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Doctoral Thesis

Duality and Integrability in String Theory

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Declaration of Authorship

I, Giacomo PICCININI, declare that this thesis titled "Duality and Integrability in String Theory" and the work presented in it are based on my research projects in collaboration with Prof. Dr. Daniel C. THOMPSON, Dr. Saskia DEMULDER, Prof. Dr. Falk HASSLER, Dr. Neil B. COPLAND, and Dr. Camille ELOY. In particular, this thesis is based on [1–3], the work [4] currently under completion and a set of unpublished notes [5].

I confirm that:

- This work has not previously been accepted in substance for any degree and is not being concurrently submitted for any degree.
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"Once more unto the breach, dear friends, once more."

William Shakespeare, *Henry V*

SWANSEA UNIVERSITY

Abstract

Physics Department
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Doctor of Philosophy

Duality and Integrability in String Theory

by Giacomo Piccinini

This thesis investigates aspects of duality and integrable deformations in String Theory. In the first two chapters we review standard material in Mathematics and Physics, laying the ground to the novel contributions later reported. In Chapter 4 we introduce generalised cosets, on which we are able to provide a canonical construction for a generalised frame field and spin connection that together furnish an algebra under the generalised Lie derivative. In Chapter 5 we study the geometric properties of the Yang-Baxter deformation of $\mathbb{CP}^n$, showing that it constitutes an exemplar of Generalised Kähler Geometry. For $\mathbb{CP}^2$ we compute the generalised Kähler potential. Tangentially, we furnish a closed form for the metric and B-field of the Yang-Baxter deformed sphere S^n, for every n. In Chapter 7 we address the problem of two-loop renormalisation of the Tseytlin doubled string for cosmological spacetimes. Whilst the results do satisfy a number of key consistency criteria, we find however that the two-loop counter-terms are incompatible with $O(n, n)$ symmetry, pointing perhaps to the presence of scheme changes. In Chapter 8 we build on this work and set the stage for a two-loop calculation for a Poisson-Lie T-duality covariant theory.

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If a Ph.D. is widely recognised as a solitary journey, it is nevertheless a collective effort, even when people ignore being part of the picture. This is my humble attempt to give back all the love I received.

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Chapter 1

Introduction

Physics and Art have evolved in radically opposite ways throughout the centuries. With scientists trying to organise a vast, scattered body of notions under a few guiding principles, painters systematically and deliberately carved rationality out of the original Renaissance order. If a progressive abandonment of the concept of harmony took over the aesthetics of the XX century, in 1972 the Physics Nobel laureate P.W. Anderson wrote "It is only slightly overstating the case to say that Physics is the study of symmetry" [6].

In either subjects, symmetry is usually regarded as some invariance property of a model, be it the facade of a church or a Quantum Field Theory, that simplifies its description. As one might expect, the more symmetric a model, the more constrained is the dynamics, sometimes to the point where it is possible to solve it exactly. Solvable systems are called *integrable*, and are incredibly hard to come by: they represent the opposite of chaos, a lush oasis of *kósmos*.

As rare gems, integrable theories need to be handled with care, wisely cutting their facets to maximise brilliance. In String Theory, the goldsmith toolkit consists of two essential utensils: on the one hand, a number of rigorous mathematical methods proper to integrability, developed to extract otherwise inaccessible information; on the other hand, smooth modifications of a model − called *integrable deformations* − that while preserving integrability furnish new highly non-trivial examples.

Integrable deformations, and in fact String Theory, are hardly discussed without a mention of *dualities*. If making an appropriate artistic parallelism would certainly be a stretch, we could nevertheless extend the fairly intuitive idea of symmetry so as to incorporate that of duality. By that, we mean a relation between two a priori distinct theories that turn out to describe the same physics. Qualitatively, this is extremely surprising as it draws unforeseen connections among theories with completely different degrees of freedom. If this were the end of the story, though, we would have certainly gained a gratification of our aesthetic sense, but not much in terms of quantitative results and predictions (that supposedly represent the ultimate goal of Science).

Even though duality per se does not imply any kind of *formal* solvability – e.g. there is no arsenal of dedicated techniques and methods as we had for integrability – it nevertheless bears a more subtle and perhaps effective form of simplification. Credits go to its simple proposal: if the description of a phenomenon is too convoluted in theory A, it might well be simple enough in the dual theory B. The prototype is when A is too strongly coupled to be tackled with perturbation theory, but standard Quantum Field Theory techniques readily apply to B. In truth, dualities do *not* require String Theory at all to be formulated (think, for example, of Seiberg duality in supersymmetric gauge theories [7]) but there is a huge body of evidence indicating their role as primary actors whenever strings are involved.

Rather than concentrating on the weak/strong duality mentioned above (also called S-duality), we shall focus on a more geometric (and thus stringy) type of implementation known as T-duality. It was first observed in [8], but only later formalised by Buscher in [9, 10] and its whole program is easily stated: String Theory does not quite agree with Riemann as to what we mean by "geometry". After all, that XIX century differential geometry was not really satisfactory in describing string backgrounds was already pretty clear: the metric alone is not the sole focus any longer, and needs to be supplemented at least with fluxes (differential forms) and dilaton (scalar). But T-duality goes way beyond that: T-dual geometries, i.e. the different backgrounds giving rise to the same physics, are *not* related by ordinary diffeomorphisms but through a completely new type of transformation, the eponymous T-duality. What's even more astonishing (or perhaps exciting?) is that in some cases the whole concept of differentiable manifolds collapses and one needs to resort to generalisations, such as T-folds [11], to address the non-geometric aspects entailed by T-duality.

Even when restricting our interest to T-duality, ruefully accepting the inadequacy of some of the lessons painstakingly learnt and brought to us by Einstein, the situation remains involved. Buscher's duality, most often called "Abelian", does not certainly encompass all relevant cases, as it only strictly applies to tori, arguably the simplest (compact) manifolds strings are allowed to propagate on. Whilst the importance of toric compactifications for our comprehension of String Theory can not be overstated, the need for more challenging, and yet dualisable, manifolds has become clear.

Conceptually, Abelian T-duality is based on the presence of Abelian isometries for the target space, thereby justifying the relevance of tori in this context. Evidently, more complicated manifolds shall come at least with non-Abelian isometries or, even more likely, no isometries at all. In these cases, the hypothesis of the Buscher's derivation go astray, and one is required to cook up some generalisation. In the light of Ockham's razor, we could try and upgrade Abelian to non-Abelian isometries, replicating the steps that lead to the discovery of Abelian T-duality. This works, and, as one might guess, the result is known as non-Abelian T-duality [12]. Still, it is not quite satisfactory, as the case with no isometries is completely missing from the picture. Here there is no philosophical tool to our rescue: we simply can not give up on isometries *and* retain the Buscher's procedure. The resolution of the conundrum, called Poisson-Lie T-duality, was put forward by Klimčík and Ševera in [13, 14]: while admitting that dualisable manifolds can have no isometries, they need to obey an additional constraint, known as the Poisson-Lie condition. This last form of duality, that comprises of both Abelian and non-Abelian as special cases, will represent a central topic for this thesis.

At this point it is perfectly legitimate to ask how duality and deformations relate to one another. We have so far remained purposefully vague on *explicit* implementations of either concepts, but we shall now partially remedy that. Our starting point is the Principal Chiral Model (PCM), the two-dimensional non-linear σ-model describing the motion of a Bosonic string on a group manifold G[1]. Albeit perhaps not very realistic, the PCM is extremely relevant, for it is arguably the simplest non-trivial example of integrable system in this context. Other than that, it displays many of the features typical of QCD, such as a dynamically generated mass-gap, asymptotic freedom and confinement.

Since the 1990's, many integrable deformations of the Principal Chiral Model have been found[2]; in essence, they all rely on modifying and/or extending its action through the introduction of some parameters and operators, so that the PCM is eventually recovered in a certain limit. Of paramount importance for this thesis are Yang-Baxter (or η-) deformations, first conjectured by Klimčík in [16]. They are based upon the introduction of a deformation operator based on the so-called Yang-Baxter matrix (whose relevance for the model is controlled by a real parameter η), and were only proven integrable in [17]. A few years later, Sfetsos [18] proposed a new integrable deformation of the Wess-Zumino-Witten (WZW) model (a conformal extension of the PCM [19]), the λ-deformation. Crucially for us, η- and λ-deformations are related by Poisson-Lie T-duality [20, 21][3].

Up to now we have only mentioned group manifolds and we should really plead guilty to oversimplification for that: even in the theoretician's spirit of easing the description of the universe, Lie groups are too narrow of a subclass of manifolds to consider. At the

[1] Even though String Theory admittedly constrains the number of spacetime dimensions, we will mostly neglect this aspect and free G of any imposition so as to broaden the discussion.

[2] It is possible to consider integrable deformations that do *not* require the PCM as a starting point, for instance $T\overline{T}$-deformations [15]. However, they will not be part of this thesis.

[3] Plus, in fact, an analytic continuation.

very least we should be able include spheres, (anti-) de Sitter and projective spaces if we want to have a shot at putting forward integrable deformations as a topic worth of explaining some of the features of String Theory. Actually, all of these manifolds have a common denominator: they are symmetric spaces, i.e. quotients of a Lie group G by a subgroup $H \subset G$ with particular properties. Starting from the PCM on G, we can tweak it so as to describe the dynamics of a Bosonic string on G/H by opportunely removing the degrees of freedom associated to H: the lesson previously learnt is not to discard after all! Furthermore, this model is easily proven integrable[4] and, up to some detail, allows for integrable deformations, too.

Despite this progress, many problems both at the classical and quantum level remain open.

We have already mentioned that Riemannian geometry has some serious shortcomings when it comes to describing string backgrounds. For instance, Abelian T-duality for the NS-NS sector is characterised by some non-linear transformation involving metric and B-field that makes T-duality hard to detect. To draw a parallelism, the formulation of Electromagnetism with electric and magnetic fields overshadowed the underpinning Lorentz covariance for decades. Building on previous experience, it would be advantageous to have a re-formulation of String Theory with T-duality made explicit. In fact, this exists and consists, roughly speaking, in embedding the two T-dual manifolds in a new fictitious space of doubled dimensions. In the same spirit, space and time were reunited in a single four-dimensional entity in Special Relativity. The mathematical framework into which this formulation fits is that of Generalised Geometry as introduced by Hitchin and Gualtieri [24, 25].

The first task we shall embark upon is precisely the description of coset models in Generalised Geometry. We shall not be concerned with the dynamics but rather with their (generalised-) geometric properties. In practice, this point will be addressed through a detailed and canonical construction of generalised frame fields, the analogues of the vierbein lying at the heart of the tetrad formalism so widely employed in General Relativity. The purpose is twofold: on the one hand, we achieve a great deal of simplification, for arbitrarily complicated objects will be traded for constant quantities which are much easier to handle. On the other, generalised cosets lend themselves to Supergravity and consistent truncations, and we shall provide the necessary backbone to those useful applications. This will be the subject of Chapter 4.

Building on that, we will consider Yang-Baxter deformations of complex projective spaces in Chapter 5. Even though originally motivated by the investigation of some extensions of

[4] Here we are really making statements only about classical integrability; one should anticipate that quantum integrability necessitates modifications, e.g. the inclusion of appropriate fermionic content for $\mathbb{CP}^n$ [22, 23].

the AdS/CFT correspondence [26][5], e.g. the deformation of the $AdS_4 \times \mathbb{CP}^3$ background, we shall unravel a beautiful geometrical structure, Generalised Kähler Geometry, underpinning $\mathbb{CP}^n_\eta$. This legitimates an outright interest, even devoid of possible holographic applications, repaid with the explicit construction of a generalised Kähler potential for $\mathbb{CP}^2_\eta$, an extremely rare object. Furthermore, the same technology will be applied – with minor modifications – to the study of η-deformed spheres and AdS spaces, for which we conjecture closed form for metric and B-field in every dimension. The expressions are simple and can be used straight out of the box, without resorting any longer to a lengthy and somewhat involved algebraic construction. Finally, in passing, we shall resolve a puzzle that dates back to [20] and concerns the possibility of a double Yang-Baxter deformation of $\mathbb{CP}^n$.

We will then start exploring the quantum properties of the doubled formalism from the worldsheet point of view. To this end, in Chapter 7 we study the two-loop renormalisation of the "doubled string" (i.e. an explicit Abelian T-duality covariant formulation of String Theory compactified on an n-torus) in a simplified setting known as cosmological spacetime. Notwithstanding the apparent reduced complexity, we will face a large number of technical challenges which we will try to address. In doing so, we will develop and discuss novel techniques for evaluating loop integrals and create a Mathematica notebook for automatising the entire calculation. In Chapter 8 we will report on a conceptually similar computation, this time adapted to Poisson-Lie T-duality on a Drinfel'd double (a specific type of Lie group we will introduce soon). We will be able to infer a number of constraints on the shape of the final result – also providing a graphical interpretation to it –, as well as to furnish an all-loop expansion of the interacting Lagrangian. We anticipate that the treatment of this chapter will not be complete, for the project is still under investigation at the time of writing.

Finally, we will draw conclusions on this work and, more broadly speaking, on the whole field. We will indicate a few interesting avenues for feature research that we hope might materialise in useful results and outline the main technical and conceptual challenges one is expected to face. We complement the entire thesis with a number of appendices.

[5]The AdS/CFT correspondence, a particular materialisation of the holographic principle proposed by Susskind [27], conjectures the equivalence of String Theory on anti-de Sitter spaces and a *dual* Conformal Field Theory living on the boundary of said space.

