Global well-posedness and regularity of stochastic 3D Burgers equation with multiplicative noise

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Abstract: By utilising the so-called Doss-Sussman transformation, we link our stochastic 3D Burgers equation with linear multiplicative noise to a random 3D Burger equation. With the help of techniques from partial differential equations (PDEs) and probability, we establish the global well-posedness of stochastic 3D Burgers with the diffusion coefficient being constant. Next, by developing a solution which is orthogonal with the gradient of coefficient of the noise, we extend the global well-posedness to a more general case in which the diffusion coefficient is spatial dependent, i.e., it is a function of the spatial variable.

Our results and methodology pave a way to extend some regularity results of stochastic 1D Burgers equation to stochastic 3D Burgers equations.

AMS Subject Classification: 60H15, 35Q35.

Keywords: stochastic 3D Burgers equations; regularity; maximum principle.

1 Introduction

In this paper, we are concerned with 3D positive viscosity stochastic Burgers equation driven by linear multiplicative noise on the three dimensional torus $\mathbb{T}^3 := \mathbb{R}^3/2\pi\mathbb{Z}^3$. To be more precise, fix any $T > 0$, let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]})$ be a stochastic basis which is given through out the whole paper, wherein the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is assumed to fulfil the usual conditions and $(\{B(t)\}_{t\in[0,T]})$ is a one-dimensional standard $\{\mathcal{F}_t\}_{t\in[0,T]}$ adapted-Brownian motions. We use $\mathbb{E}$ to denote the expectation with respect to $\mathbb{P}$. We consider the following Cauchy problem

$$du(t,x) = (\nu\Delta u(t,x) - (u\cdot\nabla)u)(t,x)dt + u(t,x)\circ b(x)dB(t), \quad \text{on } [0,T]\times\mathbb{T}^3, \qquad (1.1)$$
$$u(0,x) = u_0(x), \quad x = (x_1,x_2,x_3)\in\mathbb{T}^3,$$

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for a 3D vector valued random field $u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x)) \in \mathbb{R}^3, t \in [0,T]$ and $x \in \mathbb{T}^3$, where the parameter $\nu > 0$ in the system (1.1) stands for the viscosity, $b(x) : \mathbb{T}^3 \to \mathbb{R}$ is a given smooth function, and $\circ$ denotes the Stratonovich integral. To simplify the notations, we set $\partial_i = \partial x_i, \ \ i = 1,2,3$. Moreover, $\Delta := \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator, $\nabla := (\partial_1, \partial_2, \partial_3)$ is the gradient operator. From Section 3 to Section 5, we consider (1.1) when $b(x)$ is only a constant. In Section 6, we establish the global well-posedness of (1.1) with $b(x)$ being a smooth function of the spatial variable.

The Burgers equation was first introduced by H. Bateman [9] and Forsyth [30] in 1915 to describe both nonlinear propagation effects and diffusive effects, occurring in various areas of applied mathematics, such as gas dynamics, fluid mechanics, nonlinear acoustics, and (more recently) traffic flow. This equation was studied mathematically by J. M. Burgers (see [18]) in the 1940s. Adding a random force so that the equation becomes a stochastic Burgers equation, the corresponding result is totally different, see for instance [19, 23, 36, 33]. Moreover, the stochastic Burgers equations have also been applied to study the dynamics of interfaces in the seminal work [37].

One dimensional stochastic Burgers equation has been fairly well studied. By an adaptation of the celebrated Hopf-Cole transformation, Bertini, Cancrini, and Jona-Lasinio [8] solve the one dimensional modified Burgers equation with additive space-time white noise, where the nonlinearity in the equation was formulated in terms of Wick product. Moreover, Chan [20] utilises Hopf-Cole transformation to study the scaling limit of Wick ordered KPZ equation involving additive space-time white noise. Later, Da Prato, Debussche, Temam [23] study the Burgers equation based on semigroup property for the heat equation on a bounded domain. In the paper [23], the authors establish the existence of an invariant measure for the corresponding transition semigroup. In [5], Bakhtin, Cator, and Khanin study the long-term behavior of the Burgers dynamics for the situation where the forcing is a space-time stationary random process. In particular, they construct space-time stationary global solutions for the Burgers equation on the real line and show that they can be viewed as one-point attractors. In [3], Bakhtin consider the Burgers equation with random boundary conditions. Concerning the one-dimensional Burgers equation with viscosity coefficient defined on a bounded domain driven by multiplicative Gaussian noise, Da Prato and Debussche [22] succeed to obtain its global well-posedness. Furthermore, Gyöngy and Nualart [32] extend the results of Da Prato and Debussche to the Burgers equation defined on the whole line. When the Gaussian force is replaced by Lévy jumps, Dong and Xu prove its global well-posedness of the strong, weak and mild solutions as well as the ergodicity in [26, 25].

For the multidimensional Burgers equations, Kiselev and Ladyzhenskaya [40] prove the existence and uniqueness of solution in the class of functions $L^\infty(0,T; L^\infty(O)) \cap L^2(0,T; H_0^1(O))$. Inspired by [40], Pooley, Robinson [46] prove the global well-posedness for 3D Burgers equations in $H^{\frac{1}{2}}$. When the viscosity tends to zero and the initial condition is zero, Bui [17] prove the convergence of solutions to the inviscid Burgers equations on a small time interval. In the higher dimensional inviscid stochastic case, the stationary solution and a stationary distribution were constructed by Gomes, Iturriaga, Khanin and Padilla in [35] based on a very delicate use of the Lagrangian formalism and the Hamilton-Jacobi equation. Based on the stochastic version of Lax formula for solutions to the initial and final value problems for the viscous Hamilton-Jacobi equation, Gomes, Iturriaga, Khanin and Padilla in [31] prove convergence of stationary distributions for the randomly forced multi-dimensional Burgers and Hamilton-Jacobi equations in the limit when viscosity tends to zero. Utilising the maximum principle, Brzezniak, Goldys and Neklyudov [13] establish the global existence and uniqueness for the mild solutions to multidimensional Burgers equations with additive noise. Furthermore, the asymptotic behavior of solutions to multidimensional stochastic Burgers equations is studied when the viscosity tends to zero. For the multidimensional generalised stochastic Burgers equation in the space-periodic setting, Boritchev [10] prove that if the solution u of this equation is a gradient, then each of Sobolev norms of u averaged in time and in ensemble behaves as a given negative power of the viscosity coefficient μ, which gives the sharp upper and lower bounds for natural analogues of quantities characterising the hydrodynamical turbulence. Recently, Khanin and Zhang [39] generalized the results of an important paper [28] to arbitrary dimensional Burgers equation by using Green bundles method, which is complete different from the approach used by [28].

There are extensive works dealing with the stochastic partial differential equations driven by linear multiplicative noise, see e.g., [1, 16, 14, 15, 43, 11, 12, 6, 7, 42, 48, 49, 51] and references therein. The so-called Doss-Sussman transformation is extensively used, see [27, 49]. And one can also refer to Barbu, Röckner and Zhang [6, 7, 51] where the same method was used for stochastic nonlinear Schrödinger equations under the name of rescaling approach. There are also other methods to solve stochastic partial differential equations driven by linear noise, let us mention that Brzézniak, Flandoli, and Maurelli [12] implement a Lagrangian approach to solve the strong existence and the pathwise uniqueness of solutions of the stochastic 2D Euler equations. Some inventive and general method are introduced by Brzézniak, Hausenblas and Razafimandimby [14] to solve stochastic penalised nematic liquid crystals with linear Gaussian noise. A Wong-Zakai approximation for stochastic Landau-Lifshitz-Gilbert equations with linear noise is addressed, see Brzézniak, Manna, Mukherjee and Panda [16, 43]. One can also find similar results in Chugreeva and Melcher [21], Röger and Weber [48]. A natural question is how about Wong-Zakai approximation for stochastic 3D Burgers equation. Inspired by the celebrated work [13], one can adapt the argument of [16, 43] to establish Wong-Zakai approximation for stochastic 3D Burgers equation (1.1).

In what follows, let us explicate the essential difficulties we encountered in treating this high nonlinear multidimensional stochastic system without incompressibility.

1. The 3D Burgers equation is a high nonlinear model without cancellation property. When the noise is introduced into this equation new difficulties emerge. The first difficulty is how to adapt the frame of PDEs to the stochastic frame such that the maximum principle is available for Itô equations, i.e., stochastic 3D Burgers equation. To address the global well-posedness of (1.1), the key point is how to utilize the probabilistic techniques and deterministic techniques to control the effect of the noise such that the maximum principle can be applied to stochastic 3D Burgers equation. Based on this strategy, the noise in (1.1) is linear. If the noise is nonlinear, the maximum principle is unavailable.

2. To solve (1.1) globally, our ideal is that we firstly perform the Doss-Sussman transformation on (1.1), then apply the maximum principle to the Galerkin approximations of 3D random Burgers equation. Here, we should mention that different from the deterministic cases, random 3D Burgers equation loses some regularity with respect to time, which forces us to establish the local well-posedness for the random 3D Burgers equation in a more regular space $\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$ than the global strong solution space $\mathbb{H}^1(\mathbb{T}^3)$, see the proof of Theorem 4.1.

3. In Theorem 4.5, we establish a maximum principle for the relevant random 3D Burgers equations. Here, the random maximum principle enable us to derive several important moment estimates for stochastic 3D Burgers equations.

4. In Section 6, we construct a global solution u for (1.1) which is orthogonal with $\nabla b(x)$, please see Theorem 6.6. The results and methods presented here seem to be new even for the deterministic PDEs and might pave a way to extend the results of [45, 44] to higher dimensions.

The paper is organised as follows. Arguments from Section 2 to Section 5 are devoted to the regularities of (1.1) with $b(x)$ being only a constant. Section 6 further establishes the global well-posedness of (1.1) with $b(x)$ being a smooth function. More precisely, preliminaries are presented in Section 2, the local existence and uniqueness of the solutions to (1.1) are given in Section 3, the global existence and regularities of solutions to (1.1) are established in Section 4. Moment estimates are derived in Section 5 for (1.1) in various functional spaces. In section 6, we establish the global well-posedness of 3D Burgers equation with the noise $u \circ b(x)dB(t)$, where $b(x)$ is a given smooth function. Appendix states some illustrations about the model considered in Section 6.

Throughout the paper, we use $c > 0$ for a generic constant with possibly different values at each appearance. Unless a specific description is given, we denote by $c(a) > 0$ a constant which depends on parameter a.

2 Mathematical preliminaries

2.1 Notational conventions

For $1 \le p \le \infty$, let $\mathbb{L}^p(\mathbb{T}^3)$ be the usual Lebesgue spaces $\mathbb{L}^p(\mathbb{T}^3;\mathbb{R}^3)$ with the norm $|\cdot|_p$. When $p = 2$, we denote by $\langle\cdot,\cdot\rangle$ the inner product in $\mathbb{L}^2(\mathbb{T}^3)$. Similarly, without confusion, we denote by $L^p(\mathbb{T}^3)$ the usual Lebesgue spaces $L^p(\mathbb{T}^3;\mathbb{R})$ with the norm $|\cdot|_p$. For $s \ge 0$, we introduce an operator Λ^s acting on $\mathbb{H}^s(\mathbb{T}^3)$ which is a Sobolev space $\mathbb{H}^s(\mathbb{T}^3;\mathbb{R}^3)$. For the detail, please see the following. Assuming $f \in \mathbb{H}^s(\mathbb{T}^3)$ with the Fourier series

$$f(x) = \sum_{k\in\mathbb{Z}^3} \hat{f}_k e^{ik\cdot x} \in \mathbb{H}^s(\mathbb{T}^3), \tag{2.2}$$

we define

$$\Lambda^s f(x) = \sum_{k\in\mathbb{Z}^3} |k|^s \hat{f}_k e^{ik\cdot x} \in \mathbb{L}^2(\mathbb{T}^3).$$

Denote by $\|\cdot\|_s$ the seminorm $|\Lambda^s \cdot|_2$, then the Sobolev norm $\|\cdot\|_{\mathbb{H}^s}$ of $\mathbb{H}^s(\mathbb{T}^3)$ is equivalent to $|\cdot|_2 + \|\cdot\|_s$. Hence, we can define the norm on $\mathbb{H}^s$ by

$$\|f\|_{\mathbb{H}^s} = \Big(\sum_{k\in\mathbb{Z}^3}(1+|k|^{2s})|\hat{f}_k|^2\Big)^{1/2}.$$

Obviously, for $0 < s_1 \le s_2$, we have $\|f\|_{s_1} \le \|f\|_{s_2}$ and $\Lambda^2 = -\Delta$. For $s \in \mathbb{R}^+$, set $\dot{\mathbb{H}}^s(\mathbb{T}^3) = \{f \in \mathbb{L}^2(\mathbb{T}^3) : \sum_{k\in\mathbb{Z}^3} |k|^{2s}|\hat{f}_k|^2 < \infty\}$. Then, we have $\mathbb{H}^s(\mathbb{T}^3) \subset \dot{\mathbb{H}}^s(\mathbb{T}^3)$. Obviously, $\|\cdot\|_s$ is the seminorm in $\dot{\mathbb{H}}^s(\mathbb{T}^3)$.

Similarly, we can define $H^s(\mathbb{T}^3)$, which is a Sobolev space $H^s(\mathbb{T}^3;\mathbb{R})$ with $s \ge 0$. Let $\{e_n(x), x \in \mathbb{T}^3, n \ge 1\}$ be an orthogonal basis of $L^2(\mathbb{T}^3)$, then we know $\{e_n(x), n \ge 1\} = \{1, \sin x, \cos x, \sin 2x, \cos 2x, ..., \sin nx, \cos nx, ...\}$. Assuming $g \in H^s(\mathbb{T}^3)$ with the Fourier series

$$g(x) = \sum_{n=1}^{\infty} \hat{g}_n e_n(x) \in H^s(\mathbb{T}^3),$$

then

$$\Lambda^s g(x) = \sum_{n=1}^{\infty} |n|^s \hat{g}_n e_n(x) \in L^2(\mathbb{T}^3).$$

Without confusion, we still denote by $\|\cdot\|_s$ the seminorm $|\Lambda^s \cdot|_2$, then the Sobolev norm $\|\cdot\|_{H^s}$ of $H^s(\mathbb{T}^3)$ is equivalent to $|\cdot|_2 + \|\cdot\|_s$. Hence, we can define the norm on $H^s(\mathbb{T}^3)$ by

$$\|g\|_{H^s} = \Big(\sum_{n=1}^{\infty}(1+|n|^{2s})|\hat{g}_n|^2\Big)^{1/2}.$$

Without loss of generality, we simply take the viscosity parameter $\nu = 1$. In fact, we only need ν to be any strictly positive number. In the following, we first introduce notations of local solution, maximum solution and global solution to (2.3) and (2.4). Since (2.3) and (2.4) is discussed by pathwise, so these definitions are from deterministic PDEs.

Definition 2.1 (**Local strong solutions to (2.3) and (2.4)**). *Suppose u_0 is an $\mathbb{H}^1(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable, T is an arbitrary positive constant.*

1. *A pair (v,τ) is a local strong pathwise solution to (2.3) and (2.4) if τ is a strictly positive random variable taking values in $(0,\infty)$ and $v(\cdot\wedge\tau)$ satisfies (2.3) and (2.4) in a weak sense so that the following regularities hold almost surely,*

$$v(\cdot\wedge\tau) \in C([0,T];\mathbb{H}^1(\mathbb{T}^3)) \cap L^2([0,T];\mathbb{H}^2(\mathbb{T}^3)),$$

and

$$\partial_t v(\cdot\wedge\tau) \in L^1([0,T];\mathbb{L}^2(\mathbb{T}^3)).$$

2. *Strong pathwise solutions of (2.3) and (2.4) are said to be pathwise unique up to a random positive time $\tau > 0$ if given any pair of solutions $(v^1,\tau),(v^2,\tau)$ which coincide at $t = 0$ on the event $\tilde{\Omega} = \{v^1(0) = v^2(0)\} \subset \Omega$, then*

$$\mathbb{P}(I_{\tilde{\Omega}}(v^1(t\wedge\tau) - v^2(t\wedge\tau)) = 0; \forall t \in [0,T]) = 1.$$

Definition 2.2 (**Maximal and global strong solutions to (2.3) and (2.4)**).

(i) *Let ξ be a positive random variable which may take ∞ at some $\omega \in \Omega$. We say the pair (v, ξ) is a maximal pathwise strong solution if for each random variable $\tau \in (0, \xi)$, (v, τ) is a local strong pathwise solution satisfying*

$$\sup_{t\in[0,\tau]} \|v(t)\|_1 < \infty, \quad \text{and} \quad \limsup_{t\to\xi} I_{[\xi<\infty]}\|v(t)\|_1 = \infty$$

almost surely.

(ii) *If (v, ξ) is a maximum pathwise strong solution and $\xi = \infty$ a.s., then we say the solution v is global.*

Definition 2.3 (**Local weak solutions to (2.3) and (2.4)**). *Suppose u_0 is an $\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable, T is an arbitrary positive constant.*

(i) A pair (v, τ) is a local weak pathwise solution to (2.3) and (2.4) if τ is a strictly positive random variable taking values in $(0, \infty)$ and $v(\cdot \wedge \tau)$ satisfies (2.3) and (2.4) in a weak sense so that the following regularities hold almost surely,

$$v(\cdot \wedge \tau) \in C([0, T]; \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)) \cap L^2([0, T]; \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)),$$

and

$$\partial_t v(\cdot \wedge \tau) \in L^1([0, T]; \mathbb{L}^2(\mathbb{T}^3)).$$

(ii) Weak pathwise solutions of (2.3) and (2.4) are said to be pathwise unique up to a random positive time $\tau > 0$ if given any pair of solutions $(v^1, \tau), (v^2, \tau)$ which coincide at $t = 0$ on the event $\tilde{\Omega} = \{v^1(0) = v^2(0)\} \subset \Omega$, then

$$\mathbb{P}(I_{\tilde{\Omega}}(v^1(t \wedge \tau) - v^2(t \wedge \tau)) = 0; t \in [0, T]) = 1.$$

Definition 2.4 (**Maximal and global weak solutions to (2.3) and (2.4)**).

