Advances in Applied Mathematics 142 (2023) 102412

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Algebraic methods to study the dimension of supersmooth spline spaces

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A R T I C L E I N F O

Article history:
Received 14 July 2021
Received in revised form 26 July 2022
Accepted 29 July 2022
Available online xxxx

MSC:
13D02
65D07
41A15

Keywords:
Spline functions
Superspline spaces on triangulations
Dimension of spline spaces
Supersmoothness
Intrisic supersmoothness

A B S T R A C T

Multivariate piecewise polynomial functions (or splines) on polyhedral complexes have been extensively studied over the past decades and find applications in diverse areas of applied mathematics including numerical analysis, approximation theory, and computer aided geometric design. In this paper we address various challenges arising in the study of splines with enhanced mixed (super-)smoothness conditions at the vertices and across interior faces of the partition. Such supersmoothness can be imposed but can also appear unexpectedly on certain splines depending on the geometry of the underlying polyhedral partition. Using algebraic tools, a generalization of the Billera–Schenck–Stillman complex that includes the effect of additional smoothness constraints leads to a construction which requires the analysis of ideals generated by products of powers of linear forms in several variables. Specializing to the case of planar triangulations, a combinatorial lower bound on the dimension of splines with supersmoothness at the vertices is presented, and we also show that this lower bound gives the exact dimension in

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[1] D. Toshniwal was supported by project number 212.150 funded through the Veni research programme by the Dutch Research Council (NWO).
[2] N. Villamizar was supported by the UK Engineering and Physical Sciences Research Council (EPSRC) New Investigator Award EP/V012835/1.

https://doi.org/10.1016/j.aam.2022.102412

high degree. The methods are further illustrated with several examples.

1. Introduction

A multivariate spline is a piecewise polynomial function defined on a partition Δ of a domain in $\Omega \subseteq \mathbb{R}^n$ such that, as a function on Ω, it is continuously differentiable up to a fixed order $r \geqslant 0$. A more general definition arises when additional smoothness conditions are imposed on specific faces of the partition Δ. Such splines are called *supersmooth splines* or *supersplines*, and in this article we study them using algebraic tools.

Spline spaces with supersmoothness are used for spline-based finite elements or isogeometric analysis applications [18]. On a general planar triangulation, the dimension of the space of C^r-continuous splines of polynomial degree at most d may depend on the geometry of the partition for small d. This is undesirable for finite elements as it complicates, for instance, the efficient construction of locally supported basis functions. However, enhanced supersmoothness can be employed to eliminate this geometric-dependence and yield more tractable spline spaces; e.g., see Speleers [34] and Grošelj and Speleers [17]. Given this, developing an understanding of spline spaces with (enhanced) supersmoothness has both theoretical and practical relevance. In this article, we present an application of homological methods toward this task.

Classically, splines have been studied using Bernstein–Bézier representations and the construction of minimal determining sets, see [20] and the references therein. These methods were first applied to superspline spaces on triangulations by Chui in [7], where a special order of supersmoothness $r + \lfloor (d - 2r - 1)/2 \rfloor$ was imposed on the vertices of the partition for C^r-spline spaces of degree $d \geqslant 3r + 2$. The motivation to construct this spline space came from the construction of locally supported basis functions and optimal finite element approximation. Splines with arbitrary uniform supersmoothness were introduced by Schumaker in [29]; and splines with varying orders of supersmoothness at the vertices by Ibrahim and Schumaker in [19]. See also [20, Chapter 5] where Bernstein–Bézier methods for splines on triangulation and well-known results on superspline spaces have been collected and summarized. Alfeld and Schumaker in [2] introduced the notion of smoothness functionals and provided lower and upper bounds for bivariate spline spaces with enhanced smoothness conditions across interior edges of the underlying triangulation. This led to a more general notion of supersmoothness, which can also be found in [20, Chapter 9].

Supersmoothness properties can be imposed but they can also appear unexpectedly on certain splines with only uniform global smoothness constraints. Splines with such unexpected smoothness are said to have *intrinsic supersmoothness*. This feature was first observed by Farin in [13] in the case of cubic C^1-continuous splines on the Clough–Tocher split, which is the triangulation of a triangle with a single interior vertex and three interior edges. Farin observed that the second order derivatives of the C^1-splines supported on this triangulation are also continuous at the interior vertex. A detailed proof of this case as well as its trivariate analog can be found in [1]. It is now known that on a given triangulation, for certain combinations of degrees and global smoothness, the dimension of a spline space can be determined combinatorially if additional smoothness constraints on the faces of the partition are revealed and appropriately addressed. The latter has been studied via Bernstein–Bézier methods to prove results on dimension of spline spaces by Sorokina in [31,32], and Shekhtman and Sorokina in [30]. Recently, in this direction, Floater and Hu in [14] determine the maximal order of intrinsic supersmoothness at vertices for various simplicial complexes with a single interior vertex.

Algebraic methods developed for studying C^r-continuous splines [3,4,23,25,26] on polyhedral complexes were explored by Geramita and Schenck in [15] to study spline spaces with varying order of smoothness across the codimension-1 faces of a simplicial complex in $\mathbb{R}^n$. In this approach, the connection between spline functions and fat point ideals is used to derive a dimension formula for mixed spline spaces on planar triangulations in sufficiently high polynomial degree. This connection is further explored by DiPasquale in [9] for splines on polytopal complexes, and for splines with mixed supersmoothness conditions on the edges of planar quadrangular and T-meshes in [35,36].

The application of algebraic methods to the study of splines with mixed smoothness (i.e., with differing orders of smoothness across different codimension-1 faces of an n-dimensional complex) are the ones closest in spirit to the focus of this article. We extend these algebraic methods to the setting where supersmoothness can be imposed at any arbitrary i-dimensional faces, $i \leqslant n-1$, of such a complex. This is a very general setting which can be used to further our understanding of both superspline and classical spline spaces. Indeed, the two are related by the notion of intrinsic supersmoothness, identification of which has been shown to yield a better understanding of the dimension of classical splines [31,32]. The latter is an open problem in spline theory in general and algebraic methods have provided new results, for instance, see the recent developments in [10,11,27,38].

The paper is organized as follows. In Section 2 we set up notation, giving the definition of mixed splines and superspline spaces. In Section 3 we present the relevant homological and algebraic background to study the dimension of superspline spaces. In Section 4 and following we consider the case of splines on planar triangulations. First, we study certain ideals that arise when considering mixed supersmoothness conditions at edges and vertices of planar domains. Next, we derive a lower bound on the dimension of superspline spaces in Section 5 and we prove that the lower bound coincides with the exact dimension in large degree. Finally, we devote Section 6 to specific examples of superspline spaces that appear in the literature [6,14,21,34] before concluding. All examples utilize Macaulay2 [16] for computations and the scripts for the same can be downloaded from https://github.com/dtoshniwal/M2_supersmoothness.

2. Splines with mixed and supersmoothness conditions

In this section we set notation and important definitions concerning the spline spaces that we will study in the rest of the paper.

We denote by Δ an n-dimensional simplicial complex embedded in $\mathbb{R}^n$. We will assume that Δ is pure, i.e., each simplex in Δ is the face of an n-simplex in Δ. If there is no confusion about the embedding, we identify Δ with its embedding and write $\Delta \subseteq \mathbb{R}^n$. If $n = 2$ we refer to Δ as a triangulation, and as a tetrahedral complex if $n = 3$. We write Δ° and $\partial\Delta = \Delta \setminus \Delta^\circ$ for the collection of interior and boundary faces of Δ, respectively. The set of i-dimensional faces of Δ, also called i-faces, is denoted Δ_i, and $\Delta_i^\circ \subseteq \Delta_i$ is the set of the interior i-faces, for $i = 0, \dots, n-1$. The number of elements of Δ_i and Δ_i° is denoted f_i and f_i°, respectively.

Denote by $\mathtt{R} = \mathbb{R}[x_1, \dots, x_n]$ the polynomial ring in n-variables, and by $\mathtt{R}_{\leqslant d}$ the vector space of polynomials in $\mathtt{R}$ of total degree at most d. We write $C^r(\Delta)$ for the set of all functions $F\colon \Delta \to \mathbb{R}$ which are continuously differentiable of order r on Δ. We call these functions C^r-continuous, or C^r-smooth, on Δ.

Definition 2.1. Let $\Delta \subseteq \mathbb{R}^n$ be a simplicial complex, and $0 \leqslant r \leqslant d$ be integers. The set $\mathcal{S}_d^r(\Delta)$ of C^r-continuous splines on Δ is defined as the set of all piecewise polynomial functions on Δ of degree at most d that are continuously differentiable up to order r on Δ. More precisely,

$$\mathcal{S}_d^r(\Delta) = \left\{ f \in C^r(\Delta)\colon f|_\sigma \in \mathtt{R}_{\leqslant d} \text{ for all } \ \sigma \in \Delta_n \right\}.$$

If $f \in \mathcal{S}_d^r(\Delta)$ we say that f is a C^r-spline, or a C^r-continuous (or -smooth) spline, on Δ. The collection of all C^r-splines on Δ is denoted $\mathcal{S}^r(\Delta) = \bigcup_{d\geqslant 0} S_d^r(\Delta)$.

For a given simplicial complex Δ, we extend Definition 2.1 and consider spline functions with variable smoothness conditions at the vertices or across the interior faces of Δ. If $\beta \in \Delta_i^\circ$, let us denote by Δ_β the *star of* β *in* Δ, that is the simplicial complex composed of all simplices σ of Δ which satisfy either $\beta \subset \sigma$ or there is a simplex σ' so that $\sigma, \beta \subset \sigma'$. Following the notation in [20] and [15], we first define the space of splines with *mixed smoothness* conditions across the interior codimension-1 faces of Δ.

Definition 2.2 *(Spline functions with mixed smoothness).* For a simplicial complex $\Delta \subseteq \mathbb{R}^n$ and a non-negative integer d, let $\boldsymbol{r} = \left\{ r_\tau\colon \tau \in \Delta_{n-1}^\circ \right\}$ be a set of integers, $0 \leqslant \max\left\{ r_\tau\colon \tau \in \Delta_{n-1}^\circ \right\} \leqslant d$. The space $\mathcal{S}_d^{\boldsymbol{r}}(\Delta)$ of splines with *mixed smoothness* $\boldsymbol{r}$ on Δ is defined as the set of all C^0-continuous functions on Δ which are splines with smoothness r_τ across the face τ for each $\tau \in \Delta_{n-1}^\circ$. Namely,

$$\mathcal{S}_d^{\boldsymbol{r}}(\Delta) = \left\{ f \in C^0(\Delta)\colon f|_{\Delta_\tau} \in \mathcal{S}_d^{r_\tau}(\Delta_\tau) \text{ for all } \ \tau \in \Delta_{n-1}^\circ \right\},$$

where Δ_τ is the star of the face τ in Δ. Similarly as before, we denote $\mathcal{S}^{\boldsymbol{r}}(\Delta) = \bigcup_{d\geqslant 0} S_d^{\boldsymbol{r}}(\Delta)$. If $r_\tau = r \in \mathbb{Z}_{\geqslant 0}$ for all $\tau \in \Delta_{n-1}^\circ$, then $\mathcal{S}_d^{\boldsymbol{r}}(\Delta)$ coincides with $\mathcal{S}_d^r(\Delta)$ in Definition 2.1. In this case we write $\mathcal{S}_d^{\boldsymbol{r}}(\Delta) = \mathcal{S}_d^r(\Delta)$.

We now define spline functions with variable order of smoothness at the i-faces in Δ_i for $i = 0, \dots, n-2$. We follow the notation in [20] for planar Δ and call the sets of these functions *superspline spaces*. We say that a spline $f \in \mathcal{S}_d^0(\Delta)$ is C^s-continuous at a face $\beta \in \Delta_i$ provided that, for all $\sigma \in \Delta_n$ such that β is a face of σ, all polynomials $f|_\sigma$ have common derivatives up to order s on β. In this case we say that f has *supersmoothness* s at β and, following the convention introduced in [20] for a simplex β, write $f|_{\Delta_\beta} \in C^s(\beta)$, or simply $f \in C^s(\beta)$.

Definition 2.3 *(Superspline functions).* Suppose $\Delta \subseteq \mathbb{R}^n$ is a simplicial complex and r_τ, $\tau \in \Delta_{n-1}^\circ$ and d are integers such that $0 \leqslant r_\tau \leqslant d$ for each $\tau \in \Delta_{n-1}^\circ$. For a fixed $0 \leqslant i \leqslant n-2$, let $\boldsymbol{s} = \{s_\beta : \beta \in \Delta_i\}$ be a sequence of integers s_β with $0 \leqslant s_\beta \leqslant d$. The *superspline space* $\mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$ is defined as the set of all $C^{\boldsymbol{r}}$-continuous splines on Δ with supersmoothness s_β at β for each face $\beta \in \Delta_i$ i.e.,

$$\mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta) = \left\{ f \in \mathcal{S}_d^{\boldsymbol{r}}(\Delta) \colon f \in C^{s_\beta}(\beta) \text{ for all } \ \beta \in \Delta_i \right\}.$$

We denote $\mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}(\Delta) = \bigcup_{d\geqslant 0} S_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$. If $s_\beta = s \in \mathbb{Z}_{\geqslant 0}$ for all $\beta \in \Delta_i$, we write $\mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta) = \mathcal{S}_d^{\boldsymbol{r},s}(\Delta)$; if $r_\tau = r \in \mathbb{Z}_{\geqslant 0}$ for all $\tau \in \Delta_{n-1}^\circ$ we simply write $\mathcal{S}_d^{r,\boldsymbol{s}}(\Delta)$, and if $s = r$ then we write $\mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta) = \mathcal{S}_d^r(\Delta)$.

Remark 2.1. Notice that if $\gamma \in \Delta_i$ for $0 \leqslant i \leqslant n-2$ is a face of $\tau \in \Delta_{n-1}$ and $f \in \mathcal{S}_d^r(\Delta)$, then $f \in C^s(\gamma)$ does not necessarily imply $f|_{\Delta_\tau} \in \mathcal{S}_d^s(\Delta_\tau)$. Conversely, if $f|_{\Delta_\tau} \in \mathcal{S}_d^s(\Delta_\tau)$ holds for all $(n-1)$-face $\tau \in \Delta_\gamma$ then $f \in C^s(\gamma)$ for each face $\gamma \subseteq \tau$.

Note that in the following we will fix an index $0 \leqslant i \leqslant n-2$ and only consider supersplines that posses enhanced smoothness at the i-faces of the simplicial complex. Therefore, in the case $n = 2$ (which will also comprise the majority of our discussion), the only superspline space will be that of splines with supersmoothness at the vertices of the triangulation. Similarly, in the case $n = 3$, we can consider two superspline spaces, one composed of splines with supersmoothness across the edges and the other of splines with supersmoothness at the vertices of the given tetrahedral partition.

3. Supersplines as the homology of a chain complex

In this section we review the necessary results from [3,4,15,26], and extend these results to the setting of superspline spaces $\mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$ introduced in Section 2.

First we recall that for any pair of integers $r, d \geqslant 0$, the study of the splines $\mathcal{S}_d^r(\Delta)$ on Δ of degree at most d and global smoothness r can be reduced to the study of splines on a simplicial complex whose polynomial pieces are homogeneous polynomials of degree d.

In fact, if $\Delta \subseteq \mathbb{R}^n$ is the star of a vertex (i.e., if all simplices in Δ share a common vertex), then

$$\mathcal{S}^r(\Delta) \cong \bigoplus_{i\geq 0} \mathcal{S}^r(\Delta)_i, \text{ and } \quad \mathcal{S}^r_d(\Delta) \cong \bigoplus_{i=0}^{d} \mathcal{S}^r(\Delta)_i, \tag{1}$$

where $\mathcal{S}^r(\Delta)_i$ denotes the splines on Δ of degree exactly i, and the isomorphism is as $\mathbb{R}$-vector spaces.

