left[\frac{-2,-2}{0,2}\right] + 8|p|\left[\frac{-2,-1}{2,2}\middle|\cos\theta_{pq}\right] - 2p^2\left[\frac{-2,-2}{2,2}\right] - 2\left[\frac{0,-2}{2,2}\right] \\ & - 8|p|^3\left[\frac{-2,-1}{2,4}\middle|\cos\theta_{pq}\right] + \left[\frac{-2,2}{2,4}\right] - 2\left[\frac{0,-2}{0,4}\right] - 2\left[\frac{-2,0}{0,4}\right] - 2\left[\frac{-2,0}{2,2}\right]\Bigg\}.\end{aligned} \tag{B.68}$$

Using the formulae in Section D.4, there is a non-trivial cancellation in divergences

$$\Sigma^{(6;1),1}_{\overline{\phi}\phi}(p|\alpha^4) \sim 0. \tag{B.69}$$

Next

$$\Sigma^{(6;2)}_{\overline{\phi}\phi}(p|\alpha^4) = \alpha^4 \int \frac{\mathrm{d}^3k\mathrm{d}^3q}{(2\pi)^6} \frac{(2q+k)_a\Pi_{(ab)}(k)(2p+k)_b(2p+k)_c\Pi_{(cd)}(k)(2q+k)_d}{(p+k)^2(k+q)^2q^4}$$
$$\equiv \Sigma^{(6;2),1}_{\overline{\phi}\phi}(p|\alpha^4) - (1-\xi)\Sigma^{(6;2),2}_{\overline{\phi}\phi}(p|\alpha^4) + (1-\xi)^2\Sigma^{(6;2),3}_{\overline{\phi}\phi}(p|\alpha^4), \tag{B.70}$$

where

$$\begin{aligned}\Sigma^{(6;2),1}_{\bar{\phi}\phi}(p|\alpha^4) = \alpha^4 \Bigg\{ & \left[\frac{2,-4}{2,2}\right] + 2p^2\left[\frac{0,-4}{2,2}\right] + p^4\left[\frac{-2,-4}{2,2}\right] - 8|p|\left[\frac{0,-3}{2,2}\middle| \cos\theta_{pq}\right] \\ & + 2\left[\frac{0,-2}{2,2}\right] + 2p^2\left[\frac{-2,-2}{2,2}\right] + 16p^2\left[\frac{-2,-2}{2,2}\middle|\cos^2\theta_{pq}\right] - 8|p|\left[\frac{-2,-1}{2,2}\middle|\cos\theta_{pq}\right] \\ & + \left[\frac{-2,-4}{-2,2}\right] + \left[\frac{-2,-4}{2,-2}\right] + 8|p|\left[\frac{-2,-3}{0,2}\middle|\cos\theta_{pq}\right] - 2p^2\left[\frac{-2,-4}{0,2}\right] - 2\left[\frac{0,-4}{0,2}\right] \\ & + 2\left[\frac{-2,-4}{0,0}\right] + 8|p|\left[\frac{-2,-3}{2,0}\middle|\cos\theta_{pq}\right] - 2p^2\left[\frac{-2,-4}{2,0}\right] - 2\left[\frac{0,-4}{2,0}\right] \\ & - 8|p|^3\left[\frac{-2,-3}{2,2}\middle|\cos\theta_{pq}\right] + \left[\frac{-2,0}{2,2}\right] - 2\left[\frac{-2,-2}{0,2}\right] - 2\left[\frac{-2,-2}{2,0}\right] \Bigg\}. \end{aligned} \tag{B.71}$$

Using the formulae in Section D.4, there is again a non-trivial cancellation in divergences

$$\Sigma^{(6;2),1}_{\bar{\phi}\phi}(p|\alpha^4) \sim 0. \tag{B.72}$$

Finally

$$\begin{aligned}\Sigma^{(6;3)}_{\bar{\phi}\phi}(p|\alpha^4) &= \alpha^4 \int \frac{d^3k d^3q}{(2\pi)^6} \frac{(2q+k)_a \Pi_{(ab)}(k)(2p+k)_b(2p+k)_c\Pi_{(cd)}(k)(2q+k)_d}{(p+k)^4(k+q)^2q^2} \\ &\equiv \Sigma^{(6;3),1}_{\bar{\phi}\phi}(p|\alpha^4) - (1-\xi)\Sigma^{(6;3),2}_{\bar{\phi}\phi}(p|\alpha^4) + (1-\xi)^2\Sigma^{(6;3),3}_{\bar{\phi}\phi}(p|\alpha^4). \end{aligned} \tag{B.73}$$

where

$$\begin{aligned}\Sigma^{(6;3),1}_{\bar{\phi}\phi}(p|\alpha^4) = \alpha^4 \Bigg\{ & \left[\frac{2,-2}{4,2}\right] + 2p^2\left[\frac{0,-2}{4,2}\right] + p^4\left[\frac{-2,-2}{4,2}\right] - 8|p|\left[\frac{0,-1}{4,2}\middle| \cos\theta_{pq}\right] \\ & + 2\left[\frac{0,0}{4,2}\right] + 2p^2\left[\frac{-2,0}{4,2}\right] + 16p^2\left[\frac{-2,0}{4,2}\middle|\cos^2\theta_{pq}\right] - 8|p|\left[\frac{-2,1}{4,2}\middle|\cos\theta_{pq}\right] \\ & + \left[\frac{-2,-2}{0,2}\right] + \left[\frac{-2,-2}{4,-2}\right] + 8|p|\left[\frac{-2,-1}{2,2}\middle|\cos\theta_{pq}\right] - 2p^2\left[\frac{-2,-2}{2,2}\right] \\ & + 2\left[\frac{-2,-2}{2,0}\right] + 8|p|\left[\frac{-2,-1}{4,0}\middle|\cos\theta_{pq}\right] - 2p^2\left[\frac{-2,-2}{4,0}\right] - 2\left[\frac{0,-2}{4,0}\right] - 2\left[\frac{-2,0}{4,0}\right] \\ & - 8|p|^3\left[\frac{-2,-1}{4,2}\middle|\cos\theta_{pq}\right] + \left[\frac{-2,2}{4,2}\right] - 2\left[\frac{0,-2}{2,2}\right] - 2\left[\frac{-2,0}{2,2}\right] \Bigg\}. \end{aligned} \tag{B.74}$$

Again there is a non-trivial cancellation

$$\Sigma^{(6;3),1}_{\bar{\phi}\phi}(p|\alpha^4) \sim 0. \tag{B.75}$$

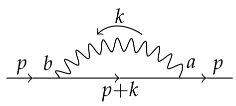
C Feynman diagram analysis of the fermionic BCFT

C.1 One-loop analysis

C.1.1 Fermion self-ernergy.

To get used to the slight difference in analyzing the fermionic diagrams, we shall provide the details for computing the one-loop fermion self-energy graph

$$\Sigma_{\psi}^{(1)}(p|\alpha) = \tag{C.1}$$



Dimensional regularization.