Chapter 2

Mathematical Preliminaries

In this introductory chapter we shall review the main mathematical concepts that will be used on a number of occasions as the thesis unfolds. Rather than letting them complement the Physics review of Chapter 3, we have preferred to furnish here a more cohesive presentation that can be consulted quickly whenever needed. Given the wide breadth of the topics touched here, we will not attempt to deliver an exhaustive treatment, but only report results that are pertinent to our goals. In this sense, proofs will be omitted but the interested reader will nevertheless be referred to various resources where they can be easily found.

2.1 Drinfel'd Double

The algebraic structure known as Drinfel'd double, which we shall introduce shortly, is key to the understanding of Poisson-Lie T-duality.

Given a Lie algebra $\mathfrak{g}$ with Lie product $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, if it is possible to further endow it with a *coproduct*[1] which is also a cocycle, then we call $(\mathfrak{g}, [\cdot,\cdot], \Delta)$ a *bialgebra*. Remarkably, we can immediately declare the dual vector space $\mathfrak{g}^*$ a bialgebra if we equip it with the transposed products Δ^t and $[\cdot,\cdot]^t$ [28].

We now take one step further and consider the *vector space* $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$. On $\mathfrak{d}$ we have the notion of natural scalar product $\langle\langle \cdot,\cdot \rangle\rangle$: for $x, y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$ it is given by $\langle\langle x, y\rangle\rangle = 0$, $\langle\langle \xi, \eta\rangle\rangle = 0$ and $\langle\langle \xi, x\rangle\rangle = \langle x, \xi\rangle$, where $\langle \cdot,\cdot\rangle$ denotes the natural pairing between a vector space and its dual. As it is, however, $\mathfrak{d}$ is not (yet) a Lie algebra. It turns out [28] that every Lie bracket on $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ preserving the natural scalar product $\langle\langle \cdot,\cdot \rangle\rangle$ and such

[1] That is, an anticommutative map $\Delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ that obeys co-Jacobi identity $(\Delta \otimes \mathrm{id}) \circ \Delta + \text{cyclic} = 0$.

that both $\mathfrak{g}$ and $\mathfrak{g}^*$ are subalgebrae is given by

$$\begin{cases} [x,y]_{\mathfrak{d}} = [x,y] & \forall x,y \in \mathfrak{g} \\ [\xi,\eta]_{\mathfrak{d}} = [\xi,\eta]_{\mathfrak{g}^*} & \forall \xi,\eta \in \mathfrak{g}^* \\ [x,\xi]_{\mathfrak{d}} = \mathrm{ad}^*_x\xi - \mathrm{ad}^*_\xi x & \forall x \in \mathfrak{g},\, \forall \xi \in \mathfrak{g}^* \end{cases} \tag{2.1}$$

where ad^* denotes the co-adjoint action[2]. The Lie algebra $(\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*, [\cdot,\cdot]_{\mathfrak{d}})$ is called the Drinfel'd double of the Lie bialgebra $(\mathfrak{g}, [\cdot,\cdot], \Delta)$ [28]. Equivalently, $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ is known as a *Manin triple* [29, 30], and will appear often when treating integrable deformations. To complete the picture, one can prove that in fact $\mathfrak{g}$ is a Lie bialgebra if and only if the Jacobi identity holds for all triples $\alpha \in \mathfrak{g}^*, x, y \in \mathfrak{g}$, i.e.

$$[\alpha,[x,y]]_{\mathfrak{d}} + [x,[y,\alpha]]_{\mathfrak{d}} + [y,[\alpha,x]]_{\mathfrak{d}} = 0\,. \tag{2.2}$$

Let us make an example to clarify the notation. Take generators $T_{\mathbb{a}}$ and $T^{\mathbb{a}}$ to span $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively (conventions for indices are reported in Appendix A). These will induce different types of structure constants, namely $[T_{\mathbb{a}}, T_{\mathbb{b}}] = f_{\mathbb{a}\mathbb{b}}{}^{\mathbb{c}} T_{\mathbb{c}}$ and $[T^{\mathbb{a}}, T^{\mathbb{b}}] = \tilde{f}^{\mathbb{a}\mathbb{b}}{}_{\mathbb{c}} T^{\mathbb{c}}$. For the mixed commutator $[T_{\mathbb{a}}, T^{\mathbb{b}}]$, a straightforward calculation using (2.1) yields

$$[T_{\mathbb{a}}, T^{\mathbb{b}}] = -f_{\mathbb{a}\mathbb{c}}{}^{\mathbb{b}} T^{\mathbb{c}} + \tilde{f}^{\mathbb{b}\mathbb{c}}{}_{\mathbb{a}} T_{\mathbb{c}}\,. \tag{2.3}$$

Given (2.3), the Jacobi identity (2.2) will comprise two parts, one directed along $\mathfrak{g}$ and one along $\mathfrak{g}^*$. The latter automatically vanishes: it corresponds to the Jacobi identity for $\mathfrak{g}$. The other part, instead, results in the non-trivial constraint

$$f_{\mathbb{a}\mathbb{b}}{}^{\mathbb{c}} \tilde{f}^{\mathbb{d}\mathbb{e}}{}_{\mathbb{c}} = f_{\mathbb{a}\mathbb{c}}{}^{\mathbb{d}} \tilde{f}^{\mathbb{c}\mathbb{e}}{}_{\mathbb{b}} + f_{\mathbb{a}\mathbb{c}}{}^{\mathbb{e}} \tilde{f}^{\mathbb{d}\mathbb{c}}{}_{\mathbb{b}} + f_{\mathbb{c}\mathbb{b}}{}^{\mathbb{d}} \tilde{f}^{\mathbb{c}\mathbb{e}}{}_{\mathbb{a}} + f_{\mathbb{c}\mathbb{b}}{}^{\mathbb{e}} \tilde{f}^{\mathbb{d}\mathbb{c}}{}_{\mathbb{a}}\,. \tag{2.4}$$

For a semi-simple Lie algebra $\mathfrak{g}$ – the case we will always be interested in – because of Whitehead's lemma the bialgebra structure is necessarily specified by an R-matrix [28], i.e. an endomorphism R of $\mathfrak{g}$ obeying the modified Classical Yang-Baxter Equation (mCYBE)

$$[R(x), R(y)] - R([R(x), y] + [x, R(y)]) + c^2[x,y] = 0\,, \tag{2.5}$$

for some $c \in \mathbb{C}$ and for all $x, y \in \mathfrak{g}$. Upon picking a basis $\{T_{\mathbb{a}}\}$ for $\mathfrak{g}$, the explicit action of R is determined by $R(x^{\mathbb{a}} T_{\mathbb{a}}) = x^{\mathbb{a}} R_{\mathbb{a}}{}^{\mathbb{b}} T_{\mathbb{b}}$. Thanks to $\mathfrak{g}$ being semi-simple and thus equipped with a non degenerate pairing $\kappa_{\mathbb{a}\mathbb{b}}$, it is possible to raise (or lower)

[2] Recall that, given a Lie algebra $\mathfrak{g} \ni x, y$ and a dual vector space $\mathfrak{g}^* \ni \xi$, the co-adjoint action is defined as the map $\mathrm{ad}^*_x = -\mathrm{ad}^t_x$ satisfying $\langle \xi, \mathrm{ad}_x(y) \rangle = -\langle \mathrm{ad}^*_x(\xi), y \rangle$. The adjoint action on the algebra is simply $\mathrm{ad}_x(y) = [x,y]$.

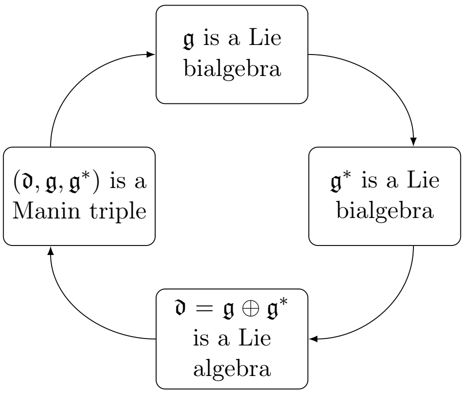


FIGURE 2.1: The relation between a bialgebra and a Manin triple.

the indices of R, resulting in a skew-symmetric tensor[3] $R^{\mathbb{ab}} = \kappa^{\mathbb{ac}} R_{\mathbb{c}}{}^{\mathbb{b}} = -R^{\mathbb{ba}}$. One consequence of the presence of R is that we can introduce a second Lie-bracket on $\mathfrak{g}$ given by $[x, y]_R := [R(x), y] + [x, R(y)]$: anti-symmetry is manifest and the Jacobi identity is automatically implied by the mCYBE. We will indicate with $\mathfrak{g}_R$ the vector space $\mathfrak{g}$ endowed with the Lie bracket $[\cdot, \cdot]_R$.

Naively, the mCYBE for $c^2 = -1$ is identical in form to the vanishing of the Nijenhuis tensor for an almost complex structure. Guided by this observation, one could try and turn a Lie algebra $\mathfrak{g}$ of even *real* dimension into a complex Lie algebra. As expected, R will be required to obey $R^2 = -1$ but this alone is not sufficient: the additional requirement $R([x, y]) = [R(x), y]$ for all $x, y \in \mathfrak{g}$ is necessary [32]. On coset spaces the situation is slightly more delicate. Consider a real Lie group G and a subgroup $H = \exp(\mathfrak{h}) \subset G$, giving rise to the (reductive) homogeneous space $M = G/H$. Indicate with $\mathfrak{m}$ the linear space such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. If R is an R-matrix on $\mathfrak{g}$ with the aforementioned properties, it would be tempting to first restrict it to $\mathfrak{m}$ and then uplift it, through the action of a vielbein, to a complex structure on M. However, as proved by Koszul [33–35], additional requirements are needed.

Theorem 2.1.1 (Koszul theorem)**.** *Let G be a real connected Lie group with Lie algebra $\mathfrak{g}$ and $H \subset G$, a closed subgroup of G with Lie algebra $\mathfrak{h}$. Suppose that $\mathfrak{g}$ has a decomposition such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, with $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. The coset space G/H has a G-invariant complex structure if and only if there is a linear operator R on $\mathfrak{g}$ (the Koszul operator) such that*

1. $R|_{\mathfrak{h}} = 0$ *and* $R|_{\mathfrak{m}}{}^2 = -1$;
2. $\mathrm{ad}(x) \circ R = R \circ \mathrm{ad}(x)\,, \forall x \in \mathfrak{h}$;
3. $[R(x), R(y)] - R([R(x), y] + [x, R(y)]) - [x, y] = 0 \,(\mathrm{mod}\, \mathfrak{h})\,, \forall x, y \in \mathfrak{g}$.

[3] An R-matrix is more formally defined [31] as an element of $\mathfrak{g} \otimes \mathfrak{g}$ that comprises of a skew-symmetric and a symmetric part. For our purposes, we will restrict to purely anti-symmetric R-matrices and most often view them as endomorphisms thanks to the canonical isomorphism provided by κ.

2.1.1 Example: Symmetric Space and Coisotropic Subalgebra

Let us conclude this section with an example aiming at encompassing some of the concepts encountered thus far. Let $\mathfrak{g} = \text{Lie}(G)$ be a semi-simple Lie algebra. As such, $\mathfrak{g}$ is endowed with a non-degenerate pairing κ, the Killing form. Non-degeneracy implies that κ induces an isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}^*$ explicitly given by

$$\phi(x) = \kappa(x, \cdot) \in \mathfrak{g}^* \quad \forall x \in \mathfrak{g}\,. \tag{2.6}$$

Let $R : \mathfrak{g} \to \mathfrak{g}$ be an R-matrix obeying the mCYBE. Upon rescaling R, we can always set c^2 to either $+1$, -1 or 0. Here, we shall fix $c^2 = -1$. If we choose $\{T_{\mathbb{a}}\}$ as a basis for $\mathfrak{g}$, we can raise the indices of R through the action of the Killing form,

$$R^{\mathbb{a}\mathbb{b}} = \kappa^{\mathbb{a}\mathbb{c}} R_{\mathbb{c}}{}^{\mathbb{b}} = -R^{\mathbb{b}\mathbb{a}}\,. \tag{2.7}$$

Indicating with $f_{\mathbb{a}\mathbb{b}}{}^{\mathbb{c}}$ the structure constants on $\mathfrak{g}$ induced by the canonical Lie bracket $[\cdot,\cdot]$, the algebra $\mathfrak{g}_R$ with Lie bracket $[\cdot,\cdot]_R$ will have structure constants $\tilde{f}_{\mathbb{a}\mathbb{b}}{}^{\mathbb{c}}$,

$$\tilde{f}_{\mathbb{a}\mathbb{b}}{}^{\mathbb{c}} = -2R_{[\mathbb{a}}{}^{\mathbb{d}} f_{\mathbb{b}]\mathbb{d}}{}^{\mathbb{c}}\,. \tag{2.8}$$