If $\Delta \subseteq \mathbb{R}^n$ is not the star of a vertex, then the isomorphism (1) does not hold for $\mathcal{S}^r(\Delta)$, but one can associate to Δ a star of a vertex $\hat{\Delta} \subseteq \mathbb{R}^{n+1}$ and (1) will still be valid for $\hat{\Delta}$. This new complex $\hat{\Delta}$ can be constructed as follows. If $x_1, \ldots, x_n$ are the coordinates of $\mathbb{R}^n$, consider the embedding $\phi\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ in the hyperplane $\{x_0 = 1\} \subseteq \mathbb{R}^{n+1}$ given by $\phi(x_1, \ldots, x_n) = (1, x_1, \ldots, x_n)$. If σ is a simplex in $\mathbb{R}^n$, the *cone over* σ, denoted $\hat{\sigma}$, is the simplex in $\mathbb{R}^{n+1}$ which is the convex hull of the origin in $\mathbb{R}^{n+1}$ and $\phi(\sigma)$. If $\Delta \subseteq \mathbb{R}^n$ is a simplicial complex, the *cone over* Δ, denoted $\hat{\Delta}$, is the simplicial complex consisting of the simplices $\{\hat{\beta} : \beta \in \Delta\}$ along with the origin in $\mathbb{R}^n$. Then, by construction, $\hat{\Delta} \subseteq \mathbb{R}^{n+1}$ is the star of the origin and (1) yields $\mathcal{S}^r(\hat{\Delta}) \cong \bigoplus_{i\geq 0} \mathcal{S}^r(\hat{\Delta})_i$ and $\mathcal{S}^r_d(\hat{\Delta}) \cong \bigoplus_{i=0}^{d} \mathcal{S}^r(\hat{\Delta})_i$. The following result from Billera and Rose [4] links these two spline spaces.

Theorem 3.1 *([4, Theorem 2.6]). If $\Delta \subseteq \mathbb{R}^n$ is a simplicial complex and $\hat{\Delta}$ is the cone over Δ in $\mathbb{R}^{n+1}$ then $\mathcal{S}^r_d(\Delta) \cong \mathcal{S}^r(\hat{\Delta})_d$.*

In the following we extend Theorem 3.1 to the superspline functions introduced in Definition 2.3.

3.1. Superspline ideals

Suppose $\Delta \subseteq \mathbb{R}^n$ is an n-dimensional simplicial complex. As defined above, let $\hat{\Delta} \subseteq \mathbb{R}^{n+1}$ be the cone over Δ, and denote by $\mathtt{S} = \mathbb{R}[x_0, x_1, \ldots, x_n]$ the polynomial ring associated to $\hat{\Delta}$. Given a polynomial $f \in \mathtt{R} = \mathbb{R}[x_1, \ldots, x_n]$ of degree d, its homogenization in $\mathtt{S}$ is defined as

$$\hat{f}(x_0, x_1, \ldots, x_n) = x_0^d f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

Conversely, if $f \in \mathtt{S}$, its dehomogenized counterpart in $\mathtt{R}$ is defined by taking $x_0 = 1$ and will be denoted by $\check{f} \in \mathtt{R}$.

For homogeneous polynomials $f_1, \ldots, f_k \in \mathtt{S}$, we denote by $\langle f_i \rangle \subseteq \mathtt{S}$ the ideal generated by f_i and $\langle f_1, \ldots, f_k \rangle = \sum_{i=1}^{k} \langle f_i \rangle$ the ideal of $\mathtt{S}$ generated by $f_1, \ldots, f_k$. We write $\mathbf{V}(f_1, \ldots, f_k) \subseteq \mathbb{R}^{n+1}$ for the set of points $\boldsymbol{p} \in \mathbb{R}^{n+1}$ such that $f_i(\boldsymbol{p}) = 0$ for all $i = 1, \ldots, k$. Similarly, we define $\mathbf{V}(\check{f}_1, \ldots, \check{f}_k) \subseteq \mathbb{R}^n$ for $\check{f}_i \in \mathtt{R}$.

Fix $0 \leqslant i \leqslant n-2$, and take two sets of integers $\boldsymbol{r} = \{r_\tau : \tau \in \Delta^\circ_{n-1}\}$ and $\boldsymbol{s} = \{s_\beta : \beta \in \Delta_i\}$ such that $r_\tau, s_\beta \geqslant 0$ for each $\tau \in \Delta^\circ_{n-1}$ and $\beta \in \Delta_i$. To each face of Δ we associate an (homogeneous) ideal in $\mathtt{S}$ as follows.

- If $\sigma \in \Delta_n$ define $\mathtt{J}(\sigma) = 0$.
- If $\tau \in \Delta^\circ_{n-1}$, let $\ell_\tau \in \mathtt{S}$ be (a choice of) a linear form vanishing on $\hat{\tau}$. For each i-face $\beta \subset \tau$ let $\mathfrak{m}_\beta = \{\hat{f} \in \mathtt{S} : f \in \check{\mathfrak{m}}_\beta\}$, where $\check{\mathfrak{m}}_\beta \subseteq \mathtt{R}$ is the ideal of all polynomials vanishing at β. In other words, $\mathfrak{m}_\beta$ is the ideal of all polynomials vanishing on $\hat{\beta}$. We define

$$\mathtt{J}(\tau) = \left\langle \ell_\tau^{r_\tau+1} \right\rangle \cap \left(\bigcap_{\beta \in \Delta_i,\, \beta \subset \tau} \mathfrak{m}_\beta^{s_\beta+1} \right). \tag{2}$$

- If $\gamma \in \Delta_j$ for $0 \leqslant j \leqslant n-2$, take

$$\mathtt{J}(\gamma) = \textstyle\sum_{\tau \ni \gamma,\, \tau \in \Delta^\circ_{n-1}} \mathtt{J}(\tau). \tag{3}$$

Additionally, we denote by $\check{\mathtt{J}}(\tau)$ the ideal in $\mathtt{R}$ corresponding to the edge $\tau \in \Delta^\circ_1$, namely

$$\check{\mathtt{J}}(\tau) = \left\langle \check{\ell}_\tau^{r_\tau+1} \right\rangle \cap \left(\bigcap_{\beta \in \Delta_i,\, \beta \subset \tau} \check{\mathfrak{m}}_\beta^{s_\beta+1} \right),$$

where $\check{\ell}_\tau \in \mathtt{R}$ is a linear polynomial vanishing at τ. The ideal $\mathtt{J}(\tau)$ can be equivalently defined as the homogenization of $\check{\mathtt{J}}(\tau)$ in $\mathtt{S}$.

Note that, if $r_\tau = s_\beta$ for all i-faces $\beta \subset \tau$, the ideal $\mathtt{J}(\tau)$ associated to $\tau \in \Delta^\circ_{n-1}$ reduces to $\mathtt{J}(\tau) = \langle \ell_\tau^{r_\tau+1} \rangle$, and we recover the ideals defined by Schenck and Stillman in [26].

3.2. A chain complex of supersplines

Recall that a simplicial complex $\Delta \subseteq \mathbb{R}^n$ is *pure* if all its maximal faces (with respect to inclusion) are of dimension n; and it is *hereditary* if for all pairs of faces $\sigma, \sigma' \in \Delta_n$ such that $\sigma \cap \sigma' = \beta \in \Delta_i$ there is a sequence of n-faces $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_{m-1}, \sigma_m = \sigma'$ such that $\beta \in \sigma_i$ for all i and $\sigma_{i-1} \cap \sigma_i \in \Delta^\circ_{n-1}$ for each $i = 1, \ldots, m$.

For a pure and hereditary n-dimensional simplicial complex Δ, Billera proved in [3] the following algebraic criterion for a piecewise polynomial function on a simplicial complex Δ to be C^r-smooth on Δ.

Theorem 3.2 *([3, Theorem 2.4]). Suppose $\Delta \subseteq \mathbb{R}^n$ is a pure and hereditary simplicial complex and $r \geqslant 0$ is an integer. Then $f \in \mathcal{S}^r(\Delta)$ if and only if $\hat{f}|_{\hat{\sigma}} - \hat{f}|_{\hat{\sigma}'} \in \mathtt{J}(\tau)$ or, equivalently, if and only if $f|_\sigma - f|_{\sigma'} \in \langle \check{\ell}_\tau^{r+1} \rangle$, for every pair $\sigma, \sigma' \in \Delta_n$ satisfying $\sigma \cap \sigma' = \tau \in \Delta^\circ_{n-1}$.*

Remark 3.1. In the case $\Delta \subseteq \mathbb{R}^2$, Wang in [37] and Chui in [5, Theorem 4.2] provided earlier proofs of Theorem 3.2.

An alternate proof of Theorem 3.2 given in [4, Proposition 1.2] yields an analogous criterion for supersmoothness at the i-faces of Δ for all $i < n$. We combine these results into the following statement for splines with smoothness $\boldsymbol{r}$ across the codimension-1 faces and supersmoothness $\boldsymbol{s}$ at the i-faces of the partition.

Theorem 3.3. *Suppose $\Delta \subseteq \mathbb{R}^n$ is a pure and hereditary simplicial complex and $\mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$ denotes the set of splines with smoothness $\boldsymbol{r} = \{r_\tau : \tau \in \Delta^\circ_{n-1}\}$ at the codimension-1 faces and supersmoothness $\boldsymbol{s} = \{s_\beta : \beta \in \Delta_i\}$ across all the i-faces of Δ, for a fixed $0 \leqslant i \leqslant n-2$. Then $f \in \mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$ if and only if $\hat{f}|_{\hat{\sigma}} - \hat{f}|_{\hat{\sigma}'} \in \mathtt{J}(\tau)$ or, equivalently, if and only if $f|_\sigma - f|_{\sigma'} \in \check{\mathtt{J}}(\tau)$, for all $\tau \in \Delta^\circ_{n-1}$ and $\sigma, \sigma' \in \Delta_n$ satisfying $\sigma \cap \sigma' = \tau$.*

Proof. Let $\sigma, \sigma' \in \Delta_n$ such that $\sigma \cap \sigma' = \tau \in \Delta^\circ_{n-1}$. Suppose $f|_\sigma - f|_{\sigma'} \in \check{\mathtt{J}}(\tau)$. In particular, $f|_\sigma - f|_{\sigma'} \in \langle \check{\ell}_\tau^{r_\tau+1} \rangle$ and clearly the restriction of the derivatives up to order r_τ of $f|_{\hat{\sigma}} - f|_{\hat{\sigma}'}$ to the edge $\hat{\tau}$ are zero. On the other hand, $f|_\sigma - f|_{\sigma'} \in \check{\mathfrak{m}}_\beta^{s_\beta+1}$ for each $\beta \in \Delta_i$ such that $\beta \subset \tau$, so the polynomial $f|_\sigma - f|_{\sigma'}$, and all its derivatives up to order s_β, vanish at β.

By hypothesis Δ is hereditary, then there is a sequence of n-faces $\sigma_0, \sigma_1, \ldots, \sigma_m$ such that $\sigma_j \supset \beta$ for all j and $\sigma_{j-1} \cap \sigma_j \in \Delta^\circ_{n-1}$. Applying the previous argument to each pair of faces σ_{j-1} and σ_j, we get that all the derivatives up to order s_β of $f|_{\sigma_{j-1}}$ and $f|_{\sigma_j}$ coincide at β for every $j = 1, \ldots, m$, and hence $f \in C^{s_\beta}(\beta)$ for each $\beta \in \Delta_i$. It follows that $f \in \mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$.

Conversely, if $f \in \mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$ then by Theorem 3.2 $f|_\sigma - f|_{\sigma'} \in \langle \check{\ell}_\tau^{r_\tau+1} \rangle$ for all $\tau \in \Delta^\circ_{n-1}$. Let β be one of the i-faces of τ. The ideal $\check{\mathfrak{m}}_\beta = \{g \in \mathtt{R} : g(\beta) = 0\}$ is generated by $n - i$ linearly independent linear polynomials, each of them vanishing at β. By hypothesis, the function $f|_\sigma - f|_{\sigma'}$, and all its derivatives up to order s_β, are zero when restricted to β. If follows $f|_\sigma - f|_{\sigma'} \in \check{\mathfrak{m}}_\beta$, and by induction (on the order of the derivatives) we get that $f|_\sigma - f|_{\sigma'} \in \check{\mathfrak{m}}_\beta^{s_\beta+1}$. This argument applies to every i-face $\beta \subset \tau$ and leads to $f|_\sigma - f|_{\sigma'} \in \check{\mathtt{J}}(\tau)$ for each $\tau \in \Delta^\circ_{n-1}$, as required. $\square$

We now extend the construction by Billera [3] and refined by Schenck and Stillman in [26] to the context of superspline spaces.

If $\Delta \subseteq \mathbb{R}^n$ is a simplicial complex, let $\bigoplus_{\beta \in \Delta_i} \mathtt{S}$ be the direct sum of the polynomial ring $\mathtt{S}$. If ∂_i is the simplicial boundary map relative to the boundary $\partial\Delta$, we denote by $\mathcal{R}$ the chain complex

$$\mathcal{R}: \quad 0 \to \bigoplus_{\sigma \in \Delta_n} \mathtt{S} \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{i+1}} \bigoplus_{\beta \in \Delta^\circ_i} \mathtt{S} \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_1} \bigoplus_{\gamma \in \Delta^\circ_0} \mathtt{S} \to 0\,.$$

The restriction of the maps ∂_i to the ideals $\bigoplus_{\beta \in \Delta_i} \mathtt{J}(\beta)$ yields the subcomplex $\mathcal{J}$ given by

$$\mathcal{J}\colon 0 \to \bigoplus_{\tau\in\Delta^\circ_{n-1}} \mathtt{J}(\tau) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} \bigoplus_{\gamma\in\Delta^\circ_0} \mathtt{J}(\gamma) \to 0\,; \tag{4}$$

and taking the quotient leads to chain complex $\mathcal{R}/\mathcal{J}$ given by

$$\mathcal{R}/\mathcal{J}\colon 0 \to \bigoplus_{\sigma\in\Delta_n} \mathtt{S} \xrightarrow{\overline{\partial}_n} \cdots \xrightarrow{\overline{\partial}_{i+1}} \bigoplus_{\beta\in\Delta^\circ_i} \mathtt{S}/\mathtt{J}(\beta) \xrightarrow{\overline{\partial}_i} \cdots \xrightarrow{\overline{\partial}_1} \bigoplus_{\gamma\in\Delta^\circ_0} \mathtt{S}/\mathtt{J}(\gamma) \to 0\,. \tag{5}$$

If we take $s_\beta = r_\tau = r$, for some $r \in \mathbb{Z}_{\geqslant 0}$, for all i-faces β and all codimension-1 faces τ, the complex $\mathcal{R}/\mathcal{J}$ reduces to that in [26].

We recall that for a chain complex $\mathcal{C}$ with boundary maps ∂_i, the i-th homology module $H_i(\mathcal{C})$ is defined as $H_i(\mathcal{C}) = \ker(\partial_i)/\mathrm{im}(\partial_{i-1})$. It was shown by Billera in [3] that $\mathcal{S}^r(\hat{\Delta}) \cong H_n(\mathcal{R}/\mathcal{J}) = \ker \overline{\partial}_n$. This isomorphism also holds in our setting by the algebraic criterion in Theorem 3.3. However, in contrast to the case of splines with uniform global smoothness conditions $r = s$, in our setting we need to specify the superspline space we consider on Δ and the corresponding one on $\hat{\Delta}$. Namely, if we take the set $\mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$ of C^r-continuous splines on Δ with supersmoothness $\boldsymbol{s}$ on the i-faces $\beta \in \Delta_i$, the corresponding spline space on $\hat{\Delta}$, denoted $\mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}(\hat{\Delta})$, is the set of C^r-splines on $\hat{\Delta}$ with supersmoothness $\boldsymbol{s}$ at the $(i+1)$-faces $\hat{\beta}$ of $\hat{\Delta}$. Following this notation we have the following two results.

