Some new fixed point results under constraint inequalities in comparable complete partially ordered Menger PM-spaces

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Abstract In this paper, we introduce the concept of comparable $\mathscr{T}$-completeness of a partially ordered Menger PM-space and discuss the existence of fixed points for mappings satisfying certain conditions in the framework of a comparable $\mathscr{T}$-complete partially ordered Menger PM-space. We obtain some new results which generalize many known ones in the literature. Moreover, we derive some consequent results and give an example to illustrate our main result.

Key Words: Menger PM-space; fixed point; constraint inequalities; partial order; implicit contraction

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1 Introduction and preliminaries

The concept of a probabilistic metric space (PM-space) was first raised by Menger and revisited by Schweizer and Sklar [1, 2]. The fundamental theory of PM-spaces has been established and developed during the second half of the 20th century [3, 4]. Specifically, fixed point theory and nonlinear operator theory in PM-spaces has attracted much attention and a large number of papers are focused on such field [5–13].

It was Turinici who first suggested imposing a partial order on the structure of a metric space and discussed fixed point problems in this framework [14], which inspired many consequent work in this regard [15–18]. It is a natural idea to consider fixed point problems in a partially ordered Menger PM-space, and many results were also obtained in such spaces in recent years [19–22]. On the other hand, the notion of α admissible mapping has been defined in [23], and the fixed point results for α-ψ contractive mappings, generalized α-ψ contractive mappings and α-ψ-Meir-Keeler contractive mappings have been obtained in [23–25]. In particular, it has been shown in [24] that the fixed

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point results in standard metric spaces, metric spaces endowed with a partial order and metric spaces where mappings are cyclic can be obtained by proper choice of α from the main results of [24].

Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space and X be endowed with two partial orders $\preceq_1$ and $\preceq_2$, and $T, A, B, C, D : X \to X$ be five self-mappings. Consider the following problem: Find $x \in X$, such that

$$\begin{cases} x = Tx, \\ Ax \preceq_1 Bx, \\ Cx \preceq_2 Dx. \end{cases} \tag{1.1}$$

Jleli and Samet discussed in [26] the existence of solutions to (1.1) in metric spaces by introducing the concepts of d-regularity and $(A, B, C, D, \preceq_1, \preceq_2)$-stability. In [27], Ansari *et al.* revisited the results in [26] and proved the uniqueness of the solution to (1.1) by assuming that only A and B are continuous (or only C and D are continuous). The main results of [26] and [27] were generalized to the setting of Menger PM-spaces in [28]. In [29], the authors investigated the existence of solution to problem (1.1) by replacing the completeness of the metric space by introducing the so-called comparable completeness and considering a more general contractive condition.

In this paper, we will revisit problem (1.1) in partially ordered Menger PM-spaces and discuss its solution by introducing $\mathscr{T}$-completeness of partially ordered Menger PM-spaces and a more general contractive condition. Our results are the generalizations of the results in [29] and many other literatures.

We now recall some basic definitions in the theory of Menger PM-spaces.

A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a *distribution function* if it is nondecreasing left-continuous with $\sup_{t \in \mathbb{R}} F(t) = 1$ and $\inf_{t \in \mathbb{R}} F(t) = 0$.

We will denote by $\mathscr{D}$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Let $F_1, F_2 \in \mathscr{D}$. The algebraic sum $F_1 \oplus F_2$ is defined by [30]

$$(F_1 \oplus F_2)(t) = \sup_{t_1 + t_2 = t} \min\{F_1(t_1), F_2(t_2)\} \quad \text{for all } t \in \mathbb{R}.$$

Definition 1.1 **[6]** A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a *triangular norm* (for short, a t-norm) if the following conditions are satisfied: $\Delta(a, 1) = a$; $\Delta(a, b) = \Delta(b, a)$; $\Delta(a, c) \geq \Delta(b, d)$ for $a \geq b, c \geq d$;$\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$.

A typical example of a t-norm is Δ_{min} which is defined by $\Delta_{min}(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$.

Definition 1.2 [**6**] A triplet $(X, \mathscr{F}, \Delta)$ is called a *Menger probabilistic metric space* (for short, a *Menger PM-space*) if X is a nonempty set, Δ is a t-norm and $\mathscr{F}$ is a mapping from $X \times X$ into $\mathscr{D}$ satisfying the following conditions (we denote $\mathscr{F}(x, y)$ by $F_{x,y}$):

(MPM-1) $F_{x,y}(t) = H(t)$ for all $t \in \mathbb{R}$ if and only if $x = y$;

(MPM-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in \mathbb{R}$;

(MPM-3) $F_{x,y}(t + s) \geq \Delta(F_{x,z}(t), F_{z,y}(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

Remark 1.1 [**6**] If $\sup_{0<t<1} \Delta(t, t) = 1$, then $(X, \mathscr{F}, \Delta)$ is a Hausdorff topological space in the (ϵ, λ)-topology $\mathscr{T}$, *i.e.*, the family of sets $\{U_x(\epsilon, \lambda) : \epsilon > 0, \lambda \in (0, 1]\}(x \in X)$ is a basis of neighborhoods of a point x for $\mathscr{T}$, where $U_x(\epsilon, \lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda)\}$.

By virtue of the topology $\mathscr{T}$, a sequence $\{x_n\}$ is said to be $\mathscr{T}$-convergent to $x \in X$(we write $x_n \stackrel{\mathscr{T}}{\to} x(n \to \infty)$) if for any given $\epsilon > 0$ and $\lambda \in (0, 1]$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$, which is equivalent to $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$; $\{x_n\}$ is called a $\mathscr{T}$-Cauchy sequence in $(X, \mathscr{F}, \Delta)$ if for any given $\epsilon > 0$ and $\lambda \in (0, 1]$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ whenever $n, m \geq N$; $(X, \mathscr{F}, \Delta)$ is said to be $\mathscr{T}$-complete if each $\mathscr{T}$-Cauchy sequence in X is $\mathscr{T}$-convergent in X. It is worth noting that in a Menger PM-space, $\lim_{n\to\infty} x_n = x$ implies that $x_n \stackrel{\mathscr{T}}{\to} x(n \to \infty)$.

Remark 1.2 [**6**] Let (X, d) be a metric space and $\mathscr{F} : X \times X \to \mathscr{D}$ be defined by

$$\mathscr{F}(x, y)(t) = F_{x,y}(t) = H(t - d(x, y)), \forall x, y \in X \text{ and } t > 0. \tag{1.2}$$

Then $(X, \mathscr{F}, \Delta_{min})$ is a $\mathscr{T}$-complete Menger PM-space induced by (X, d).

We next recall the definition of F-regularity and $(A, B, C, D, \preceq_1, \preceq_2)$-stability.

Definition 1.3 [**28**] Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space and $\preceq$ be partial order on X. $\preceq$ is called *F-regular*, if for any sequences $\{a_n\}, \{b_n\} \subset X$, we have

$$\lim_{n\to\infty} F_{a_n,a}(t) = \lim_{n\to\infty} F_{b_n,b}(t) = 1 \text{ and } a_n \preceq b_n \text{ for all } n \in \mathbb{N} \text{ and } t > 0 \Longrightarrow a \preceq b,$$

where $(a, b) \in X \times X$.

Definition 1.4 [**26**] Let X be a nonempty set endowed with two partial orders $\preceq_1$ and $\preceq_2$. Let $T, A, B, C, D : X \to X$ be five self-mappings. The mapping T is called $(A, B, C, D, \preceq_1, \preceq_2)$-*stable*, if the following condition is satisfied:

$$x \in X, Ax \preceq_1 Bx \Longrightarrow CTx \preceq_2 DTx.$$

The concept of an α-admissible mapping with respect to η on a Menger PM-space has been proposed in [8] as follows.

Definition 1.5 [**8**] Let T be a self-mapping on a Menger PM-space $(X, \mathscr{F}, \Delta)$ and $\alpha, \eta : X \times X \times (0, +\infty) \to (0, +\infty)$ be two functions. T is called an *α-admissible mapping with respect to η*, if for all $t > 0$, we have

$$\alpha(x, y, t) \leq \eta(x, y, t) \Longrightarrow \alpha(Tx, Ty, t) \leq \eta(Tx, Ty, t), x, y \in X.$$

Remark 1.3 [8] T is called an α-admissible mapping, if $\eta(x,y,t) \equiv 1$. In this case, it coincides with Definition 3.2 in [31]. T is called an η-subadmissible mapping, if $\alpha(x,y,t) \equiv 1$. In this case, it coincides with Definition 2.2 in [31].

We can further give the notion of a triangular α-admissible mapping with respect to η on a Menger PM-space in the following way.