Take $M = \exp(\mathfrak{m})$ to be the coset $M = G/H$ and further require it to be a symmetric space [36], so that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and $\kappa(\mathfrak{m}, \mathfrak{h}) = 0$, where $\mathfrak{h}$ and $\mathfrak{m}$ are, respectively, the $+1$ and -1 eigenspace of the $\mathbb{Z}_2$ involution. We shall further impose that $\mathfrak{h}$ is a *coisotropic subalgebra*. Recall that a subalgebra $\mathfrak{h}$ of a Lie bialgebra $\mathfrak{g}$ is called coisotropic if its annihilator $\mathfrak{h}^\perp$, i.e. the space of functionals $\xi \in \mathfrak{g}^*$ such that $\langle \xi, x\rangle = 0\ \forall x \in \mathfrak{h}$, is a Lie subalgebra in $\mathfrak{g}^*$ [37]. Defining $\mathfrak{m}^* = \phi(\mathfrak{m})$ and $\mathfrak{h}^* = \phi(\mathfrak{h})$ (so that $\mathfrak{g}^* = \mathfrak{m}^* \oplus \mathfrak{h}^*$), orthogonality implies $0 = \kappa(\mathfrak{m}, \mathfrak{h}) = \langle \mathfrak{m}^*, \mathfrak{h}\rangle$. There can be no $\xi \in \mathfrak{h}^*$ obeying $\langle \xi, \mathfrak{h}\rangle = 0$, or otherwise the restriction of κ to $\mathfrak{h}$ would be degenerate, hence $\mathfrak{h}^\perp = \mathfrak{m}^*$. Without further constraints, $\mathfrak{m}^*$ is not a subalgebra of $\mathfrak{g}^*$, as needed for $\mathfrak{h}$ to be coisotropic. Requiring $[\mathfrak{m}^*, \mathfrak{m}^*]_{\mathfrak{g}^*} \subset \mathfrak{m}^*$ is equivalent to imposing $[\mathfrak{m}, \mathfrak{m}]_R|_{\mathfrak{h}} = 0$. We obtain the coisotropy condition

$$([Rx, y] + [x, Ry])|_{\mathfrak{h}} = 0 \qquad \forall x, y \in \mathfrak{m}\,. \tag{2.9}$$

If $\mathfrak{h}$ is coisotropic and the coset G/H is a symmetric space, $H^* = \exp(\mathfrak{h}^*)$ is a subgroup of $G^* = \exp(\mathfrak{g}^*)$ and the coset $M^* = G^*/H^*$ is a symmetric space, provided we endow $\mathfrak{g}^*$ with the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}^*}$ induced by the Drinfel'd-Jimbo R-matrix[4] [38], obtained

[4]In general, the Drinfel'd-Jimbo procedure is not the unique possibility for constructing an R-matrix. However, it is most useful when building Poisson bi-vectors out of Yang-Baxter matrices, as in our case.

by taking the wedge product of (properly normalised) positive and negative roots[5]

$$R = \frac{1}{2} \sum_{\lambda \in \Delta^+} X_\lambda \wedge X_{-\lambda} \qquad \in \mathfrak{g} \wedge \mathfrak{g}. \tag{2.10}$$

This is most easily seen using the dual bracket $[\cdot,\cdot]_R$. More precisely, grading $\mathfrak{g}$ into ± 1-eigenspaces, it follows from the definition of the Cartan-Chevalley basis that, for a fixed root λ, the ladder operators X_λ and $X_{-\lambda}$ belong to the same subspace, while the Cartan subalgebra belongs to the $+1$ eigenspace. The Drinfel'd-Jimbo construction then implies that the Yang-Baxter matrix has no mixed components, $R(\mathfrak{h})|_{\mathfrak{m}} = 0$ and $R(\mathfrak{m})|_{\mathfrak{h}} = 0$. This fact, together with coisotropy and symmetric space decomposition, yields

$$[\mathfrak{m},\mathfrak{m}]_R = 0\,, \qquad [\mathfrak{m},\mathfrak{h}]_R \subset \mathfrak{m}\,, \qquad [\mathfrak{h},\mathfrak{h}]_R \subset \mathfrak{h}\,. \tag{2.11}$$

Lifting these conditions to the dual algebra $\mathfrak{g}^*$ we get $[\mathfrak{h}^*,\mathfrak{h}^*]_{\mathfrak{g}^*} \subset \mathfrak{h}^*$, $[\mathfrak{h}^*,\mathfrak{m}^*]_{\mathfrak{g}^*} \subset \mathfrak{m}^*$ and $[\mathfrak{m}^*,\mathfrak{m}^*]_{\mathfrak{g}^*} = 0$, the defining relations for a (particular type of) symmetric space M^*. It can be checked for $G/H = SU(2)/U(1)$: assuming $\mathfrak{m} = \mathrm{Span}(\sigma_1,\sigma_2)$, where σ_i are the Pauli matrices, the Drinfel'd-Jimbo R-matrix acts as $R(\sigma_1) = \sigma_2$, $R(\sigma_2) = -\sigma_1$ and $R(\sigma_3) = 0$; the relations (2.11) follow.

2.2 Poisson Manifolds

Together with Drinfel'd doubles, Poisson manifolds represent the mathematical backbone of integrable deformations of bosonic String Theory. Whilst the reader is certainly familiar with the concept of Poisson brackets, we shall nevertheless recapitulate some less-known facts building in particular on the interplay between Poisson and group structure.

A Poisson manifold M is a Riemannian manifold endowed with a Poisson structure, i.e. an $\mathbb{R}$-linear map called Poisson bracket $\{\cdot,\cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ satisfying the properties of anti-symmetry, Jacobi identity and Leibniz rule [39]. We call Poisson bi-vector the skew-symmetric two-tensor $\pi \in TM \otimes TM$ that implements the Poisson structure

$$\{f,g\} = \langle \mathrm{d}f \otimes \mathrm{d}g, \pi \rangle\,, \tag{2.12}$$

where $f, g \in \mathcal{C}^\infty(M)$, d is the exterior derivative and $\langle\cdot,\cdot\rangle$ the natural pairing between dual vector spaces. Notice that the Jacobi identity for the Poisson bracket is equivalent

[5] We adopt a Cartan-Chevalley basis $\{H_\lambda, X_\lambda, X_{-\lambda}\}$, where $\lambda \in \Delta_+$ is a positive root, H_λ span the Cartan subalgebra and $X_{\pm\lambda}$ are ladder operators. We choose the normalisation with respect to the Killing form $\kappa(X_\lambda, X_{-\lambda}) = \frac{2}{\langle\lambda,\lambda\rangle}$.

to the vanishing of the Schouten bracket for π, reading in local coordinates x^i

$$\pi^{[i|l|}\partial_l \pi^{jk]} = 0 \,. \tag{2.13}$$

Every Poisson manifold admits a trivial Poisson structure, $\{f,g\} = 0 \ \forall f,g \in \mathcal{C}^\infty(M)$ or, equivalently, a *maximally degenerate* π. On the opposite side of the spectrum, if the bi-vector is everywhere non-degenerate, it can be inverted yielding a non-degenerate symplectic two-form ω,

$$\omega = \pi^{-1} \,. \tag{2.14}$$

The vanishing of the Schouten bracket for π translates to ω being closed. In this situation, M is called a *symplectic manifold*. Hence, all symplectic manifolds are Poisson but the converse is, in general, false. Nevertheless, any (not necessarily symplectic) Poisson manifold can be foliated with symplectic submanifolds called *symplectic leaves*.

Poisson-Lie groups are particular instances of Lie groups where the multiplication map is required to preserve the Poisson bracket [31][6]. In broad strokes, Poisson structures on a Poisson-Lie group G come in two flavours $-$ multiplicative and affine $-$ depending on whether they obey $\pi(gh) = \lambda_g \pi(h) + \rho_h \pi(g)$ or $\pi(gh) = \lambda_g \pi(h) + \rho_h \pi(g) - \lambda_g r_h \pi(e)$ for every $g, h \in G$ [30]. Here and henceforth, λ_g and ρ_g represent the *differential* of left and right translations, respectively[7].

Theorem 2.2.1. *Every multiplicative Poisson structure π on a connected semi-simple Lie group G is of the form*

$$\pi(g) = \rho_g(R) - \lambda_g(R), \qquad g \in G \tag{2.15}$$

where $R \in \bigwedge^2 \mathfrak{g}$ is a bivector at the identity $e \in G$.

Factorising a left-action out, the multiplicative Poisson structure above becomes $\pi(g) = \lambda_g(\mathrm{Ad}_{g^{-1}}{}_* R - R)$, where $\mathrm{Ad}_{g^{-1}}{}_*$ indicates the differential of the adjoint action[8]. In a basis $\{T_{\mathbb{a}}\}$ for $\mathfrak{g} = \mathrm{Lie}(G)$, the term inside bracket is

$$\mathrm{Ad}_{g^{-1}}{}_* R - R = (\mathrm{Ad}_g)^{\mathbb{a}}{}_{\mathbb{c}}\, R^{\mathbb{cd}} (\mathrm{Ad}_{g^{-1}})_{\mathbb{d}}{}^{\mathbb{b}}\, T_{\mathbb{a}} \otimes T_{\mathbb{b}} - R^{\mathbb{ab}} T_{\mathbb{a}} \otimes T_{\mathbb{b}} \,, \tag{2.16}$$

[6]A map $\varphi : M \to N$ between Poisson manifolds respects the Poisson brackets (i.e. it is a *Poisson map*) if $\{f,g\}_N(\varphi(x)) = \{f \circ \varphi, g \circ \varphi\}_M(x)$ for every $x \in M$ and $f, g \in \mathcal{C}^\infty(N)$.

[7]If $x \in \mathfrak{g}$, then $\lambda_g x = gx$ and $\rho_g x = xg$. Notice that, with abuse of notation, when $x = x_1 \otimes x_2 \in \mathfrak{g} \otimes \mathfrak{g}$ we will still use the same symbols to denote e.g. $\lambda_g(x_1 \otimes x_2) = \lambda_g x_1 \otimes \lambda_g x_2$.

[8]We define $\mathrm{Ad}_{g_1} g_2 = g_1 g_2 g_1^{-1}$.

where we have used Ad-invariance of the Killing form to replace $\mathrm{Ad}_{g^{-1}} = \mathrm{Ad}_g^t$. It is common to rewrite $\mathrm{Ad}_{g^{-1}*}R - R = R_g - R$ introducing the notation[9]

$$R_g = \mathrm{Ad}_{g^{-1}} \cdot R \cdot \mathrm{Ad}_g \,. \tag{2.17}$$

Given a Poisson-Lie group G with Poisson structure π, a subgroup $H \subset G$ is not necessarily Poisson-Lie as it is not granted that, for $f, g \in \mathcal{C}^\infty(H)$, the bracket $\{f, g\}$ is closed in $\mathcal{C}^\infty(H)$. Hence, a Lie subgroup H of a Poisson-Lie group G is called a Poisson-Lie subgroup if it has its own Poisson-Lie structure and the inclusion $\iota : H \hookrightarrow G$ is a Poisson map [31]. A Poisson-Lie subgroup H can be characterised more explicitly using properties of its algebra $\mathfrak{h}$. Recall that a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called *coisotropic* if its annihilator $\mathfrak{h}^\perp$ is a Lie subalgebra of $\mathfrak{g}^*$. However, it can be proven that H is a Poisson-Lie subgroup of G if the annihilator $\mathfrak{h}^\perp$ is an ideal in $\mathfrak{g}^*$. Given that any ideal is also a subalgebra, a Poisson-Lie subgroup H is necessarily a coisotropic subgroup. Quotient spaces G/H with Poisson structures will be our main focus in Chapter 5; these are called *Poisson homogeneous spaces*, provided the projection map $\rho : G \to G/H$ is Poisson [31]. Whilst H being a Poisson-Lie subgroup of G guarantees that G/H is Poisson homogeneous, it is possible to relax this condition and require that H be only coisotropic.

Proposition 2.2.2. *Given a Poisson-Lie group G and a subgroup $H \subset G$, a sufficient condition for the coset space G/H to be a Poisson homogenous space is for H to be a coisotropic subgroup of G. Moreover, there exists a unique Poisson structure, known as Poisson-Bruhat, on the coset*

$$\pi_{\mathrm{B}} = (\mathrm{d}\rho^* \otimes \mathrm{d}\rho^*)\, \pi \tag{2.18}$$

such that the projection map $\rho : G \to G/F$ is Poisson.

Coadjoint orbits serve useful examples of Poisson homogeneous spaces. Given a compact semi-simple Poisson-Lie group G, its coadjoint orbits are: i) obtained as the quotient by a Poisson-Lie subgroup; ii) homogeneous Kähler manifolds [40, 41]. The first condition tell us that the orbit $\mathcal{O}$ inherits from G a unique Poisson-Bruhat structure. The second condition implies the presence of a second Poisson structure ω^{-1} obtained from inverting the Kähler form. Contrary to the Poisson-Bruhat structure π_{B}, ω^{-1} is not *multiplicative* but rather *affine*. We now have two different, and equally valid, Poisson structures on an orbit. We might wonder whether the two can be combined into a "larger" object, given by a linear combination $\pi_\tau = \pi_{\mathrm{B}} - \tau\omega^{-1}$, with $\tau \in \mathbb{R}$. In general, this object need not be a Poisson structure itself, for π_{B} and ω^{-1} might not be compatible, i.e. the Schouten bracket $[\pi_{\mathrm{B}}, \omega^{-1}]$ might not vanish. When this is the case we call π_τ *Poisson pencil.*

[9]Here $\cdot$ denotes composition of operators. The actual matrix product is reversed in order, cf. (2.16).

As we will eventually restrict to $\mathbb{CP}^n$, which is both a codajoint orbit and a Hermitian symmetric space, it is important to recall that the following theorem holds [42].

Theorem 2.2.3. *For any hermitian symmetric space, the Poisson structures ω^{-1} and π_{B} are compatible.*

2.3 Generalised Geometry

Riemannian geometry is not completely satisfactory when it comes to the description of (bosonic) String Theory. Certainly Riemannian metrics, symplectic forms, complex structures and diffeomorphisms (just to name a few) remain key players; however, the inclusion of the B-field with its associated gauge transformations, the dilaton and the RR fluxes all point to some sort of extension of conventional Riemannian geometry able to put these objects on the same footing. Also, a genuinely stringy feature such as T-duality (to be discussed in the next chapter) relates manifolds through transformations that are simply not ascribable to diffeomorphisms. Generalised Geometry has emerged as the mathematical framework more suited to addressing these issues, resulting in a clearer understanding of the theory as a whole [43–48]. Reviewing its main aspects we will lay out the foundations for several concepts (Double Field Theory, the geometry of integrable deformations etc.) that will constitute the core of the thesis. The main mathematical corpus of Generalised Geometry was developed by Hitchin, Gualtieri and Cavalcanti in an impressive series of papers [24, 25, 49–53] but, in exposing it, we shall also make use of the useful physical review provided by Koerber [54].