Corollary 3.4. *Let $\Delta \subseteq \mathbb{R}^n$ be a pure and hereditary simplicial complex and let $0 \leqslant r_\tau \leqslant s_\beta$ be integers for each $\tau \in \Delta^\circ_{n-1}$ and $\beta \in \Delta^\circ_i$, for a fixed $0 \leqslant i \leqslant n-2$. Then, $\mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}(\hat{\Delta}) \cong \ker\big(\overline{\partial}_n\big)$, where $\mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}(\hat{\Delta})$ is the set of C^r-splines with supersmoothness $\boldsymbol{s}$ at the $(i+1)$-faces $\hat{\beta}$, and $\overline{\partial}_n$ is the differential map in the chain complex $\mathcal{R}/\mathcal{J}$ in Equation (5).*

Proof. By Theorem 3.3, we have that $f \in \mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}(\hat{\Delta})$ if and only if $\partial_n(f)|_\tau = f|_{\hat{\sigma}} - f|_{\hat{\sigma}'} \in \mathtt{J}(\tau)$ for each $\tau \in \Delta^\circ_{n-1}$, or equivalently, if and only if $f \in \ker(\overline{\partial}_n)$, as required. □

Proposition 3.5. *If $\Delta \subseteq \mathbb{R}^n$ is a pure and hereditary simplicial complex, then $\mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}_d(\Delta) \cong \mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}(\hat{\Delta})_d$, as real vector spaces.*

Proof. We follow the argument used to prove the corresponding statement for $\mathcal{S}^r_d(\Delta)$ in [4, Theorem 2.6]. We define the map $\varphi\colon \mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}(\hat{\Delta})_d \to \mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}_d(\Delta)$ by $\varphi(f)|_{\hat{\sigma}} = \check{f}|_\sigma$ for each $\sigma \in \Delta_n$, where $\check{f}|_\sigma$ is the dehomogenization of $f|_\sigma$ and $\hat{\sigma}$ is the cone over σ. It is easy to see that φ is an $\mathbb{R}$-linear map. Theorem 3.3 applied to both $\mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$ (with supersmoothness $\boldsymbol{s}$ at the i-faces β of Δ) and $\mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}(\hat{\Delta})$ (with supersmothness $\boldsymbol{s}$ at the $(i+1)$-faces $\hat{\beta}$ of $\hat{\Delta}$) implies that φ is an isomorphism of real vector spaces. □

Let $\mathcal{C}\colon 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \to 0$ be a chain complex of graded modules C_i with boundary maps ∂_i. Denoting the homology modules as $H_i(\mathcal{C})$, the Euler–Poincaré characteristic of $\mathcal{C}$ at degree d is denoted by $\chi(\mathcal{C}, d) = \sum_{i=0}^n (-1)^i \dim(C_{n-i})_d = \sum_{i=0}^n (-1)^i \dim H_{n-i}(\mathcal{C})_d$. (This result from homological algebra can be found in [33,

§4], for instance.) We apply this equality to the complex $\mathcal{R}/\mathcal{J}$, which together with Corollary 3.4 and Proposition 3.5, leads to

$$\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta) = \sum_{i=0}^{n}(-1)^i \dim \bigoplus_{\beta\in\Delta_{n-i}^\circ} \mathtt{S}/\mathtt{J}(\beta)_d - \sum_{i=1}^{n}(-1)^i \dim H_{n-i}(\mathcal{R}/\mathcal{J})_d\,. \tag{6}$$

In Equation (6), we consider all maximal n-faces of Δ to be interior, so $\Delta_n^\circ = \Delta_n$.

4. Supersmooth ideals at edges and vertices in planar domains

In this section we assume Δ is a simplicial complex in $\mathbb{R}^2$, and study the dimension of the modules on the right hand side of Equation (6). The objective is to get an explicit formula for $\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$ for special cases of Δ, which we use in Section 5 to prove a lower bound on $\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$ for arbitrary triangulations homeomorphic to a disk.

If $\Delta \subseteq \mathbb{R}^2$, Equation (6) simplifies to

$$\begin{aligned}\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta) = &\dim \bigoplus_{\sigma\in\Delta_2} \mathtt{S}_d - \dim \bigoplus_{\tau\in\Delta_1^\circ} \mathtt{S}/\mathtt{J}(\tau)_d + \dim \bigoplus_{\gamma\in\Delta_0^\circ} \mathtt{S}/\mathtt{J}(\gamma)_d \\ &+ \dim H_1(\mathcal{R}/\mathcal{J})_d - \dim H_0(\mathcal{R}/\mathcal{J})_d.\end{aligned} \tag{7}$$

The short exact sequence of complexes $0 \to \mathcal{J} \to \mathcal{R} \to \mathcal{R}/\mathcal{J} \to 0$ leads to a long exact sequence of homology modules $H_i(\mathcal{J})$, $H_i(\mathcal{R})$ and $H_i(\mathcal{R}/\mathcal{J})$. In particular, if $\Delta \subseteq \mathbb{R}^2$ is homeomorphic to a disk then $f_2 - f_1^\circ - f_0^\circ = 1$, and so $\dim \oplus_{\sigma\in\Delta_2}\mathtt{S}_d - \dim \oplus_{\sigma\in\Delta_1^\circ}\mathtt{S}_d + \dim \oplus_{\sigma\in\Delta_0^\circ}\mathtt{S}_d = \dim \mathtt{S}_d$. Moreover, because modulo the image of ∂_1 (respectively, ∂_2), every vertex of Δ is equivalent to a boundary vertex (respectively, every cycle of edges of Δ is equivalent to a cycle formed by the boundary edges of Δ), then $H_0(\mathcal{R}) = 0$ (respectively, $H_1(\mathcal{R}) = 0$). The latter implies $H_0(\mathcal{R}/\mathcal{J}) = 0$ and $H_1(\mathcal{R}/\mathcal{J}) \cong H_0(\mathcal{J})$, respectively. Therefore, in the case $\Delta \subseteq \mathbb{R}^2$ is homeomorphic to a disk, Equation (7) can be written as

$$\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta) = \binom{d+2}{2} + \sum_{\tau\in\Delta_1^\circ} \dim \mathtt{J}(\tau)_d - \sum_{\gamma\in\Delta_0^\circ} \dim \mathtt{J}(\gamma)_d + \dim H_0(\mathcal{J})_d\,. \tag{8}$$

4.1. Ideals of edges and vertices

If $\tau = [\gamma, \gamma'] \in \Delta_1^\circ$ is an interior edge of Δ with vertices γ and γ', we write $\mathtt{J}(\tau) \subseteq \mathtt{S}$ for the ideal of τ defined in (2). We start by discussing the generators of this ideal.

Lemma 4.1. *Let $[\gamma, \gamma'] = \tau \in \Delta_1^\circ$ be an edge with vertices γ and γ', and let integers r_τ and $s_\gamma, s_{\gamma'} \geqslant r_\tau$ denote the desired orders of smoothness and supersmoothness from τ and γ, γ', respectively. Then, $\mathtt{J}(\tau) = \left\langle \ell_\tau^{r_\tau+1}\right\rangle \cap \mathfrak{m}_\gamma^{s_\gamma+1} \cap \mathfrak{m}_{\gamma'}^{s_{\gamma'}+1}$, as defined in (2), can be expressed as*

$$\mathsf{J}(\tau) = \left\langle \ell_\tau^i\, \ell_{\tau,\gamma}^j\, \ell_{\tau,\gamma'}^k : i,j,k \geqslant 0\ ,\ \ i \geqslant r_\tau + 1\ ,\ \ i+j \geqslant s_\gamma + 1\ ,\ \ i+k \geqslant s_{\gamma'} + 1 \right\rangle ,$$

for any linear forms $\ell_{\tau,\gamma}$ and $\ell_{\tau,\gamma'}$ such that $\mathbf{V}(\ell_\tau, \ell_{\tau,\gamma})$ and $\mathbf{V}(\ell_\tau, \ell_{\tau,\gamma'})$ are the lines (in $\mathbb{R}^3$) containing the faces $\hat{\gamma}$ and $\hat{\gamma}'$ of $\hat{\Delta}$, respectively.

Proof. Choose ℓ_τ and $\ell_{\tau,\gamma}$ as the generators of $\mathfrak{m}_\gamma = \left\langle \ell_\tau, \ell_{\tau,\gamma} \right\rangle$, and similarly $\mathfrak{m}_{\gamma'} = \left\langle \ell_\tau, \ell_{\tau,\gamma'} \right\rangle$. Then, by a change of coordinates so that $\ell_\tau = x$, $\ell_{\tau,\gamma} = y$ and $\ell_{\tau,\gamma'} = z$, we have

$$\mathsf{J}(\tau) = \left\langle x^{r_\tau+1} \right\rangle \cap \left\langle x, y \right\rangle^{s_\gamma+1} \cap \left\langle x, z \right\rangle^{s_{\gamma'}+1}.$$

The claim follows and, in particular, it does not depend on a specific choice of $\ell_{\tau,\gamma}$ and $\ell_{\tau,\gamma'}$. Indeed, if ℓ_τ, $\ell_{\tau,\gamma}$ and ℓ are three distinct linear forms vanishing at $\hat{\gamma}$, then it is easy to see that ℓ can be written as a linear combination $\ell = a\ell_\tau + b\ell_{\tau,\gamma}$, for $a, b \in \mathbb{R}$. A generator of the ideal $\mathfrak{m}_\gamma^k = \left\langle \ell_\tau, \ell_{\tau,\gamma} \right\rangle^k$, for some $k \geqslant 1$, has the form $\ell_\tau^i \ell_{\tau,\gamma}^j$, with $i+j = k$, and $\ell_\tau^i \ell^j = \ell_\tau^i (a\ell_\tau + b\ell_{\tau,\gamma})^j$, which is clearly an element of $\mathfrak{m}_\gamma^k$. Hence $\left\langle \ell_\tau, \ell \right\rangle^k \subseteq \mathfrak{m}_\gamma^k$, and the converse trivially follows writing $\ell_{\tau,\gamma}$ in terms of ℓ_τ and ℓ. A similar argument shows the corresponding statement for the ideal $\mathfrak{m}_{\gamma'}$. $\square$

From the above description of $\mathsf{J}(\tau)$, a dimension formula for the graded pieces $\mathsf{J}(\tau)_d$ follows immediately, this is shown in the next lemma. In the following lemmas, and throughout this paper, we define $\binom{a}{b} = 0$ whenever $a < b$.

Lemma 4.2. *Let $[\gamma, \gamma'] = \tau \in \Delta_1^\circ$ be an edge with vertices γ and γ', and let integers r and $s_\gamma, s_{\gamma'} \geqslant r_\tau$ denote the desired orders of smoothness and supersmoothness from τ and γ, γ', respectively. If $d \leqslant \max\{s_\gamma, s_{\gamma'}\}$ then $\mathsf{J}(\tau)_d = 0$, and else*

$$\dim \mathsf{J}(\tau)_d = \binom{d - r_\tau + 1}{2} - \binom{s_\gamma + 1 - r_\tau}{2} - \binom{s_{\gamma'} + 1 - r_\tau}{2} + \binom{s_\gamma + s_{\gamma'} + 1 - d - r_\tau}{2}.$$

Proof. By a change of coordinates in $\mathbb{R}^3$ we can assume that the linear polynomial vanishing at $\hat{\tau}$ is $\ell_\tau = x$, and the ideals of polynomials vanishing at $\hat{\gamma}$ and $\hat{\gamma}'$ are $\mathfrak{m}_\gamma = \langle x, y \rangle$ and $\mathfrak{m}_{\gamma'} = \langle x, z \rangle$, respectively. Then $\mathsf{J}(\tau) = \left\langle x^{r_\tau+1} \right\rangle \cap \langle x, y \rangle^{s_\gamma+1} \cap \langle x, z \rangle^{s_{\gamma'}+1}$. Since this is a monomial ideal, $\mathsf{J}(\tau)_d$ is the span of the monomials $m = x^i y^j z^k$ so that i, j, k are non-negative integers satisfying all of the following conditions:

- $i + j + k = d$,
- $i \geqslant r_\tau + 1$,
- $i + j \geqslant s_\gamma + 1$, and
- $i + k \geqslant s_{\gamma'} + 1$.

The claim follows by simply counting the number of such triplets (i, j, k). $\square$

Remark 4.1. In the previous result, two specific simplifications may be of interest in applications. One is when the supersmoothness at both vertices of τ is the same and the other is when only one of the vertices has supersmoothness. For the first case, if $s_\gamma = s_{\gamma'}$ and $d > s_\gamma \geqslant r_\tau$, then

$$\dim\big(\mathtt{S}/\mathtt{J}(\tau)\big)_d = \binom{d+2}{2} + \binom{d-2s_\gamma + r_\tau}{2} - (d-s_\gamma)^2\,. \tag{9}$$

Similarly, for the second case, if $s_{\gamma'} = r_\tau$ and $d > s_\gamma \geqslant r_\tau$, we get

$$\dim\big(\mathtt{S}/\mathtt{J}(\tau)\big)_d = \binom{d+2}{2} - (s_\gamma - r_\tau + 1)\binom{d-s_\gamma+1}{2} + (s_\gamma - r_\tau)\binom{d-s_\gamma}{2}. \tag{10}$$

If $\gamma \in \Delta_0$ is a vertex in Δ, we write $\mathtt{J}(\gamma)$ for the ideal of γ defined in (3); in our case, $\Delta \subseteq \mathbb{R}^2$ and

$$\mathtt{J}(\gamma) = \sum_{\tau \supset \gamma, \tau \in \Delta_1^\circ} \mathtt{J}(\tau) = \sum_{\gamma' \in \Delta_0\,, \tau = [\gamma,\gamma'] \in \Delta_1} \langle \ell_\tau^{r_\tau+1} \rangle \cap \mathfrak{m}_\gamma^{s_\gamma+1} \cap \mathfrak{m}_{\gamma'}^{s_{\gamma'}+1}. \tag{11}$$

Moreover, if there is supersmoothness only at γ, i.e., $s'_\gamma = r_\tau$ in (11), then we will denote this ideal as

$$\overline{\mathtt{J}}(\gamma) = \sum_{\tau \ni \gamma\,,\, \tau \in \Delta_1^\circ} \big(\langle \ell_\tau^{r_\tau+1} \rangle \cap \mathfrak{m}_\gamma^{s_\gamma+1} \big). \tag{12}$$

We introduce this different notation for this special case because it will be useful for obtaining explicit bounds on the superspline space dimension. The reason is that, in general, it is not clear how to compute $\dim \mathtt{J}(\gamma)$ as defined in Equation (11) while an explicit dimension formula for the simpler ideal $\overline{\mathtt{J}}(\gamma)$ can be found. We show this in the following results and use it in Section 4.2 to compute the dimension of supersplines on triangulations with only one interior vertex, and in Section 5 to compute bounds on the dimension of $\mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$ for any triangulation Δ homeomorphic to a planar disk (see Theorem 5.4).

