Averaging principle for stochastic tidal dynamics equations

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Abstract: In this paper, we aim to establish a strong averaging principle for stochastic tidal dynamics equations. The averaging principle is an effective method for studying the qualitative analysis of nonlinear dynamical systems. Under suitable assumptions, utilizing Khasminkii's time discretization approach, we derive a strong averaging principle showing that the solution of stochastic tidal dynamics equations can be approximated by solutions of the system of averaged stochastic equations in the sense of convergence in mean square.

Keywords: stochastic tidal dynamics equations; Averaging principle; Strong convergence
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1 Introduction

For hundreds of years, ocean tides have been a source of interest for many physicists and mathematicians. Historically, Newton first gave a mathematical explanation of ocean tides and Laplace established the hydrodynamic equations for ocean tides, we refer the readers to the literature [12] for a complete history and theoretical description of tides. Over the last few decades, this field has developed further with the help of computer and satellite technology and is used in a wide and diverse range of fields such as geophysics, atmospheric science and communications, just to mention a few. In this paper, we will consider the tidal dynamics model proposed by Marchuk and Kagan [15]. In the monograph [15], Marchuk and Kagan constructed the tidal dynamics model from the three-dimensional Navier-Stokes equations by integrating along the z-axis (the vertical direction) and then by considering the model on a rotating sphere, which is a generalisation of the Laplace model.

Due to modelling complexity, it is intrinsically difficult and challenge to study the tidal dynamics equations which are highly nonlinear partial differential equations of parabolic-hyperbolic type. Let us give a brief review of results in the literature. In Manna et al. [14], the authors obtained the existence and uniqueness of weak solutions of the deterministic tide dynamics equations and the existence and uniqueness of strong solutions of the stochastic tide dynamics equations with additive Gaussian white noise. The existence, uniqueness, large deviation principle and moderate deviation principle for stochastic tidal dynamics equations driven by multiplying Gaussian noise have been

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studied in [9,19]. The authors in [1] established the existence of optimal controls for stochastic tidal dynamics equations driven by Lévy noise. For further studies regarding stochastic tidal dynamics equations, interested readers are referred to [16,17,23] and references therein.

On the other hand, averaging principle is an effective approach for studying dynamical systems involving highly oscillating components. Under certain assumptions, the highly oscillating components can be averaged out to generate an averaged dynamical system, which is comparably easier for analysis which governs the evolution of the original system over long time scales. The averaging principle for deterministic dynamics initially established by Krylov and Bogolyubov indeed provides a powerful and efficient tool for investigating the properties of highly complex and nonlinear dynamical systems. Averaging principle for stochastic differential equations was first derived by Khasminskii in [11]. The fundamental idea of the stochastic averaging principle is to derive averaged stochastic differential equations and establish approximation of the averaging solutions to the solutions of the original equations, so that the original complex stochastic differential equations could be analysed via the corresponding easier averaged equations. To date, there are extensive literatures concerning stochastic averaging principle for finitely dimensional and infinitely dimensional stochastic systems, see, e.g., [2–4, 7, 8, 10, 18, 20–22] and the references therein. Motivated by all the above mentioned works, in this paper, we want to establish a strong averaging principle for the stochastic tidal dynamics equations in which we derive averaging stochastic tidal dynamical equations as easier modelling equations for tidal dynamical equations.

Let us first describe the model we are concerned with in this paper. Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^2$ with smooth boundary. We are concerned with the following stochastic tidal dynamics equation

$$\begin{cases} \mathrm{d}u^\varepsilon(t) + [Au^\varepsilon(t) + B(u^\varepsilon(t)) + \nabla z^\varepsilon(t)]\mathrm{d}t \\ \qquad = f(\frac{t}{\varepsilon}, u^\varepsilon(t))\mathrm{d}t + \sigma(\frac{t}{\varepsilon}, u^\varepsilon(t))\mathrm{d}W(t),\ (t,x) \in [0,T] \times \mathcal{O}, \\ \mathrm{d}z^\varepsilon(t) + \mathrm{div}(h(x)u^\varepsilon(t))\mathrm{d}t = 0, \quad (t,x) \in [0,T] \times \mathcal{O}, \\ u^\varepsilon(t,x) = 0, \quad x \in \partial\mathcal{O}, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x),\ z^\varepsilon(0,x) = z_0^\varepsilon(x), \quad x \in \mathcal{O}, \end{cases} \tag{1.1}$$

for $\varepsilon > 0$, where W is a Q-Wiener process. The operators A and B are defined by

$$A := \begin{pmatrix} -\alpha\Delta & -\beta \\ \beta & -\alpha\Delta \end{pmatrix},$$
$$B(u) := \gamma|u + w^0|(u + w^0),$$

where α and (the Coriolis parameter) β are positive constants and $w^0(t,x)$ is a known deterministic function on the boundary, which is extended to the whole domain as a smooth function. The function $h(x)$ is defined to be the depth of the sea at x in the region $\mathcal{O}$ and we assume that it is a continuously differentiable function nowhere becoming zero, so that

$$\kappa := \min_{x\in\mathcal{O}} h(x), \quad \mu := \max_{x\in\mathcal{O}} h(x), \quad M := \max_{x\in\mathcal{O}} |\nabla h(x)|,$$

are positive constants. Then $\gamma(x) := r/h(x), r > 0$, is a strictly positive smooth function. Our aim of the present paper is to establish an averaging principle for the stochastic tidal dynamics equation (1.1).

The remainder of the paper is organised as follows. Section 2 presents some preliminaries for our later use. Section 3 is devoted to formulating and proving the strong averaging principle.

2 Preliminaries

For simplicity, throughout this paper, C denotes a positive constant whose value may change from line to line. We will also write the dependence of a constant on parameters explicitly if it is essential. Let $L^p = L^p(\mathcal{O}) = L^p(\mathcal{O};\mathbb{R}^2)$, $p \geq 1$, be the vector valued L^p space equipped with the norm $\|\cdot\|_{L^p}$. The inner product and the norm in $L^2(\mathcal{O})$ are denote by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Let $H^1 = H^1(\mathcal{O}) = H^1(\mathcal{O};\mathbb{R}^2)$ denotes the Sobolev space with the norm $\|u\|_{H^1}^2 = \|u\|^2 + \|\nabla u\|^2$, $u \in H^1$. We also let $H_0^1 = H_0^1(\mathcal{O};\mathbb{R}^2)$ be the closure of $C_c^\infty(\mathcal{O})$ in $H^1(\mathcal{O})$ norm. According to Poincaré inequality, for any $u \in H_0^1$, $\|u\|_{H^1} \simeq \|\nabla u\| =: \|u\|_{H_0^1}$. We denote the dual of $H_0^1(\mathcal{O})$ by $H^{-1}(\mathcal{O})$. Then, we have the following continuous and dense embedding:

$$H_0^1(\mathcal{O}) \subseteq L^2(\mathcal{O}) \subseteq H^{-1}(\mathcal{O}).$$

The induced duality between the spaces $H_0^1(\mathcal{O})$ and $H^{-1}(\mathcal{O})$ is denoted by $\langle\cdot,\cdot\rangle$. For any $u \in L^2$ and $v \in H_0^1$, it follows that $\langle u, v\rangle = (u, v)$.