(i) *Let ξ be a positive random variable which may take ∞ at some $\omega \in \Omega$. We say the pair (u, ξ) is a maximal weak pathwise solution if for each random variable $\tau \in (0, \xi)$, (v, τ) is a local strong pathwise solution satisfying*

$$\sup_{t\in[0,\tau]} \|v(t)\|_{\frac{1}{2}} < \infty, \quad \text{and} \quad \limsup_{t\to\xi} I_{[\xi<\infty]}\|v(t)\|_{\frac{1}{2}} = \infty$$

almost surely.

(ii) *If (v, ξ) is a maximum weak pathwise solution and $\xi = \infty$ a.s., then we say the solution v is global.*

Definition 2.5 (**Global strong solutions to (1.1)**). *Suppose u_0 is a $\mathbb{H}^1(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable. A stochastic process u is said to be a global strong solution to (1.1) if*

(i) for arbitrary $T > 0$ and $t \in [0, T]$, $u(t)$ is an $\mathcal{F}_t$ adapted process satisfying $u \in C([0, T]; \mathbb{H}^1(\mathbb{T}^3)) \cap \mathbb{L}^2([0, T]; \mathbb{H}^2(\mathbb{T}^3))$ almost surely;

(ii) u solves the stochastic 3D Burgers equation in the following sense:

$$u(t)-\int_0^t \Delta u ds+\int_0^t (u\cdot\nabla u)ds = u(0)+\int_0^t b(x)u\circ dB(s), a.s.,$$

with the equality understood in $\mathbb{H}$ *and* $t\in[0,T]$. *Furthermore, let u and ũ be two global strong solutions to (1.1). If* $u(0)=\tilde{u}(0)$ *a.s., we have*

$$\mathbb{P}(u(t)=\tilde{u}(t),\ for\ all\ t\in[0,T])=1,$$

then we say the strong solution u to (1.1) is unique.

Definition 2.6 (Global weak solutions to (1.1)). *Suppose* u_0 *is a* $\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ *valued,* $\mathcal{F}_0$ *measurable random variable. A stochastic process u is said to be a global weak solution to (1.1) if*

(i) for arbitrary $T>0$ *and* $t\in[0,T]$, $u(t)$ *is an* $\mathcal{F}_t$ *adapted process satisfying* $u\in C([0,T];\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3))\cap\mathbb{L}^2([0,T];\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$ *almost surely;*

(ii) u solves the stochastic 3D Burgers equation in the following sense:

$$\begin{aligned}&\langle u(t),\phi\rangle+\int_0^t\langle u(s),\Lambda^2\phi\rangle ds+\int_0^t\langle (u\cdot\nabla u)(s),\phi\rangle ds\\ =\ &\langle u(0),\phi\rangle+\int_0^t\langle b(x)u(s,x),\phi\rangle\circ dB(s),\end{aligned}$$

for all $t\in[0,T]$ *and* $\phi\in D(\Lambda^2)$. *Furthermore, let u and ũ be two strong solutions to (1.1). If* $u(0)=\tilde{u}(0)$ *a.s., we have*

$$\mathbb{P}(u(t)=\tilde{u}(t),\ for\ all\ t\in[0,T])=1,$$

then we say the weak solution u to (1.1) is unique.

2.2 Reformulation of stochastic 3D Burgers equations

From this part to Section 5, we only consider the case that $b(x)$ is only a constant. Let $W(t)=bB(t)$, and $\alpha(t)=\exp(-W(t)), t\in[0,T]$. As we know the characteristic function of $-W(t)$ is

$$\mathbb{E}\exp(-ixW(t))=\exp(\frac{-1}{2}b^2tx^2), x\in\mathbb{R}, i \textit{ is the imaginary unit},$$

which implies that $\exp(-W(t)-\frac{1}{2}b^2t)$ is a martingale. Hence, by Doob's maximum inequality we have

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\alpha^n(t) &= \mathbb{E}\sup_{t\in[0,T]}\exp\Big(-nW(t)\Big)\\ &\le \mathbb{E}\sup_{t\in[0,T]}\exp\Big(-nW(t)-\frac{n^2}{2}b^2t\Big)\exp\Big(\frac{n^2}{2}b^2T\Big)\\ &\le \Big(\mathbb{E}\sup_{t\in[0,T]}\exp\Big(-2nW(t)-2n^2b^2t\Big)\Big)^{\frac{1}{2}}\exp\Big(n^2b^2T\Big)\\ &\le 2\Big(\mathbb{E}\exp\Big(-2nW(T)-2n^2b^2T\Big)\Big)^{\frac{1}{2}}\exp\Big(n^2b^2T\Big)=2\exp\Big(n^2b^2T\Big),\end{aligned}$$

where $n \geq 1$. Similarly, we also have

$$\mathbb{E} \sup_{t\in[0,T]} \alpha^{-n}(t) \leq 2\exp\left(n^2 b^2 T\right).$$

Set $v = \alpha u$, then equations (1.1) is equivalent to the following

$$dv(t,x) = \Delta v(t,x)dt - \alpha^{-1}(t)[(v\cdot\nabla v)(t,x)]dt, \quad \text{on } [0,T]\times\mathbb{T}^3, \tag{2.3}$$

$$v(0,x) = u(0,x), \quad x = (x_1,x_2,x_3)\in\mathbb{T}^3. \tag{2.4}$$

We firstly consider the Galerkin approximation of (2.3)-(2.4). For $n\in\mathbb{N}$ let P_n denote the projection on to the Fourier modes of order up to n, that is

$$P_n\left(\sum_{k\in\mathbb{Z}^3}\hat{u}_k e^{ix\cdot k}\right) = \sum_{|k|\leq n}\hat{u}_k e^{ix\cdot k}.$$

Then we obtain the Galerkin approximation of (2.3)-(2.4) as the following

$$dv_n(t,x) = \Delta v_n(t,x)dt - \alpha^{-1}(t)P_n[(v_n\cdot\nabla v_n)(t,x)]dt, \quad \text{on } [0,T]\times\mathbb{T}^3, \tag{2.5}$$

$$v_n(0,x) = u_n(0,x) = P_n u(0,x), \quad x = (x_1,x_2,x_3)\in\mathbb{T}^3. \tag{2.6}$$

Since (2.5)-(2.6) is a locally-Lipschitz system of random ODEs, we set v_n to be the unique local solution to (2.5)-(2.6) with $v_n(0,x)\in\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$. Define

$$\tau_n = \inf\{t\in\mathbb{R}^+ : \sup_{0\leq s\leq t}\|v_n(s)\|_{\mathbb{H}^{\frac{1}{2}}} = \infty\}.$$

Obviously, $v_n \in C([0,\tau_n)\times\mathbb{T}^3)$.

For the multidimensional Burgers equations, if the initial data has zero average, the solutions are not necessary to have zero average for positive times. This leads to that, for positive s, $\|\cdot\|_s$ is smaller than $\|\cdot\|_{\mathbb{H}^s}$. Hence, $\|\cdot\|_s$ is not equivalent to $\|\cdot\|_{\mathbb{H}^s}$ for the multidimensional Burgers equations. This is different from the case of Navier-Stokes equations. Further more, due to the absence of the incompressible property and high nonlinearity of 3D Burgers equations, one can not obtain the *a priori* estimates in $\mathbb{L}^2(\mathbb{T}^3)$. To overcome the difficulty, we need to use estimates in $\dot{\mathbb{H}}^s(\mathbb{T}^3)$ and $\mathbb{L}^1(\mathbb{T}^3)$ norm of initial data to dominate the energy in $\mathbb{L}^2(\mathbb{T}^3)$, see (2.8) and Lemma 2.1. In fact, one can see that estimate (2.7) in Lemma 2.1 is vital to establish the uniqueness of the solutions to the stochastic 3D Burgers equations, see derivation of (3.11), (3.11), (4.29) and (4.30).

2.3 Some lemmas

Lemma 2.1. *Let u, v be the local solutions of (2.5) up to a random positive time $\tau > 0$, with initial data $u_0 \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ and $v_0\in\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$, respectively. Let $\xi := u - v$ and $\xi_0 := u_0 - v_0$, then for $t\in[0,\tau]$, we have*

$$\left|\int_{\mathbb{T}^3}\left(\xi(t)-\xi_0\right)dx\right| \leq 8\pi^3\alpha^{-1}\int_0^t \|\xi\|_{\frac{1}{2}}(\|u(s)\|_{\frac{1}{2}} + \|v(s)\|_{\frac{1}{2}})ds. \tag{2.7}$$

In particular, taking $v \equiv 0$ yields the following

$$\left|\int_{\mathbb{T}^3} u(x,t)dx\right| \le 8\pi^3 \int_0^t \alpha^{-1}(s)\|u(s)\|_{\frac{1}{2}}^2 ds + \left|\int_{\mathbb{T}^3} u_0(x)dx\right|.$$

Proof. For $k \in \mathbb{Z}^3$, let $\hat{u}_k, \hat{v}_k$ and $\hat{\xi}_k$ be the kth Fourier coefficients of u, v and ξ, respectively. In view of the equations of by u and v, we derive the following

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^3} \xi(t,x)dx &= -\alpha^{-1}\int_{\mathbb{T}^3} \Big((u\cdot\nabla)\xi(t,x) + (\xi\cdot\nabla)v(t,x)\Big)dx \\ &= -8\pi^3 i\alpha^{-1}\sum_{k\in\mathbb{Z}^3}\Big\{\Big(\overline{\hat{u}_k(t)}\cdot k\Big)\hat{\xi}_k(t) + \Big(\overline{\hat{\xi}_k(t)}\cdot k\Big)\hat{v}_k(t)\Big\}.\end{aligned}$$

Hence

$$\begin{aligned}\left|\frac{d}{dt}\int_{\mathbb{T}^3} \xi(t,x)dx\right| &\le 8\pi^3\alpha^{-1}\sum_{k\in\mathbb{Z}^3}|\hat{\xi}_k||k|(|\hat{u}_k| + |\hat{v}_k|) \\ &\le 8\pi^3\alpha^{-1}\|\xi(t)\|_{\frac{1}{2}}(\|u(t)\|_{\frac{1}{2}} + \|v(t)\|_{\frac{1}{2}}),\end{aligned}$$

and (2.7) then follows from the integration of the above estimate with respect to t. □

In view of Lemma 2.1, we can obtain Corollary 2.1 where the formula (2.8) will play important roles in the proofs of global existence and uniqueness results for solutions to (2.3)-(2.4), see Theorem 4.1 and Theorem 4.3.

Corollary 2.1. *Let v_n be the solution to (2.5)-(2.6) with $v_n(0)(\in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3))$ and τ being its initial data and existence time, respectively. For any $s > 0$ and $t \in [0,\tau]$, we have*

$$\|v_n(t)\|_s \le \|v_n(t)\|_{\mathbb{H}^s} \le \|v_n(t)\|_s + c\int_0^t \|v_n(s)\|_{\frac{1}{2}}^2 ds + c|u_0|_1, \tag{2.8}$$

for some $c = \max\{(2\pi)^{9/2}\alpha^{-1}(t), (2\pi)^{3/2}\}$.

Proof. Define $\bar{v}_n = \bar{v}_n(t) = \int_{\mathbb{T}^3} v_n(t,x)dx, t \in [0,\tau]$. If we make a decomposition of v_n as in (2.2), then we find that $\bar{v}_n$ is the first component of v_n. Hence, for any $s > 0$, we have

$$\begin{aligned}|v_n|_2 &\le |v_n - \bar{v}_n|_2 + (2\pi)^{3/2}|\bar{v}_n| \le \|v_n\|_s + (2\pi)^{3/2}|\bar{v}_n| \\ &\le \|v_n\|_s + (2\pi)^{9/2}\alpha^{-1}(t)\int_0^t \|v_n\|_{\frac{1}{2}}^2 ds + (2\pi)^{3/2}|u_0|_1,\end{aligned}$$

where the last inequality follows by Lemma2.1. Consequently, we have

$$\|v_n(t)\|_s \le \|v_n(t)\|_{\mathbb{H}^s} \le \|v_n(t)\|_s + c\int_0^t \|v_n(s)\|_{\frac{1}{2}}^2 ds + c|u_0|_1,$$

for some $c = \max\{(2\pi)^{9/2}\alpha^{-1}(t), (2\pi)^{3/2}\}$. □

To prove Theorem 4.1 and Theorem 4.3, we further need the following two classical lemmas from [50] and [41] respectively.

Lemma 2.2. *Let B_0, B, B_1 be Banach spaces such that B_0, B_1 are reflexive and $B_0 \overset{c}{\subset} B \subset B_1$, where $B_0 \overset{c}{\subset} B$ stands for compact imbedding. Define, for $0 < T < \infty$,*

$$X := \left\{ h \middle| h \in L^2([0,T]; B_0), \frac{dh}{dt} \in L^2([0,T]; B_1) \right\}.$$

Then X is a Banach space equipped with the norm $|h|_{L^2([0,T];B_0)} + |h'|_{L^2([0,T];B_1)}$. Moreover,

$$X \overset{c}{\subset} L^2([0,T]; B).$$

Lemma 2.3. *Let V, H, V' be three Hilbert spaces such that $V \subset H = H \subset V'$, where H' and V' are the dual spaces of H and V respectively. Suppose $u \in L^2(0,T;V)$ and $u' \in L^2(0,T;V')$. Then u is almost everywhere equal to a function continuous from $[0,T]$ into H.*

One can refer to Temam [50] and other references for the proof of the Lemma 2.2. The Lemma 2.3, a special case of a general result of Lions and Magenes [41], will help us to verify the continuity of the solution to stochastic Burgers equations with respect to time. For the proof of the Lemma 2.3, one can also see [50]. In fact, this regularity is important for us to establish the global existence of solutions to stochastic equations (2.3)-(2.4). As we know, the maximum principle should be applied to classical solutions to differential equations. But there is no classical solutions to stochastic partial differential equations. Therefore, our ideal is that we apply the maximum principle to random Galerkin approximations. Then, we combine the compactness argument with the regularity of the local solutions to show that the global well-posedness of (2.3)-(2.4) holds, see the proof of Theorem 4.1 for details.

3 Local existence of the solutions to (1.1)

We will use the approach of Galerkin approximations to show the existence of a local strong solution to equation (2.3)-(2.4). In fact, it is sufficient to establish the existence of a local strong solution on time interval $[0,1]$ as what we do in Proposition 3.1. Because, in view of Proposition 3.1, we can extend the existence time of the local solutions to a more broad time interval than $[0,1]$ by repeating the proof of Proposition 3.1. Through the iterative extension, we can seek the maximum existence time for the local strong solutions to (2.3)-(2.4). If the maximum existence time equals to infinite almost surely, then the local strong solutions are the global strong solutions.

Proposition 3.1. *Suppose u_0 is an $\mathbb{H}^1(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable. Then, there exists a unique local strong pathwise solution v to equation (2.3)-(2.4) on the time interval $[0,1]$ satisfying*

$$\sup_{t\in[0,\tau_0^*]} \|v(t)\|_{\mathbb{H}^1}^2 + \int_0^{\tau_1^*} \|v(t)\|_2^2 dt < \infty, \mathbb{P} - a.e. \omega \in \Omega,$$

where τ_0^* is a positive random variable which is smaller than 1. Moreover, the local strong pathwise solution v to equation (2.3)-(2.4) is Lipschitz continuous with respect to the initial data u_0 in $\mathbb{H}^1(\mathbb{T}^3)$.

Proof. For $t \in (0, \tau_n \wedge 1)$, taking inner product of (2.5) in $L^2([0,t] \times \mathbb{T}^3)$ with $\Lambda^2 v_n$ yields

$$
\begin{aligned}
& \|v_n(t)\|_1^2 + 2\int_0^t \|v_n(s)\|_2^2 ds \\
\leq\ & \|v_n(0)\|_1^2 + \int_0^t \alpha^{-1}(s) \int_{\mathbb{T}^3} |(v_n \cdot \nabla) v_n(s,x)| \times |\Lambda^2 v_n(s,x)| dx ds.
\end{aligned}
$$

Then by the Hölder inequality, the Sobolev imbedding theorem and the interpolation inequality, we have

$$
\begin{aligned}
& \|v_n(t)\|_1^2 + 2\int_0^t \|v_n(s)\|_2^2 ds \\
\leq\ & \|v_n(0)\|_1^2 + \epsilon \int_0^t \|v_n\|_2^2 ds + c(\epsilon) \int_0^t \alpha^{-2}(s) |\nabla v_n|_3^2 |v_n|_6^2 ds \\
\leq\ & \|v_n(0)\|_1^2 + \epsilon \int_0^t \|v_n\|_2^2 ds + c(\epsilon) \int_0^t \alpha^{-4}(s) \|v_n(s)\|_1^2 |v_n(s)|_6^4 ds \\
\leq\ & \|v_n(0)\|_1^2 + \epsilon \int_0^t \|v_n\|_2^2 ds + c(\epsilon) \int_0^t \alpha^{-4}(s) \|v_n(s)\|_1^2 \|v_n(s)\|_{\mathbb{H}^1}^4 ds.
\end{aligned}
$$

Note that

$$
\|v_n(s)\|_{\mathbb{H}^1}^4 = (\|v_n(s)\|_1 + |v_n(s)|_2)^4
$$

and

$$
|v_n(s)|_2 \leq \|v_n(s)\|_1 + c\int_0^s \|v_n(r)\|_{\frac{1}{2}}^2 dr + |u_0|_1,
$$

where c only depends on the dimain $\mathbb{T}^3$. In view of the estimates above, we arrive at

$$
\begin{aligned}
& \|v_n(t)\|_1^2 + \int_0^t \|v_n(s)\|_2^2 ds \\
\leq\ & \|v_n(0)\|_1^2 + c\int_0^t \alpha^{-4}(s) \|v_n(s)\|_1^2 \Big(\|v_n(s)\|_1 + c\int_0^s \|v_n(r)\|_{\frac{1}{2}}^2 dr + |u_0|_1 \Big)^4 ds \\
\leq\ & \|v_n(0)\|_1^2 + c\int_0^t \alpha^{-4}(s) \|v_n(s)\|_1^2 \Big[\|v_n(s)\|_1^4 + c\Big(\int_0^s \|v_n(r)\|_{\frac{1}{2}}^2 dr + |u_0|_1 \Big)^4 \Big] ds \\
\leq\ & \|v_n(0)\|_1^2 + c\int_0^t \alpha^{-4}(s) \|v_n(s)\|_1^2 \Big[\|v_n(s)\|_1^4 + c\Big(\int_0^t \|v_n(r)\|_{\frac{1}{2}}^2 dr + |u_0|_1 \Big)^4 \Big] ds.
\end{aligned}
$$