2.3.1 The Generalised Tangent Bundle

Conceptually, Generalised Geometry moves from a shift of focus, from the tangent bundle to the *generalised* tangent bundle. Given a Riemannian manifold M of dimension $\dim M = d$ with tangent bundle T, the generalised tangent bundle E is the $2d$-dimensional direct sum bundle $E = T \oplus T^*$. An element X of the section $\Gamma(E)$, called generalised vector, is the formal sum of a vector v and a one-form ξ, $X = v + \xi$. The natural pairing between T and its dual T^* can be used to introduce an inner product on E via

$$(v + \xi, w + \zeta) = \frac{1}{2}(\iota_v \zeta + \iota_w \xi)\,, \tag{2.19}$$

where ι indicates contraction[10]. Using a matrix representation for the generalised vectors $X = (v, \xi)$ and $Y = (w, \zeta)$, the inner product is realised through a pairing η of signature

[10] We will sometimes use the alternative notations $\iota_v \xi = \xi(v) = \langle \xi, v\rangle$.

(d, d)

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{such that} \qquad (X, Y) = \eta(X, Y)\,, \tag{2.20}$$

where 0 and 1 are understood to be $(d \times d)$-dimensional matrices.

The extension of the Lie bracket between vector fields to the generalised tangent bundle is known as the Courant bracket. For two sections $v + \xi$ and $w + \zeta$ of $T \oplus T^*$ the Courant bracket is

$$[v + \xi, w + \zeta] = [v, w] + L_v\zeta - L_w\xi - \frac{1}{2}\mathrm{d}(\iota_v\zeta - \iota_w\xi)\,, \tag{2.21}$$

where L is the ordinary Lie derivative. Sometimes it is useful to consider instead the H-twisted bracket $[\cdot, \cdot]_H$, consisting in a modification of the Courant bracket so as to include the contribution from a closed three-form H,

$$[v + \xi, w + \zeta]_H = [v + \xi, w + \zeta] + \iota_v\iota_w H\,. \tag{2.22}$$

Partially related to this, we shall sometimes make use of the generalised Lie derivative $\mathscr{L}$ which is described by

$$\mathscr{L}_X Y = [v, w] + (L_v\zeta - \iota_w \mathrm{d}\xi)\,. \tag{2.23}$$

2.3.2 $O(d, d)$ Elements and Generalised Diffeomorphisms

Consider the pairing η. Given its signature, it defines an element g of the Lie group $O(d, d)$ through $\eta g^t \eta = g^{-1}$. If $g = \exp(x)$, then x is constrained:

$$x = \underbrace{\begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}}_{x_A} + \underbrace{\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}}_{x_\beta} + \underbrace{\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}}_{x_B}, \tag{2.24}$$

for $A \in \mathfrak{gl}(d)$ and skew-symmetric β, B. Notice that exponentiating x_A and x_B and letting them act on a generalised vector results in

$$e^{x_A} \cdot \begin{pmatrix} v \\ \xi \end{pmatrix} = \begin{pmatrix} e^A v \\ e^{-A^t}\xi \end{pmatrix}, \qquad e^{x_B} \cdot \begin{pmatrix} v \\ \xi \end{pmatrix} = \begin{pmatrix} v \\ \xi + \iota_v B \end{pmatrix}. \tag{2.25}$$

As $e^A \in GL(d)$, the first transformation implements diffeomorphisms on vectors and one-forms. Conversely, the second operation (sometimes known as B-action) does not affect vectors but shifts one-forms, just as gauge transformations do. In this sense, Generalised Geometry is seen to treat diffeomorphisms and gauge transformations on a similar footing. In fact, the B-action leaves the inner product (2.19) invariant and is an automorphism of the Courant bracket (2.21) provided B is a closed two-form. The global symmetry of the theory is given by the semi-direct product of closed two-forms and diffeomorphisms

$$\Omega^2(M)_{\text{cl}} \rtimes \text{Diff}(M)\,, \tag{2.26}$$

sometimes known as *generalised diffeomorphisms*.

2.3.3 Generalised Complex Geometry

Drawing inspiration from the complexification of Riemannian manifolds, Generalised Geometry can be complefixied with the introduction of a (generalised) complex structure [25]. As a first step, the generalised bundle itself needs complexification, obtaining $(T \oplus T^*)_{\mathbb{C}} \equiv (T \oplus T^*) \otimes \mathbb{C}$. Then, a generalised almost complex structure is defined as an endomorphism $\mathcal{J}$ of $T \oplus T^*$, with $\mathcal{J}^2 = -1$, that defines a maximally isotropic sub-bundle[11] L of $(T \oplus T^*)_{\mathbb{C}}$ with $L \cap \bar{L} = 0$[12]. Hence

$$(T \oplus T^*)_{\mathbb{C}} = L \oplus \bar{L}\,. \tag{2.27}$$

To obtain a full-fledged generalised complex structure, an integrability condition is necessary. In Riemannian geometry, the latter is encoded in the vanishing of the Nijenhuis tensor for the complex structure J, expressed in terms of Lie brackets of vector fields. On a generalised tangent bundle, we can impose an identical condition upon replacing the Lie bracket with the Courant bracket and vector fields with generalised vector fields,

$$[\mathcal{J}X, \mathcal{J}Y] - \mathcal{J}([\mathcal{J}X, Y] + [X, \mathcal{J}Y]) = [X, Y]\,. \tag{2.28}$$

Alternatively, and equivalently, $\mathcal{J}$ is said to be integrable if it $+i$-eigenspace L is closed with respect to the Courant bracket.

2.3.3.1 Form of Generalised Complex Structures

A generalised complex structure has to satisfy three type of constraints. Two are algebraic, namely $\mathcal{J}^2 = -1$ and $\mathcal{J}\eta\mathcal{J}^t = \eta$, and one differential, i.e. the integrability condition. Assume $\mathcal{J}$ has the form

$$\mathcal{J} = \begin{pmatrix} I & P \\ L & K \end{pmatrix}\,, \tag{2.29}$$

[11] A subspace L of $E = T \oplus T^*$ is called isotropic if, for any $X, Y \in \Gamma(L)$, $(X, Y) = 0$. If the dimension of L is maximal, i.e. $\dim(L) = \dim(T)$, L is called maximally isotropic or Lagrangian.

[12] L is the $+i$-eigenspace in $(T \oplus T^*)_{\mathbb{C}}$ and $\bar{L}$ the $-i$-eigenspace.

where no conditions are placed on the blocks. The algebraic constraints, together with a few simple manipulations, lead to $P^t = -P$, $I = -K^t$ and $L^t = -L$. Additional information is contained in the differential constraint. One can try and plug the generic form for $\mathcal{J}$ into the integrability condition (2.28) and workout the requirements for the blocks. Crucially, the one for P reads $[P, P] = 0$ where $[\cdot, \cdot]$ is the Schouten bracket. Hence $P \equiv \pi$ is a Poisson structure [55]. The conclusion is that the generic form of an integrable generalised complex structure is

$$\mathcal{J} = \begin{pmatrix} I^t & \pi \\ L & -I \end{pmatrix} \qquad \text{with} \qquad L\pi = -1 - I^2\,, \tag{2.30}$$

plus the left-over differential constraints. Notice that two possibilities stand out. Namely, if J and ω are respectively a complex structure and a symplectic form,

$$\mathcal{J}_J = \begin{pmatrix} J^t & 0 \\ 0 & -J \end{pmatrix}\,, \qquad \mathcal{J}_\omega = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}\,. \tag{2.31}$$

Generalised Complex Geometry thus encompasses symplectic and complex geometry.

2.3.3.2 Pure Spinors

Generalised complex structures can be rephrased in terms of polyforms. Given a manifold M, take a polyform[13] $\phi \in \Omega^\bullet(M)$; the natural action (indicated with $\cdot$) of a generalised vector $X = v + \xi$ on it is

$$X \cdot \phi = \iota_v \phi + \xi \wedge \phi\,. \tag{2.32}$$

Letting $\{X, Y\} \cdot \bullet \equiv (X \cdot Y + Y \cdot X) \cdot \bullet$, we have

$$\{Y, X\} \cdot \phi = 2\eta(Y, X)\,\phi\,, \tag{2.33}$$

so that the $\cdot$ action is in fact a Clifford action. The presence of a Clifford algebra suggests that polyforms transform in the spin representation of $\mathrm{Spin}(d, d)$ and are identified with (generalised) spinors. Actually, the precise statement [54] is that the positive and negative chirality spin bundles $S^\pm$ are isomorphic to

$$S^\pm \cong \bigwedge^{\text{even/odd}} T^* \otimes |\det T^*|^{-1/2}\,. \tag{2.34}$$

That is, a spin representation of positive (negative) chirality corresponds to a polyform of all even (odd) forms, up to a volume form $\epsilon \in \Gamma(\det T^* M)$ entering as $\phi = \epsilon^{1/2}\psi$, where ψ represents the spinor. Preference of one form over the other is just a matter of

[13]A polyform is the formal sum of differential forms of different degree.

convenience. For instance, it is somewhat useful to introduce the Mukai pairing between two polyforms

$$\langle \phi_1, \phi_2 \rangle = (\phi_1 \wedge \sigma(\phi_2))|_{\text{top}} \,, \tag{2.35}$$

where σ is a reversing operator defined by

$$\sigma : \frac{1}{p!}\omega_{i_1 \dots i_p} \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_p} \mapsto \frac{1}{p!}\omega_{i_1 \dots i_p} \mathrm{d}x^{i_p} \wedge \dots \wedge \mathrm{d}x^{i_1} \,, \tag{2.36}$$

and "top" is used to select the top-form (i.e. the form with degree $d = \dim M$).

To reconnect with generalised complex structures we shall start by considering *pure spinors*, i.e. spinors for which the null space $L_\phi = \{X \in T \oplus T^* | X \cdot \phi = 0\}$ as a sub-bundle is maximally isotropic. The *type* of ϕ, i.e. the integer k corresponding to the lowest degree of the forms it is made up of, is useful to constrain its form: as proved by Gualtieri [25], the general form of a non-degenerate (complex) pure spinor is $\phi = \Xi \wedge \rho$, where ρ is a complex two-form, Ξ a decomposable k-form and k its type.

Now, a generalised *almost* complex structure $\mathcal{J}$ comes with a natural maximally isotropic sub-bundle, the $+i$ eigenspace L. A spinor ϕ can be constructed and associated to $\mathcal{J}$ requiring that $L \cong L_\phi$. For all polyforms $\phi \in \Omega^\bullet(M)$ and sections X, Y of the generalised tangent bundle the identity

$$[X, Y] \cdot \phi = [\{X, \mathrm{d}\}, Y] \cdot \phi - \mathrm{d}(\eta(X, Y)) \wedge \phi \tag{2.37}$$

holds. If X, Y are restricted to be sections of the null space L_ϕ, the last term vanishes. On top of that, sections of L_ϕ annihilate ϕ by definition and the identity boils down to

$$[X, Y] \cdot \phi = X \cdot Y \cdot \mathrm{d}\phi \,. \tag{2.38}$$

Integrability of $\mathcal{J}$ requires $L \cong L_\phi$ to be Courant involutive, thereby imposing via (2.38) that $\mathrm{d}\phi = Z \cdot \phi$ for some $Z \in \Gamma(E \otimes \mathbb{C})$. The converse is also true.

Theorem 2.3.1. *A pure spinor ψ defines a generalised complex structure if and only if, for some $X \in \Gamma(E \otimes \mathbb{C})$,*

$$\mathrm{d}\psi = X \cdot \psi \,. \tag{2.39}$$

2.3.3.3 Generalised Kähler Geometry

One of the perks of having a Kähler structure in Riemannian geometry is the ability to deduce the metric g from the complex structure J and Kähler form ω via $g = -J\omega$. As seen, Generalised Complex Geometry already puts ω and J on the same footing as they can be associated to two generalised complex structure $\mathcal{J}_\omega$ and $\mathcal{J}_J$. In particular, we can observe that $\mathcal{H} = \mathrm{diag}(g^{-1}, g) = \mathcal{J}_J \mathcal{J}_\omega \eta$, i.e. the generalised metric of a Kähler manifold, is completely determined by the two generalised complex structures. Clearly this is an extremely special case, but one can nevertheless try and generalise this construction: the result is known as Generalised Kähler Geometry [52] in which the generalised metric can be decomposed in terms of two *commuting* integrable generalised complex structures, $\mathcal{H} = \mathcal{J}_1 \mathcal{J}_2 \eta$.

Even more surprisingly, Gualtieri was able to show that Generalised Kähler geometry is in fact equivalent to bi-Hermitian geometry. The latter was theorised much earlier as the target space of a two-dimensional non-linear σ-model with $\mathcal{N} = (2, 2)$ supersymmetry is required to be bi-Hermitian [56–58]: that is, the metric g should be Hermitian with respect to two complex structures $J_\pm$ each of which is covariantly constant with respect to the torsionful connections $0 = \nabla^{(\pm)} J_\pm = (\partial + \Gamma \pm H) J_\pm$ with $H = \mathrm{d}b$. More precisely the map between Generalised Kähler and bi-Hermitian geometry is[14]

$$\mathcal{J}_{1,2} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ t^{-1} b & 1 \end{pmatrix} \begin{pmatrix} (J_+^t \pm J_-^t) & -t(\omega_+^{-1} \mp \omega_-^{-1}) \\ t^{-1}(\omega_+ \mp \omega_-) & -(J_+ \pm J_-) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t^{-1} b & 1 \end{pmatrix} , \tag{2.40}$$

with $\omega_\pm = J_\pm g$.

