FRACTIONAL DAMPING IN A MONOATOMIC CHAIN WITH CUBIC NONLINEARITY

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Abstract. *In this communication, we show the model of a nonlinear periodic chain with fractional-order damping included. The model includes point masses connected through cubic nonlinear and linear springs and fractional spring-pot elements. The motion equation is solved by using the first-order multiple scales perturbation analysis. The case with weak nonlinearity and damping is observed and the corresponding dispersion equation derived. A parametric study is performed to examine the effects of different parameters on dispersion curves. Given the wavenumber, the cubic nonlinearity and fractional damping parameter have shown to produce lower frequencies than the corresponding undamped linear system.*

Keywords: *fractional damping, cubic nonlinearity, monoatomic chain, dispersion relation.*

1. INTRODUCTION

The interest for phonic crystals, acoustic and elastic metamaterials in their mechanical setup has significantly grown over the past decade. The main feature of these materials is geometric or material periodicity, which can cause formation of band gaps. It is a well known fact that band gaps existing in elastic metamaterials can block propagation of incident waves in certain frequency ranges. This can be achieved by manipulating with their periodicity, geometry and material density or properties of local resonators. Therefore, one can design different types of phonic crystals or mechanical metamaterials having desired wave propagation characteristics based on application demands. Some recent studies gave a comprehensive review of available works in this field and possible directions for future research Hussein *et al.* (2013); Banerjee *et al.* (2019). Two main mechanisms that are used to induce band gaps in phononic crystals and elastic metamaterials are Bragg scattering and local resonance. However, some recent findings revealed the existence of amplitude-induced band gaps Bae and Oh (2020). This property is specific for nonlinear mechanical systems, where different kinds of nonlinearity such as material or geometric one can cause amplitude dependent behavior to come into effect.

Narisetti *et al.* (2011) explored the amplitude-dependent dispersion and band-gap behavior of discrete periodic systems with cubic nonlinearities demonstrating that the boundary of the dispersion curve shifts with the amplitude for a single plane wave. However, the subject of wave propagation in nonlinear periodic structures has a longer history and some researchers suggested the application of multiple scales perturbation method for studying the problems with weak nonlinearity Asfar and Nayfeh (1983); Chakraborty and Mallik (2001). Later, this method was applied to solve some other problems such as wave-wave interactions in cubically nonilnear monoatomic chain Manktelow *et al.* (2011), one-dimensional periodic structures with quadratic nonlinearity Panigrahi *et al.* (2017b) or nonlinear dispersion in one-dimensional diatomic lattice with cubic inter-atomic coupling Lepidi and Bacigalupo (2019). Panigrahi *et al.* (2017a) also analysed the low-amplitude travelling waves in a periodic chain with both quadratic and cubic nonlinearity. Moreover, the interaction of two waves in diatomic chain and monoatomic two-dimensional lattice with a cubic nonlinearity were studied by Manktelow *et al.* (2014). Some other approaches are used in the literature to study strongly nonlinear systems of locally resonant granular crystals Vorotnikov *et al.* (2018) and one-dimensional chains and two-dimensional lattices Narisetti *et al.* (2012).

However, dissipation in phononic structures and metamaterials can cause significant shift of band gaps and introduction of damping, also known as metadamping, is important for constructing reliable models of such materials, Hussein *et al.* (2013). The problem of metadamping in multiresonator metamaterials was recently investigated by Aladwani and Nouh (2021), where viscoelastic damping in one-dimensional multibandgap metamaterials is studied by combining the linear hereditary theory of viscoelasticity and the Floquet-Bloch theory. Besides the classical approaches to describe energy dissipation in materials, in the literature one can find more general fractional viscoelasticity models used in mechanics and structural dynamics to describe this phenomenon for linear, Rossikhin and Shitikova (2010), as well as for nonlinear problems, Rossikhin and Shitikova (2009). Despite a number of applications of fractional damping models, only a recent work by Cajić *et al.* (2020) provided a detailed study of fractional metadamping in linear metamaterials and phononic crystals.

In this study, our aim is to introduce the fractional damping into the model of one-dimensional monoatomic chain with cubic nonlinearity. The solution is sought by using the multiple scales method with corresponding fractional derivative expansion. Obtained approximated solution is used to perform the parametric study and investigate the influence of different parameters on nonlinear dispersion. The effect of the order of fractional derivative and damping parameter are investigated separately to demonstrate the generality of the fractional damping models. Effect of nonlinear cubic and linear stiffness are also examined to show their impact on dispersion.