Split the last term into two terms, we get

$$
\begin{aligned}
& \|v_n(t)\|_1^2 + \int_0^t \|v_n(s)\|_2^2 ds \\
\leq \ & \|v_n(0)\|_1^2 + c\int_0^t \alpha^{-4}(s)\|v_n(s)\|_1^6 ds + c\int_0^t \alpha^{-4}(s)\|v_n(s)\|_1^2 ds\Big(\int_0^t \|v_n(s)\|_{\frac{1}{2}}^2 ds + |u_0|_1\Big)^4 \\
\leq \ & \|v_n(0)\|_1^2 + c\int_0^t \alpha^{-4}(s)\|v_n(s)\|_1^6 ds + c|u_0|_1^4\int_0^t \alpha^{-4}(s)\|v_n(s)\|_1^2 ds \\
& +c\int_0^t \alpha^{-4}(s)\|v_n(s)\|_1^2 ds\Big(\int_0^t \|v_n(s)\|_{\frac{1}{2}}^2 ds\Big)^4 \\
\leq \ & \|v_n(0)\|_1^2 + c\int_0^t \alpha^{-4}(s)\|v_n(s)\|_1^6 ds + c|u_0|_1^4\int_0^t \alpha^{-4}(s)\|v_n(s)\|_1^2 ds \\
& +c\Big(\int_0^t \alpha^{-4}(s)\|v_n(s)\|_1^2 ds\Big)^5 + c\Big(\int_0^t \|v_n(s)\|_1^2 ds\Big)^5 \\
\leq \ & \|v_n(0)\|_1^2 + c\int_0^t \alpha^{-4}(s)\|v_n(s)\|_1^6 ds + c|u_0|_1^4\int_0^t \alpha^{-4}(s)\|v_n(s)\|_1^2 ds \\
& +c\int_0^t t^4\alpha^{-20}(s)\|v_n(s)\|_1^{10} ds + c\int_0^t t^4\|v_n(s)\|_1^{10} ds.
\end{aligned}
$$

At the beginning of the proof, we know that $t \in [0, 1]$, so we obtain

$$
\begin{aligned}
\|v_n(t)\|_1^2 + \int_0^t \|v_n(s)\|_2^2 ds \ \leq \ & \|v_n(0)\|_1^2 + \int_0^t \Big(c(1+\alpha^{-4}(s))(1+|u_0|_1^4)^{1/5}\|v_n(s)\|_1^2\Big)^5 ds \\
& + \int_0^t \Big(c(1+|u_0|_1^4)^{1/5}(1+\alpha^{-4}(s))\Big)^5 ds
\end{aligned}
$$

Let $f(s) := c(1+\alpha^{-4}(s))$ and $g(s) := c(1+|u_0|_1^4)^{1/5}(1+\alpha^{-4}(s))$. Then

$$
\begin{aligned}
\|v_n(t)\|_1^2 + \int_0^t \|v_n(s)\|_2^2 ds \ = \ & \|v_n(0)\|_1^2 + \int_0^t \Big(f(s)\|v_n(s)\|_1^2\Big)^5 ds + \int_0^t g^5(s) ds \\
\leq \ & \|v_n(0)\|_1^2 + \int_0^t (f(s)\|v_n(s)\|_1^2 + g(s))^5 ds \\
\leq \ & \|u_0\|_1^2 + \int_0^t (\|v_n(s)\|_1^2 \sup_{s\in[0,1]} f(s) + \sup_{s\in[0,1]} g(s))^5 ds.
\end{aligned}
$$

For simplicity, we set $A := \sup_{s\in[0,1]} f(s)$ and $B := \sup_{s\in[0,1]} g(s)$, then we have

$$
\|v_n(t)\|_1^2 + \int_0^t \|v_n(s)\|_2^2 ds \leq \|v_n(0)\|_1^2 + \int_0^t (A\|v_n(s)\|_1^2 + B)^5 ds. \tag{3.9}
$$

By the comparison theorem (see Theorem III-5-1 in page 59 of [34]), it follows that

$$
\|v_n(t)\|_1^2 \ \leq \ \frac{A\|u_0\|_1^2 + B}{A(1-4At(A\|u_0\|_1^2+B)^4)^{1/4}} - \frac{B}{A}. \tag{3.10}
$$

The estimates above rules out a blowup of v_n in $\mathbb{H}^1(\mathbb{T}^3)$ before the time

$$\tau^* := \frac{1}{4A(A\|u_0\|_1^2 + B)^4}.$$

It follows that one can choose $\tau_0^* = \frac{\tau^*}{2} > 0$ such that τ_0^* is independent of $n \in \mathbb{N}$, which together with (3.9) and (3.10) implies that v_n are uniformly bounded in $L^\infty([0, \tau_0^*]; \mathbb{H}^1) \cap L^2([0, \tau_0^*]; \mathbb{H}^2)$. From (2.3), by virtue of the Hölder inequality, the Sobolev imbedding theorem and Young's inequality we have

$$\begin{aligned}
|\partial_t v_n|_2 &\le \alpha^{-1}|v_n \cdot \nabla v_n|_2 + |\Delta v_n|_2 \\
&\le \alpha^{-1}|v_n|_\infty |\nabla v_n|_2 + \|v_n\|_2 \\
&\le c\alpha^{-1}\|v_n\|_{\mathbb{H}^1}^{\frac{1}{2}}\|v_n\|_{\mathbb{H}^2}^{\frac{1}{2}}\|v_n\|_1 + \|v_n\|_2 \\
&\le c\alpha^{-1}(\|v_n\|_1^{\frac{3}{2}}\|v_n\|_2^{\frac{1}{2}} + \|v_n\|_1^{\frac{3}{2}}|v_n|_2^{\frac{1}{2}} \\
&\quad + |v_n|_2^{\frac{1}{2}}\|v_n\|_1\|v_n\|_2^{\frac{1}{2}} + |v_n|_2\|v_n\|_1) + \|v_n\|_2,
\end{aligned}$$

where c is independent of dimension n and random time τ. In view of (2.8), we note that

$$|v_n(t)|_2 \le c\Big(\int_0^t \|v_n(s)\|_{\frac{1}{2}}^2 ds + |v_0|_1\Big),$$

where c is independent of n and random time τ. Hence, from (3.9)-(3.10), we know $\partial_t v_n$ are uniformly bounded in $L^2([0, \tau_0^*]; \mathbb{L}^2(\mathbb{T}^3))$. From Lemma 2.2 and Lemma 2.3, we conclude that there exists a subsequence of v_n, which is still denoted by v_n, such that v_n converges to v in $L^2([0, \tau_0^*]; \mathbb{H}^1(\mathbb{T}^3))$ and $v \in C([0, \tau_0^*]; \mathbb{H}^1(\mathbb{T}^3))$. Following a standard argument as in [50], one can show that v is the local strong solution to (2.3)-(2.4).

In the following, we will prove the uniqueness of v in $C([0, \tau_0^*]; \mathbb{H}^1(\mathbb{T}^3))$. Let v_1 and v_2 be two local strong solutions to (2.3)-(2.4). Denote by $\hat{v} := v_1 - v_2$. Then, for $t \in [0, \tau_0^*]$, we have

$$\begin{aligned}
\frac{1}{2}\partial_t|\hat{v}|_2^2 + \|\hat{v}\|_1^2 &\le -\alpha^{-1}\langle \hat{v} \cdot \nabla v_1, \hat{v}\rangle - \alpha^{-1}\langle v_2 \cdot \nabla \hat{v}, \hat{v}\rangle \\
&\le \alpha^{-1}|\hat{v}|_2^{\frac{1}{2}}\|\hat{v}\|_{\mathbb{H}^1}^{\frac{3}{2}}\|v_1\|_1 + \varepsilon\|\hat{v}\|_1^2 + c\alpha^{-2}|v_2|_\infty^2|\hat{v}|_2^2 \\
&\le \varepsilon\|\hat{v}\|_1^2 + \alpha^{-4}|\hat{v}|_2^2\|v_1\|_1^4 + c\alpha^{-2}\Big(\|v_2\|_1 + \int_0^t \|v_2\|_{\frac{1}{2}}^2 ds + |u_0|_1\Big) \\
&\quad \times\Big(\|v_2\|_2 + \int_0^t \|v_2\|_{\frac{1}{2}}^2 ds + |u_0|_1\Big)|\hat{v}|_2^2,
\end{aligned}$$

which implies via the Gronwall inequality and $\hat{v}(0) = 0$ that $|\hat{v}(t)|_2 = 0, t \in [0, \tau_0^*]$. Taking inner product of (2.5) in $\mathbb{L}^2(\mathbb{T}^3)$ with $(-\Delta v_n)$ and using interpolation inequality further yields,

$$\begin{aligned}
\frac{1}{2}\partial_t\|\hat{v}\|_1^2 + \|\hat{v}\|_2^2 &\le \alpha^{-1}\langle \hat{v} \cdot \nabla v_1, \Delta\hat{v}\rangle + \alpha^{-1}\langle v_2 \cdot \nabla\hat{v}, \Delta\hat{v}\rangle \\
&\le \alpha^{-1}\|\hat{v}\|_2\|\hat{v}\|_1^{\frac{1}{2}}\|\hat{v}\|_2^{\frac{1}{2}}\|v_1\|_1 + \alpha^{-1}\|v_2\|_{\mathbb{H}^1}^{\frac{1}{2}}\|v_2\|_{\mathbb{H}^2}^{\frac{1}{2}}\|\hat{v}\|_1\|\hat{v}\|_2 \\
&\le \varepsilon\|\hat{v}\|_2^2 + c(\varepsilon)\alpha^{-4}\|\hat{v}\|_1^2\|v_1\|_1^4 + c(\varepsilon)\alpha^{-2}\|v_2\|_{\mathbb{H}^1}\|v_2\|_{\mathbb{H}^2}\|\hat{v}\|_1^2,
\end{aligned}$$

which implies via the Gronwall inequality and $\hat{v}(0) = 0$ that $\|\hat{v}(t)\|_1 = 0$ for $t \in [0, \tau_0^*]$. The above two estimates about $\hat{v}$ imply the Lipschitz continuity of the local strong solution v with respect to the initial data in $\mathbb{H}^1(\mathbb{T}^3)$. $\square$

We should emphasise here that the following Lemma is key to establish the global well-posedness for strong solutions and weak solutions to (1.1) in $C([0,T];\mathbb{H}^1(\mathbb{T}^3))$ and $C([0,T];\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3))$, respectively. The local strong solutions should be smoother than the global solutioms. That is what the Lemma 3.1 does.

The proof of Lemma 3.1 relies on *commutator estimates*, see Theorem A.8 of [38] for more details of commutator estimates.

Lemma 3.1. *Suppose u_0 is an $\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable. Then, there exists a unique local strong pathwise solution v to equation (2.3)-(2.4) on* $[0,1]$ *satisfying*

$$\sup_{t\in[0,\tau_1]} \|v(t)\|^2_{\mathbb{H}^{\frac{3}{2}}} + \int_0^{\tau_1} \|v(t)\|^2_{\frac{5}{2}} dt < \infty, \mathbb{P} - a.e. \omega \in \Omega.$$

where the positive random variable τ_1 is the local existence time for v. Moreover, the local strong pathwise solution v to equation (2.3)-(2.4) is Lipschitz continuous with respect to the initial data in $\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$.

Proof. For $t \in (0, \tau_n)$, taking inner product of (2.5) in $L^2([0,t]\times\mathbb{T}^3)$ with $\Lambda^3 v_n$ yields

$$\frac{1}{2}\partial_t \|v_n\|^2_{\frac{3}{2}} + \|v_n\|^2_{\frac{5}{2}} = -\alpha^{-1}\langle \Lambda^{1/2}(v_n\cdot\nabla v_n), \Lambda^{5/2} v_n\rangle$$

which implies

$$\begin{aligned} \|v_n(t)\|^2_{\frac{3}{2}} + 2\int_0^t \|v_n(s)\|^2_{\frac{5}{2}} ds \quad &\le \quad \|u_0\|^2_{\frac{3}{2}} + \varepsilon \int_0^t \|v_n(s)\|^2_{\frac{5}{2}} ds \\ &\quad + c \sup_{s\in[0,t]} \alpha^{-2}(s) \int_0^t \int_{\mathbb{T}^3} |\Lambda^{1/2}(v_n\cdot\nabla v_n)|^2 dx ds. \end{aligned} \tag{3.11}$$

In order to bound the last term on the right hand side of (3.11), we will use Theorem A.8 in [38]. Without loss of generality, we assume $t \le 1$. Then the estimates of the last term follows

$$\begin{aligned} & \int_0^t \int_{\mathbb{T}^3} |\Lambda^{1/2}(v_n\cdot\nabla v_n)|^2 dx ds \\ \le \quad & 3\int_0^t \int_{\mathbb{T}^3} |v_n\cdot(\Lambda^{3/2} v_n)|^2 dx ds + 3\int_0^t \int_{\mathbb{T}^3} |(\Lambda^{1/2} v_n)\cdot(\Lambda v_n)|^2 dx ds \\ & + c\Big\{\int_0^t \Big(\int_{\mathbb{T}^3} |\Lambda^{3/8} v_n|^6 dx\Big)^{2/3} ds\Big\}^{1/2} \Big\{\int_0^t \Big(\int_{\mathbb{T}^3} |\Lambda^{9/8} v_n|^3 dx\Big)^{4/3} ds\Big\}^{1/2} \\ =: \quad & I_1 + I_2 + I_3^{1/2} \times I_4^{1/2} \\ \le \quad & I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Using the Holder inequality, the interpolation inequality and Young's inequality, we get

$$\begin{aligned} I_2 &\le \int_0^t |\Lambda v_n(s)|_2^2 |\Lambda^{1/2} v_n(s)|_\infty^2 ds \\ &\le c\int_0^t \|v_n(s)\|_1^2 \|v_n(s)\|_{\frac{3}{2}} \|v_n(s)\|_{\frac{5}{2}} ds \\ &\le \varepsilon \int_0^t \|v_n(s)\|_{\frac{5}{2}}^2 ds + c\int_0^t \|v_n(s)\|_{\frac{3}{2}}^6 ds. \end{aligned}$$

In view of the Holder inequality, the Sobolev embedding theorem and (2.8), we have

$$\begin{aligned} I_1 &\le c\int_0^t |v_n|_\infty^2 \|v_n\|_{\frac{3}{2}}^2 ds \le c\int_0^t \|v_n\|_{\mathbb{H}^{\frac{3}{2}}}^4 ds \\ &\le c\int_0^t \|v_n\|_{\frac{3}{2}}^4 ds + c\int_0^t |v_n|_2^4 ds \\ &\le c\int_0^t \|v_n\|_{\frac{3}{2}}^4 ds + ct\Big(\int_0^t \|v_n(s)\|_{\frac{1}{2}}^2 ds\Big)^4 + \int_0^t |u_0|_1^4 ds \\ &\le c\int_0^t \|v_n\|_{\frac{3}{2}}^4 ds + c\int_0^t \|v_n(s)\|_{\frac{1}{2}}^8 ds + ct|u_0|_1^4. \end{aligned}$$

Utilising the interpolation inequality and (2.8), we then derive

$$\begin{aligned} I_4 &\le c\int_0^t |\Lambda^{9/8} v_n|_3^4 ds \le c\int_0^t \|v_n\|_{\mathbb{H}^{\frac{3}{2}}}^{7/2} \|v_n\|_{\mathbb{H}^{\frac{5}{2}}}^{1/2} ds \\ &\le \varepsilon \int_0^t \|v_n\|_{\mathbb{H}^{\frac{5}{2}}}^2 ds + c\int_0^t \|v_n\|_{\mathbb{H}^{\frac{3}{2}}}^{14/3} ds \\ &\le \varepsilon \int_0^t \|v_n(s)\|_{\frac{5}{2}}^2 ds + c\int_0^t \|v_n(s)\|_{\frac{3}{2}}^{14/3} ds \\ &\quad + c\int_0^t \|v_n(s)\|_{\frac{1}{2}}^4 ds + ct|u_0|_1^2 + c\int_0^t \|v_n(s)\|_{\frac{1}{2}}^{28/3} ds + ct|u_0|_1^{14/3}. \end{aligned}$$

By the Sobolev imbedding theorem, we obtain

$$I_3 \le c\int_0^t \|v_n\|_{\frac{3}{2}}^4 ds.$$

Combing the argument above, we get the new estimates for (3.11), that is,

$$\begin{aligned} &\|v_n(t)\|_{\frac{3}{2}}^2 + \int_0^t \|v_n(s)\|_{\frac{5}{2}}^2 ds \\ \le\ & c\|u_0\|_{\frac{3}{2}}^2 + c\sup_{s\in[0,1]} \alpha^{-2}(s) \int_0^t \Big[(1+|u_0|_1^2) + \|v_n(s)\|_{\frac{3}{2}}^2\Big]^{14} ds. \end{aligned}$$

Define $A = 1 + |u_0|_1^2$. Then we have

$$\|v_n(t)\|_{\frac{3}{2}}^2 + \int_0^t \|v_n(s)\|_{\frac{5}{2}}^2 ds \le c\|u_0\|_{\frac{3}{2}}^2 + c\sup_{s\in[0,1]} \alpha^{-2}(s) \int_0^t (A + \|v_n(s)\|_{\frac{3}{2}}^2)^{14} ds. \tag{3.12}$$

Again, by the comparison theorem (see Theorem III-5-1 in page 59 of [34])

$$\|v_n(t)\|^2_{\frac{3}{2}} \le \frac{A + \|u_0\|^2_{\frac{3}{2}}}{\left[1 - 13c \sup\limits_{s\in[0,1]} \alpha^{-2}(s) t (A + \|u_0\|^2_{\frac{3}{2}})^{13}\right]^{1/13}} - A. \tag{3.13}$$

Hence the estimates (3.13) rules out a blowup of v_n in $\mathbb{H}^{\frac{3}{2}}$ before the time $\tau_1^* = \frac{1}{13c \sup\limits_{s\in[0,1]} \alpha^{-2}(s)(A+\|u_0\|^2_{\frac{3}{2}})^{13}}$. It follows that there exists $\tau_1 > 0$, we can for example take $\tau_1 = \tau_1^*/2$, such that $\tau_n \ge \tau_1$ for all n. From (3.12) and (3.13), we have uniform bounds for v_n in $L^\infty([0,\tau_1];\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$ and in $L^2([0,\tau_1];\mathbb{H}^{\frac{5}{2}}(\mathbb{T}^3))$. It is easy to show that $\partial_t v_n$ is uniformly bounded in $L^2([0,\tau_1];\mathbb{L}^2(\mathbb{T}^3))$. By Lemma 2.2 and Lemma 2.3, there exists a subsequence of v_n, which converges to v in $L^2([0,\tau_1];\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$ with $v \in C([0,\tau_1];\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$. By a standard argument one knows v is a local strong solution to (2.3)-(2.4). Taking a similar argument as in Proposition 3.1, we can show that v is Lipschitz continuous with respect to the initial data in $\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$.