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration. Let Q be a positive, symmetric, trace class operator on L^2 and $e_k, k \in \mathbb{N}$, be the complete orthonormal basis of L^2 such that $Qe_k = \mu_k e_k$, $k \in \mathbb{N}$. We assume that $W(\cdot)$ is an L^2-valued Q-Wiener process. The process $W(\cdot)$ can be expressed as

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\mu_k} e_k(x) \beta_k(t),$$

where $(\beta_k, k \in \mathbb{N})$ is a sequence of independent, one-dimensional Brownian motions. We denote the collection of Hilbert-Schmidt operators from $\mathcal{H}_0$ to L^2 by $\mathcal{L}_2(\mathcal{H}_0; L^2)$, where $\mathcal{H}_0 = Q^{1/2}L^2$. Defining the norm on the space $\mathcal{L}_2(\mathcal{H}_0; L^2)$ by $\|\Phi\|_{\mathcal{L}_Q}^2 = \mathrm{Tr}(Q\Phi Q^*)$. For any $\mathcal{L}_2(\mathcal{H}_0; L^2)$ valued predictable process $\Phi(t)$, $0 \leq t \leq T$ satisfying $\mathbb{E}\int_0^T \mathrm{Tr}(\Phi Q \Phi^*)dt < +\infty$, one can then define the stochastic integral $\int_0^T \Phi(t)dW(t)$. For more details, we refer the reader, e.g., to [5, 6, 13].

Now we introduce the following conditions which will be used in the sequel.

Assumption (H1). There exist positive constants $L_f, K_f > 0$, such that

$$\|f(t,u) - f(t,v)\| \leq L_f\|u - v\|, \ \|f(t,0)\| \leq K_f, \quad u, v \in L^2.$$

Assumption (H2). There exist positive constants $L_\sigma, K_\sigma > 0$, such that

$$\|\sigma(t,u) - \sigma(t,v)\|_{\mathcal{L}_Q} \leq L_\sigma\|u - v\|, \ \|\sigma(t,0)\| \leq K_\sigma, \quad u, v \in L^2.$$

Under the above assumptions, we can state the existence and uniqueness of solutions to (1.1).

Theorem 2.1 *([19]) Let $w^0 \in L^4([0,T]; H_0^1), u_0^\varepsilon \in L^2$, $z_0^\varepsilon \in L^2$. Then there exists a pathwise unique, strong solution $(u^\varepsilon(\cdot), z^\varepsilon(\cdot))$ to the equations (1.1) and the solution satisfies the following*

$$(u^\varepsilon, z^\varepsilon) \in C([0,T]; L^2) \cap L^2([0,T]; H_0^1) \times C([0,T]; L^2), \quad \mathbb{P} - a.s.$$

3 Main result

In this section, we shall derive the averaging principle for equations (1.1). Specifically, we want to show that the original equations (1.1) converges, as $\varepsilon \to 0$, to the following averaged equations:

$$\begin{cases} \mathrm{d}\bar{u}(t) + [A\bar{u}(t) + B(\bar{u}(t)) + \nabla \bar{z}(t)]\mathrm{d}t \\ \qquad = \bar{f}(\bar{u}(t))\mathrm{d}t + \bar{\sigma}(\bar{u}(t))\mathrm{d}W(t), \ (t,x) \in [0,T] \times \mathcal{O}, \\ \mathrm{d}\bar{z}(t) + \mathrm{div}(h(x)\bar{u}(t))\mathrm{d}t = 0, \quad (t,x) \in [0,T] \times \mathcal{O}, \\ \bar{u}(t,x) = 0, \quad x \in \partial\mathcal{O}, \\ \bar{u}(0,x) = \bar{u}_0(x) \in L^2, \ \bar{z}(0,x) = \bar{z}_0(x) \in L^2, \quad x \in \mathcal{O}, \end{cases} \tag{3.1}$$

where the coefficients $\bar{f} : L^2 \to L^2$ and $\bar{\sigma} : L^2 \to \mathcal{L}_2(\mathcal{H}_0; L^2)$ satisfy the following averaging conditions:

Assumption (H3). For any $T > 0, x \in L^2$,

$$\frac{1}{T}\Big\|\int_0^T (f(t,x) - \bar{f}(x))\mathrm{d}t\Big\| \le \kappa_1(T)(1 + \|x\|),$$

where $\kappa_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded function with $\lim\limits_{T\to\infty} \kappa_1(T) = 0$.

Assumption (H4). For any $T > 0, x \in L^2$,

$$\frac{1}{T}\int_0^T \|\sigma(t,x) - \bar{\sigma}(x)\|^2_{\mathcal{L}_Q}\mathrm{d}t \le \kappa_2(T)(1 + \|x\|),$$

where $\kappa_2 : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded function with $\lim\limits_{T\to\infty} \kappa_2(T) = 0$.

Before proceeding the main result of the paper, we need to show the following lemmas. For convenience, for a given process φ, we define $\tilde{\varphi}$ such that $\tilde{\varphi}(\sigma) = \varphi(s + k\delta)$ for any $\sigma \in [k\delta, (k+1)\delta), \ k \ge 0$. For simplicity, we set $f_\varepsilon(t,x) := f(\frac{t}{\varepsilon}, x), \ \sigma_\varepsilon(t,x) := \sigma(\frac{t}{\varepsilon}, x), \ \forall\, t > 0, x \in L^2$.

Lemma 3.1 *For any $\varepsilon > 0$. Let $(u^\varepsilon(t), z^\varepsilon(t))$ be the solution of (1.1), then for any $p \ge 1$, we have*

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\le t\le T}\Big(\|u^\varepsilon(s)\|^{2p} + \|z^\varepsilon(s)\|^{2p}\Big)\right] + 2\alpha^p\mathbb{E}\left[\int_0^T \|\nabla u^\varepsilon(t)\|^2 dt\right]^p \\ &\qquad \le C\Big[1 + \|u_0^\varepsilon\|^{2p} + \|z_0^\varepsilon\|^{2p}\Big]. \end{aligned} \tag{3.2}$$

Proof Using Itô formula, we have

$$\begin{aligned} &\mathrm{d}\|u_n^\varepsilon(t)\|^2 + 2\left[\alpha\|\nabla u^\varepsilon\|^2 + \langle B(u^\varepsilon(t)), u^\varepsilon(t)\rangle + \langle \nabla z^\varepsilon(t), u^\varepsilon(t)\rangle\right]\mathrm{d}t \\ &\quad = 2\langle f_\varepsilon(t), u^\epsilon(t)\rangle\mathrm{d}t + \|\sigma_\varepsilon(t, u^\varepsilon(t))\|^2_{\mathcal{L}_Q}\mathrm{d}t + 2\langle \sigma_\varepsilon(t, u^\varepsilon(t)\mathrm{d}W(t), u^\varepsilon(t)\rangle. \end{aligned}$$

Then according to (2.15) in [19], one gets

$$\begin{aligned} &\|u^\varepsilon(t)\|^2 + 2\alpha\int_0^t \|\nabla u^\varepsilon(s)\|\mathrm{d}s \\ &\quad \le \|u_0^\varepsilon\|^2 + \frac{r}{\kappa}\int_0^t (\|w^0\|^4_{L^4} + \|u^\varepsilon(s)\|^2)\mathrm{d}s + \int_0^t \left(\frac{4}{\alpha}\|z^\varepsilon(s)\|^2 + \frac{\alpha}{2}\|\nabla u^\varepsilon(s)\|\right)\mathrm{d}s \\ &\qquad + \int_0^t \left[(2L_f^2 + 2L_\sigma^2 + 1)\|u^\varepsilon(s)\|^2 + 2(K_f^2 + K_\sigma^2)\right] \\ &\qquad + 2\int_0^t \langle \sigma_\varepsilon(s, u^\varepsilon(s)\mathrm{d}W(s), u^\varepsilon(s)\rangle. \end{aligned} \tag{3.3}$$

Moreover, taking inner product of $(1.1)_2$ with z^ε, we get

$$\mathrm{d}\|z^\varepsilon(t)\|^2 + 2(\mathrm{div}(hu^\varepsilon(t)), z^\varepsilon(t))\mathrm{d}t = 0.$$

Then, it follows from Lemma 2.1 (iii) in [9] that

$$\begin{aligned}\|z^\varepsilon(t)\|^2 \le \|z_0^\varepsilon\|^2 + \frac{\alpha}{2}\int_0^t \|\nabla u^\varepsilon(s)\|^2 \mathrm{d}s + M\int_0^t \|u^\varepsilon(s)\|^2\mathrm{d}s \\ + \left(\frac{4\mu^2}{\alpha} + M\right)\int_0^t \|z^\varepsilon(s)\|^2\mathrm{d}s.\end{aligned} \tag{3.4}$$

