On a class of distribution dependent stochastic differential equations driven by time-changed Brownian motions

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Abstract

In this paper, a class of distribution dependent stochastic differential equations driven by time-changed Brownian motion is studied. The existence and uniqueness theorem of strong solutions for the distribution dependent stochastic differential equations is established. Then, sufficient conditions are provided to guarantee the solutions to be stable in several different senses in terms of Lyapunov function. Finally, we show that the solutions of the distribution dependent stochastic differential equations can be approximated by solutions of the associated averaged stochastic differential equations in mean square convergence.

Keywords: Distribution dependent stochastic differential equations, time-changed Brownian motions, stability, averaging principle.

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1 Introduction

Inspired by the seminal work [21] of Kac, McKean in [31] studied nonlinear Fokker-Planck equations using stochastic differential equations with distribution dependent drifts. Thereafter, distribution dependent stochastic differential equations (DDSDEs, for short) of the following form

$$dX(t) = b(t, X(t), \mathscr{L}(X(t)))dt + \sigma(t, X(t), \mathscr{L}(X(t)))dB_t, \tag{1.1}$$

have received vast attention, where $\mathscr{L}(X(t))$ stands for the distribution (i.e., the law) of the random variable $X(t)$. These equations have also been named as McKean-Vlasov stochastic differential equations or mean-field stochastic differential equations in the literature. A distinct feature of such stochastic differential equations is the appearance of probability laws in

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the coefficients of the equations. Owing to the investigation of mean-field control, mean-field games and complex networked systems (see, for example, Bensoussan et al. [3], Huang et al. [16], Lasry et al. [25]), there are many papers devoted to the study of distribution dependent systems and the relevant theories have been well developed. A thorough illustration of the general theory of DDSDEs and their particle approximations can be found in [43], and the survey article [18] by Huang et al. summarises recent progresses on DDSDEs. There are many interesting investigations on existence and uniqueness for solutions of DDSDEs under various conditions, we would like to mention a few here. Wang [45] established strong well-posedness of DDSDEs with one-sided Lipschitz continuous drifts and Lipschitz-continuous dispersion coefficients. Li et al. [26] obtained existence and uniqueness for McKean-Vlasov stochastic differential equations under local Lipschitz conditions of state variables. Further studies of DDSDEs can be found in Bao et al. [2], Huang and Wang [17], Ren and Wang [37], Röckner and Zhang [40], Mishura and Veretennikov [32], Chaudru de Raynal [6], Hammersley et al [13], Ding and Qiao [8], Fan et al [10], Hong et al [14] just mention a few.

For stochastic differential equations, based on the existence and uniqueness of their solutions, stability of the solutions is an important topic. It means that the trajectories do not change too much under small perturbations. The stability has been studied widely in various different senses, such as stochastically stable, stochastically asymptotically stable, moment exponentially stable, almost surely stable, mean square polynomial stable and so on (see for example Mao [30] for systematic introduction of stabilities). Recently, Bahlali et al. [1] derived various stability properties of McKean-Vlasov stochastic differential equations with respect to initial data, coefficients and driving processes. Ding and Qiao [9] considered the exponential stability of second moments, almost surely asymptotic stability for a type of stochastic McKean-Vlasov equations. Gong and Qiao [11] investigated the stability and path-independence of additive functionals for a type of multivalued McKean-Vlasov stochastic differential equations under non-Lipschitz conditions.

On the other hand, time-changed semimartingales have attracted considerable attention, and their various generalisations have been widely used to model anomalous diffusions arising in physics, finance, hydrology, and cell biology (see recent monograph Umarov, Hahn and Kobayashi [44] and references therein). In [23], Kobayashi introduced the duality theorem between time-changed stochastic differential equations and the corresponding non-time-changed SDEs, and established the Itô formula for time-changed stochastic differential equations. When the original semimartingale is a standard Brownian motion, then it is well known that the transition probability density of the time-changed Brownian motion satisfies a time-fractional partial differential equation (Nane and Ni [35]). Deng and Liu [7] approximated a class of time-changed stochastic differential equations using semi-implicit Euler-Maruyama method and discussed the convergence rate. Liu et al [28] used Truncated Euler-Maruyama method to study time-changed nonautonomous stochastic differential equations. Wu [46] considered stabilities in different senses for SDEs driven by time-changed Brownian motion. Li and Ren [27] studied the practical stability with regard to a part of the variables for DDSDEs driven by time-changed Brownian motion. For the stabilities and related results when the driven process is a time-changed Lévy process, we can see Nane and Yi [33, 34], Shen et al. [42].

Inspired by the aforementioned works, in this paper for arbitrarily fixed $d, m \in \mathbb{N}$, we are concerned with the following class of DDSDEs driven by time-changed Brownian motions:

$$dX(t) = b(t, E_t, X(t), \mathscr{L}(X(t)))dE_t + \sigma(t, E_t, X(t), \mathscr{L}(X(t)))dB_{E_t} \tag{1.2}$$

on a given complete, filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a standard filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, where the coefficients $b : [0,T] \times [0,\infty) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \mapsto \mathbb{R}^d$ and $\sigma : [0,T] \times [0,\infty) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \mapsto \mathbb{R}^{d\times m}$ are Borel measurable functions, $\mathscr{L}(X(t))$ stands for the probability law or distribution of the random variable $X(t)$ and $\mathcal{P}(\mathbb{R}^d)$ denotes the space of all probability measures on the Borel measurable space $\mathbb{R}^d$ equipped with the weak topology, the initial value X_0 is an $\mathcal{F}_0$-measurable random variable satisfying $\mathbb{E}|X_0|^2 < \infty$, $B(t), t \geq 0$, is an m-dimensional $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion, E_t is a random time-change denoting a new clock and it is defined as

$$E_t := \inf\{s > 0 : D(s) > t\},$$

the generalised inverse of an increasing, α-stable $\{\mathcal{F}_t\}_{t\geq 0}$-adapted Lévy process $D(t), t \geq 0$, with Lévy stable index $\alpha \in (1,2)$. The process $\{D(t), t \geq 0\}$ is also named as a subordinator starting from 0 with the Laplace transform

$$\mathbb{E}(e^{-\lambda D(t)}) = e^{-t\phi(\lambda)}, \quad \lambda > 0,$$

for the Laplace exponent

$$\phi(\lambda) := \int_0^\infty (1 - e^{-\lambda x})\mu(dx)$$

associated with a given σ-finite measure μ on $(0,\infty)$ such that $\int_0^\infty (1 \wedge x)\mu(dx) < \infty$. We assume that μ is infinite, i.e., $v(0,\infty) = \infty$. The time change E_t is continuous and nondecreasing, however, it is not Markovian. In this paper, we assume that B_t is independent of D_t. Without loss of generality, we further specify that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is defined (determined) by

$$\mathcal{F}_t := \bigcap_{s>t} \left\{ \sigma_1[B_r : 0 \leq r \leq s] \vee \sigma_2[E_r : r \geq 0] \right\}, \tag{1.3}$$

where the notation $\sigma_1 \vee \sigma_2$ denotes the σ-algebra generated by the union of the two σ-algebras σ_1 and σ_2. Magdziarz [29] named the composition $B \circ E = (B_{E_t})_{t\geq 0}$ as a time-changed Brownian motion, it is a square integrable martingale with respect to the filtration $\{\mathcal{G}_t\}_{t\geq 0}$ where $\mathcal{G}_t := \mathcal{F}_{E_t}, t \geq 0$. Without loss of generality, we simply assume that $b(t, E_t, 0, \delta_0) = 0$ and $\sigma(t, E_t, 0, \delta_0) = 0$ for all $t \geq 0$ and $E_t \in [0,\infty)$, where δ_0 denotes the Dirac measure at 0. Thus, Equation (1.2) has the trivial solution $X(t) = 0$ corresponding to the initial condition $X(0) = 0$.

The primary objectives of this paper is to establish the existence and uniqueness theorem of strong solutions for (1.2), and then to investigate stability of the solutions in several different senses in terms of Lyapunov functions. Our final objective is to derive an averaging principle to show that the solutions of (1.2) can be efficiently approximated by solutions of the associated averaged stochastic differential equations in mean square convergence.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries for this paper. In Section 3, we prove that there is a unique strong solution of Eq.(1.2) under some conditions. In addition, we extend the classical Itô's formula from stochastic differential equations to DDSDEs driven by time-changed Brownian motions, which is a powerful tool for our stability research. Based on the generalised Itô formula, in Section 4, we give sufficient conditions to ensure that the solutions of time-changed DDSDEs (1.2) is stable in several different senses, including stochastic stability, stochastically asymptotic stability and globally stochastically asymptotic stability, we also provide an example to illustrate the obtained results. As a related problem, in Section 5, under certain averaging condition, we show that the solutions of DDSDEs can be approximated by the solutions of the associated averaged DDSDEs in the sense of the mean square convergence.

2 Preliminaries

In this section, we briefly give preliminaries which will be used in the sequel. For technical reasons, we will work on the following subspace of $\mathcal{P}(\mathbb{R}^d)$ for any fixed $\theta \in [2, \infty)$

$$\mathcal{P}_\theta(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \mu(|\cdot|^\theta) := \int_{\mathbb{R}^d} |x|^\theta \mu(dx) < \infty \right\}$$

which is a Polish space under the L^θ-Wasserstein distance

$$\mathbb{W}_\theta(\mu_1, \mu_2) := \inf_{\pi \in \mathscr{C}(\mu_1,\mu_2)} \Big(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^\theta \pi(dx, dy) \Big)^{\frac{1}{\theta}}, \mu_1, \mu_2 \in \mathcal{P}_\theta(\mathbb{R}^d),$$

where $\mathscr{C}(\mu_1, \mu_2)$ is the set of all couplings for μ_1 and μ_2.

Note that for any $x \in \mathbb{R}^d$, the Dirac measure δ_x belongs to $\mathcal{P}_\theta(\mathbb{R}^d)$ for any $\theta \in [2, \infty)$ and if $\mu_1 = \mathscr{L}(X)$, $\mu_2 = \mathscr{L}(Y)$ are the corresponding distributions of random variables X and Y respectively, then

$$\mathbb{W}_\theta(\mu_1, \mu_2)^\theta \le \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^\theta \mathscr{L}((X, Y))(dx, dy) = \mathbb{E}|X - Y|^\theta, \tag{2.1}$$

in which $\mathscr{L}((X, Y))$ represents the joint distribution of the random pair (X, Y). For any $T > 0$, let $C([0, T]; \mathbb{R}^d)$ be the collection of all $\mathbb{R}^d$-valued continuous functions on $[0, T]$, endowing with the supremum norm. Furthermore, we let $\mathcal{S}^\theta(\Omega; C([0, T]; \mathbb{R}^d))$ be the totality of $C([0, T]; \mathbb{R}^d)$-valued random variables X satisfying $\mathbb{E}[\sup_{0\le t\le T} |X(t)|^\theta] < \infty$. Then, $\mathcal{S}^\theta(\Omega; C([0, T]; \mathbb{R}^d))$ is a Banach space under the norm

$$||X||_{\mathcal{S}^\theta} := (\mathbb{E}[\sup_{0\le t\le T} |X(t)|^\theta])^{\frac{1}{\theta}}.$$

Throughout this paper, the letter C will denote a positive constant whose value may change in different occasions. We will write the dependence of constant on parameters explicitly if it is essential. For convenience, we shall use $|\cdot|$ and $||\cdot||$ for norms of vectors and matrices, respectively. Let $\mathcal{C}^{1,1,2}([0, T] \times [0, \infty) \times \mathbb{R}^d)$ denote the family of all functions $V(t_1, t_2, x)$ from $[0, T] \times \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}$ which are continuously once differentiable in t_1 and t_2 as well as continuously twice differentiable in x. Let $\mathcal{K}$ denote the family of all nondecreasing functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(r) > 0$ for all $r > 0$. Also let $S_h := \{x \in \mathbb{R}^d : |x| < h\}$ and $\bar{S}_h := \{x \in \mathbb{R}^d : |x| \le h\}$ for any $h > 0$.

Now we recall the definition of L-derivative for functions (we can see Ren and Wang [38]).

Definition 2.1 *(1) Let functional* $V : \mathscr{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ *and* $\mu \in \mathscr{P}_2(\mathbb{R}^d)$*. If the functional*

$$L^2(\mathbb{R}^d, \mu; \mathbb{R}^d) \ni \phi \mapsto V(\mu \circ (I + \phi)^{-1}),$$

is Fréchet differentiable at $0 \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$*, that is, there is a unique* $\psi \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$ *such that*

$$\lim_{\mu(|\phi|^2)\to 0} \frac{V(\mu \circ (I + \phi)^{-1}) - V(\mu) - \mu(\langle \psi, \phi \rangle)}{\sqrt{\mu(|\phi|^2)}} = 0,$$

where $\mu(\phi) := \int_{\mathbb{R}^d} \phi(x)\mu(dx)$*. Then* V *is L-differentiable at* $\mu \in \mathscr{P}_2(\mathbb{R}^d)$*. Denote* $\partial_\mu V(\mu) := \psi$*, which is termed the L-derivative of* V *at* μ*.*

(2) Suppose that $V : \mathscr{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$*. If the L-derivative* $\partial_\mu V(\mu)$ *exists for every* $\mu \in \mathscr{P}_2(\mathbb{R}^d)$*, then we say that* V *is L-differentiable on* $\mathcal{P}_2(\mathbb{R}^d)$*.*

Definition 2.2 *(1) A function* $V : \mathscr{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ *is said to be in the space* $\mathcal{C}^2(\mathscr{P}_2(\mathbb{R}^d))$*, if* V *is L-differentiable on* $\mathscr{P}_2(\mathbb{R}^d)$*, and its derivative* $\partial_\mu V(y) : \mathscr{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \mapsto \mathbb{R}^d$ *is continuous at every* (μ, y) *for* $y \in supp(\mu)$*.* $\partial_\mu V(y)$ *is differentiable in* y *and* $\partial_y \partial_\mu V : \mathscr{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$ *is continuous at all* (μ, y) *satisfying* $y \in supp(\mu)$*. In addition,* $\partial_\mu^2 V : \mathscr{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$ *exists and is continuous.*