2. Mathematical model of a nonlinear periodic chain

2.1 Governing equation

Let us consider the mass–spring-spring-pot chain that is given is such way that each mass is separated by a distance L from its nearest neighbour at the equilibrium state, see Fig. 1. We assume that all the masses are equal $m_j = m$ and only the adjacent masses have direct influence on each other.

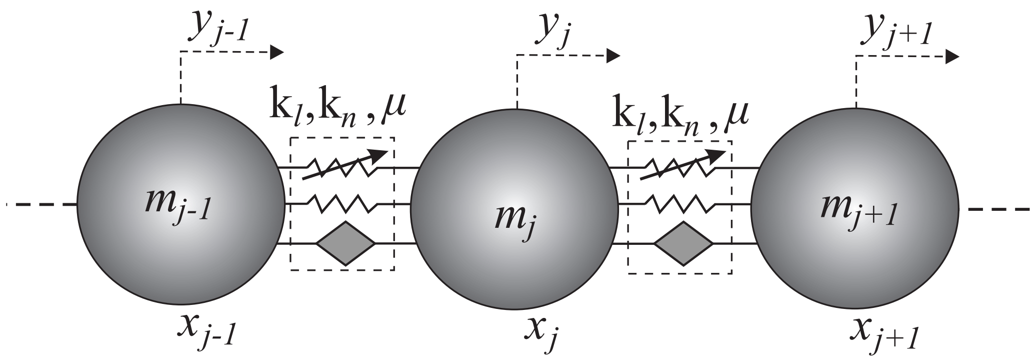


Figure 1: Illustration of an infinite mass-spring-spring-pot chain.

The equation of motion of the considered system is given as

$$\ddot{y}_j + \bar{k}_l(2y_j - y_{j-1} - y_{j+1}) + \epsilon\bar{\mu}D_t^{\alpha}(2y_j - y_{j-1} - y_{j+1}) + \epsilon\bar{k}_n\left[(y_{j+1} - y_j)^3 + (y_j - y_{j-1})^3\right] = 0, \tag{1}$$

for $j = -\infty, \ldots, 1, 0, 1, \ldots, \infty, \bar{k}_l = k_l/m$ with k_l denoting the stiffness coefficient of the linear spring, $\bar{k}_n = k_n/m$ with k_n denoting the stiffness of the nonlinear spring and $\bar{\mu} = \mu/m$ with μ denoting the damping parameter of the fractional-order term representing the spring-pot element. Here, D_t^{α} denotes the operator of Riemann–Liouville fractional derivative of order α for $0 < \alpha < 1$, Rossikhin and Shitikova (2009). Small parameter ϵ is introduced in order to have a small damping and small nonlinearity.

2.2 Wave dispersion analysis

Before continuing with the analysis let as first introduce the fractional Riemann-Liouville derivative in time as given by Rossikhin and Shitikova (2009)

$$D_+^{\alpha} = \frac{d}{dt}\int_{-\infty}^{t} \frac{x(t-t')}{\Gamma(1-\alpha)t'^{\alpha}}\,dt'. \tag{2}$$

This allows us to use the following formula for the exponential function when lower limit is equal to zero

$$D_{0+}^{\alpha} = (i\omega)^{\alpha}e^{i\omega t} + \frac{\sin\pi\alpha}{\pi}\int_0^{\infty} \frac{u^{\alpha}e^{-ut}du}{u+i\omega}, \tag{3}$$

where for $t \to +\infty$ the equation reduces to the first term only. Also, in Rossikhin and Shitikova (2009) it was noted that under certain conditions the improper integral term can be neglected. In our case, the improper integral term will be neglected in all presented cases.