Adding (3.3) and (3.4), we find that there exists two constants $C_1, C_2 > 0$, such that

$$\begin{aligned}&\|u^\varepsilon(t)\|^2 + \|z^\varepsilon(t)\|^2 + \alpha\int_0^t \|\nabla u^\varepsilon(s)\|^2\mathrm{d}s \\ &\quad\le \|u_0^\varepsilon\|^2 + \|z_0^\varepsilon\|^2 + C_1 t + C_2\int_0^t (\|u^\varepsilon(s)\|^2 + \|z^\varepsilon(s)\|^2)\mathrm{d}s \\ &\qquad + 2\int_0^t \langle \sigma_\varepsilon(s, u^\varepsilon(s)\mathrm{d}W(s), u^\varepsilon(s)\rangle.\end{aligned}$$

It follows that for any $p \ge 1$,

$$\begin{aligned}&\|u^\varepsilon(t)\|^{2p} + \|z^\varepsilon(t)\|^{2p} + \alpha^p\left[\int_0^t \|\nabla u^\varepsilon(s)\|^2\mathrm{d}s\right]^p \\ &\le C_{p,T}\left[\|u_0^\varepsilon\|^{2p} + \|z_0^\varepsilon\|^{2p} + 1 + \left(\int_0^t (\|u^\varepsilon(s)\|^2 + \|z^\varepsilon(s)\|^2)\mathrm{d}s\right)^p\right] \\ &\quad + C_{p,T}\left|\int_0^t \langle \sigma_\varepsilon(s, u^\varepsilon(s)\mathrm{d}W(s), u^\varepsilon(s)\rangle\right|^p \\ &\le C_{p,T}\left[\|u_0^\varepsilon\|^{2p} + \|z_0^\varepsilon\|^{2p} + 1 + \int_0^t (\|u^\varepsilon(s)\|^{2p} + \|z^\varepsilon(s)\|^{2p})\mathrm{d}s\right] \\ &\quad + C_{p,T}\left|\int_0^t \langle \sigma_\varepsilon(s, u^\varepsilon(s)\mathrm{d}W(s), u^\varepsilon(s)\rangle\right|^p.\end{aligned}$$

Taking supremum on $[0, T]$ and expectation, it holds that

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T}\Big(\|u^\varepsilon(t)\|^{2p} + \|z^\varepsilon(t)\|^{2p}\Big)\right] + \alpha^p\mathbb{E}\left[\int_0^T \|\nabla u^\varepsilon(s)\|^2\mathrm{d}s\right]^p \\ &\le C_{p,T}[\|u_0^\varepsilon\|^{2p} + \|z_0^\varepsilon\|^{2p} + 1] + C_{p,T}\mathbb{E}\int_0^T (\|u^\varepsilon(s)\|^{2p} + \|z^\varepsilon(s)\|^{2p})\mathrm{d}s \\ &\quad + C_{p,T}\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t \langle \sigma_\varepsilon(s, u^\varepsilon(s)\mathrm{d}W(s), u^\varepsilon(s)\rangle\right|^p.\end{aligned} \tag{3.5}$$

According to Burkholder-Davis-Gundy inequality and Young's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\langle\sigma_\varepsilon(s,u^\varepsilon(s)\mathrm{d}W(s),u^\varepsilon(s)\rangle\right|^p\\
&\le C_p\mathbb{E}\left(\int_0^T\|u^\varepsilon(s)\|^2\|\sigma_\varepsilon(s,u^\varepsilon(s))\|^2_{\mathcal{L}_Q}\mathrm{d}s\right)^{\frac{p}{2}}\\
&\le C_p\mathbb{E}\left(\sup_{0\le t\le T}\|u^\varepsilon(t)\|^2\int_0^T\|\sigma_\varepsilon(s,u^\varepsilon(s))\|^2_{\mathcal{L}_Q}\mathrm{d}s\right)^{\frac{p}{2}}\\
&\le\frac{1}{2}\mathbb{E}\left(\sup_{0\le t\le T}\|u^\varepsilon(t)\|^{2p}\right)+C_{p,T}\mathbb{E}\int_0^T\|\sigma_\varepsilon(s,u^\varepsilon(s))\|^{2p}_{\mathcal{L}_Q}\mathrm{d}s\\
&\le\frac{1}{2}\mathbb{E}\left(\sup_{0\le t\le T}\|u^\varepsilon(t)\|^{2p}\right)+C_{p,T}+C_{p,T}\mathbb{E}\int_0^T\|u^\varepsilon(s)\|^{2p}\mathrm{d}s.
\end{aligned}\tag{3.6}$$

Combing with (3.5) and (3.6), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left(\|u^\varepsilon(t)\|^{2p}+\|z^\varepsilon(t)\|^{2p}\right)\right]+2\alpha^p\mathbb{E}\left[\int_0^T\|\nabla u^\varepsilon(s)\|^2\mathrm{d}s\right]^p\\
&\le C_{p,T}\left[1+\|u_0^\varepsilon\|^2+\|z_0^\varepsilon\|^2\right]+C_{p,T}\mathbb{E}\int_0^T(\|u^\varepsilon(s)\|^{2p}+\|z^\varepsilon(s)\|^{2p})\mathrm{d}s.
\end{aligned}$$

An application of Gronwall's inequality yields that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left(\|u^\varepsilon(s)\|^{2p}+\|z^\varepsilon(s)\|^{2p}\right)\right]+2\alpha^p\mathbb{E}\left[\int_0^T\|\nabla u^\varepsilon(t)\|^2\mathrm{d}t\right]^p\\
&\le C\Big[1+\|u_0^\varepsilon\|^{2p}+\|z_0^\varepsilon\|^{2p}\Big].
\end{aligned}$$

The proof is completed. □

Remark 3.1 *It can be easily verified that $\bar{f}$ and $\bar{\sigma}$ satisfy the assumptions (H1) and (H2). Moreover, the solution to the equation (3.1) also satisfies Lemma 3.1.*

Lemma 3.2 *Under the assumptions $(H1)-(H4)$, let $(u^\varepsilon,z^\varepsilon)$ and $(\bar{u},\bar{z})$ be, respectively, the solution of equation (1.1) and (3.1). Then for any $T>0$, we have*

$$\mathbb{E}\int_0^T\|u^\varepsilon(t)-\tilde{u}^\varepsilon(t)\|^2dt\le C(T,|u_0^\varepsilon\|^2,\|z_0^\varepsilon\|^2)\delta^{\frac{1}{2}},\tag{3.7}$$

$$\mathbb{E}\int_0^T\|\bar{u}(t)-\tilde{\bar{u}}(t)\|^2dt\le C(T,\|u_0^\varepsilon\|^2,\|z_0^\varepsilon\|^2)\delta^{\frac{1}{2}}.\tag{3.8}$$

Proof Since the proofs of (3.7) and (3.8) are similar, we only prove that (3.7) holds. Let $T(\delta)=$