Applying Feynman rules

$$\Sigma_{\psi}^{(1)}(p|\alpha) = (\mu^{\epsilon}\alpha)^2 \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\gamma^a(\not{p}+\not{k})\gamma^b}{(p+k)^2} \cdot \frac{1}{|k|}\left(\delta_{ab} - (1-\xi)\frac{k_a k_b}{k^2}\right). \tag{C.2}$$

On general grounds we can write the fermion self-energy as [222, 243]

$$\Sigma_{\psi}(p) = \not{p}\sigma(p) \tag{C.3}$$

where $\sigma(p)$ is a c-number. It is then straightforward to solve for σ as

$$\sigma(p) = \frac{1}{2p^2}\,\mathrm{Tr}(\not{p}\Sigma_{\psi}). \tag{C.4}$$

In particular, this holds order-by-order in α^2, so

$$\Sigma_{\psi}^{(1)}(p|\alpha^2) = \not{p}\sigma^{(1)}(p|\alpha^2). \tag{C.5}$$

Using the gamma matrix identities in Section A.5 and reducing the numerator, we get

$$\begin{aligned}\sigma^{(1)}(p|\alpha^2) = \frac{(\mu^\epsilon \alpha)^2}{2p^2}\Bigg\{&(-1+2\epsilon)\Big(p^2 I(1,1/2) + I(0,1/2) - I(1,-1/2)\Big)\\ &-(1-\xi)\Big(I(-1,3/2) - 2p^2 I(0,3/2) + p^4 I(1,3/2) - I(0,1/2) - p^2 I(1,1/2)\Big)\Bigg\},\end{aligned} \tag{C.6}$$

where the class of integrals

$$I(n,m) := \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{((p+k)^2)^n (k^2)^m}, \tag{C.7}$$

were originally defined in Section (B.4). $I(n,m)$ can be expressed as a rational product $G(n,m)$ of Gamma functions (c.f. equations (D.4) and (D.5))

$$\begin{aligned}\sigma^{(1)}(p|\alpha^2) = \frac{\alpha^2}{16\pi^{3/2}}\left(\frac{4\pi\mu^2}{p^2}\right)^\epsilon \Bigg\{&(-1+2\epsilon)\Big(G(1,1/2) - G(1,-1/2)\Big)\\ &-(1-\xi)\Big(G(1,3/2) - G(1,1/2)\Big)\Bigg\}.\end{aligned} \tag{C.8}$$

Performing an ϵ-expansion

$$\sigma^{(1)}(p|\alpha^2) = \frac{\alpha^2}{4\pi^2}\left\{\left(\frac{2}{3}-\xi\right)\frac{1}{\epsilon} + \left(\frac{2}{3}-\xi\right)\log\frac{\overline{\mu}^2}{p^2} + \frac{14}{9} - 2\xi\right\}, \tag{C.9}$$

where $\overline{\mu} := \sqrt{\pi}\,\mathrm{e}^{-\gamma}\mu$ was defined in (6.15). Including the counterterm $\Sigma_{\mathrm{ct}}(p|\alpha^2) = -\not{p}\,\delta Z_\psi$ we thus get

$$\delta Z_\psi = \frac{\overline{\alpha}^2}{4\pi^2}\left(\frac{2}{3}-\xi\right)\frac{1}{\epsilon}. \tag{C.10}$$

Momentum cutoff.

Repeating the steps which led to (C.6), now using out cutoff regularization scheme, we are left with

$$\sigma^{(1)}(p|\alpha^2) = \frac{\alpha^2}{2p^2}\Bigg\{ - p^2 I_{(\Lambda)}(1,1/2) - I_{(\Lambda)}(0,1/2) + I_{(\Lambda)}(1,-1/2) \\ - (1-\xi)\Big[I_{(\Lambda)}(-1,3/2) + p^4 I_{(\Lambda)}(1,3/2) - 2p^2 I_{(\Lambda)}(0,3/2) - I_{(\Lambda)}(0,1/2)\Big]\Bigg\}, \tag{C.11}$$

where $I_{(\Lambda)}(n,m)$ was defined in (B.10). The relevant integrals are recorded in equations (D.7)-(D.12). We get

$$\sigma_1(p|\alpha^2) = \frac{\alpha^2}{4\pi^2}\left\{\left(\frac{2}{3} - \xi\right)\log\frac{\Lambda^2}{p^2} - \frac{17}{9} + \xi\right\}. \tag{C.12}$$

C.2 Two loop fermion self-energy.

C.2.1 First diagram.

The approach is essentially identical to that of the scalar case in Section B, only with the extra work of performing some gamma matrix algebra. Applying the Feynman rules to the graph in (6.26)

$$\Sigma_\psi^{(1)}(p|\alpha^4) = \alpha^4 \int \frac{\mathrm{d}^3k \mathrm{d}^3q}{(2\pi)^6} \frac{\gamma^c(\not p + \not q)\gamma^a(\not p + \not k + \not q)\gamma^d(\not p + \not k)\gamma^b \Pi_{ab}(k)\Pi_{cd}(q)}{(p+q)^2(p+k+q)^2(p+k)^2} \\ \equiv \Sigma_\psi^{(1),1}(p|\alpha^4) - (1-\xi)\Sigma_\psi^{(1),2}(p|\alpha^4) + (1-\xi)^2\Sigma_\psi^{(1),3}(p|\alpha^4). \tag{C.13}$$

By (C.3) this can be written

$$\Sigma_\psi^{(1),1}(p|\alpha^4) = \not p\,\sigma^{(1),1}(p|\alpha^4) \tag{C.14}$$

where $\sigma^{(1),1}(p|\alpha^4)$ is a c-number, with $\sigma^{(1),2}(p|\alpha^4)$ and $\sigma^{(1),3}(p|\alpha^4)$ defined in the analogous way. Using the general expression (C.4), the gamma matrix algebra in Section A.5, and the reduction formulae (B.3) and (B.15), we get

$$\begin{aligned}\sigma^{(1),1}(p|\alpha^4) = \frac{\alpha^4}{2p^2}\Bigg\{ & -\frac{3}{2}\left[\frac{3,-1}{2,2,2}\right] + \frac{3p^2}{2}\left[\frac{1,-1}{2,2,2}\right] - 5\left[\frac{1,1}{2,2,2}\right] + 2p^2\left[\frac{-1,1}{2,2,2}\right] - \left[\frac{-1,3}{2,2,2}\right] \\ & + \frac{3}{2}\left[\frac{-1,-1}{0,2,0}\right] + \frac{3}{2}\left[\frac{1,-1}{0,2,2}\right] - 2p^2\left[\frac{-1,-1}{0,2,2}\right] + 2\left[\frac{-1,1}{0,2,2}\right] + \frac{3}{2}\left[\frac{1,-1}{2,2,0}\right] \\ & - \frac{3p^2}{2}\left[\frac{-1,-1}{2,2,0}\right] + 2\left[\frac{-1,1}{2,2,0}\right] - \frac{3}{2}\left[\frac{-1,-1}{0,0,2}\right] - 2\left[\frac{-1,-1}{2,0,0}\right] + \frac{3}{2}\left[\frac{1,-1}{2,0,2}\right] \\ & + \frac{3p^2}{2}\left[\frac{-1,-1}{2,0,2}\right] + 2\left[\frac{-1,1}{2,0,2}\right]\Bigg\}. \end{aligned} \tag{C.15}$$