[14] We have added here a dependence on the (inverse) string tension t for later convenience.

Chapter 3

Physical Preliminaries

The purpose of the mathematical technology introduced thus far was to lay the ground for several concepts in Theoretical Physics. String Theory is an enormously vast and rich subject, modelled in many facets that most often do overlap or interact with one another. Amongst these many aspects, the present thesis will concentrate on the intersection of two – *duality* and *integrability* – and will revolve mostly around *Poisson-Lie T-duality* and *integrable deformations*. Even so, the amount of literature is somewhat extended but we will try and sharpen our focus on a handful of concepts. As they are mostly built on one another, we will try and follow this logic in the presentation. In doing so, it is perhaps worth remarking that we are really only scratching the surface of these topics: only vital tools are discussed, refraining from furnishing a more thorough overview.

3.1 Classical Integrability

Albeit integrability is a building block of our work, it is most certainly not limited to String Theory by any means. It was first developed in Classical Mechanics and, remarkably, a number of famous models turn out to be integrable, e.g. the Kepler problem or the harmonic oscillator.

Perhaps with a misnomer, integrable models are sometimes called "solvable". Even though integrability and solvability *do appear* together very often, this is far from being a general rule: integrability per se refers to a number of properties a system might have that usually make it the opposite of a *chaotic* system. When this is the case, one has many different tools to extract information, possibly leading to the exact solution. On the other hand, solvability merely refers to one's ability (and, perhaps, technology) to retrieve some sort of solution, regardless of the intrinsic properties of the system.

As we will see shortly, the formalism employed to prove the integrability of classical problems does not quite translate to field theories: passing from a finite to an infinite number of degrees of freedom is tricky, and will prompt us to rethink the way we assess integrability. The introduction of quantum mechanics, leading to the concept of *quantum integrability*, is yet another thing requiring different and advanced mathematics. As we shall not make use of it anywhere in the thesis, we will simply restrict to its classical counterpart. A standard book on the subject, to which the interested reader is referred to for a thorough discussion, is [59].

3.1.1 Liouville Integrability in Classical Mechanics

Integrability in the realm of Classical Mechanics is mostly known as Liouville (or Liouville-Arnold) theory [60, 61]. Classically, dynamics is encoded in the Hamiltonian function H and coordinates on the phase-space are given by positions x^i and momenta p_i with canonical Poisson brackets $\{x^i, p_j\} = \delta^i{}_j$. A system describing the motion of a particle on a D-dimensional manifold M is said Liouville integrable if there are exactly D *independent* conserved charges Q_i in involution, that is, obeying

$$\{Q_i, Q_j\} = 0\,, \quad \forall i, j = 1, \ldots, D\,. \tag{3.1}$$

For a Liouville-integrable theory, it is always possible to solve the equations of motion by quadrature, i.e. through a finite number of algebraic manipulations and integrations. In practice, however, these steps are difficult to perform.

An equivalent way for assessing Liouville integrability of a classical system is through Lax pairs. A Lax pair (L, M) consists of two matrices such that the evolution equation

$$\dot{L} = [L, M] \tag{3.2}$$

encodes the equations of motion of the entire dynamical system[1]. If we define a set of charges as $Q_j = \mathrm{Tr}(L^j)$, these are conserved thanks to (3.2),

$$\dot{Q}_j = j\,\mathrm{Tr}(L^{j-1}[L, M]) = 0\,. \tag{3.3}$$

However, these need not be independent nor in involution. Independence can be either checked explicitly (when possible, e.g. lower-dimensional systems) or just assumed true based on group-theoretical arguments.

[1] We will indicate with a dot the time-derivative, $\dot{L} \equiv \frac{\mathrm{d}}{\mathrm{d}t} L$.

Involution is instead assessed exploiting R-matrices. Let us introduce some notation which will be used throughout the thesis. Given a $\mathfrak{g}$-valued object L ($\mathfrak{g}$ being a Lie algebra), we denote with L_1 and L_2 the tensor products $L_1 = L \otimes 1$ and $L_2 = 1 \otimes L$. Similarly, an element $R = u \otimes v \in \mathfrak{g} \otimes \mathfrak{g}$ can be extended to $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ in a number of ways, e.g. $R_{12} = R \otimes 1$, $R_{13} = u \otimes 1 \otimes v$ etc. As it turns out [59], a sufficient condition for the charges to be in involution is that the Poisson bracket for the Lax matrix L can be written in the form

$$\{L_1, L_2\} = [R_{12}, L_1] - [R_{21}, L_2]\,, \tag{3.4}$$

where r satisfies the classical Yang-Baxter equation which, in tensor notation, reads $[R_{12}, R_{13}] + [R_{12}, R_{23}] + [R_{32}, R_{13}] = 0$.

3.1.2 Classical Integrability in Two-Dimensional Field Theories

Replicating the Liouvillian construction of classical mechanics in two-dimensional field theories (and, in particular, non-linear σ-models [62]) is a delicate matter. In fact, the infinite amount of degrees of freedom of field theory would somehow suggest the presence of an infinite amount of conserved charges (in involution) for an integrable theory. However, this notion of infinity clashes with the limited tools currently at our disposal.

To remedy that, we shall take the notion of Lax pair and upgrade it introducing a *spectral parameter* z, that is, a $\mathbb{C}$-valued scalar the Lax pair is required to (smoothly) depend on. In doing so, we dodge the counting issue and condense the infinity in the continuous parameter z. In fact, for the purposes of field theory, it is better to first consider the Lax connection $\mathcal{L}$, a $\mathfrak{g}$-valued one-form obeying the flatness condition[2]

$$\partial_+ \mathcal{L}_- - \partial_- \mathcal{L}_+ + [\mathcal{L}_+, \mathcal{L}_-] = 0\,. \tag{3.5}$$

To reconnect with the canonical Lax formalism, we can identify the temporal and spatial components of the Lax connection with the Lax matrices, $\mathcal{L}_\sigma = L$ and $\mathcal{L}_\tau = M$. Equation (3.5) then implies

$$\partial_\tau L - \partial_\sigma M + [M, L] = 0\,. \tag{3.6}$$

Any system admitting a Lax connection whose flatness implies the equations of motion is called *weakly* integrable, i.e. admits an infinite number of charges. As in Classical Mechanics, these need not be in involution and further requirements have to be placed.

[2]Recall the conventions for worldsheet coordinates: $\sigma^\pm = \tau \pm \sigma$, $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$.

3.1.3 Maillet Algebra

A sufficient condition for addressing the problem of the involutive conserved charges in two-dimensional field theories was put forward by Maillet in a number of papers [63, 64]. His proposal is reminiscent of that encountered in the previous section: it states that a sufficient condition for the infinite charges obtained from the Lax connection to be in involution is that the *equal time* Poisson brackets are, in tensor notation, given by

$$\begin{aligned}\{L_1(\sigma,z),L_2(\sigma',w)\} = [\mathcal{R}_{12}(z,w),L_1(\sigma,z)]\delta(\sigma-\sigma') - [\mathcal{R}_{21}(w,z),L_2(\sigma',w)]\delta(\sigma-\sigma') \\ -(\mathcal{R}_{12}(z,w)+\mathcal{R}_{21}(w,z))\delta'(\sigma-\sigma')\,.\end{aligned} \tag{3.7}$$

Some comments are in order. Whenever derivatives of the Dirac delta appear, the theory is said to be *non-ultra local*: this is the case we shall be interested in. Conversely, if the theory is *ultra-local* (i.e. it only contains the Dirac delta), a sufficient condition was already suggested by Sklyanin [65]. Notice that, in this sense, the $\mathfrak{g}\otimes\mathfrak{g}$-valued matrix $\mathcal{R}$ need not be skew-symmetric, i.e. $\mathcal{R}_{12}(z,w)\neq\mathcal{R}_{21}(w,z)$. If so, the non-ultra local term would drop out. Hence, we shall assume that $\mathcal{R}$ has a symmetric part s and a skew-symmetric part r,

$$r_{12}(z,w)=\frac{1}{2}(\mathcal{R}_{12}(z,w)-\mathcal{R}_{21}(w,z))\,,\qquad s_{12}(z,w)=\frac{1}{2}(\mathcal{R}_{12}(z,w)+\mathcal{R}_{21}(w,z))\,. \tag{3.8}$$

Therefore, we can rephrase the Maillet condition as the following

$$\begin{aligned}\{L_1(\sigma,z),L_2(\sigma',w)\} =&[r_{12}(z,w),L_1(\sigma,z)+L_2(\sigma,w)]\delta(\sigma-\sigma') \\ &+[s_{12}(z,w),L_1(\sigma,z)-L_2(\sigma,w)]\delta(\sigma-\sigma')-2s_{12}(z,w)\delta'(\sigma-\sigma')\,.\end{aligned} \tag{3.9}$$

Observe that the Poisson bracket on the left-hand side should obey the Jacobi identity; this requires $\mathcal{R}$ to satisfy the classical Yang-Baxter equation, making it an R-matrix.

3.1.4 Application to NLSM on Group Manifolds

As long as integrability is concerned, this thesis will revolve around (integrable) non-linear σ-models with (deformations of) group manifolds or cosets as target spaces. In such cases the general discussion from the preceding section can be made more specific as we shall now explain[3]. If the target space is a Lie group G with semi-simple Lie algebra $\mathfrak{g}$, we have at our disposal a canonical construction for the r and s matrices (provided the model is weakly integrable, of course). More precisely, we can consider the split Casimir

[3] Even though integrable deformations do affect the geometry to a great extent, most of the general results here carry on unaltered, as they only rely on having an underlying group/coset structure.

element of $\mathfrak{g}$,

$$\mathcal{C}_{12} = \kappa^{\mathbb{ab}} T_{\mathbb{a}} \otimes T_{\mathbb{b}} \,, \tag{3.10}$$

where κ is the non-degenerate Killing form, and $\{T_{\mathbb{a}}\}$ a basis of generators for $\mathfrak{g}$. It is possible to show (see [66] for a review) that the quantity

$$\mathcal{R}^0_{12}(z,w) = \frac{1}{w-z}\mathcal{C}_{12} \tag{3.11}$$

obeys the CYBE. This solution for $\mathcal{R}$ is called the standard *non-twisted* R-matrix on the loop algebra of $\mathfrak{g}$. In particular, $\mathcal{R}^0_{12}$ is skew-symmetric as swapping w and z produces a minus sign. Furthermore, given any function $\varphi : \mathbb{C} \to \mathbb{C}$ the matrix $\mathcal{R}_{12}(z,w) = \mathcal{R}^0_{12}(z,w)\varphi^{-1}(w)$ satisfies the CYBE. This new $\mathcal{R}$ is called a *twisted* R-matrix and φ the *twist function*. The r and s matrices are affected by φ according to

$$r_{12}(z,w) = -\frac{1}{2}\frac{\varphi^{-1}(w)+\varphi^{-1}(z)}{z-w}\mathcal{C}_{12}\,, \qquad s_{12}(z,w) = -\frac{1}{2}\frac{\varphi^{-1}(w)-\varphi^{-1}(z)}{z-w}\mathcal{C}_{12}\,. \tag{3.12}$$

The twist function encodes crucial details for the theory at the hand and its computation should just represent our primary concern[4]. Ideally, we would start from a set of Poisson brackets, compute the left-hand side of (3.9), infer r and s and finally extrapolate the twist function. Actually, the matter is slightly more delicate. Suppose we decompose the Lax matrix according to

$$L(z) = \sum_{Q\in\mathcal{O}} A_Q(z) Q\,, \tag{3.13}$$

where Q are the operators/fields in our theory and A_Q coefficients possibly depending on the spectral parameter. On the right-hand side of (3.9), we can separate ultra-local terms (named collectively P^{UL}_{12}) from non-ultra-local terms (called P^{NUL}_{12}) so that

$$\{L_1(\sigma,z), L_2(\sigma',w)\} = P^{\text{UL}}_{12}(z,w,\sigma)\delta(\sigma-\sigma') + P^{\text{NUL}}_{12}(z,w)\delta'(\sigma-\sigma')\,. \tag{3.14}$$

Given the form for s as in (3.12), it is evident that $P^{\text{NUL}}_{12}(z,w) = -\frac{\varphi^{-1}(z)-\varphi^{-1}(w)}{z-w}\mathcal{C}_{12}$. As for the ultra-local part we observe that $[\mathcal{C}_{12}, Q_1 + Q_2] = 0$, for structure constants are completely antisymmetric objects. Making use of this fact when plugging the expansion (3.13) for L into (3.9) we arrive at

$$P^{\text{UL}}_{12}(z,w,\sigma) = \sum_Q \frac{A_Q(z)\varphi^{-1}(w) - A_Q(w)\varphi^{-1}(z)}{z-w}[\mathcal{C}_{12}, Q_2(\sigma)]\,. \tag{3.15}$$

This step does not determine r or s at all, but it is nevertheless useful as it singles out the dependence on the twist function.

[4]It has been recently proven in [67] that it is possible to extract one-loop β-functions out of the twist function (see also [68]).

3.1.5 Example 1: Principal Chiral Model

It is now time to apply the technology developed so far to some actual examples. Given its centrality for the remainder of this thesis, we shall start from the Principal Chiral Model on a Lie group G. We will use this example as an excuse to introduce the model and study its main features.