Definition 1.6 Let T be a self-mapping on a Menger PM-space $(X, \mathscr{F}, \Delta)$ and $\alpha, \eta : X \times X \times (0,+\infty) \to (0,+\infty)$ be two functions. T is called a *triangular α-admissible mapping with respect to η*, if it is an α-admissible mapping with respect to η, and

$$\alpha(x,y,t) \leq \eta(x,y,t) \text{ and } \alpha(y,z,t) \leq \eta(y,z,t) \Longrightarrow \alpha(x,z,t) \leq \eta(x,z,t), x,y,z \in X.$$

Now, we introduce some new definitions that will be used in the next section. These concepts are generalized from a metric space to the setting of a Menger PM-space.

Definition 1.7 Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space and $\alpha, \eta : X \times X \times (0,+\infty) \to (0,+\infty)$ be two functions. A sequence $\{x_n\}$ is called *α-regular with respect to η* if the following conditions is satisfied: if $\{x_n\}$ satisfies that $\alpha(x_n, x_{n+1}, t) \leq \eta(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N}$ and $t > 0$ with $x_n \xrightarrow{\mathscr{T}} x \in X(n \to \infty)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x, t) \leq \eta(x_{n_k}, x, t)$ for all $k \in \mathbb{N}$ and $t > 0$.

Definition 1.8 A partially ordered Menger PM-space $(X, \mathscr{F}, \Delta, \preceq)$ is called *regular* if for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \xrightarrow{\mathscr{T}} x \in X(n \to \infty)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all k.

Definition 1.9 [29] Let $(X, \preceq)$ be an ordered space. A sequence $\{x_n\}$ is called a *comparable sequence*, if

$$(x_n \preceq x_{n+k} \text{ for all } n,k) \text{ or } (x_{n+k} \preceq x_n \text{ for all } n,k).$$

Definition 1.10 A partially ordered Menger PM-space $(X, \mathscr{F}, \Delta, \preceq)$ is said to be *comparable $\mathscr{T}$-complete* if every $\mathscr{T}$-Cauchy comparable sequence is $\mathscr{T}$-convergent in X.

It is claimed that every complete metric space is comparable complete and that the converse is not true by giving an example [29]. It is also easy to see that every $\mathscr{T}$-complete Menger PM-space is comparable $\mathscr{T}$-complete but the converse is not true.

Definition 1.11 Let $(X, \mathscr{F}, \Delta, \preceq)$ be a partially ordered Menger PM-space. A mapping $f : X \to X$ is said to be *comparable $\mathscr{T}$-continuous in $a \in X$*, if for each comparable sequence $\{a_n\}$ in X with $a_n \xrightarrow{\mathscr{T}} a(n \to \infty)$, we have $f(a_n) \xrightarrow{\mathscr{T}} f(a)(n \to \infty)$. f is comparable $\mathscr{T}$-continuous on X if f is comparable $\mathscr{T}$-continuous in each $a \in X$.

Definition 1.12 [29] Let $(X, \preceq)$ be a partially ordered space and $T : X \to X$ be a mapping. $x_0 \in X$ is said to be *T-comparable* if for all $n \in \mathbb{N}$, x_0 and $T^n x_0$ are comparable. We denote

$$\mathfrak{T}_T = \{x_0 \in X : (x_0 \preceq T^n x_0 \text{ for all } n \in \mathbb{N}) \text{ or } (T^n x_0 \preceq x_0 \text{ for all } n \in \mathbb{N})\}.$$

Definition 1.13 [29] Let $(X, \preceq)$ be a partially ordered space. A mapping $T : X \to X$ is said to be *$\preceq$-preserving*, if $x \preceq y$ implies $Tx \preceq Ty$.

Proposition 1.1 [29] Let $(X, \preceq)$ be a partially ordered space and $T : X \to X$ be $\preceq$-preserving. Let $\{x_n\}$ be a Picard iterative sequence with initial point $x_0 \in \mathfrak{T}_T$, i.e., $x_n = T^n(x_0)$. Then $\{x_n\}$ is a comparable sequence.

Denote by Φ the set of functions $\varphi : (0,1] \to [0,+\infty)$ satisfying the following conditions:

(Φ_1) φ is continuous and nonincreasing;

(Φ_2) $\varphi(x) = 0$ if and only if $x = 1$.

Denote by $\mathcal{H}(X)$ the class of mappings $h : X \times X \times (0,+\infty) \to [0,1)$ satisfying the following condition:

$$\lim_{n\to\infty} h(x_n, y_n, t) = 1 \text{ for all } t > 0 \Longrightarrow \lim_{n\to\infty} F_{x_n,y_n}(t) = 1 \text{ for all } t > 0,$$

for all sequences $\{x_n\}, \{y_n\}$ in X such that the sequence $\{F_{x_n,y_n}(t)\}$ is increasing and convergent for each $t > 0$.

2 Main results

We are now ready to prove our main result.

Theorem 2.1 Let $(X, \mathscr{F}, \Delta_{min}, \preceq)$ be a comparable $\mathscr{T}$-complete Menger PM-space and $\preceq_1$ and $\preceq_2$ be two partial orders on X. Also, let $T, A, B, C, D : X \to X$ be self-mappings and $\alpha, \eta : X \times X \times (0,+\infty) \to (0,+\infty)$ be two functions. Suppose that the following conditions are satisfied:

(i) $\preceq_i$ is F-regular $(i = 1,2)$, and T is $\preceq$-preserving and α-admissible with respect to η;

(ii) A, B and T are comparable $\mathscr{T}$-continuous or C, D and T are comparable $\mathscr{T}$-continuous;

(iii) there exists $x_0 \in \mathfrak{T}_T$, such that $Ax_0 \preceq_1 Bx_0$, $Cx_0 \preceq_2 Dx_0$ and $\alpha(x_0, Tx_0, t) \leq \eta(x_0, Tx_0, t)$ for all $t > 0$;

(iv) T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable and $(C, D, A, B, \preceq_2, \preceq_1)$-stable;

(v) there exists $h \in \mathcal{H}(X)$ and $\varphi \in \Phi$ such that for $x, y \in X$,

$$Ax \preceq_1 Bx, Cy \preceq_2 Dy \Longrightarrow \eta(x,y,t)\varphi(F_{Tx,Ty}(t)) \leq \alpha(x,y,t)h(x,y,t)\varphi(M_{x,y}(t)) \text{ for all } t > 0,$$

where $M_{x,y}(t) = \min\{F_{x,y}(t), [F_{x,Tx} \oplus F_{y,Ty}](2t), [F_{x,Ty} \oplus F_{y,Tx}](2t)\}$.

Then the sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a solution to (1.1).

Proof. Without loss of generality, we can assume that A, B and T are comparable $\mathscr{T}$-continuous for assumption (ii).

Step 1. By assumption (iii), there exists $x_0 \in \mathfrak{T}_T$ such that

$$Ax_0 \preceq_1 Bx_0 \text{ and } \alpha(x_0, Tx_0, t) \leq \eta(x_0, Tx_0, t) \text{ for all } t > 0.$$

Define the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. It follows from Proposition 1.1 that $\{x_n\}$ is a comparable sequence. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of T. Now, suppose that $x_n \neq x_{n+1}$ for $n \in \mathbb{N} \cup \{0\}$. By assumption (iv), we have $CTx_0 \preceq_2 DTx_0$, that is, $Cx_1 \preceq_2 Dx_1$. By assumption (iv), we have $ATx_1 \preceq_1 BTx_1$, that is, $Ax_2 \preceq_1 Bx_2$. Again, by assumption (iv), we obtain $CTx_2 \preceq_2 DTx_2$, that is, $Cx_3 \preceq_2 Dx_3$. Continuing this process, we obtain

$$Ax_{2n} \preceq_1 Bx_{2n} \text{ and } Cx_{2n+1} \preceq_2 Dx_{2n+1}, n = 0, 1, 2, \cdots .$$

From $Cx_0 \preceq_1 Dx_0$ and condition (iv), we can similarly obtain

$$Cx_{2n} \preceq_1 Dx_{2n} \text{ and } Ax_{2n+1} \preceq_2 Bx_{2n+1}, n = 0, 1, 2, \cdots .$$

Thus we have

$$Ax_n \preceq_1 Bx_n \text{ and } Cx_n \preceq_2 Dx_n, n = 0, 1, 2, \cdots . \tag{2.1}$$

Since T is α-admissible with respect to η, by (iii), we have

$$\alpha(x_0, x_1, t) \leq \eta(x_0, x_1, t), \forall t > 0 \Longrightarrow \alpha(Tx_0, Tx_1, t) \leq \eta(Tx_0, Tx_1, t) \text{ for all } t > 0.$$

Inductively, we obtain

$$\alpha(x_{n-1}, x_n, t) \leq \eta(x_{n-1}, x_n, t) \text{ for all } n \in \mathbb{N} \text{ and } t > 0. \tag{2.2}$$