$\left[\frac{T}{\delta}\right]$, where $[x]$ is the integer part of x. Then

$$
\begin{aligned}
\mathbb{E}\int_0^T \|u^\varepsilon(t)-\tilde{u}^\varepsilon(t)\|^2\mathrm{d}t &= \mathbb{E}\int_0^\delta \|u^\varepsilon(t)-u_0^\varepsilon\|^2\mathrm{d}t + \mathbb{E}\sum_{k=1}^{T(\delta)-1}\int_{k\delta}^{(k+1)\delta}\|u^\varepsilon(t)-u^\varepsilon(k\delta)\|^2\mathrm{d}t \\
&\quad + \mathbb{E}\int_{T(\delta)\delta}^T \|u^\varepsilon(t)-u^\varepsilon(T(\delta)\delta)\|^2\mathrm{d}t \\
&\le C(T,\|u_0^\varepsilon\|^2)\delta + 2\mathbb{E}\sum_{k=1}^{T(\delta)-1}\int_{k\delta}^{(k+1)\delta}\|u^\varepsilon(t)-u^\varepsilon(t-\delta)\|^2\mathrm{d}t \\
&\quad + 2\mathbb{E}\sum_{k=1}^{T(\delta)-1}\int_{k\delta}^{(k+1)\delta}\|u^\varepsilon(t-\delta)-u^\varepsilon(k\delta)\|^2\mathrm{d}t \\
&=: C(T,\|u_0^\varepsilon\|^2,\|z_0^\varepsilon\|^2)\delta + 2\sum_{k=1}^{T(\delta)-1}\mathrm{I}_k + 2\sum_{k=1}^{T(\delta)-1}\mathrm{II}_k.
\end{aligned}
\tag{3.9}
$$

Given $1 \le k \le T(\delta)-1$ and $k\delta \le t < (k+1)\delta$, by using Itô formula, we have

$$
\begin{aligned}
\|u^\varepsilon(t)-u^\varepsilon(t-\delta)\|^2 &= -2\int_{t-\delta}^t \langle Au^\varepsilon(\tau), u^\varepsilon(\tau)-u^\varepsilon(t-\delta)\rangle\mathrm{d}\tau \\
&\quad - 2\int_{t-\delta}^t \langle B(u^\varepsilon(\tau)), u^\varepsilon(\tau)-u^\varepsilon(t-\delta)\rangle\mathrm{d}\tau \\
&\quad - 2\int_{t-\delta}^t \langle \nabla z^\varepsilon, u^\varepsilon(\tau)-u^\varepsilon(t-\delta)\rangle\mathrm{d}\tau \\
&\quad + 2\int_{t-\delta}^t \langle f_\varepsilon(\tau,u^\varepsilon(\tau)), u^\varepsilon(\tau)-u^\varepsilon(t-\delta)\rangle\mathrm{d}\tau \\
&\quad + \int_{t-\delta}^t \|\sigma_\varepsilon(\tau,u^\varepsilon(\tau))\|_{\mathcal{L}_Q}^2\mathrm{d}\tau \\
&\quad + 2\int_{t-\delta}^t \langle u^\varepsilon(\tau)-u^\varepsilon(t-\delta), \sigma_\varepsilon(\tau,u^\varepsilon(\tau))\mathrm{d}W(\tau)\rangle \\
&=: \mathrm{J}_1(t)+\mathrm{J}_2(t)+\mathrm{J}_3(t)+\mathrm{J}_4(t)+\mathrm{J}_5(t)+\mathrm{J}_6(t).
\end{aligned}
$$

We estimate the above terms separately. For the term $\mathrm{J}_1(t)$, we have

$$
\begin{aligned}
\mathrm{J}_1(t) &= -2\int_{t-\delta}^t \langle Au^\varepsilon(\tau), u^\varepsilon(\tau)\rangle\mathrm{d}\tau + 2\int_{t-\delta}^t \langle Au^\varepsilon(\tau), u^\varepsilon(t-\delta)\rangle\mathrm{d}\tau \\
&= -2\alpha\int_{t-\delta}^t \|u^\varepsilon(\tau)\|_{H^1}^2\mathrm{d}\tau + 2\int_{t-\delta}^t \langle Au^\varepsilon(\tau), u^\varepsilon(t-\delta)\rangle\mathrm{d}\tau \\
&\le C_\alpha\int_{t-\delta}^t \|u^\varepsilon(\tau)\|_{H^1}\|u^\varepsilon(t-\delta)\|_{H^1}\mathrm{d}\tau \\
&\le \alpha\int_{t-\delta}^t \|u^\varepsilon(\tau)\|_{H^1}^2\mathrm{d}\tau + C_\alpha\int_{t-\delta}^t \|u^\varepsilon(t-\delta)\|_{H^1}^2\mathrm{d}\tau,
\end{aligned}
$$

For the term J_2, using the property of the operator $B(\cdot)$ (Lemma 2.2 in [19]) and the Ladyzhenskaya's inequality, we have

$$\begin{aligned}
\mathrm{J}_2(t) &\leq 2\int_{t-\delta}^{t} \|B(u^\varepsilon(\tau))\|\|u^\varepsilon(\tau)-u^\varepsilon(t-\delta)\|\mathrm{d}\tau \\
&\leq C\int_{t-\delta}^{t} (\|u^\varepsilon(\tau)\|_{L^4}+1)^2\|u^\varepsilon(\tau)-u^\varepsilon(t-\delta)\|\mathrm{d}\tau \\
&\leq C\int_{t-\delta}^{t} (\|u^\varepsilon(\tau)\|_{H^1}^2\|u^\varepsilon(\tau)\|^2+1)\mathrm{d}\tau + C\int_{t-\delta}^{t}(\|u^\varepsilon(\tau)\|^2+\|u^\varepsilon(t-\delta)\|^2)\mathrm{d}\tau
\end{aligned}$$

Similarity, we have

$$\begin{aligned}
\mathrm{J}_3(t) &= 2\int_{t-\delta}^{t} \langle z^\varepsilon(\tau), \mathrm{div}(u^\varepsilon(\tau)-u^\varepsilon(t-\delta))\rangle\mathrm{d}\tau \\
&\leq C\int_{t-\delta}^{t} \|z^\varepsilon(\tau)\|\|\mathrm{div}(u^\varepsilon(\tau)-u^\varepsilon(t-\delta))\|\mathrm{d}\tau \\
&\leq C\int_{t-\delta}^{t} \|z^\varepsilon(\tau)\|\|u^\varepsilon(\tau)\|_{H^1}\mathrm{d}\tau + C\int_{t-\delta}^{t}\|z^\varepsilon(\tau)\|\|u^\varepsilon(t-\delta)\|_{H^1}\mathrm{d}\tau
\end{aligned}$$

For the terms $\mathrm{J}_4(t)$ and $\mathrm{J}_5(t)$, it follows from assumptions (H1) and (H2) that

$$\begin{aligned}
\mathrm{J}_4(t)+\mathrm{J}_5(t) &\leq 2\int_{t-\delta}^{t} \|f_\varepsilon(\tau,u^\varepsilon(\tau))\|\|u^\varepsilon(\tau)-u^\varepsilon(t-\delta)\|\mathrm{d}\tau \\
&\qquad + 2C\int_{t-\delta}^{t}(L_\sigma\|u^\varepsilon(\tau)\|^2+K_\sigma^2)\mathrm{d}\tau \\
&\leq 2\int_{t-\delta}^{t}(L_f\|u^\varepsilon(\tau)\|+K_f)\|u^\varepsilon(\tau)-u^\varepsilon(t-\delta)\|\mathrm{d}\tau \\
&\qquad + 2C\int_{t-\delta}^{t}(L_\sigma\|u^\varepsilon(\tau)\|^2+K_\sigma^2)\mathrm{d}\tau \\
&\leq C\int_{t-\delta}^{t}[\|u^\varepsilon(\tau)\|^2+\|u^\varepsilon(t-\delta)\|^2+1]\mathrm{d}\tau.
\end{aligned}$$

Combing with all the above estimates, we find

$$\begin{aligned}
\mathrm{I}_k &= \mathbb{E}\int_{k\delta}^{(k+1)\delta} \|u^\varepsilon(t)-u^\varepsilon(t-\delta)\|_{H^1}^2\mathrm{d}t \\
&\leq C\mathbb{E}\int_{k\delta}^{(k+1)\delta}\int_{t-\delta}^{t}\Big[\|u^\varepsilon(\tau)\|_{H^1}^2+\|u^\varepsilon(\tau)\|_{H^1}^2\|u^\varepsilon(\tau)\|^2+\|u^\varepsilon(\tau)\|^2+\|u^\varepsilon(\tau)\|_{H^1}\|z^\varepsilon(\tau)\| \\
&\quad +\|u^\varepsilon(t-\delta)\|_{H^1}^2+\|u^\varepsilon(t-\delta)\|^2+\|u^\varepsilon(t-\delta)\|_{H^1}\|z^\varepsilon(\tau)\|+1\Big]\mathrm{d}\tau\mathrm{d}t \\
&\quad +2\mathbb{E}\int_{k\delta}^{(k+1)\delta}\int_{t-\delta}^{t}\langle u^\varepsilon(\tau)-u^\varepsilon(t-\delta),\sigma_\varepsilon(\tau,u^\varepsilon(\tau))\mathrm{d}W(\tau)\rangle\mathrm{d}t \\
&=: \mathrm{I}_k^1+\mathrm{I}_k^2,
\end{aligned}$$