The above equation of motion Eq. (1) can be analysed by using the first-order multiple scales method, where we introduce the fast $t = T_0$ and slow $t = \epsilon T_1$ time scales. Based on methodology from Rossikhin and Shitikova (2009), we introduce the following assumptions for time derivatives in terms of new time scales

$$\frac{d}{dt} = D_0 + \epsilon D_1 + ..., \quad \frac{d^2}{dt^2} = D_0^2 + \epsilon(2DD_1) + ..., \quad \frac{d^\alpha}{dt^\alpha} = D_{0+}^\alpha + \epsilon\alpha D_{0+}^{\alpha-1}D_1 + ..., \tag{4}$$

where $D_n = \frac{\partial}{\partial T_n}$. Further, we assume the approximate solution for small amplitudes in terms of different time scales as $y_j = q_{(j)0}(T_0, T_1) + \epsilon q_{(j)1}(T_0, T_1)$. Substituting this into Eq. (1) and equating like powers of ϵ we have

$$\epsilon^0: \quad D_0^2 q_{(j)0} + \bar{k}_l(2q_{(j)0} - y_{(j-1)0} - y_{(j+1)0}) = 0, \tag{5a}$$

$$\begin{aligned} \epsilon^1: \quad & D_0^2 q_{(j)1} + \bar{k}_l(2q_{(j)1} - y_{(j-1)1} - y_{(j+1)1}) = -2D_0D_1q_{(j)0} + 3\bar{k}_n q_{(j)0}q_{(j-1)0}^2 \\ & + \bar{\mu}D_{0+}^\alpha(2q_{(j)0} - y_{(j-1)0} - y_{(j+1)0}) + 2\bar{k}_n q_{(j)0}^3 - 3\bar{k}_n q_{(j)0}^2 q_{(j-1)0} - \bar{k}_n q_{(j-1)0}^3 \\ & - 3\bar{k}_n q_{(j)0}^2 q_{(j+1)0} + 3\bar{k}_n q_{(j)0} q_{(j+1)0}^2 - \bar{k}_n q_{(j+1)0}^3 \end{aligned} \tag{5b}$$

We assume a travelling dispersive wave solution as

$$q_{(j)0} = Ae^{i(\kappa x_j - \omega_0 T_0)} + c.c. \tag{6}$$

where κ is the wave number and $x_{j\pm1} = x_j \pm L$. By replacing this assumed solution into Eq. (5a) we get the following

$$\omega_0 = (2\bar{k}_l(1 - \cos\kappa L))^{1/2}. \tag{7}$$

If we again use the assumed solution for ϵ^0 in Eq. (5b), we can find the solvability condition resulting from the secularity terms associated with $e^{i(\kappa x_j - \omega_0 T_0)}$ while taking the polar form of complex amplitude $A = 1/2ae^{-i\varphi}$. Elimination of secular terms, solving equations for a and φ and taking that $e^{i(\kappa x_j - \omega t)} = e^{i(\kappa x_j - \omega_0 t - \varphi)}$ gives the following dispersion relation

$$\omega = \omega_0 - \bar{\mu}\epsilon\omega_0^{\alpha-1}\cos\frac{\alpha\pi}{2}(1 - \cos\kappa L) + \frac{3a_0^2\bar{k}_n\sin^2(\frac{\kappa L}{2})}{2\omega_0^\alpha\bar{\mu}t\sin(\frac{\alpha\pi}{2})}(e^{-4\bar{\mu}\epsilon t\omega_0^{\alpha-1}\sin^2(\frac{\kappa L}{2})\sin(\frac{\alpha\pi}{2})} - 1) \tag{8}$$

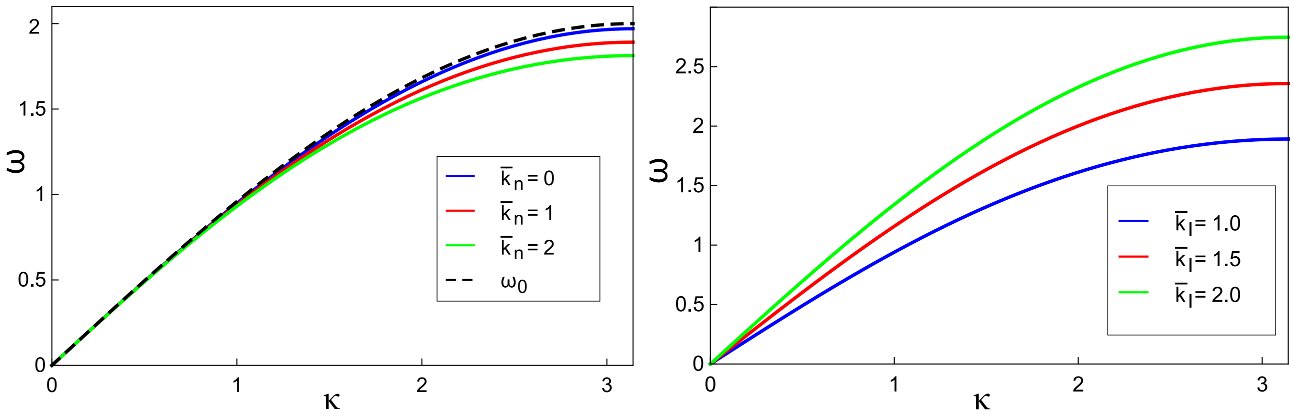
where amplitude a is defined as $a = a_0e^{(\beta t)}$ for $\beta = -\bar{\mu}\epsilon\omega_0^{\alpha-1}\sin\frac{\alpha\pi}{2}(1 - \cos\kappa L)$. According to Rossikhin and Shitikova (2009), ratio of the dissipation coefficient β and the linear part of the difference in frequencies $\omega - \omega_0 = \bar{\mu}\epsilon\omega_0^{\alpha-1}\cos\frac{\alpha\pi}{2}(1 - \cos\kappa L)$ is equal to $\beta/\omega-\omega_0 \approx \tan(\alpha\pi/2)$ i.e. the coefficient of dissipation depends approximately linearly on the difference of damped and undamped frequencies. On the other hand, the nonlinear part of $\omega - \omega_0$ is dependent on the squared amplitude.