By (2.1), (2.2) and (v), it holds for all $n \in \mathbb{N}$ and $t > 0$ that

$$\varphi(F_{x_n,x_{n+1}}(t)) \leq h(x_{n-1}, x_n, t)\varphi(M_{x_{n-1},x_n}(t)) < \varphi(M_{x_{n-1},x_n}(t)), \tag{2.3}$$

where

$$\begin{aligned} M_{x_{n-1},x_n}(t) &= \min\{F_{x_{n-1},x_n}(t), [F_{x_{n-1},Tx_{n-1}} \oplus F_{x_n,Tx_n}](2t), [F_{x_{n-1},Tx_n} \oplus F_{x_n,Tx_{n-1}}](2t)\} \\ &= \min\{F_{x_{n-1},x_n}(t), [F_{x_{n-1},x_n} \oplus F_{x_n,x_{n+1}}](2t), [F_{x_{n-1},x_{n+1}} \oplus F_{x_n,x_n}](2t)\}. \end{aligned}$$

Note that for all $n \in \mathbb{N}$ and $t > 0$, for any $\delta \in (0, 2t)$, we have

$$\begin{aligned} [F_{x_{n-1},x_{n+1}} \oplus F_{x_n,x_n}](2t) &\geq \min\{F_{x_{n-1},x_{n+1}}(2t - \delta), F_{x_n,x_n}(\delta)\} \\ &= \min\{F_{x_{n-1},x_{n+1}}(2t - \delta), 1\}. \end{aligned}$$

Letting $\delta \to 0$, by the left-continuity of the distribution function, we obtain

$$[F_{x_{n-1},x_{n+1}} \oplus F_{x_n,x_n}](2t) \geq F_{x_{n-1},x_{n+1}}(2t) \text{ for all } n \in \mathbb{N} \text{ and } t > 0.$$

For all $n \in \mathbb{N}$ and $t > 0$, for each $t_1, t_2 \in (0, 2t)$ with $t_1 + t_2 = 2t$, we have

$$F_{x_{n-1},x_{n+1}}(2t) \geq \Delta_{min}(F_{x_{n-1},x_n}(t_1), F_{x_n,x_{n+1}}(t_2)) = \min\{F_{x_{n-1},x_n}(t_1), F_{x_n,x_{n+1}}(t_2)\},$$

and thus we obtain

$$F_{x_{n-1},x_{n+1}}(2t) \geq [F_{x_{n-1},x_n} \oplus F_{x_n,x_{n+1}}](2t) \text{ for all } n \in \mathbb{N} \text{ and } t > 0.$$

Therefore, it holds for all $n \in \mathbb{N}$ and $t > 0$ that

$$\begin{aligned} M_{x_{n-1},x_n}(t) &= \min\{F_{x_{n-1},x_n}(t), [F_{x_{n-1},Tx_{n-1}} \oplus F_{x_n,Tx_n}](2t)\} \\ &\geq \min\{F_{x_{n-1},x_n}(t), F_{x_n,x_{n+1}}(t)\}. \end{aligned}$$

If $\min\{F_{x_{n-1},x_n}(t), F_{x_n,x_{n+1}}(t)\} = F_{x_n,x_{n+1}}(t)$, then

$$\varphi(F_{x_n,x_{n+1}}(t)) < \varphi(M_{x_{n-1},x_n}(t)) \leq \varphi(F_{x_n,x_{n+1}}(t)) \text{ for all } n \in \mathbb{N} \text{ and } t > 0$$

which is a contradiction. Thus, we conclude that $\min\{F_{x_{n-1},x_n}(t), F_{x_n,x_{n+1}}(t)\} = F_{x_{n-1},x_n}(t)$, and thus by (2.3), we obtain

$$\varphi(F_{x_n,x_{n+1}}(t)) < \varphi(F_{x_{n-1},x_n}(t)) \text{ for all } n \in \mathbb{N} \text{ and } t > 0.$$

By the monotonicity of φ, we have

$$F_{x_n,x_{n+1}}(t) \geq F_{x_{n-1},x_n}(t) \text{ for all } n \in \mathbb{N} \text{ and } t > 0.$$

Thus, $\{F_{x_{n+1},x_n}(t)\}$ is an increasing sequence of positive numbers for each $t > 0$. Therefore, there exists some $r(t) \in [0,1]$, such that

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(t) = r(t) \text{ for all } t > 0.$$

Suppose that there exists $t_0 > 0$ such that $r(t_0) < 1$. Then by (2.3), we have

$$\frac{\varphi(F_{x_n,x_{n+1}}(t_0))}{\varphi(F_{x_{n-1},x_n}(t_0))} \geq h(x_{n-1}, x_n, t_0),$$

which implies that

$$\lim_{n\to\infty} h(x_{n-1}, x_n, t_0) = 1.$$

Noting that $h \in \mathcal{H}(X)$, we thus obtain

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(t_0) = r(t_0) = 1,$$

which is a contradiction. Therefore, we have $r(t) = 1$ for all $t > 0$, that is,

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(t) = 1 \text{ for all } t > 0. \tag{2.4}$$

Step 2. We now show that $\{x_n\}$ is a $\mathscr{T}$-Cauchy comparable sequence in $(X, \mathscr{F}, \Delta, \preceq)$. Suppose that this is not true. Then there exists $\epsilon_0 > 0$ and $\lambda_0 \in (0,1]$, for which we can find two sequences of positive integers $\{m_k\}$ and $\{n_k\}$, such that for all positive integers k, we have

$$n_k > m_k > k, \;\; F_{x_{m_k},x_{n_k}}(\epsilon_0) \leq 1 - \lambda_0, \;\; F_{x_{m_k},x_{n_k-1}}(\epsilon_0) > 1 - \lambda_0. \tag{2.5}$$

For any $\delta \in (0, \epsilon_0)$, we have

$$F_{x_{m_k},x_{n_k}}(\epsilon_0) \geq \Delta(F_{x_{m_k},x_{n_k-1}}(\epsilon_0 - \delta), F_{x_{n_k-1},x_{n_k}}(\delta)).$$

Letting $k \to \infty$, by (2.4), we have

$$\liminf_{k\to\infty} F_{x_{m_k},x_{n_k}}(\epsilon_0) \geq \Delta(\liminf_{k\to\infty} F_{x_{m_k},x_{n_k-1}}(\epsilon_0 - \delta), 1) = \liminf_{k\to\infty} F_{x_{m_k},x_{n_k-1}}(\epsilon_0 - \delta).$$

Letting $\delta \to 0$, by the left-continuity of the distribution function and (2.5), we obtain

$$\liminf_{k\to\infty} F_{x_{m_k},x_{n_k}}(\epsilon_0) \geq \liminf_{k\to\infty} F_{x_{m_k},x_{n_k-1}}(\epsilon_0) \geq 1 - \lambda_0.$$

On the other hand, it can be seen easily from (2.5) that

$$\limsup_{k\to\infty} F_{x_{m_k},x_{n_k}}(\epsilon_0) \leq 1 - \lambda_0.$$

So we obtain

$$\lim_{k\to\infty} F_{x_{m_k},x_{n_k}}(\epsilon_0) = 1 - \lambda_0. \tag{2.6}$$

Similar arguments show that

$$\lim_{k\to\infty} F_{x_{n_k+1},x_{m_k}}(\epsilon_0) = \lim_{k\to\infty} F_{x_{n_k},x_{m_k-1}}(\epsilon_0) = \lim_{k\to\infty} F_{x_{n_k+1},x_{m_k+1}}(\epsilon_0) = 1 - \lambda_0. \tag{2.7}$$

Note that for all $k \in \mathbb{N}$, there exists a positive integer $i_k \in \{0, 1\}$ such that

$$n_k - m_k + i_k \equiv 1(2).$$

By (2.1), for all $k > 1$, we have

$$Ax_{n_k} \preceq_1 Bx_{n_k} \text{ and } Cx_{m_k-i_k} \preceq_2 Dx_{m_k-i_k}$$

or

$$Ax_{m_k-i_k} \preceq_1 Bx_{m_k-i_k} \text{ and } Cx_{n_k} \preceq_2 Dx_{n_k}.$$

By (vi), for $k \in \mathbb{N}$, we have

$$\varphi(F_{x_{n_k+1},x_{m_k-i_k+1}}(\epsilon_0)) \leq h(x_{n_k}, x_{m_k-i_k}, \epsilon_0)\varphi(M_{x_{n_k},x_{m_k-i_k}}(\epsilon_0)), \tag{2.8}$$

where

$$\begin{aligned} M_{x_{n_k},x_{m_k-i_k}}(\epsilon_0) &= \min\{F_{x_{n_k},x_{m_k-i_k}}(\epsilon_0), F_{x_{n_k},Tx_{n_k}}(\epsilon_0), F_{x_{m_k-i_k},Tx_{m_k-i_k}}(\epsilon_0), \\ &\qquad [F_{x_{n_k},Tx_{m_k-i_k}} \oplus F_{x_{m_k-i_k},Tx_{n_k}}](2\epsilon_0)\} \\ &= \min\{F_{x_{n_k},x_{m_k-i_k}}(\epsilon_0), F_{x_{n_k},x_{n_k+1}}(\epsilon_0), F_{x_{m_k-i_k},x_{m_k-i_k+1}}(\epsilon_0), \\ &\qquad [F_{x_{n_k},x_{m_k-i_k+1}} \oplus F_{x_{m_k-i_k},x_{n_k+1}}](2\epsilon_0)\}. \end{aligned}$$