For the term I_k^1, we have

$$\begin{aligned}
\mathrm{I}_k^1 \leq{}& C\mathbb{E}\int_{k\delta}^{(k+1)\delta}\int_{t-\delta}^{t}\Big[\|u^\varepsilon(\tau)\|_{H^1}^2+\left(\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^2\right)\|u^\varepsilon(\tau)\|_{H^1}^2 \\
&\qquad\qquad +\|u^\varepsilon(\tau)\|^2+\left(\sup_{0\leq t\leq T}\|z^\varepsilon(t)\|^2\right)\|u^\varepsilon(\tau)\|_{H^1}^2\Big]\mathrm{d}\tau\mathrm{d}t \\
&+C\mathbb{E}\int_{k\delta}^{(k+1)\delta}\int_{t-\delta}^{t}\Big[\|u^\varepsilon(t-\delta)\|_{H^1}^2+\|u^\varepsilon(t-\delta)\|^2+\left(\sup_{0\leq t\leq T}\|z^\varepsilon(t)\|^2\right)\|u^\varepsilon(t-\delta)\|_{H^1}^2+1\Big]\mathrm{d}\tau\mathrm{d}t \\
\leq{}& C\delta\mathbb{E}\int_{(k-1)\delta}^{(k+1)\delta}\Big[\|u^\varepsilon(t)\|_{H^1}^2+\left(\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^2+\sup_{0\leq t\leq T}\|z^\varepsilon(t)\|^2\right)\|u^\varepsilon(t)\|_{H^1}^2+\|u^\varepsilon(t)\|^2\Big]\mathrm{d}t \\
&+C\delta\mathbb{E}\int_{(k-1)\delta}^{(k+1)\delta}\Big[\|u^\varepsilon(t)\|_{H^1}^2+\|u^\varepsilon(t)\|^2+\left(\sup_{0\leq t\leq T}\|z^\varepsilon(t)\|^2\right)\|u^\varepsilon(t)\|_{H^1}^2+1\Big]\mathrm{d}t \\
\leq{}& C\delta\mathbb{E}\int_{(k-1)\delta}^{(k+1)\delta}\Big[\|u^\varepsilon(t)\|_{H^1}^2+\|u^\varepsilon(t)\|^2+1 \\
&\qquad\qquad +\left(\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^2\right)\|u^\varepsilon(t)\|_{H^1}^2+\left(\sup_{0\leq t\leq T}\|z^\varepsilon(t)\|^2\right)\|u^\varepsilon(t)\|_{H^1}^2\Big]\mathrm{d}t.
\end{aligned} \tag{3.10}$$

For the term I_k^2, using Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned}
\mathrm{I}_k^2 ={}& 2\mathbb{E}\int_{k\delta}^{(k+1)\delta}\int_{t-\delta}^{t}\langle u^\varepsilon(\tau)-u^\varepsilon(t-\delta),\sigma_\varepsilon(\tau,u^\varepsilon(\tau))\mathrm{d}W(\tau)\rangle\,\mathrm{d}t \\
\leq{}& C\int_{k\delta}^{(k+1)\delta}\mathbb{E}\left(\int_{t-\delta}^{t}\|\sigma_\varepsilon(\tau,u^\varepsilon(\tau))\|_{\mathcal{L}_Q}^2\|u^\varepsilon(\tau)-u^\varepsilon(t-\delta)\|^2\,\mathrm{d}\tau\right)^{\frac{1}{2}}\,\mathrm{d}t \\
\leq{}& C\int_{k\delta}^{(k+1)\delta}\mathbb{E}\left(\int_{t-\delta}^{t}(\|u^\varepsilon(\tau)\|^2+1)\|u^\varepsilon(\tau)-u^\varepsilon(t-\delta)\|^2\,\mathrm{d}\tau\right)^{\frac{1}{2}}\,\mathrm{d}t \\
\leq{}& C\delta^{\frac{1}{2}}\left[\int_{k\delta}^{(k+1)\delta}\mathbb{E}\int_{t-\delta}^{\sigma}(\|u^\varepsilon(\tau)\|^4+\|u^\varepsilon(t-\delta)\|^4+1)\mathrm{d}\tau\mathrm{d}t\right]^{\frac{1}{2}} \\
\leq{}& C\delta^{\frac{1}{2}}\left(\mathbb{E}\int_{(k-1)\delta}^{(k+1)\delta}\delta\|u^\varepsilon(\tau)\|^4\,\mathrm{d}\tau+\delta^2\right)^{\frac{1}{2}}\leq C\delta\left(\mathbb{E}\int_{(k-1)\delta}^{(k+1)\delta}\delta\|u^\varepsilon(\tau)\|^4\,\mathrm{d}\tau\right)^{\frac{1}{2}}+C\delta^{\frac{3}{2}}.
\end{aligned} \tag{3.11}$$

Consequently, combing with (3.10) and (3.11), we obtain

$$
\begin{aligned}
2\sum_{k=1}^{T(\delta)-1}\mathrm{I}_k &= 2\sum_{k=1}^{T(\delta)-1}\mathrm{I}_k^1 + 2\sum_{k=1}^{T(\delta)-1}\mathrm{I}_k^2 \\
&\le C\delta\mathbb{E}\int_0^T \Big[\|u^\varepsilon(t)\|_{H^1}^2 + \|u^\varepsilon(t)\|^2 + 1 \\
&\qquad + \Big(\sup_{0\le t\le T}\|u^\varepsilon(t)\|^2\Big)\|u^\varepsilon(t)\|_{H^1}^2 + \Big(\sup_{0\le t\le T}\|z^\varepsilon(t)\|^2\Big)\|u^\varepsilon(t)\|_{H^1}^2\Big]\mathrm{d}t \\
&\quad + C\delta\sum_{k=1}^{T(\delta)-1}\Big(\mathbb{E}\int_{(k-1)\delta}^{(k+1)\delta}\delta\|u^\varepsilon(\tau)\|^4\,\mathrm{d}\tau\Big)^{\frac{1}{2}} + C_T\delta^{\frac{3}{2}} \\
&\le C\delta\mathbb{E}\int_0^T \|u^\varepsilon(t)\|_{H^1}^2\mathrm{d}t + CT\delta\Big(\mathbb{E}\sup_{0\le t\le T}\|u^\varepsilon(t)\|^2 + 1\Big) \\
&\quad + C\delta\Big(\mathbb{E}\sup_{0\le t\le T}\|u^\varepsilon(t)\|^4\Big)^{1/2}\Big[\mathbb{E}\Big(\int_0^T\|u^\varepsilon(t)\|_{H^1}^2\mathrm{d}t\Big)^2\Big]^{1/2} \\
&\quad + C\delta\Big(\mathbb{E}\sup_{0\le t\le T}\|z^\varepsilon(t)\|^4\Big)^{1/2}\Big[\mathbb{E}\Big(\int_0^T\|u^\varepsilon(t)\|_{H^1}^2\mathrm{d}t\Big)^2\Big]^{1/2} \\
&\quad + C\delta[T(\delta)]^{\frac{1}{2}}\Bigg(\sum_{k=1}^{T(\delta)-1}\mathbb{E}\int_{(k-1)\delta}^{(k+1)\delta}\|u^\varepsilon(\tau)\|_{L^4(\mathbb{T}^d)}^4\,\mathrm{d}\tau\Bigg)^{\frac{1}{2}} + C_T\delta^{3/2} \\
&\le C_T(1+\|u_0^\varepsilon\|^2 + \|z_0^\varepsilon\|^2)\delta^{1/2}.
\end{aligned} \tag{3.12}
$$

Similarly, we can prove that

$$
2\sum_{k=1}^{T(\delta)-1}\mathrm{II}_k \le C_T(1+\|u_0^\varepsilon\|^2 + \|z_0^\varepsilon\|^2)\delta^{1/2}. \tag{3.13}
$$

Combining with (3.9), (3.12) and (3.13), we know (3.7) holds. The proof is completed. □

Now we are in the position to prove the following main result of the current paper.