The integrals here were originally defined in (B.18). Using the formulae in Section D.3 to extract the logarithmic divergence, we obtain

$$\sigma^{(1),1}(p|\alpha^4) \sim \frac{\alpha^2}{\pi^4}\left\{ -\frac{13}{216}\log\frac{\Lambda^2}{p^2} - \frac{1}{144}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}. \tag{C.16}$$

Compared with [108]

$$\sigma^{(1),1}(p|\alpha^4) \sim \frac{\alpha^2}{\pi^4}\left\{ -\frac{13}{432}\frac{1}{\epsilon} - \frac{1}{144}\frac{1}{\epsilon^2}\right\}. \qquad \text{(DR)} \tag{C.17}$$

We have only analyzed the integrals which appear in the gauge $\xi = 1$; this corresponds to dropping $\Sigma_\psi^{(1),2}$ and $\Sigma_\psi^{(1),3}$ in (C.13). For completeness we shall quote them. $\sigma^{(1),2}$ is expressed as a sum

$$\sigma^{(1),2}(p|\alpha^4) = \sigma_1^{(1),2}(p|\alpha^4) + \sigma_2^{(1),2}(p|\alpha^4), \tag{C.18}$$

with

$$\begin{aligned}\sigma_1^{(1),2}(p|\alpha^4) = \frac{\alpha^4}{4p^2}\Bigg\{ & p^2\left[\frac{-3,-1}{2,-2,2}\right] + \left[\frac{1,1}{2,2,2}\right] + p^2\left[\frac{-3,1}{2,2,0}\right] + p^2\left[\frac{-1,1}{2,2,2}\right] + p^4\left[\frac{-3,1}{2,2,2}\right] \\ & + \left[\frac{-3,-1}{-2,2,0}\right] + \left[\frac{-3,1}{-2,2,2}\right] + \left[\frac{-3,-1}{0,2,-2}\right] - \left[\frac{-1,1}{0,2,2}\right] - 2p^2\left[\frac{-3,1}{0,2,2}\right] - 2\left[\frac{-1,-1}{0,2,0}\right] \\ & - p^2\left[\frac{-3,-1}{0,2,0}\right] - \left[\frac{-3,1}{0,2,0}\right] + p^4\left[\frac{-3,-1}{2,0,2}\right] - p^2\left[\frac{-3,-1}{2,0,0}\right] - 2p^2\left[\frac{-1,-1}{2,0,2}\right] \\ & - p^2\left[\frac{-3,1}{2,0,2}\right] + \left[\frac{-3,1}{0,0,2}\right] - p^2\left[\frac{-3,-1}{0,0,2}\right] - \left[\frac{-3,-1}{0,0,0}\right]\Bigg\}, \end{aligned} \tag{C.19}$$

and

$$\begin{aligned}\sigma_2^{(1),2}(p|\alpha^4) = \frac{\alpha^4}{2p^2}\Bigg\{ & \left[\frac{-1,-3}{-2,2,0}\right] + p^2\left[\frac{-1,-3}{2,-2,2}\right] + \left[\frac{-1,-3}{0,2,-2}\right] + p^2\left[\frac{1,-3}{0,2,2}\right] - \left[\frac{1,-3}{0,2,0}\right] \\ & - p^2\left[\frac{-1,-3}{0,2,0}\right] - \left[\frac{-1,-1}{0,2,0}\right] + p^4\left[\frac{-1,-3}{2,0,2}\right] - p^2\left[\frac{1,-3}{2,0,2}\right] - p^2\left[\frac{-1,-1}{2,0,2}\right] \\ & - p^2\left[\frac{-1,-3}{0,0,2}\right] - \left[\frac{-1,-3}{0,0,0}\right] + \left[\frac{1,-3}{2,0,0}\right] - p^2\left[\frac{-1,-3}{2,0,0}\right]\Bigg\}. \end{aligned} \tag{C.20}$$

Finally

$$\begin{aligned}\sigma^{(1),3}(p|\alpha^4) = \frac{\alpha^4}{4p^2}\Bigg\{ & p^2\left[\frac{1,-1}{2,2,2}\right] + p^4\left[\frac{-1,-1}{2,2,2}\right] + \left[\frac{1,1}{2,2,2}\right] + 2p^2\left[\frac{-1,1}{2,2,2}\right] + p^4\left[\frac{-3,1}{2,2,2}\right] \\ & - \left[\frac{1,-1}{2,2,0}\right] - p^2\left[\frac{-1,-1}{2,2,0}\right] + 2\left[\frac{-3,-3}{-2,2,-2}\right] - \left[\frac{-3,-1}{-2,2,0}\right] + \left[\frac{-3,1}{-2,2,2}\right] \\ & + p^2\left[\frac{-3,-1}{0,2,0}\right] - p^2\left[\frac{-1,-1}{0,2,2}\right] - 2\left[\frac{-1,1}{0,2,2}\right] - 2p^2\left[\frac{-3,1}{0,2,2}\right] + 2p^4\left[\frac{-3,-3}{2,-2,2}\right] \\ & - 3p^2\left[\frac{-1,-1}{2,0,2}\right] - p^4\left[\frac{-3,-1}{2,0,2}\right] - 4p^2\left[\frac{-3,-3}{0,0,0}\right] + p^2\left[\frac{-3,-1}{0,0,2}\right]\Bigg\}. \end{aligned} \tag{C.21}$$

C.2.2 Second self-energy diagram

Applying the Feynman rules to the graph in (6.27)

$$\begin{aligned}\Sigma_\psi^{(2)}(p|\alpha^4) &= \alpha^4 \int \frac{\mathrm{d}^3k\mathrm{d}^3q}{(2\pi)^6} \frac{\gamma^a(\not{p}+\not{k})\gamma^c(\not{p}+\not{k}+\not{q})\gamma^d(\not{p}+\not{k})\gamma^b\Pi_{ab}(k)\Pi_{cd}(q)}{(p+k)^4(p+k+q)^2} \\ &\equiv \Sigma_\psi^{(2),1}(p|\alpha^4) - (1-\xi)\Sigma_\psi^{(2),2}(p|\alpha^4) + (1-\xi)^2\Sigma_\psi^{(2),3}(p|\alpha^4).\end{aligned} \tag{C.22}$$

Repeating the steps which led to (C.15), we get

$$\begin{aligned}\sigma^{(2),1}(p|\alpha^4) = \frac{\alpha^4}{2p^2}\Bigg\{ &-\left[\frac{-1,-1}{2,2,-2}\right] + \left[\frac{-1,-1}{2,0,0}\right] + p^2\left[\frac{-1,-1}{2,2,0}\right] - \left[\frac{1,-1}{4,0,0}\right] + p^2\left[\frac{-1,-1}{4,0,0}\right] \\ &+ \left[\frac{1,1}{4,2,0}\right] - p^2\left[\frac{-1,1}{4,2,0}\right]\Bigg\}.\end{aligned} \tag{C.23}$$

Using the formulae in Section D.3 to extract the logarithmic divergence, we obtain

$$\sigma^{(2),1}(p|\alpha^4) \sim \frac{\alpha^2}{\pi^4}\left\{\frac{7}{216}\log\frac{\Lambda^2}{p^2} + \frac{1}{288}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}. \tag{C.24}$$

Compared with [108]

$$\sigma^{(2),1}(p|\alpha^4) \sim \frac{\alpha^2}{\pi^4}\left\{\frac{5}{432}\frac{1}{\epsilon} + \frac{1}{288}\frac{1}{\epsilon^2}\right\} \qquad \text{(DR)}. \tag{C.25}$$