Note that

$$[F_{x_{n_k},x_{m_k-i_k+1}} \oplus F_{x_{m_k-i_k},x_{n_k+1}}](2\epsilon_0) \geq \min\{F_{x_{n_k},x_{m_k-i_k+1}}(\epsilon_0), F_{x_{m_k-i_k},x_{n_k+1}}(\epsilon_0)\}. \tag{2.9}$$

For any $\delta \in (0, \epsilon_0)$, we have

$$F_{x_{n_k}, x_{m_k - i_k + 1}}(\epsilon_0) \geq \Delta(F_{x_{n_k}, x_{m_k - i_k}}(\epsilon_0 - \delta), F_{x_{m_k - i_k}, x_{m_k - i_k + 1}}(\delta)).$$

It follows from (2.4) that

$$\liminf_{k\to\infty} F_{x_{n_k}, x_{m_k - i_k + 1}}(\epsilon_0) \geq \liminf_{k\to\infty} F_{x_{n_k}, x_{m_k - i_k}}(\epsilon_0 - \delta).$$

Letting $\delta \to 0$, by the left-continuity of the distribution function, we obtain

$$\liminf_{k\to\infty} F_{x_{n_k}, x_{m_k - i_k + 1}}(\epsilon_0) \geq \liminf_{k\to\infty} F_{x_{n_k}, x_{m_k - i_k}}(\epsilon_0). \tag{2.10}$$

Similarly, we can prove that

$$\liminf_{k\to\infty} F_{x_{m_k - i_k}, x_{n_k + 1}}(\epsilon_0) \geq \liminf_{k\to\infty} F_{x_{n_k}, x_{m_k - i_k}}(\epsilon_0). \tag{2.11}$$

Combining (2.9), (2.10) and (2.11), we obtain

$$\liminf_{k\to\infty} [F_{x_{n_k}, x_{m_k - i_k + 1}} \oplus F_{x_{m_k - i_k}, x_{n_k + 1}}](2\epsilon_0) \geq \liminf_{k\to\infty} F_{x_{n_k}, x_{m_k - i_k}}(\epsilon_0). \tag{2.12}$$

And thus

$$\liminf_{k\to\infty} M_{x_{n_k}, x_{m_k - i_k}}(\epsilon_0) \geq \liminf_{k\to\infty} F_{x_{n_k}, x_{m_k - i_k}}(\epsilon_0). \tag{2.13}$$

It follows from (2.8) and (2.13) that

$$\varphi(\liminf_{k\to\infty} F_{x_{n_k+1}, x_{m_k - i_k + 1}}(\epsilon_0)) \leq \liminf_{k\to\infty} h(x_{n_k}, x_{m_k - i_k}, \epsilon_0) \varphi(\liminf_{k\to\infty} F_{x_{n_k}, x_{m_k - i_k}}(\epsilon_0)),$$

which by (2.7) and the continuity of φ implies that

$$\liminf_{k\to\infty} h(x_{n_k}, x_{m_k - i_k}, \epsilon_0) \geq 1.$$

Noting that $\limsup_{k\to\infty} h(x_{n_k}, x_{m_k - i_k}, \epsilon_0) \leq 1$ holds, we obtain

$$\lim_{k\to\infty} h(x_{n_k}, x_{m_k - i_k}, \epsilon_0) = 1,$$

which yields that

$$\lim_{k\to\infty} F_{x_{n_k}, x_{m_k - i_k}}(\epsilon_0) = 1.$$

This is in contradiction to (2.6) or (2.7). Therefore, $\{x_n\}$ is a $\mathscr{T}$-Cauchy comparable sequence in $(X, \mathscr{F}, \Delta, \preceq)$.

Step 3. Since $(X, \mathscr{F}, \Delta, \preceq)$ is comparable $\mathscr{T}$-complete, from Step 2, we know that there exists $x^* \in X$ such that $x_n \xrightarrow{\mathscr{T}} x^* (n \to \infty)$. Since T is comparable $\mathscr{T}$-continuous, we get $x_{n+1} = Tx_n \xrightarrow{\mathscr{T}} Tx^* (n \to \infty)$. So we obtain

$$Tx^* = x^*. \tag{2.14}$$

Since A and B are comparable $\mathscr{T}$-continuous and $\{x_{2n}\}$ is a comparable sequence, we have

$$\lim_{n\to\infty} F_{Ax_{2n},Ax^*}(t) = \lim_{n\to\infty} F_{Bx_{2n},Bx^*}(t) = 1 \text{ for all } t > 0. \tag{2.15}$$

Noting that $\preceq_1$ is F-regular, it follows from (2.1) and (2.15) that

$$Ax^* \preceq_1 Bx^*. \tag{2.16}$$

By assumption (iv) and (2.16), we obtain

$$CTx^* \preceq_2 DTx^*,$$

which implies that

$$Cx^* \preceq_2 Dx^*. \tag{2.17}$$

Combining (2.14), (2.16) and (2.17), we conclude that x^* is a solution to problem (1.1). We can similarly prove the theorem by alternatively assuming that C, D and T are comparable $\mathscr{T}$-continuous. This completes the proof.

The next result removes the $\mathscr{T}$-continuity assumption of the mapping T in Theorem 2.1 by utilizing α-regularity with respect to η assumption of a sequence.

Theorem 2.2 Let $(X, \mathscr{F}, \Delta_{min}, \preceq)$ be a comparable $\mathscr{T}$-complete Menger PM-space and $\preceq_1$ and $\preceq_2$ be two partial orders on X. Also, let $T, A, B, C, D : X \to X$ be self-mappings and $\alpha, \eta : X \times X \times (0, +\infty) \to (0, +\infty)$ be two functions. Suppose that the following conditions are satisfied:

(i) $\preceq_i$ is F-regular ($i = 1, 2$), and T is $\preceq$-preserving and α-admissible with respect to η;

(ii) A and B are comparable $\mathscr{T}$-continuous or C and D are comparable $\mathscr{T}$-continuous;

(iii) there exists $x_0 \in \mathfrak{X}_T$, such that $Ax_0 \preceq_1 Bx_0$, $Cx_0 \preceq_2 Dx_0$ and $\alpha(x_0, Tx_0, t) \le \eta(x_0, Tx_0, t)$ for all $t > 0$;

(iv) the sequence $\{T^{2n}x_0\}$ is α-regular with respect to η;

(v) T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable and $(C, D, A, B, \preceq_2, \preceq_1)$-stable;

(vi) there exists $h \in \mathcal{H}(X)$ and $\varphi \in \Phi$ such that for $x, y \in X$,

$$Ax \preceq_1 Bx, Cy \preceq_2 Dy \Longrightarrow \eta(x, y, t)\varphi(F_{Tx,Ty}(t)) \le \alpha(x, y, t)h(x, y, t)\varphi(M_{x,y}(t)), \forall t > 0,$$

where $M_{x,y}(t) = \min\{F_{x,y}(t), [F_{x,Tx} \oplus F_{y,Ty}](2t), [F_{x,Ty} \oplus F_{y,Tx}](2t)\}$.

Then the sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a solution to (1.1).

Proof. Without loss of generality, we assume that A, B are comparable $\mathscr{T}$-continuous. The proof for the case that C, D are comparable $\mathscr{T}$-continuous is similar.

By assumption (iii), there exists $x_0 \in \mathfrak{X}_T$ such that

$$Ax_0 \preceq_1 Bx_0 \text{ and } \alpha(x_0, Tx_0, t) \le \eta(x_0, Tx_0, t) \text{ for all } t > 0.$$

Define the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Following the same arguments in Theorem 2.1, we can prove that

$$Ax_{2n} \preceq_1 Bx_{2n} \text{ and } Cx_{2n+1} \preceq_2 Dx_{2n+1}, n = 0, 1, 2, \cdots . \tag{2.18}$$

and

$$\alpha(x_{n-1}, x_n, t) \leq \eta(x_{n-1}, x_n, t) \text{ for all } n \in \mathbb{N} \text{ and } t > 0. \tag{2.19}$$

Also, we can prove that there exists $x^* \in X$ such that $x_n \xrightarrow{\mathscr{T}} x^*(n \to \infty)$ and

$$Ax^* \preceq_1 Bx^*. \tag{2.20}$$

Now, we prove that $Tx^* = x^*$. Suppose this is not true, that is, $Tx^* \neq x^*$. Then we claim that there exists $t_0 > 0$, such that

$$F_{x^*,Tx^*}(2t_0) > F_{x^*,Tx^*}(t_0). \tag{2.21}$$

In fact, if (2.21) is not true, then for all $t > 0$, we have

$$F_{x^*,Tx^*}(t) = F_{x^*,Tx^*}(2t) = \cdots = F_{x^*,Tx^*}(2^n t) \to 1(n \to \infty).$$

This implies that $F_{x^*,Tx^*}(t) = 1, \forall t > 0$, which is in contradiction to $Tx^* \neq x^*$, and thus (2.21) holds.