Lemma 4.3. *Let $\check{\mathfrak{m}}$ be the maximal ideal in $\mathtt{R}$ of all polynomials vanishing at γ, and for each edge $\tau \in \Delta_1^\circ$ let ℓ_τ be a linear form vanishing at $\hat{\tau}$. Given $s_\gamma \geqslant \max\{r_\tau : \tau \ni \gamma\}$, we get*

$$\sum_{\tau \ni \gamma} \big(\mathfrak{m}^{s_\gamma+1} \cap \langle \ell_\tau^{r_\tau+1} \rangle \big) = \mathfrak{m}^{s_\gamma+1} \cap \sum_{\tau \ni \gamma} \langle \ell_\tau^{r_\tau+1} \rangle\,.$$

Proof. Let $f \in \mathfrak{m}^{s_\gamma+1} \cap \sum_{\tau \ni \gamma} \langle \ell_\tau^{r_\tau+1} \rangle$. Then, there exist $g_\tau \in \mathtt{S}$ such that $f = \sum_{\tau \ni \gamma} g_\tau \ell_\tau^{r_\tau+1}$. Notice that $\ell_\tau^{r_\tau+1} \in \mathfrak{m}^{r_\tau+1}$ for all edges τ containing γ. Since $f \in \mathfrak{m}^{s_\gamma+1}$, then we may assume $g_\tau \in \mathfrak{m}^{s_\gamma - r_\tau}$. Indeed, we may write $f = \sum_{\tau \ni \gamma} g_\tau \ell_\tau^{r_\tau+1} =$

$\sum_{\tau\ni\gamma}\big(h_\tau \ell_\tau^{r_\tau+1} + q_\tau \ell_\tau^{r_\tau+1}\big)$ with $h_\tau \in \mathfrak{m}^{s_\gamma - r_\tau}$ and either $q_\tau = 0$ or $q_\tau \notin \mathfrak{m}^{s_\gamma - r_\tau}$; but then $f \in \mathfrak{m}^{s_\gamma+1}$ implies $\sum_{\tau\ni\gamma} q_\tau \ell_\tau^{r_\tau+1} = 0$. In particular, for each $\tau \ni \gamma$ we have $g_\tau \ell_\tau^{r_\tau+1} \in \mathfrak{m}^{s_\gamma+1} \cap \langle \ell_\tau^{r_\tau+1}\rangle$. It implies $f \in \sum_{\tau\ni\gamma}\big(\mathfrak{m}^{s_\gamma+1} \cap \langle \ell_\tau^{r_\tau+1}\rangle\big)$.

Conversely, if $f = \sum_{\tau\ni\gamma} f_\tau$, with $f_\tau \in \mathfrak{m}^{s_\gamma+1} \cap \langle \ell_\tau^{r_\tau+1}\rangle$, then $f_\tau \in \mathfrak{m}^{s_\gamma+1}$ for each edge $\tau \ni \gamma$, and so $f \in \mathfrak{m}^{s_\gamma+1} \cap \sum_{\tau\ni\gamma}\langle \ell_\tau^{r_\tau+1}\rangle$. So the containment follows, and this proves the equality. $\square$

Corollary 4.4. *Let $\Delta \subseteq \mathbb{R}^2$, $\gamma \in \Delta_0^\circ$, and take ℓ_τ and $\mathfrak{m}$ as in Lemma 4.3. Define $E_\gamma = \{(\tau, r_\tau) \,:\, \gamma \in \tau\}$ and let $\overline{E}_\gamma \subset E_\gamma$ be the largest subset such that all linear forms ℓ_τ associated to $(\tau,\cdot) \in \overline{E}_\gamma$ are distinct, and such that $(\tau, r_\tau) \in \overline{E}_\gamma$ implies that $r_\tau \leqslant r_{\tau'}$ for any $(\tau', r_{\tau'}) \in E_\gamma$ if $\ell_{\tau'} = \ell_\tau$. Let t be the cardinality of $\overline{E}_\gamma$. Then,*

$$s_\gamma \geqslant \Omega - 1 = \left\lfloor \frac{\sum_{(\cdot,r_\tau)\in\overline{E}_\gamma} r_\tau}{t-1} \right\rfloor \quad\Longrightarrow\quad \overline{\mathtt{J}}(\gamma) = \sum_{\tau\ni\gamma}\big(\langle \ell_\tau^{r_\tau+1}\rangle \cap \mathfrak{m}^{s_\gamma+1}\big) = \mathfrak{m}^{s_\gamma+1}\,.$$

Proof. Assume, without loss of generality, that γ is at the origin of $\mathbb{R}^2$. Thus, the linear forms ℓ_τ are polynomials in two variables in $\check{\mathfrak{m}}$. By Lemma 4.3 we have $\overline{\mathtt{J}}(\gamma) = \mathfrak{m}^{s_\gamma+1} \cap \langle \ell_\tau^{r_\tau+1} \colon \tau \ni \gamma, \tau \in \Delta_1^\circ\rangle$. If $\mathtt{I}(\gamma) = \big\langle \ell_\tau^{r_\tau+1} \colon \tau \ni \gamma, \tau \in \Delta_1^\circ \big\rangle \subseteq \mathbb{R}[x,y]$, then we can write $\overline{\mathtt{J}}(\gamma) = \big(\check{\mathfrak{m}}^{s_\gamma+1} \cap \mathtt{I}(\gamma)\big) \otimes_{\mathbb{R}} \mathbb{R}[z]$. By [15, Theorem 2.6], the socle degree of $\mathbb{R}[x,y]/\mathtt{I}(\gamma)$ is $\Omega - 1 = \left\lfloor \frac{\sum_{(\cdot,r_\tau)\in\overline{E}_\gamma} r_\tau}{t-1} \right\rfloor$. Thus, if $s_\gamma \geqslant \Omega - 1$ then $\check{\mathfrak{m}}^{s_\gamma+1} \subseteq \mathtt{I}(\gamma)$, and so $\overline{\mathtt{J}}(\gamma) = \mathfrak{m}^{s_\gamma+1}$. $\square$

We now compute the dimension of the ideal $\overline{\mathtt{J}}(\gamma)$ in (12) at degree d for any $d \geqslant 0$.

Lemma 4.5. *Following the notation in Corollary 4.4, take $d > s_\gamma$. If $s_\gamma < \Omega - 1$, then*

$$\dim(\mathtt{S}/\overline{\mathtt{J}}(\gamma))_d = \binom{d+2}{2} - \sum_{(\cdot,r_\tau)\in\overline{E}_\gamma} \frac{1}{2}(d - s_\gamma)(d + s_\gamma - 2r_\tau + 1) + b\binom{d+2-\Omega}{2} + a\binom{d+1-\Omega}{2},$$

where

$$a = \sum_{(\cdot,r_\tau)\in\overline{E}_\gamma} (r_\tau + 1) + (1-t)\Omega, \ \text{and} \ b = t - a - 1.$$

If $s_\gamma \geqslant \Omega - 1$, then $\dim(\mathtt{S}/\overline{\mathtt{J}}(\gamma))_d = \binom{s_\gamma+2}{2}$. In the case $0 \leqslant d \leqslant s_\gamma$, we have

$$\dim(\mathtt{S}/\overline{\mathtt{J}}(\gamma))_d = \binom{d+2}{2}.$$

Proof. If $d \leqslant s_\gamma$, then the ideal $\overline{\mathtt{J}}(\gamma)$ is trivial and the dimension is $\dim \mathtt{S}_d$. Therefore, let $d > s_\gamma$ and assume without loss of generality that γ is at the origin of $\mathbb{R}^2$. As in proof of Corollary 4.8, take $\mathtt{I}(\gamma) = \left\langle \ell_\tau^{r_\tau+1} \colon \tau \ni \gamma, \tau \in \Delta_1^\circ \right\rangle \subseteq \mathbb{R}[x,y]$, then $\check{\mathfrak{m}} = \langle x, y\rangle$ and $\mathtt{I}(\gamma) \subseteq \check{\mathfrak{m}}$. Since, $\overline{\mathtt{J}}(\gamma) = \left(\mathtt{I}(\gamma) \cap \check{\mathfrak{m}}^{s_\gamma+1}\right) \otimes_{\mathbb{R}} \mathbb{R}[z]$, then $\dim(\mathtt{S}/\overline{\mathtt{J}}(\gamma))_d = \dim \mathtt{S}_d - \sum_{k=s_\gamma+1}^{d} \dim \mathtt{I}(\gamma)_k = \dim \mathtt{S}_d - \dim(\mathtt{I}(\gamma) \otimes_{\mathbb{R}} \mathbb{R}[z])_d + \dim(\mathtt{I}(\gamma) \otimes_{\mathbb{R}} \mathbb{R}[z])_{s_\gamma}$. From [15, Theorem 2.7] we know the dimension of $\mathtt{I}(\gamma)_k$ for any $k \geqslant \max\{r_\tau \colon \tau \ni \gamma\}$, and we have

$$\dim\left(\mathtt{S}/\mathtt{I}(\gamma) \otimes_{\mathbb{R}} \mathbb{R}[z]\right)_k = \binom{k+2}{2} - \sum_{(\cdot, r_\tau)\in \overline{E}_\gamma} \binom{k - r_\tau + 1}{2} + b\binom{k+2-\Omega}{2} + a\binom{k-\Omega+1}{2}. \tag{13}$$

The statement follows directly by applying (13) with $k = d$ and $k = s_\gamma$. Notice that if $s_\gamma = k < \Omega - 1$ the binomial coefficients $\binom{s_\gamma+2-\Omega}{2}$ and $\binom{s_\gamma+1-\Omega}{2}$ in (13) vanish; if $s_\gamma \geqslant \Omega - 1$, by Corollary 4.4 we have $\overline{\mathtt{J}}(\gamma) = \mathfrak{m}^{s_\gamma+1}$, and $\dim\left(\mathfrak{m}^{s_\gamma+1}\right)_d = \binom{d+2}{2} - \binom{s_\gamma+2}{2}$. □

The following lemma relates dimension of the ideals $\overline{\mathtt{J}}(\gamma)$ and $\mathtt{J}(\gamma)$ in degree d. We show this result following the ideas in the proof of [26, Lemma 3.2]. Recall that the *link* of a vertex γ in $\Delta \subseteq \mathbb{R}^2$, denoted $\mathrm{Lk}(\gamma)$, is the set of all edges (and their vertices) in $\mathrm{star}(\gamma)$ which do not contain γ.

Lemma 4.6. *Let $\Delta \subseteq \mathbb{R}^2$, and $\gamma \in \Delta_0^\circ$. Then, $\dim \mathtt{J}(\gamma)_d \leqslant \dim \overline{\mathtt{J}}(\gamma)_d$ for every $d \geqslant 0$, equality holds if $d \gg 0$.*

Proof. If $\gamma \in \Delta_0^\circ$, the ideal $\mathtt{J}(\gamma)$ in (11) can be written as

$$\mathtt{J}(\gamma) = \sum_{\tau=[\gamma,\nu]} \langle \ell_\tau^{r_\tau+1}\rangle \cap \mathfrak{m}_\gamma^{s_\gamma+1} \cap \mathfrak{m}_\nu^{s_\nu+1}, \tag{14}$$

where $\mathfrak{m}_\gamma$ and $\mathfrak{m}_\nu$ are the ideals in $\mathtt{S}$ of polynomials vanishing on $\hat{\gamma}$ and $\hat{\nu}$, respectively, for every vertex $\nu \in \mathrm{Lk}(\gamma)$. Then, clearly, for any set of non-negative integers $\boldsymbol{r} = \{r_\tau \colon \tau \in \mathrm{star}(\gamma)_1^\circ\}$ and $\boldsymbol{s} = \{s_\gamma \colon \gamma \in \mathrm{star}(\gamma)_0^\circ\}$ we have $\mathtt{J}(\gamma) \subseteq \overline{\mathtt{J}}(\gamma)$ proving the first claim.

The second claim can be proved by showing that $\langle x, y, z\rangle^N$ annihilates $\overline{\mathtt{J}}(\gamma)/\mathtt{J}(\gamma)$ for a large enough N. Let $s = \max\{s_\nu, r_\tau \colon \nu, \tau \in \mathrm{star}(\gamma)\}$ and let $\tau' = [\gamma, \gamma']$ and $\tau'' = [\gamma, \gamma'']$ be two edges with distinct slope that contain the vertex γ. Take $p = \prod_{\tau \in \mathrm{Lk}(\gamma)} \ell_\tau$, where ℓ_τ denotes a choice of a linear form in $\mathtt{S}$ vanishing on $\hat{\tau}$. Since $\ell_{\tau'} \in \mathfrak{m}_\gamma \cap \mathfrak{m}_{\gamma'}$, $\ell_{\tau''} \in \mathfrak{m}_\gamma \cap \mathfrak{m}_{\gamma''}$, and $p \in \mathfrak{m}_\nu$ for any $\nu \in \mathrm{Lk}(\gamma)$, then for any $f \in \overline{\mathtt{J}}(\gamma)$ we have

$$\ell_{\tau'}^{s+1} f,\ \ell_{\tau''}^{s+1} f,\ p^{s+1} f \in \mathtt{J}(\gamma)\,.$$

But $\langle x, y, z\rangle^N \subseteq \left\langle \ell_{\tau'}^{s+1}, \ell_{\tau''}^{s+1}, p^{s+1}\right\rangle$ for some $N \gg 0$, and the claim follows. □

4.2. Supersplines on vertex stars

We devote this section to triangulations $\Delta \subseteq \mathbb{R}^2$ which are the star of a vertex i.e., all the triangles $\sigma \in \Delta$ share a common vertex γ. In this case, we write $\Delta = \mathrm{star}(\gamma)$ and say that Δ is a vertex star, or the star of the vertex γ.

Definition 4.1. Let $\Delta \subseteq \mathbb{R}^2$ be the star of the vertex γ, with interior edges $\tau \in \Delta_1^\circ$, and take integers $0 \leqslant r_\tau \leqslant d$, and $s_\gamma \geqslant \max\{r_\tau : \tau \in \Delta_1^\circ\}$. We write $\boldsymbol{r} = \{r_\tau : \tau \in \Delta_1^\circ\}$ and define $\mathcal{S}_d^{\boldsymbol{r}, s_\gamma}(\Delta^\circ)$ as the set of splines of degree at most d on a vertex star $\Delta = \mathrm{star}(\gamma)$ with smoothness r_τ across the edge τ, and supersmoothness s_γ at the vertex γ.

In terms of Definition 2.3, we have $\mathcal{S}_d^{\boldsymbol{r}, s_\gamma}(\Delta^\circ) = \mathcal{S}_d^{\boldsymbol{r}, \boldsymbol{s}}(\Delta)$ where $\boldsymbol{s}$ assigns supersmoothness s_γ to γ and smoothness $s_{\gamma'} = r_\tau$ to γ' for $\tau = [\gamma, \gamma']$ and $\gamma' \in \partial\Delta$. In particular, the ideal $\mathsf{J}(\tau) = \langle \ell_\tau^{r_\tau} + 1\rangle \cap \mathfrak{m}_\gamma^{s_\gamma+1} \cap \mathfrak{m}_{\gamma'}^{r_\tau+1} = \langle \ell_\tau^{r_\tau} + 1\rangle \cap \mathfrak{m}_\gamma^{s_\gamma+1}$, as in Remark 4.1, Equation (10).