Similarly

$$
\begin{aligned}
\sigma^{(2),2}(p|\alpha^4) = \frac{\alpha^4}{2p^2}\Bigg\{ & -\left[\frac{-3,-1}{-2,2,0}\right] + p^2\left[\frac{-1,-1}{4,0,0}\right] - p^4\left[\frac{-3,-1}{4,0,0}\right] - p^2\left[\frac{-1,1}{4,2,0}\right] + p^4\left[\frac{-3,1}{4,2,0}\right] \\
& - \left[\frac{-3,-1}{0,2,-2}\right] + 2p^2\left[\frac{-3,-1}{0,2,0}\right] + \left[\frac{-1,-1}{0,2,0}\right] + \left[\frac{-3,1}{0,2,0}\right] + p^2\left[\frac{-3,-1}{2,2,-2}\right] \\
& + p^2\left[\frac{-3,-1}{2,0,0}\right] - p^4\left[\frac{-3,-1}{2,2,0}\right] - 2p^2\left[\frac{-3,1}{2,2,0}\right] - \left[\frac{-1,-3}{-2,2,0}\right] + \left[\frac{1,-3}{4,-2,0}\right] \\
& - p^2\left[\frac{-1,-3}{4,-2,0}\right] + \left[\frac{-1,-3}{2,0,-2}\right] - \left[\frac{-1,-3}{2,-2,0}\right] + p^2\left[\frac{-1,-3}{2,0,0}\right] - 2\left[\frac{1,-3}{2,0,0}\right] \\
& - \left[\frac{-1,-3}{0,2,-2}\right] + 2\left[\frac{-1,-3}{0,0,0}\right] + \left[\frac{1,-3}{0,2,0}\right] + \left[\frac{-1,-1}{0,2,0}\right] + p^2\left[\frac{-1,-1}{4,0,0}\right] - \left[\frac{1,-1}{4,0,0}\right]\Bigg\},
\end{aligned}
\tag{C.26}
$$

and

$$
\begin{aligned}
\sigma^{(2),3}(p|\alpha^4) = \frac{\alpha^4}{2p^2}\Bigg\{ & -\left[\frac{-3,-3}{-2,2,-2}\right] - p^2\left[\frac{-3,-3}{2,-2,0}\right] - p^2\left[\frac{-1,-3}{4,-2,0}\right] + p^4\left[\frac{-3,-3}{4,-2,0}\right] \\
& + p^2\left[\frac{-3,-3}{0,2,-2}\right] - p^2\left[\frac{-1,-3}{0,2,0}\right] + \left[\frac{-1,-1}{0,2,0}\right] + p^2\left[\frac{-1,-1}{4,0,0}\right] - p^4\left[\frac{-3,-1}{4,0,0}\right] \\
& + \left[\frac{-3,-3}{0,0,-2}\right] + p^2\left[\frac{-3,-3}{0,0,0}\right] - \left[\frac{-3,-1}{0,0,0}\right] - p^2\left[\frac{-3,-3}{2,0,-2}\right] + 2p^2\left[\frac{-1,-3}{2,0,0}\right] \\
& - p^4\left[\frac{-3,-3}{2,0,0}\right] + 2p^2\left[\frac{-3,-1}{2,0,0}\right]\Bigg\}.
\end{aligned}
\tag{C.27}
$$

C.2.3 Third diagram

Applying the Feynman rules to the graph in (6.27)

$$
\begin{aligned}
\Sigma_\psi^{(3)}(p|\alpha^4) &= -\int \frac{d^3k d^3q}{(2\pi)^6} \frac{\gamma^c(\not{p}+\not{k})\gamma^b \,\mathrm{Tr}[\gamma^d \not{q}\gamma^a(\not{k}+\not{q})]}{(p+k)^2(k+q)^2q^2}\Pi_{ab}(k)\Pi_{cd}(k) \\
&\equiv \Sigma^{(3),1}(p|\alpha^4) - (1-\xi)\Sigma^{(3),2}(p|\alpha^4) + (1-\xi)^2\Sigma^{(3),3}(p|\alpha^4).
\end{aligned}
\tag{C.28}
$$

Following similar steps to before, now using the reduction formulae (B.37), we obtain

$$
\begin{aligned}
\sigma^{(3),1} = -\frac{\alpha^4}{2p^2}\Bigg\{ & \left[\frac{2,-2}{2,2}\right] + 3p^2\left[\frac{0,-2}{2,2}\right] - 8|p|\left[\frac{0,-1}{2,2}\middle|\cos\theta_{pq}\right] \\
& - 8|p|^3\left[\frac{-2,-1}{2,2}\middle|\cos\theta_{pq}\right] + 3\left[\frac{0,0}{2,2}\right] + p^2\left[\frac{-2,0}{2,2}\right] + 16p^2\left[\frac{-2,0}{2,2}\middle|\cos^2\theta_{pq}\right] \\
& - 8|p|\left[\frac{-2,1}{2,2}\middle|\cos\theta_{pq}\right] - \left[\frac{0,-2}{2,0}\right] - 3p^2\left[\frac{-2,-2}{2,0}\right] + 8|p|\left[\frac{-2,-1}{2,0}\middle|\cos\theta_{pq}\right] \\
& + \left[\frac{-2,-2}{0,0}\right] - \left[\frac{0,-2}{0,2}\right] + 8|p|\left[\frac{-2,-1}{0,2}\middle|\cos\theta_{pq}\right] - 3\left[\frac{-2,0}{0,2}\right]\Bigg\},
\end{aligned}
\tag{C.29}
$$

where the integrals were defined in (B.39) and (B.40). Right off the bat we can see that all the $\left[\frac{a,b}{c,d}\right]$'s occurring in $\sigma^{(3),1}$ with $d = 0$ do not contribute a purely logarithmic divergence:

$$
\left[\frac{0,-2}{2,0}\right], \left[\frac{-2,-2}{2,0}\right], \left[\frac{-2,-1}{2,0}\middle|\cos\theta_{pq}\right], \left[\frac{-2,-2}{0,0}\right] \sim 0. \tag{C.30}
$$

Using the formulae in Section D.4 on the remaining integrals, we find the logarithmic divergence

$$
\sigma^{(3),1} \sim -\frac{\alpha^4}{96\pi^2}\log\frac{\Lambda^2}{p^2}. \tag{C.31}
$$

Compared with

$$
\sigma^{(3),1} \sim -\frac{\alpha^4}{192\pi^2}\frac{1}{\epsilon} \qquad \text{(DR)}. \tag{C.32}
$$

C.3 Two-loop mass-squared operator renormalization.

C.3.1 First diagram.