(2) Assume that $V : [0, T] \times [0, \infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$*. If* $V(\cdot, \cdot, \cdot, \mu)$ *is in* $\mathcal{C}^{1,1,2}([0, T] \times [0, \infty) \times \mathbb{R}^d)$ *for any* μ*, and* $V(t, E, X, \cdot) \in \mathcal{C}^2(\mathcal{P}_2(\mathbb{R}^d))$ *for every* (t, E, X)*, and all partial derivatives* $\partial_{t_1} V(t, E, X, \mu)$*,* $\partial_{t_2} V(t, E, X, \mu)$*,* $\partial_x V(t, E, X, \mu)$*,* $\partial_x^2 V(t, E, X, \mu)$*,* $\partial_\mu V(t, E, X, \mu)(y)$*,* $\partial_y \partial_\mu V(t, E, X, \mu)(y)$ *and* $\partial_\mu^2 V(t, E, X, \mu)(y, y')$ *are continuous with respect to* (t, E, X, μ)*,* (t, E, X, μ, y) *or* (t, E, X, μ, y, y')*. Then,* V *is said to be in the set* $\mathcal{C}^{1,1,2,(2)}([0, T] \times [0, \infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$*.*

(3) A function $V : [0, T] \times [0, \infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ *is said to be in* $\mathscr{C}_b([0, T] \times [0, \infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$*, if* $V \in \mathcal{C}^{1,1,2,(2)}([0, T] \times [0, \infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ *and all its derivatives are uniformly bounded on* $[0, T] \times [0, \infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$*. In addition, if* $V \in \mathscr{C}_b([0, T] \times [0, \infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ *and* $V \geq 0$*, we say that* $V \in \mathscr{C}_{b,+}([0, T] \times [0, \infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$*.*

Definition 2.3 *(1) The trivial solution of Eq. (1.2) is said to be stochastically stable or stable in probability if for every pair of* $\epsilon \in (0, 1)$ *and* $r > 0$*, there exists a* $\delta = \delta(\epsilon, r) > 0$ *such that*

$$\mathbb{P}\{|X(t, x_0)| < r \text{ for all } t \geq 0\} \geq 1 - \epsilon,$$

whenever $|x_0| < \delta$*.*

(2) The trivial solution of Eq. (1.2) is said to be stochastically asymptotically stable if it is stochastically stable and, moreover, for every $\epsilon \in (0, 1)$*, there exists a* $\delta_0 = \delta_0(\epsilon) > 0$ *such that*

$$\mathbb{P}\{\lim_{t\to\infty} X(t, x_0) = 0\} \geq 1 - \epsilon,$$

whenever $|x_0| < \delta_0$*.*

(3) The trivial solution of Eq. (1.2) is said to be globally stochastically asymptotically stable or stochastically asymptotically stable in the large if it is stochastically stable and for all $x_0 \in \mathbb{R}^d$*,*

$$\mathbb{P}\{\lim_{t\to\infty} X(t, x_0) = 0\} = 1.$$

3 The generalised Itô formula for DDSDEs driven by time-changed Brownian motions

In this section, we establish the existence and uniqueness of strong solutions to Eq.(1.2) under monotonicity condition. In addition, we extend the classical Itô formula from stochastic differential equations to DDSDEs driven by time-changed Brownian motions. Moreover, the Lyapunov functions not only contain state of the solution but also the distributions of the solution, which is the essential difference from the classical Lyapunov functions. It is a powerful tool for our study of stability.

In order to derive the main results of this section, we require that the coefficients $b(t_1, t_2, x, \mu)$ and $\sigma(t_1, t_2, x, \mu)$ satisfy the following assumptions.

Assumption 3.1 **(H1)** *The function b, σ satisfy for $t_1 \in [0, T]$, $t_2 \in [0, \infty)$ and (x, μ), $(y, v) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$*

$$2\langle x - y, b(t_1, t_2, x, \mu) - b(t_1, t_2, y, \nu)\rangle + ||\sigma(t_1, t_2, x, \mu) - \sigma(t_1, t_2, y, \nu)||^2 \leq K_1(|x-y|^2 + \mathbb{W}_2(\mu, v)^2),$$

where $K_1 > 0$ is a positive constant.
(H2) *The function b is bounded on bounded sets in $[0, T] \times [0, \infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ and for every $t \geq 0$, $b(t, E_t, x, \mu)$ is continuous on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$. Moreover, there exist a constant $K_2 > 0$ such that*

$$|b(t_1, t_2, x, \mu)|^2 + ||\sigma(t_1, t_2, x, \mu)||^2 \leq K_2(1 + |x|^2 + \mathbb{W}_2(\mu, \delta_0)^2),$$

where δ_0 denotes the Dirac measure at 0.
(H3)(Technical condition) *If a stochastic process $X(t)$ is right continuous with left limits and $\mathcal{G}_t$-adapted, then*

$$b(t, E_t, X(t), \mathscr{L}(X(t))), \sigma(t, E_t, X(t), \mathscr{L}(X(t))) \in \mathbb{L}(\mathcal{G}_t),$$

where $\mathbb{L}(\mathcal{G}_t)$ denotes the totality of càglàd (i.e., sample paths which are left continuous with right limits) and $\mathcal{G}_t$-adapted processes.

According to Jin and Kobayashi [20], we know that since the Brownian motion B and the subordinator D are assumed independent, it is possible to set up B and D on a product space with product measure $\mathbb{P} = \mathbb{P}_B \times \mathbb{P}_D$ with obvious notations. Let $\mathbb{E}_B, \mathbb{E}_D$ and $\mathbb{E}$ denote the expectations under the probability $\mathbb{P}_B$, $\mathbb{P}_D$ and $\mathbb{P}$, respectively.

Theorem 3.2 *Suppose that Assumption 3.1 holds. Then, for any initial value X_0 satisfying $\mathbb{E}|X_0|^2 < \infty$, there is a unique solution $X(t), t \in [0, T]$ of Equation (1.2). Moreover, this solution satisfies*

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X(t)|^2\right] \leq C.$$

Proof. We will split the proof in the following three steps.

(I) Existence.

We set the following iteration sequence: Let $X^0(t) = X_0$, $X^n(0) = X_0$, for all $n \geq 0$, $(X^{n+1}(t))_{t\in[0,T]}$ solve the DDSDE

$$X^{n+1}(t) = X_0 + \int_0^t b(s, E_s, X^{n+1}(s), \mathscr{L}(X^n(s)))dE_s + \int_0^t \sigma(s, E_s, X^{n+1}(s), \mathscr{L}(X^n(s)))dB_{E_s}, \tag{3.1}$$

for all $t \in [0, T]$, where $X^{n+1}(0) = X_0$.

First, we show $X^{n+1}(t)$ is bounded.

By Itô's formula, we have

$$\begin{aligned} d|X^{n+1}(t)|^2 = {} & 2\left\langle X^{n+1}(t), b(t, E_t, X^{n+1}(t), \mathscr{L}(X^n(t)))\right\rangle dE_t \\ & + \left\|\sigma(t, E_t, X^{n+1}(t), \mathscr{L}(X^n(t)))\right\|^2 dE_t \\ & + 2\left\langle X^{n+1}(t), \sigma(t, E_t, X^{n+1}(t), \mathscr{L}(X^n(t)))dB_{E_t}\right\rangle, \end{aligned} \tag{3.2}$$

then we get

$$\begin{aligned} & \mathbb{E}_B\big(\sup_{s\in[0,t]} |X^{n+1}(s)|^2\big) \\ & \leq \mathbb{E}_B|X_0|^2 + 2\mathbb{E}_B\left(\sup_{s\in[0,t]} \int_0^s \left\langle X^{n+1}(u), b(u, E_u, X^{n+1}(u), \mathscr{L}(X^n(u)))\right\rangle dE_u\right) \\ & \quad + \mathbb{E}_B\left(\int_0^t \left\|\sigma(s, E_s, X^{n+1}(s), \mathscr{L}(X^n(s)))\right\|^2 dE_s\right) \\ & \quad + 2\mathbb{E}_B\left(\sup_{s\in[0,t]} \Big[\int_0^s \left\langle X^{n+1}(u), \sigma(u, E_u, X^{n+1}(u), \mathscr{L}(X^n(u)))dB_{E_u}\right\rangle\Big]\right) \\ & =: \mathbb{E}_B|X_0|^2 + I_1 + I_2 + I_3. \end{aligned}$$

For term I_1, by Assumption 3.1 and Hölder inequality, we obtain

$$\begin{aligned} I_1 & \leq 2\mathbb{E}_B\left[\left(\int_0^t |X^{n+1}(u)|^2 dE_u\right)\left(\int_0^t |b(u, E_u, X^{n+1}(u), \mathscr{L}(X^n(u)))|^2 dE_u\right)\right]^{\frac{1}{2}} \\ & \leq \mathbb{E}_B\left(\int_0^t |X^{n+1}(u)|^2 dE_u\right) + \mathbb{E}_B\left(\int_0^t |b(u, E_u, X^{n+1}(u), \mathscr{L}(X^n(u)))|^2 dE_u\right) \\ & \leq \mathbb{E}_B\left(\int_0^t |X^{n+1}(u)|^2 dE_u\right) + K_2\mathbb{E}_B\left(\int_0^t (1 + |X^{n+1}(u)|^2 + \mathbb{W}_2(\mathscr{L}(X^n(u)), \delta_0)^2) dE_u\right) \\ & \leq K_2\mathbb{E}_B E_t + (1 + K_2)\mathbb{E}_B\left(\int_0^t \sup_{u\in[0,s]} |X^{n+1}(u)|^2 dE_s\right) + K_2\mathbb{E}_B\left(\int_0^t \mathbb{E}[\sup_{u\in[0,s]} |X^n(u)|^2] dE_s\right) \\ & \leq K_2 E_T + (1 + K_2)\int_0^t \mathbb{E}_B\big(\sup_{u\in[0,s]} |X^{n+1}(u)|^2\big) dE_s + K_2 E_t \mathbb{E}\big(\sup_{s\in[0,t]} |X^n(s)|^2\big), \end{aligned}$$

where we use the independence between B and E, meanwhile we apply (2.1) in the last inequality.

Next, for term I_2, by Assumption 3.1, (2.1) and the independence between B and E, we have

$$\begin{aligned} I_2 &\le K_2 \int_0^t \mathbb{E}_B(1+|X^{n+1}(s)|^2 + \mathbb{W}_2(\mathscr{L}(X^n(s)),\delta_0)^2) dE_s \\ &\le K_2 E_T + K_2 \int_0^t \mathbb{E}_B\big(\sup_{u\in[0,s]} |X^{n+1}(u)|^2\big) dE_s + K_2 E_t \mathbb{E}(\sup_{s\in[0,t]} |X^n(s)|^2). \end{aligned}$$

Besides, for term I_3, by Assumption 3.1, the Burkholder-Davis-Gundy inequality (Jin and Kobayashi [20]) and Hölder inequality we have

$$\begin{aligned} I_3 &\le 2\sqrt{32}\mathbb{E}_B\bigg(\int_0^t |X^{n+1}(u)|^2 ||\sigma(u,E_u,X^{n+1}(u),\mathscr{L}(X^n(u)))||^2 dE_u\bigg)^{\frac{1}{2}} \\ &\le \frac{1}{2}\mathbb{E}_B\big(\sup_{s\in[0,t]}|X^{n+1}(s)|^2\big) + 64\mathbb{E}_B\bigg(\int_0^t ||\sigma(u,E_u,X^{n+1}(u),\mathscr{L}(X^n(u)))||^2 dE_u\bigg) \\ &\le \frac{1}{2}\mathbb{E}_B\big(\sup_{s\in[0,t]}|X^{n+1}(s)|^2\big) + 64K_2\mathbb{E}_B\bigg(\int_0^t \big(1+|X^{n+1}(u)|^2 + \mathbb{W}_2(\mathscr{L}(X^n(u)),\delta_0)^2\big) dE_u\bigg) \\ &\le \frac{1}{2}\mathbb{E}_B\big(\sup_{s\in[0,t]}|X^{n+1}(s)|^2\big) + 64K_2 E_T \\ &\quad + 64K_2 \int_0^t \mathbb{E}_B(\sup_{u\in[0,s]}|X^{n+1}(u)|^2) dE_s + 64K_2 E_T \mathbb{E}(\sup_{s\in[0,t]}|X^n(u)|^2). \end{aligned}$$

Thus we obtain

$$\begin{aligned} \mathbb{E}_B(\sup_{s\in[0,t]}|X^{n+1}(s)|^2) \le C_{K_2,X_0}E_T + C_{K_2}\int_0^t \mathbb{E}_B(\sup_{u\in[0,s]}|X^{n+1}(u)|^2) dE_s \\ + C_{K_2}E_T\mathbb{E}(\sup_{s\in[0,t]}|X^n(s)|^2). \end{aligned}$$

For $n=0$, by the time-changed Gronwall's inequality [19] and $\mathbb{E}|X_0|^2<\infty$ we have

$$\mathbb{E}_B\big(\sup_{s\in[0,t]}|X^1(s)|^2\big) \le \big(C_{K_2,X_0} + C_{K_2}\mathbb{E}(|X_0|^2)\big) E_T e^{C_{K_2}E_T}.$$

Taking $\mathbb{E}_D$ on both sides, we have

$$\mathbb{E}\big(\sup_{s\in[0,t]}|X^1(s)|^2\big) \le \big(C_{K_2,X_0} + C_{K_2}\mathbb{E}(|X_0|^2)\big)\mathbb{E}(E_T e^{C_{K_2}E_T}).$$

The condition $\mu(0,\infty)=\infty$ guarantees that the inverse E of D has a finite exponential moment, i.e, for all $\lambda\in\mathbb{R}$, $t>0$

$$\mathbb{E}(e^{\lambda E_t}) < \infty.$$

For any $t\ge 0$ and $x>0$, by Markov's inequality, we have

$$P(E_t > s) \le P(D(s) < t) = P(e^{-xD(s)} \ge e^{-xt}) \le e^{xt}\mathbb{E}(e^{-xD(s)}) = e^{xt}e^{-s\phi(x)},$$

it follows that

$$\mathbb{E}(E_t)^2 = 2\int_0^\infty P(E_t>s) s ds \le 2e^{xt}\int_0^\infty e^{-s\phi(x)} s ds = 2e^{xt}\frac{1}{\phi^2(x)} < \infty,$$

then by Hölder inequality we have

$$\mathbb{E}[\sup_{s\in[0,t]}|X^1(s)|^2] \le (C_{K_2,X_0}+C_{K_2}\mathbb{E}|X_0|^2)(\mathbb{E}[E_T]^2)^{\frac{1}{2}}[\mathbb{E}e^{2C_{K_2}E_T}]^{\frac{1}{2}} < \infty.$$

Assume that the case of $n=k$ is right, we can prove that the assertion holds for $n=k+1$ by the same way. Therefore, we know that

$$\mathbb{E}\bigg[\sup_{0\le t\le T}|X^n(t)|^2\bigg] \le C. \tag{3.3}$$

Next, we show the sequence $\{X^{n+1}(t), t\in[0,T]\}_{n\ge 0}$ is Cauchy.