Theorem 4.7. *Let $\Delta = \mathrm{star}(\gamma) \subseteq \mathbb{R}^2$ for an interior vertex γ and interior edges $\tau \in \Delta_1^\circ$. If $0 \leqslant r_\tau \leqslant d$ and $s_\gamma \geqslant \max\{r_\tau : \tau \in \Delta_1^\circ\}$ are integers, then*

$$\dim \mathcal{S}_d^{\boldsymbol{r}, s_\gamma}(\Delta^\circ) = \binom{d+2}{2} + \sum_{\tau \in \Delta_1^\circ} \left[(s_\gamma - r_\tau + 1)\binom{d - s_\gamma + 1}{2} - (s_\gamma - r_\tau)\binom{d - s_\gamma}{2} \right] - \dim \mathsf{J}(\gamma)_d,$$

where $\dim \mathsf{J}(\gamma)_d$ is given by the formula in Lemma 4.5.

Proof. Put $\mathsf{J}(\tau) = \langle \ell_\tau^{r_\tau+1}\rangle \cap \mathfrak{m}^{s_\gamma+1}$ and $\mathsf{J}(\gamma) = \sum_{\tau \in \Delta_1^\circ} \mathsf{J}(\tau)_d$. Consider the complex

$$0 \to \bigoplus_{\sigma \in \Delta_2} \mathsf{S} \xrightarrow{\overline{\partial}_2} \bigoplus_{\tau \in \Delta_1^\circ} \mathsf{S}/\mathsf{J}(\tau) \xrightarrow{\overline{\partial}_1} \mathsf{S}/\mathsf{J}(\gamma) \to 0\,. \tag{15}$$

Using similar arguments to those in Corollary 3.4 and Proposition 3.5, we get $\dim \mathcal{S}_d^{\boldsymbol{r}, s_\gamma}(\Delta^\circ) = \ker(\overline{\partial}_2)_d$. The Euler–Poincaré characteristic of the complex (15) leads to $\dim \mathcal{S}_d^{\boldsymbol{r}, s_\gamma}(\Delta^\circ) = \binom{d+2}{2} + \dim \sum_{\tau \in \Delta_1^\circ} \mathsf{J}(\tau)_d - \dim \mathsf{J}(\gamma)_d$. Notice that in this case, $\mathsf{J}(\gamma) = \overline{\mathsf{J}}(\gamma)$ as defined in (12). Thus, the formula in the statement follows by applying Lemma 4.2 and Lemma 4.5 to the previous equality. $\square$

Corollary 4.8. *Let Δ be as in Theorem 4.7, and t be the number of edges with different slopes containing γ as a vertex. If $r_\tau = r$ for all $\tau \in \Delta_1^\circ$, and $d > s_\gamma \geqslant r + \lfloor \frac{r}{t-1} \rfloor$, then*

$$\dim \mathcal{S}_d^{r, s_\gamma}(\Delta^\circ) = f_1^\circ (s_\gamma - r + 1)\binom{d - s_\gamma + 1}{2} - f_1^\circ (s_\gamma - r)\binom{d - s_\gamma}{2} + \binom{s_\gamma + 2}{2}$$

$$= f_1^\circ(s_\gamma - r + 1)\left[\binom{d-r+1}{2} - \binom{s_\gamma - r + 1}{2}\right] + \binom{s_\gamma + 2}{2},$$

where f_1° is the number of interior edges of Δ; if $d \leqslant s_\gamma$ then $\dim \mathcal{S}_d^{r,s}(\Delta^\circ) = \binom{d+2}{2}$.

Proof. Following the notation in Corollary 4.8, if $r_\tau = r$ for all $\tau \in \Delta_1^\circ$, then $\Omega - 1 = \lfloor \frac{tr}{t-1} \rfloor = r + \lfloor \frac{r}{t-1} \rfloor$. The statement follows by Theorem 4.7 and the case $s_\gamma \geqslant \Omega - 1$ in Lemma 4.5. If $d \leqslant s_\gamma$ then the only splines in the space are those which are global polynomials of degree $\leqslant d$. $\square$

5. A lower bound on the dimension of superspline spaces on triangulations

Throughout this section we assume Δ is a pure and hereditary simplicial complex in $\mathbb{R}^2$ isomorphic to a disk.

Since $\dim H_0(\mathcal{J})_d \geqslant 0$ for any degree $d \geqslant 0$, then by Equation (8) for any choice of smoothness $\boldsymbol{r} = \{r_\tau : \tau \in \Delta_1^\circ\}$ and supersmoothness $\boldsymbol{s} = \{s_\gamma : \gamma \in \Delta_0\}$ we have

$$\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta) \geqslant \binom{d+2}{2} + \sum_{\tau \in \Delta_1^\circ} \dim \mathsf{J}(\tau)_d - \sum_{\gamma \in \Delta_0^\circ} \dim \mathsf{J}(\gamma)_d\,. \tag{16}$$

In fact, it can be shown that the homology module $H_0(\mathcal{J})$ has finite length, i.e., $H_0(\mathcal{J})_d = 0$ for degree $d \gg 0$. The proof of this result follows by a slight modification of the proof by Schenck and Stillman in [26, Lemma 3.2] which considered the case of splines with global uniform smoothness. We include here the proof of this result for completeness. First, we recall the following lemma.

Lemma 5.1 *([26, Lemma 3.3]). If $\Delta \subseteq \mathbb{R}^2$ is a triangulation, then there exists a total order $\succ$ on Δ_0 such that for every $\gamma \in \Delta_0^\circ$ there exist vertices γ' and γ'' adjacent to γ, with $\gamma \succ \gamma', \gamma''$ and such that the edges $\tau' = [\gamma, \gamma']$ and $\tau'' = [\gamma, \gamma'']$ have different slopes.*

Lemma 5.2. *Let $\Delta \subseteq \mathbb{R}^2$ and $\mathcal{J}$ be the complex of ideals associated to $\boldsymbol{r} = \{r_\tau : \tau \in \Delta_1^\circ\}$ and $\boldsymbol{s} = \{s_\gamma : \gamma \in \Delta_0\}$. Then, $H_0(\mathcal{J})_d = 0$ for all $d \gg 0$.*

Proof. We show the claim by proving that $H_0(\mathcal{J})$ has finite length. For a vertex $\gamma \in \Delta_0^\circ$ and $f \in \mathsf{J}(\gamma)$, we denote by $f[\gamma]$ the corresponding element in $H_0(\mathcal{J})$, where $\mathcal{J}$ is the complex of ideals defined in (4). If γ is a boundary vertex we write $f[\gamma] = 0$ for any $f \in \mathsf{S}$. We prove that $H_0(\mathcal{J})$ has finite length by showing that there exist a sufficiently large integer M such that $\langle x, y, z\rangle^M f[\gamma] = 0$ in $H_0(\mathcal{J})$ for all $\gamma \in \Delta_0^\circ$ and $f \in \mathsf{J}(\gamma)$.

First, fix a total ordering $\succ$ on the vertices of Δ as in Lemma 5.1, i.e., such that for each vertex $\gamma \in \Delta_0^\circ$ there are two edges $\tau' = [\gamma, \gamma']$ and $\tau'' = [\gamma, \gamma'']$ with different slopes such that each of the vertices γ' and γ'' is either on the boundary of Δ or is $\prec$ than γ. Take such a vertex $\gamma \in \Delta_0^\circ$, and suppose $\langle x, y, z\rangle^N g[\nu] = 0$ for all vertices $\nu \prec \gamma$ and $g \in \mathsf{J}(\nu)$, for some integer $N > 0$.

If $\tau' = [\gamma, \gamma']$, let $s = \max\{s_{\gamma'}, s_\gamma, r_{\tau'}\}$. By Equation (2), the ideal associated to the edge τ' is given by $\mathtt{J}(\tau') = \langle \ell_{\tau'}^{r_{\tau'}+1} \rangle \cap \mathfrak{m}_\gamma^{s_\gamma+1} \cap \mathfrak{m}_{\gamma'}^{s_{\gamma'}+1}$, where $\mathfrak{m}_\gamma$ and $\mathfrak{m}_{\gamma'}$ are the ideals of polynomials in $\mathtt{S}$ vanishing at $\hat{\gamma}$ and $\hat{\gamma}'$, respectively. In particular, $\ell_{\tau'}^{s+1} \in \mathtt{J}(\tau')$ and thus

$$\ell_{\tau'}^{s+1} f[\gamma] = \ell_{\tau'}^{s+1} f[\gamma'] \tag{17}$$

in $H_0(\mathcal{J})$. Since $\ell_{\tau'}$ is a linear form in $\mathtt{S}$ and $\gamma' \in \partial\Delta$ or $\gamma' \prec \gamma$, by Equation (17) it follows that $\ell_{\tau'}^N$ annihilates $f[\gamma]$. Similarly, some power of $\ell_{\tau''}$ annihilates $f[\gamma]$.

On the other hand, if $f \in \mathtt{J}(\gamma)$ is given by $f = \sum_{\tau \ni \gamma} f_\tau$ for $f_\tau \in \mathtt{J}(\tau)$, we have

$$f[\gamma] = \sum_{\tau = [\gamma, \theta(\tau)]} f_\tau[\theta(\tau)]\,,$$

in $H_0(\mathcal{J})$, where $\theta(\tau) \in \tau$ denotes the vertex adjacent to γ on the edge τ. For an edge $\tau \in \mathrm{Lk}(\gamma)$, denote by ℓ_τ a choice of a linear form vanishing on $\hat{\tau}$. Define

$$p = \prod_{\tau \in \mathrm{Lk}(\gamma)} \ell_\tau,$$

and $s' = \max\{s_\nu, r_\tau \colon \nu, \tau \in \mathrm{star}(\gamma)\}$. Then, by construction $p(\gamma) \neq 0$ and $p^{s'+1} f[\gamma] = p^{s'+1} f[\nu]$ in $H_0(\mathcal{J})$ for any vertex $\nu \in \mathrm{Lk}(\gamma)$. In particular, if $\nu = \theta(\tau') = \gamma'$ we have $p^N f[\gamma'] = 0$, and therefore $p^N f[\gamma] = 0$.

Hence, some power of $\ell_{\tau'}$, $\ell_{\tau''}$, and p annihilate $f[\gamma]$. But $\langle x, y, z\rangle^M \subseteq \langle \ell_{\tau'}^N, \ell_{\tau''}^N, p^N\rangle$ for a sufficiently large integer M, and thus it follows $\langle x, y, z\rangle^M f[\gamma] = 0$. $\square$

Lemma 5.2 and Equation (8) lead directly to the following theorem.

Theorem 5.3. *If $\Delta \subseteq \mathbb{R}^2$, and $d \gg 0$, equality holds in (16), i.e.,*

$$\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta) = \binom{d+2}{2} + \sum_{\tau \in \Delta_1^\circ} \dim \mathtt{J}(\tau)_d - \sum_{\gamma \in \Delta_0^\circ} \dim \mathtt{J}(\gamma)_d\,.$$

We now use the results on vertex stars in Section 4.2, and prove a lower bound formula on $\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$ for any $d \geqslant 0$. As before, for a vertex $\gamma \in \Delta_0^\circ$, we put

$$\overline{\mathtt{J}}(\gamma) = \sum_{\tau \ni \gamma} \langle \ell_\tau^{r_\tau+1} \rangle \cap \mathfrak{m}_\gamma^{s_\gamma+1},$$

where $\mathfrak{m}_\gamma \subseteq \mathtt{S}$ is the ideal of polynomials vanishing at $\hat{\gamma}$.

Theorem 5.4. *Let $\Delta \subseteq \mathbb{R}^2$ be a simplicial complex homeomorphic to a disk, then*

$$\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta) \geqslant \binom{d+2}{2} + \sum_{\tau \in \Delta_1^\circ} \dim \mathtt{J}(\tau)_d - \sum_{\gamma \in \Delta_0^\circ} \dim \overline{\mathtt{J}}(\gamma)_d\,, \tag{18}$$

for every $d \geqslant 0$, and equality holds if $d \gg 0$. The dimension of $\mathtt{J}(\tau)_d$ follows from Lemma 4.2 and that of $\overline{\mathtt{J}}(\gamma)_d$ follows from Lemma 4.5.

Proof. The claim follows from Theorem 5.3 and Lemma 4.6. □

In the examples in Section 6 we compare the lower bounds (16) and (18) for specific triangulations; we also consider the homology modules $H_0(\mathcal{J})$ and give their explicit description.

We briefly comment that an upper bound can be proved on $\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$ following a similar argument to that used in the case of splines $\mathcal{S}_d^r(\Delta)$ with global uniform smoothness r by Mourrain and Villamizar in [22]. Namely, we fix a numbering $\gamma_1, \dots, \gamma_{f_0^\circ}$ on the interior vertices of Δ. For each vertex γ_i, denote by $N(\gamma_i)$ the set of edges that connect γ_i to any of the first $i-1$ vertices in the list or to a vertex on the boundary, and define the ideal $\widetilde{\mathtt{J}}(\gamma_i) = \sum_{\tau \in N(\gamma_i)} \mathtt{J}(\tau)$.

Proposition 5.5. *The dimension of $\mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$ is bounded above by*

$$\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta) \leqslant \binom{d+2}{2} + \sum_{\tau \in \Delta_1^\circ} \dim \mathtt{J}(\tau)_d - \sum_{i=1}^{f_0^\circ} \dim \widetilde{\mathtt{J}}(\gamma_i)_d \,. \tag{19}$$

Proof. The argument used in [22, Theorem 2] is independent of the ideals $\mathtt{J}(\tau)$ associated to the edges $\tau \in \Delta_1^\circ$, and therefore it immediately leads to the upper bound in Equation (19). □

An explicit upper bound formula requires the computation of $\dim \widetilde{\mathtt{J}}(\gamma)$, but the following result follows immediately by comparing the lower and the upper bound in (16) and (19), respectively.

Corollary 5.6. *If $\Delta \subseteq \mathbb{R}^2$ is a simplicial complex homeomorphic to a disk such that $\dim \mathtt{J}(\gamma)_d = \dim \widetilde{\mathtt{J}}(\gamma)_d$ for all $\gamma \in \Delta_0^\circ$ then equality holds in (16). In particular, this implies that $H_0(\mathcal{J})_d = 0$.*

Example 1 *(Optimality of lower bounds).* We generate a random triangulation Δ, shown in Fig. 1, for $r = 2$ we consider the space $\mathcal{S}_d^{r,\boldsymbol{s}}(\Delta)$ of C^r-continuous splines on Δ with supersmoothness $\boldsymbol{s} = \{s_\gamma : \gamma \in \Delta_0^\circ\}$ with $s_\gamma \in \{2,3,4\}$. We compare the lower bound (18) in Theorem 5.4 with the exact dimension of $\mathcal{S}_d^{r,\boldsymbol{s}}(\Delta)$ which is a subspace of $\mathcal{S}_d^2(\Delta)$. In particular, we randomly assign supersmoothness $s_\gamma \in \{2,3,4\}$ to vertices $\gamma \in \Delta_0$. With reference to Fig. 1, the colored vertices correspond to $s_\gamma = 3$, the ones colored and encircled correspond to $s_\gamma = 4$, and the others correspond to $s_\gamma = 2$. As shown in Table 1, the explicit bound from Theorem 5.4 coincides with the lower bound in Equation (16) as well as the dimension of $\mathcal{S}_d^{r,\boldsymbol{s}}(\Delta)$ in large degree. In fact, in this case the equality between the dimension of the vertex ideals (12) and (14) in Lemma 4.5 holds for every $d \geqslant 6$.