□

Lemma 3.2. *The maximum existence times τ^* and τ^{**} for local solutions in Proposition 3.1 and Lemma 3.1, satisfy*

$$\mathbb{P}\Big(I_{(\tau^{**}<\infty)}(\tau^* - \tau^{**}) \ge 0\Big) = 1.$$

Proof. In order to prove the result, it is equivalent to prove

$$\mathbb{P}(\tau^{**} = \infty) + \mathbb{P}(\tau^{**} < \infty, \tau^* - \tau^{**} \ge 0) = 1.$$

If $\mathbb{P}(\tau^{**} = \infty) = 1$, then the result follows. Or else, we assume $\mathbb{P}(\tau^{**} < \infty) > 0$, then the above equality is equivalent to

$$\mathbb{P}(\tau^* - \tau^{**} \ge 0 | \tau^{**} < \infty) = 1.$$

On $(\tau^{**} < \infty)$, for arbitrary $t \in (0, \tau^{**})$, we have

$$v \in C([0,t];\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)).$$

Note that $\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3) \subset \mathbb{H}^1(\mathbb{T}^3)$, which implies $t < \tau^*$. By the arbitrariness of t, one gets that on $(\tau^{**} < \infty)$, $\tau^{**} \le \tau^*$ holds. It completes the proof. □

4 Global well-posedness to (1.1)

The key tool in our study of the regularity of the solutions to stochastic 3D Burgers equations is the maximum principle, stated as Lemma 4.1.

Lemma 4.1. *If v_n is a solution to the random Burgers equation (2.5)-(2.6) on the time interval $[0,t]$, then*

$$\sup_{s\in[0,t]} |v_n(s)|_\infty \quad \le \quad |v_n(0)|_\infty, \;\; \mathbb{P} - a.s. \omega \in \Omega. \tag{4.14}$$

Proof. Let $\beta > 0$ and set $f(s,x) := e^{-\beta s} v_n(s,x)$ for all $s \in [0,t]$ and $x \in \mathbb{T}^3$. Then, multiplying v on both sides of (2.3) yields

$$\partial_s |v_n(s)|^2 + \alpha^{-1}(s) v_n(s) \cdot \nabla |v_n(s)|^2 - 2(\Delta v_n \cdot v_n)(s) = 0.$$

Note that $|v_n(s)|^2 = |f(s)|^2 e^{2\beta s}$ satisfies

$$(e^{2\beta s} \partial_s |f(s)|^2 + 2\beta e^{2\beta s} |f(s)|^2) + e^{3\beta s} \alpha^{-1}(s) f(s) \cdot \nabla |f(s)|^2 - 2e^{2\beta s} \Delta f(s) \cdot f(s) = 0,$$

which implies

$$\partial_s |f(s)|^2 + 2\beta |f(s)|^2 + e^{\beta s} \alpha^{-1}(s) f(s) \cdot \nabla |f(s)|^2 - 2\Delta f(s) \cdot f(s) = 0.$$

On the other hand, since

$$2\Delta f(s) \cdot f(s) = \Delta |f(s)|^2 - 2|\nabla f|^2,$$

then we have

$$\partial_s |f(s)|^2 + 2\beta |f(s)|^2 + e^{\beta s} \alpha^{-1}(s) f(s) \cdot \nabla |f(s)|^2 - \Delta |f(s)|^2 + 2|\nabla f|^2 = 0.$$

We observe that if $|f|$ has local maximum at $(t,x) \in (t_0, t] \times \mathbb{T}^3$, then the left hand side of the above equality is positive unless $|f(t,x)| \equiv 0$. Therefore,

$$|f(s)|_\infty \le |f(0)|_\infty,$$

which implies

$$|v_n(s)|_\infty \le e^{\beta s} |v_n(0)|_\infty, \text{ for } s \in (0,t].$$

Let β tends to 0, we get the desired result. □

In the following, we will use Lemma 4.1 as well as compactness and regularity arguments to complete the proof of the global well-posedness of (2.3)-(2.4).

Theorem 4.1. *Suppose u_0 is an $\mathbb{H}^1(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable. Then, for any $T > 0$, there exists a unique global strong pathwise solution v to (2.3)-(2.4) in the sense of Definition 2.1 satisfying*

$$\sup_{t\in[0,T]} \|v(t)\|_{\mathbb{H}^1}^2 + \int_0^T \|v(s)\|_{\mathbb{H}^2}^2 ds \le \|u_0\|_{\mathbb{H}^1}^2 + c\|v(\epsilon)\|_{\mathbb{H}^1}^2 \exp\left(c\|v(\epsilon)\|_{\mathbb{H}^{\frac{3}{2}}}^2 \int_0^T \alpha^{-2}(s) ds\right) < \infty,$$

almost surely, where ϵ is some positive random variable in $(0,T)$. Moreover, the strong pathwise solution v to equation (2.3)-(2.4) is Lipschitz continuous with respect to the initial data in $\mathbb{H}^1(\mathbb{T}^3)$.

Proof. Let τ^* be the maximum existence time for the unique local strong solution v to (2.3)-(2.4). For $t \in (0, \tau^*)$, taking inner product of (2.5) with $\Lambda^2 v_n$ in $\mathbb{L}^2(\mathbb{T}^3)$ yields

$$\partial_t \|v_n\|_1^2 \leq 2\alpha^{-1} \left| \int_{\mathbb{T}^3} (v_n \cdot \nabla) v_n \Delta v_n dx \right| - 2\|v_n\|_2^2 \leq \alpha^{-2} |v_n|_\infty^2 \|v_n\|_1^2 + \|v_n\|_2^2.$$

For $0 < \epsilon < t < \tau^*$, by Lemma 4.1, we have

$$\|v_n(t)\|_1^2 + \int_\epsilon^t \|v_n(s)\|_2^2 ds \leq c\|v_n(\epsilon)\|_1^2 \exp\left(c\|v_n(\epsilon)\|_{\mathbb{H}^{\frac{3}{2}}}^2 \int_0^t \alpha^{-2}(r) dr \right). \tag{4.15}$$

From the proof of Proposition3.1, there exists (a subsequence of) v_n converging to v in $L^2([0, \hat{\tau}]; \mathbb{H}^1(\mathbb{T}^3))$, where the random variable $\hat{\tau} \in (0, \tau^*)$ and $v \in C([0, \hat{\tau}]; \mathbb{H}^1(\mathbb{T}^3)) \cap L^2([0, \hat{\tau}]; \mathbb{H}^2(\mathbb{T}^3))$. It implies that we can choose a subsequence of v_n still denoted by itself such that

$$v_n(t) \to v(t) \text{ in } \mathbb{H}^1(\mathbb{T}^3) \text{ almost every with respect to time } t \in [0, \tau^*),$$

Without loss of generality, we assume $v(\epsilon) \in \mathbb{H}^2(\mathbb{T}^3) \subset \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$. Regarding ϵ as the initial time of (2.3)-(2.4), by Lemma 3.1, there exists a maximum local solution (v, τ^{**}). By the proof of Lemma 3.1, there exists a subsequence of v_n still denoted by the above sequence such that

$$v_n(r) \to v(r) \text{ in } \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3) \text{ almost every with respect to time } r \in [\epsilon, \tau^* \wedge \tau^{**}).$$

Letting n tend to infinite in (4.15) yields

$$\|v(t)\|_1^2 \leq c\|v(r)\|_1^2 \exp\left(c\|v(r)\|_{\mathbb{H}^{\frac{3}{2}}}^2 \int_0^t \alpha^2(s) ds \right), \tag{4.16}$$

where (4.16) holds for t and r almost everywhere in $[0, \tau^*)$ and $[\epsilon, \tau^* \wedge \tau^{**})$ respectively. Keeping in mind that $v \in C([0, \tau^*); \mathbb{H}^1(\mathbb{T}^3)) \cap C([\epsilon, \tau^* \wedge \tau^{**}); \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$, then from (4.16) we get that

$$\|v(t)\|_1^2 \leq \|v(\epsilon)\|_1^2 \exp\left(\|v(\epsilon)\|_{\mathbb{H}^{\frac{3}{2}}}^2 \int_0^t \alpha^2(s) ds \right), \text{ for arbitrary } t \in [0, \tau^*)$$

which implies

$$\sup_{t \in [0,T]} \|v(t)\|_1^2 \leq \|v(\epsilon)\|_1^2 \exp\left(\|v(\epsilon)\|_{\mathbb{H}^{\frac{3}{2}}}^2 \int_0^T \alpha^2(s) ds \right), \; for\ arbitrary\ T \in [0, \infty). \tag{4.17}$$

As the arguments as above, we can choose $s \in \left[\epsilon, \tau^* \wedge \tau^{**}\right)$ such that there exists a subsequence $(v_{n'})_{n' \in \mathbb{N}'}$ of $(v_n)_{n \in \mathbb{N}}$ with $\mathbb{N}' \subset \mathbb{N}$ satisfying

$$v_{n'}(s) \to v(s) \text{ in } \mathbb{H}^1(\mathbb{T}^3), \text{ for almost everywhere } s \in \left[\epsilon, \tau^* \wedge \tau^{**}\right),$$

and

$$v_{n'}(s) \to v(s) \text{ in } \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3), \text{ for almost everywhere } s \in \left[\epsilon, \tau^* \wedge \tau^{**}\right).$$

From (4.15), for $t \in (0, \tau^*)$, we arrive at

$$\sup_{n' \in \mathbb{N}'} \int_s^t \|v_{n'}(s)\|_2^2 ds \leq c \sup_{n' \in \mathbb{N}'} \|v_{n'}(s)\|_1^2 \exp\left(c\|v_{n'}(s)\|_{\mathbb{H}^{\frac{3}{2}}}^2 \int_0^T \alpha^{-2}(r)dr \right) < \infty, \mathbb{P} - a.e.\omega \in \Omega.$$

Hence, there exists a subsequence $(v_{n''})_{n'' \in \mathbb{N}''}$ of $(v_{n'})_{n' \in \mathbb{N}'}$ with $\mathbb{N}'' \subset \mathbb{N}'$ such that $v_{n''}$ converges to v weakly in $L^2\Big(\Big[\epsilon, T\Big]; \mathbb{H}^2(\mathbb{T}^3)\Big)$. That is,

$$v_{n''} \rightharpoonup v, \ in \ L^2\Big(\Big[\epsilon, T\Big]; \mathbb{H}^2(\mathbb{T}^3)\Big), \ as \ n'' \to \infty,$$

where $\rightharpoonup$ stands for weak convergence. Let ϕ be the test function in $L^2([0, T]; \mathbb{H}^2(\mathbb{T}^3))$ with $\int_0^T \|\phi(s)\|_2^2 ds \leq 1$. Then, we obtain

$$\begin{aligned}
\int_s^t \langle \Delta v(r), \Delta\phi \rangle dr &= \lim_{n'' \to \infty} \int_s^t \langle \Delta v_{n''}(r), \Delta\phi \rangle dr \\
&\leq c \lim_{n'' \to \infty} \|v_{n''}(s)\|_1 \exp\Big(c\|v_{n''}(s)\|_{\mathbb{H}^{\frac{3}{2}}}^2 \int_0^T \alpha^{-2}(r)dr \Big) \\
&= c\|v(s)\|_1 \exp\Big(c\|v(s)\|_{\mathbb{H}^{\frac{3}{2}}}^2 \int_0^T \alpha^{-2}(r)dr \Big) < \infty, \mathbb{P} - a.s.,
\end{aligned}$$

which implies that

$$\int_s^t \|v(r)\|_2^2 dr \leq c\|v(s)\|_1^2 \exp\left(c\|v(s)\|_{\mathbb{H}^{\frac{3}{2}}}^2 \int_0^T \alpha^{-2}(r)dr \right) < \infty, \mathbb{P} - a.s.$$

For simplicity, we set

$$\int_s^t \|v(r)\|_2^2 dr = h_1(s) \text{ and } c\|v(s)\|_1^2 \exp\left(c\|v(s)\|_{\mathbb{H}^{\frac{3}{2}}}^2 \int_0^T \alpha^{-2}(r)dr \right) := h_2(s).$$

Then

$$h_1(s) \leq h_2(s), \ \text{for almost everywhere } s \in \Big[\epsilon, \tau^* \wedge \tau^{**}\Big).$$

From Proposition 3.1 and Lemma 3.1, we know that

$$v \in C\Big([0, \tau^*); \mathbb{H}^1(\mathbb{T}^3)\Big) \cap L^2_{loc}\Big([0, \tau^*); \mathbb{H}^2(\mathbb{T}^3)\Big) \text{ and } v \in C\Big(\Big[\epsilon, \tau^* \wedge \tau^{**}\Big); \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)\Big) \cap L^2_{loc}\Big(\Big[\epsilon, \tau^* \wedge \tau^{**}\Big); \mathbb{H}^{\frac{5}{2}}(\mathbb{T}^3)\Big).$$

It means $h_i(s), i = 1, 2$, is continuous in $\Big[\epsilon, \tau^* \wedge \tau^{**}\Big)$. Consequently, we get that $h_1(\epsilon) \leq h_2(\epsilon)$. That is,

$$\int_\epsilon^t \|v(r)\|_2^2 dr \leq c\|v(\epsilon)\|_1^2 \exp\left(c\|v(\epsilon)\|_{\mathbb{H}^{\frac{3}{2}}}^2 \int_0^T \alpha^{-2}(r)dr \right) < \infty, \mathbb{P} - a.s. \tag{4.18}$$

In Proposition 3.1, we have establish the existence of local strong solutions to (2.3)-(2.4). Hence, we define

$$t^* = \inf\Big\{ t \in [0, \tau^*) | \int_0^t \|v(r)\|_2^2 dr \geq \|u_0\|_1^2 \Big\}.$$

Then we have

$$\int_0^{t^*} \|v(r)\|_2^2 dr \le \|u_0\|_1^2, \mathbb{P} - a.s. \omega \in \Omega.$$

If ϵ in (4.18) is small enough such that $\epsilon \le t^*$, $\mathbb{P}$-a.e., then we have

$$\int_0^t \|v(r)\|_2^2 dr \le \|u_0\|_1^2 + c\|v(\epsilon)\|_1^2 \exp\left(c\|v(\epsilon)\|^2_{\mathbb{H}^{\frac{3}{2}}} \int_0^T \alpha^{-2}(r)dr\right) < \infty, \mathbb{P} - a.s. \omega \in \Omega.$$

where $t \in [0, \tau^*)$. Therefore, the results of this theorem follows. The uniqueness is given in Proposition3.1. □

In view of Theorem 4.1 and $v = \alpha u$, it is clearly true that

Theorem 4.2. *Suppose u_0 is an $\mathbb{H}^1(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable. Then, there exists a unique global strong pathwise solution u to (1.1) in the sense of Definition 2.1.*

The following theorem states the uniqueness and global existence for the weak solutions to (2.3)-(2.4). Our idea is that if the initial data $u_0 \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$, taking advantage of the parabolic structure of the Burgers equation we know that for arbitrary positive constant ϵ and some $t \in (0, \epsilon)$, the local weak solution $v(t) \in \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3) \subset \mathbb{H}^1(\mathbb{T}^3)$. Then the global existence of the strong solutions to (2.3)-(2.4) can be applied to the case of weak solutions.