The principal chiral model (PCM) with "radius" k and tension t is expressed by the action

$$S = \frac{k^2}{\pi t} \int_\Sigma \mathrm{d}^2\sigma \, \langle g^{-1}\partial_+ g, g^{-1}\partial_- g \rangle \,, \tag{3.16}$$

where $g : \Sigma \to G$ are maps from the worldsheet Σ to the target space G, a (semi-simple) Lie group. As in Chapter 2, the pairing $\langle \cdot, \cdot \rangle$ denotes the Killing form on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Famously, this action shows a *global* $G_L \times G_R$ symmetry, corresponding to multiplication of g by a constant group element either on the left or on the right. Introducing the currents (left-invariant forms) $j_\pm = g^{-1}\partial_\pm g$, the equations of motion and Bianchi identity are

$$\partial_\tau j_\tau - \partial_\sigma j_\sigma = 0 \,, \tag{3.17}$$

$$\partial_\tau j_\sigma - \partial_\sigma j_\tau + [j_\tau, j_\sigma] = 0 \,. \tag{3.18}$$

It is easy to check that the flatness of the Lax connection

$$\mathcal{L}_\pm(z) = \frac{j_\pm}{1 \mp z} \tag{3.19}$$

implies both equations of motion *and* Bianchi identity, thereby making the model weakly integrable.

Proving strong integrability requires knowing the Poisson brackets involving j_τ and j_σ. If we place coordinates $x^{\mathbb{i}}$ on G, dual to a set of momenta $p_{\mathbb{i}}$, their (equal time) Poisson brackets are canonical $\{x^{\mathbb{i}}(\sigma), p_{\mathbb{j}}(\sigma')\} = \delta^{\mathbb{i}}{}_{\mathbb{j}}\delta(\sigma - \sigma')$. Letting $e = g^{-1}\mathrm{d}g$ be a frame field, the momentum is $p_{\mathbb{i}} = \frac{1}{2\pi} e_{\mathbb{a}\mathbb{i}} j^{\mathbb{a}}_\tau$, making $X = e^{\mathbb{a}\mathbb{i}} p_{\mathbb{i}} T_{\mathbb{a}} = \frac{1}{2\pi} j_\tau$ a convenient object to introduce[5]. In the same spirit, it is sensible to introduce $Y = \frac{1}{2\pi} j_\sigma$. Since $\{g_1(\sigma), g_2(\sigma')\} = 0$,

$$\{Y_1(\sigma), Y_2(\sigma')\} = 0 \,. \tag{3.20}$$

[5] In the case of PCM, some of these definitions are redundant. For instance, the frame field coincides with the current, $e^{\mathbb{a}}{}_{\mathbb{i}} = j^{\mathbb{a}}{}_{\mathbb{i}}$ and X is just a multiple of j_τ. However, in view of more complicated cases, we shall keep these objects separate.

Recalling that vector fields dual to left-invariant one-forms $v_{\mathbb{a}} = e^{\mathbb{i}}{}_{\mathbb{a}}\partial_{\mathbb{i}}$ generate the Lie algebra of G, $[v_{\mathbb{a}}, v_{\mathbb{b}}] = f_{\mathbb{a}\mathbb{b}}{}^{\mathbb{c}} v_{\mathbb{c}}$, we also have

$$\{X_1(\sigma), X_2(\sigma')\} = -[\mathcal{C}_{12}, X_2(\sigma)]\delta(\sigma - \sigma')\,. \tag{3.21}$$

The algebra is completed with

$$\{X_1(\sigma), Y_2(\sigma')\} = -[\mathcal{C}_{12}, Y_2(\sigma)]\delta(\sigma - \sigma') - \frac{1}{2\pi}\mathcal{C}_{12}\partial_\sigma\delta(\sigma - \sigma')\,. \tag{3.22}$$

Now, the Lax matrix is

$$L = \mathcal{L}_+ - \mathcal{L}_- = \frac{2\pi}{1 - z^2}(zX + Y) \equiv A_X(z)X + A_Y(z)Y\,, \tag{3.23}$$

where, following the notation outlined above, we have identified $A_X(z) = \frac{2\pi z}{1-z^2}$ and $A_Y(z) = \frac{2\pi}{1-z^2}$. Computing $\{L_1(\sigma, z), L_2(\sigma', w)\}$ making use of the brackets given above and exploiting e.g. (3.15), we find the twist function for the PCM:

$$\varphi(z) = \frac{1}{2\pi}\frac{z^2}{1 - z^2}\,. \tag{3.24}$$

3.1.6 Example 2: Principal Chiral Model on Symmetric Spaces

The construction of the PCM action can be modified so as to accommodate the case of cosets. There are a number of additional intricacies entailed in this scenario (mostly when including deformations), but for the Principal Chiral Model they do in fact remain easily under control. We will concentrate on a specific type of cosets − symmetric spaces − to which will return on multiple occasions in what follows.

Consider a semi-simple Lie group G and a subgroup $H \subset G$ such that G/H is a symmetric space. That is, there exists an automorphism of $\mathfrak{g} = \text{Lie}(G)$ realizing a $\mathbb{Z}_2$-gradation of the algebra,

$$\mathfrak{g} \cong \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}\,. \tag{3.25}$$

The subalgebra $\mathfrak{h} = \text{Lie}(H) \cong \mathfrak{g}^{(0)}$ and the linear space $\mathfrak{m} \cong \mathfrak{g}^{(1)}$ correspond to the $+1$ and -1 eigenspace, respectively. The $\mathbb{Z}_2$-gradation enforces the (schematic) commutation relations $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. The projection of $x \in \mathfrak{g}$ onto $\mathfrak{g}^{(i)}$ is obtained via projectors $\mathcal{P}_i$ and denoted with $x^{(i)}$.

If $j_\pm = g^{-1}\partial_\pm g$ with $g \in G$, the action for the Principal Chiral Model on the G/H symmetric space is

$$S = \frac{1}{\pi}\int \mathrm{d}^2\sigma \langle j_+^{(1)}, j_-^{(1)}\rangle\,. \tag{3.26}$$

The PCM on a group manifold had a global symmetry $G_L \times G_R$, corresponding to left and right multiplication by a *constant* element of G. Now, considering the right-multiplication by a *non constant* element $h \in H$,

$$j_\pm \to h^{-1} j_\pm h + h^{-1} \partial_\pm h \,. \tag{3.27}$$

The second term on the right-hand side is $\mathfrak{h}$-valued and drops out upon applying the projector $\mathcal{P}_1$. Ad-invariance of the inner product $\langle \cdot, \cdot \rangle$ then ensures that H_R is a gauge symmetry for the action (3.26). The left global symmetry G_L is left untouched.

The equations of motion and Bianchi identity for the system are given by

$$0 = \partial_\pm j_\mp^{(1)} + [j_\pm^{(0)}, j_\mp^{(1)}] \,, \tag{3.28}$$

$$0 = \partial_+ j_-^{(0)} - \partial_- j_+^{(0)} + [j_+^{(0)}, j_-^{(0)}] + [j_+^{(1)}, j_-^{(1)}] \,, \tag{3.29}$$

and can be encoded in the Lax connection

$$\mathcal{L}_\pm = j_\pm^{(0)} + z^{\pm 1} j_\pm^{(1)} \,. \tag{3.30}$$

Having assessed weak integrability, we can try and ascertain strong integrability of the model. This poses a challenge, for the H_R local symmetry makes the theory *constrained*: different solutions of the equations of motion are related to one another via gauge transformations. This subtle point is overshadowed in the Lagrangian formalism (weak integrability) but most certainly kicks in when the Hamiltonian formalism is involved (strong integrability).

The theory of constrained Hamiltonian systems is a vast topic in the Mathematical Physics literature and we will not aim at providing a complete treatment of the subject which can be found in the standard references [69, 70]. Rather, we shall focus on aspects pertinent to our case. A Lagrangian with a gauge invariance group becomes a constrained Hamiltonian system. Gauge symmetry implies the existence of first class constraints, i.e. quantities having weakly vanishing Poisson brackets with *all* other constraints[6]. The model might also require second class constraints, that is having some non-weakly vanishing Poisson bracket. We will label the set of N constraints generically by Φ_k, $k = 1, \dots, N$. Accordingly, the Lax matrix as well as the Hamiltonian can, and as we will see should, be *extended* by adding terms proportional to these constraints.

As in the group case, the algebra-valued momentum X takes the form

$$X^{(1)} = \frac{1}{2\pi} j_\tau^{(1)} \,. \tag{3.31}$$

[6]A function F of the phase space variables is said to vanish *weakly* if it does when the constraints are applied. In Dirac's notation, this situation is indicated with $F \approx 0$.

As a consequence of the H_R-invariance, its subgroup-directed counterpart $X^{(0)}$ implies the primary constraint[7] $\Phi_1 : X^{(0)} \approx 0$, which as expected is first class. Replicating the construction of the previous section, we shall complete the picture with the spatial component of the current, $Y^{(1)} = \frac{1}{2\pi} j_\sigma^{(1)}$. The Legendre transformation of the Lagrangian leads to the Hamiltonian

$$\text{Ham} = \frac{\pi}{2} \int \mathrm{d}\sigma \left(\langle X^{(1)}, X^{(1)} \rangle + \langle Y^{(1)}, Y^{(1)} \rangle \right) . \tag{3.32}$$

As argued above, in a constrained Hamiltonian system, the correct object to consider is instead the *extended* Hamiltonian

$$\text{Ham}_E = \text{Ham} + \int \mathrm{d}\sigma \, \langle \chi, X^{(0)} \rangle \,, \tag{3.33}$$

where χ is an algebra-valued Lagrange multiplier. Using the extended Hamiltonian one can check that no secondary constraints arise from the stability under time evolution of the constraint, $\{\text{Ham}_E, X^{(0)}\} = 0$.

Replicating the reasoning above, one ought to consider a suitably extended Lax matrix $L_E(z) = L(z) + f(z)\Phi$ obeying the usual Maillet algebra. The function $f(z)$ can be fixed by requiring the closure of the Maillet algebra. In the coset case, the untwisted R-matrix $\mathcal{R}^0$ also assumes a new form [66],

$$\mathcal{R}^0_{12}(z, w) = \frac{w}{w^2 - z^2} \mathcal{C}^{(00)}_{12} + \frac{z}{w^2 - z^2} \mathcal{C}^{(11)}_{12} \,, \tag{3.34}$$

where now $\mathcal{C}^{(ii)}_{12}$ are the graded components of the Casimir operator. The explicit calculation reveals that, for the PCM on a symmetric space, the twist function and $f(z)$ are given by

$$\varphi(z) = -\frac{1}{\pi} \frac{z}{(1 - z^2)^2} \qquad \text{and} \qquad f(z) = \pi(1 - z^2) \,. \tag{3.35}$$

3.2 T-duality

Dualities relate two seemingly unrelated (string) theories by ensuring that they describe the same physics only, so to speak, from different viewpoints. They come in a number of flavours [71–73], but we shall restrict to the so-called *target space* duality or, for short, T-duality.

[7] Primary constraints are relations between dynamical variables that do not require the equations of motion to hold. However these need to be preserved over time. If this is not immediately the case, one has to provide additional constraints, known as secondary, that depend on the application of the equations of motion.

Its first formalisation was to due to Buscher [9, 10], and was then extensively studied and extended in various directions. At the core it states that, provided some criteria are met, two string backgrounds with different geometries, dilaton and fluxes are physically equivalent, and the choice of one over the other is just a matter of convenience. Most importantly, the explicit relation between the two is known, making us able to move with relative ease between the two descriptions.

As a matter of fact, T-duality itself comes in a number of flavours depending on the geometric features of the background we are trying to T-dualise. The easiest (and original) incarnation is *Abelian* T-duality: in this case the target space should display *Abelian* symmetries in the form of isometries and the relationships with its dual go down in the literature as "Buscher rules". Growing in complexity we encounter *non-Abelian* T-duality [74–76] where isometries are now required to furnish a representation of a *non-Abelian* group. Even more general, we find *Poisson-Lie* T-duality [14, 77, 78], where one tries to relax the notion of symmetry (and, in particular, isometry) in favour of what it is called a *Poisson-Lie condition*.

We shall begin with an introduction to Abelian T-duality via the Buscher procedure, i.e. the series of steps which ultimately lead to the Buscher rules. We will show how this set-up might benefit from a doubled formulation (in a sense akin to that of Generalised Geometry), following a recent review [79]. We will then explain how Poisson-Lie T-duality generalises this construction whilst retaining both Abelian and non-Abelian T-dualities as particular cases.

3.2.1 The Buscher Procedure

Take an n-dimensional manifold M. Consider the bosonic Polyakov σ-model Lagrangian having M as target space (hence neglect the overall tension, as well as factors of π); in particular, gauge-fix the worldsheet metric to the constant two-dimensional Minkowski metric and neglect dilation contributions in the form of a Fradkin-Tseytlin term [80–82]. This reads

$$\mathcal{L} = \partial_+ x^\mu (g_{\mu\nu} + b_{\mu\nu}) \partial_- x^\nu \equiv \partial_+ x^\mu E_{\mu\nu} \partial_- x^\nu \, , \tag{3.36}$$

with Greek indices running from 1 to n and light-cone coordinates $\sigma^\pm = \tau \pm \sigma$. We require M to have $D < n$ Abelian isometries realised as D commuting vector fields $k_{\mathbb{i}} = k_{\mathbb{i}}^\mu \partial_\mu$, $\mathbb{i} = 1, \dots, D$. Even though the generalisation is straightforward, we shall take $n = D+1$, i.e. a one-dimensional base. These vectors are Killing and, furthermore, shall preserve fluxes, $L_{k_{\mathbb{i}}} H = 0$, for $H = \mathrm{d}b$. In practice, on M we can choose adapted coordinates $x^\mu = (x^{\mathbb{i}}, y)$ so that neither g nor b depend on x-coordinates, thus realising the symmetry explicitly. The $x^{\mathbb{i}}$- and y-coordinates are called "isometries" and "spectators", respectively.