Without loss of generality, we can assume that t_0 is a continuous point of $F_{x^*,Tx^*}(\cdot)$. In fact, since the distribution function is left-continuous, by (2.21), there exists $\theta > 0$, such that

$$F_{x^*,Tx^*}(2t) > F_{x^*,Tx^*}(t) \;\; \forall t \in (t_0 - \theta, t_0].$$

Since the distribution function is nondecreasing, the discontinuous points are at most a countable set. Thus, when t_0 is not a continuous point of $F_{x^*,Tx^*}(\cdot)$, we can always choose a point t_1 in $(t_0 - \delta, t_0]$ to replace t_0.

Since $\{x_{2n}\}$ is α-regular with respect to η, by (2.19), there exists a subsequence $\{x_{2n_k}\}$ such that

$$\alpha(x_{2n_k}, x^*, t) \leq \eta(x_{2n_k}, x^*, t) \text{ for all } k \in \mathbb{N} \text{ and } t > 0. \tag{2.22}$$

By (2.18), (2.20), (2.21) and (vi), it holds for all $k \in \mathbb{N}$ that

$$\varphi(F_{x_{2n_k+1},Tx^*}(t_0)) = \varphi(F_{Tx_{2n_k},Tx^*}(t_0)) \leq h(x_{2n_k}, x^*, t)\varphi(M_{x_{2n_k},x^*}(t_0)), \tag{2.23}$$

where

$$\begin{aligned} M_{x_{2n_k},x^*}(t_0) &= \min\{F_{x_{2n_k},x^*}(t_0), [F_{x_{2n_k},Tx_{2n_k}} \oplus F_{x^*,Tx^*}](2t_0), [F_{x_{2n_k},Tx^*} \oplus F_{x^*,Tx_{2n_k}}](2t_0)\} \\ &= \min\{F_{x_{2n_k},x^*}(t_0), [F_{x_{2n_k},x_{2n_k+1}} \oplus F_{x^*,Tx^*}](2t_0), [F_{x_{2n_k},Tx^*} \oplus F_{x^*,x_{2n_k+1}}](2t_0)\}. \end{aligned}$$

Note that for any $\delta \in (0, 2t_0)$, we have

$$[F_{x_{2n_k},Tx^*} \oplus F_{x^*,x_{2n_k+1}}](2t_0) \geq \min\{F_{x_{2n_k},Tx^*}(2t_0 - \delta), F_{x^*,x_{2n_k+1}}(\delta)\} \text{ for all } k \in \mathbb{N}.$$

Since $x_n \xrightarrow{\mathscr{T}} x^*(n \to \infty)$, we get

$$\liminf_{k\to\infty}[F_{x_{2n_k},Tx^*} \oplus F_{x^*,x_{2n_k+1}}](2t_0) \geq F_{x^*,Tx^*}(2t_0).$$

Similarly, we have

$$\liminf_{k\to\infty}[F_{x_{2n_k},x_{2n_k+1}} \oplus F_{x^*,Tx^*}](2t_0) \geq F_{x^*,Tx^*}(2t_0).$$

Therefore, we obtain

$$\liminf_{k\to\infty} M_{x_{2n_k},x^*}(t_0) \geq F_{x^*,Tx^*}(2t_0). \tag{2.24}$$

It follows from (2.23) that

$$\varphi(F_{x_{2n_k+1},Tx^*}(t_0)) < \varphi(M_{x_{2n_k},x^*}(t_0)) \text{ for all } k \in \mathbb{N},$$

which by the monotonicity of φ implies that

$$F_{x_{2n_k+1},Tx^*}(t_0)) \geq M_{x_{2n_k},x^*}(t_0) \text{ for all } k \in \mathbb{N}. \tag{2.25}$$

Since $x_n \xrightarrow{\mathscr{T}} x^*(n \to \infty)$, and t_0 is a continuous point of $F_{x^*,Tx^*}(\cdot)$, combining (2.24) and (2.25) yields that $F_{x^*,Tx^*}(t_0) \geq F_{x^*,Tx^*}(2t_0)$, which is in contradiction with (2.21). Therefore, we proved that

$$Tx^* = x^*. \tag{2.26}$$

Since T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable, by (2.20), we obtain

$$CTx^* \preceq_2 DTx^*,$$

which implies that

$$Cx^* \preceq_2 Dx^*. \tag{2.27}$$

Combining (2.20), (2.26) and (2.27), we thus conclude that x^* is a solution to (1.1).

Next, we discuss the uniqueness of the solution to problem (1.1). Denote by Fix(T) the set of all fixed points of the mapping T. Consider the following assumptions.

(H_1) For all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$, such that $Az \preceq_1 Bz$, $Cz \preceq_2 Dz$, $\alpha(x, z, t) \leq \eta(x, z, t)$ and $\alpha(y, z, t) \leq \eta(y, z, t)$ for all $t > 0$.

(H_2) For all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$, such that $\alpha(x, z, t) \leq \eta(x, z, t)$ and $\alpha(z, y, t) \leq \eta(z, y, t)$ for all $t > 0$.

Theorem 2.3 Suppose that the hypotheses of Theorem 2.1 (resp. Theorem 2.2) remain true. Suppose further that one of the following conditions is satisfied:

(i) assumption (H_1) holds;

(ii) assumption (H_2) holds, and T is triangular α-admissible with respect to η.

Then problem (1.1) has a unique solution x^*.

Proof. Suppose that y^* is another solution to (1.1), that is,

$$Ty^* = y^*, \ Ay^* \preceq_1 By^*, \ Cy^* \preceq_2 Dy^*. \tag{2.28}$$

We next show that $x^* = y^*$. First, we assume that condition (i) holds. By assumption (H_1), there exists $z \in X$ such that

$$Az \preceq_1 Bz, Cz \preceq_2 Dz, \alpha(x^*, z, t) \leq \eta(x^*, z, t) \text{ and } \alpha(y^*, z, t) \leq \eta(y^*, z, t) \text{ for all } t > 0. \tag{2.29}$$

Since T is α-admissible with respect to η, from (2.28), we have

$$\alpha(x^*, T^n z, t) \leq \eta(x^*, T^n z, t) \ \text{ and } \alpha(y^*, T^n z, t) \leq \eta(y^*, T^n z, t) \text{ for all } n \in \mathbb{N} \text{ and } t > 0. \tag{2.30}$$

Define the sequence $\{z_n\}$ by $z_{n+1} = Tz_n$ for $n \in \mathbb{N} \cup \{0\}$ with $z_0 = z$. It follows from $Az \preceq_1 Bz$, $Cz \preceq_2 Dz$ and condition (iv) of Theorem 2.1 (or (v) of Theorem 2.2) that $Cz_n \preceq_2 Dz_n$ for all $n \in \mathbb{N} \cup \{0\}$. Noting that $Ax^* \preceq_1 Bx^*$, from (2.30), it holds for all $n \in \mathbb{N}$ and $t > 0$ that

$$\varphi(F_{x^*, z_{n+1}}(t)) \leq h(x^*, z_n, t)\varphi(M_{x^*, z_n}(t)) < \varphi(M_{x^*, z_n}(t)), \tag{2.31}$$

where

$$\begin{aligned} M_{x^*, z_n}(t) &= \min\{F_{x^*, z_n}(t), [F_{x^*, Tx^*} \oplus F_{z_n, Tz_n}](2t), [F_{x^*, Tz_n} \oplus F_{z_n, Tx^*}](2t)\} \\ &= \min\{F_{x^*, z_n}(t), [F_{x^*, x^*} \oplus F_{z_n, z_{n+1}}](2t), [F_{x^*, z_{n+1}} \oplus F_{z_n, x^*}](2t)\}. \end{aligned}$$

Note that for any $\delta \in (0, 2t)$, we have

$$\begin{aligned} [F_{x^*, x^*} \oplus F_{z_n, z_{n+1}}](2t) &\geq \min\{F_{x^*, x^*}(\delta), F_{z_n, z_{n+1}}(2t - \delta)\} \\ &= \min\{1, F_{z_n, z_{n+1}}(2t - \delta), \}, \ \text{ for all } n \in \mathbb{N} \text{ and } t > 0. \end{aligned}$$

Letting $\delta \to 0$, by the left-continuity of the distribution function, we obtain

$$[F_{x^*, x^*} \oplus F_{z_n, z_{n+1}}](2t) \geq F_{z_n, z_{n+1}}(2t), \}, \ \text{ for all } n \in \mathbb{N} \text{ and } t > 0.$$