Theorem 3.1 *Let $(u^\varepsilon, z^\varepsilon)$ and $(\bar{u}, \bar{z})$ be, respectively, the solution of equation (1.1) and (3.1). If we assume that $\lim_{\varepsilon\to 0}\Big(\|u_0^\varepsilon - \bar{u}_0\|^2 + \|z_0^\varepsilon - \bar{z}_0\|^2\Big) = 0$ and assumptions (H1)-(H4) hold. Then for any $T > 0$ and $p \ge 1$, we have*

$$
\lim_{\varepsilon\to 0}\Big(\mathbb{E}\sup_{0\le t\le T}\|u^\varepsilon(t) - \bar{u}(t)\|^{2p} + \mathbb{E}\sup_{0\le t\le T}\|z^\varepsilon(t) - \bar{z}(t)\|^{2p}\Big) = 0.
$$

Proof Observe that

$$
\begin{aligned}
\mathrm{d}(u^\varepsilon(t) - \bar{u}(t)) = &-[A(u^\varepsilon(t) - \bar{u}(t)) + (B(u^\varepsilon(t)) - B(\bar{u}(t)))]\mathrm{d}t \\
&- (\nabla z^\varepsilon(t) - \nabla\bar{z}(t))]\mathrm{d}t + (f_\varepsilon(t, u^\varepsilon(t)) - \bar{f}(\bar{u}(t)))\mathrm{d}t \\
&+ (\sigma_\varepsilon(t, u^\varepsilon(t)) - \bar{\sigma}(\bar{u}(t)))\mathrm{d}W(t).
\end{aligned}
$$

Using Itô formula, we obtain

$$
\begin{aligned}
\|u^\varepsilon(t)-\bar{u}(t)\|^2 = \|u_0^\varepsilon-\bar{u}_0\|^2 &- 2\int_0^t \langle A(u^\varepsilon(\sigma)-\bar{u}(\sigma)), u^\varepsilon(\sigma)-\bar{u}(\sigma)\rangle \mathrm{d}\sigma \\
&- 2\int_0^t \langle B(u^\varepsilon(\sigma))-B(\bar{u}(\sigma)), u^\varepsilon(\sigma)-\bar{u}(\sigma)\rangle \mathrm{d}\sigma \\
&- 2\int_0^t \langle \nabla(z^\varepsilon(\sigma)-\bar{z}(\sigma)), u^\varepsilon(\sigma)-\bar{u}(\sigma)\rangle \mathrm{d}\sigma \\
&+ 2\int_0^t \langle f_\varepsilon(\sigma, u^\varepsilon(\sigma))-\bar{f}(\sigma,\bar{u}(\sigma)), u^\varepsilon(\sigma)-\bar{u}(\sigma)\rangle \mathrm{d}\sigma \\
&+ \int_0^t \|\sigma_\varepsilon(t, u^\varepsilon(\sigma))-\bar{\sigma}(\bar{u}(\sigma))\|^2_{\mathcal{L}_Q} \mathrm{d}\sigma \\
&+ 2\int_0^t \langle u^\varepsilon(\sigma)-\bar{u}(\sigma), \sigma_\varepsilon(t, u^\varepsilon(\sigma))-\bar{\sigma}(\bar{u}(\sigma))\mathrm{d}W(\sigma)\rangle \\
=:& \|u_0^\varepsilon-\bar{u}_0\|^2 + \sum_{i=6}^{11} \mathrm{J}_i(t).
\end{aligned}
$$

Observe that

$$
\mathrm{J}_6(t)+\mathrm{J}_7(t) \le -2\alpha\int_0^t \|\nabla(u^\varepsilon(\sigma)-\bar{u}(\sigma))\|^2 \mathrm{d}\sigma,
$$

and

$$
\begin{aligned}
\mathrm{J}_8(t) &= -2\int_0^t \langle \nabla(z^\varepsilon(\sigma)-\bar{z}(\sigma)), u^\varepsilon(\sigma)-\bar{u}(\sigma)\rangle \mathrm{d}\sigma \\
&= 2\int_0^t \langle z^\varepsilon(\sigma)-\bar{z}(\sigma), \mathrm{div}(u^\varepsilon(\sigma)-\bar{u}(\sigma))\rangle \mathrm{d}\sigma \\
&\le 2\sqrt{2}\int_0^t \|z^\varepsilon(\sigma)-\bar{z}(\sigma)\|\|\nabla(u^\varepsilon(\sigma)-\bar{u}(\sigma))\| \mathrm{d}\sigma \\
&\le \frac{4}{\alpha}\int_0^t \|z^\varepsilon(\sigma)-\bar{z}(\sigma)\|^2 \mathrm{d}\sigma + \frac{\alpha}{2}\int_0^t \|\nabla(u^\varepsilon(\sigma)-\bar{u}(\sigma))\|^2 \mathrm{d}\sigma.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|u^\varepsilon(t)-&\bar{u}(t)\|^2 + 2\alpha\int_0^t \|\nabla(u^\varepsilon(\sigma)-\bar{u}(\sigma))\|^2 \mathrm{d}\sigma \le \|u_0^\varepsilon-\bar{u}_0\|^2 \\
&+ \frac{4}{\alpha}\int_0^t \|z^\varepsilon(\sigma)-\bar{z}(\sigma)\|^2 \mathrm{d}\sigma + \frac{\alpha}{2}\int_0^t \|\nabla(u^\varepsilon(\sigma)-\bar{u}(\sigma))\|^2 \mathrm{d}\sigma + \mathrm{J}_9(t)+\mathrm{J}_{10}(t)+\mathrm{J}_{11}(t).
\end{aligned} \tag{3.14}
$$

Moreover, we also have

$$
\begin{aligned}
\|z^\varepsilon(t)-\bar{z}(t)\|^2 \le& \|z_0^\varepsilon-\bar{z}_0\|^2 + \frac{\alpha}{2}\int_0^t \|\nabla(u^\varepsilon(\sigma)-\bar{u}(\sigma))\|^2 \mathrm{d}\sigma \\
&+ \left(\frac{4\mu^2}{\alpha}+M\right)\int_0^t \|z^\varepsilon(\sigma)-\bar{z}(\sigma)\|^2 \mathrm{d}\sigma + M\int_0^t \|u^\varepsilon(\sigma)-\bar{u}(\sigma)\|^2 \mathrm{d}\sigma.
\end{aligned} \tag{3.15}
$$

Adding (3.15) to (3.14) yields

$$
\begin{aligned}
\Xi(t) &+ \alpha\int_0^t \|\nabla(u^\varepsilon(\sigma)-\bar{u}(\sigma))\|^2 \mathrm{d}\sigma \\
&\le \|u_0^\varepsilon-\bar{u}_0\|^2 + \|z_0^\varepsilon-\bar{z}_0\|^2 + C\int_0^t \Xi(\sigma)\mathrm{d}\sigma + \mathrm{J}_9(t)+\mathrm{J}_{10}(t)+\mathrm{J}_{11}(t).
\end{aligned}
$$

where

$$\Xi(t)=\|u^\varepsilon(t)-\bar u(t)\|^2+\|z^\varepsilon(t)-\bar z(t)\|^2.$$

Applying Gronwall's inequality, we obtain

$$\sup_{0\le t\le T}\Xi(t)\le C_T\left[\|u_0^\varepsilon-\bar u_0\|^2+\|z_0^\varepsilon-\bar z_0\|^2+\sup_{0\le t\le T}\mathrm{J}_9(t)+\sup_{0\le t\le T}\mathrm{J}_{10}(t)+\sup_{0\le t\le T}\mathrm{J}_{11}(t)\right].$$