Adopting matrix notation and shorthand

$$E_{\mathbb{i}\mathbb{j}} \equiv E\,, \quad E_{n\mathbb{j}} \equiv N\,, \quad E_{\mathbb{i}n} \equiv M\,, \quad E_{nn} = K\,, \tag{3.37}$$

the Lagrangian expanded according to the coordinate-split reads[8]

$$\mathcal{L} = \partial_+ x^t E \partial_- x + \partial_+ x^t M \partial_- y + \partial_+ y^t N \partial_- x + \partial_+ y^t K \partial_- y\,. \tag{3.38}$$

Notice that M and N serve as "connection terms" between the base manifold and the torus (identified with the $x^{\mathbb{i}}$-coordinates) fibred above it. This observation can be made more precise introducing two connections B and $\widetilde{B}$ that obey

$$M\partial y = EB + \widetilde{B}\,, \qquad \partial y^t N = B^t E - \widetilde{B}^t\,. \tag{3.39}$$

Upon inserting (3.39) into (3.38), the Lagrangian with covariant derivative $\nabla x = \partial x + B$

$$\mathcal{L} = \nabla_+ x^t E \nabla_- x + \partial_+ x^t \widetilde{B}_- - \partial_- x^t \widetilde{B}_+ - B_+^t E B_- + \partial_+ y^t K \partial_- y \tag{3.40}$$

is obtained. Now, the $U(1)^D$ global isometries, which correspond to the constant shifts $x^{\mathbb{i}} \to x^{\mathbb{i}} + \zeta^{\mathbb{i}}$, can be gauged by making ζ a local parameter. Whilst $\mathcal{L}$ enjoys no such local symmetry, gauge invariance is restored inserting $2 \times D$ gauge fields $A_\pm$ transforming as

$$A_\pm \to A'_\pm = A_\pm - \partial_\pm \zeta\,, \tag{3.41}$$

and promoting $\partial_\pm \to \partial_\pm + A_\pm$. In the Buscher procedure we shall require the connection A to be flat, i.e. of vanishing field strength $F = \partial_+ A_- - \partial_- A + = 0$. The easiest way to accomodate this constraint is through the use of a Lagrange multiplier $\tilde{x}_{\mathbb{i}}$,

$$\begin{aligned}\mathcal{L}_{\text{Gauged}} = (\nabla_+ x + A_+)^t E (\nabla_- x + A_-) + (\partial_+ x + A_+)^t \tilde{B}_- - (\partial_- x + A_-)^t \tilde{B}_+ \\ -B_+^t E B_- + \partial_+ y^t K \partial_- y + \tilde{x}^t (\partial_+ A_- - \partial_- A_+)\,.\end{aligned} \tag{3.42}$$

If we integrate over $\tilde{x}$, the gauge field A becomes pure gauge and can be consistently set to zero, thereby recovering the original model (3.40). Conversely, we could try and integrate by parts the term containing the Lagrange multiplier. In this case, A comes with no kinetic term and can thus be integrated out via its equations of motion

$$A_- = -\nabla_- x - E^{-1} \nabla_- \tilde{x}\,, \tag{3.43}$$

$$A_+ = -\nabla_+ x + E^{-t} \nabla_+ \tilde{x}\,, \tag{3.44}$$

[8]From now on, x without indices is understood to represent the isometries $x^{\mathbb{i}}$. Also, despite having a single y-coordinate, we persist on using transposition for a better comparison with $x^{\mathbb{i}}$.

where the covariant derivative has been changed to $\nabla\tilde{x} = \partial\tilde{x} + \widetilde{B}$. What we obtain is the "dual" Lagrangian

$$\mathcal{L}_{\text{Dual}} = \nabla_+\tilde{x}^t\widetilde{E}\nabla_-\tilde{x} - \partial_-\tilde{x}^tB_+ + \partial_+\tilde{x}^tB_- - \widetilde{B}^t_+\widetilde{E}\widetilde{B}_- + \partial_+y^t\widetilde{K}\partial_-y\,, \tag{3.45}$$

where we have identified the combinations

$$\widetilde{E} = E^{-1}\,, \qquad \widetilde{K} = K - NE^{-1}M\,, \qquad \widetilde{M} = E^{-1}M\,, \qquad \widetilde{N} = -NE^{-1}\,. \tag{3.46}$$

The form of (3.45) is identical to that of (3.40), upon using the so-called "Buscher rules" (3.46): they relate metric and B-field of two seemingly different models that, however, describe the same physics i.e. are T-dual.

3.2.2 A Generalised Geometry Perspective

The procedure outlined above provides us with a mechanical way to obtain the Lagrangians (3.38) and (3.45) in the two duality frames. However, T-duality is not really manifest. Our starting point to remedy that is the gauged Lagrangian (3.42): albeit not explicitly, $\mathcal{L}_{\text{Gauged}}$ already hints at a "doubled" torus, for it contains both $x^{\mathbb{i}}$ and $\tilde{x}_{\mathbb{i}}$. For a single scalar field f (with $\partial_\pm f \neq 0$ everywhere), we impose on A the gauge-fixing condition

$$\partial_+fA_- = \partial_-fA_+\,. \tag{3.47}$$

A possible parametrisation is to take $A_\pm = A\partial_\pm f$, with A a $D\times 1$ matrix of scalar fields. The equations of motion (3.43) can be in fact used to solve for A: operating a symmetric choice we find that

$$A = -\frac{1}{2\partial_-f}g^{-1}(E\nabla_-x + \nabla_-\tilde{x}) - \frac{1}{2\partial_+f}g^{-1}(E^t\nabla_+x - \nabla_+\tilde{x})\,. \tag{3.48}$$

We shall now plug this choice for $A_\pm$ back into the gauged action (applying integration by parts on the term containing $\tilde{x}$). Once the dust settles, a number of $O(D,D)$-covariant quantities can be introduced,

$$\mathbb{X} = \begin{pmatrix}\tilde{x}\\ x\end{pmatrix}\,, \quad \mathbb{B} = \begin{pmatrix}\widetilde{B}\\ B\end{pmatrix}\,, \quad \mathcal{H} = \begin{pmatrix}g^{-1} & g^{-1}b\\ -bg^{-1} & g-bg^{-1}b\end{pmatrix}\,, \quad \omega = \begin{pmatrix}0 & -1\\ +1 & 0\end{pmatrix}\,,$$
$$\eta = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\,, \qquad P_\pm = \frac{1}{2}(\eta \pm \mathcal{H})\,, \tag{3.49}$$

as well as an $O(D,D)$-invariant metric

$$\hat{K} = K - \frac{1}{2}Ng^{-1}M - \frac{1}{4}M^tg^{-1}M - \frac{1}{4}Ng^{-1}N^t\,, \tag{3.50}$$

so as to rewrite the gauged Lagrangian in a doubled fashion,

$$\begin{aligned}\mathcal{L}_{\text{Doubled}} = \frac{1}{2}\nabla_+\mathbb{X}^t\mathcal{H}\nabla_-\mathbb{X} - \frac{1}{2}\partial_+\mathbb{X}^t\omega\partial_-\mathbb{X} \\ -\frac{1}{2}\frac{\partial_- f}{\partial_+ f}\nabla_+\mathbb{X}^t\mathcal{H}P_-\nabla_+\mathbb{X} - \frac{1}{2}\frac{\partial_+ f}{\partial_- f}\nabla_-\mathbb{X}^t\mathcal{H}P_+\nabla_-\mathbb{X} \\ +\frac{1}{2}\partial_+\mathbb{X}^t\eta\mathbb{B}_- - \frac{1}{2}\partial_-\mathbb{X}^t\eta\mathbb{B}_+ \\ +\partial_+ y^t\hat{K}\partial_- y\,.\end{aligned} \tag{3.51}$$

One of the novelties with respect to the treatment of Generalised Geometry as provided in Chapter 2 is the presence of ω. The coordinates x and $\tilde{x}$ span the $2D$-dimensional torus T^{2D}, a Kähler manifold. As $T^{2D} \cong \mathbb{C}^D/\Lambda$ for some lattice Λ, we can assume that locally its Kähler form is precisely ω. Hence, the contribution of the new term to the action is simply the pull-back of the Kähler to the worldsheet

$$S_\omega = -\frac{1}{2}\int_\Sigma \mathrm{d}^2\sigma\,\partial_+\mathbb{X}^t\omega\partial_-\mathbb{X} = -\frac{1}{2}\int_\Sigma \mathbb{X}^*(\omega) = -\frac{1}{2}\int_{\mathbb{X}(\Sigma)}\omega\,. \tag{3.52}$$

By definition, ω is a closed two-form and its integral will only depend on the homology class of $\mathbb{X}(\Sigma)$. For a fixed homology class β of $\mathbb{X}(\Sigma)$, S_ω contributes with a pre-factor to the path-integral. In particular, it is insensitive to the worldsheet metric and is thus *topological*. Nevertheless, it is important not to discard this term, as it is vital in the path integral approach [83, 84].

Now, a particularly simple gauge-fixing for the doubled Lagrangian is $\partial_+ f = \partial_- f$. Using $\mathcal{H}P_\pm = \frac{1}{2}(\mathcal{H}\pm\eta)$ and trading the light-cone coordinates for the ordinary τ and σ, an explicit calculation reveals that

$$\begin{aligned}\mathcal{L}_{\text{Doubled}} = -\frac{1}{4}\partial_1\mathbb{X}^t\mathcal{H}\partial_1\mathbb{X} + \frac{1}{4}\partial_1\mathbb{X}^t\eta\partial_0\mathbb{X} + \frac{1}{2}\partial_1\mathbb{X}^t\eta\mathbb{B}_0 - \frac{1}{2}\partial_1\mathbb{X}^t\mathcal{H}\mathbb{B}_1 \\ +\frac{1}{4}\mathbb{B}_1^t\eta\mathbb{B}_0 - \frac{1}{4}\mathbb{B}_1^t\mathcal{H}\mathbb{B}_1 + \frac{1}{4}\partial_\mu y^t\hat{K}^t\partial^\mu y + \frac{1}{4}\partial_0\mathbb{X}^t\omega\partial_1\mathbb{X}\,.\end{aligned} \tag{3.53}$$

This is the Hull-Tseytlin action [11, 85]. Its minimal formulation – the one we shall use in Chapter 7 – is recovered for $\mathbb{B} = 0$[9]. Interestingly, for vanishing connection the dual metric on the base boils down to K,

$$\mathcal{L}_{\text{Minimal}} = -\frac{1}{4}\partial_1\mathbb{X}^t\mathcal{H}\partial_1\mathbb{X} + \frac{1}{4}\partial_1\mathbb{X}^t\eta\partial_0\mathbb{X} + \frac{1}{4}\partial_\mu y^t K^t\partial^\mu y\,. \tag{3.54}$$

[9]We will also neglect the topological term for it only contributes an overall factor, as discussed.

3.2.3 Abelian T-Duality in Generalised Geometry

Abelian T-duality and the Buscher rules are better understood in a generalised-geometric formalism [43]. For simplicity, we shall restrict to the case of a $U(1)$ isometry group, even though a generalisation is straightforward. Recall that the Buscher rules hold provided we have an Abelian isometry in our background, meaning that there is a vector v such that

$$L_v g = 0\,, \qquad L_v H = 0\,, \tag{3.55}$$

for a Riemannian metric g and an H-flux $H = \mathrm{d}b$. In general, $L_v H = 0$ does not imply that $L_v b = 0$. A simple counterexample is when $v = \partial_\theta$, $H = \mathrm{d}\theta \wedge \mathrm{d}x \wedge \mathrm{d}y$ and $b = \theta \mathrm{d}x \wedge \mathrm{d}y$. Still, the invariance of H has implications for the invariance of b. Given that H is exact, we have $L_v H = \mathrm{d}\iota_v H$, meaning that we can rewrite, at least locally, $\iota_v H = \mathrm{d}\alpha$, for some two-form α. In a similar fashion, we have that

$$L_v b = \mathrm{d}\iota_v b + \iota_v H = \mathrm{d}(\iota_v b + \alpha)\,. \tag{3.56}$$

The B-field is defined up to gauge transformations, though. Hence, we could define a new field $b' = b + \mathrm{d}\xi'$ such that now $L_v b' = \mathrm{d}(\iota_v b + \alpha + \iota_v \mathrm{d}\xi')$. In practice, we can tune ξ' so as to make the former expression vanish. Or, in terms of b, we can equivalently write

$$L_v b = \mathrm{d}(\iota_v \mathrm{d}\xi' + \mathrm{d}f) \equiv \mathrm{d}\xi\,, \tag{3.57}$$

where we have introduced yet another scalar function f thanks to properties of the exterior derivative. So, to sum up, Abelian T-duality is possible whenever

$$L_v g = 0\,, \qquad L_v b = \mathrm{d}\xi\,. \tag{3.58}$$

As we can notice, these conditions rely on a vector field v and on a one-form ξ. It is natural to gather them together in a unified object − a generalised vector $V = v + \xi$. As one can check, our conditions can be rephrased as the invariance of the generalised metric under the generalised Lie derivative $\mathscr{L}_V \mathcal{H} = 0$.