2.3 Numerical results

Here, we will perform the parametric study to investigate the behavior of dispersion curves for changes of certain parameters of the model. The case of a monoatomic chain with unit cells having a single degree of freedom is investigated (see Fig. 1). The aim of this analysis is to show both the effect of nonlinearity and fractional damping on dispersion. For the simulation purposes we use the following values of parameters in the model $L = 1$, $\bar{k}_l = 1$, $\bar{k}_n = 1$, $\bar{\mu} = 0.5$, $\alpha = 0.5$, $\epsilon = 0.3$ and the initial amplitude constant $a_0 = 0.3$ if not given differently in figures. The results are obtained by evaluating the expression (8) over the entire First Brillouin Zone (FBZ).

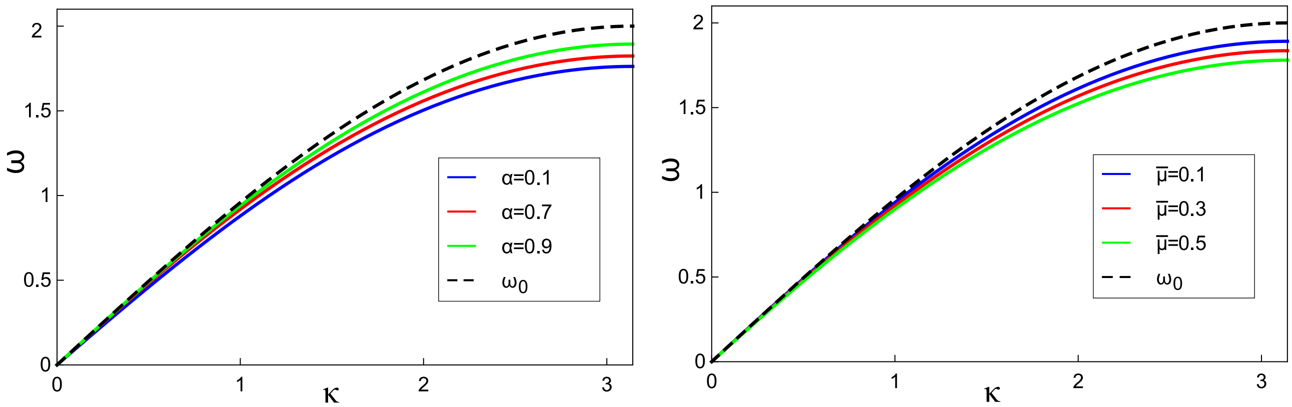
Fig. 2 shows the influence of the linear and nonlinear stiffness coefficient on dispersion. In Fig. 2a one can notice that dispersion curve is very sensitive to changes in linear stiffness, where frequency significantly increases, especially at the edges of the FBZ. On the other side, in Fig. 2b one can notice a decrease of the frequency for an increase of the nonlinear stiffness parameter. This change is less then the previous one but it is also more pronounced at higher values of the wave number. When comparing frequencies of nonlinear and damped dispersion to that of linear and undamped dispersion (black dashed curve) we can notice that they are lower.