By Itô's formula we obtain

$$\begin{aligned} & d|X^{n+1}(t)-X^n(t)|^2 \\ &= 2\big\langle X^{n+1}(t)-X^n(t), b(t,E_t,X^{n+1}(t),\mathscr{L}(X^n(t)))-b(t,E_t,X^n(t),\mathscr{L}(X^{n-1}(t)))\big\rangle dE_t \\ &\quad + ||\sigma(t,E_t,X^{n+1}(t),\mathscr{L}(X^n(t)))-\sigma(t,E_t,X^n(t),\mathscr{L}(X^{n-1}(t)))||^2 dE_t \\ &\quad + 2\big\langle X^{n+1}(t)-X^n(t), \sigma(t,E_t,X^{n+1}(t),\mathscr{L}(X^n(t)))-\sigma(t,E_t,X^n(t),\mathscr{L}(X^{n-1}(t)))dB_{E_t}\big\rangle, \end{aligned}$$

by Itô's formula, Assumption 3.1, Burkholder-Davis-Gundy inequality (Jin and Kobayashi [20]) and (2.1) , we have

$$\begin{aligned} & \mathbb{E}_B\big(\sup_{s\in[0,t]}|X^{n+1}(s)-X^n(s)|^2\big) \\ &\le K_1\int_0^t \mathbb{E}_B\big[|X^{n+1}(s)-X^n(s)|^2+\mathbb{W}_2(\mathscr{L}(X^n(s)),\mathscr{L}(X^{n-1}(s)))^2\big]dE_s \\ &\quad + 2\sqrt{32}\mathbb{E}_B\bigg(\int_0^t |X^{n+1}(s)-X^n(s)|^2\big\|\sigma(s,E_s,X^{n+1}(s),\mathscr{L}(X^n(s)) \\ &\qquad\qquad - \sigma(s,E_s,X^n(s),\mathscr{L}(X^{n-1}(s)))\big\|^2 dE_s\bigg)^{\frac{1}{2}} \\ &\le K_1\int_0^t \mathbb{E}_B\big(\sup_{u\in[0,s]}|X^{n+1}(u)-X^n(u)|^2\big)dE_s \\ &\quad + K_1\int_0^t \mathbb{E}_B\Big(\mathbb{E}\big(\sup_{u\in[0,s]}|X^n(u)-X^{n-1}(u)|^2\big)\Big)dE_s \\ &\quad + \frac{1}{2}\mathbb{E}_B\big(\sup_{s\in[0,t]}|X^{n+1}(s)-X^n(s)|^2\big) \\ &\quad + 64\mathbb{E}_B\bigg(\int_0^t \big\|\sigma(s,E_s,X^{n+1}(s),\mathscr{L}(X^n(s)))-\sigma(s,E_s,X^n(s),\mathscr{L}(X^{n-1}(s)))\big\|^2 dE_s\bigg) \\ &\le \frac{1}{2}\mathbb{E}_B\big(\sup_{s\in[0,t]}|X^{n+1}(s)-X^n(s)|^2\big)+65K_1\int_0^t \mathbb{E}_B\big(\sup_{u\in[0,s]}|X^{n+1}(u)-X^n(u)|^2\big)dE_s \\ &\quad + 65K_1E_t\mathbb{E}\big(\sup_{s\in[0,t]}|X^n(s)-X^{n-1}(s)|^2\big). \end{aligned}$$

Hence, we have

$$\begin{aligned}\mathbb{E}_B\big(\sup_{s\in[0,t]}|X^{n+1}(s)-X^n(s)|^2\big)\leq C_{K_1}\int_0^t\mathbb{E}_B\big(\sup_{u\in[0,s]}|X^{n+1}(u)-X^n(u)|^2\big)dE_s\\+C_{K_1}E_t\mathbb{E}\big(\sup_{s\in[0,t]}|X^n(s)-X^{n-1}(s)|^2\big),\end{aligned}$$

therefore, by the time-changed Gronwall's inequality [19], it holds that

$$\mathbb{E}_B\big(\sup_{s\in[0,t]}|X^{n+1}(s)-X^n(s)|^2\big)\leq C_{K_1}E_te^{E_tC_{K_1}}\mathbb{E}\big(\sup_{s\in[0,t]}|X^n(s)-X^{n-1}(s)|^2\big).$$

Taking $\mathbb{E}_D$ on both sides, using (3.3) and Hölder inequality we have

$$\mathbb{E}\big(\sup_{s\in[0,t]}|X^{n+1}(s)-X^n(s)|^2\big)\leq C_{K_1}\mathbb{E}[E_te^{E_TC_{K_1}}]\mathbb{E}\big(\sup_{s\in[0,t]}|X^n(s)-X^{n-1}(s)|^2\big).$$

By Monotone convergence theorem we obtain

$$\begin{aligned}\lim_{t\to0}\mathbb{E}(E_t)^2&=2\lim_{t\to0}\int_0^\infty sP(E_t>s)ds\\&=2\int_0^\infty\lim_{t\to0}sP(E_t>s)ds\\&=0.\end{aligned}$$

Then we can take $t_0>0$ such that $C_{K_1}\mathbb{E}E_te^{E_TC_{K_1}}\leq C_{K_1}(\mathbb{E}[E_t]^2)^{\frac{1}{2}}(\mathbb{E}[e^{2E_TC_{K_1}}])^{\frac{1}{2}}\leq e^{-1}$, we arrive at

$$\begin{aligned}\mathbb{E}\big(\sup_{s\in[0,t_0]}|X^{n+1}(s)-X^n(s)|^2\big)&\leq e^{-n}\mathbb{E}\big(\sup_{s\in[0,t_0]}|X^1(s)-X_0|^2\big)\\&\leq 2e^{-n}\mathbb{E}\big(\sup_{s\in[0,t_0]}|X^1(s)|^2\big)+2e^{-n}\mathbb{E}|X_0|^2\\&\leq 4e^{-n}\mathbb{E}\big(\sup_{s\in[0,t_0]}|X^1(s)|^2\big),\end{aligned}$$

then we have

$$\lim_{n\to\infty}\mathbb{E}\big(\sup_{s\in[0,t_0]}|X^{n+1}(s)-X^n(s)|^2\big)=0.$$

Thus $\{X^{n+1}(t)\}_{t\in[0,t_0]}$ is a Cauchy sequence in $\mathcal{S}^2(\Omega;C([0,t_0];\mathbb{R}^d))$ and then the limit, denoted by $X(t)$. By (2.1) we obtain

$$\lim_{n\to\infty}\sup_{t\in[0,t_0]}\mathbb{W}_2(\mathscr{L}(X^n(t)),\mathscr{L}(X(t)))^2\leq\lim_{n\to\infty}\mathbb{E}\big(\sup_{t\in[0,t_0]}|X^n(t)-X(t)|^2\big)=0.$$

It follows from Assumption 3.1 and the dominated convergence theorem imply that $\mathbb{P}-a.s.$

$$X(t)=X_0+\int_0^tb(s,E_s,X(s),\mathscr{L}(X(s)))dE_s+\int_0^t\sigma(s,E_s,X(s),\mathscr{L}(X(s)))dB_{E_s},t\in[0,t_0].$$

Therefore, $\{X(t)\}_{t\in[0,t_0]}$ is a solution to (1.1). Since $t_0>0$ is independent of X_0, we conclude that (1.1) has a solution $\{X(t)\}_{t\in[0,T]}$ with

$$\mathbb{E}\big(\sup_{s\in[0,T]}|X(s)|^2\big)<\infty.$$

(II) Uniqueness.

Assume that $(X(t))_{t\in[0,T]}$ and $(Y(t))_{t\in[0,T]}$ are two solutions of Eq.(1.2) with $X(0)=Y(0)=X_0$, that is,

$$X(t)=X_0+\int_0^t b(s,E_s,X(s),\mathscr{L}(X(s)))dE_s+\int_0^t \sigma(s,E_s,X(s),\mathscr{L}(X(s)))dB_{E_s},$$

and

$$Y(t)=X_0+\int_0^t b(s,E_s,Y(s),\mathscr{L}(Y(s)))dE_s+\int_0^t \sigma(s,E_s,Y(s),\mathscr{L}(Y(s)))dB_{E_s}.$$

By Itô's formula, we have

$$\begin{aligned} d|X(t)-Y(t)|^2 &= 2\left\langle X(t)-Y(t), b(t,E_t,X(t),\mathscr{L}(X(t)))-b(t,E_t,Y(t),\mathscr{L}(Y(t)))\right\rangle dE_t\\ &\quad+\|\sigma(t,E_t,X(t),\mathscr{L}(X(t)))-\sigma(t,E_t,Y(t),\mathscr{L}(Y(t)))\|^2\,dE_t\\ &\quad+2\left\langle X(t)-Y(t),\sigma(t,E_t,X(t),\mathscr{L}(X(t)))-\sigma(t,E_t,Y(t),\mathscr{L}(Y(t)))dB_{E_t}\right\rangle. \end{aligned}$$

By Assumption 3.1 and Burkholder-Davis-Gundy inequality (Jin and Kobayashi [20]), we obtain

$$\begin{aligned} &\mathbb{E}_B\big(\sup_{0\le s\le t}|X(s)-Y(s)|^2\big)\\ &\le K_1\mathbb{E}_B\bigg(\int_0^t \big(|X(s)-Y(s)|^2+\mathbb{W}_2(\mathscr{L}(X(s)),\mathscr{L}(Y(s)))^2\big)dE_s\bigg)\\ &\quad+2\sqrt{32}\mathbb{E}_B\bigg(\int_0^t|X(s)-Y(s)|^2||\sigma(s,E_s,X(s),\mathscr{L}(X(s)))-\sigma(ts,E_s,Y(s),\mathscr{L}(Y(s)))||^2dE_s\bigg)^{\frac{1}{2}}\\ &\le 65K_1\int_0^t\mathbb{E}_B\big(\sup_{0\le u\le s}|X(u)-Y(u)|^2\big)dE_s+65K_1E_B\bigg(\int_0^t\mathbb{E}\big(\sup_{0\le u\le s}|X(u)-Y(u)|^2\big)dE_s\bigg)\\ &\quad+\frac{1}{2}\mathbb{E}_B\big(\sup_{0\le s\le t}|X(s)-Y(s)|^2\big)\\ &\le 65K_1\int_0^t\mathbb{E}_B\big(\sup_{0\le u\le s}|X(u)-Y(u)|^2\big)dE_s+65K_1E_t\mathbb{E}\big(\sup_{0\le s\le t}|X(s)-Y(s)|^2\big)\\ &\quad+\frac{1}{2}\mathbb{E}_B\big(\sup_{0\le s\le t}|X(s)-Y(s)|^2\big), \end{aligned}$$

by the time-changed Gronwall's inequality, it holds that

$$\mathbb{E}_B\big(\sup_{0\le s\le t}|X(s)-Y(s)|^2\big)\le C_{K_1}E_t\mathbb{E}\big(\sup_{0\le s\le t}|X(s)-Y(s)|^2\big)e^{C_{K_1}E_t}.$$

Taking $\mathbb{E}_D$ on both sides, we obtain

$$\mathbb{E}\big(\sup_{0\le s\le t}|X(s)-Y(s)|^2\big)\le C_{K_1}\mathbb{E}\big(\sup_{0\le s\le t}|X(s)-Y(s)|^2\big)\mathbb{E}E_te^{C_{K_1}E_t},$$

since $\lim_{t\to 0}\mathbb{E}E_te^{C_{K_1}E_t}=0$, we can take a $t_0>0$ such that $C_{K_1}\mathbb{E}E_te^{C_{K_1}E_t}<1$, then we have

$$\mathbb{E}\big(\sup_{0\le t\le t_0}|X(t)-Y(t)|^2\big)=0.$$

By the same steps we obtain

$$\begin{aligned}&\mathbb{E}_B\big(\sup_{t_0\le s\le t}|X(s)-Y(s)|^2\big)\\&\le 65K_1\int_{t_0}^t\mathbb{E}_B\big(\sup_{t_0\le u\le s}|X(u)-Y(u)|^2\big)dE_s+65K_1(E_t-E_{t_0})\mathbb{E}\big(\sup_{t_0\le s\le t}|X(s)-Y(s)|^2\big)\\&\quad+\frac{1}{2}\mathbb{E}_B\big(\sup_{t_0\le s\le t}|X(s)-Y(s)|^2\big),\end{aligned}$$

by the time-changed Gronwall's inequality, it holds that

$$\mathbb{E}_B\big(\sup_{t_0\le s\le t}|X(s)-Y(s)|^2\big)\le C_{K_1}(E_t-E_{t_0})\mathbb{E}\big(\sup_{t_0\le s\le t}|X(s)-Y(s)|^2\big)e^{C_{K_1}(E_t-E_{t_0})}.$$

Taking $\mathbb{E}_D$ on both sides, we obtain

$$\mathbb{E}\big(\sup_{t_0\le s\le t}|X(s)-Y(s)|^2\big)\le C_{K_1}\mathbb{E}\big[(E_t-E_{t_0})e^{C_{K_1}(E_t-E_{t_0})}\big]\mathbb{E}\big(\sup_{t_0\le s\le t}|X(s)-Y(s)|^2\big).$$

Since $\lim_{t\to t_0}C_{K_1}\mathbb{E}[(E_t-E_{t_0})e^{C_{K_1}(E_t-E_{t_0})}]=0$, we can take a $t_1>t_0$ such that

$$C_{K_1}\mathbb{E}[(E_t-E_{t_0})e^{C_{K_1}(E_t-E_{t_0})}]<1,$$

then we have

$$\mathbb{E}\big(\sup_{t_0\le t\le t_1}|X(t)-Y(t)|^2\big)=0.$$

Thus, repeat the above steps, we have

$$\mathbb{E}\big(\sup_{0\le t\le T}|X(t)-Y(t)|^2\big)=0.$$

■

Wu [46] provided an Itô formula for stochastic differential equations driven by time-changed Brownian motions. In the rest of this section, we will extend the Itô formula from SDEs to DDSDEs driven by time-changed Brownian motions. First of all, we strengthen the condition $(H1)$ to the following assumption:
(H1$'$) The function b, σ satisfy for $t_1\in[0,T]$, $t_2\in[0,\infty)$ and (x,μ), $(y,v)\in\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$

$$|b(t_1,t_2,x,\mu)-b(t_1,t_2,y,\nu)|^2+||\sigma(t_1,t_2,x,\mu)-\sigma(t_1,t_2,y,\nu)||^2\le K_1(|x-y|^2+\mathbb{W}_2(\mu,\nu)^2).$$

An interesting problem coming from the work is whether Itô formula can be further improved in the sense of allowing the coefficient b to satisfy the condition $(H1)$. Our techniques are currently not enough to give an affirmative answer. We will leave this topic for future work.