Applying the Feynman rules to the graph in (6.48)

$$
\begin{aligned}
\Sigma^{(1;1)}_{\overline{\psi}\psi} &= \alpha^2 \int \frac{d^3k d^3q}{(2\pi)^6} \frac{\gamma^c(\not p + \not q)\gamma^a(\not p + \not k + \not q)\gamma^d(\not p + \not k)^2\gamma^b}{(p+k)^4(p+k+q)^2(p+q)^2}\Pi_{ab}(k)\Pi_{cd}(q) \\
&= \frac{\alpha^2}{(2\pi)^6}\int \frac{d^3k d^3q}{|k||q|} \frac{\gamma^c(\not p + \not q)\gamma^a(\not p + \not k + \not q)\gamma^c\gamma^a}{(p+k)^2(p+k+q)^2(p+q)^2},
\end{aligned}
$$

where we have gone to the gauge $\xi = 1$ in the second line. Similarly

$$\Sigma^{(1;2)}_{\overline{\psi}\psi} = \frac{\alpha^2}{(2\pi)^6} \int \frac{d^3k d^3q}{|k||q|} \frac{\gamma^c(\not{p} + \not{q})\gamma^a \gamma^c(\not{p} + \not{k})\gamma^a}{(p+k)^2(p+k+q)^2(p+q)^2}, \tag{C.33}$$

$$\Sigma^{(1;3)}_{\overline{\psi}\psi} = \frac{\alpha^2}{(2\pi)^6} \int \frac{d^3k d^3q}{|k||q|} \frac{\gamma^c \gamma^a(\not{p} + \not{k} + \not{q})\gamma^c(\not{p} + \not{k})\gamma^a}{(p+k)^2(p+k+q)^2(p+q)^2}. \tag{C.34}$$

In general

$$\Sigma_{\overline{\psi}\psi} = \sigma_{\overline{\psi}\psi} \mathbb{I}, \tag{C.35}$$

where $\mathbb{I}$ is the 2×2 identity matrix in spinor space. In a hopefully obvious notation, we find

$$\sigma^{(1;1)}_{\overline{\psi}\psi} = \frac{\alpha^2}{128\pi^6} \left(\left[\frac{-1, -1}{2, 2, 0} \right] + \left[\frac{-1, -1}{2, 0, 2} \right] - \left[\frac{1, -1}{2, 2, 2} \right] \right) \tag{C.36}$$

$$\sigma^{(1;2)}_{\overline{\psi}\psi} \sim \frac{5\alpha^2}{128\pi^6} \left(\left[\frac{-1, -1}{2, 0, 2} \right] - 2 \left[\frac{-1, 1}{2, 2, 2} \right] \right), \tag{C.37}$$

and $\sigma^{(1;3)}_{\overline{\psi}\psi} = \sigma^{(1;1)}_{\overline{\psi}\psi}$. Using formulae in Section D.3 we find the logarithmic divergences

$$\Sigma^{(1;1)}_{\overline{\psi}\psi} \sim \frac{\alpha^2}{128\pi^4} \left\{ 16 \log \frac{\Lambda^2}{p^2} + 6 \left(\log \frac{\Lambda^2}{p^2} \right)^2 \right\}, \tag{C.38}$$

$$\Sigma^{(1;2)}_{\overline{\psi}\psi} \sim -\frac{5\alpha^2}{128\pi^4} \cdot \frac{1}{4} \log \frac{\Lambda^2}{p^2}. \tag{C.39}$$

Summing them

$$\Sigma^{(1)}_{\overline{\psi}\psi} \sim \frac{\alpha^2}{\pi^4} \left\{ -\frac{3}{8} \log \frac{\Lambda^2}{p^2} + \frac{1}{16} \left(\log \frac{\Lambda^2}{p^2} \right)^2 \right\}. \tag{C.40}$$

Compared with

$$\Sigma^{(1)}_{\overline{\psi}\psi} \sim \frac{\alpha^2}{\pi^4} \left\{ -\frac{1}{16} \frac{1}{\epsilon} + \frac{1}{16} \frac{1}{\epsilon^2} \right\}. \qquad \text{(DR)}. \tag{C.41}$$

The analysis for the second and third diagrams is similar. The necessary divergences are listed in Sections D.3 and D.4, respectively.

D Loop integrals

D.1 One-loop integrals.

D.1.1 Dimensional regularization.

All one loop integrals can be reduced to

$$I(n,m) := \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{((p+k)^2)^n (k^2)^m}, \tag{D.1}$$

originally defined in (B.4). Clearly $I(n,m)$ is symmetric in n and m. For general d one finds

$$I(n,m) = \frac{(p^2)^{d/2-(n+m)}}{(4\pi)^{d/2}} G(n,m), \tag{D.2}$$

where

$$G(n,m) := \frac{\Gamma(n+m-d/2)\Gamma(d/2-n)\Gamma(d/2-m)}{\Gamma(n)\Gamma(m)\Gamma(d-n-m)}. \tag{D.3}$$

This is derived in e.g. [243]. Plugging in for $d = 3-2\epsilon$, which is of interest to us in this work, this reads

$$I(n,m) = \frac{(p^2)^{3/2-(n+m)}}{(4\pi)^{3/2}} \cdot \left(\frac{p^2}{4\pi}\right)^{-\epsilon} G(n,m), \tag{D.4}$$

with

$$G(n,m) = \frac{\Gamma(n+m-3/2+\epsilon)\Gamma(3/2-n-\epsilon)\Gamma(3/2-m-\epsilon)}{\Gamma(n)\Gamma(m)\Gamma(3-n-m-2\epsilon)}. \tag{D.5}$$

D.1.2 Momentum cutoff.

The analogous integral to (D.1) with a momentum cutoff is

$$I_{(\Lambda)}(n,m) := \int_{|k|<\Lambda} \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{((p+k)^2)^n (k^2)^m}. \tag{D.6}$$

This can be done on a case-by-case basis by changing to spherical coordinates. Some examples relevant to the computation of the self-energy in Section B.1 are

$$I_{(\Lambda)}(1,-1/2) = \frac{1}{(2\pi)^2}\left(\Lambda^2 + \frac{p^2}{3}\log\frac{\Lambda^2}{p^2} + \frac{2p^2}{9}\right), \tag{D.7}$$

$$I_{(\Lambda)}(1,1/2) = \frac{1}{(2\pi)^2}\left(2 + \log\frac{\Lambda^2}{p^2}\right), \tag{D.8}$$

$$I_{(\Lambda)}(1,3/2) = \frac{1}{(2\pi)^2 p^2}\left(2 + \log\frac{p^2}{\delta^2}\right), \tag{D.9}$$

$$I_{(\Lambda)}(0,1/2) = \frac{1}{(2\pi)^2}\cdot\Lambda^2, \tag{D.10}$$

$$I_{(\Lambda)}(0,3/2) = \frac{1}{(2\pi)^2}\cdot\log\frac{\Lambda^2}{\delta^2}, \tag{D.11}$$

$$I_{(\Lambda)}(-1,3/2) = \frac{1}{(2\pi)^2}\left(\Lambda^2 + p^2\log\frac{\Lambda^2}{\delta^2}\right). \tag{D.12}$$

D.2 Two loop integrals in dimensional regularization

In this section we review some of the known results on the massless two loop propagator graphs, and the issue we had in applying it to the scalar BCFT case. This is defined by the class of integrals (this is reviewed in [243–246] for example)

$$\begin{aligned} I(n_1,n_2,n_3,n_4,n_5) &\doteq \int \frac{\mathrm{d}^d k \mathrm{d}^d q}{(2\pi)^{2d}} \frac{1}{((p+k)^2)^{n_1}((p+q)^2)^{n_2}(k^2)^{n_3}(q^2)^{n_4}((k-q)^2)^{n_5}} \\ &= \frac{(p^2)^{d-|\vec{n}|}}{(4\pi)^d} G(n_1,n_2,n_3,n_4,n_5), \end{aligned} \tag{D.13}$$

where G is a dimensionless function independent of p. The dependence on the external momentum p follows from rotational invariance and homogeneity. Relabelling dummy integration variables, $G(n_1,\dots,n_5)$ is invariant under the permutations