For all $n \in \mathbb{N}$ and $t > 0$, for each $t_1, t_2 \in (0, 2t)$ with $t_1 + t_2 = 2t$, we have

$$F_{z_n, z_{n+1}}(2t) \geq \Delta_{min}(F_{z_n, x^*}(t_1), F_{x^*, z_{n+1}}(t_2)) = \min\{F_{z_n, x^*}(t_1), F_{x^*, z_{n+1}}(t_2)\},$$

and thus we obtain

$$F_{z_n, z_{n+1}}(2t) \geq [F_{z_n, x^*} \oplus F_{x^*, z_{n+1}}](2t), \ \text{ for all } n \in \mathbb{N} \text{ and } t > 0.$$

Therefore, it holds for all $n \in \mathbb{N}$ and $t > 0$ that

$$\begin{aligned} M_{x^*,z_n}(t) &= \min\{F_{x^*,z_n}(t), [F_{x^*,z_{n+1}} \oplus F_{z_n,x^*}](2t)\} \\ &\geq \min\{F_{x^*,z_n}(t), F_{x^*,z_{n+1}}(t)\}. \end{aligned}$$

If $\min\{F_{x^*,z_n}(t), F_{x^*,z_{n+1}}(t)\} = F_{x^*,z_{n+1}}(t)$, then

$$\varphi(F_{x^*,z_{n+1}}(t)) < \varphi(M_{x^*,z_n}(t)) \leq \varphi(F_{x^*,z_{n+1}}(t)),$$

which is a contradiction. Thus, we conclude that $\min\{F_{x^*,z_n}(t), F_{x^*,z_{n+1}}(t)\} = F_{x^*,z_n}(t)$, for all $n \in \mathbb{N}$ and $t > 0$, and thus by (2.31), we obtain

$$\varphi(F_{x^*,z_{n+1}}(t)) < \varphi(F_{x^*,z_n}(t)), \text{ for all } n \in \mathbb{N} \text{ and } t > 0.$$

By the monotonicity of φ, we have

$$F_{x^*,z_{n+1}}(t) \geq F_{x^*,z_n}(t), \text{ for all } n \in \mathbb{N} \text{ and } t > 0.$$

Thus, $\{F_{x^*,z_n}(t)\}$ is an increasing sequence of positive numbers for each $t > 0$. Imitating the proof of Theorem 2.1, we can prove that

$$\lim_{n\to\infty} F_{x^*,z_n}(t) = 1, \ \forall t > 0.$$

Similarly, it can be deduced that

$$\lim_{n\to\infty} F_{y^*,z_n}(t) = 1, \ \forall t > 0.$$

Therefore, we get $x^* = y^*$, which implies the solution to (1.1) is unique.

Now assume that condition (ii) holds. Suppose $x^* \neq y^*$. By assumption (H_2), there exists $z \in X$ such that

$$\alpha(x^*, z, t) \leq \eta(x^*, z, t) \text{ and } \alpha(z, y^*, t) \leq \eta(z, y^*, t), \text{for all } t > 0.$$

Since T is triangular α-admissible with respect to η, we have $\alpha(x^*, y^*, t) \leq \eta(x^*, y^*, t)$ for all $t > 0$. Noting that $Ax^* \preceq_1 Bx^*$ and $Cy^* \preceq_2 Dy^*$, by (v) of Theorem 2.1 (resp. (vi) of Theorem 2.2), we obtain

$$\varphi(F_{x^*,y^*}(t)) = \varphi(F_{Tx^*,Ty^*}(t)) \leq h(x^*, y^*, t)\varphi(M_{x^*,y^*}(t)), \tag{2.32}$$

where $M_{x^*,y^*}(t) = \min\{F_{x^*,y^*}(t), [F_{x^*,Tx^*} \oplus F_{y^*,Ty^*}](2t), [F_{x^*,Ty^*} \oplus F_{y^*,Tx^*}](2t)\} = F_{x^*,y^*}(t)$. This implies that $h(x^*, y^*, t) \geq 1$, which is a contradiction. Therefore, we have $x^* = y^*$. This completes the proof.

Example 2.1 Let $X = [-4, 6)$ and define the partial order " $\preceq$ " on X as follows:

$$x \preceq y \Longleftrightarrow [x] = [y] \text{ and } x \geq y.$$

Define $\mathscr{F}: X \times X \to \mathscr{D}$ by

$$\mathscr{F}(x,y)(t) = F_{x,y}(t) = \begin{cases} 0, & t \leq 0, \\ e^{-\frac{d(x,y)}{t}}, & t > 0. \end{cases}$$

Then $(X, \mathscr{F}, \Delta_{min})$ is not a $\mathscr{T}$-complete Menger PM-space, but it is a comparable $\mathscr{T}$-complete Menger PM-space. Take $\preceq_1 = \preceq_2 = \leq$. Then " $\preceq_i$ " is F-regular for $i = 1, 2$. Define the mapping $T : X \to X$ by

$$Tx = \frac{1}{3}(x - [x]) \text{ for all } x \in X$$

and $A, B, C, D : X \to X$ by

$$Ax = \begin{cases} \frac{1}{2}x + 1, & 0 \leq x < 6, \\ -\frac{1}{2}x + 2, & -4 \leq x < 0, \end{cases}$$

$$Bx = \begin{cases} \frac{7}{4}, & 1 \leq x < 6, \\ \frac{5}{4}, & -4 \leq x < 1, \end{cases}$$

$$Cx = \begin{cases} \frac{1}{3}x + 2, & 1 \leq x < 6, \\ \frac{1}{2}, & -4 \leq x < 1, \end{cases}$$

$$Dx = \begin{cases} -\frac{1}{2}x + \frac{3}{4}, & 0 \leq x < 6, \\ x - \frac{1}{2}, & -4 \leq x < 0. \end{cases}$$

It is easy to verify that T is $\preceq$-preserving, and A, B and T are comparable $\mathscr{T}$-continuous. Moreover, routine calculations show that T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable and $(C, D, A, B, \preceq_1, \preceq_2)$-stable. Define $\alpha, \eta : X \times X \times (0, +\infty) \to [0, +\infty)$ by

$$\alpha(x,y,t) = \begin{cases} \frac{5}{8}, & \text{if } [x] = [y], \\ 2, & \text{otherwise.} \end{cases} \quad t > 0,$$

$$\eta(x,y,t) = \begin{cases} \frac{3}{4}, & \text{if } [x] = [y], \\ \frac{3}{2}, & \text{otherwise.} \end{cases} \quad t > 0,$$

and $h : X \times X \times (0, +\infty) \to [0, 1)$ by $h(x,y,t) = \frac{2}{5}$ for all $x, y \in X$ and $t > 0$. We can easily check that T is triangular α-admissible with respect to η. Also, note that there exists $x_0 = 0.4 \in \mathfrak{T}_T$, such that $Ax_0 \leq Bx_0$ and $\alpha(x_0, Tx_0, t) \leq \eta(x_0, Tx_0, t)$ for all $t > 0$. If $Ax \leq Bx, Cy \leq Dy$, we have $x, y \in [0, \frac{1}{2}]$. Take $\varphi(x) = -\ln x$. Thus

$$\begin{aligned} \eta(x,y,t)\varphi(F_{Tx,Ty}(t)) &= \frac{3}{4}(-lne^{-\frac{|x-y|}{3t}}) = \frac{3}{4} \cdot \frac{|x-y|}{3t} = \frac{5}{8} \cdot \frac{2}{5} \cdot \frac{|x-y|}{3t} \\ &= \alpha(x,y,t)h(x,y,t)(-\ln e^{-\frac{|x-y|}{t}}) \\ &= \alpha(x,y,t)h(x,y,t)\varphi(F_{x,y}(t)) \\ &\leq \alpha(x,y,t)h(x,y,t)\varphi(M_{x,y}(t)). \end{aligned}$$

The conditions of Theorem 2.1 are all satisfied. Therefore there exists at least one solution to (1.1). Also, we can verify that (H_1) or (H_2) holds, and so the solution is unique. In fact, $x^* = 0$ is the unique solution to (1.1).

3 Some consequences

In this section, we will derive some corollaries of our main results in Section 2.

3.1 Standard fixed point results under constraint inequalities

Taking $\alpha(x, y, t) = \eta(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ in Theorem 2.3, we have the following result.

Corollary 3.1 Let $(X, \mathscr{F}, \Delta_{min}, \preceq)$ be a comparable $\mathscr{T}$-complete Menger PM-space and $\preceq_1$ and $\preceq_2$ be two partial orders on X. Also, let $T, A, B, C, D : X \to X$ be self-mappings. Suppose that the following conditions are satisfied:

(i) $\preceq_i$ is F-regular, $i = 1, 2$, and T is $\preceq$-preserving;

(ii) A and B are comparable $\mathscr{T}$-continuous or C and D are comparable $\mathscr{T}$-continuous;

(iii) there exists $x_0 \in \mathfrak{X}_T$, such that $Ax_0 \preceq_1 Bx_0$ and $Cx_0 \preceq_2 Dx_0$;

(iv) T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable and $(C, D, A, B, \preceq_2, \preceq_1)$-stable;

(v) there exists $h \in \mathcal{H}(X)$ and $\varphi \in \Phi$ such that for $x, y \in X$,

$$Ax \preceq_1 Bx, Cy \preceq_2 Dy \Longrightarrow \varphi(F_{Tx,Ty}(t)) \leq h(x, y, t)\varphi(M_{x,y}(t)) \text{ for all } t > 0,$$

where $M_{x,y}(t) = \min\{F_{x,y}(t), [F_{x,Tx} \oplus F_{y,Ty}](2t), [F_{x,Ty} \oplus F_{y,Tx}](2t)\}$.