Proposition 3.3 *(The generalised Itô formula). Let us denote $\mu_t := \mathscr{L}(X(t))$. Suppose that the function $V : [0,\infty) \times \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ belongs to $\mathscr{C}_b([0,\infty) \times \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d))$. Then, under conditions (H1$'$), (H2) and (H3), for any $\mathcal{F}_0$-measurable random variable x_0 with $\mathbb{E}|x_0|^2 < \infty$, the following Itô formula holds*

$$\begin{aligned}
&V(E_t, X(t), \mu_t) - V(0, x_0, \mu_0)\\
&\quad = \int_0^t \Big[\partial_{t_2} V(E_s, X(s), \mu_0) + V_x(E_s, X(s), \mu_0) b(s, E_s, X(s), \mathscr{L}(X(s)))\\
&\qquad\qquad + \frac{1}{2} trace[V_{xx}(E_s, X(s), \mu_0)(\sigma\sigma^T)(s, E_s, X(s), \mathscr{L}(X(s)))]\Big] dE_s\\
&\quad + \mathbb{E}\Big[\int_0^t \partial_\mu V(E_t, X(t), \mu_s)(X(s)) b(s, E_s, X(s), \mathscr{L}(X(s)))\\
&\qquad\qquad + \frac{1}{2} trace[\partial_u \partial_\mu V(E_t, X(t), \mu_s)(X(s))(\sigma\sigma^T)(s, E_s, X(s), \mathscr{L}(X(s)))] dE_s\Big]\\
&\quad + \int_0^t V_x(E_s, X(s), \mu_0)\sigma(s, E_s, X(s), \mathscr{L}(X(s))) dB_{E_s}.
\end{aligned} \tag{3.4}$$

Proof. **Step I.** Suppose that both b and σ are bounded. Fix $(E, X) \in [0,\infty) \times \mathbb{R}^d$ and define $v(\mu) := V(E, X, \mu)$. Now, we prove the Itô formula for $v(\mathscr{L}(X(t)))$.
For any integer $N \geq 1$, the empirical projection of v onto $\mathbb{R}^d$ was defined as the function:

$$v^N : (\mathbb{R}^d)^N \ni (X_1(t), \cdots, X_N(t)) \mapsto v\left(\frac{1}{N}\sum_{i=1}^N \eta_{X_i(t)}\right) = v(\bar{\mu}_t^N),$$

where we use the notation η_x to denote the unit mass at the point $x \in \mathbb{R}^d$.
It follows from Proposition 3.1 in [5], we know that v^N is $\mathcal{C}^2$ on $\mathbb{R}^{d\times N}$ and

$$\begin{aligned}
&\partial_{X_i} v^N(X_1, \cdots, X_N) = \frac{1}{N}\partial_\mu v\left(\frac{1}{N}\sum_{l=1}^N \eta_{X_l}\right)(X_i),\\
&\partial^2_{X_i X_j} v^N(X_1, \cdots, X_N) = \frac{1}{N}\partial_u\left[\partial_\mu v\left(\frac{1}{N}\sum_{l=1}^N \eta_{X_l}\right)\right](X_i)\delta_{i,j} + \frac{1}{N^2}\partial_\mu^2 v\left(\frac{1}{N}\sum_{l=1}^N \eta_{X_l}\right)(X_i, X_j).
\end{aligned} \tag{3.5}$$

Denote by $\{X^\ell(t)\}_{t\in[0,T]}$ a sequence of independent identically distributed (i.i.d. for short) copies of $\{X(t)\}_{t\in[0,T]}$. That is, for any $\ell \geq 1$,

$$dX^\ell(t) = b(t, E_t, X^\ell(t), \mathscr{L}(X^\ell(t)))dE_t + \sigma(t, E_t, X^\ell(t), \mathscr{L}(X^\ell(t)))dB^\ell_{E_t}, \qquad t \in [0,T],$$

where $B^\ell(\cdot), \ell = 1, 2, \cdots, N$ are i.i.d. copies of $B(\cdot)$.

The Itô formula [[46], Lemma 3.2, Page 4], together with (3.5) yields, $\forall t \in [0,T], \mathbb{P}$-a.s.

$$
\begin{aligned}
v^N(X_1(t),\cdots,X_N(t)) &= v^N(X_1(0),\cdots,X_N(0)) \\
&+ \frac{1}{N}\sum_{\ell=1}^{N}\int_0^t \partial_\mu v(\bar{\mu}_s^N)(X^\ell(s))b(s,E_s,X^\ell(s),\mathscr{L}(X^\ell(s)))dE_s \\
&+ \frac{1}{N}\sum_{\ell=1}^{N}\int_0^t \partial_\mu v(\bar{\mu}_s^N)(X^\ell(s))\sigma(s,E_s,X^\ell(s),\mathscr{L}(X^\ell(s)))dB^\ell_{E_s} \\
&+ \frac{1}{2N}\sum_{\ell=1}^{N}\int_0^t trace[\partial_u\partial_\mu v(\bar{\mu}_s^N)(X^\ell(s))(\sigma\sigma^T)(s,E_s,X^\ell(s),\mathscr{L}(X^\ell(s)))]dE_s \\
&+ \frac{1}{2N^2}\sum_{\ell=1}^{N}\int_0^t trace[\partial_\mu^2 v(\bar{\mu}_s^N)(X^\ell(s),X^\ell(s))(\sigma\sigma^T)(s,E_s,X^\ell(s),\mathscr{L}(X^\ell(s)))]dE_s.
\end{aligned}
$$

Taking the expectation on both sides of this equality, we obtain that for $t \in [0,T]$

$$
\begin{aligned}
\mathbb{E}[v^N(X_1(t),\cdots,X_N(t))] &= \mathbb{E}[v^N(X_1(0),\cdots,X_N(0))] \\
&+ \frac{1}{N}\sum_{\ell=1}^{N}\mathbb{E}\bigg[\int_0^t \partial_\mu v(\bar{\mu}_s^N)(X^\ell(s))b(s,E_s,X^\ell(s),\mathscr{L}(X^\ell(s)))dE_s\bigg] \\
&+ \frac{1}{2N}\sum_{\ell=1}^{N}\mathbb{E}\bigg[\int_0^t trace[\partial_u\partial_\mu v(\bar{\mu}_s^N)(X^\ell(s))(\sigma\sigma^T)(s,E_s,X^\ell(s),\mathscr{L}(X^\ell(s)))]dE_s\bigg] \\
&+ \frac{1}{2N^2}\sum_{\ell=1}^{N}\mathbb{E}\bigg[\int_0^t trace[\partial_\mu^2 v(\bar{\mu}_s^N)(X^\ell(s),X^\ell(s))(\sigma\sigma^T)(s,E_s,X^\ell(s),\mathscr{L}(X^\ell(s)))]dE_s\bigg].
\end{aligned}
$$

Using the fact that the processes $\{(X^\ell(t))_{t\in[0,T]}\}_{\ell\in\{1,\cdots,N\}}$ are i.i.d., we deduce that

$$
\begin{aligned}
\mathbb{E}[v^N(X_1(t),\cdots,X_N(t))] &= \mathbb{E}[v^N(X_1(0),\cdots,X_N(0))] \\
&+ \mathbb{E}\bigg[\int_0^t \partial_\mu v(\bar{\mu}_s^N)(X^1(s))b(s,E_s,X^1(s),\mathscr{L}(X^1(s)))dE_s\bigg] \\
&+ \frac{1}{2}\mathbb{E}\bigg[\int_0^t trace[\partial_u\partial_\mu v(\bar{\mu}_s^N)(X^1(s))(\sigma\sigma^T)(s,E_s,X^1(s),\mathscr{L}(X^1(s)))]dE_s\bigg] \\
&+ \frac{1}{2N}\mathbb{E}\bigg[\int_0^t trace[\partial_\mu^2 v(\bar{\mu}_s^N)(X^1(s),X^1(s))(\sigma\sigma^T)(s,E_s,X^1(s),\mathscr{L}(X^1(s)))]dE_s\bigg].
\end{aligned}
$$

Next, we take the limit on both sides of the above equality. Note that for any $t \in [0,T]$,

$$
\mathbb{P}\bigg[\lim_{N\to\infty}\mathbb{W}_2(\bar{\mu}_t^N,\mu_t)^2 = 0\bigg] = 1.
$$

Then as $N \to \infty$, by continuity and boundedness of v, $\partial_\mu v$, $\partial_u\partial_\mu v$, and boundedness of $\partial_\mu^2 v$, b and σ, it follows from the dominated convergence theorem that

$$
\begin{aligned}
v(\mathscr{L}(X(t))) &= v(\mathscr{L}(X(0))) \\
&+ \mathbb{E}\bigg[\int_0^t \partial_\mu v(\mathscr{L}(X(s)))(X^1(s))b(s,E_s,X^1(s),\mathscr{L}(X^1(s)))dE_s\bigg] \\
&+ \frac{1}{2}\mathbb{E}\bigg[\int_0^t trace[\partial_u\partial_\mu v(\mathscr{L}(X(s)))(X^1(s))(\sigma\sigma^T)(s,E_s,X^1(s),\mathscr{L}(X^1(s)))]dE_s\bigg].
\end{aligned}
\tag{3.6}
$$

Step II. Supposed that (H1$'$), (H2) and (H3) hold. We will derive the Itô formula for $v(\mathscr{L}(X(t)))$.

For each $n\in\mathbb{N}$, we take a smooth function $\phi_n:\mathbb{R}^d\mapsto\mathbb{R}^d$ fulfilling

$$\begin{cases}\phi_n(x)=x, & |x|\le n,\\ \phi_n(x)=0, & |x|>2n,\end{cases}$$

such that

$$|\phi_n(x)|\le C,\quad |\partial\phi_n(x)|\le C$$

for some constant $C>0$. Set

$$b^{(n)}(t,r,x,\mu):=b(t,r,\phi_n(x),\mu),\quad \sigma^{(n)}(t,r,x,\mu):=\sigma(t,r,\phi_n(x),\mu).$$

Then as $n\to\infty$,

$$b^{(n)}(t,r,x,\mu)\to b(t,r,x,\mu),\quad \sigma^{(n)}(t,r,x,\mu)\to\sigma(t,r,x,\mu).$$

Moreover, by condition (H2), we know that $b^{(n)}$ and $\sigma^{(n)}$ are bounded, and both $b^{(n)}$ and $\sigma^{(n)}$ satisfy (H1$'$), (H3). Therefore, the following equation

$$dX^{(n)}(t)=b^{(n)}(t,E_t,X^{(n)}(t),\mathscr{L}(X^{(n)}(t)))dE_t+\sigma^{(n)}(t,E_t,X^{(n)}(t),\mathscr{L}(X^{(n)}(t)))dB_{E_t},\tag{3.7}$$

has a unique solution $X^{(n)}(t)$, where $X^n(0)=X(0)$. Then by **Step I**, it holds that for $t\in[0,T]$,

$$\begin{aligned}&v(\mathscr{L}(X^{(n)}(t)))=v(\mathscr{L}(X^{(n)}(0)))\\&\quad+\mathbb{E}\Big[\int_0^t\partial_\mu v(\mathscr{L}(X^{(n)}(s)))(X^{(n)}(s))b(s,E_s,X^{(n)}(s),\mathscr{L}(X^{(n)}(s)))dE_s\Big]\\&\quad+\frac{1}{2}\mathbb{E}\Big[\int_0^t trace[\partial_u\partial_\mu v(\mathscr{L}(X^{(n)}(s)))(X^{(n)}(s))(\sigma\sigma^T)(s,E_s,X^{(n)}(s),\mathscr{L}(X^{(n)}(s)))]dE_s\Big].\end{aligned}\tag{3.8}$$

Applying the Itô formula and Burkholder-Davis-Gundy inequality (Jin and Kobayashi [20]) we have

$$\begin{aligned}&\mathbb{E}_B\big(\sup_{0\le s\le t}|X^{(n)}(s)-X(s)|^2\big)\\&\le 2\mathbb{E}_B\bigg(\sup_{0\le s\le t}\int_0^s\big\langle X^n(r)-X(r),b^{(n)}(r,E_r,X^{(n)}(r),\mathscr{L}(X^{(n)}(r)))\\&\qquad\qquad-b(r,E_r,X(r),\mathscr{L}(X(r)))\big\rangle dE_r\bigg)\\&\quad+2\mathbb{E}_B\bigg(\sup_{0\le s\le t}\int_0^s\big\langle X^n(r)-X(r),\sigma^{(n)}(r,E_r,X^{(n)}(r),\mathscr{L}(X^{(n)}(r)))\\&\qquad\qquad-\sigma(r,E_r,X(r),\mathscr{L}(X(r)))dB_{E_r}\big\rangle\bigg)\\&\quad+\mathbb{E}_B\bigg(\int_0^t||\sigma^{(n)}(r,E_r,X^{(n)}(r),\mathscr{L}(X^{(n)}(r)))-\sigma(r,E_r,X^{(n)}(r),\mathscr{L}(X^{(n)}(r)))||^2dE_r\bigg)\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\mathbb{E}_B \int_0^t |b^{(n)}(r, E_r, X^{(n)}(r), \mathscr{L}(X^{(n)}(r))) - b(r, E_r, X(r), \mathscr{L}(X(r)))|^2 dE_r\right) \\
&\quad + \mathbb{E}_B\bigg(\int_0^t ||\sigma^{(n)}(r, E_r, X^{(n)}(r), \mathscr{L}(X^{(n)}(r))) - \sigma(r, E_r, X(r), \mathscr{L}(X(r)))||^2 dE_r\bigg) \\
&\quad + 2\sqrt{32}\mathbb{E}_B\bigg(\int_0^t |X^{(n)}(r) - X(r)|^2 ||\sigma^{(n)}(r, E_r, X^{(n)}(r), \mathscr{L}(X^{(n)}(r))) \\
&\qquad\qquad\qquad - \sigma(r, E_r, X(r), \mathscr{L}(X(r)))||^2 dE_r\bigg)^{\frac{1}{2}} \\
&\quad + \mathbb{E}_B\bigg(\int_0^t |X^{(n)}(r) - X(r)|^2 dE_r\bigg) \\
&\leq C_{K_1}\mathbb{E}_B\bigg(\int_0^t |\phi_n(X^{(n)}(s)) - X(s)|^2 dE_s\bigg) + C_{K_1}E_t\mathbb{E}\big(\sup_{0\leq s\leq t}|X^{(n)}(s) - X(s)|^2\big) \\
&\quad + \int_0^t \mathbb{E}_B\big(\sup_{0\leq r\leq s}|X^{(n)}(r) - X(r)|^2\big)dE_s + \frac{1}{2}\mathbb{E}_B\big(\sup_{0\leq s\leq t}|X^{(n)}(s) - X(s)|^2\big),
\end{aligned}
$$

by Gronwall inequality we have

$$
\begin{aligned}
\mathbb{E}_B\big(\sup_{0\leq s\leq t}|X^{(n)}(t) - X(t)|^2\big) \leq \bigg(&C_{K_1}\mathbb{E}_B\big(\int_0^t |\phi_n(X^{(n)}(s)) - X(s)|^2 dE_s\big) \\
&+ C_{K_1}E_t\mathbb{E}\big(\sup_{0\leq s\leq t}|X^{(n)}(s) - X(s)|^2\big)\bigg)e^{E_t}.
\end{aligned}
$$