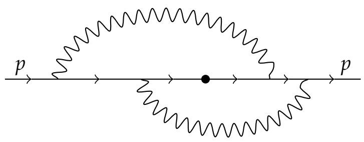
$$(n_1 \leftrightarrow n_2, n_3 \leftrightarrow n_4), \quad (n_1 \leftrightarrow n_3, n_2 \leftrightarrow n_4), \quad (n_1 \leftrightarrow n_4, n_2 \leftrightarrow n_3). \tag{D.14}$$

If one of the indices vanishes, $I(n_1,\dots,n_5)$ reduces to a product of one loop integrals (B.4). The problem becomes more difficult when all of the indices are non-vanishing. A particularly powerful method for computing them in this case is to use integration by parts, first introduced in [247–249]. Using these identities allows us to compute $I(n_1, \ldots, n_5)$ in the case where all of the n_i's are integer, since they can eventually be reduced to a sum of integrals with at least one vanishing n_i.

In our BCFT setup the non-local photon propagator $\sim \frac{1}{|p|}$ complicates matters, as we have to consider integrals with half-integer n_i's. The authors of [250] found a closed form expression for $G(n_1, n_2, n_3, 1, 1)$ with arbitrary real n_1, n_2, n_3 in terms of the generalized hypergeometric function ${}_3F_2$.

The combination of integration by parts along with the result of [250] is enough to determine the $\frac{1}{\epsilon}$ and $\frac{1}{\epsilon^2}$ poles in all but one of the integrals which occur. The problematic integral is $G(1, \frac{1}{2}, \frac{1}{2}, 1, 2)$, which occurs in renormalizing the mass-squared operator. Specifically it appears in the diagram

$$\Sigma^{(1;2)}_{\bar{\phi}\phi}(p|\alpha^4) =$$



(D.15)

D.3 Two-loop integrals with a cutoff.

Here we record our findings for the logarithmic divergences of the class of integrals defined in (B.18)

$$\begin{bmatrix} \frac{a, b}{c, d, e} \end{bmatrix} := \int \frac{\mathrm{d}^3k \mathrm{d}^3q}{(2\pi)^6} \frac{|k|^a |q|^b}{(p+k)^c (p+k+q)^d (p+q)^e}. \tag{D.16}$$

Some integrals exhibit an IR divergence from regions where the loop momenta are collinear. We regulate this by a parameter $\varepsilon > 0$ which we eventually take to zero.

D.3.1 $\left[\frac{*,*}{6,2,0}\right]$ Integrals.

$$\left[\frac{1,1}{6,2,0}\right] \sim' \left(-\frac{1}{144\pi^4} + \frac{1}{96\pi^3}\frac{1}{\varepsilon}\right)\log\frac{\Lambda^2}{p^2}, \tag{D.17}$$

$$\left[\frac{-1,1}{6,2,0}\right] \sim' \frac{1}{p^2}\left(-\frac{1}{288\pi^4} + \frac{1}{384\pi^3}\frac{1}{\varepsilon^3} + \frac{1}{768\pi^3}\frac{1}{\varepsilon}\right)\log\frac{\Lambda^2}{p^2}. \tag{D.18}$$

D.3.2 $\left[\frac{*,*}{4,4,0}\right]$ Integrals

$$\left[\frac{1,1}{4,4,0}\right] \sim' \left(-\frac{1}{8\pi^4} + \frac{1}{16\pi^3}\frac{1}{\varepsilon}\right)\log\frac{\Lambda^2}{p^2} + \frac{1}{32\pi^4}\left(\log\frac{\Lambda^2}{p^2}\right)^2, \tag{D.19}$$

$$\left[\frac{-1,1}{4,4,0}\right] \sim' \frac{1}{p^2}\left(-\frac{1}{16\pi^4} + \frac{1}{32\pi^3}\frac{1}{\varepsilon}\right)\log\frac{\Lambda^2}{p^2}. \tag{D.20}$$

D.3.3 $\left[\frac{*,*}{4,2,2}\right]$ Integrals

$$\left[\frac{3,-1}{4,2,2}\right] \sim' \frac{1}{4\pi^4}\log\frac{\Lambda^2}{p^2} + \frac{1}{32\pi^4}\left(\log\frac{\Lambda^2}{p^2}\right)^2, \tag{D.21}$$

$$\left[\frac{1,1}{4,2,2}\right] \sim' \frac{1}{16\pi^4}\log\frac{\Lambda^2}{p^2} + \frac{1}{32\pi^4}\left(\log\frac{\Lambda^2}{p^2}\right)^2 + \frac{1}{32\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}, \tag{D.22}$$

$$\left[\frac{-1,3}{4,2,2}\right] \sim' \frac{17}{144}\log\frac{\Lambda^2}{p^2} + \frac{1}{94\pi^4}\left(\log\frac{\Lambda^2}{p^2}\right)^2 + \frac{1}{96\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}, \tag{D.23}$$

$$\left[\frac{-1,1}{4,2,2}\right] \sim' -\frac{1}{16p^2\pi^4}\log\frac{\Lambda^2}{p^2} + \frac{1}{32p^2\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}. \tag{D.24}$$

D.3.4 $\left[\frac{*,*}{2,4,2}\right]$

$$\left[\frac{1,1}{2,4,2}\right] \sim' \left(\frac{1}{16\pi^4} + \frac{1}{32\pi^4}\log\frac{2}{\varepsilon} - \frac{1}{16\pi^3}\frac{1}{\varepsilon}\right)\log\frac{\Lambda^2}{p^2}, \tag{D.25}$$

$$\left[\frac{-1,3}{2,4,2}\right] \sim' \left(\frac{1}{16\pi^4} + \frac{1}{32\pi^3}\frac{1}{\varepsilon}\right)\log\frac{\Lambda^2}{p^2} + \frac{1}{32\pi^4}\left(\log\frac{\Lambda^2}{p^2}\right)^2, \tag{D.26}$$

$$\left[\frac{-1,1}{2,4,2}\right] = \text{UV finite}. \tag{D.27}$$

D.3.5 $\left[\frac{*,*}{4,2,0}\right]$

$$\left[\frac{1,1}{4,2,0}\right] \sim' \frac{p^2}{\pi^4}\left\{\frac{13}{432}\log\frac{\Lambda^2}{p^2}+\frac{1}{288}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}, \tag{D.28}$$

$$\left[\frac{1,-1}{4,2,0}\right] \sim' \frac{1}{16\pi^4}\log\frac{\Lambda^2}{p^2}+\frac{1}{32\pi^4}\left(\log\frac{\Lambda^2}{p^2}\right)^2+\frac{1}{32\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}, \tag{D.29}$$

$$\left[\frac{-1,1}{4,2,0}\right] \sim' \frac{1}{\pi^4}\left\{\frac{1}{18}\log\frac{\Lambda^2}{p^2}+\frac{1}{96}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}, \tag{D.30}$$

$$\left[\frac{-1,-1}{4,2,0}\right] \sim' -\frac{1}{16p^2\pi^4}\log\frac{\Lambda^2}{p^2}+\frac{1}{32p^2\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}. \tag{D.31}$$