Then the sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a unique solution to (1.1).

3.2 Fixed point results under constraint inequalities in comparable $\mathscr{T}$-complete Menger PM-spaces endowed with a partial order

We can obtain the following two results.

Corollary 3.2 Let $(X, \mathscr{F}, \Delta_{min}, \preceq)$ be a comparable $\mathscr{T}$-complete Menger PM-space and $\preceq_1$ and $\preceq_2$ be two partial orders on X. Also, let $T, A, B, C, D : X \to X$ be self-mappings. Suppose that the following conditions are satisfied:

(i) $\preceq_i$ is F-regular, $i = 1, 2$, and T is $\preceq$-preserving;

(ii) A, B and T are comparable $\mathscr{T}$-continuous or C, D and T are comparable $\mathscr{T}$-continuous;

(iii) there exists $x_0 \in X$, such that $x_0 \preceq Tx_0$, $Ax_0 \preceq_1 Bx_0$ and $Cx_0 \preceq_2 Dx_0$;

(iv) T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable and $(C, D, A, B, \preceq_2, \preceq_1)$-stable;

(v) there exists $h \in \mathcal{H}(X)$ and $\varphi \in \Phi$ such that for $x, y \in X$,

$$Ax \preceq_1 Bx, Cy \preceq_2 Dy, x \preceq y \Longrightarrow \varphi(F_{Tx,Ty}(t)) \leq h(x, y, t)\varphi(M_{x,y}(t)) \text{ for all } t > 0,$$

where $M_{x,y}(t) = \min\{F_{x,y}(t), [F_{x,Tx} \oplus F_{y,Ty}](2t), [F_{x,Ty} \oplus F_{y,Tx}](2t)\}$.

Then the sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a solution to (1.1). Moreover, if one of the following conditions holds:

(a) for all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $Az \preceq_1 Bz$, $Cz \preceq_2 Dz$ and $x \preceq y \preceq z$;

(b) for all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $x \preceq z$ and $z \preceq y$.

Then the solution to (1.1) is unique.

Proof. Define the mappings $\alpha, \eta : X \times X \times (0, +\infty) \to [0, +\infty)$ by

$$\alpha(x,y,t) = \begin{cases} 1, & \text{if } x \preceq y, \\ 3, & \text{otherwise,} \end{cases} \quad t > 0$$

and

$$\eta(x,y,t) = \begin{cases} 1, & \text{if } x \preceq y, \\ 2, & \text{otherwise.} \end{cases} \quad t > 0$$

It follows from condition (v) of Corollary 3.2 that (v) of Theorem 2.1 holds. Since $x_0 \preceq Tx_0$, we have $\alpha(x_0, Tx_0, t) \leq \eta(x_0, Tx_0, t)$ for all $t > 0$, and it is easy to check that $\{x_n\}$ which is defined by $x_n = T^n x_0$ is a comparable sequence. Moreover, for all $x, y \in X$ and $t > 0$, since T is $\preceq$-preserving, we have

$$\alpha(x,y,t) \leq \eta(x,y,t) \Longrightarrow x \preceq y \Longrightarrow Tx \preceq Ty \Longrightarrow \alpha(Tx,Ty,t) \leq \eta(Tx,Ty,t).$$

So T is α-admissible with respect to η. The existence of a solution to (1.1) follows from Theorem 2.1.

Now, we prove the uniqueness of the solution to (1.1). First, assume that (a) holds. Let $x, y \in \mathrm{Fix}(T)$. Then there exists $z \in X$ such that $Az \preceq_1 Bz$, $Cz \preceq_1 Dz$, and $x \preceq y \preceq z$. From the definition of α and η, it is easy to see that $\alpha(x,z,t) \leq \eta(x,z,t)$ and $\alpha(y,z,t) \leq \eta(y,z,t)$ for all $t > 0$. This implies that assumption (H_1) holds.

Next assume that (b) holds. Let $x, y \in$ Fix(T). Then there exists $z \in X$ such that $x \preceq z$ and $z \preceq y$. From the definition of α and η, it is easy to see that $\alpha(x,z,t) \leq \eta(x,z,t)$ and $\alpha(z,y,t) \leq \eta(z,y,t)$ for all $t > 0$. This implies that assumption (H_2) holds. Also, for all $x, y, z \in X$ and $t > 0$, it holds that

$$\begin{cases} \alpha(x,y,t) \leq \eta(x,y,t) \Longrightarrow x \preceq y \\ \alpha(y,z,t) \leq \eta(y,z,t) \Longrightarrow y \preceq z \end{cases} \Longrightarrow x \preceq z \Longrightarrow \alpha(x,z,t) \leq \eta(x,z,t),$$

which implies that T is triangular α-admissible with respect to η.

In either case, the uniqueness of the solution can thus be derived from Theorem 2.3.

Corollary 3.3 Let $(X, \mathscr{F}, \Delta_{min}, \preceq)$ be a comparable $\mathscr{T}$-complete Menger PM-space and $\preceq_1$ and $\preceq_2$ be two partial orders on X. Also, let $T, A, B, C, D : X \to X$ be self-mappings. Suppose that the following conditions are satisfied:

(i) $\preceq_i$ is F-regular, $i = 1, 2$, and T is $\preceq$-preserving;

(ii) A and B are comparable $\mathscr{T}$-continuous or C and D are comparable $\mathscr{T}$-continuous;

(iii) there exists $x_0 \in X$, such that $x_0 \preceq Tx_0$, $Ax_0 \preceq_1 Bx_0$ and $Cx_0 \preceq_2 Dx_0$;

(iv) $(X, \mathscr{F}, \Delta, \preceq)$ is regular;

(v) T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable and $(C, D, A, B, \preceq_2, \preceq_1)$-stable;

(vi) there exists $h \in \mathcal{H}(X)$ and $\varphi \in \Phi$ such that for $x, y \in X$,

$$Ax \preceq_1 Bx, Cy \preceq_2 Dy \Longrightarrow \varphi(F_{Tx,Ty}(t)) \leq h(x, y, t)\varphi(M_{x,y}(t)) \text{ for all } t > 0,$$

where $M_{x,y}(t) = \min\{F_{x,y}(t), [F_{x,Tx} \oplus F_{y,Ty}](2t), [F_{x,Ty} \oplus F_{y,Tx}](2t)\}$.

Then the sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a solution to (1.1). Moreover, if one of the following conditions holds:

(a) for all $x, y \in \text{Fix}(T)$, there exists $z \in X$ such that $Az \preceq_1 Bz$, $Cz \preceq_2 Dz$ and $x \preceq y \preceq z$;

(b) for all $x, y \in \text{Fix}(T)$, there exists $z \in X$ such that $x \preceq z$ and $z \preceq y$.

Then the solution to (1.1) is unique.

Proof. Define the mappings $\alpha, \eta : X \times X \times (0, +\infty) \to [0, +\infty)$ as the ones in Corollary 3.2. It follows from condition (vi) of Corollary 3.4 that (vi) of Theorem 2.2 holds. By the proof of Corollary 3.3, it is shown that $\alpha(x_0, Tx_0, t) \leq \eta(x_0, Tx_0, t)$ for all $t > 0$, $\{x_n = T^n x_0\}$ is a comparable sequence, and T is triangular α-admissible with respect to η.

From condition (iv), $(X, \mathscr{F}, \Delta, \preceq)$ is regular. Suppose that $\{x_{2n}\}$ satisfies that $\alpha(x_{2n}, x_{2n+1}, t) \leq \eta(x_{2n}, x_{2n+1}, t)$ for all $n \in \mathbb{N}$ and $t > 0$ with $x_{2n} \xrightarrow{\mathscr{T}} x \in X (n \to \infty)$. Then it follows from the regularity of $(X, \mathscr{F}, \Delta, \preceq)$ that there exists a subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$ such that $x_{2n_k} \preceq x$ for all k. Thus, from the definition of α and η, we have $\alpha(x_{2n_k}, x, t) \leq \eta(x_{2n_k}, x, t)$ for all $k \in \mathbb{N}$ and $t > 0$. Therefore, the sequence $\{T^{2n} x_0\}$ is α-regular with respect to η. So the conclusion follows from Theorem 2.2. The proof of the uniqueness is the same as the deductions in Corollary 3.2.