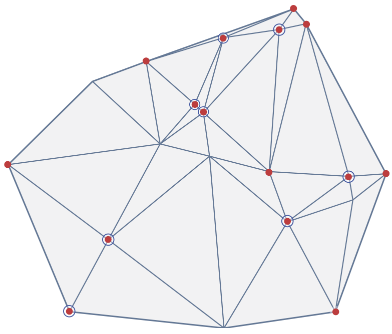

Fig. 1. A randomly generated triangulation, the smoothness across all edges is $r = 2$, and additional smoothness $s - r \in \{0, 1, 2\}$ is assigned randomly to all vertices. Above, $s_\gamma - r = 1$ for vertices with only a red disk on them, $s_\gamma - r = 2$ for vertices with an encircled red disk on them, and $s_\gamma - r = 0$ otherwise. The exact dimensions of $\mathcal{S}_d^{r,\boldsymbol{s}}$ and the lower bounds LB(16) and LB(18) are given in Table 1 for different choices of d. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Table 1
Lower bounds and dimension for the superspline space $\mathcal{S}_d^{2,\boldsymbol{s}}(\Delta)$ in Example 1, where Δ is the triangulation shown in Fig. 1. Here, LB(16) and LB(18) are the lower bounds from (16) and (18), respectively.

d	$\dim H_0(\mathcal{J})_d$	$\max\left(\binom{d+2}{2}, \mathrm{LB}(18)\right)$	$\max\left(\binom{d+2}{2}, \mathrm{LB}(16)\right)$	$\dim \mathcal{S}_d^{2,\boldsymbol{s}}(\Delta)$
4	4	15	15	15
5	0	30	31	31
6	0	108	108	108
7	0	223	223	223

6. Examples

6.1. Argyris superspline space

Let $\Delta \subseteq \mathbb{R}^2$ be a triangulation homeomorphic to a disk, and $r \geqslant 0$ an integer. In this example we compute the dimension of the superspline space $\mathcal{S}_{4r+1}^{r,2r}(\Delta)$. The particular case of $r = 1$ is called the Argyris element $\mathcal{S}_5^{1,2}(\Delta)$ and was introduced in the finite-element literature in [39,40]. A description of the Argyris space, and the general case $\mathcal{S}_{4r+1}^{r,2r}(\Delta)$ using Bernstein–Bézier techniques is included in [20, Chapter 6–8].

Following Definition 2.3, the space $\mathcal{S}_{4r+1}^{r,2r}(\Delta)$ corresponds to the set

$$\mathcal{S}_{4r+1}^{r,2r}(\Delta) = \left\{ f \in \mathcal{S}_{4r+1}^{r}(\Delta) \colon f \in C^{2r}(\gamma) \text{ for all } \ \gamma \in \Delta_0 \right\}.$$

If $\gamma \in \Delta_0^\circ$ is an interior vertex, then there are at least three edges having γ as one of their vertices, and at least two of them, say τ and τ', have different slopes. Let ℓ be a linear form vanishing on the plane containing $\hat{\gamma}$ and $\hat{\gamma}'$. After a suitable change of coordinates we can write

$$\begin{aligned}\mathsf{J}(\gamma) \supseteq \mathsf{J}(\tau) + \mathsf{J}(\tau') &= \left\langle \ell_\tau^{2r+1-i}\ell_{\tau'}^{i}\ell^{i} \ , \ \ell_{\tau'}^{2r+1-i}\ell_{\tau}^{i}\ell^{i} \colon 0 \leqslant i \leqslant r \right\rangle \\ &= \left\langle x^{2r+1-i}y^{i}z^{i} \ , \ x^{i}y^{2r+1-i}z^{i} \colon 0 \leqslant i \leqslant r \right\rangle.\end{aligned}$$

Then, every monomial $x^a y^b z^c \in \mathtt{S}$, with $a+b+c=4r+1$ and $0 \leqslant c \leqslant 2r$, is contained in $\mathtt{J}(\gamma)_{4r+1}$. Thus

$$\mathtt{J}(\gamma)_{4r+1} \cong \mathtt{S}_{4r+1}/\langle z^{2r+1}\rangle_{4r+1}, \tag{20}$$

and $\dim \mathtt{J}(\gamma)_{4r+1} = \binom{4r+3}{2} - \binom{2r+2}{2} = (2r+1)(3r+2)$.

By [26, Lemma 3.3] (see Lemma 5.1), we know that there exists a numbering of the vertices of Δ such that every interior vertex $\gamma \in \Delta_0^\circ$ is connected to two vertices with smaller index by edges which have distinct slopes. Taking such an ordering on the vertices of Δ, if $\gamma \in \Delta_0^\circ$, denote by $\widetilde{\mathtt{J}}(\gamma)$ the sum of ideals $\mathtt{J}(\tau)$ associated to the edges τ containing γ and whose other vertex is of smaller index than γ. Since the number of those edges with different slope is at least two, then $\widetilde{\mathtt{J}}(\gamma)_{4r+1} = \mathtt{J}(\gamma)_{4r+1}$. Thus, Corollary 5.6 implies $\dim H_0(\mathcal{J})_{4r+1} = 0$.

On the other hand, for any edge $\tau \in \Delta_1^\circ$, the edge ideal $\mathtt{J}$ can be written as $\mathtt{J}(\tau) = \langle x^{2r+1-i} y^i z^i \colon 0 \leqslant i \leqslant r\rangle$. Then $\dim \mathtt{J}(\tau)_{4r+1}$ is given in Lemma 4.2. Since Δ is homeomophic to a disk, then $f_2 - f_1^\circ - f_0^\circ = 1$, and applying the dimension formula (8), together with (9) and (20), we get

$$\begin{aligned}\dim \mathcal{S}_{4r+1}^{r,2r}(\Delta) &= \binom{4r+3}{2} + f_1^\circ (2r+1)^2 - f_1^\circ \binom{r+1}{2} - f_0^\circ (2r+1)(3r+2) \\ &= \binom{2r+2}{2} f_0 + \binom{r+1}{2} f_1 + \binom{r}{2} f_2 \,.\end{aligned} \tag{21}$$

The last equality follows by the Euler relation $3f_2 = f_1 + f_1^\circ$. A proof of (21) using Bernstein–Bézier methods is in [20, Theorem 8.1].

6.2. Intrinsic supersmoothness and degenerate spaces on vertex stars

Let us consider $\Delta = \operatorname{star}(\gamma) \subseteq \mathbb{R}^2$ be the star of the vertex γ. For any pair of integers $0 \leqslant r \leqslant d$ we have

$$\dim \mathcal{S}_d^r(\Delta) = \binom{d+2}{2} + (f_1^\circ - t)\binom{d-r+1}{2} + b\binom{d+2-\Omega}{2} + a\binom{d-\Omega+1}{2}, \tag{22}$$

where t is the number of different slopes of the edges containing γ, $\Omega = \lfloor \frac{tr}{t-1} \rfloor + 1$, $a = t(r+1) + (1-t)\Omega$, and $b = t - a - 1$.

The dimension formula (22) was proved by Schumaker [28]. The notation we use here follows the algebraic approach to prove this formula by Schenck and Stillman in [25] and Mourrain and Villamizar in [22].

Notice that for any $s_\gamma \geqslant r$, we have $\mathcal{S}_d^{r,s_\gamma}(\Delta) \subseteq \mathcal{S}_d^{r,s_\gamma}(\Delta^\circ) \subseteq \mathcal{S}_d^r(\Delta)$, where as before, $\mathcal{S}_d^{r,s_\gamma}(\Delta^\circ)$ is the set of C^r-splines on Δ with supersmoothness s_γ at γ. It is clear that the set $\mathcal{S}_d^{r,s_\gamma}(\Delta)$ contains all trivial splines, also called global polynomials, on Δ, i.e., the splines F on Δ whose restriction $F|_\sigma = f$ to each face $\sigma \in \Delta$ is the same polynomial

$f \in \mathtt{R}$. Therefore if $\dim \mathcal{S}^r_d(\Delta) = \binom{d+2}{2}$ then both $\mathcal{S}^{r,s_\gamma}_d(\Delta)$ and $\mathcal{S}^{r,s_\gamma}_d(\Delta^\circ)$ only contain trivial splines. From (22) it is easy to see that $\dim \mathcal{S}^r_d(\Delta) = \binom{d+2}{2}$ for all $d \leqslant \Omega$ when $f_1^\circ > t$, and for all $d \leqslant r$ in the generic case.

The dimension formula for supersplines spaces proved in Section 3 can be used to identify unexpected (also called intrinsic) supersmoothness in spaces of C^r-splines. For example, by computing the exact dimension of the spaces we can provide a short alternative proof of the result by Sorokina in [31, Theorem 3.1]. Namely, we will show that the C^r-splines on any generic vertex star all possess supersmoothness $\lfloor \frac{r+1}{t-1} \rfloor + r$ at the interior vertex.

Suppose $f_1^\circ = t$, and take $s_\gamma = \lfloor \frac{r+1}{t-1} \rfloor + r$. Following the notation in Equation (22), we have that $s_\gamma = \Omega$ if $\frac{r+1}{t-1} \in \mathbb{Z}$, and $s_\gamma = \Omega - 1$ otherwise. By Corollary 4.8, if $d > s_\gamma$ we get

$$\dim \mathcal{S}^{r,s_\gamma}_d(\Delta^\circ) = \begin{cases} t(\Omega - r + 1)\binom{d-\Omega+1}{2} - t(\Omega - r)\binom{d-\Omega}{2} + \binom{\Omega+2}{2}\,; & \text{if } \frac{r+1}{t-1} \in \mathbb{Z} \\ t(\Omega - r)\binom{d-\Omega+2}{2} - t(\Omega - 1 - r)\binom{d-\Omega+1}{2} + \binom{\Omega+1}{2}\,; & \text{otherwise}. \end{cases} \tag{23}$$

If $d \leqslant s_\gamma$, then $\dim \mathcal{S}^{r,s_\gamma}_d(\Delta^\circ) = \binom{d+2}{2}$.

On the other hand, if $\frac{r+1}{t-1} \in \mathbb{Z}$ we have $a = t - 1$, $b = 0$, and Equation (22) leads to

$$\dim \mathcal{S}^r_d(\Delta) = \begin{cases} \binom{d+2}{2} + (t-1)\binom{d-\Omega+1}{2}\,; & \text{if } \frac{r+1}{t-1} \in \mathbb{Z} \\ \binom{d+2}{2} + (t-a-1)\binom{d+2-\Omega}{2} + a\binom{d-\Omega+1}{2}\,; & \text{otherwise}. \end{cases} \tag{24}$$

A straightforward computation shows that $\dim \mathcal{S}^r_d(\Delta) = \mathcal{S}^{r,s_\gamma}_d(\Delta^\circ)$ in both cases in (23) and (24), and also when $d \leqslant s_\gamma$.

Similarly, we can show that for vertex stars $\mathcal{S}^{r,s}(\Delta^\circ) = \mathcal{S}^r(\Delta)$ if and only if $\dim \mathcal{S}^r_s(\Delta) = \binom{s+2}{2}$. This criterion corresponds to the planar case of the result proved by Floater and Hu in [14, Theorem 1] for vertex stars in $\mathbb{R}^n$, $n \geqslant 2$; they call such trivial spline spaces *degenerated*.

If we assume that $\mathcal{S}^r(\Delta) \subseteq \mathcal{S}^{r,s_\gamma}(\Delta^\circ)$, by Theorem 4.7 we know that $\dim \mathcal{S}^{r,s_\gamma}_{s_\gamma}(\Delta^\circ) = \binom{s_\gamma+2}{2}$. Then $\mathcal{S}^r_{s_\gamma}(\Delta)$ is also degenerated. Conversely, if $\dim \mathcal{S}^r_{s_\gamma}(\Delta) = \binom{s_\gamma+2}{2}$ for $0 \leqslant r < s_\gamma$, then by (22) we have

$$\dim \mathcal{S}^r_{s_\gamma}(\Delta) = \binom{s_\gamma + 2}{2} + (f_1^\circ - t)\binom{s_\gamma - r + 1}{2} + b\binom{s_\gamma + 2 - \Omega}{2} + a\binom{s_\gamma - \Omega + 1}{2},$$

and this implies that the triangulation is generic i.e., $f_1^\circ = t$, and that $s_\gamma + 2 - \Omega \leqslant 1$, or $s_\gamma + 1 - \Omega \leqslant 1$ and $b = 0$.

First, suppose that $s_\gamma + 2 - \Omega \leqslant 1$. It follows $\Omega \geqslant s_\gamma + 1$, which is equivalent to say that the generators of the module of syzygies of the forms $\{\ell_\tau^{r+1} : \tau \in \Delta_1^\circ\}$ have degree strictly greater than $s_\gamma - (r+1)$. If we assume γ is at the origin then the linear forms $\ell_\tau \in \mathtt{S}$, and therefore the generators of their module of syzygies, only involve the variables x, y.

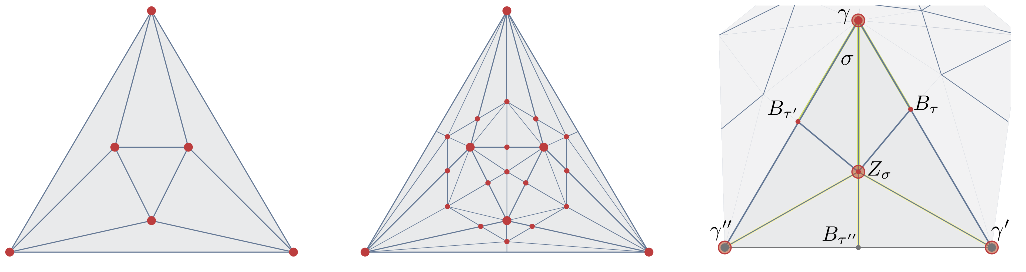

Fig. 2. Symmetric Morgan–Scott triangulation (left), and the corresponding Powell–Sabin 6-split applied to each triangle of this triangulation (center). The notation in the 6-split (right) is used in Example 6.3; in this case, the vertices γ' and γ'' are assumed to be on the boundary. The smoothness across the edges $[Z_\sigma, B_\tau]$ and $[Z_\sigma, B_{\tau'}]$ and at the vertices γ, γ' and γ'' is $s \geqslant r$.

As a graded module over $\mathtt{S} = \mathbb{R}[x, y, z]$, the set $\mathcal{S}^r(\Delta)$ is generated by trivial splines and splines of the form $G = (g_1\ell_1^{r+1}, g_1\ell_1^{r+1} + g_2\ell_2^{r+1}, \ldots, g_1\ell_1^{r+1} + \cdots + g_t\ell_t^{r+1})$, where $g_1\ell_1^{r+1} + \cdots + g_t\ell_t^{r+1} = 0$ is a syzygy of the forms $\{\ell_\tau^{r+1} : \tau \in \Delta_1^\circ\}$. (An introduction to splines as modules over a ring can be found in [8, Chapter 8].) Since all the polynomials g_i are homogeneous in x, y of degree strictly greater than $s_\gamma - (r+1)$, then each polynomial (piece) $g_i\ell_i^{r+1}$ is zero up to order s_γ at γ. Hence $G \in \mathcal{S}^{s_\gamma}(\gamma)$, which implies that every spline in $\mathcal{S}^r(\Delta)$ is in $\mathcal{S}^{s_\gamma}(\gamma)$.

Alternatively, if $b = 0$, then the smallest degree of a syzygy is $\Omega+1$. The condition $\Omega \geqslant s_\gamma$ implies $\deg(g_i) \geqslant s_\gamma+1-(r+1)$, hence also in this case $\mathcal{S}^r(\Delta) \subseteq C^{s_\gamma}(\gamma)$ and it follows that $\mathcal{S}^r(\Delta) \subseteq \mathcal{S}^{r,s_\gamma}(\Delta^\circ)$. Therefore $\mathcal{S}^r(\Delta) \subseteq \mathcal{S}^{r,s_\gamma}(\Delta^\circ)$ if and only if $\mathcal{S}^r_{s_\gamma}(\Delta)$ contains only trivial splines. In particular, this criterion combined with the result by Sorokina [31, Theorem 3.1] implies that $s_\gamma = \lfloor \frac{r+1}{t-1} \rfloor + r$ is the largest order of supersmoothness such that $\mathcal{S}^r(\Delta) \subseteq \mathcal{S}^{r,s_\gamma}(\Delta^\circ)$.