D.3.6 $\left[\frac{*,*}{4,0,2}\right]$ Integrals

$$\left[\frac{1,-1}{4,0,2}\right] \sim' \frac{1}{16\pi^4}\log\frac{\Lambda^2}{p^2}+\frac{1}{16\pi^4}\left(\log\frac{\Lambda^2}{p^2}\right)^2+\frac{1}{32\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}, \tag{D.32}$$

$$\left[\frac{-1,1}{4,0,2}\right] \sim' -\frac{1}{48\pi^4}\log\frac{\Lambda^2}{p^2}+\frac{1}{96\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}, \tag{D.33}$$

$$\left[\frac{-1,-1}{4,0,2}\right] \sim' -\frac{1}{16p^2\pi^4}\log\frac{\Lambda^2}{p^2}+\frac{1}{32p^2\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}. \tag{D.34}$$

D.3.7 $\left[\frac{*,*}{2,4,0}\right]$

$$\left[\frac{1,-1}{2,4,0}\right] \sim' \left(\frac{1}{16\pi^4}+\frac{1}{32\pi^4}\log\frac{2}{\varepsilon}-\frac{1}{16\pi^3}\frac{1}{\varepsilon}\right)\log\frac{\Lambda^2}{p^2}, \tag{D.35}$$

$$\left[\frac{-1,1}{2,4,0}\right] \sim' \left(\frac{1}{16\pi^4}+\frac{1}{32\pi^3}\frac{1}{\varepsilon}\right)\log\frac{\Lambda^2}{p^2}+\frac{1}{32\pi^4}\left(\log\frac{\Lambda^2}{p^2}\right)^2, \tag{D.36}$$

$$\left[\frac{-1,0}{2,4,0}\right], \left[\frac{-1,-1}{2,4,0}\right] = \text{UV finite}. \tag{D.37}$$

D.3.8 $\left[\frac{*,*}{2,2,2}\right]$

$$\left[\frac{1,1}{2,2,2}\right] \sim' \frac{p^2}{\pi^4}\left\{\frac{13}{144}\log\frac{\Lambda^2}{p^2} + \frac{1}{48}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}, \tag{D.38}$$

$$\left[\frac{-1,3}{2,2,2}\right] \sim' \frac{p^2}{\pi^4}\left\{\frac{61}{432}\log\frac{\Lambda^2}{p^2} + \frac{7}{288}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}, \tag{D.39}$$

$$\left[\frac{-1,1}{2,2,2}\right] \sim' \frac{1}{\pi^4}\left\{\frac{1}{4}\log\frac{\Lambda^2}{p^2} + \frac{1}{32}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}. \tag{D.40}$$

D.3.9 $\left[\frac{*,*}{4,2,-2}\right]$ Integrals

$$\left[\frac{-1,-1}{4,2,-2}\right] \sim' -\frac{13}{144\pi^4}\log\frac{\Lambda^2}{p^2} + \frac{1}{96\pi^4}\left(\log\frac{\Lambda^2}{p^2}\right)^2 + \frac{1}{32\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}. \tag{D.41}$$

D.3.10 $\left[\frac{*,*}{4,-2,2}\right]$ Integrals

$$\left[\frac{-1,-1}{4,-2,2}\right] \sim' \frac{11}{48\pi^4}\log\frac{\Lambda^2}{p^2} + \frac{1}{16\pi^4}\left(\log\frac{\Lambda^2}{p^2}\right)^2 + \frac{1}{96\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}. \tag{D.42}$$

D.3.11 $\left[\frac{*,*}{2,4,-2}\right]$ Integrals

$$\left[\frac{-1,-1}{2,4,-2}\right] \sim' \left(\frac{1}{16\pi^4} + \frac{1}{32\pi^3}\frac{1}{\varepsilon}\right)\log\frac{\Lambda^2}{p^2} + \frac{1}{32\pi^4}\left(\log\frac{\Lambda^2}{p^2}\right)^2. \tag{D.43}$$

D.3.12 $\left[\frac{*,*}{0,4,0}\right]$ Integrals

$$\left[\frac{-1,-1}{0,4,0}\right] \sim' p^2\left(-\frac{1}{192\pi^4} + \frac{1}{384\pi^3}\frac{1}{\varepsilon}\right)\log\frac{\Lambda^2}{p^2}. \tag{D.44}$$

D.3.13 $\left[\frac{*,*}{2,2,0}\right]$

$$\left[\frac{1,-1}{2,2,0}\right] \sim' \frac{p^2}{\pi^4}\left\{\frac{17}{144}\log\frac{\Lambda^2}{p^2} + \frac{1}{96}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}, \tag{D.45}$$

$$\left[\frac{-1,1}{2,2,0}\right] \sim' -\frac{p^2}{144\pi^4}\log\frac{\Lambda^2}{p^2}, \tag{D.46}$$

$$\left[\frac{-1,-1}{2,2,0}\right] \sim' \frac{1}{\pi^4}\left\{\frac{1}{4}\log\frac{\Lambda^2}{p^2} + \frac{1}{32}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}. \tag{D.47}$$

D.3.14 $\left[\frac{*,*}{2,0,2}\right]$

$$\left[\frac{1,-1}{2,0,2}\right] \sim' \frac{p^2}{\pi^4}\left\{\frac{1}{18}\log\frac{\Lambda^2}{p^2} + \frac{1}{48}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}, \tag{D.48}$$

$$\left[\frac{-1,-1}{2,0,2}\right] \sim' \frac{1}{\pi^4}\left\{\frac{1}{4}\log\frac{\Lambda^2}{p^2} + \frac{1}{16}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}. \tag{D.49}$$

D.3.15 $\left[\frac{*,*}{2,2,-2}\right]$

$$\left[\frac{-1,-1}{2,2,-2}\right] \sim' \frac{p^2}{\pi^4}\left\{\frac{92}{9}\log\frac{\Lambda^2}{p^2} + \frac{10}{9}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}. \tag{D.50}$$

D.3.16 $\left[\frac{*,*}{2,-2,2}\right]$

$$\left[\frac{-1,-1}{2,-2,2}\right] \sim' \frac{p^2}{\pi^4}\left\{-\frac{7}{108}\log\frac{\Lambda^2}{p^2} - \frac{7}{144}\left(\log\frac{\Lambda^2}{p^2}\right)^2\right\}. \tag{D.51}$$

D.3.17 $\left[\frac{*,*}{0,2,0}\right]$

$$\left[\frac{-1,-1}{0,2,0}\right] \sim' -\frac{p^2}{48\pi^4}\log\frac{\Lambda^2}{p^2}. \tag{D.52}$$

D.4 More two loop integrals with a cutoff

Here we record our findings for the logarithmic divergences of the class of integrals defined in (B.39)

$$\begin{bmatrix} \frac{a,b}{c,d} \end{bmatrix} := \int \frac{d^3k d^3q}{(2\pi)^6} \frac{|k|^a |q|^b}{(p+k)^c (k+q)^d}, \tag{D.53}$$

and (B.40)

$$\left[\frac{a,b}{c,d} \middle| \cos\theta_{pq} \right] := \int \frac{d^3k d^3q}{(2\pi)^6} \frac{|k|^a |q|^b \cos\theta_{pq}}{(p+k)^c (k+q)^d}. \tag{D.54}$$

Like in the previous section, some integrals exhibit an IR divergence from regions where the loop momenta are collinear. We regulate this by a parameter $\varepsilon > 0$ which we eventually take to zero.