3.3 Fixed point results under constraint inequalities in comparable $\mathscr{T}$-complete Menger PM-spaces for cyclic contractive mappings

Corollary 3.4 Let A_1 and A_2 be two nonempty $\mathscr{T}$-closed subsets of a comparable $\mathscr{T}$-complete Menger PM-space $(X, \mathscr{F}, \Delta_{min}, \preceq)$ and $\preceq_1$ and $\preceq_2$ be two partial orders on X. Also, let $A, B, C, D : X \to X$ and $T : Y \to Y$ be self-mappings, where $Y = A_1 \cup A_2$. Suppose that the following conditions are satisfied:

(i) $\preceq_i$ is F-regular, $i = 1, 2$, T is $\preceq$-preserving, and $T(A_1) \subset A_2, T(A_2) \subset A_1$;

(ii) A, B and T are comparable $\mathscr{T}$-continuous or C, D and T are comparable $\mathscr{T}$-continuous;

(iii) there exists $x_0 \in \mathfrak{T}_T$, such that $Ax_0 \preceq_1 Bx_0$ and $Cx_0 \preceq_2 Dx_0$;

(iv) T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable and T is $(C, D, A, B, \preceq_2, \preceq_1)$-stable;

(v) there exists $h \in \mathcal{H}(X)$ and $\varphi \in \Phi$ such that for $(x, y) \in A_1 \times A_2$,

$$Ax \preceq_1 Bx, Cy \preceq_2 Dy \Longrightarrow \varphi(F_{Tx,Ty}(t)) \leq h(x, y, t)\varphi(M_{x,y}(t)) \text{ for all } t > 0,$$

where $M_{x,y}(t) = \min\{F_{x,y}(t), [F_{x,Tx} \oplus F_{y,Ty}](2t), [F_{x,Ty} \oplus F_{y,Tx}](2t)\}$.
Suppose further that there exists $z \in X$, such that $Az \preceq_1 Bz$ and $Cz \preceq_2 Dz$. Then the sequence $\{T^n x_0\}$ converges to some $x^* \in A_1 \cap A_2$, which is a unique solution to (1.1).

Proof. Since A_1 and A_2 be two nonempty $\mathscr{T}$-closed subsets of a comparable $\mathscr{T}$-complete Menger PM-space $(X, \mathscr{F}, \Delta_{min}, \preceq)$, we have $(Y, \mathscr{F}, \Delta_{min}, \preceq)$ is comparable $\mathscr{T}$-complete. Define the mappings $\alpha, \eta : X \times X \times (0, +\infty) \to [0, +\infty)$ by

$$\alpha(x, y, t) = \begin{cases} 1, & \text{if } (x, y) \in (A_1 \times A_2) \cup (A_2 \times A_1), \\ 3, & \text{otherwise,} \end{cases} \quad t > 0$$

and

$$\eta(x, y, t) = \begin{cases} 1, & \text{if } (x, y) \in (A_1 \times A_2) \cup (A_2 \times A_1), \\ 2, & \text{otherwise.} \end{cases} \quad t > 0$$

From (v) of Corollary 3.4 and the definition of α and η, we obtain that (v) of Theorem 2.1 holds.

Let $(x, y) \in Y \times Y$ such that $\alpha(x, y, t) \leq \eta(x, y, t)$ for all $t > 0$. Then $(x, y) \in (A_1 \times A_2) \cup (A_2 \times A_1)$. If $(x, y) \in A_1 \times A_2$, from (i) of Corollary 3.4, $(Tx, Ty) \in A_2 \times A_1$. If $(x, y) \in A_2 \times A_1$, from (i) of Corollary 3.4, $(Tx, Ty) \in A_1 \times A_2$. Thus, $(Tx, Ty) \in (A_1 \times A_2) \cup (A_2 \times A_1)$, which implies that $\alpha(Tx, Ty, t) \leq \eta(Tx, Ty, t)$ for all $t > 0$, and so T is α-admissible with respect to η.

Also, from (i) of Corollary 3.4, for any $a \in A_1$, we have $(a, Ta) \in A_1 \times A_2$, and thus $\alpha(a, Ta, t) \leq \eta(a, Ta, t)$ for all $t > 0$.

Finally, let $x, y \in \text{Fix}(T)$. It follows from condition (i) that $x, y \in A_1 \cap A_2$, and thus for any $z \in Y$, we have $\alpha(x, z, t) \leq \eta(x, z, t)$ and $\alpha(y, z, t) \leq \eta(y, z, t)$ for all $t > 0$. Also, note that there exists $z \in X$, such that $Az \preceq_1 Bz$, $Cz \preceq_2 Dz$. This implies that assumption (H_1) holds. The conclusion follows from Theorem 2.3.

Corollary 3.5 Let A_1 and A_2 be two nonempty $\mathscr{T}$-closed subsets of a comparable $\mathscr{T}$-complete Menger PM-space $(X, \mathscr{F}, \Delta_{min}, \preceq)$ and $\preceq_1$ and $\preceq_2$ be two partial orders on X. Also, let $A, B, C, D : X \to X$ and $T : Y \to Y$ be self-mappings, where $Y = A_1 \cup A_2$. Suppose that the following conditions are satisfied:

(i) $\preceq_i$ is F-regular, $i = 1, 2$, T is $\preceq$-preserving, and $T(A_1) \subset A_2, T(A_2) \subset A_1$;

(ii) A and B or C and D are comparable $\mathscr{T}$-continuous;

(iii) there exists $x_0 \in \mathfrak{T}_T$, such that $Ax_0 \preceq_1 Bx_0$ and $Cx_0 \preceq_2 Dx_0$;

(iv) T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable and T is $(C, D, A, B, \preceq_2, \preceq_1)$-stable;

(v) there exists $h \in \mathcal{H}(X)$ and $\varphi \in \Phi$ such that for $x, y \in X$,

$$Ax \preceq_1 Bx, Cy \preceq_2 Dy \Longrightarrow \varphi(F_{Tx,Ty}(t)) \leq h(x,y,t)\varphi(M_{x,y}(t)) \text{ for all } t > 0,$$

where $M_{x,y}(t) = \min\{F_{x,y}(t), [F_{x,Tx} \oplus F_{y,Ty}](2t), [F_{x,Ty} \oplus F_{y,Tx}](2t)\}$.
Suppose further that there exists $z \in X$, such that $Az \preceq_1 Bz$ and $Cz \preceq_2 Dz$. Then the sequence $\{T^n x_0\}$ converges to some $x^* \in A_1 \cap A_2$, which is a unique solution to (1.1).

Proof. Define the mappings $\alpha, \eta : X \times X \times (0, +\infty) \to [0, +\infty)$ as the ones in Corollary 3.4. It follows from condition (v) of Corollary 3.5 that (vi) of Theorem 2.2 holds. By the proof of Corollary 3.4, it is shown that $\alpha(a, Ta, t) \leq \eta(a, Ta, t)$ for all $a \in A_1$ and $t > 0$, and T is α-admissible with respect to η.

Suppose that $\{x_{2n}\}$ satisfies that $\alpha(x_{2n}, x_{2n+1}, t) \leq \eta(x_{2n}, x_{2n+1}, t)$ for all $n \in \mathbb{N}$ and $t > 0$ with $x_{2n} \stackrel{\mathscr{T}}{\to} x \in X(n \to \infty)$. By the definition of α, we have

$$(x_{2n}, x_{2n+1}) \in (A_1 \times A_2) \cup (A_2 \times A_1) \text{ for all } n \in \mathbb{N}.$$

Since $(A_1 \times A_2) \cup (A_2 \times A_1)$ is $\mathscr{T}$-closed, we obtain

$$(x, x) \in (A_1 \times A_2) \cup (A_2 \times A_1),$$

which implies that $x \in A_1 \cap A_2$. From the definition of α, we get $\alpha(x_{2n}, x, t) \leq \eta(x_{2n}, x, t)$ for all $n \in \mathbb{N}$ and $t > 0$. Therefore, the sequence $\{T^{2n}x_0\}$ is α-regular with respect to η. It can be similarly shown that assumption (H_1) holds. So the conclusion follows from Theorem 2.3.

Remark 3.1 Setting $\preceq_1 = \preceq_2$, $C = B$ and $D = A$ in Theorem 2.1 (resp. Theorem 2.2, Theorem 2.3), we can obtain some other corollaries. Furthermore, by setting $\preceq_1 = \preceq_2$, $C = B$ and $D = A = I_X$, where I_X denotes the identity mapping on X, we get the existence and uniqueness results for common fixed points of the mappings B and T. For the sake of brevity, we omit them here.

4 Conclusions

Inspired by [29], we have introduced the concept of comparable $\mathscr{T}$-completeness of an ordered Menger PM-space, and utilized some functions to give a more generalized contractive condition under constraints for the mapping T. Based on these, we have revisited problem (1.1) proposed in [26], and have obtained some new results which guarantee the existence of the solution to problem (1.1) under certain conditions.

Recently, many authors devoted themselves to studying problem (1.1) and other related ones, such as best proximity point problems under constraint inequalities and so on. It would be interesting to further consider relaxing assumptions to obtain more general results concerning these problems in different types of spaces.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally. All authors read and approved the final manuscript.

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