Theorem 4.3. *Suppose u_0 is an $\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable. Then, there exists a unique global weak pathwise solution v to (2.3)-(2.4) in the sense of Definition 2.4. Moreover, the weak pathwise solution v to equation (2.3)-(2.4) is Lipschitz continuous with respect to the initial data in $\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$.*

Proof. Let $\mathbf{z}$ be the periodic solution to the linear heat equation with initial data u_0, then $\mathbf{z}_n := P_n\mathbf{z}$ satisfies

$$\partial_t \mathbf{z}_n - \Delta \mathbf{z}_n = 0, \;\; \mathbf{z}_n(0) = P_n u_0. \tag{4.19}$$

Let $\hat{v}_n := v_n - \mathbf{z}_n$, then $\hat{v}_n$ satisfies

$$\partial_t \hat{v}_n + \alpha^{-1} P_n[(v_n \cdot \nabla)v_n] - \Delta \hat{v}_n = 0, \;\; \hat{v}_n(0) = 0. \tag{4.20}$$

Let τ_n be the maximal existence time for v_n to (2.5)-(2.6). Then for $t \in [0, \tau_n)$, taking inner product of equation (4.19) in $\mathbb{H}$ yields,

$$\frac{1}{2}\partial_t |\mathbf{z}_n(t)|_2^2 + \|\mathbf{z}_n(t)\|_1^2 = 0.$$

which implies

$$|\mathbf{z}_n(t)|_2^2 + 2\int_0^t \|\mathbf{z}_n(s)\|_1^2 ds = 2|P_n u_0|_2^2.$$

Multiplying (4.19) with $\Lambda \mathbf{z}_n(t)$, taking integration with respect to spatial variables and time yields

$$\|\mathbf{z}_n(t)\|_{\frac{1}{2}}^2 + 2\int_0^t \|\mathbf{z}_n(s)\|_{\frac{3}{2}}^2 ds = 2\|P_n u_0\|_{\frac{1}{2}}^2.$$

Combining the estimates above for $\mathbf{z}_n$ yields,

$$\sup_{s\in[0,t]} \|\mathbf{z}_n(s)\|_{\mathbb{H}^{\frac{1}{2}}}^2 + 2\int_0^t \|\mathbf{z}_n\|_{\mathbb{H}^{\frac{3}{2}}}^2 \leq 2\|P_n u_0\|_{\mathbb{H}^{\frac{1}{2}}}^2. \tag{4.21}$$

Multiplying (4.20) with $\Lambda \hat{v}_n$ and integrating over $\mathbb{T}^3$ gives

$$\begin{aligned}
&\|\hat{v}_n(s)\|_{\frac{1}{2}}^2 + 2\int_0^t \|\hat{v}_n(s)\|_{\frac{3}{2}}^2 ds \\
\leq\ & \int_0^t \alpha^{-1}(s)|v_n(s)|_6 |\nabla v_n(s)|_2 |\Lambda \hat{v}_n(s)|_3 ds \\
\leq\ & c\int_0^t \alpha^{-1}(s)\|v_n(s)\|_{\mathbb{H}^1}\|v_n(s)\|_1\|\hat{v}_n(s)\|_{\frac{3}{2}} ds,
\end{aligned}\tag{4.22}$$

where the last inequality follows by the Sobolev imbedding theorem. To estimate (4.22), by Lemma 2.1, we have

$$\begin{aligned}
&\alpha^{-1}(s)\|v_n(s)\|_{\mathbb{H}^1}\|v_n(s)\|_1 \\
\leq\ & c\alpha^{-1}(s)\Big(\|v_n(s)\|_1 + \int_0^s \|v_n(s)\|_{\frac{1}{2}}^2 ds + |u_0|_1\Big)\|v_n(s)\|_1 \\
\leq\ & c\alpha^{-1}(s)(\|\mathbf{z}_n(s)\|_1^2 + \|\hat{v}_n(s)\|_1^2) \\
& + c\alpha^{-1}(s)(\|\mathbf{z}_n(s)\|_1 + \|\hat{v}_n(s)\|_1)\Big(\int_0^s \|v_n(s)\|_{\frac{1}{2}}^2 ds + |u_0|_1\Big).
\end{aligned}$$

In view of (4.22) and Young's inequality we have

$$\begin{aligned}
&\alpha^{-1}(s)(\|\mathbf{z}_n(s)\|_1^2 + \|\hat{v}_n(s)\|_1^2)\|\hat{v}_n(s)\|_{\frac{3}{2}} \\
\leq\ & \varepsilon\|\hat{v}_n(s)\|_{\frac{3}{2}}^2 + c(\varepsilon)\alpha^{-2}(s)\|\mathbf{z}_n(s)\|_1^4 + \alpha^{-1}(s)\|\hat{v}_n(s)\|_{\frac{3}{2}}\|\hat{v}_n(s)\|_1^2 \\
\leq\ & \varepsilon\|\hat{v}_n(s)\|_{\frac{3}{2}}^2 + c(\varepsilon)\alpha^{-2}(s)\|\mathbf{z}_n(s)\|_1^4 + \alpha^{-1}(s)\|\hat{v}_n(s)\|_{\frac{3}{2}}^2\|\hat{v}_n(s)\|_{\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
&\alpha^{-1}(s)(\|\mathbf{z}_n(s)\|_1 + \|\hat{v}_n(s)\|_1)\Big(\int_0^s \|v_n(s)\|_{\frac{1}{2}}^2 ds + |u_0|_1\Big)\|\hat{v}_n(s)\|_{\frac{3}{2}} \\
\leq\ & \frac{1}{2}\alpha^{-2}(s)\|\hat{v}_n(s)\|_{\frac{3}{2}}(\|\mathbf{z}_n(s)\|_1^2 + \|\hat{v}_n(s)\|_1^2) \\
& + \alpha^{-2}(s)\|\hat{v}_n(s)\|_{\frac{3}{2}}\Big(\int_0^s \|v_n(s)\|_{\frac{1}{2}}^2 ds + |u_0|_1\Big)^2 \\
\leq\ & \varepsilon\|\hat{v}_n(s)\|_{\frac{3}{2}}^2 + c(\varepsilon)\alpha^{-4}(s)\|\mathbf{z}_n(s)\|_1^4 + \frac{1}{2}\alpha^{-2}(s)\|\hat{v}_n(s)\|_{\frac{3}{2}}\|\hat{v}_n(s)\|_1^2 \\
& + c(\varepsilon)\alpha^{-4}(s)\Big(\int_0^s \|v_n(s)\|_{\frac{1}{2}}^2 ds + |u_0|_1\Big)^4 \\
\leq\ & \varepsilon\|\hat{v}_n(s)\|_{\frac{3}{2}}^2 + c(\varepsilon)\alpha^{-4}(s)\|\mathbf{z}_n(s)\|_1^4 + c(\varepsilon)\alpha^{-2}(s)\|\hat{v}_n(s)\|_{\frac{3}{2}}^2\|\hat{v}_n(s)\|_{\frac{1}{2}} \\
& + c(\varepsilon)\alpha^{-4}(s)\Big(\int_0^s \|v_n(s)\|_{\frac{1}{2}}^2 ds + |u_0|_1\Big)^4.
\end{aligned}$$

Rearranging the argument below (4.22) yields,

$$\begin{aligned}
&\sup_{s\in[0,t]}\|\hat{v}_n(s)\|^2_{\frac{1}{2}}+2\int_0^t\|\hat{v}_n(s)\|^2_{\frac{3}{2}}ds\\
\leq\;& c\int_0^t\alpha^{-4}(s)\|\mathbf{z}_n(s)\|_1^4ds+c\sup_{s\in[0,t]}\|\hat{v}_n(s)\|_{\frac{1}{2}}\int_0^t\alpha^{-2}(s)\|\hat{v}_n(s)\|^2_{\frac{3}{2}}ds\\
&+c\int_0^t\alpha^{-4}(s)\Big(\int_0^s\|v_n\|^2_{1/2}ds+|u_0|_1\Big)^4ds+c\int_0^t\alpha^{-2}(s)\|\mathbf{z}_n(s)\|_1^4ds\\
&+\sup_{s\in[0,t]}\|\hat{v}_n(s)\|_{\frac{1}{2}}\int_0^t\alpha^{-1}(s)\|\hat{v}_n(s)\|^2_{\frac{3}{2}}ds\\
\leq\;& \varepsilon\sup_{s\in[0,t]}\|\hat{v}_n(s)\|^2_{\frac{1}{2}}+c(\varepsilon)\Big(\int_0^t(\alpha^{-2}(s)+1)\|\hat{v}_n(s)\|^2_{\frac{3}{2}}ds\Big)^2\\
&+c\int_0^t\alpha^{-4}(s)\Big(\int_0^s\|v_n\|^2_{1/2}ds+|u_0|_1\Big)^4ds+c\int_0^t(1+\alpha^{-4}(s))\|\mathbf{z}_n(s)\|_1^4ds.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\sup_{s\in[0,t]}\|\hat{v}_n(s)\|^2_{\frac{1}{2}}+\int_0^t\|\hat{v}_n(s)\|^2_{\frac{3}{2}}ds\\
\leq\;& ct\sup_{s\in[0,t]}\alpha^{-4}(s)|u_0|_1^4+c\Big(1+\sup_{s\in[0,t]}\alpha^{-4}(s)\Big)\int_0^t\|\mathbf{z}(s)\|_1^4ds\\
&+ct\sup_{s\in[0,t]}\alpha^{-4}(s)\Big(\int_0^t\|\mathbf{z}(s)\|^2_{1/2}ds\Big)^4+ct^5\sup_{s\in[0,t]}\alpha^{-4}(s)\|\hat{v}_n(s)\|^8_{\frac{1}{2}}\\
&+c\sup_{s\in[0,t]}\Big(1+\alpha^{-4}(s)\Big)\Big(\int_0^t\|\hat{v}_n(s)\|^2_{\frac{3}{2}}ds\Big)^2.
\end{aligned}$$

To simplify the notations, we introduce

$$h(t):=c\sup_{s\in[0,t]}\Big(1+\alpha^{-4}(s)\Big)\int_0^t\|\hat{v}_n(s)\|^2_{\frac{3}{2}}ds+ct^5\sup_{s\in[0,t]}\alpha^{-4}(s)\|\hat{v}_n(t)\|^6_{\frac{1}{2}}$$

and

$$I(t)\;:=\;ct\sup_{s\in[0,t]}\alpha^{-4}(s)|u_0|_1^4+c\Big(1+\sup_{s\in[0,t]}\alpha^{-4}(s)\Big)\int_0^t\|\mathbf{z}(s)\|_1^4ds+ct\sup_{s\in[0,t]}\alpha^{-4}(s)\Big(\int_0^t\|\mathbf{z}(s)\|^2_{\frac{1}{2}}ds\Big)^4.$$

Then we have

$$\sup_{s\in[0,t]}\|\hat{v}_n(s)\|^2_{\frac{1}{2}}+\int_0^t\|\hat{v}_n(s)\|^2_{\frac{3}{2}}ds\leq I(t)+h(t)\Big(\sup_{s\in[0,t]}\|\hat{v}_n(s)\|^2_{\frac{1}{2}}+\int_0^t\|\hat{v}_n(s)\|^2_{\frac{3}{2}}ds\Big).\qquad(4.23)$$

In the following, we will find a uniform lower bound for the maximal existence time τ_n, then we can show that the local solutions exist. Set

$$\tau_n^*:=\sup\{t\in[0,\tau_n):h(t)\leq 1/2\}.$$

Since h is continuous, we have $\tau_n^* < \tau_n$. Obviously, $h(t) \uparrow \infty$ as $t \uparrow \tau_n$ and $h(\tau_n^*) = \frac{1}{2}$. It is easy to see that $I(t)$ is continuous, increasing and positive except at $t = 0$. Denote

$$\kappa := \sup\Big\{t \in [0,\infty) : I(t) < \min\Big(\frac{1}{8c \sup_{s\in[0,t]} (1+\alpha^{-4}(s))}, \frac{1}{(32ct^5 \sup_{s\in[0,t]} \alpha^{-4}(s))^{1/3}}\Big)\Big\}.$$

Obviously, $\kappa > 0$ and is independent of n. We will show that $\tau_n \geq \kappa$ for all n. Suppose, for contradiction, that $\tau_n^* < \kappa$ for some n, then by (4.23),

$$\sup_{s\in[0,\tau_n^*]} \|\hat{v}_n(s)\|_{\frac{1}{2}}^2 + \int_0^{\tau_n^*} \|\hat{v}_n(s)\|_{\frac{3}{2}}^2 ds \leq 2I(\tau_n^*),$$

which implies

$$h(\tau_n^*) := c \sup_{s\in[0,\tau_n^*]} (1+\alpha^{-4}(s)) \int_0^{\tau_n^*} \|\hat{v}_n(s)\|_{\frac{3}{2}}^2 ds + c(\tau_n^*)^5 \sup_{s\in[0,\tau_n^*]} \alpha^{-4}(s) \|\hat{v}_n(\tau_n^*)\|_{1/2}^6 < \frac{1}{2}.$$

This results in a contradiction. So, we obtain that $\tau_n \geq \kappa$ for all n. Furthermore, we have that

$$\sup_{s\in[0,\kappa]} \|\hat{v}_n(s)\|_{\frac{1}{2}}^2 + \int_0^{\kappa} \|\hat{v}_n(s)\|_{\frac{3}{2}}^2 ds \leq 2I(\kappa). \tag{4.24}$$

Therefore, $(\hat{v}_n)_{n=1}^\infty$ is uniformly bounded in $L^2([0,\kappa]; \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$ and $L^\infty([0,\kappa]; \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3))$; From (4.20), let $\varphi \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$, we have

$$\begin{aligned}
\langle \partial_s \hat{v}_n, \varphi \rangle &= \alpha^{-1} \langle P_n[(v_n \cdot \nabla) v_n(s)], \varphi \rangle - \langle \Lambda^{3/2} \hat{v}_n, \Lambda^{1/2} \varphi \rangle \\
&\leq \alpha^{-1} |\varphi|_3 |\nabla v_n(s)|_2 |v_n(s)|_6 + \|\hat{v}_n(s)\|_{\frac{3}{2}} \|\varphi\|_{\mathbb{H}^{\frac{1}{2}}} \\
&\leq c\alpha^{-1} \|\varphi\|_{\mathbb{H}^{\frac{1}{2}}} \|v_n(s)\|_{\mathbb{H}^{\frac{1}{2}}} \|v_n(s)\|_{\mathbb{H}^{\frac{3}{2}}} + \|\hat{v}_n(s)\|_{\frac{3}{2}} \|\varphi\|_{\mathbb{H}^{\frac{1}{2}}},
\end{aligned}$$

where the last inequality follows from the interpolation inequality. Therefore,

$$\begin{aligned}
\int_0^\kappa \langle \partial_s \hat{v}_n(s), \varphi \rangle^2 dt &\leq c \int_0^\kappa \alpha^{-2}(s) \|\varphi\|_{\mathbb{H}^{\frac{1}{2}}}^2 \|v_n(s)\|_{\mathbb{H}^{\frac{1}{2}}}^2 \|v_n(s)\|_{\mathbb{H}^{\frac{3}{2}}}^2 ds \\
&\quad + 2 \int_0^\kappa \|\hat{v}_n(s)\|_{\frac{3}{2}}^2 \|\varphi\|_{\mathbb{H}^{\frac{1}{2}}}^2 ds.
\end{aligned} \tag{4.25}$$

From (4.24), (4.25) and (2.8), we have $\partial_t \hat{v}_n \in L^2([0,\kappa]; \mathbb{H}^{-\frac{1}{2}}(\mathbb{T}^3))$. In view of Lemma2.2 and Lemma2.3, we have $(v_n)_{n\geq 1}$ converges to v in $L^2([0,\kappa]; \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3))$ and $v \in C([0,\kappa]; \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)) \cap L^2([0,\kappa]; \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$. Obviously, following a standard argument (see [50]), we see that v is the local weak pathwise solution to (2.3)-(2.4) according to Definition 2.3.

Let τ_v be the maximum existence time of v. In order to prove the global existence of v, in view of Definition 2.4, it is sufficient to show that

$$\mathbb{P}\{\tau_v < \infty\} = 0.$$

Let us prove it by contradiction. In deed, if we assume

$$\mathbb{P}\{\tau_v < \infty\} > 0,$$

then for arbitrary $\epsilon \in (0, \kappa)$, we have

$$\mathbb{P}\Big(\sup_{t\in[\epsilon,\tau_v)} \|v(t)\|_{\mathbb{H}^{\frac{1}{2}}} = \infty | \tau_v < \infty \Big) = 1. \tag{4.26}$$

By the local existence of weak solutions established above, we can choose

$$v(\epsilon) \in \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3) \subset \mathbb{H}^1(\mathbb{T}^3).$$

If we regard $v(\epsilon) \in \mathbb{H}^1(\mathbb{T}^3)$ as the initial data of (2.3)-(2.4), by Theorem 4.1, we know that the unique global strong solution exists on $[\epsilon, \infty)$, $\mathbb{P}$-a.e.$\omega \in \Omega$. More precisely, for arbitrary $T > 0$, we have

$$\mathbb{P}\Big(\sup_{t\in[\epsilon,T]} \|v(t)\|_{\mathbb{H}^1} < \infty \Big) = 1,$$

or

$$\mathbb{P}\Big(\sup_{t\in[\epsilon,T]} \|v(t)\|_{\mathbb{H}^1} = \infty \Big) = 0. \tag{4.27}$$

We will apply (4.27) to (4.26) to derive a contradiction. In deed,

$$\begin{aligned}
& \mathbb{P}\Big(\sup_{t\in[\epsilon,\tau_v)} \|v(t)\|_{\mathbb{H}^{\frac{1}{2}}} = \infty, \tau_v < \infty \Big) \\
= & \mathbb{P}\Big(\sup_{t\in[\epsilon,\tau_v)} \|v(t)\|_{\mathbb{H}^{\frac{1}{2}}} = \infty, \cup_{n=1}^{\infty}(\tau_v < n) \Big) \\
\leq & \sum_{n=1}^{\infty} \mathbb{P}\Big(\sup_{t\in[\epsilon,n)} \|v(t)\|_{\mathbb{H}^{\frac{1}{2}}} = \infty \Big) \\
\leq & \sum_{n=1}^{\infty} \mathbb{P}\Big(\sup_{t\in[\epsilon,n]} \|v(t)\|_{\mathbb{H}^1} = \infty \Big) = 0,
\end{aligned}$$

which implies

$$\mathbb{P}\Big(\sup_{t\in[\epsilon,\tau_v)} \|v(t)\|_{\mathbb{H}^{\frac{1}{2}}} = \infty | \tau_v < \infty \Big) = \frac{\mathbb{P}\Big(\sup_{t\in[\epsilon,\tau_v)} \|v(t)\|_{\mathbb{H}^{\frac{1}{2}}} = \infty, \tau_v < \infty \Big)}{\mathbb{P}(\tau_v < \infty)} = 0.$$

The contradiction folllows. Therefore, we arrive at $P\{\tau_v < \infty\} = 0$, which implies the global existence of the weak pathwise solutions.