It is well known that attenuation can occur both in space or time. In the case of temporal attenuation, the frequency is complex, interpreted as damping, and energy is lost as time evolves i.e. it is related to the energy loss. However, in the case of spatial attenuation the wave number is a complex value and this is characterized as a geometric attenuation without energy loss. The case of fractional damping that is studied in this work belongs to the temporal type of attenuation. Fig. 3



(a) The effect of nonlinear stiffness parameter (b) The effect of linear stiffness parameter

Figure 2: The effect of linear and nonlinear stiffness parameters on dispersion



(a) The effect of order of fractional derivative (b) The effect of damping parameter

Figure 3: The effect of the order of fractional derivative and damping parameter on dispersion

shows the effect of fractional order damping term on nonlinear dispersion. In Fig. 3a one can notice that an increase of damping parameter causes a decrease of the nonlinear dispersion frequency for fixed values of other parameters. Using the terminology from Manktelow *et al.* (2011), we can notice that in all presented cases we have the so-called softening chain effect i.e. frequency is lower than the frequency of the linear undamped chain (black dashed line). Further, an increase of the order of fractional parameter shows unusual behavior of increased frequency for an increase of the order of fractional derivative and fixed value of damping parameter. This means that effect of the order of fractional derivative is strong and gives different results from the usual integer damping model. This effect is even more enhanced for larger values of the damping parameter $\bar{\mu}$. This can be seen in Fig. 3b, where for fixed value of α the frequency decreases for an increase of the damping parameter. More details on linear and nonlinear fractional damping oscillators one can find in Rossikhin and Shitikova (2010).

Finally, Fig. 4 shows the effect of small parameter ϵ on dispersion curves. It is a well known fact that the perturbation analysis is valid for small displacements. We depicted three different cases where ϵ is changed in the range $0.1 - 0.3$ to reveal the effect of cubic nonlinearity. Despite the asymptotic analysis usually means a small value of ϵ, the simulations shows that as we increase the value of small parameter nonlinear behavior becomes more enhanced i.e. the frequency decreases in a manner similar to those in Fig. 2a. When compared to the the dispersion of the linear and undamped chain (black dashed line), one can notice that frequencies are lower due to introduced both fractional damping and cubic nonlinearity within the monoatomic chain.

2.4 Conclusion

In this work, dispersion characteristics of a discrete nonlinear medium represented by a monoatomic chain with a cubic type material nonlinearity is investigated. The main contribution includes development of the model that includes fractional order derivative damping and solution of the corresponding nonlinear fractional differential equation by the multiple time scales perturbation method. This enabled us to study both the effects of nonlinearity and fractional damping. The presented parametric study uncovered the effects of cubic nonlinearity showing that the cubic nonlinearity lead to lower frequencies when compared to the linear case. On the other side, an increase of damping parameter lead to lower

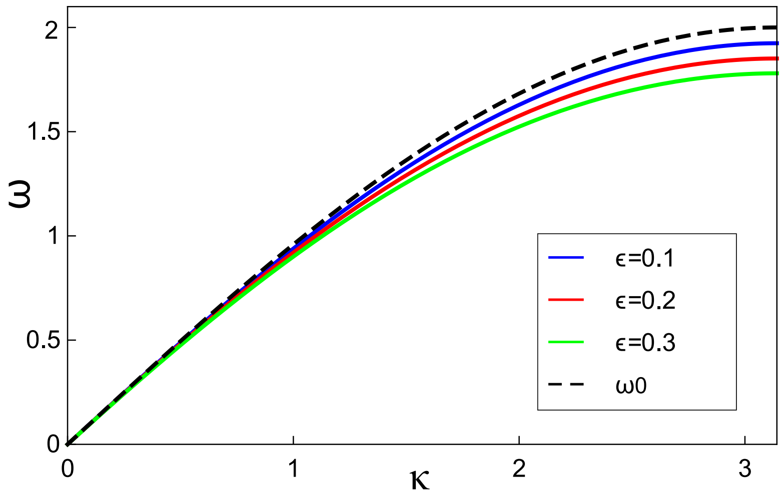


Figure 4: The effect of the small scale parameter on dispersion.

frequency while an increase of frequency can be observed for an increase of the order of fractional derivative. The presented methodology and analysis can be used in future studies of more complex nonlinear periodic systems with fractional damping.

3. ACKNOWLEDGEMENTS

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