It follows from Burkholder-Davis-Gundy inequality that

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\Xi(t)\le C_T\left[\|u_0^\varepsilon-\bar u_0\|^2+\|z_0^\varepsilon-\bar z_0\|^2\right]+\mathbb{E}\left(\sup_{0\le t\le T}\mathrm{J}_9(t)\right)\\ +C_T\mathbb{E}\int_0^T\|\sigma_\varepsilon(t,u^\varepsilon(\sigma))-\bar\sigma(\bar u(\sigma))\|^2_{\mathcal{L}_Q}\mathrm{d}\sigma.\end{aligned}\tag{3.16}$$

Now, we estimate the term $\mathbb{E}\sup\limits_{0\le t\le T}\mathrm{J}_9(t)$. The method is very similar to [4], for the convenience of the reader, we give the proof here. Observe that

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le t\le T}\mathrm{J}_9(t)\Big)\le{}&2\mathbb{E}\int_0^T\|f_\varepsilon(s,u^\varepsilon(s))-f_\varepsilon(s,\bar u(s))\|\|u^\varepsilon(s)-\bar u(s)\|\mathrm{d}s\\ &+2\mathbb{E}\sup_{0\le t\le T}\int_0^t\langle f_\varepsilon(s,\bar u(s))-\bar f(\bar u(s)),u^\varepsilon(s)-\bar u(s)\rangle\mathrm{d}s\\ \le{}&2L_f\mathbb{E}\int_0^T\|u^\varepsilon(s)-\bar u(s)\|^2\mathrm{d}s\\ &+2\mathbb{E}\sup_{0\le t\le T}\int_0^t\langle f_\varepsilon(s,\bar u(s))-\bar f(\bar u(s)),u^\varepsilon(s)-\tilde u^\varepsilon(s)\rangle\mathrm{d}s\\ &+2\mathbb{E}\sup_{0\le t\le T}\int_0^t\langle f_\varepsilon(s,\bar u(s))-\bar f(\bar u(s)),\tilde u^\varepsilon(s)-\tilde{\bar u}(s)\rangle\mathrm{d}s\\ &+2\mathbb{E}\sup_{0\le t\le T}\int_0^t\langle f_\varepsilon(s,\bar u(s))-\bar f(\bar u(s)),\tilde{\bar u}(s)-\bar u(s)\rangle\mathrm{d}s\\ =:{}&2L_f\mathbb{E}\int_0^T\|u^\varepsilon(s)-\bar u(s)\|^2\mathrm{d}s+\mathrm{J}_9^1+\mathrm{J}_9^2+\mathrm{J}_9^3.\end{aligned}\tag{3.17}$$

For the term J_9^1, by Lemma 3.1, Lemma 3.2 and assumption (H1),

$$\begin{aligned}\mathrm{J}_9^1={}&2\mathbb{E}\sup_{0\le t\le T}\int_0^t\langle f_\varepsilon(s,\bar u(s))-\bar f(\bar u(s)),u^\varepsilon(s)-\tilde u^\varepsilon(s)\rangle\mathrm{d}s\\ \le{}&2\mathbb{E}\int_0^T(\|f_\varepsilon(s,\bar u(s))\|+\|\bar f(\bar u(s))\|)\|u^\varepsilon(s)-\tilde u^\varepsilon(s)\|\mathrm{d}s\\ \le{}&4\mathbb{E}\int_0^T(L_f\|\bar u(s)\|+K_f)\|u^\varepsilon(s)-\tilde u^\varepsilon(s)\|\mathrm{d}s\\ \le{}&4\Big[\mathbb{E}\int_0^T(L_f\|\bar u(s)\|+K_f)^2\mathrm{d}s\Big]^{\frac12}\Big[\mathbb{E}\int_0^T\|u^\varepsilon(s)-\tilde u^\varepsilon(s))\|^2\mathrm{d}s\Big]^{\frac12}\\ \le{}&C(T,\|u_0^\varepsilon\|^2,\|z_0^\varepsilon\|^2)\delta^{\frac14}.\end{aligned}\tag{3.18}$$

Similarly, for J_9^3, we can get

$$\mathrm{J}_9^3\le C(T,\|u_0^\varepsilon\|^2,\|z_0^\varepsilon\|^2)\delta^{\frac14}.\tag{3.19}$$

For the term J_9^2, we have

$$\begin{aligned}
\mathrm{J}_9^2 &\leq 2\mathbb{E}\int_0^T \|f_\varepsilon(t,\bar{u}(t)) - f_\varepsilon(t,\tilde{\bar{u}}(t))\|\|\tilde{u}^\varepsilon(t) - \tilde{\bar{u}}(t)\|\mathrm{d}t \\
&\quad + 2\mathbb{E}\sup_{0\leq t\leq T}\int_0^t \langle f_\varepsilon(s,\tilde{\bar{u}}(s)) - \bar{f}(\tilde{\bar{u}}(s)), \tilde{u}^\varepsilon(s) - \tilde{\bar{u}}(s)\rangle \mathrm{d}s \\
&\quad + 2\mathbb{E}\int_0^T \|\bar{f}(\tilde{\bar{u}}(s)) - \bar{f}(\bar{u}(s))\|\|\tilde{u}^\varepsilon(s) - \tilde{\bar{u}}(s)\|\mathrm{d}s \\
&\leq 4K_f\mathbb{E}\int_0^T \|\bar{u}(t) - \tilde{\bar{u}}(t)\|\|\tilde{u}^\varepsilon(t) - \tilde{u}(t)\|\mathrm{d}t \\
&\quad + 2\mathbb{E}\sup_{0\leq t\leq T}\int_0^t \langle f_\varepsilon(s,\tilde{\bar{u}}(s)) - \bar{f}(\tilde{\bar{u}}(s)), \tilde{u}^\varepsilon(s) - \tilde{\bar{u}}(s)\rangle \mathrm{d}s.
\end{aligned}$$

By Lemma 3.2, we can see

$$\begin{aligned}
&\mathbb{E}\int_0^T \|\bar{u}(t) - \tilde{\bar{u}}(t)\|\|\tilde{u}^\varepsilon(t) - \tilde{\bar{u}}(t)\|\mathrm{d}t \\
&\leq \Big[\mathbb{E}\int_0^T \|\bar{u}(t) - \tilde{\bar{u}}(t)\|^2\mathrm{d}t\Big]^{\frac{1}{2}}\Big[\mathbb{E}\int_0^T \|\tilde{u}^\varepsilon(t) - \tilde{\bar{u}}(t)\|^2\mathrm{d}t\Big]^{\frac{1}{2}} \\
&\leq C(T,\|\bar{u}_0\|^2,\|\bar{z}_0\|^2)\delta^{\frac{1}{4}}.
\end{aligned}$$