Using $\phi_n(X)\to X$ for $X \in \mathbb{R}^d$, and the estimate $\mathbb{E}\bigg(\sup_{0\leq t\leq T}|X(t)|^2\bigg) < +\infty$, it comes from the dominated convergence theorem that

$$
\overline{\lim_{n\to\infty}} \int_0^t \mathbb{E}_B\big(|\phi_n(X(s)) - X(s)|^2\big)dE_s = 0.
$$

Then by Fatou lemma we obtain

$$
\begin{aligned}
\overline{\lim_{n\to\infty}}\,\mathbb{E}\big(\sup_{0\leq s\leq t}|X^{(n)}(s) - X(s)|^2\big) &\leq \mathbb{E}_D\,\overline{\lim_{n\to\infty}}\,\mathbb{E}_B\big(\sup_{0\leq s\leq t}|X^{(n)}(s) - X(s)|^2\big) \\
&\leq C_{K_1}\mathbb{E}E_t e^{E_t}\,\overline{\lim_{n\to\infty}}\,\mathbb{E}\big(\sup_{0\leq s\leq t}|X^{(n)}(s) - X(s)|^2\big).
\end{aligned}
$$

Using the method of uniqueness proof, we get

$$
\overline{\lim_{n\to\infty}}\,\mathbb{E}\big(\sup_{0\leq s\leq t}|X^{(n)}(s) - X(s)|^2\big) = 0.
$$

Moreover, we obtain that

$$
\mathbb{W}_2(\mathscr{L}(X^{(n)}(t)), \mathscr{L}(X(t)))^2 \leq \mathbb{E}|X^{(n)}(t) - X(t)|^2 = 0.
$$

Taking the limit on two sides of (3.8), by the dominated convergence theorem, one can conclude (3.6).

Step III. We prove the Itô formula (3.4). By the Itô formula [[46], Lemma 3.2, Page4] and (3.6), it holds that

$$
\begin{aligned}
&V(E_t, X(t), \mu_t) - V(0, x_0, \mu_0) \\
&\quad = V(E_t, X(t), \mu_t) - V(E_t, X(t), \mu_0) + V(E_t, X(t), \mu_0) - V(0, x_0, \mu_0) \\
&\quad = \int_0^t \Big[\partial_{t_2} V(E_s, X(s), \mu_0) + V_x(E_s, X(s), \mu_0) b(s, E_s, X(s), \mathscr{L}(X(s))) \\
&\qquad\qquad + \frac{1}{2} trace[V_{xx}(E_s, X(s), \mu_0)(\sigma\sigma^T)(s, E_s, X(s), \mathscr{L}(X(s)))] \Big] dE_s \\
&\quad + \mathbb{E}\Big[\int_0^t \partial_\mu V(E_t, X(t), \mu_s)(X(s)) b(s, E_s, X(s), \mathscr{L}(X(s))) \\
&\qquad\qquad + \frac{1}{2} trace[\partial_u \partial_\mu V(E_t, X(t), \mu_s)(X(s))(\sigma\sigma^T)(s, E_s, X(s), \mathscr{L}(X(s)))] dE_s \Big] \\
&\quad + \int_0^t V_x(E_s, X(s), \mu_0) \sigma(s, E_s, X(s), \mathscr{L}(X(s))) dB_{E_s}.
\end{aligned}
$$

■

Remark 3.1 *(1) When the time-change E_t degenerates into time t, the generalised Itô formula becomes the Itô formula for distribution-dependent functions provided by [[4], Proposition 5.102, p.485] and [[13], Proposition A.8].*

(2) If V is independent of the measure μ, then $V \in \mathscr{C}_b([0,\infty) \times \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d))$ is simplified to $\mathcal{C}^{1,2}([0,\infty) \times \mathbb{R}^d)$ and revelent Itô formula is reduced to the standard Itô formula provided by [46].

4 Stability results

As we know, the stability of solutions of stochastic differential equations has always been a very important problem. Lyapunov method can help us study various stability of solutions, but it often needs the help of Itô formula. In the previous section, we have already addressed generalisation of relevant Itô formula, so in this section, we will study the stability of the solutions of the time-changed DDSDEs (1.2). Using the Lyapunov method, some sufficient criteria are proposed to derive stochastic stability, stochastically asymptotical stability and globally stochastically asymptotical stability of the solutions, respectively. Without loss of generality, we consider the initial value to be a deterministic constant $X(0) = x_0 \in \mathbb{R}^d$. The initial value being a random variable seems more general but in effect it is equivalent to having a deterministic constant initial value.

Theorem 4.1 *Suppose that there exist a function $V(t_2, x, \mu) \in \mathscr{C}_{b,+}([0,\infty) \times S_h \times \mathscr{P}_2(\mathbb{R}^d))$ and another function $\varphi \in \mathcal{K}$ such that for all $(t_2, x, \mu) \in [0,\infty) \times S_h \times \mathscr{P}_2(\mathbb{R}^d)$:*

$$
\begin{aligned}
&(1) V(t_2, 0, \delta_0) = 0, \\
&(2) \varphi(|x|) \leq V(t_2, x, \mu) \ for\ all\ (t_2, x, \mu) \in [0,\infty) \times S_h \times \mathscr{P}_2(\mathbb{R}^d), \\
&(3) L^\mu V(t_2, x, \mu) \leq 0.
\end{aligned}
\tag{4.1}
$$

Then the trivial solution of Equ.(1.2) is stochastically stable or stable in probability, where $L^\mu V$ is defined as follows:

$$
\begin{aligned}
&L^\mu V(t_2,x,\mu)\\
&:= \partial_{t_2} V(t_2,x,\mu_0) + V_x(t_2,x,\mu_0) b(t,t_2,x,\mu)\\
&\quad + \frac{1}{2} trace[V_{xx}(t_2,x,\mu_0)(\sigma\sigma^T)(t,t_2,x,\mu)]\\
&\quad + \partial_\mu V(E_{t_1}, X(E_{t_1}),\mu)(x) b(t,t_2,x,\mu)\\
&\quad + \frac{1}{2} trace[\partial_u \partial_\mu V(E_{t_1}, X(E_{t_1}),\mu)(x)(\sigma\sigma^T)(t,t_2,x,\mu)].
\end{aligned}
$$

Proof. Choose $\epsilon \in (0,1)$ and $0 < r < h$ arbitrary. Owing to the continuity of $V(t_2,x,\mu)$ on $[0,\infty) \times S_h \times \mathscr{P}_2(\mathbb{R}^d)$ and $V(t_2,0,\delta_0) = 0$, there exists a $\theta = \theta(\epsilon, r) > 0$ such that

$$
\frac{1}{\epsilon} \sup_{x_0 \in S_\theta} V(0,x_0,\mu_0) \le \varphi(r). \tag{4.2}
$$

From (4.2) and the condition (2), we get

$$
\theta < r.
$$

Fix any initial value $x_0 \in S_\theta$ and define the following stopping time

$$
\tau_r := \inf\{t \ge 0 : |X(t,x_0)| \ge r\} \tag{4.3}
$$

and

$$
U_k := k \wedge \inf\{t \ge 0 : |\int_0^{\tau_r \wedge t} V_x(E_s, X(s),\mu_0)\sigma(s,E_s,X(s),\mathscr{L}(X(s)))dB_{E_s}| \ge k\} \tag{4.4}
$$

for $k = 1,2,\cdots$. Obviously, $U_k \to \infty$ as $k \to \infty$. Applying the generalised Itô formula in Proposition 3.3 to Eq.(1.2) yields

$$
\begin{aligned}
&V(E_{\tau_r \wedge U_k}, X(\tau_r \wedge U_k), \mu_{\tau_r \wedge U_k}) - V(0,x_0,\mu_0)\\
&= \int_0^{\tau_r \wedge U_k} \Big[\partial_{t_2} V(E_s, X(s),\mu_0) + V_x(E_s,X(s),\mu_0) b(s,E_s,X(s),\mathscr{L}(X(s)))\\
&\qquad + \frac{1}{2} trace[V_{xx}(E_s,X(s),\mu_0)(\sigma\sigma^T)(s,E_s,X(s),\mathscr{L}(X(s)))]\Big] dE_s\\
&\quad + \mathbb{E}\Big[\int_0^{\tau_r \wedge U_k} \partial_\mu V(E_{\tau_r \wedge U_k}, X(\tau_r \wedge U_k),\mu_s)(X(s)) b(s,E_s,X(s),\mathscr{L}(X(s)))\\
&\qquad + \frac{1}{2} trace[\partial_u\partial_\mu V(E_{\tau_r \wedge U_k}, X(\tau_r \wedge U_k),\mu_s)(X(s))(\sigma\sigma^T)(s,E_s,X(s),\mathscr{L}(X(s)))] dE_s\Big]\\
&\quad + \int_0^{\tau_r \wedge U_k} V_x(E_s,X(s),\mu_0)\sigma(s,E_s,X(s),\mathscr{L}(X(s)))dB_{E_s}.
\end{aligned} \tag{4.5}
$$

By Kuo [24] and Magdziarz [29], we conclude that

$$
\int_0^t V_x(E_s,X(s),\mu_0)\sigma(s,E_s,X(s),\mathscr{L}(X(s)))dB_{E_s}, \quad t \ge 0
$$

is a mean 0, square integrable martingale with respect the filtration $\mathcal{G}_t = \mathcal{F}_{E_t}$. With the condition (3) and taking expectations on both sides of Eq.(4.5), we have

$$\mathbb{E}V(E_{\tau_r \wedge U_k}, X(\tau_r \wedge U_k), \mu_{\tau_r \wedge U_k}) \leq V(0, x_0, \mu_0). \tag{4.6}$$

Let $k \to \infty$, then

$$\mathbb{E}V(E_{\tau_r}, X(\tau_r), \mu_{\tau_r}) \leq V(0, x_0, \mu_0).$$

Since $V(t_2, x, \mu)$ is a nonnegative function,

$$\mathbb{E}\left\{V(E_{\tau_r}, X(\tau_r), \mu_{\tau_r})1_{\{\tau_r<\infty\}}\right\} \leq \mathbb{E}V(E_{\tau_r}, X(\tau_r), \mu_{\tau_r}) \leq V(0, x_0, \mu_0). \tag{4.7}$$

By the condition (2), we have

$$\mathbb{E}\left\{\varphi(|X(\tau_r|)1_{\{\tau_r<\infty\}}\right\} \leq \mathbb{E}\left\{V(E_{\tau_r}, X(\tau_r), \mu_{\tau_r})1_{\{\tau_r<\infty\}}\right\}. \tag{4.8}$$

Also since the function φ is nondecreasing, by (4.3) and (4.8) we have

$$|X(\tau_r)| \geq r$$

and

$$\varphi(r)\mathbb{P}\left\{\tau_r < \infty\right\} \leq \mathbb{E}\left\{V(E_{\tau_r}, X(\tau_r), \mu_{\tau_r})1_{\{\tau_r<\infty\}}\right\}. \tag{4.9}$$

Combine (4.2), (4.7) and (4.9), we have

$$\mathbb{P}\left\{\tau_r < \infty\right\} \leq \epsilon,$$

which implies

$$\mathbb{P}\left\{\tau_r = \infty\right\} \geq 1 - \epsilon.$$

Equivalently,

$$\mathbb{P}\{|X(t, x_0)| < r \ \textit{for all}\ t \geq 0\} \geq 1 - \epsilon,$$

so $X(t, x_0)$ is stochastically stable. This completes the proof. ∎

Theorem 4.2 *Suppose that there exist a function $V(t_2, x, \mu) \in \mathscr{C}_{b,+}([0, \infty) \times S_h \times \mathscr{P}_2(\mathbb{R}^d))$ and another function $\varphi \in \mathcal{K}$ such that*

$$\begin{aligned} &(1) V(t_2, 0, \delta_0) = 0, \\ &(2) \varphi(|x|) \leq V(t_2, x, \mu) \ \textit{for all}\ (t_2, x, \mu) \in [0, \infty) \times S_h \times \mathscr{P}_2(\mathbb{R}^d), \\ &(3) \textit{for any}\ \alpha \in (0, h),\ x \in S_h - \bar{S}_\alpha,\ L^\mu V(t_2, x, \mu) \leq -\gamma(\alpha)\ a.s., \end{aligned} \tag{4.10}$$

where $\gamma(\alpha) \geq 0$, and $\gamma(\alpha) \neq 0$ when $\alpha \neq 0$. Then the trivial solution of Eq.(1.2) is stochastically asymptotically stable.