D.4.1 $\left[\frac{*,*}{4,2}\right]$ Integrals

$$\begin{bmatrix} \frac{2,-2}{4,2} \end{bmatrix} \sim' \frac{1}{32\pi^2} \log \frac{\Lambda^2}{p^2}, \tag{D.55}$$

$$\begin{bmatrix} \frac{0,0}{4,2} \end{bmatrix} \sim' 0, \tag{D.56}$$

$$\left[\frac{0,-1}{4,2} \middle| \cos\theta_{pq} \right] \sim' 0, \tag{D.57}$$

$$\begin{bmatrix} \frac{-2,2}{4,2} \end{bmatrix} \sim' 0, \tag{D.58}$$

$$\left[\frac{-2,1}{4,2} \middle| \cos\theta_{pq} \right] \sim' 0. \tag{D.59}$$

D.4.2 $\left[\frac{*,*}{2,4}\right]$ Integrals

$$\left[\frac{2,-2}{2,4}\right] \sim' \frac{1}{32\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}, \tag{D.60}$$

$$\left[\frac{0,0}{2,4}\right] \sim' \frac{1}{32\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}, \tag{D.61}$$

$$\left[\frac{-2,1}{2,4}\middle|\cos\theta_{pq}\right] \sim' 0, \tag{D.62}$$

$$\left[\frac{-2,2}{2,4}\right] \sim' \frac{1}{32\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}. \tag{D.63}$$

D.4.3 $\left[\frac{*,*}{2,2}\right]$ Integrals

$$\left[\frac{2,-2}{2,2}\right] \sim' \frac{p^2}{96\pi^2}\log\frac{\Lambda^2}{p^2}, \tag{D.64}$$

$$\left[\frac{2,-4}{2,2}\right] \sim' 0, \tag{D.65}$$

$$\left[\frac{0,0}{2,2}\right] \sim' 0, \tag{D.66}$$

$$\left[\frac{0,-1}{2,2}\middle|\cos\theta_{pq}\right] \sim' \frac{|p|}{96\pi^2}\log\frac{\Lambda^2}{p^2}, \tag{D.67}$$

$$\left[\frac{0,-2}{2,2}\right] \sim' \frac{1}{32\pi^2}\log\frac{\Lambda^2}{p^2}, \tag{D.68}$$

$$\left[\frac{-2,2}{2,2}\right] \sim' 0, \tag{D.69}$$

$$\left[\frac{-2,1}{2,2}\middle|\cos\theta_{pq}\right] \sim' 0, \tag{D.70}$$

$$\left[\frac{-2,0}{2,2}\right] \sim' \left[\frac{-2,0}{2,2}\middle|\cos^2\theta_{pq}\right] \sim' 0, \tag{D.71}$$

$$\left[\frac{-2,-1}{2,2}\middle|\cos\theta_{pq}\right] \sim' 0, \tag{D.72}$$

D.4.4 $\left[\frac{*,*}{0,4}\right]$ Integrals

$$\left[\frac{-2,0}{0,4}\right] \sim' \frac{1}{32\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}. \tag{D.73}$$

D.4.5 $\left[\frac{*,*}{4,-2}\right]$ Integrals

$$\left[\frac{-2,-2}{4,-2}\right] \sim' 0. \tag{D.74}$$

D.4.6 $\left[\frac{*,*}{2,0}\right]$ Integrals

$$\left[\frac{0,-4}{2,0}\right] \sim' 0, \tag{D.75}$$

$$\left[\frac{-2,-1}{2,0}\middle|\cos\theta_{pq}\right] \sim' 0, \tag{D.76}$$

$$\left[\frac{-2,-2}{2,0}\right] \sim' 0, \tag{D.77}$$

$$\left[\frac{-2,-3}{2,0}\middle|\cos\theta_{pq}\right] \sim' 0. \tag{D.78}$$

D.4.7 $\left[\frac{*,*}{0,2}\right]$ Integrals

$$\left[\frac{0,-4}{0,2}\right] \sim' 0, \tag{D.79}$$

$$\left[\frac{-2,0}{0,2}\right] \sim' 0, \tag{D.80}$$

$$\left[\frac{-2,-1}{0,2}\middle|\cos\theta_{pq}\right] \sim' 0, \tag{D.81}$$

$$\left[\frac{-2,-2}{0,2}\right] \sim' \frac{1}{32\pi^2}\log\frac{\Lambda^2}{p^2}. \tag{D.82}$$

D.4.8 $\left[\frac{*,*}{-2,4}\right]$ Integrals

$$\left[\frac{-2,-2}{-2,4}\right] \sim' \frac{1}{32\pi^3}\frac{1}{\varepsilon}\log\frac{\Lambda^2}{p^2}. \tag{D.83}$$

D.4.9 $\left[\frac{*,*}{2,-2}\right]$ Integrals

$$\left[\frac{-2,-2}{2,-2}\right] \sim' 0, \tag{D.84}$$

$$\left[\frac{-2,-4}{2,-2}\right] \sim' 0. \tag{D.85}$$

D.4.10 $\left[\frac{*,*}{-2,2}\right]$ Integrals

$$\left[\frac{-2,-2}{-2,2}\right] \sim' \frac{p^2}{32\pi^2} \log \frac{\Lambda^2}{p^2}, \tag{D.86}$$

$$\left[\frac{-2,-4}{-2,2}\right] \sim' 0. \tag{D.87}$$

E Some Fourier transforms

The following Fourier transform is rather useful (c.f. e.g. [244]):

$$\int \mathrm{d}^d p \, \frac{\mathrm{e}^{\mathrm{i}\, px}}{|p|^{d-2\alpha}} = \frac{2^{2\alpha}\pi^{d/2}\Gamma(\alpha)}{\Gamma(\frac{d}{2}-\alpha)} \cdot \frac{1}{|x|^{2\alpha}}, \tag{E.1}$$

Analyzing the two loop diagrams we shall also need the Fourier transform

$$\int \mathrm{d}^3 x \, e^{-\mathrm{i}\, px} \frac{1}{|x|} \log \frac{1}{|x|} = \frac{4\pi}{p^2} \log(\mathrm{e}^{\gamma}\, |p|). \tag{E.2}$$

This formula can be proven by considering the integral

$$\int \mathrm{d}^d x \; \mathrm{e}^{-\mathrm{i}\, px} \frac{1}{|x|^{2\alpha}}. \tag{E.3}$$

On the one hand, we can set $\alpha = (1+\varepsilon)/2$ in the above, Taylor expand $\frac{1}{|x|^{1+\varepsilon}}$ for $\varepsilon \approx 0$ and integrate term by term. This generates a series in ε. On the other hand, we can compute it exactly (similar to (E.1)) and expand this answer as a series in ε. Comparing the $\mathrm{O}(\varepsilon)$ terms we obtain (E.2).

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