Let $t(\delta) := \left[\frac{t}{\delta}\right]\delta$, we have

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T}\int_0^t \langle f_\varepsilon(s,\tilde{\bar{u}}(s)) - \bar{f}(\tilde{\bar{u}}(s)), \tilde{u}^\varepsilon(s) - \tilde{\bar{u}}(s)\rangle \mathrm{d}s \\
&\leq \mathbb{E}\sup_{0\leq t\leq T}\Big\{\sum_{k=0}^{[\frac{t}{\delta}]-1}\int_{k\delta}^{(k+1)\delta} \langle f_\varepsilon(s,\bar{u}(k\delta)) - \bar{f}(\bar{u}(k\delta)), u^\varepsilon(k\delta) - \bar{u}(k\delta)\rangle \mathrm{d}s \\
&\quad + \int_{t(\delta)}^t \langle f_\varepsilon(s,\bar{u}(t(\delta))) - \bar{f}(\bar{u}(t(\delta))), u^\varepsilon(t(\delta)) - \bar{u}(t(\delta))\rangle \mathrm{d}s\Big\} \\
&\leq \mathbb{E}\sup_{0\leq t\leq T}\Big\{\sum_{k=0}^{[\frac{t}{\delta}]-1}\Big\langle\int_{k\delta}^{(k+1)\delta} f_\varepsilon(s,\bar{u}(k\delta)) - \bar{f}(\bar{u}(k\delta))\mathrm{d}s, u^\varepsilon(k\delta) - \bar{u}(k\delta)\Big\rangle \\
&\quad + \int_{t(\delta)}^t \|f_\varepsilon(s,\bar{u}(t(\delta))) - \bar{f}(\bar{u}(t(\delta)))\|\|u^\varepsilon(t(\delta)) - \bar{u}(t(\delta))\|\mathrm{d}s\Big\} \\
&\leq \mathbb{E}\sup_{0\leq t\leq T}\left\{\sum_{k=0}^{[\frac{t}{\delta}]-1}\Big\|\int_{k\delta}^{(k+1)\delta} f_\varepsilon(s,\bar{u}(k\delta)) - \bar{f}(\bar{u}(k\delta))\mathrm{d}s\Big\|\|u^\varepsilon(k\delta) - \bar{u}(k\delta)\|\right\} \\
&\quad + C(T,\|\bar{u}_0\|^2,\|\bar{z}_0\|^2)\delta \\
&\leq \frac{C_T}{\delta}\sup_{0\leq k\leq T(\delta)-1}\Big[\mathbb{E}\Big\|\int_{k\delta}^{(k+1)\delta} f_\varepsilon(s,\bar{u}(k\delta)) - \bar{f}(\bar{u}(k\delta))\mathrm{d}s\Big\|^2\Big]^{\frac{1}{2}} \\
&\quad + C(T,\|\bar{u}_0\|^2,\|\bar{z}_0\|^2)\delta \\
&\leq \frac{C_T}{\delta}\sup_{0\leq k\leq T(\delta)-1}\delta\kappa_1\big(\delta/\varepsilon\big)\Big[\mathbb{E}(1+\|\bar{u}(k\delta)\|^2)\Big]^{\frac{1}{2}} + C(T,\|u_0\|_{H^1})\delta \\
&\leq C(T,\|\bar{u}_0\|^2,\|\bar{z}_0\|^2)\Big(\delta + \kappa_1\big(\delta/\varepsilon\big)\Big).
\end{aligned}$$

Consequently,

$$\mathrm{J}_9^2 \le C(T, \|\bar{u}_0\|^2, \|\bar{z}_0\|^2)\Big(\delta^{1/4} + \kappa_1\big(\delta/\varepsilon\big)\Big). \tag{3.20}$$

Combing with (3.10)-(3.20), then using Gronwall's inequality, we find that

$$\mathbb{E}\Big(\sup_{0\le t\le T} \mathrm{J}_9(t)\Big) \le C\mathbb{E}\int_0^T \|u^\varepsilon(s)-\bar{u}(s)\|^2 \mathrm{d}s + C(T, \|\bar{u}_0\|^2, \|\bar{z}_0\|^2)\Big(\delta^{1/4} + \kappa_1\big(\delta/\varepsilon\big)\Big). \tag{3.21}$$

Similarly, we can obtain

$$\begin{aligned} &\mathbb{E}\int_0^T \|\sigma_\varepsilon(t, u^\varepsilon(\sigma)) - \bar{\sigma}(\bar{u}(\sigma))\|^2_{\mathcal{L}_Q} \mathrm{d}\sigma \\ &\le C\mathbb{E}\int_0^T \|u^\varepsilon(s)-\bar{u}(s)\|^2 \mathrm{d}s + C(T, \|\bar{u}_0\|^2, \|\bar{z}_0\|^2)\Big(\delta^{1/4} + \kappa_2\big(\delta/\varepsilon\big)\Big). \end{aligned} \tag{3.22}$$

Combining with (3.16), (3.21) and (3.22), we get

$$\begin{aligned} \mathbb{E}\sup_{0\le t\le T} \Xi(t) \le{}& C_T\left[\mathbb{E}\|u_0^\varepsilon - \bar{u}_0\|^2 + \mathbb{E}\|z_0^\varepsilon - \bar{z}_0\|^2\right] + C\int_0^T \mathbb{E}\sup_{0\le s\le t} \Xi(s)\mathrm{d}t \\ &+ C(T, \|\bar{u}_0\|^2, \|\bar{z}_0\|^2)\Big(\delta^{1/4} + \kappa_1\big(\delta/\varepsilon\big) + \kappa_2\big(\delta/\varepsilon\big)\Big). \end{aligned}$$

With the help of Gronwall's inequality, we have

$$\begin{aligned} &\mathbb{E}\sup_{0\le t\le T} \Xi(t) \\ &\le C(T, \|\bar{u}_0\|^2, \|\bar{z}_0\|^2)\left[\|u_0^\varepsilon - \bar{u}_0\|^2 + \|z_0^\varepsilon - \bar{z}_0\|^2 + \delta^{1/4} + \kappa_1\big(\delta/\varepsilon\big) + \kappa_2\big(\delta/\varepsilon\big)\right]. \end{aligned} \tag{3.23}$$

Taking $\delta = \sqrt{\varepsilon}$ and letting $\varepsilon \to 0$ in (3.23), we obtain

$$\lim_{\varepsilon\to 0} \mathbb{E}\sup_{0\le t\le T}\Big(\|u^\varepsilon(t) - \bar{u}(t)\|^2 + \|z^\varepsilon(t) - \bar{z}(t)\|^2\Big) = 0.$$

Finally, for any $p \ge 1$, we have

$$\begin{aligned} &\lim_{\varepsilon\to 0}\left[\mathbb{E}\sup_{0\le t\le T}\|u^\varepsilon(t)-\bar{u}(t)\|^{2p} + \mathbb{E}\sup_{0\le t\le T}\|z^\varepsilon(t)-\bar{z}(t)\|^{2p}\right] \\ &\le \lim_{\varepsilon\to 0}\left[\mathbb{E}\Big(\sup_{0\le t\le T}\|u^\varepsilon(t)-\bar{u}(t)\|^{4p-2}\Big)\right]^{1/2}\left[\mathbb{E}\Big(\sup_{0\le t\le T}\|u^\varepsilon(t)-\bar{u}(t)\|^{2}\Big)\right]^{1/2} \\ &\quad + \lim_{\varepsilon\to 0}\left[\mathbb{E}\Big(\sup_{0\le t\le T}\|z^\varepsilon(t)-\bar{z}(t)\|^{4p-2}\Big)\right]^{1/2}\left[\mathbb{E}\Big(\sup_{0\le t\le T}\|z^\varepsilon(t)-\bar{z}(t)\|^{2}\Big)\right]^{1/2} \\ &\le C(1+\|u_0^\varepsilon\|^{2p-1} + \|z_0^\varepsilon\|^{2p-1})\lim_{\varepsilon\to 0}\left[\mathbb{E}\Big(\sup_{0\le t\le T}\|u^\varepsilon(t)-\bar{u}(t)\|^{2}\Big)\right]^{1/2} \\ &\quad + C(1+\|u_0^\varepsilon\|^{2p-1} + \|z_0^\varepsilon\|^{2p-1})\lim_{\varepsilon\to 0}\left[\mathbb{E}\Big(\sup_{0\le t\le T}\|z^\varepsilon(t)-\bar{z}(t)\|^{2}\Big)\right]^{1/2} \\ &= 0. \end{aligned}$$

The proof is complete. □

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