Proof. According to Theorem 4.1, the trivial solution of Eq.(1.2) is stochastically stable. Thus, for any fixed $\epsilon \in (0,1)$, there exists a $\theta = \theta(\epsilon) > 0$ such that

$$\mathbb{P}\{|X(t,x_0)| < h \ for\ all\ t \geq 0\} \geq 1 - \frac{\epsilon}{5}, \tag{4.11}$$

when $x_0 \in S_\theta$. Fix $x_0 \in S_\theta$ and put $0 < \alpha < \beta < |x_0|$ arbitrarily. And we define the stopping times as follows:

$$\begin{aligned} &\tau_h := \inf\{t \geq 0 : |X(t,x_0)| \geq h\}, \\ &\tau_\alpha := \inf\{t \geq 0 : |X(t,x_0)| \leq \alpha\}, \\ &U_k := k \wedge \inf\{t \geq 0 : |\int_0^{\tau_h \wedge \tau_\alpha \wedge t} V_x(E_s, X(s), \mu_0)\sigma(s, E_s, X(s), \mathscr{L}(X(s)))dB_{E_s}| \geq k\} \end{aligned} \tag{4.12}$$

for $k = 1, 2, \cdots$. $U_k \to \infty$ as $k \to \infty$, obviously. Applying the generalised Itô formula in Proposition 3.3 to Eq.(1.2) yields

$$\begin{aligned} 0 &\leq \mathbb{E}V(E_{t\wedge\tau_h\wedge\tau_\alpha\wedge U_k}, X(t\wedge\tau_h\wedge\tau_\alpha\wedge U_k), \mu_{t\wedge\tau_h\wedge\tau_\alpha\wedge U_k}) \\ &= V(0, x_0, \mu_0) + \mathbb{E}\int_0^{t\wedge\tau_h\wedge\tau_\alpha\wedge U_k} L^\mu V(E_s, X(s), \mu_s)dE_s \\ &\leq V(0, x_0, \mu_0) - \gamma(\alpha)\mathbb{E}E_{t\wedge\tau_h\wedge\tau_\alpha\wedge U_k}, \end{aligned}$$

while for the last inequality we use the condition (3). Let $k \to \infty$ and $t \to \infty$, we have

$$\gamma(\alpha)\mathbb{E}E_{\tau_h\wedge\tau_\alpha} \leq V(0, x_0, \mu_0).$$

Since $E_t \to \infty$ as $t \to \infty$ and $\gamma(\alpha) \neq 0$,

$$\mathbb{P}\Big\{\tau_h\wedge\tau_\alpha < \infty\Big\} = 1. \tag{4.13}$$

From (4.11) and (4.12)

$$\mathbb{P}\Big\{\tau_h < \infty\Big\} \leq \frac{\epsilon}{5}. \tag{4.14}$$

Thus, we have

$$1 = \mathbb{P}\Big\{\tau_h\wedge\tau_\alpha < \infty\Big\} \leq \mathbb{P}\Big\{\tau_h < \infty\Big\} + \mathbb{P}\Big\{\tau_\alpha < \infty\Big\} \leq \mathbb{P}\Big\{\tau_\alpha < \infty\Big\} + \frac{\epsilon}{5}.$$

Consequently,

$$\mathbb{P}\Big\{\tau_\alpha < \infty\Big\} \geq 1 - \frac{\epsilon}{5}.$$

So we can find a positive constant $\bar{\theta} = \bar{\theta}(\alpha)$ such that

$$\mathbb{P}\Big\{\tau_\alpha < \bar{\theta}\Big\} \geq 1 - \frac{2\epsilon}{5}.$$

Besides,

$$
\begin{aligned}
&\mathbb{P}\Big\{\tau_\alpha < \tau_h \wedge \bar\theta\Big\} \\
&\geq \mathbb{P}\Big\{\{\tau_\alpha < \bar\theta\} \bigcap \{\tau_h = \infty\}\Big\} = \mathbb{P}\Big\{\tau_\alpha < \bar\theta\Big\} - \mathbb{P}\Big\{\{\tau_\alpha < \bar\theta\} \bigcap \{\tau_h < \infty\}\Big\} \\
&\geq \mathbb{P}\Big\{\tau_\alpha < \bar\theta\Big\} - \mathbb{P}\Big\{\tau_h < \infty\Big\} \geq 1 - \frac{3\epsilon}{5}.
\end{aligned}
\tag{4.15}
$$

Next, we define stopping times as following

$$
\pi := \begin{cases} \tau_\alpha, & if\ \tau_\alpha < \tau_h \wedge \bar\theta, \\ \infty, & otherwise. \end{cases} \tag{4.16}
$$

$$
\begin{aligned}
&\tau_\beta := \inf\{t \geq \pi : |X(t, x_0)| \geq \beta\}, \\
&\tau_i := \inf\{t \geq \pi : |\int_\pi^{\tau_\beta \wedge t} V_x(E_s, X(s), \mu_0)\sigma(s, E_s, X(s), \mathscr{L}(X(s)))dB_{E_s}| \geq i\}
\end{aligned}
\tag{4.17}
$$

for $i = 1, 2, \cdots$. Similarly, we have $\tau_i \to \infty$ as $i \to \infty$. With condition (3) we use the generalised Itô formula again

$$
\mathbb{E}V(E_{\pi\wedge\tau_h\wedge t}, X(\pi \wedge \tau_h \wedge t), \mu_{\pi\wedge\tau_h\wedge t}) \geq \mathbb{E}V(E_{\tau_\beta\wedge\tau_i\wedge\tau_h\wedge t}, X(\tau_\beta \wedge \tau_i \wedge \tau_h \wedge t), \mu_{\tau_\beta\wedge\tau_i\wedge\tau_h\wedge t})
$$

for all $i = 1, 2, \cdots$, and $t \geq 0$. Let $i \to \infty$ and then $t \to \infty$,

$$
\mathbb{E}V(E_{\pi\wedge\tau_h}, X(\pi \wedge \tau_h), \mu_{\pi\wedge\tau_h}) \geq \mathbb{E}V(E_{\tau_\beta\wedge\tau_h}, X(\tau_\beta \wedge \tau_h), \mu_{\tau_\beta\wedge\tau_h}). \tag{4.18}
$$

Combine (4.16), (4.17) and (4.18),

$$
\mathbb{E}\Big\{V(E_{\pi\wedge\tau_h}, X(\pi \wedge \tau_h), \mu_{\pi\wedge\tau_h})1_{\{\pi<\infty\}}\Big\} \geq \mathbb{E}\Big\{V(E_{\tau_\beta\wedge\tau_h}, X(\tau_\beta \wedge \tau_h), \mu_{\tau_\beta\wedge\tau_h})1_{\{\tau_\beta<\infty\}}\Big\}, \tag{4.19}
$$

which indicates from (4.15) that

$$
\mathbb{E}\Big\{V(E_{\tau_\alpha}, X(\tau_\alpha), \mu_{\tau_\alpha})1_{\{\tau_\alpha<\tau_h\wedge\bar\theta\}}\Big\} \geq \mathbb{E}\Big\{V(E_{\tau_\beta}, X(\tau_\beta), \mu_{\tau_\beta})1_{\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\}}\Big\}. \tag{4.20}
$$

Furthermore, we define

$$
B_\alpha := \sup\Big\{V(t_2, x, \mu) : (t_2, x, \mu) \in [0, \infty) \times \bar S_\alpha \times \mathscr{P}_2(\mathbb{R}^d)\Big\}. \tag{4.21}
$$

By condition (1) we have $\lim_{\alpha\to 0} B_\alpha = 0$. Thus, we can find a small α such that

$$
\frac{B_\alpha}{\mu(\beta)} < \frac{\epsilon}{5}. \tag{4.22}
$$

From (4.20), (4.21) and (4.22)

$$
\mathbb{P}\Big\{\{\tau_\beta < \infty\} \cap \{\tau_h = \infty\}\Big\} \leq \frac{B_\alpha}{\mu(\beta)} < \frac{\epsilon}{5}. \tag{4.23}
$$

With the same as (4.15), we can derive

$$\mathbb{P}\Big\{\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\}\Big\}\geq\mathbb{P}\Big\{\tau_\beta<\infty\Big\}-\mathbb{P}\Big\{\tau_h<\infty\Big\}$$
$$\geq\mathbb{P}\Big\{\tau_\beta<\infty\Big\}-\frac{\epsilon}{5}. \tag{4.24}$$

From (4.23) and (4.24) we have

$$\mathbb{P}\Big\{\tau_\beta<\infty\Big\}<\frac{2\epsilon}{5}. \tag{4.25}$$

Furthermore, combine (4.15) and (4.25) we can derive

$$\begin{aligned}\mathbb{P}\Big\{\pi<\infty\ and\ \tau_\beta=\infty\Big\}&\geq\mathbb{P}\Big\{\pi<\infty\Big\}-\mathbb{P}\Big\{\tau_\beta<\infty\Big\}\\&\geq\mathbb{P}\Big\{\tau_\alpha<\tau_h\wedge\bar{\theta}\Big\}-\frac{2\epsilon}{5}\\&>1-\epsilon,\end{aligned}$$

which implies

$$\mathbb{P}\Big\{\omega:\limsup{}_{t\to\infty}|X(t,x_0)|\leq\beta\Big\}>1-\epsilon.$$

Finally, since β is arbitrary, let $\beta\to 0$ to arrive the following

$$\mathbb{P}\Big\{\omega:\lim_{t\to\infty}|X(t,x_0)|=0\Big\}\geq 1-\epsilon.$$

This completes the proof. ■

Theorem 4.3 *Suppose that there exist a function* $V(t_2,x,\mu)\in\mathscr{C}_{b,+}([0,\infty)\times\mathbb{R}^d\times\mathscr{P}_2(\mathbb{R}^d))$ *and another function* $\varphi\in\mathcal{K}$ *such that*

(1) $V(t_2,0,\delta_0)=0$,

(2) $\varphi(|x|)\leq V(t_2,x,\mu)$ *for all* $(t_2,x,\mu)\in[0,\infty)\times\mathbb{R}^d\times\mathscr{P}_2(\mathbb{R}^d)$,

(3) for all $(t_2,x,\mu)\in[0,\infty)\times\mathbb{R}^d\times\mathscr{P}_2(\mathbb{R}^d)$,

$$\lim_{|x|\to\infty}\inf_{t_2\geq 0,\mu\in\mathscr{P}_2(\mathbb{R}^d)}V(t_2,x,\mu)=\infty,$$

(4) for all $(t_2,x,\mu)\in[0,\infty)\times\mathbb{R}^d\times\mathscr{P}_2(\mathbb{R}^d)$,

$$L^\mu V(t_2,x,\mu)\leq-\gamma(x)\ a.s.,$$

where $\gamma(x)\geq 0$, *and* $\gamma(x)\neq 0$ *when* $x\neq 0$. *Then the trivial solution of Equation(1.2) is globally stochastically asymptotically stable.*

Proof. From Theorem 4.1, we know that the trivial solution of Eq.(1.2) is stochastically stable. To show it is globally stochastically asymptotically stable, we only need to prove

$$\mathbb{P}\{\lim_{t\to\infty} X(t,x_0)=0\}=1,$$

for all $x_0 \in \mathbb{R}^d$. In fact, for any $x_0 \in \mathbb{R}^d$ and arbitrary $\epsilon \in (0,1)$, from the condition (3), there exists a sufficiently large h such that

$$\inf_{|x|\geq h, t_2\geq 0, \mu\in \mathscr{P}_2(\mathbb{R}^d)} V(t_2,x,\mu) \geq \frac{5V(0,x_0,\mu_0)}{\epsilon}. \tag{4.26}$$

Define the following stopping time

$$\tau_h := \inf\{t\geq 0 : |X(t,x_0)| \geq h\}.$$

With Condition (4), we can use the generalised Itô formula to yield the following

$$\mathbb{E}V(E_{\tau_h\wedge t}, X(\tau_h\wedge t), \mu_{\tau_h\wedge t}) \leq V(0,x_0,\mu_0). \tag{4.27}$$

By (4.26),

$$\mathbb{E}V(E_{\tau_h\wedge t}, X(\tau_h\wedge t), \mu_{\tau_h\wedge t}) \geq \frac{5V(0,x_0,\mu_0)}{\epsilon}\mathbb{P}\{\tau_h < t\}. \tag{4.28}$$

From (4.27) and (4.28) we have

$$\mathbb{P}\{\tau_h < t\} \leq \frac{\epsilon}{5}.$$

Let $t\to\infty$

$$\mathbb{P}\{\tau_h < \infty\} \leq \frac{\epsilon}{5},$$

which implies

$$\mathbb{P}\{|X(t,x_0)| < h \ for \ all \ t\geq 0\} \geq 1-\frac{\epsilon}{5}. \tag{4.29}$$

According to the proof of Theorem 4.2, (4.29) implies,

$$\mathbb{P}\{\lim_{t\to\infty} X(t,x_0)=0\} \geq 1-\epsilon.$$

Since ϵ is arbitrary,

$$\mathbb{P}\{\lim_{t\to\infty} X(t,x_0)=0\} = 1$$

for all $x_0 \in \mathbb{R}^d$. The proof is complete. ∎

To end this section, we give an example to explicate our results.

Example 4.4 *Let us consider the following one-dimensional McKean-Vlasov stochastic differential equation driven by the time-changed Brownian motion*

$$dX(t) = b(t,E_t,X(t),\mathscr{L}(X(t)))dE_t + \sigma(t,E_t,X(t),\mathscr{L}(X(t)))dB_{E_t}, \tag{4.30}$$

where

$$b(t,E_t,X(t),\mu_t) = -f(t,E_t)\cos(\int_{\mathbb{R}} y\mu_t(dy))|X(t)|, \quad \sigma(t,E_t,X(t),\mu_t) = g(t,E_t)X(t),$$

with initial value $X(0)=x_0$, where $f(t_1,t_2)$ and $g(t_1,t_2)$ are real-valued functions on $\mathbb{R}_+\times\mathbb{R}_+$ satisfying the following condition

$$0\le f(t_1,t_2),g(t_1,t_2)\le L \ for\ all\ t_1,t_2\in\mathbb{R}_+,$$

with positive constant L. Also, assume $f(t_1,t_2)$ and $g(t_1,t_2)$ are measurable with respect to the filtration $\mathcal{G}_t=\mathcal{F}_{E_t}$. Let us consider $V(E_t,X(t),\mu_t)=(2+\sin(\int_{\mathbb{R}}y\mu_t(dy)))|X(t)|^\alpha$ for some $\alpha\in(0,1)$, we have

$$V_x(E_t,X(t),\mu_t)=\alpha(2+\sin(\int_{\mathbb{R}}y\mu_t(dy)))|X(t)|^{\alpha-1},$$

$$V_{xx}(E_t,X(t),\mu_t)=\alpha(\alpha-1)(2+\sin(\int_{\mathbb{R}}y\mu_t(dy)))|X(t)|^{\alpha-2},$$

$$\partial_\mu V(E_t,X(t),\mu_t)(z)=\cos(\int_{\mathbb{R}}y\mu_t(dy))|X(t)|^\alpha,\quad \partial_z\partial_\mu V(E_t,X(t),\mu_t)(z)=0.$$

Then

$$\begin{aligned}
&L^\mu V(E_t,X(t),\mu_t)\\
&=-\alpha\cos(\int_{\mathbb{R}}y\mu_t(dy))(2+\sin(\int_{\mathbb{R}}y\mu_0(dy)))f(t,E_t)|X(t)|^\alpha\\
&\quad+\frac{\alpha(\alpha-1)}{2}(2+\sin(\int_{\mathbb{R}}y\mu_0(dy)))g^2(t,E_t)|X(t)|^\alpha\\
&\quad-\cos^2(\int_{\mathbb{R}}y\mu_t(dy))f(t,E_t)\int_{\mathbb{R}}|y|\mu_t(dy)|X(T)|^\alpha|X(t)|.
\end{aligned}$$

Therefore, if

$$\alpha(2+\sin(\int_{\mathbb{R}}y\mu_0(dy)))\left[\frac{\alpha-1}{2}g^2(t,E_t)-\cos(\int_{\mathbb{R}}y\mu_t(dy))f(t,E_t)\right]\le 0$$

and

$$-\cos^2(\int_{\mathbb{R}}y\mu_t(dy))f(t,E_t)\int_{\mathbb{R}}|y|\mu_t(dy)\le 0$$

hold a.s. for all $t,E_t\in\mathbb{R}_+$, from Theorem 4.1 we know that the trivial solution of Eq.(4.30) is stochastically stable. If the above inequalities hold strictly, Theorem 4.3 indicates that the trivial solution of Eq.(4.30) is globally stochastically asymptotically stable.