Let v_1 and v_2 be weak pathwise solutions to equation (2.3). Then we denote by v the difference of v_1 and v_2, i.e., $v = v_1 - v_2$. Taking inner product of the equation satisfied by v with Λv in $\mathbb{L}^2(\mathbb{T}^3)$ yields,

$$\begin{aligned}
& \|v(t)\|_{\frac{1}{2}}^2 + 2\int_0^t \|v(s)\|_{\frac{3}{2}}^2 ds \\
\leq \ & c\int_0^t \alpha^{-1}(s)|v_1(s)|_6 |\nabla v(s)|_2 |\Lambda v(s)|_3 ds + c\int_0^t \alpha^{-1}(s)|v(s)|_6 |\nabla v_1(s)|_3 |\Lambda v(s)|_2 ds \\
\leq \ & c\int_0^t \alpha^{-1}(s)|v_1(s)|_6 \|v(s)\|_1 \|v(s)\|_{\frac{3}{2}} ds + c\int_0^t \alpha^{-1}(s)|v(s)|_6 \|v_1(s)\|_{\frac{3}{2}} \|v(s)\|_1 ds \\
=: \ & K_1 + K_2,
\end{aligned} \tag{4.28}$$

where in the last inequality, we have used $|\nabla v_1(s)|_3 \le c\|v_1(s)\|_{\frac{3}{2}}$. Since by the interpolation inequality, the Sobolev inequality and Young's inequality, we have

$$\begin{aligned} K_1 &\le c\int_0^t \alpha^{-1}(s)\|v_1(s)\|_{\mathbb{H}^1}\|v(s)\|_{\frac{1}{2}}^{1/2}\|v(s)\|_{\frac{3}{2}}^{3/2}ds \\ &\le c(\varepsilon)\int_0^t \alpha^{-4}(s)\|v_1(s)\|_{\mathbb{H}^1}^4\|v(s)\|_{\frac{1}{2}}^2 ds + \varepsilon\int_0^t \|v(s)\|_{\frac{3}{2}}^2 ds, \end{aligned}$$

and

$$\begin{aligned} K_2 &\le c\int_0^t \alpha^{-1}(s)|v(s)|_6\|v_1(s)\|_{\frac{3}{2}}\|v(s)\|_1 ds \\ &\le c\int_0^t \alpha^{-1}(s)\Big(\|v(s)\|_1 + \int_0^s \|v(r)\|_{\frac{1}{2}}^2 dr\Big)\|v(s)\|_1\|v_1(s)\|_{\frac{3}{2}} ds \\ &\le c\int_0^t \alpha^{-1}(s)\|v(s)\|_{\frac{1}{2}}\|v\|_{\frac{3}{2}}\|v_1(s)\|_{\frac{3}{2}} ds \\ &\quad +c\int_0^t \alpha^{-1}(s)\Big(\|v_1(s)\|_{\frac{3}{2}}^2 + \|v_2(s)\|_{\frac{3}{2}}^2\Big)ds\int_0^t \|v(s)\|_{\frac{1}{2}}^2 ds \\ &\le \varepsilon\int_0^t \|v(s)\|_{\frac{3}{2}}^2 ds + c(\varepsilon)\int_0^t \alpha^{-2}(s)\|v\|_{\frac{1}{2}}^2\|v_1(s)\|_{\frac{3}{2}}^2 ds \\ &\quad +c\int_0^t \alpha^{-1}(s)\Big(\|v_1(s)\|_{\frac{3}{2}}^2 + \|v_2(s)\|_{\frac{3}{2}}^2\Big)ds\int_0^t \|v(s)\|_{\frac{1}{2}}^2 ds. \end{aligned}$$

From (4.28) and estimates of K_1 and K_2, we have

$$\begin{aligned} &\|v(t)\|_{\frac{1}{2}}^2 + 2\int_0^t \|v(s)\|_{\frac{3}{2}}^2 ds \\ \le\ & c\int_0^t \alpha^{-4}(s)\|v_1(s)\|_{\mathbb{H}^1}^4\|v(s)\|_{\frac{1}{2}}^2 ds + c\int_0^t \alpha^{-2}(s)\|v(s)\|_{\frac{1}{2}}^2\|v_1(s)\|_{\frac{3}{2}}^2 ds \\ &+c\int_0^t \alpha^{-1}(s)\Big(\|v_1(s)\|_{\frac{3}{2}}^2 + \|v_2(s)\|_{\frac{3}{2}}^2\Big)ds\int_0^t \|v(s)\|_{\frac{1}{2}}^2 ds. \end{aligned} \tag{4.29}$$

From (2.8), we know that

$$\begin{aligned} &\int_0^t \alpha^{-4}(s)\|v_1(s)\|_{\mathbb{H}^1}^4 ds \\ \le\ & \int_0^t \alpha^{-4}(s)\|v_1(s)\|_{\frac{1}{2}}^2\|v_1(s)\|_{\frac{3}{2}}^2 ds + c\sup_{s\in[0,t]}\alpha^{-4}(s)\Big(\int_0^t \|v_1(s)\|_1^2 ds + |u_0|\Big)^4 \end{aligned}$$

for arbitrary $t(>0)$. Therefore, by (4.29) and the Gronwall inequality, we obtain that $\|v(t)\|_{\frac{1}{2}} = 0$ for arbitrary $t(>0)$. Then in view of (2.8) that

$$\begin{aligned} |v(t)|_2 &\le |v(t)-\bar{v}(t)|_2 + (2\pi)^{\frac{3}{2}}|\bar{v}(t)| \le \|v(t)-\bar{v}(t)\|_{\frac{1}{2}} + (2\pi)^{\frac{3}{2}}|\bar{v}(t)| \\ &\le \|v(t)\|_{\frac{1}{2}} + \|\bar{v}(t)\|_{\frac{1}{2}} + (2\pi)^{\frac{3}{2}}|\bar{v}(t)| = \|v(t)\|_{\frac{1}{2}} + (2\pi)^{\frac{3}{2}}|\bar{v}(t)| \\ &\le \|v(t)\|_{\frac{1}{2}} + (2\pi)^{\frac{9}{2}}\int_0^t \|v(s)\|_{\frac{1}{2}}\Big(\|v_1(s)\|_{\frac{1}{2}} + \|v_2(s)\|_{\frac{1}{2}}\Big)ds = 0. \end{aligned} \tag{4.30}$$

where $\bar{v} = \int_{\mathbb{T}^3} v(x)dx$ and $\|\bar{v}(t)\|_{\frac{1}{2}} = 0$. Hence, in view of (4.30) we arrive at that, for each $t \geq 0$, $v(t,x) = 0$ for a.e.$x \in \mathbb{T}^3$. Moreover, from (4.29) and (4.30), we can establish the Lipschitz continuity of v with respect to the initial data in $\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$. The uniqueness of the weak solutions to (2.3)-(2.4) follows from the continuous dependence of v with respect to the initial data. □

In view of Theorem 4.3 and $v = \alpha u$, it is clearly true that

Theorem 4.4. *Suppose u_0 is an $\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable. Then, there exists a unique global weak pathwise solution u to (1.1) in the sense of Definition 2.4.*

Theorem 4.5. *For any $\mathcal{F}_0-$adapted initial value $u_0 \in \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$, let (v,ξ) be the maximum strong solution. Then for any $t \in (0,\xi)$, the solution v to (2.3)-(2.4) satisfies*

$$\sup_{s\in[0,t]} |v(s)|_\infty \leq |v(0)|_\infty = |u(0)|_\infty, \mathbb{P} - a.s.\omega \in \Omega.$$

Proof. In view of Remark 1, there exists a subsequence of solutions v_n to (2.5)and (2.6), which is still denoted by v_n such that

$$v_n(s) \to v(s) \text{ in } L^2([0,t];\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)).$$

Then we can choose a subsequence of v_n still denoted by v_n satisfying

$$v_n(s) \to v(s) \text{ in } \mathbb{L}^\infty(\mathbb{T}^3) \text{ for almost every } s \in [0,t].$$

Let $\varphi \in \mathbb{L}^1(\mathbb{T}^3)$ with $|\varphi|_1 \leq 1$, we have

$$\langle v(s),\varphi\rangle = \lim_{n\to\infty}\langle v_n(s),\varphi\rangle \leq \lim_{n\to\infty}|v_n(s)|_\infty \leq \lim_{n\to\infty}|v_n(0)|_\infty \leq |v(0)|_\infty,$$

where the first equality above holds for almost every $s \in [0,t]$ and the second inequality follows by Lemma 4.1. Hence, by Theorem 4.6 we arrive at

$$\sup_{s\in[0,t]} |v(t)|_\infty \leq |v(0)|_\infty = |u(0)|_\infty.$$

□

Theorem 4.6. *Suppose u_0 is an $\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable. Then, for any $T > 0$, there exists a unique global strong pathwise solution v to (2.3)-(2.4) satisfying $v \in C([0,T];\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)) \cap L^2([0,T];\mathbb{H}^{\frac{5}{2}}(\mathbb{T}^3))$ and v is Lipschitz continuous with respect to the initial data in $\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$.*

Proof. Let (v,ξ) be the maximum strong solution to (2.3)-(2.4) with $u_0 \in \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$. For $t \in (0,\xi)$, taking inner product of (2.3) in $L^2([0,t]\times\mathbb{T}^3)$ with $\Lambda^3 v$ yields,

$$\frac{1}{2}\partial_t\|v\|^2_{\frac{3}{2}} + \|v\|^2_{\frac{5}{2}} = -\alpha^{-1}\langle\Lambda^{1/2}(v\cdot\nabla v),\Lambda^{5/2}v\rangle$$

which implies

$$\begin{aligned}\|v(t)\|_{\frac{3}{2}}^2 + 2\int_0^t \|v(s)\|_{\frac{5}{2}}^2 ds &\leq \|u_0\|_{\frac{3}{2}}^2 + \varepsilon \int_0^t \|v(s)\|_{\frac{5}{2}}^2 ds \\ &\quad + c \sup_{s\in[0,t]} \alpha^{-2}(s) \int_0^t \int_{\mathbb{T}^3} |\Lambda^{1/2}(v\cdot\nabla v)|^2 dxds. \qquad (4.31)\end{aligned}$$

In order to bound the last term on the right hand side of (4.31), we will use Theorem A.8 in [38]. Without loss of generality, we assume $t \in (0, \xi)$. Then the estimates of the last term follows

$$\begin{aligned}&\int_0^t \int_{\mathbb{T}^3} |\Lambda^{1/2}(v\cdot\nabla v)|^2 dxds \\ \leq\ & 3\int_0^t \int_{\mathbb{T}^3} |v\cdot(\Lambda^{3/2}v)|^2 dxds + 3\int_0^t \int_{\mathbb{T}^3} |(\Lambda^{1/2}v)\cdot(\Lambda v)|^2 dxds \\ & + c\Big\{\int_0^t \Big(\int_{\mathbb{T}^3} |\Lambda^{3/8}v|^6 dx\Big)^{2/3} ds\Big\}^{1/2} \Big\{\int_0^t \Big(\int_{\mathbb{T}^3} |\Lambda^{9/8}v|^3 dx\Big)^{4/3} ds\Big\}^{1/2} \\ =:\ & I_1 + I_2 + I_3^{1/2}\times I_4^{1/2} \\ \leq\ & I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Using the Holder inequality, the interpolation inequality, Young's inequality and Theorem 4.1, we get

$$\begin{aligned}I_2 &\leq \int_0^t |\Lambda v(s)|_2^2 |\Lambda^{1/2} v(s)|_\infty^2 ds \\ &\leq c\int_0^t \|v(s)\|_1^2 \|v(s)\|_{\frac{3}{2}} \|v(s)\|_{\frac{5}{2}} ds \\ &\leq \varepsilon \int_0^t \|v(s)\|_{\frac{5}{2}}^2 ds + c\int_0^t \|v(s)\|_{\frac{3}{2}}^2 ds.\end{aligned}$$

In view of the Holder inequality, the Sobolev embedding theorem and (2.8), we have

$$I_1 \leq c\int_0^t |v(s)|_\infty^2 \|v(s)\|_{\frac{3}{2}}^2 ds \leq c\|u_0\|_{\mathbb{H}^{\frac{3}{2}}} \int_0^t \|v(s)\|_{\frac{3}{2}}^2 ds$$

Utilising the interpolation inequality and (2.8), we then derive

$$\begin{aligned}I_4 &\leq c\int_0^t |\Lambda^{9/8} v|_3^4 ds \leq c\int_0^t \|v\|_{\mathbb{H}^{\frac{3}{2}}}^{7/2} \|v\|_{\mathbb{H}^{\frac{5}{2}}}^{1/2} ds \\ &\leq \varepsilon \int_0^t \|v\|_{\mathbb{H}^{\frac{5}{2}}}^2 ds + c(\varepsilon)\int_0^t \|v\|_{\mathbb{H}^{\frac{3}{2}}}^{14/3} ds \\ &\leq \varepsilon \int_0^t \|v(s)\|_{\frac{5}{2}}^2 ds + c\int_0^t \|v(s)\|_{\frac{3}{2}}^{14/3} ds \\ &\quad + c\int_0^t \|v(s)\|_{\frac{1}{2}}^4 ds + ct|u_0|_1^2 + c\int_0^t \|v(s)\|_{\frac{1}{2}}^{28/3} ds + ct|u_0|_1^{14/3}.\end{aligned}$$

By virtue of the interpolation inequality, the Hölder inequality and Theorem 4.1, we obtain

$$\begin{aligned} I_4 &\le \varepsilon\int_0^t \|v(s)\|^2_{\frac{5}{2}}ds + c\int_0^t \|v(s)\|_1^{7/3}\|v(s)\|_2^{7/3}ds + cT \\ &\le \varepsilon\int_0^t \|v(s)\|^2_{\frac{5}{2}}ds + c\int_0^t \|v(s)\|_{\frac{3}{2}}^{7/6}\|v(s)\|_{\frac{5}{2}}^{7/6}ds + cT \\ &\le \varepsilon\int_0^t \|v(s)\|^2_{\frac{5}{2}}ds + c\int_0^t \|v(s)\|_{\frac{3}{2}}^{14/5}ds + cT \\ &\le \varepsilon\int_0^t \|v(s)\|^2_{\frac{5}{2}}ds + c\int_0^t \|v(s)\|_1^{7/5}\|v(s)\|_2^{7/5}ds + cT \le cT. \end{aligned}$$

By the Sobolev imbedding theorem and Theorem 4.1, we arrive at

$$I_3 \le c\int_0^t \|v(s)\|^2_{\mathbb{H}^1}\|v(s)\|^2_{\mathbb{H}^2}ds < cT.$$

The global existence of v in $C([0,T];\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$ follows by (4.31), estimates of $I_1,...,I_4$ and the Gronwall inequality. The Lipschitz continuity with respect to the initial data in $\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$ is proved in Lemma 3.1 or we can follow the argument in Proposition 3.1 to establish it. □

Remark 1. *In view of Theorem 4.6, repeating the argument in Proposition3.1, one can choose a subsequence of v_n, which is still denoted by v_n, such that v_n is uniformly bounded in $L^\infty([0,T];\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$ and converges to v in $L^2([0,T];\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$.*

Noticing the argument above and $v=\alpha u$, we arrive at

Theorem 4.7. *Suppose u_0 is an $\mathbb{H}^m(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable. Then, for any $T>0$, there exists a unique global strong pathwise solution u to (1.1) satisfying $u\in C([0,T];\mathbb{H}^m(\mathbb{T}^3))\cap L^2([0,T];\mathbb{H}^{m+1}(\mathbb{T}^3))$.*

5 Moment estimates to (1.1)

This section is devoted to moment estimates for stochastic 3D Burgers equations with multiplicative noise.

Proposition 5.1. *For any $\mathcal{F}_0-$adapted initial value $u_0\in\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$ satisfying $\mathbb{E}\|u_0\|^{q+\delta}_{\mathbb{H}^{\frac{3}{2}}}<\infty, q\ge 1$, and δ is an arbitrary small positive constant. Then for any $T>0$, the unique global strong solution u to (1.1) satisfies*

$$\mathbb{E}\sup_{t\in[0,T]}|u(t)|_p^q \le c\exp cT,$$

where the constant c is independent of T.

Proof. From Lemma 4.1 and the Sobolev imbedding theorem, we know

$$\sup_{t\in[0,T]} |v_n(t)|_p \le \sup_{t\in[0,T]} |v_n(t)|_\infty \le \sup_{n\in\mathbb{N}} |v_n(0)|_\infty \le c\|v_n(0)\|_1 \le c\|v(0)\|_1 = c\|u(0)\|_1,$$

where the constant c is independent of T. Hence, there exists a subsequence of v_n still denoted by v_n such that

$$v_n \rightharpoonup^* v \quad in\ L^\infty([0,T];\mathbb{L}^p(\mathbb{T}^3)), \mathbb{P}-a.e.\omega \in \Omega,$$

where v is the strong solution to (2.3)-(2.4), see Theorem 4.1.

Let $\phi \in L^1([0,T];\mathbb{L}^q(\mathbb{T}^3))$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\int_0^T |\phi(s)|_q ds \le 1$. Then we have that

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^3} \phi(t,x)v(t,x)dxdt &= \lim_{n\to\infty} \int_0^T \int_{\mathbb{T}^3} \phi(t,x)v_n(t,x)dxdt \\ &\le \int_0^T |\phi(s)|_q ds \sup_{t\in[0,T]} |v_n(t)|_p \\ &\le \|u(0)\|_1.\end{aligned}$$

Hence, for arbitrary $m \ge 1$, we obtain that

$$\mathbb{E} \sup_{t\in[0,T]} |v(t)|_p^q \le c\mathbb{E}\|u(0)\|_1^q < \infty.$$

For positive constants p' and q' satisfying $\frac{1}{p'}+\frac{1}{q'}=1$, we have

$$\begin{aligned}\mathbb{E} \sup_{t\in[0,T]} |u(t)|_p^q &\le \Big(\mathbb{E} \sup_{t\in[0,T]} |v(t)|_p^{qp'}\Big)^{\frac{1}{p'}} \Big(\mathbb{E} \sup_{t\in[0,T]} \alpha^{-qq'}(t)\Big)^{1/q'} \\ &\le c\Big(\mathbb{E}\|u(0)\|_{\mathbb{H}^{\frac{3}{2}}}^{qp'}\Big)^{\frac{1}{p'}} \Big(\mathbb{E} \sup_{t\in[0,T]} \alpha^{-qq'}(t)\Big)^{1/q'}.\end{aligned}$$

Let $Q = qq'$, recall that $W(t) = bB(t), t \in [0,T]$. Then $W(t) \sim N(0, b^2t)$, where $N(0,b^2t)$ denotes the normal distribution with mean 0 and variance b^2t. In the following, we will compute $\mathbb{E} \sup_{t\in[0,T]} \alpha^{-qq'}(t)$. By the Doob's maximal inequality,

$$\begin{aligned}\mathbb{E} \sup_{t\in[0,T]} \alpha^{-qq'}(t) &= \mathbb{E} \sup_{t\in[0,T]} \Big(\exp W(t)\Big)^Q \\ &\le \Big(\frac{Q}{Q-1}\Big)^Q \mathbb{E}\exp QW(T) \\ &= \Big(\frac{Q}{Q-1}\Big)^Q \frac{1}{\sqrt{2\pi b^2 T}} \int_{-\infty}^{\infty} \exp(-\frac{x^2}{2b^2T} + Qx)dx \\ &= \sqrt{2}\Big(\frac{Q}{Q-1}\Big)^Q \exp(b^2TQ^2/2). \end{aligned} \tag{5.32}$$

Hence

$$\mathbb{E} \sup_{t\in[0,T]} |u(t)|_p^q \le \Big(\mathbb{E}\|u(0)\|_{\mathbb{H}^{\frac{3}{2}}}^{qp'}\Big)^{\frac{1}{p'}} 2^{\frac{1}{2q'}} \Big(\frac{qq'}{qq'-1}\Big)^q \exp(b^2Tq^2q'/2).$$

□

Proposition 5.2. *For any $\mathcal{F}_0-$adapted initial value $u_0 \in \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$ satisfying $\mathbb{E}\|u_0\|^{q+\delta}_{\mathbb{H}^{\frac{3}{2}}} < \infty, q \geq 1$ and δ is an arbitrary small positive constant. Then for any $T > 0$, the unique global strong solution u to (1.1) satisfies*

$$\mathbb{E} \sup_{t\in[0,T]} |u(t)|^q_\infty \leq c \exp cT,$$

where the constant c is independent of T.