6.3. *Supersmooth splines on Powell–Sabin 6-split refinements*

Let $\Delta \subseteq \mathbb{R}^2$ be a triangulation, and let $\Delta^\star$ be a triangulation obtained from Δ via a Powell–Sabin six split. Namely, we choose a point Z_σ in the interior of each triangle $\sigma \in \Delta$ so that if two triangles $\sigma, \sigma' \in \Delta$ share a common edge $\tau = \sigma \cap \sigma'$, then the line joining Z_σ and $Z_{\sigma'}$ intersects τ at a point B_τ that lies at the interior of τ. If $\tau \in \Delta_1$ is an edge on the boundary, we choose an interior point on τ and denote it by B_τ. The set of vertices Δ_0 of Δ together with the points Z_σ and B_τ, for all $\sigma \in \Delta_2$ and $\tau \in \Delta_1$, are the vertices of the new triangulation $\Delta^\star$. If $\sigma \in \Delta_2$ is a triangle of Δ, we join Z_σ to each vertex of σ, and to each vertex B_τ on the edges $\tau \in \sigma$. Thus, the Powell–Sabin triangulation $\Delta^\star$ is a refinement of Δ, where each triangle in Δ has been subdivided into six smaller triangles. An example of a partition along with its Powell–Sabin 6-split is in Fig. 2.

In the following, given integers $r \geqslant 0$, $s \geqslant \max\{r, 2r-1\}$ and $d \geqslant 2s - r + 1$, we compute $\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta^\star)$, where $\boldsymbol{r} = \{r_\tau : \tau \in (\Delta^\star)_1^\circ\}$ and $\boldsymbol{s} = \{s_\gamma : \gamma \in (\Delta^\star)_0\}$ are defined by

$$r_\tau = \begin{cases} s & \text{if } \tau = [Z_\sigma, B_\beta] \text{ for some } \sigma \in \Delta_2, \text{and } \sigma \supseteq \beta \in \Delta_1\,, \\ r & \text{otherwise,} \end{cases}$$

$$s_\gamma = \begin{cases} s & \text{if } \gamma \in \Delta_0 \cup \{Z_\sigma \colon \sigma \in \Delta_2\}\,, \\ r & \text{if } \gamma \in \{B_\tau \colon \tau \in \Delta_1^\circ\}\,. \end{cases}$$

The specific choice $r \geqslant 0$, $s = 2r-1$ and $d = 3r-1$ is studied by Speleers in [34] using Bernstein–Bézier methods.

In our setting, from the dimension formula in Equation (8) we get

$$\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta^\star) = \binom{d+2}{2} + \sum_{\tau \in (\Delta^\star)_1^\circ} \dim \mathtt{J}(\tau)_d - \sum_{\gamma \in (\Delta^\star)_0^\circ} \dim \mathtt{J}(\gamma)_d + \dim H_0(\mathcal{J})_d\,, \quad (25)$$

where the ideal $\mathtt{J}(\tau)$, for each $\tau \in (\Delta^\star)_1^\circ$, is defined by

$$\mathtt{J}(\tau) = \begin{cases} \langle \ell_\tau^{s+1} \rangle & \text{if } \tau = [Z_\sigma, B_\beta] \text{ for } \sigma \in \Delta_2, \text{and } \beta \in \Delta_1, \\ \langle \ell_\tau^{r+1} \rangle \cap \mathfrak{m}_\gamma^{s+1} \cap \mathfrak{m}_{Z_\sigma}^{s+1} & \text{if } \tau = [Z_\sigma, \gamma] \text{ for } \sigma \in \Delta_2, \text{and } \gamma \in \Delta_0, \\ \langle \ell_\tau^{r+1} \rangle \cap \mathfrak{m}_\gamma^{s+1} & \text{if } \tau = [B_\beta, \gamma] \text{ for } \beta \in \Delta_1^\circ, \text{and } \gamma \in \Delta_0\,, \end{cases} \quad (26)$$

and $\mathtt{J}(\gamma) = \sum_{\tau \in \Delta_1^\circ, \gamma \in \tau} \mathtt{J}(\tau)$, for each vertex $\gamma \in (\Delta^\star)_0^\circ$. Here, as before, if $\tau \in \Delta_1^\star$ and $\gamma \in \Delta_0^\star$, then ℓ_τ is a linear form vanishing on $\hat{\tau}$, and $\mathfrak{m}_\gamma$ is the ideal of all polynomials in $\mathtt{S}$ vanishing at $\hat{\gamma}$.

Notice that for the ideals $\mathtt{J}(\tau)$ in (26), we have $\dim \mathtt{J}(\tau) = \binom{d-s+1}{2}$ if $\tau = [Z_\sigma, B_\beta]$, and $\dim \mathtt{J}(\tau)$ in the other two cases follows directly from Equations (9) and (10), respectively. The dimension of the ideal $\mathtt{J}(\gamma)$ associated to the vertices can be computed as follows. We consider the three types of vertices separately. Thereafter, we show that $H_0(\mathcal{J})_d = 0$ for every polynomial degree $d \geqslant 2s - r + 1$.

Case 1. We show that $\dim \mathtt{J}(Z_\sigma)_d = \binom{d+2}{2} - \binom{s+2}{2}$ for every $d \geqslant 2s - r + 1$, $r \geqslant 0$, and $s \geqslant \max\{r, 2r-1\}$. By construction, $\mathtt{J}(Z_\sigma)$ is the sum of three ideals of the form $\langle \ell_\tau^{s+1} \colon \tau = [Z_\sigma, B_\beta] \rangle \subseteq \mathfrak{m}_{Z_\sigma}^{s+1}$ where B_β is the vertex on the edge $\beta \subseteq \sigma$, and three ideals of the form $\langle \ell_\tau^{r+1} \rangle \cap \mathfrak{m}_{Z_\sigma}^{s+1} \cap \mathfrak{m}_\nu^{s+1}$ for the edges $\tau = [Z_\sigma, \nu]$ for vertices $\nu \in \sigma$, $\nu \in \Delta_0$. Then, in particular $\mathtt{J}(Z_\sigma) \subseteq \mathfrak{m}_{Z_\sigma}^{s+1}$.

By a change of coordinates, we may assume $\ell_{[Z_\sigma,\gamma]} = x$, $\ell_{[Z_\sigma,\gamma']} = y$, and $\mathfrak{m}_\gamma = \langle x, z \rangle$. Then, $\mathfrak{m}_{Z_\sigma} = \langle x, y \rangle$ and $\mathfrak{m}_{\gamma'} = \langle y, z \rangle$. We want to show that $x^i y^j z^k \in \mathtt{J}(Z_\sigma)$ for all monomials of degree $d = i + j + k$ for $d \geqslant 2s - r + 1$, such that $i + j = s + 1$.

Since

$$\big\langle \ell_{[Z_\sigma,\gamma]}^{r+1} \big\rangle \cap \mathfrak{m}_{Z_\sigma}^{s+1} \cap \mathfrak{m}_\gamma^{s+1} + \big\langle \ell_{[Z_\sigma,\gamma']}^{r+1} \big\rangle \cap \mathfrak{m}_{Z_\sigma}^{s+1} \cap \mathfrak{m}_{\gamma'}^{s+1} \subseteq \mathtt{J}(Z_\sigma)\,,$$

then $x^{s+1-i} y^i z^i$ and $y^{s+1-i} x^i z^i$ are elements in $\mathtt{J}(Z_\sigma)$, for all $i = 0, \ldots, s - r$. Thus, if $s \geqslant 2r - 1$ this implies that $x^i y^j z^k \in \mathtt{J}(Z_\sigma)$ for all $i + j = s + 1$ in degree $d \geqslant 2s - r + 1$, except for $x^r y^r z^k$ when $s = 2r - 1$. But in the latter case, since $\ell_{[Z_\sigma, B_{\tau''}]}^{s+1} \in \mathtt{J}(Z_\sigma)$ with $s+1 = 2r$ and $\ell_{[Z_\sigma, B_{\tau''}]} \in \mathfrak{m}_{Z_\sigma} = \langle x, y\rangle$, it follows $x^r y^r z^k \in \mathtt{J}(Z_\gamma)$. Consequently, $(\mathfrak{m}_{Z_\sigma}^{s+1})_d \subseteq \mathtt{J}(Z_\sigma)_d$ and the dimension formula follows.

Case 2. Let $\gamma \in \Delta_0^\circ$. Similarly as in Case 1, we have $\dim \mathtt{J}(\gamma)_d = \binom{d+2}{2} - \binom{s+2}{2}$ for $d \geqslant 2s - r + 1$. Indeed, the ideal $\mathtt{J}(\gamma)$ is the sum of at least two ideals of the form $\langle \ell_\tau^{r+1}\rangle \cap \mathfrak{m}_\gamma^{s+1}$ and three of the form $\langle \ell_\tau^{r+1}\rangle \cap \mathfrak{m}_\gamma^{s+1} \cap \mathfrak{m}_{Z_\sigma}^{s+1}$, for at least three linearly independent forms ℓ_τ, for faces $\sigma \in \Delta$ and $\tau \in \Delta_1^0$ containing γ. Then, also in this case $\mathtt{J}(\gamma) \subseteq \mathfrak{m}_\gamma^{s+1}$, and the argument used in Case 1 leads to the dimension formula for $\mathtt{J}(Z_\sigma)_d$.

Case 3. Let B_τ be the vertex on the (interior of the) edge $\tau \in \Delta_1^\circ$. The ideal $\mathtt{J}(B_\tau)$ is generated by the sum of four ideals, two of the form $\langle \ell_\tau^{s+1}\rangle \cap \mathfrak{m}_{Z_\sigma}^{s+1} = \langle \ell_\tau^{s+1}\rangle$, for $\tau = [B_\tau, Z_\sigma]$, and two of the form $\langle \ell_\tau^{r+1}\rangle \cap \mathfrak{m}_\gamma^{s+1}$, for $\tau = [B_\tau, \gamma]$. By a change of coordinates we may assume that $\ell_{[B_\tau,\gamma]} = x$, $\ell_{[B_\tau, Z_\sigma]} = y$ and $\ell_{[Z_\sigma, \gamma']} = z$. Then,

$$\mathtt{J}(B_\tau) = \langle y^{s+1}, x^{s+1-i}(y+ax)^i, x^{s+1-i}z^i \colon 0 \leqslant i \leqslant s-r\rangle\,, \tag{27}$$

for some $a \in \mathbb{R}$. We use the following lemma to compute the dimension of this ideal in degree $d \geqslant 2s - r + 1$.

Lemma 6.1. *Let $\mathtt{J}(B_\tau)$ be the ideal in (27) and $d \geqslant 2s - r + 1$. Then*

$$\mathtt{J}(B_\tau) = \langle y^{s+1}, x^{s+1-i}y^i, x^{s+1-i}z^i \colon 0 \leqslant i \leqslant s-r\rangle\,,$$

and

$$\dim \mathtt{J}(B_\tau)_d = \binom{d+1-r}{2} + \binom{d+1-s}{2} - \binom{d-s-r}{2}. \tag{28}$$

Proof. Let $\mathtt{J} = \langle y^{s+1}, x^{s+1-i}y^i, x^{s+1-i}z^i \colon 0 \leqslant i \leqslant s-r\rangle$. It is clear that $\mathtt{J}(B_\tau) \subseteq \mathtt{J}$. By induction we show that $x^{s+1-i}y^i \in J(B_\tau)$ for all $i = 0, \dots, s-r$. In fact, if $i = 0$ then $x^{s+1} \in \mathtt{J}(B_\tau)$, also if $i = 1$ we have $x^s(y+ax) \in \mathtt{J}(B_\tau)$, but $x^{s+1} \in \mathtt{J}(B_\tau)$ so $x^s y \in \mathtt{J}(B_\tau)$. Suppose $x^{s+1-i}y^i \in \mathtt{J}(B_\tau)$ for every $0 \leqslant i \leqslant k < s-r$, then $x^{s-k}(y+ax)^{k+1} \in \mathtt{J}(B_\tau)$, and by induction hypothesis we easily see that $x^{s-k}y^{k+1} \in \mathtt{J}(B_\tau)$. Hence $\mathtt{J} \subseteq \mathtt{J}(B_\tau)$, and so $\mathtt{J}(B_\tau) = \mathtt{J}$.

Take $d \geqslant 2s - r + 1$ and $x^{r+1}\big(x^a y^b z^c\big) \in \mathtt{S}_d$, for non-negative integers a, b, c. If $a + b \geqslant s - r$ then $x^{r+1}\big(x^a y^b z^c\big) \in \mathtt{S}_d$. Suppose $a + b < s - r$. Since $d \geqslant 2s - r + 1$ then $c = d - r + 1 - (r+1) - (s-r) \geqslant s - r$, and so $x^{r+1}\big(x^a y^b z^c\big) \in \mathtt{J}_d$. This shows that $\mathtt{J}_d = \langle x^{r+1}, y^{s+1}\rangle_d$ is a complete intersection, and therefore its dimension is given by (28). $\square$

Vanishing homology:

We now prove that $H_0(\mathcal{J})_d = 0$ for every $d \geqslant 2s - r + 1$. Recall that by [26, Lemma 3.3] (see Lemma 5.1) we can always choose a triangle in Δ with two vertices on the boundary. Let $\sigma \in \Delta$ be such a triangle, we denote its vertices as in Fig. 2 (right), with the edge $[\gamma', \gamma'']$ lying on the boundary of Δ.

First, for each interior vertex of $\Delta^\star$ contained in σ we select a subset of interior edges in $\Delta_1^\star$ such that the ideal of the vertex in degree d can be generated by the sum of these edge ideals in degree d, for every $d \geqslant 2s - r + 1$. Specifically, for the vertex Z_σ we take the three edges connecting Z_σ to the boundary, for γ we choose the edges $[B_\tau, \gamma]$, $[B_{\tau'}, \gamma]$ and $[Z_\sigma, \gamma]$, for B_τ we take the edges $[B_\tau, \gamma]$, $[B_\tau, \gamma']$, and $[Z_\sigma, B_\tau]$, and the three corresponding ones at $B_{\tau'}$. From Case 1, Case 2, and Case 3, we know that $\mathtt{J}(Z_\sigma)_d$, $\mathtt{J}(\gamma)_d$, $\mathtt{J}(B_\tau)_d$, and $\mathtt{J}(B_\tau)_d$ can be generated by the ideals of these edges at degree d, for every $d \geqslant 2s - r + 1$.

Denote by $\sigma' \in \Delta_2$ the triangle adjacent to σ such that $\sigma \cap \sigma' = \tau$, and $\tau = [\gamma, \gamma']$. Up to a change of coordinates, we may assume that $\ell_\tau = x$ and $\mathtt{J}(B_\tau)$ is the sum of the ideals

$$\begin{aligned}
\mathtt{J}([B_\tau, Z_\sigma]) = \mathtt{J}([B_\tau, Z_{\sigma'}]) &= \langle (y + az)^{s+1} \rangle, \text{ for some } a \in \mathbb{R},\, a \neq 0, \\
\mathtt{J}([B_\tau, \gamma]) &= \langle x^{s+1-i} y^i \colon 0 \leqslant i \leqslant s - r \rangle, \text{ and} \\
\mathtt{J}([B_\tau, \gamma']) &= \langle x^{s+1-i} z^i \colon 0 \leqslant i \leqslant s - r \rangle.
\end{aligned}$$

By Case 1 and Case 2, we know that $\mathtt{J}(Z_\sigma)_d = \mathfrak{m}_{Z_\sigma}^{s+1}$ and $\mathtt{J}(\gamma)_d = \mathfrak{m}_\gamma^{s+1}$, respectively. For our choice of coordinates, we have $\mathfrak{m}_{Z_\sigma} = \langle y, z \rangle$ and $\mathfrak{m}_\gamma = \langle x, y \rangle$. Also, $\mathtt{J}([Z_\sigma, \gamma]) = \langle y^{s+1-i} x^i z^i \colon 0 \leqslant i \leqslant s - r \rangle$, and $\mathtt{J}\big([B_{\tau'}, \gamma]\big) = \big\langle (x + by)^{r+1} \big\rangle \cap \mathfrak{m}_\gamma^{s+1}$, where $\ell_{\tau'} = x + by$, for some $b \in \mathbb{R}$, $b \neq 0$.