5 Averaging principle for time-changed DDSDEs

Averaging principle is a powerful and efficient tool in studying of the qualitative properties of dynamical systems. It has a very rich theory on both deterministic and stochastic differential equations. Indeed, averaging principle is an effective method for studying dynamical systems with highly oscillating components. Under certain suitable conditions, the highly oscillating components can be "averaged out" to produce an averaged system. The averaged system is easier for analysis which governs the evolution of the original system over long time scales. The fundamental idea of the stochastic averaging principle is to study complex stochastic equations with related averaging stochastic equations, which paves a convenient and easy way to study many important properties. The theory of stochastic averaging principle started since the seminal work [22] of Khasminskii who established the averaging principle for stochastic differential equations driven by Gaussian white noise in a weak sense. There are many works devoted to extending Khasminski's classical result to various stochastic (partial) differential equations (Guo et al [12], Röckner and Xie [39], Röckner et al [41], Xu et al [47], Hong et al [15], Pei et al [36] just mention a few). Along this line, in this final section, we want to study averaging principle for DDSDEs driven by time-changed Brownian motions. Comparing to the classical SDEs driven by Brownian motion, Lévy processes, the DDSDEs driven by time-changed Brownian motions are much more complex, therefore, a stochastic averaging principle for such SDEs is naturally interesting and would also be very useful. The integral formulation of Equation (1.2) involving parameter $\epsilon \in (0, \epsilon_0]$ is as follows:

$$X^\epsilon(t) = X^\epsilon(0) + \int_0^t b(\frac{s}{\epsilon}, E_{\frac{s}{\epsilon}}, X^\epsilon(s), \mathscr{L}(X^\epsilon(s)))dE_s + \int_0^t \sigma(\frac{s}{\epsilon}, E_{\frac{s}{\epsilon}}, X^\epsilon(s), \mathscr{L}(X^\epsilon(s)))dB_{E_s}, \tag{5.1}$$

with the initial value $X^\epsilon(0) = x_0$ satisfying $\mathbb{E}|x_0|^2 < \infty$, where $\epsilon_0 > 0$ is a fixed real number.

According to Khasminskii type averaging principle ([22]), we consider the following averaged DDSDEs associated with the integral formulation (5.1)

$$\hat{X}(t) = X^\epsilon(0) + \int_0^t \bar{b}(\hat{X}(s), \mathscr{L}(\hat{X}(s)))dE_s + \int_0^t \bar{\sigma}(\hat{X}(s), \mathscr{L}(\hat{X}(s)))dB_{E_s}, \tag{5.2}$$

where the both coefficients $\bar{b} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}^d$ and $\bar{\sigma} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}^{d\times m}$ are Borel measurable functions. The DDSDEs (5.2) enjoys a unique solution $\hat{X}(t)$ under (H1$'$), (H2) and (H3). In addition, we assume that the following holds.

Assumption 5.1 *(Averaging condition): For $h \in [0, T]$, $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, there exist positive bounded functions $\varphi_i(h) \le C_i, i = 1, 2$ such that*

$$\frac{1}{h}\int_0^h |b(s, E_s, x, \mu) - \bar{b}(x, \mu)|^2 dE_s \le \varphi_1(h)(|x|^2 + \mathbb{W}_2(\mu, \delta_0)^2),$$

and

$$\frac{1}{h}\int_0^h ||\sigma(s, E_s, x, \mu) - \bar{\sigma}(x, \mu)||^2 dE_s \le \varphi_2(h)(|x|^2 + \mathbb{W}_2(\mu, \delta_0)^2),$$

where $\lim_{h\to\infty}\varphi_i(h) = 0, i = 1, 2,$ and the both $\bar{b} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}^d$, $\bar{\sigma} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}^{d\times m}$ are Borel measurable functions.

Our main result of this section is

Theorem 5.2 *Suppose that (H1$'$), (H2) and (H3) and Assumptions 5.1 hold. Then for a given arbitrarily small number $\delta_1 > 0$, there exist $L > 0$, $\epsilon_1 \in (0, \epsilon_0]$ and $\beta \in (0, \alpha - 1)$, such that for any $\epsilon \in (0, \epsilon_1]$,*

$$\mathbb{E}(\sup_{t\in[0,L\epsilon^{-\beta}]} |X^\epsilon(t) - \hat{X}(t)|^2) \le \delta_1.$$

Proof. For any $t' \in [0,T]$, we have

$$\begin{aligned}
&X^\epsilon(t') - \hat{X}(t') \\
&= \int_0^{t'} [b(\frac{s'}{\epsilon}, E_{\frac{s'}{\epsilon}}, X^\epsilon(s'), \mathscr{L}(X^\epsilon(s'))) - \bar{b}(\hat{X}(s'), \mathscr{L}(\hat{X}(s')))] dE_{s'} \\
&\quad + \int_0^{t'} [\sigma(\frac{s'}{\epsilon}, E_{\frac{s'}{\epsilon}}, X^\epsilon(s'), \mathscr{L}(X^\epsilon(s'))) - \bar{\sigma}(\hat{X}(s'), \mathscr{L}(\hat{X}(s')))] dB_{E_{s'}}.
\end{aligned} \tag{5.3}$$

Let $s = \frac{s'}{\epsilon}, t = \frac{t'}{\epsilon}$, (5.3) can be rewritten

$$\begin{aligned}
&X^\epsilon(\epsilon t) - \hat{X}(\epsilon t) \\
&= \epsilon^\alpha \int_0^{t} [b(s, E_s, X^\epsilon(s\epsilon), \mathscr{L}(X^\epsilon(s\epsilon))) - \bar{b}(\hat{X}(s\epsilon), \mathscr{L}(\hat{X}(s\epsilon)))] dE_s \\
&\quad + \epsilon^{\frac{\alpha}{2}} \int_0^{t} [\sigma(s, E_s, X^\epsilon(s\epsilon), \mathscr{L}(X^\epsilon(s\epsilon))) - \bar{\sigma}(\hat{X}(s\epsilon), \mathscr{L}(\hat{X}(s\epsilon)))] dB_{E_s},
\end{aligned}$$

where we use α-self-similarity of E_t and $\frac{\alpha}{2}$-self-similarity of B_{E_t}. By Jensen's inequality, we have for any $0 < u < T$ the following

$$\begin{aligned}
&\mathbb{E}_B(\sup_{0\le t\epsilon\le u} |X^\epsilon(\epsilon t) - \hat{X}(\epsilon t)|^2) \\
&\le 2\epsilon^{2\alpha} \mathbb{E}_B \left(\sup_{0\le t\epsilon\le u} \Big| \int_0^t [b(s, E_s, X^\epsilon(s\epsilon), \mathscr{L}(X^\epsilon(s\epsilon))) - \bar{b}(\hat{X}(s\epsilon), \mathscr{L}(\hat{X}(s\epsilon)))] dE_s \Big|^2 \right) \\
&\quad + 2\epsilon^{\alpha} \mathbb{E}_B \left(\sup_{0\le t\epsilon\le u} \Big| \int_0^t [\sigma(s, E_s, X^\epsilon(s\epsilon), \mathscr{L}(X^\epsilon(s\epsilon))) - \bar{\sigma}(\hat{X}(s\epsilon), \mathscr{L}(\hat{X}(s\epsilon)))] dB_{E_s} \Big|^2 \right) \\
&=: I_1 + I_2.
\end{aligned}$$

Next, we estimations $I_i, i = 1, 2$, respectively. For the term I_1,

$$\begin{aligned}
I_1 &\le 4\epsilon^{2\alpha} \mathbb{E}_B \left(\sup_{0\le t\epsilon\le u} \Big| \int_0^t [b(s, E_s, X^\epsilon(s\epsilon), \mathscr{L}(X^\epsilon(s\epsilon))) - b(s, E_s, \hat{X}(s\epsilon), \mathscr{L}(\hat{X}(s\epsilon)))] dE_s \Big|^2 \right) \\
&\quad + 4\epsilon^{2\alpha} \mathbb{E}_B \left(\sup_{0\le t\epsilon\le u} \Big| \int_0^t [b(s, E_s, \hat{X}(s\epsilon), \mathscr{L}(\hat{X}(s\epsilon))) - \bar{b}(\hat{X}(s\epsilon), \mathscr{L}(\hat{X}(s\epsilon)))] dE_s \Big|^2 \right) \\
&=: I_{11} + I_{12}.
\end{aligned}$$

By (H1$'$), (H2) and (H3), Jensen's inequality and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
I_{11} &= 4\epsilon^{2\alpha}\mathbb{E}_B\Big(\sup_{0\le t\epsilon\le u}|\int_0^t[b(s,E_s,X^\epsilon(s\epsilon),\mathscr{L}(X^\epsilon(s\epsilon)))-b(s,E_s,\hat{X}(s\epsilon),\mathscr{L}(\hat{X}(s\epsilon)))]dE_s|^2\Big)\\
&\le 4\epsilon^{2\alpha}E_T\mathbb{E}_B\Big(\sup_{0\le t\epsilon\le u}\int_0^t K_1(|X^\epsilon(s\epsilon)-\hat{X}(s\epsilon)|^2+\mathbb{W}_2(\mathscr{L}(X^\epsilon(s\epsilon)),\mathscr{L}(\hat{X}(s\epsilon)))^2)dE_s\Big)\\
&\le 4\epsilon^{2\alpha}K_1E_T\mathbb{E}_B\Big(\sup_{0\le t\epsilon\le u}(\int_0^t|X^\epsilon(s\epsilon)-\hat{X}(s\epsilon)|^2dE_s)\\
&\quad+\sup_{0\le t\epsilon\le u}(\int_0^t\mathbb{W}_2(\mathscr{L}(X^\epsilon(s\epsilon)),\mathscr{L}(\hat{X}(s\epsilon)))^2dE_s)\Big)\\
&\le 4\epsilon^{2\alpha}K_1E_T\Big(\int_0^{\frac{u}{\epsilon}}\mathbb{E}_B(\sup_{0\le r\le s}|X^\epsilon(r\epsilon)-\hat{X}(r\epsilon)|^2)dE_s\Big)\\
&\quad+4\epsilon^{2\alpha}K_1E_T\mathbb{E}_B\Big(\int_0^{\frac{u}{\epsilon}}\mathbb{E}\big(\sup_{0\le r\le s}|X^\epsilon(r\epsilon)-\hat{X}(r\epsilon)|^2\big)dE_s\Big),
\end{aligned}$$

where the last inequality uses the following fact

$$W_2(\mathscr{L}(X^\epsilon(r\epsilon)),\mathscr{L}(\hat{X}(r\epsilon)))^2\le\mathbb{E}|X^\epsilon(r\epsilon)-\hat{X}(r\epsilon)|^2.$$

By Hölder formula and Assumption 5.1, we can get

$$\begin{aligned}
I_{12} &= 4\epsilon^{2\alpha}\mathbb{E}_B\Big(\sup_{0\le t\epsilon\le u}|\int_0^t[b(s,E_s,\hat{X}(s\epsilon),\mathscr{L}(\hat{X}(s\epsilon)))-\bar{b}(\hat{X}(s\epsilon),\mathscr{L}(\hat{X}(s\epsilon)))]dE_s|^2\Big)\\
&\le 4\epsilon^{2\alpha}E_T\mathbb{E}_B\Big(\sup_{0\le t\epsilon\le u}\int_0^t|b(s,E_s,\hat{X}(s\epsilon),\mathscr{L}(\hat{X}(s\epsilon)))-\bar{b}(\hat{X}(s\epsilon),\mathscr{L}(\hat{X}(s\epsilon)))|^2dE_s\Big)\\
&\le 4C_1\epsilon^{2\alpha-1}E_Tu\Big(\mathbb{E}_B(\sup_{0\le s\le\frac{u}{\epsilon}}|\hat{X}(s\epsilon)|^2)+\mathbb{E}(\sup_{0\le s\le\frac{u}{\epsilon}}|\hat{X}(s\epsilon)|^2)\Big),
\end{aligned}$$

where we use the fact that $\mathbb{W}_2(\mathscr{L}(\hat{X}(s\epsilon)),\delta_0)\le\mathbb{E}|\hat{X}(s\epsilon)|^2$. Next, for the term I_2, we have

$$\begin{aligned}
I_2 &= 2\epsilon^\alpha\mathbb{E}_B\Big(\sup_{0\le t\epsilon\le u}|\int_0^t[(\sigma(s,E_s,X^\epsilon(s\epsilon),\mathscr{L}(X^\epsilon(s\epsilon)))-\sigma(s,E_s,\hat{X}(s\epsilon),\mathscr{L}(\hat{X}(s\epsilon))))\\
&\quad+(\sigma(s,E_s,\hat{X}(s\epsilon),\mathscr{L}(\hat{X}(s\epsilon)))-\bar{\sigma}(\hat{X}(s\epsilon),\mathscr{L}(\hat{X}(s\epsilon))))]dB_{E_s}|^2\Big)\\
&\le 4\epsilon^\alpha\mathbb{E}_B\Big(\sup_{0\le t\epsilon\le u}|\int_0^t(\sigma(s,E_s,X^\epsilon(s\epsilon),\mathscr{L}(X^\epsilon(s\epsilon)))-\sigma(s,E_s,\hat{X}(s\epsilon),\mathscr{L}(\hat{X}(s\epsilon))))dB_{E_s}|^2\Big)\\
&\quad+4\epsilon^\alpha\mathbb{E}_B\Big(\sup_{0\le t\epsilon\le u}|\int_0^t(\sigma(s,E_s,\hat{X}(s\epsilon),\mathscr{L}(\hat{X}(s\epsilon)))-\bar{\sigma}(\hat{X}(s\epsilon),\mathscr{L}(\hat{X}(s\epsilon))))dB_{E_s}|^2\Big)\\
&=: I_{21}+I_{22}.
\end{aligned}$$

By (H1$'$), (H2) and (H3) and the Burkholder-Davis-Gundy inequality (Jin and Kobayashi