Proof. Let u and v be the unique strong solutions to equations (1.1) and (2.3) respectively. Then note that $u = \alpha^{-1}v$. Hence for $q \geq 1$ we have

$$\sup_{t\in[0,T]} |u(t)|^q_\infty \leq \sup_{t\in[0,T]} |v(t)|^q_\infty \sup_{t\in[0,T]} \alpha^{-q}(t) \leq |u(0)|^q_\infty \sup_{t\in[0,T]} \alpha^{-q}(t).$$

For positive constants p' and q' satisfying $\frac{1}{p'} + \frac{1}{q'} = 1$, we have

$$\begin{aligned}\mathbb{E} \sup_{t\in[0,T]} |u(t)|^q_\infty &\leq \Big(\mathbb{E} \sup_{t\in[0,T]} |v(t)|^{qp'}_\infty\Big)^{\frac{1}{p'}} \Big(\mathbb{E} \sup_{t\in[0,T]} \alpha^{-qq'}(t)\Big)^{1/q'} \\ &\leq c\Big(\mathbb{E}\|u(0)\|^{qp'}_{\mathbb{H}^{\frac{3}{2}}}\Big)^{\frac{1}{p'}} \Big(\mathbb{E} \sup_{t\in[0,T]} \alpha^{-qq'}(t)\Big)^{1/q'}.\end{aligned}$$

Finally, the estimate of the Theorem follows by (5.32). □

Next, we aim to obtain $\mathbb{E} \lim_{t\in[0,T]} \|u(t)\|^2_1 < \infty$ for the strong solution u of (1.1). But, it is difficult! Due to the high nonlinearity of stochastic 3D Burgers equations, we can only establish the logarithmic moments in $\mathbb{H}^1(\mathbb{T}^3)$, see the Theorem 5.3 below. We need techniques from logarithmic moments to reduce the powers arising from the nonlinear term.

Theorem 5.3. *For any $\mathcal{F}_0-$adapted initial value $u_0 \in \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$ satisfying $\mathbb{E}\|u_0\|^{2+\sigma}_{\mathbb{H}^{\frac{3}{2}}} < \infty$, and σ is an arbitrary small positive constant. Then, for any $T > 0$, the unique global strong solution u to (1.1) satisfies*

$$\mathbb{E} \sup_{t\in[0,T]} \log(1 + \|u(t)\|^2_{\mathbb{H}^1}) \leq c \exp cT,$$

where the constant c is independent of T.

Proof. Taking a similar argument as in the proof of Lemma 2.1 yields,

$$\frac{1}{2}\partial_t\|v\|^2_1 + \|v\|^2_2 \leq \alpha^{-1}|\langle v \cdot \nabla v, \Lambda^2 v\rangle \leq \varepsilon\|v\|^2_2 + c(\varepsilon)\alpha^{-2}|v|^2_\infty\|v\|^2_1.$$

Then by the maximum principle for random Burgers equations Theorem 4.5, we have

$$\partial_t \log(\|v\|^2_1 + 1) + \frac{\|v\|^2_2}{\|v\|^2_1 + 1} \quad \leq \quad c\alpha^{-2p} + c|u(0)|^{2q}_\infty,$$

where $p>1, q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. By the Gronwall inequality, we have

$$\mathbb{E}\sup_{t\in[0,T]}\log(\|v(t)\|_1^2+1)\leq \mathbb{E}\log(\|v(0)\|_1^2+1)+c\mathbb{E}\sup_{t\in[0,T]}\alpha^{-2p}(t)+c\mathbb{E}|u(0)|^{2q}_{\mathbb{H}^{\frac{3}{2}}}\leq c\exp cT,$$

where the constant c is independent of T. Note that $\log(\|u(t)\|_1^2+1)\leq\log(\|v(t)\|_1^2+1)+\log(\alpha^{-2}(t)+1)$, the result follows. □

Remark 2. *Due to the high non-linearity and the absence of incompressibility of the stochastic 3D Burgers equation, it is difficult to obtain the logarithmic moments* $\mathbb{E}\sup_{t\in[0,T]}\log\Big(1+\|u(t)\|^2_{\mathbb{H}^{\frac{3}{2}}}\Big)<\infty$.

6 Regularity of (1.1) with infinitely dimensional noise

This section establishes the global well-posedness of 3D Burgers equation with the noise having the form of $u(t,x)\circ b(x)dB(t)$, where $b: x\in\mathbb{T}^3\mapsto b(x)\in\mathbb{R}$ is a given smooth function of the spatial varialbe, i.e., $b\in C^\infty(\mathbb{T}^3)$.

We consider 3D Burgers equation (1.1) with $b(x)$ being a given smooth function. For reader's convenience, we rewrite it here.

$$\begin{aligned}&du(t,x)=\Delta u(t,x)dt-((u\cdot\nabla)u(t,x))dt+u(t,x)\circ b(x)dB(t),\quad \text{on } [0,T]\times\mathbb{T}^3, \qquad (6.33)\\ &u(0,x)=u_0(x),\quad x=(x_1,x_2,x_3)\in\mathbb{T}^3,\end{aligned}$$

where $b(x):\mathbb{T}^3\ni x\to\mathbb{R}$, is a given smooth function. To simplify the notations, let

$$\lambda=\sup_{(t,x)\in[0,T]\times\mathbb{T}^3}\Big[\Big(|\sum_{i=1}^3\partial_{x_i}b(x)B(t)|\Big)^2+|\Delta b(x)B(t)|\Big)\Big].$$

For $(t,x)\in[0,T]\times\mathbb{T}^3$, we define

$$\hat{v}(t,x)\quad=\quad u(t,x)\exp\Big(-b(x)B(t)\Big)\exp(-\lambda t)=:u(t,x)\alpha(t,x)\exp(-\lambda t).$$

Consequently, (6.33) is equivalent to the following

$$\begin{aligned}&\partial_t\hat{v}(t,x)-\Delta\hat{v}(t,x)-2\sum_{i=1}^3\Big(\partial_{x_i}b(x)B(t)\Big)\partial_{x_i}\hat{v}(t,x) \qquad (6.34)\\ &+\alpha^{-1}(t,x)\exp(\lambda t)\sum_{i=1}^3\hat{v}_i(t,x)\partial_{x_i}\hat{v}(t,x)\\ &+\Big(\lambda-\Big(\sum_{i=1}^3\partial_{x_i}b(x)B(t)\Big)^2-\Big(\Delta b(x)B(t)\Big)\Big)\hat{v}(t,x)\\ &+\alpha^{-1}(t,x)\exp(\lambda t)\Big(\sum_{i=1}^3\hat{v}_i(t,x)\partial_{x_i}b(x)B(t)\Big)\hat{v}(t,x)=0,\quad\text{on } [0,T]\times\mathbb{T}^3,\\ &\hat{v}(0,x)=u_0(x),\quad x=(x_1,x_2,x_3)\in\mathbb{T}^3,\end{aligned}$$

where $\hat{v}(t,x) = (\hat{v}_1(t,x), \hat{v}_2(t,x), \hat{v}_3(t,x)) \in \mathbb{R}^3, x \in \mathbb{T}^3,\ t \in [0,T]$. The definitions of solutions to (6.34) are given in appendix, which are very close to the definitions in Section 2. The Galerkin approximation of (6.34) is given by

$$\begin{aligned}
&\partial_t \hat{v}_n(t,x) - \Delta \hat{v}_n(t,x) - 2P_n\Big(\sum_{i=1}^{3}\Big(\partial_{x_i} b(x)B(t)\Big)\partial_{x_i}\hat{v}_n(t,x)\Big) \\
&+P_n\Big(\alpha^{-1}(t,x)\exp(\lambda t)\sum_{i=1}^{3}\hat{v}_{n,i}(t,x)\partial_{x_i}\hat{v}_n(t,x)\Big) \\
&+P_n\Big[\Big(\lambda - \Big(\sum_{i=1}^{3}\partial_{x_i} b(x)B(t)\Big)^2 - \Delta b(x)B(t)\Big)\hat{v}_n(t,x)\Big)\Big] \\
&+P_n\Big(\alpha^{-1}(t,x)\exp(\lambda t)\Big(\sum_{i=1}^{3}\hat{v}_{n,i}(t,x)\partial_{x_i} b(x)B(t)\Big)\hat{v}_n(t,x)\Big) = 0, \quad \text{on } [0,T]\times\mathbb{T}^3, \\
&\hat{v}_n(0,x) = u_n(0,x), \quad x = (x_1,x_2,x_3) \in \mathbb{T}^3.
\end{aligned} \tag{6.35}$$

where $\hat{v}_n(t,x) = (\hat{v}_{n,1}(t,x), \hat{v}_{n,2}(t,x), \hat{v}_{n,3}(t,x)) \in \mathbb{R}^3, (t,x) \in [0,T]\times\mathbb{T}^3$.

To assure the global well-posedness of (6.34), we assume

$$\textit{Any two compoments of } \nabla b(x) = (\partial_{x_1} b(x), \partial_{x_2} b(x), \partial_{x_3} b(x)),\ \textit{are linearly correlated}, \tag{6.36}$$

where $x = (x_1,x_2,x_3) \in \mathbb{T}^3$. For the reason that why we choose this condition, please see the detailed arguments in the appendix.

Remark 3. *There are lots of examples satisfying (6.36). The first example is the case that $b(x)$ is a constant, then $\nabla b(x) \equiv 0$, for all $x \in \mathbb{T}^3$. We have discussed this case from Section 3 to Section 5. The second example is, there exists constant $c \in \mathbb{R}$ such that $\partial_{x_1} b(x) = c\partial_{x_2} b(x)$, holds for all $x \in \mathbb{T}^3$, that is $(\nabla b(x) = (c\partial_{x_2} b(x), \partial_{x_2} b(x), \partial_{x_3} b(x)))$ holds for all $x \in \mathbb{T}^3$. The third example is, there is a component of $\nabla b(x)$ equaling to some constant $c \in \mathbb{R}$, i.e., $\nabla b(x) = (\partial_{x_1} b(x), \partial_{x_2} b(x), c)$ for all $x \in \mathbb{T}^3$.*

Without loss of generality, under the condition (6.36), we assume there exists some constant $a \in \mathbb{R}$, such that

$$\partial_{x_1} b(x) = a\partial_{x_2} b(x),\ \textit{holds for all } x \in \mathbb{T}^3. \tag{6.37}$$

Let $\eta = (1,-a,0) \in \mathbb{R}^3$. In the following we will find a solution $\tilde{v}_n(t,x) = g_n(t,x)\eta$ to (6.35), where $g_n : [0,T]\times\mathbb{T}^3 \to \mathbb{R}$ will be determined by (6.39). Obviously, $\tilde{v}_n(t,x)$ solves (6.35) if and only if it solves the following equation.

$$\begin{aligned}
&\partial_t \tilde{v}_n(t,x) - \Delta \tilde{v}_n(t,x) - 2P_n\Big(\sum_{i=1}^{3}\Big(\partial_{x_i} b(x)B(t)\Big)\partial_{x_i}\tilde{v}_n(t,x)\Big) \qquad (6.38)\\
&+P_n\Big(\alpha^{-1}(t,x)\exp(\lambda t)\sum_{i=1}^{3}\tilde{v}_{n,i}(t,x)\partial_{x_i}\tilde{v}_n(t,x)\Big)\\
&\Big(\lambda - \Big(\sum_{i=1}^{3}\partial_{x_i} b(x)B(t)\Big)^2 - \Delta b(x)B(t)\Big)\tilde{v}_n(t,x) = 0, \ \text{ on } [0,T]\times\mathbb{T}^3,\\
&\tilde{v}_n(0,x) = g_n(0,x)\eta \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3), \quad x=(x_1,x_2,x_3)\in\mathbb{T}^3,
\end{aligned}$$

equivalently,

$$\begin{aligned}
&\partial_t g_n(t,x) - \Delta g_n(t,x) - 2P_n\Big(\sum_{i=1}^{3}\Big(\partial_{x_i} b(x)B(t)\Big)\partial_{x_i} g_n(t,x)\Big) \qquad (6.39)\\
&+P_n\Big(\alpha^{-1}(t,x)\exp(\lambda t)(g_n(t,x)\partial_{x_1} - a g_n(t,x)\partial_{x_2})g_n(t,x)\Big)\\
&P_n\Big(\Big(\lambda - \Big(\sum_{i=1}^{3}\partial_{x_i} b(x)B(t)\Big)^2 - \Delta b(x)B(t)\Big)g_n(t,x)\Big) = 0, \ \text{ on } [0,T]\times\mathbb{T}^3,\\
&g_n(0,x)\in H^{\frac{1}{2}}(\mathbb{T}^3), \quad x=(x_1,x_2,x_3)\in\mathbb{T}^3.
\end{aligned}$$

Since (6.39) is a locally-Lipschitz system of random ODEs, we set g_n to be the unique local solution to (6.39).

Similar to Lemma 2.1, we have the following Poincaré's inequality (6.40) for (6.35).

In fact, let $(\tilde{v}_n, \tau_n)$ be the unique local solution to the (6.38). Recall that $\bar{\tilde{v}}_n = \bar{\tilde{v}}_n(t) = \int_{\mathbb{T}^3}\tilde{v}_n(t,x)dx, t\in [0,\tau_n)$. From (6.38), by integration by parts, it is not difficult to derive that

$$|\bar{\tilde{v}}_n(t)| \le c(T,\omega)\int_0^t \|\tilde{v}_n(s)\|_1 ds + c(T,\omega)\int_0^t \|\tilde{v}_n(s)\|_{\frac{1}{2}}^2 ds + c(T,\omega)\int_0^t |\bar{\tilde{v}}_n|(s)ds.$$

By the Gronwall inequality, we get

$$|\bar{\tilde{v}}_n(t)| \le c\Big(\int_0^t \|\tilde{v}_n(s)\|_1 ds + \int_0^t \|\tilde{v}_n(s)\|_{\frac{1}{2}}^2 ds\Big)\exp(ct).$$

Then we have

$$|\tilde{v}_n(t)|_2 \le |\tilde{v}_n - \bar{\tilde{v}}_n|_2 + (2\pi)^{\frac{3}{2}}|\bar{\tilde{v}}_n| \le \|\tilde{v}_n\|_{\frac{1}{2}} + c\Big(\int_0^t \|\tilde{v}_n(s)\|_1 ds + \int_0^t \|\tilde{v}_n(s)\|_{\frac{1}{2}}^2 ds\Big)\exp(ct). \qquad (6.40)$$

One can take similar arguments as in Proposition 3.1 with minor modifications to achieve the local existence of solutions to the following equation. The definitions of the local well-posedness and global well-posedness of (6.41) are very similar to the appendix.

$$
\begin{aligned}
&\partial_t g(t,x) - \Delta g(t,x) - 2\sum_{i=1}^{3}\Big(\partial_{x_i} b(x)B(t)\Big)\partial_{x_i} g(t,x) \\
&+\alpha^{-1}(t,x)\exp(\lambda t)(g(t,x)\partial_{x_1} - ag(t,x)\partial_{x_2})g(t,x) \\
&\Big(\lambda - \Big(\sum_{i=1}^{3}\partial_{x_i} b(x)B(t)\Big)^2 - \Delta b(x)B(t)\Big)g(t,x) = 0, \quad \text{on } [0,T]\times\mathbb{T}^3, \\
&g(0,x) := g_0 \in H^1(\mathbb{T}^3), \quad x = (x_1,x_2,x_3)\in\mathbb{T}^3.
\end{aligned} \tag{6.41}
$$

We state the local well-posedness (6.41) as the Proposition 6.1 without proof.