If $g \in \mathtt{J}(\beta)$ for some $\beta \in \big(\Delta^\star\big)_i^\circ$, we denote by $g[\beta]$ the element in $\bigoplus\{\mathtt{J}(\alpha)_d \colon \alpha \in (\Delta^\star)_i^\circ\}$ such that $g_\beta = g$, and $g_\alpha = 0$ for every $\alpha \neq \beta$. By an abuse of notation, for $i = 0, 1$ and $\beta \in \big(\Delta^\star\big)_i^\circ$, we will identify $\mathtt{J}(\beta)_d$ with the set $\big\{g[\beta] \colon g \in \mathtt{J}(\beta)_d\big\} \subseteq \bigoplus\big\{\mathtt{J}(\alpha)_d \colon \alpha \in (\Delta^\star)_i^\circ\big\}$.

Let ∂_1 be the boundary map in the complex $\mathcal{J}$ of $\Delta^\star$ i.e.,

$$\partial_1 \colon \bigoplus_{\tau \in (\Delta^\star)_1^\circ} \mathtt{J}(\tau) \to \bigoplus_{\gamma \in (\Delta^\star)_0^\circ} \mathtt{J}(\gamma)\,.$$

We need to show that $\big(\operatorname{im} \partial_1\big)_d = \bigoplus\big\{\mathtt{J}(\gamma)_d \colon \gamma \in (\Delta^\star)_0^\circ\big\}$, or equivalently, that $\mathtt{J}(\gamma)_d \subseteq \big(\operatorname{im} \partial_1\big)_d$ for every vertex $\gamma \in \big(\Delta^\star\big)^\circ$, and every $d \geqslant 2s - r + 1$. First, following the notation in Fig. 2, we will show this for the vertices in the triangle σ. Namely, the vertices γ, Z_σ, B_τ, and $B_{\tau'}$.

Take $g = x^i y^j z^k \in \mathtt{J}(\gamma)_d$, then by construction $i + j \geqslant s + 1$, and $i + j + k = d \geqslant 2s - r + 1$ implies $k \geqslant s - r$. Thus, if $i \geqslant r + 1$ then $g \in \mathtt{J}\big([B_\tau, \gamma]\big) \cap \mathtt{J}\big([B_\tau, \gamma']\big)$. Since $[B_\tau, \gamma]$ connects two interior vertices, then $\partial_1(g[B_\tau, \gamma]) = \pm g[\gamma] \mp g[B_\tau]$. The signs of $g[\gamma]$ and $g[B_\tau]$ depend on the orientation of $\Delta^\star$, but we can take either g or $-g$, so without loss of generality, here and henceforth, we may assume that $\partial_1\big(g[B_\tau,\gamma]\big) = g[\gamma] - g[B_\tau]$. On the other hand, the edge $[B_\tau,\gamma']$ connects B_τ to γ', and γ' is a vertex on the boundary of $\Delta^\star$. So $\partial_1\big(g[B_\tau,\gamma']\big)$ has only one non-zero component, namely $\partial_1\big(g[B_\tau,\gamma']\big) = g[B_\tau]$. Therefore, $\partial_1\big(g[B_\tau,\gamma] + g[B_\tau,\gamma']\big) = g[\gamma]$, which implies $g[\gamma] \in \mathrm{im}(\partial_1)$.

Suppose now $i \leqslant r-1$. Then $i+j+k \geqslant 2s-r+1$ and $s \geqslant 2r-1$ imply $j+k \geqslant s+1$. But by construction $i+j \geqslant s+1$, so $x^iy^jz^k \in \mathsf{J}\big(Z_\sigma,\gamma\big)_d \subseteq \mathsf{J}\big(Z_\sigma\big)_d$. Similarly, if $i=r$ and $s \geqslant 2r$, or $i=r$, $s=2r$ and $j \geqslant r+1$, we get $j+k \geqslant s+1$. The vertex Z_σ is connected to the boundary by three edges, and the generators of the correspondent ideals to these edges (by Case 1) generate $\mathsf{J}\big(Z_\sigma\big)_d$. In particular, there are polynomials $f \in \mathsf{J}\big([Z_\sigma,\gamma']\big)$, $h \in \mathsf{J}\big([Z_\sigma,\gamma'']\big)$, and $q \in \mathsf{J}\big([Z_\sigma,B_{\tau''}]\big)$, such that $f+h+q=g$. It follows,

$$\partial_1\big(f[Z_\sigma,\gamma'] + h[Z_\sigma,\gamma''] + q[Z_\sigma,B_{\tau''}]\big) = g[Z_\sigma],$$

and so

$$\partial_1\big(g[\gamma,Z_\sigma] + f[Z_\sigma,\gamma'] + h[Z_\sigma,\gamma''] + q[Z_\sigma,B_{\tau''}]\big) = g[\gamma].$$

Thus, also in this case $g = x^iy^jz^k \in \mathrm{im}(\partial_1)$.

There only remaining case to be considered is $g = x^ry^rz^{d-2r} \in \mathsf{J}(\gamma)$. This monomial is one of the terms in $f = \frac{1}{b}(x+by)^{r+1}y^{r-1}z^{d-2r} \in \mathsf{J}\big([B_{\tau'},\gamma]\big)$. For any of the other monomials $x^uy^vz^{d-2r}$ in f, either $u \geqslant r+1$ or $v \geqslant r+1$, so they are either in $\mathsf{J}\big([B_\tau,Z_\sigma]\big)$ or $\mathsf{J}\big([Z_\sigma,\gamma]\big)$. Collecting these monomials we get two polynomials $h \in \mathsf{J}([B_\tau,\gamma])$, and $q \in \mathsf{J}([Z_\sigma,\gamma])$. Up to taking the appropriate signs (either $-h$ or h, etc.), we get

$$\begin{aligned}\partial_1\big(f[B_{\tau'},\gamma] + h[Z_\sigma,\gamma] + q[B_\tau,\gamma]\big) &= f[\gamma] - h[\gamma] - q[\gamma] - f[B_{\tau'}] + h[Z_\sigma] + q[B_\tau]\\ &= g[\gamma] - f[B_{\tau'}] + h[Z_\sigma] + q[B_\tau].\end{aligned} \tag{29}$$

Similarly as above, notice that f is also a polynomial in $\mathsf{J}\big([B_{\tau'},\gamma'']\big)$, and γ'' is a vertex on the boundary of $\Delta^\star$. So, we can use $\partial_1\big(f[B_{\tau'},\gamma'']\big)$ to eliminate $f[B_{\tau'}]$ in (29). Moreover, there are polynomials in the ideals of the edges connecting Z_σ, and B_τ to the boundary which generate h and q, respectively. By applying ∂_1 to those polynomials (with the appropriate sign) we arrive to $g[\gamma] \in \mathrm{im}(\partial_1)_d$, as required.

This shows that $\mathsf{J}(\gamma)_d \subseteq \mathrm{im}(\partial_1)_d$, for every $d \geqslant 2s-r+1$, whenever $s \geqslant \max\{r, 2r-1\}$. In fact, we have shown that

$$\mathsf{J}(\gamma)_d \subseteq \partial_1\big(\oplus\{\mathsf{J}(\tau)\colon \tau \in \sigma \cap (\Delta^\star)^\circ_1\}\big)_d.$$

Furthermore, if $d \geqslant 2s-r+1$ and $s \geqslant \max\{r, 2r-1\}$, from Case 1 we get

$$\mathsf{J}(Z_\sigma)_d \subseteq \partial_1\big(\mathsf{J}([Z_\sigma,\gamma'']) \oplus \mathsf{J}([Z_\sigma,\gamma']) \oplus \mathsf{J}([Z_\sigma,B_{\tau''}])\big)_d,$$

and a similar argument as above together with Case 3 lead to

$$\mathsf{J}(B_\tau)_d \subseteq$$
$$\partial_1\big(\mathsf{J}([Z_\sigma, B_\tau]) \oplus \mathsf{J}\big([B_\tau, \gamma] \oplus \mathsf{J}([B_\tau, \gamma'] \oplus \mathsf{J}([Z_\sigma, \gamma'']) \oplus \mathsf{J}([Z_\sigma, \gamma']) \oplus \mathsf{J}([Z_\sigma, B_{\tau''}])\big)_d.$$

Therefore, for $d \geqslant 2s - r + 1$ the graded piece at degree d of each ideal associated to an interior vertex in σ is contained in $\mathrm{im}(\partial_1)_d$.

Notice that if Δ is composed of only one triangle then the only interior vertex is Z_σ and this implies $H_0(\mathcal{J})_d = 0$. If not, we take a triangle $\sigma' \in \Delta \setminus \{\sigma\}$ with two vertices on the boundary of $\Delta \setminus \{\sigma\}$, and apply the previous argument to the complex $\Delta^\star \setminus \{\sigma\}$. After f_2-steps (equal to the number of triangles in Δ), we will have considered all the interior vertices of $\Delta^\star$. We conclude that $H_0(\mathcal{J})_d = 0$ for any simplicial complex Δ with a finite number of triangles.

Then, if $s \geqslant \max\{r, 2r - 1\}$ and $d \geqslant 2s - r + 1$, the dimension formula in Equation (25) can explicitly be written as

$$\begin{aligned}\dim \mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}_d(\Delta^\star) = \binom{d+2}{2} + 3f_2\binom{d-s+1}{2} + 3f_2\left[(d-s)^2 - \binom{d-2s+r}{2}\right] \\ + 2f_1^\circ\left[(s-r+1)\binom{d-s+1}{2} - (s-r)\binom{d-s}{2}\right] \\ - (f_0^\circ + f_2)\left[\binom{d+2}{2} - \binom{s+2}{2}\right] - f_1^\circ \dim \mathsf{J}(B_\tau)_d\,,\end{aligned} \tag{30}$$

where $\dim \mathsf{J}(B_\tau)_d$ is given in Equation (28).

In particular, for $s = 2r - 1$ and $d = 3r - 1$, the Euler relations $f_1^\circ = 2f_2 - f_0 + 1$ and $f_0^\circ = f_2 - f_0 + 2$ applied to (30) lead to

$$\dim \mathcal{S}^{\boldsymbol{r},\boldsymbol{s}}_{3r-1}(\Delta^\star) = \frac{1}{2}r(r-1)f_2 + r(2r+1)f_0\,. \tag{31}$$

The dimension formula (31) was proved by Speleers in [34, Theorem 5].

In Table 2, for different choices of r, s, and d, we compare the lower bounds from Equations (16) and (18) to the exact dimension of the spline space (computed using either Equation (30) or Macaulay2). As can be seen, in these cases both lower bounds coincide with the exact dimension.

7. Concluding remarks

We have demonstrated how methods from homological algebra can be used to compute the dimension of supersmooth spline spaces on general triangulations; in particular, we have proved a combinatorial formula for the dimension of superspline spaces in sufficiently large degree. We also illustrated how homological algebra methods can be used to reproduce a variety of results from the literature [6,14,32,34], as well as generalizing some of them [34]. This opens several directions for future research.

Table 2
The triangulation Δ is the Powell–Sabin 6-split shown in Fig. 2. The lower bounds LB(16) and LB(18) coincide for the shown choices of (r, s, d), and they both coincide with $\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta^\star)$ for large enough degree. For $(r, s) = (3, 4)$, we compute $\dim \mathcal{S}_d^{r,s}(\Delta^\star)$ using Macaulay2 [16], and for the other cases we use (30) for the same. Note that the dimension of $\mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta^\star)$ was also computed in [34] for $(r, s, d) \in \{(2, 3, 5), (3, 5, 8)\}$.

(r, s)	d	$\dim H_0(\mathcal{J})_d$	$\max\left(\binom{d+2}{2}, \mathrm{LB}(18)\right)$	$\max\left(\binom{d+2}{2}, \mathrm{LB}(16)\right)$	$\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta^\star)$
$(2, 3)$	5	0	67	67	67
	6	0	160	160	160
$(3, 4)$	6	0	54	54	54
	7	0	138	138	138
$(3, 5)$	8	0	147	147	147
	9	0	285	285	285

- Supersmoothness can help define spline spaces with both stable dimension and locally supported basis functions, retaining full approximation power and avoiding prohibitively high degrees. Consequently, in the future these methods should be combined with constructive approaches to build spline spaces that are useful for the finite element method, such as triangulations and T-meshes.
- As was noted by Schenck in [24], the algebraic tools developed for the study of spline spaces on polyhedral complexes with uniform global smoothness and mixed supersmoothness across the codimension-1 faces had not been extended to the case we study in this paper. As we observed, the algebraic approach to the dimension problem of splines with mixed supersmoothness at higher codimension faces of the partition leads to the consideration of ideals generated by products of powers of linear forms in several variables. In the case of generic forms, this type of ideals has been recently studied by DiPasquale, Flores, and Peterson in [12] via apolarity. It will be interesting to extend this approach to ideals generated by arbitrary products of powers of linear forms to study full vertex ideals and derive an improved lower bound, as well as deriving an upper bound on the dimension of superspline spaces. While we have provided simple and computable lower bounds on the dimension, they only consider a simplified version of the vertex ideals at play. Considering the full vertex ideals is a first research direction that should be explored.
- The lower bound on $\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$ proved in Theorem 5.4 gives the exact dimension of the superspline space in large enough degree d and it is also clear that the derived bounds can differ from the exact dimension in small degrees, see the first row of Table 1 for instance. It would be interesting to find the smallest value of d for which the dimension formula holds; for this, results by Ibrahim and Schumaker in [19] might give a good estimate on the smallest degree for which homology term $H_0(\mathcal{J})_d$ vanishes. The analysis of the quotient of the vertex ideals $\overline{\mathtt{J}}(\gamma)/\mathtt{J}(\gamma)$ relates to the study of intrinsic smoothness properties of splines. In fact, if $\overline{\mathtt{J}}(\gamma)_d = \mathtt{J}(\gamma)_d$ then the supersmoothness conditions at the vertices ν in the link of the vertex γ are already satisfied by only imposing supersmoothness at γ. Any result which gives the exact dimension of the spline space in a particular degree d will also give an upper

bound on the last non-vanishing degree of $H_0(\mathtt{J})$ and of $\bar{\mathtt{J}}(\gamma)/\mathtt{J}(\gamma)$. An estimate on the smallest degree for which $\dim \mathtt{J}(\gamma)_d = \dim \overline{\mathtt{J}}(\gamma)_d$ will also contribute to a better understanding of $\dim \mathcal{S}_d^{\boldsymbol{r},\boldsymbol{s}}(\Delta)$, and it would be interesting to explore the implications of this algebraic approach combined with the results and techniques developed in [14,31,32] for intrinsic supersmoothness using Bernstein–Bézier methods.

Acknowledgments

We would like to thank Michael DiPasquale for providing many helpful comments and suggestions.

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