[20]), we get

$$I_{21} = 4\epsilon^{\alpha}\mathbb{E}_B\bigg(\sup_{0\le t\epsilon\le u}|\int_0^t (\sigma(s,E_s,X^{\epsilon}(s\epsilon),\mathscr{L}(X^{\epsilon}(s\epsilon))) - \sigma(s,E_s,\hat{X}(s\epsilon),\mathscr{L}(\hat{X}(s\epsilon))))dB_{E_s}|^2\bigg)$$

$$\le 4\epsilon^{\alpha}K_1b_2\mathbb{E}_B\bigg(\int_0^{\frac{u}{\epsilon}}(|X^{\epsilon}(s\epsilon)-\hat{X}(s\epsilon)|^2 + \mathbb{W}_2(\mathscr{L}(X^{\epsilon}(s\epsilon)),\mathscr{L}(\hat{X}(s\epsilon)))^2)dE_s\bigg)$$

$$\le 4\epsilon^{\alpha}K_1b_2\bigg(\int_0^{\frac{u}{\epsilon}}\mathbb{E}_B(\sup_{0\le r\le s}|X^{\epsilon}(r\epsilon)-\hat{X}(r\epsilon)|^2) + \sup_{0\le r\le s}\mathbb{W}_2(\mathscr{L}(X^{\epsilon}(r\epsilon)),\mathscr{L}(\hat{X}(r\epsilon)))^2dE_s\bigg)$$

$$\le 4\epsilon^{\alpha}K_1b_2\int_0^{\frac{u}{\epsilon}}\mathbb{E}_B(\sup_{0\le r\le s}|X^{\epsilon}(r\epsilon)-\hat{X}(r\epsilon)|^2)dE_s + 4\epsilon^{\alpha}K_1b_2E_T\mathbb{E}\big(\sup_{0\le t\le\frac{u}{\epsilon}}|X^{\epsilon}(t\epsilon)-\hat{X}(t\epsilon)|^2\big),$$

where the positive constant b_2 comes from [20]. According to Assumption 5.1 and the Burkholder-Davis-Gundy inequality, we derive

$$I_{22} = 4\epsilon^{\alpha}\mathbb{E}_B\bigg(\sup_{0\le t\epsilon\le u}|\int_0^t(\sigma(s,E_s,\hat{X}(s\epsilon),\mathscr{L}(\hat{X}(s\epsilon))) - \bar{\sigma}(\hat{X}(s\epsilon),\mathscr{L}(\hat{X}(s\epsilon))))dB_{E_s}|^2\bigg)$$

$$\le 4\epsilon^{\alpha}b_2\mathbb{E}_B\bigg(\int_0^{\frac{u}{\epsilon}}||\sigma(s,E_s,\hat{X}(s\epsilon),\mathscr{L}(\hat{X}(s\epsilon))) - \bar{\sigma}(\hat{X}(s\epsilon),\mathscr{L}(\hat{X}(s\epsilon)))||^2dE_s\bigg)$$

$$\le 4\epsilon^{\alpha-1}b_2C_2u\mathbb{E}_B\bigg(\sup_{0\le s\le\frac{u}{\epsilon}}|\hat{X}(s\epsilon)|^2 + \sup_{0\le s\le\frac{u}{\epsilon}}\mathbb{W}_2(\mathscr{L}(\hat{X}(s\epsilon)),\delta_0)^2\bigg)$$

$$\le 4C_2\epsilon^{\alpha-1}b_2u\bigg(\mathbb{E}_B(\sup_{0\le s\le\frac{u}{\epsilon}}|\hat{X}(s\epsilon)|^2) + \mathbb{E}(\sup_{0\le s\le\frac{u}{\epsilon}}|\hat{X}(s\epsilon)|^2)\bigg).$$

Then we have

$$\begin{aligned}
&\mathbb{E}_B\big(\sup_{0\le t\epsilon\le u}|X^{\epsilon}(\epsilon t)-\hat{X}(\epsilon t)|^2\big)\\
&\le \big(4\epsilon^{2\alpha}K_1E_T + 4\epsilon^{\alpha}K_1b_2\big)\bigg(\int_0^{\frac{u}{\epsilon}}\mathbb{E}_B\big(\sup_{0\le r\le s}|X^{\epsilon}(r\epsilon)-\hat{X}(r\epsilon)|^2\big)dE_s\bigg)\\
&\quad + (4\epsilon^{2\alpha}K_1{E_T}^2 + 4\epsilon^{\alpha}K_1b_2E_T)\mathbb{E}\big(\sup_{0\le t\epsilon\le u}|X^{\epsilon}(t\epsilon)-\hat{X}(t\epsilon)|^2\big)\\
&\quad + (4C_1\epsilon^{2\alpha-1}E_Tu + 4C_2\epsilon^{\alpha-1}b_2u)\mathbb{E}_B\big(\sup_{0\le s\le\frac{u}{\epsilon}}|\hat{X}(s\epsilon)|^2\big)\\
&\quad + (4C_1\epsilon^{2\alpha-1}E_Tu + 4C_2\epsilon^{\alpha-1}b_2u)\mathbb{E}\big(\sup_{0\le s\le\frac{u}{\epsilon}}|\hat{X}(s\epsilon)|^2\big),
\end{aligned}$$

by Gronwall inequality we have

$$\begin{aligned}
&\mathbb{E}_B(\sup_{0\le t\epsilon\le u}|X^{\epsilon}(\epsilon t)-\hat{X}(\epsilon t)|^2)\\
&\le e^{4\epsilon^{2\alpha}K_1{E_T}^2+4\epsilon^{\alpha}K_1b_2E_T}\bigg(\{8\epsilon^{2\alpha}K_1{E_T}^2 + 8\epsilon^{\alpha}K_1b_2E_T\}[\mathbb{E}\sup_{0\le t\le T}|X^{\epsilon}(t)|^2 + \mathbb{E}\sup_{0\le t\le T}\hat{X}(t)|^2]\\
&\quad + \{4C_1\epsilon^{2\alpha-1}E_Tu + 4C_2\epsilon^{\alpha-1}b_2u\}\mathbb{E}_B(\sup_{0\le t\le T}|\hat{X}(t)|^2)\\
&\quad + \{4C_1\epsilon^{2\alpha-1}E_Tu + 4C_2\epsilon^{\alpha-1}b_2u\}\mathbb{E}(\sup_{0\le t\le T}|\hat{X}(t)|^2)\bigg).
\end{aligned}$$

Selecting $\beta \in (0, \alpha - 1)$ and $L > 0$ such that for any $t \in [0, L\epsilon^{-\beta-1}] \subseteq [0, \frac{T}{\epsilon}]$ and taking $\mathbb{E}_D$ on both we have

$$
\begin{aligned}
&\mathbb{E}(\sup_{0\le t\epsilon\le u} |X^\epsilon(\epsilon t) - \hat{X}(\epsilon t)|^2) \\
&\le \epsilon^{\alpha-\beta-1}\mathbb{E}_D\Bigg[e^{4\epsilon^{2\alpha}K_1 E_T{}^2 + 4\epsilon^\alpha K_1 b_2 E_T}\Bigg(\{8\epsilon^{\alpha+\beta+1}K_1 E_T{}^2 + 8\epsilon^{\beta+1}K_1 b_2 E_T\} \\
&\qquad \times [\mathbb{E}\sup_{0\le t\le T}|X^\epsilon(t)|^2 + \mathbb{E}\sup_{0\le t\le T}\hat{X}(t)|^2] + L\{4C_1\epsilon^\alpha E_T + 4C_2 b_2\}\mathbb{E}_B(\sup_{0\le t\le T}|\hat{X}(t)|^2) \\
&\qquad + L\{4C_1\epsilon^\alpha E_T + 4C_2 b_2\}\mathbb{E}(\sup_{0\le t\le T}|\hat{X}(t)|^2)\Bigg)\Bigg] \\
&\le \epsilon^{\alpha-\beta-1}\Bigg(\mathbb{E}(e^{8\epsilon_0^{2\alpha}K_1 E_T{}^2 + 8\epsilon_0^\alpha K_1 b_2 E_T})\Bigg)^{\frac{1}{2}}\Bigg[\mathbb{E}_D\Bigg(\{8\epsilon_0^{\alpha+\beta+1}K_1 E_T{}^2 + 8\epsilon_0^{\beta+1}K_1 b_2 E_T\} \\
&\qquad \times [\mathbb{E}\sup_{0\le t\le T}|X^\epsilon(t)|^2 + \mathbb{E}\sup_{0\le t\le T}\hat{X}(t)|^2] + L\{4C_1\epsilon_0^\alpha E_T + 4C_2 b_2\}\mathbb{E}_B(\sup_{0\le t\le T}|\hat{X}(t)|^2) \\
&\qquad + L\{4C_1\epsilon_0^\alpha E_T + 4C_2 b_2\}\mathbb{E}(\sup_{0\le t\le T}|\hat{X}(t)|^2)\Bigg)^2\Bigg]^{\frac{1}{2}} \\
&\le \epsilon^{\alpha-\beta-1}\Bigg(\mathbb{E}\{e^{8\epsilon_0^{2\alpha}K_1 E_T{}^2 + 8\epsilon_0^\alpha K_1 b_2 E_T}\}\Bigg)^{\frac{1}{2}}\Bigg[2\mathbb{E}\Bigg(\{8\epsilon_0^{\alpha+\beta+1}K_1 E_T{}^2 + 8\epsilon_0^{\beta+1}K_1 b_2 E_T\} \\
&\qquad\quad \times [\mathbb{E}\sup_{0\le t\le T}|X^\epsilon(t)|^2 + \mathbb{E}\sup_{0\le t\le T}\hat{X}(t)|^2] + L\{4C_1\epsilon_0^\alpha E_T + 4C_2 b_2\}\mathbb{E}(\sup_{0\le t\le T}|\hat{X}(t)|^2)\Bigg)^2 \\
&\qquad + 2\mathbb{E}_D\Bigg(L\{4C_1\epsilon_0^\alpha E_T + 4C_2 b_2\}\mathbb{E}_B(\sup_{0\le t\le T}|\hat{X}(t)|^2)\Bigg)^2\Bigg]^{\frac{1}{2}} \\
&\le \xi\epsilon^{\alpha-\beta-1},
\end{aligned}
$$

the constant

$$
\begin{aligned}
\xi :=& \Bigg(\mathbb{E}\{e^{8\epsilon_0^{2\alpha}K_1 E_T{}^2 + 8\epsilon_0^\alpha K_1 b_2 E_T}\}\Bigg)^{\frac{1}{2}}\Bigg[2\mathbb{E}\Bigg(\{8\epsilon_0^{\alpha+\beta+1}K_1 E_T{}^2 + 8\epsilon_0^{\beta+1}K_1 b_2 E_T\} \\
&\quad \times [\mathbb{E}\sup_{0\le t\le T}|X^\epsilon(t)|^2 + \mathbb{E}\sup_{0\le t\le T}\hat{X}(t)|^2] + L\{4C_1\epsilon_0^\alpha E_T + 4C_2 b_2\}\mathbb{E}(\sup_{0\le t\le T}|\hat{X}(t)|^2)\Bigg)^2 \\
&+ 2\mathbb{E}_D\Bigg(L\{4C_1\epsilon_0^\alpha E_T + 4C_2 b_2\}\mathbb{E}_B(\sup_{0\le t\le T}|\hat{X}(t)|^2)\Bigg)^2\Bigg]^{\frac{1}{2}}.
\end{aligned}
$$

Consequently, for any given $\delta_1 > 0$, there exists a $\epsilon_1 \in (0, \epsilon_0]$ such that for each $\epsilon \in (0, \epsilon_1]$ and $t \in [0, L\epsilon^{-\beta}]$,

$$\mathbb{E}(\sup_{t\in[0,L\epsilon^{-\beta}]}|X^\epsilon(t) - \hat{X}(t)|^2) \le \delta_1.$$

This completes the proof. ■

Corollary 5.3 *Suppose that (H1$'$), (H2) and (H3) and 5.1 hold. Then for a given number $\delta_2 > 0$, there exist $L > 0$, $\epsilon_1 \in (0, \epsilon_0]$ and $\beta \in (0, \alpha - 1)$, such that for any $\epsilon \in (0, \epsilon_1]$,*

$$\lim_{\epsilon\to 0}\mathbb{P}\Bigg(\sup_{t\in[0,L\epsilon^{-\beta}]}|X^\epsilon(t) - \hat{X}(t)| > \delta_2\Bigg) = 0.$$

Proof. This result can be verified by Chebyshev's inequality and the result of Theorem 5.2. ■

Example 5.4 *We consider the following distribution dependent stochastic differential equations driven by time-changed Brownian motions:*

$$dx^\epsilon(t) = \int_0^t [x^\epsilon \cos^2(E_{\frac{s}{\epsilon}}) + \sin(\int_{\mathbb{R}} y\mu(dy))]dE_s + \int_0^t \lambda dB_{E_s}, \quad t \in [0,T]$$

with initial value $x^\epsilon(0) = 0$ and $\lambda \in \mathbb{R}$ being a constant. For this equation, we have

$$f(\frac{t}{\epsilon}, E_{\frac{t}{\epsilon}}, x^\epsilon(t), \mu) = x^\epsilon \cos^2(E_{\frac{t}{\epsilon}}) + \sin(\int_{\mathbb{R}} y\mu(dy)),$$

$$g(\frac{t}{\epsilon}, E_{\frac{t}{\epsilon}}, x^\epsilon(t), \mu) = \lambda.$$

Let

$$\begin{aligned}\bar{f}(\hat{x}(s), \nu) &= \int_0^1 f(s, E_s, x^\epsilon(s), \mu)dE_s \\ &= (\frac{1}{2}E_1 + \frac{\sin 2E_1}{4})x^\epsilon + E_1 \sin(\int_{\mathbb{R}} y\mu(dy)),\end{aligned}$$

and

$$\bar{g}(\hat{x}(s), \nu) = \lambda.$$

We have the following corresponding distribution dependent averaged stochastic differential equations driven by time-changed Brownian motions:

$$d\hat{x}(t) = \epsilon^\alpha \int_0^t \left[(\frac{1}{2}E_1 + \frac{\sin 2E_1}{4})\hat{x} + E_1 \sin(\int_{\mathbb{R}} y\nu(dy))\right]dE_s + \epsilon^{\frac{\alpha}{2}} \int_0^t \lambda dB_{E_s}.$$

Clearly, Assumptions 3.1 and 5.1 are satisfied, therefore, Theorem 5.2 and Corollary 5.3 hold for this example.

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Conflict of interest

The authors have no competing interests to declare that are relevant to the content of this article.

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