Proposition 6.1. *Suppose g_0 is an $H^1(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable. Then, there exists a unique maximum strong solution (g,ξ) to equation (6.41).*

Under the condition (6.36), Proposition 6.1 is equivalent to the following

Proposition 6.2. *Suppose $\tilde{v}(0,x) = (g_0, -ag_0, 0)$ is an $\mathbb{H}^1(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable. Then, there exists a unique maximum strong solution $(\tilde{v},\xi)$ to equation (6.34) in sense of Definition 7.2.*

Adapting the argument as in Lemma 3.1, one can prove that

Proposition 6.3. *Suppose $\tilde{v}(0,x) = (g_0,-ag_0,0)$ is an $\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable. Then, there exists a unique maximum strong solution $(\tilde{v},\xi)$ to equation (6.34) in sense of Definition 7.2. That is, for any positive random variable $\eta\in(0,\xi)$, $\tilde{v}\in C([0,\eta];\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$, $\mathbb{P}$-a.s..*

Remark 4. *Similarly to the proof of Lemma 3.1, in the process of proving Proposition 6.3, one obtains a subsequence of $\tilde{v}_n$, which are solutions of* (6.35)*, such that $\tilde{v}_n$ converges to $\tilde{v}$ the solution to (6.34) in* $L^2([0,\tau];\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$*, where the positive random variable τ is smaller than the maximum existence time ξ, i.e.,* $0<\tau<\xi$, $\mathbb{P}$*-a.s..*

Proposition 6.4. *Let $(\tilde{v}_n,\xi_n)$ be a maximum strong solution to (6.35) with $\mathcal{F}_0$ measurable initial data* $P_n u_0 = P_n(g_0,-ag_0,0)\in\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$*, then under the condition (6.36) or (6.37),*

$$
\sup_{s\in[0,\xi_n)} |\tilde{v}_n(s)|_\infty \quad \le \quad |u_0|_\infty, \ \ \mathbb{P}-a.s.\omega\in\Omega.
$$

Proof. Note that $(\Delta\tilde{v}_n\cdot\tilde{v}_n)(s) = \frac{1}{2}\Delta|\tilde{v}_n|^2 - |\nabla\tilde{v}_n|^2$. Hence, multiplying (6.34) by $\tilde{v}_n$ yields,

$$
\begin{aligned}
&\frac{1}{2}\partial_s|\tilde{v}_n|^2(s,x) - \frac{1}{2}\Delta|\tilde{v}_n|^2(s,x) + |\nabla\tilde{v}_n|^2(s,x) - \sum_{i=1}^{3}\Big(\partial_{x_i}b(x)B(s)\Big)\partial_{x_i}|\tilde{v}_n|^2(s,x) \\
&+\frac{1}{2}\alpha^{-1}(s,x)\exp(\lambda s)\sum_{i=1}^{3}\tilde{v}_{n,i}(s,x)\partial_{x_i}|\tilde{v}_n|^2(s,x) \\
&\Big(\lambda - \Big(\sum_{i=1}^{3}\partial_{x_i}b(x)B(s)\Big)^2 - \Big(\Delta b(x)B(s)\Big)\Big)|\tilde{v}_n|^2(s,x) \\
&+\alpha^{-1}(s,x)\exp(\lambda s)\Big(\sum_{i=1}^{3}\tilde{v}_{n,i}(s,x)\partial_{x_i}b(x)B(s)\Big)|\tilde{v}_n|^2(s,x) = 0, \quad \text{on } [0,\xi_n)\times\mathbb{T}^3,
\end{aligned}
$$

where $\tilde{v}_n = (\tilde{v}_{n,1}, \tilde{v}_{n,2}, \tilde{v}_{n,3}) \in \mathbb{R}^3$. Note that

$$\sum_{i=1}^{3} \tilde{v}_{n,i}(s,x)\partial_{x_i} b(x)B(s) = 0, \text{ on } [0,\xi_n) \times \mathbb{T}^3,$$

and

$$\Big(\lambda - \Big(\sum_{i=1}^{3} \partial_{x_i} b(x)B(s)\Big)^2 - \Big(\Delta b(x)B(s)\Big)\Big) \geq 0, \text{ on } [0,\xi_n) \times \mathbb{T}^3.$$

According to maximum principle (see Theorem 4 in page 352 of [29]), if $|\tilde{v}_n|^2$ has local maximum at $(s,x) \in (0,\xi_n) \times \mathbb{T}^3$, then $|\tilde{v}_n| \equiv 0$. Therefore,

$$\sup_{s\in[0,\xi_n)} |\tilde{v}_n(s)|_\infty \leq |u_0|_\infty, \ \mathbb{P} - a.s.\omega \in \Omega.$$

□

Theorem 6.5. *Let $(\tilde{v},\xi)$ be a maximum strong solution to (6.35) with $\mathcal{F}_0$ measurable initial data $u_0 = (g_0, -ag_0, 0) \in \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$. Then the solution $\tilde{v}$ to (6.34) with condition (6.36) satisfies*

$$\sup_{t\in[0,\xi)} |\tilde{v}(t)|_\infty \leq |v(0)|_\infty = |u(0)|_\infty, \mathbb{P} - a.s.\omega \in \Omega.$$

Proof. In view of Remark 4, there exists a subsequence of solutions $\tilde{v}_n$ to (6.35), which is still denoted by $\tilde{v}_n$ such that

$$\tilde{v}_n(t) \to \tilde{v}(t) \text{ in } L^2([0,\tau];\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)),$$

where τ is any positive random variable which is smaller than ξ. Then we can choose a subsequence of $\tilde{v}_n$ still denoted by $\tilde{v}_n$ satisfying

$$\tilde{v}_n(t) \to \tilde{v}(t) \text{ in } \mathbb{L}^\infty(\mathbb{T}^3) \text{ for almost every } t \in [0,\xi).$$

Let $\varphi \in \mathbb{L}^1(\mathbb{T}^3)$ with $|\varphi|_1 \leq 1$, we have

$$\langle \tilde{v}(t), \varphi\rangle = \lim_{n\to\infty} \langle \tilde{v}_n(t), \varphi\rangle \leq \lim_{n\to\infty} |\tilde{v}_n(t)|_\infty \leq |\tilde{v}(0)|_\infty,$$

where the first equality above holds for almost every $t \in [0,\xi)$ and the second inequality follows by Proposition 6.4. Consequently, in view the continuity of $\tilde{v}$ in $\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$, Proposition 6.3 implies that

$$\sup_{t\in[0,\xi)} |\tilde{v}(t)|_\infty \leq |v(0)|_\infty = |u(0)|_\infty.$$

□

Theorem 6.6. *Suppose $u_0 = (g_0, -ag_0, 0)$ is an $\mathbb{H}^1(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable. Then, for any $T > 0$, there exists a unique global strong solution $\tilde{v}$ to (6.34) with condition (6.36) in the sense of Definition 7.2.*

Proof. We assume $(\tilde{v}, \xi)$ is the unique maximum strong solution to (6.34) in sense of Definition 7.2. For $t \in (0, \xi)$, taking inner product of (6.34) with $\Lambda^2 \tilde{v}$ in $\mathbb{L}^2(\mathbb{T}^3)$ yields,

$$\begin{aligned} \partial_t \|\tilde{v}\|_1^2 \quad + \quad & 2\|\tilde{v}\|_2^2 \le c\lambda \|\tilde{v}\|_1 \|\tilde{v}\|_2 + c\alpha^{-1} \exp(\lambda t) |\tilde{v}|_\infty \|\tilde{v}\|_1 \|\tilde{v}\|_2 \\ & + c\lambda |\tilde{v}|_2 \|\tilde{v}\|_2 + c\lambda\alpha^{-1} \exp(\lambda t) |\tilde{v}|_\infty |\tilde{v}|_2 \|\tilde{v}\|_2 \\ \le \quad & \|\tilde{v}\|_2^2 + c(T,\omega)\|\tilde{v}\|_1^2 + c(T,\omega)|\tilde{v}|_\infty^2 \|\tilde{v}\|_1^2 + c(T,\omega)|\tilde{v}|_\infty^2 + c(T,\omega)|\tilde{v}|_\infty^4, \end{aligned} \tag{6.42}$$

where we have used $|\tilde{v}|_2 \le (2\pi)^{\frac{3}{2}} |\tilde{v}|_\infty$ and

$$\lambda = \sup_{(t,x)\in[0,T]\times\mathbb{T}^3} \Big(\Big(\sum_{i=1}^{3} \partial_{x_i} b(x) B(t) \Big)^2 + |\Delta b(x) B(t)| \Big).$$

From Proposition 6.3, we can choose $t_0 \in (0, \xi)$ such that $\tilde{v}(t_0) \in \mathbb{H}^2(\mathbb{T}^3)$. Consequently, applying Proposition 6.4 to (6.42) on $t \in [t_0, \xi)$ yields,

$$\|\tilde{v}(t)\|_1^2 \le c(T,\omega)(1 + \|\tilde{v}(t_0\|_{\mathbb{H}^2}^4) \exp\Big(c(T,\omega)(1 + \|\tilde{v}(t_0\|_{\mathbb{H}^2}^2)\Big).$$

The global existence of the strong solution $\tilde{v}$ follows. The uniqueness of $\tilde{v}$ is similar to the argument before, we omit it. □

Similar to the arguments of Theorem 4.3, we can also obtain:

Theorem 6.7. *Suppose $u_0 = (g_0, -ag_0, 0)$ is an $\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable. Then, there exists a unique global weak pathwise solution $\tilde{v}$ to (6.34) with condition (6.36) in the sense of Definition 7.4.*

Global well-posedness of weak and strong solutions to (6.33) are equivalent to (6.34), hence, we do not restate the global well-posedness of (6.33). See the details in Remark 5. Following section 5, one can also discuss the moments estimates of solutions to (6.33).

7 Appendix

In the following, we give the definitions of weak and strong solutions to (6.34), which are PDE sense.

Definition 7.1 (Local strong solutions to (6.34)). *Suppose u_0 is an $\mathbb{H}^1(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable, T is an arbitrary positive constant.*

1. *A pair (v, τ) is a local strong pathwise solution to (6.34) if τ is a strictly positive random variable taking values in $(0, \infty)$ and $v(\cdot \wedge \tau)$ satisfies (6.34) in a weak sense so that the following regularities hold almost surely,*

$$v(\cdot \wedge \tau) \in C([0,T]; \mathbb{H}^1(\mathbb{T}^3)) \cap L^2([0,T]; \mathbb{H}^2(\mathbb{T}^3)), \tag{7.43}$$

and

$$\partial_t v(\cdot \wedge \tau) \in L^1([0,T]; \mathbb{L}^2(\mathbb{T}^3)).$$

2. *Strong pathwise solutions of (6.34) are said to be pathwise unique up to a random positive time* $\tau > 0$ *if given any pair of solutions* $(v^1, \tau), (v^2, \tau)$ *which coincide at* $t = 0$ *on the event* $\tilde{\Omega} = \{v^1(0) = v^2(0)\} \subset \Omega$, *then*
$$\mathbb{P}(I_{\tilde{\Omega}}(v^1(t \wedge \tau) - v^2(t \wedge \tau)) = 0; \forall t \in [0, T]) = 1.$$

Definition 7.2 (Maximal and global strong solutions to (6.34)). *Let* ξ *be a positive random variable which may take* ∞ *at some* $\omega \in \Omega$. *We say the pair* (v, ξ) *is a maximal pathwise strong solution if for each random variable* $\tau \in (0, \xi)$, (v, τ) *is a local strong pathwise solution satisfying*

$$\sup_{t\in[0,\tau]} \|v(t)\|_1 < \infty, \quad \text{and} \quad \limsup_{t\to\xi} I_{[\xi<\infty]}\|v(t)\|_1 = \infty \tag{7.44}$$

almost surely. And ξ *is called the maximum existence time of* v.

If (v, ξ) *is a maximum pathwise strong solution and* $\xi = \infty$ *a.s., then we say the solution is global.*

Definition 7.3 (Local weak solutions to (6.34)). *Suppose* u_0 *is an* $\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ *valued,* $\mathcal{F}_0$ *measurable random variable,* T *is an arbitrary positive constant.*

(i) A pair (v, τ) *is a local weak pathwise solution to (6.34) if* τ *is a strictly positive random variable taking values in* $(0, \infty)$ *and* $v(\cdot \wedge \tau)$ *satisfies (6.34) in a weak sense so that the following regularities hold almost surely,*

$$v(\cdot \wedge \tau) \in C([0, T]; \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)) \cap L^2([0, T]; \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)), \tag{7.45}$$

and

$$\partial_t v(\cdot \wedge \tau) \in L^1([0, T]; \mathbb{L}^2(\mathbb{T}^3)).$$

(ii) Weak pathwise solutions of (6.34) are said to be pathwise unique up to a random positive time $\tau > 0$ *if given any pair of solutions* $(v^1, \tau), (v^2, \tau)$ *which coincide at* $t = 0$ *on the event* $\tilde{\Omega} = \{v^1(0) = v^2(0)\} \subset \Omega$, *then*
$$\mathbb{P}(I_{\tilde{\Omega}}(v^1(t \wedge \tau) - v^2(t \wedge \tau)) = 0; t \in [0, T]) = 1.$$

Definition 7.4 (Maximal and global weak solutions to (6.34)). *Let* ξ *be a positive random variable which may take* ∞ *at some* $\omega \in \Omega$. *We say the pair* (u, ξ) *is a maximal weak pathwise solution if for each random variable* $\tau \in (0, \xi)$, (v, τ) *is a local strong pathwise solution satisfying*

$$\sup_{t\in[0,\tau]} \|v(t)\|_{\frac{1}{2}} < \infty, \quad \text{and} \quad \limsup_{t\to\xi} I_{[\xi<\infty]}\|v(t)\|_{\frac{1}{2}} = \infty \tag{7.46}$$

almost surely. And ξ *is called the maximum existence time of* v.

If (v, ξ) *is a maximum weak pathwise solution and* $\xi = \infty$ *a.s., then we say the solution is global.*

Remark 5. *Taking advantage of maximum principle we know the global solutions to Galerking approximations of (6.34) are adapted, then so are the limit of these solutions.*

Definition 7.5 (Global strong solutions to (6.33)). *Suppose u_0 is a $\mathbb{H}^1(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable. A stochastic process u is said to be a global strong solution to (6.33) if*

(i) for arbitrary $T > 0$ and $t \in [0, T]$, $u(t)$ is an $\mathcal{F}_t$ adapted process satisfying $u \in C([0, T]; \mathbb{H}^1(\mathbb{T}^3)) \cap \mathbb{L}^2([0, T]; \mathbb{H}^2(\mathbb{T}^3))$ almost surely;

(ii) u solves the stochastic 3D Burgers equation in the following sense:

$$u(t) - \int_0^t \Delta u ds + \int_0^t (u \cdot \nabla u) ds = u(0) + \int_0^t b(x) u \circ dB(s), a.s.,$$

with the equality understood in $\mathbb{H}$ and $t \in [0, T]$. Furthermore, let u and $\tilde{u}$ be two global strong solutions to (6.33). If $u(0) = \tilde{u}(0)$ a.s., we have

$$\mathbb{P}(u(t) = \tilde{u}(t),\ for\ all\ t \in [0, T]) = 1,$$

then we say the strong solution u to (6.33) is unique.

Definition 7.6 (Global weak solutions to (6.33)). *Suppose u_0 is a $\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable. A stochastic process u is said to be a global weak solution to (6.33) if*

(i) for arbitrary $T > 0$ and $t \in [0, T]$, $u(t)$ is an $\mathcal{F}_t$ adapted process satisfying $u \in C([0, T]; \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)) \cap \mathbb{L}^2([0, T]; \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$ almost surely;

(ii) u solves the stochastic 3D Burgers equation in the following sense:

$$\begin{aligned} \langle u(t), \phi\rangle + \int_0^t \langle u(s), \Lambda^2 \phi\rangle ds + \int_0^t \langle (u \cdot \nabla u)(s), \phi\rangle ds \\ = \langle u(0), \phi\rangle + \int_0^t \langle b(x) u(s, x), \phi\rangle \circ dB(s), \end{aligned} \tag{7.47}$$

for all $t \in [0, T]$ and $\phi \in D(\Lambda^2)$. Furthermore, let u and $\tilde{u}$ be two strong solutions to (1.1). If $u(0)$= $\tilde{u}(0)$ a.s., we have

then we say the weak solution u to (6.33) is unique.

In Section 6, we solve the global well-posedness of 3D random Burgers equation (6.34) in the frame of deterministic 3D Burgers equation (see page 11 of [47]), where the maximum principle is a key tool. According to the maximum principle, the coefficient of $\hat{v}$ should be nonnegative. That is to say, one needs that

$$B(t) \sum_{i=1}^3 \hat{v}_i(t, x) \partial_{x_i} b(x) \geq 0, \text{ on } [0, T] \times \mathbb{T}^3,$$

or equivalently,

$$B(t) \sum_{i=1}^3 u_i(t, x) \partial_{x_i} b(x) \geq 0, \text{ on } [0, T] \times \mathbb{T}^3. \tag{7.48}$$

In fact, it seems very difficult to find a solution u to (6.33) (or $\hat{v}$ to (6.34)) satisfying (7.48). Because, if there is solution $\hat{v}$ satisfies

$$B(t)\sum_{i=1}^{3}\hat{v}_i(t,x)\partial_{x_i}b(x) > 0, \ \textit{on any interval } [t_1,t_2]\subset[0,T],$$

it contradicts with the fundamental properties of Brownian motion $B(t)$, in particular, the support property that with positive probability, $B(t)$ may visit everywhere. The inequality (7.48) can be only possible when

$$\hat{v}(t,x)\cdot\nabla b(x) = 0, \ \textit{on } [0,T]\times\mathbb{T}^3, \mathbb{P}-a.s.. \tag{7.49}$$

In fact, following the arguments as in Section 3, one can prove that there exists a unique local solution to (6.34). If one further requires the solution should satisfy (7.49), this may lead to the solution equaling to a trivial solution. For example, let $\nabla b(x) = (b_1(x), b_2(x), b_3(x)), x\in\mathbb{T}^3$. We try to find a function $g : [0,T]\times\Omega\to\mathbb{R}$ and construct a solution $\hat{v} = \hat{v}(t,x) = g(t,x)(-b_2(x), b_1(x), 0), (t,x)\in[0,T]\times\mathbb{T}^3$, to (6.34). Note that, here $\hat{v}$ satisfies (7.48), hence, if $\hat{v}$ is a local solution to (6.34), then it must be global. Substituting $\hat{v}$ with $g(t,x)(-b_2(x), b_1(x), 0)$ in (6.34) yields,

$$\begin{aligned} b_1(x)\partial_t g(t,x) \ - \ & (\Delta b_1(x))g(t,x) - b_1(x)\Delta g(t,x) - 2\sum_{j=1}^{3}\partial_{x_j}b_1(x)\partial_{x_j}g(t,x) \\ & + b_1(x)\partial_{x_1}b_1(x)g^2(t,x) + b_1^2(x)g(t,x)\partial_{x_1}g(t,x) \\ & + b_2(x)\partial_{x_2}b_1(x)g^2(t,x) + b_1(x)b_2(x)g(t,x)\partial_{x_2}g(t,x) = 0, \end{aligned}$$

and

$$\begin{aligned} b_2(x)\partial_t g(t,x) \ - \ & (\Delta b_2(x))g(t,x) - b_2(x)\Delta g(t,x) - 2\sum_{j=1}^{3}\partial_{x_j}b_2(x)\partial_{x_j}g(t,x) \\ & + b_1(x)\partial_{x_1}b_2(x)g^2(t,x) + b_1(x)b_2(x)g(t,x)\partial_{x_1}g(t,x) \\ & + b_2(x)\partial_{x_2}b_2(x)g^2(t,x) + b_2^2(x)g(t,x)\partial_{x_2}g(t,x) = 0. \end{aligned}$$

Obviously, if $b_1(x)\neq b_2(x)$, one can only obtain that $g\equiv 0$, on $[0,T]\times\mathbb{T}^3$. Hence, in Section 6, we find a unique global solution to (6.34) under the assumption

$$\textit{Any two compoments of } \nabla b(x) = (\partial_{x_1}b(x), \partial_{x_2}b(x), \partial_{x_3}b(x)), \ \textit{are linearly correlated}.$$

■

Acknowledgements We are grateful to Professor Z. Brzezniak for constant support and stimulating discussions. We would also like to thank Professor Tusheng Zhang and Professor B. Goldys for numerous inspiring discussions. We thank the referees for their insightful critiques and constructive suggestions which have led us to improve the presentation of the paper. This work was partially supported by National Key R&D Program of China (No. 2020YFA0712700); NNSF of China (Grant No.12090014,11931004,11971077); Key Laboratory of Random Complex Structures and Data Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences (Grant No. 2008D-P173182); Chongqing Key Laboratory of Analytic Mathematics and Applications, Chongqing University; Natural Science Foundation Project of CQ (Grant No. cstc2020jcyj-msxmX0441); Fundamental Research Funds for the Central Universities (Grant No. 2020CDJ-LHZZ-027).

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