Now, recall that the vector ξ is not completely fixed, for we have the gauge redundancy $\xi \to \xi + \mathrm{d}f$ leaving the conditions for T-duality unaltered. We could gauge fix this freedom by requiring that the generalised vector be of unit norm, $\eta(V, V) = 1$. This goes as follows. First, we can always find adapted coordinates such that $v = \partial_\theta$ for some direction θ (the isometry). Then, by construction, it follows that $V = \partial_\theta + \iota_\theta \mathrm{d}\xi' + \mathrm{d}f$. Finally, by definition $\eta(V, V) = \iota_\theta \mathrm{d}f$. If we choose $\mathrm{d}f = \mathrm{d}\theta$ then the norm is one.

Therefore, after gauge fixing, our generalised vector looks like

$$V = \partial_\theta + \mathrm{d}\theta \, - \iota_\theta \mathrm{d}\xi' \,. \tag{3.59}$$

Notice that, in most cases, the last term is not really necessary for we already have a B-field not depending at all on the isometric direction. The T-duality transformation corresponding to the Buscher rules is then implemented via an $O(D,D)$ matrix

$$T = 1 - 2V \otimes (V^t \eta) \tag{3.60}$$

acting on the generalised metric $\mathcal{H}$ as $T\mathcal{H}T^t$. The form with indices is somewhat more explicit, $T_{\mathbb{I}}{}^{\mathbb{J}} = \delta_{\mathbb{I}}{}^{\mathbb{J}} - 2V_{\mathbb{I}}V^{\mathbb{J}}$. It is simple to check that $T^2 = 1$, meaning that the application of two T-dualities along the same direction brings us back to the background we started from.

3.2.4 Poisson-Lie T-Duality

We can now try and upgrade the concept of T-duality to backgrounds other than those admitting Abelian isometries. In fact, we would like to completely give up on the requirement of isometries as a whole.

To this end, consider a D-dimensional target manifold M and take the two-dimensional non-linear σ-model with action

$$S = \frac{1}{\pi t} \int \mathrm{d}^2\sigma \, \partial_+ x^\mu E_{\mu\nu} \partial_- x^\nu \,, \tag{3.61}$$

where E is a shorthand for the generalised metric $E = G + B$. Again, we have fixed the worldsheet metric to the flat Minkowski metric and neglected the Fradkin-Tseytlin term. Given a Lie group G acting freely on M, let $v_{\mathbb{a}} = v^\mu{}_{\mathbb{a}} \partial_\mu$, $\mathbb{a} = 1, \ldots, \dim G$ be the left-invariant vector fields corresponding to the right G-action. For a set of worldsheet coordinates-dependent parameters $\epsilon^{\mathbb{a}} = \epsilon^{\mathbb{a}}(\sigma^+, \sigma^-)$, the variation of the action entailed by the diffeomorphism $x^\mu \to x^\mu + \epsilon^{\mathbb{a}} v_{\mathbb{a}}{}^\mu$ reads

$$\delta S = \frac{1}{\pi t} \int \mathrm{d}^2\sigma \, \epsilon^{\mathbb{a}} \partial_+ x^\mu (L_{v_{\mathbb{a}}} E)_{\mu\nu} \partial_- x^\nu + \frac{1}{\pi t} \int \mathrm{d}\epsilon^{\mathbb{a}} \wedge J_{\mathbb{a}} \,, \tag{3.62}$$

where we have introduced the convenient one-form

$$J_{\mathbb{a}} = -\partial_+ x^\nu E_{\nu\mu} v_{\mathbb{a}}{}^\mu \mathrm{d}\sigma^+ + v_{\mathbb{a}}{}^\mu E_{\mu\nu} \partial_- x^\nu \mathrm{d}\sigma^- \,. \tag{3.63}$$

In case of the G-action being an isometry, $L_{v_{\mathbb{a}}}(E) = 0$, the first term in δS vanishes and $\star J_{\mathbb{a}}$ is in fact a conserved current thanks to $\mathrm{d}J_{\mathbb{a}} = 0$[10]. This is just plain Noether theorem at work. However, we would like to give up on the presence of isometries in favour of a milder condition.

In [77], Klimčík and Ševera put forward the idea of requiring the current $J_{\mathbb{a}}$ not to be conserved but to instead obey a Maurer-Cartan type of equation, namely

$$\mathrm{d}J_{\mathbb{a}} + \frac{1}{2}\tilde{f}^{\mathbb{b}\mathbb{c}}{}_{\mathbb{a}} J_{\mathbb{b}} \wedge J_{\mathbb{c}} = 0\,. \tag{3.64}$$

The $\tilde{f}^{\mathbb{b}\mathbb{c}}{}_{\mathbb{a}}$ are just some constants that will be given proper justification in a moment. Judging from δS, this condition imposes a constraint on the Lie derivative of the generalised metric,

$$L_{v_{\mathbb{a}}} E_{\mu\nu} = E_{\mu\rho} E_{\sigma\nu}\, v^{\rho}{}_{\mathbb{b}} v^{\sigma}{}_{\mathbb{c}}\, \tilde{f}^{\mathbb{b}\mathbb{c}}{}_{\mathbb{a}}\,. \tag{3.65}$$

This equation is known as the *Poisson-Lie condition*. Actually, the identity $L_{[v_{\mathbb{a}}, v_{\mathbb{b}}]} = [L_{v_{\mathbb{a}}}, L_{v_{\mathbb{b}}}]$ for the Lie derivative enforces a relation between the structure constants $f_{\mathbb{a}\mathbb{b}}{}^{\mathbb{c}}$ of $\mathfrak{g} = \mathrm{Lie}(G)$ and the $\tilde{f}^{\mathbb{b}\mathbb{c}}{}_{\mathbb{a}}$,

$$f_{\mathbb{a}\mathbb{b}}{}^{\mathbb{c}}\,\tilde{f}^{\mathbb{d}\mathbb{e}}{}_{\mathbb{c}} = f_{\mathbb{a}\mathbb{c}}{}^{\mathbb{d}}\,\tilde{f}^{\mathbb{c}\mathbb{e}}{}_{\mathbb{b}} + f_{\mathbb{a}\mathbb{c}}{}^{\mathbb{e}}\,\tilde{f}^{\mathbb{d}\mathbb{c}}{}_{\mathbb{b}} + f_{\mathbb{c}\mathbb{b}}{}^{\mathbb{d}}\,\tilde{f}^{\mathbb{c}\mathbb{e}}{}_{\mathbb{a}} + f_{\mathbb{c}\mathbb{b}}{}^{\mathbb{e}}\,\tilde{f}^{\mathbb{d}\mathbb{c}}{}_{\mathbb{a}}\,. \tag{3.66}$$

We have already encountered this equation in (2.4): it expresses the necessary consistency condition between the structure constants f of a Lie bialgebra $\mathfrak{g}$, and those $\tilde{f}$ of the dual bialgebra $\mathfrak{g}^*$. At this point, it is clear that we would like our $\tilde{f}$'s to precisely be structure constants of a bialgebra $\mathfrak{g}^*$ generated by $T^{\mathbb{a}}$, so that $J = J_{\mathbb{a}} T^{\mathbb{a}} \in \mathfrak{g}^*$.

As explained in Chapter 2, the two bialgebrae $\mathfrak{g}$ and $\mathfrak{g}^* \equiv \widetilde{\mathfrak{g}}$ are in one-to-one correspondence with a Drinfel'd double $\mathfrak{d} = \mathfrak{g} \oplus \widetilde{\mathfrak{g}}$. Duality is expressed in the ability to swap roles for $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$, obtaining a dual model with currents $\widetilde{J}^{\mathbb{a}}$ and generalised metric $\widetilde{E}_{\mu\nu}$ obeying

$$\mathrm{d}\widetilde{J}^{\mathbb{a}} + \frac{1}{2} f_{\mathbb{b}\mathbb{c}}{}^{\mathbb{a}}\, \widetilde{J}^{\mathbb{b}} \wedge \widetilde{J}^{\mathbb{c}} = 0\,, \qquad L_{\tilde{v}^{\mathbb{a}}} \widetilde{E}_{\mu\nu} = \widetilde{E}_{\mu\rho} \widetilde{E}_{\sigma\nu}\, \tilde{v}^{\rho\mathbb{b}} \tilde{v}^{\sigma\mathbb{c}}\, f_{\mathbb{b}\mathbb{c}}{}^{\mathbb{a}}\,. \tag{3.67}$$

3.2.5 Poisson-Lie Models

For our purposes, we shall focus on the case where M is the group manifold G, meaning that, as a group, G will act on itself. In particular, we shall choose $\mathfrak{g} = \mathrm{Lie}(G) = \mathrm{span}(T_{\mathbb{a}})$ semi-simple, thereby having a coboundary Lie bialgebra structure specified by an R-matrix R. Convention-wise, we will adopt coordinates $x^{\mathbb{i}}$ on G, reserving x^{μ} for more general contexts (e.g. M being a G-fibration). Given $g \in G$, define the left-invariant

[10] The symbol $\star$ denotes the Hodge dual.

one-forms $g^{-1}\mathrm{d}g = e^{\mathbb{a}}T_{\mathbb{a}}$ and their dual vector fields $v_{\mathbb{a}}$[11]. Let $\pi^{\mathbb{ij}}$ be the Poisson bi-vector of the group G, with $\pi^{\mathbb{ab}} = e^{\mathbb{a}}{}_{\mathbb{i}}\pi^{\mathbb{ij}}e_{\mathbb{j}}{}^{\mathbb{b}} = (R_g - R)^{\mathbb{ab}}$ its flattened version. Also, let $(E_0)_{\mathbb{ij}}$ be a $\dim G \times \dim G$ matrix obtained from dressing an *invertible*, *constant* matrix $(E_0)_{\mathbb{ab}}$ with the frame fields $e^{\mathbb{a}}$. In the light of the above, it is straightforward to check that

$$L_{v_{\mathbb{c}}}\pi^{\mathbb{ab}} = \tilde{f}^{\mathbb{ab}}{}_{\mathbb{c}} + f_{\mathbb{dc}}{}^{\mathbb{a}}\pi^{\mathbb{db}} + \pi^{\mathbb{ad}}f_{\mathbb{dc}}{}^{\mathbb{b}}\,. \tag{3.68}$$

Now, consider a string model on G described by the generalised metric

$$E_{\mathbb{ij}} = -e_{\mathbb{j}}{}^{\mathbb{a}}(E_0^{-1} + \pi)^{-1}_{\mathbb{ab}}e^{\mathbb{b}}{}_{\mathbb{j}}\,. \tag{3.69}$$

Exploiting (3.68) we find

$$L_{v_{\mathbb{c}}}E_{\mathbb{ij}} = E_{\mathbb{ik}}E_{\mathbb{lj}}v^{\mathbb{k}}{}_{\mathbb{a}}v^{\mathbb{l}}{}_{\mathbb{b}}\tilde{f}^{\mathbb{ab}}{}_{\mathbb{c}}\,, \tag{3.70}$$

that is, the Poisson-Lie condition. Therefore, any model on G whose generalised metric can be expressed in the form (3.69) for some choice of E_0 is amenable to Poisson-Lie T-duality. Accordingly, we will call *Poisson-Lie model* [14, 77, 78, 86, 87] with tension t'[12] a bosonic non-linear σ-model on G with action

$$S = -\frac{1}{\pi t'}\int \mathrm{d}^2\sigma\, e^{\mathbb{a}}_{+}(E_0^{-1} + \pi)^{-1}_{\mathbb{ab}}e^{\mathbb{b}}_{-}\,. \tag{3.71}$$

To find the dual model, we can just swap the roles of $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$; the bialgebra $\widetilde{\mathfrak{g}}$ exponentiates to a Lie group $\widetilde{G}$ with Poisson bi-vector $\widetilde{\pi}$ which in turns defines a new action

$$\widetilde{S} = -\frac{1}{\pi t'}\int \mathrm{d}^2\sigma\, \widetilde{e}_{\mathbb{a}+}(E_0 + \widetilde{\pi})^{-1\,\mathbb{ab}}\widetilde{e}_{\mathbb{b}-}\,. \tag{3.72}$$

Depending on the properties of G and $\widetilde{G}$, Abelian and non-Abelian T-duality can be recovered. For the sake of simplicity, let us discuss the Principal Chiral Model on G. Recall that its action is given by

$$S = \frac{k^2}{\pi t}\int \mathrm{d}^2\sigma\,\langle g^{-1}\partial_+ g, g^{-1}\partial_- g\rangle\,. \tag{3.73}$$

(Non-)Abelian T-duality is based on having a (non-)Abelian group of *isometries*. In virtue of (3.64), this requires the current $J_{\mathbb{a}}$ to be closed, i.e. the dual structure constants to vanish. Hence, the dual group should be Abelian, $\widetilde{G} = U(1)^D$ with $D = \dim G$. For Abelian T-duality, we fix $G = U(1)^D$; because of (3.67), the dual currents $\widetilde{J}^{\mathbb{a}}$ are closed (that is, $\widetilde{G} = U(1)^D$ generates the D isometries of the dual background). In this case,

[11] That is, they obey $\iota_{v_{\mathbb{a}}}e^{\mathbb{b}} = \delta_{\mathbb{a}}{}^{\mathbb{b}}$, where ι indicates contraction.

[12] We adopt a different tension t' as this might not necessarily coincide with that of the ordinary non-linear σ-model, t.

$E_0 = k^2\kappa$ for κ the Killing form on $\mathfrak{g}$ and, since $\widetilde{\pi} = 0$, we recover the Buscher rule $k \to k^{-1}$. For non-Abelian T-duality, we need G to be non-Abelian, but $\widetilde{G} = U(1)^D$ again. The Poisson structure π still vanishes, but $\widetilde{\pi}$ does not and coincides with the Kirillov-Kostant-Souriau form [88].

3.2.6 $\mathcal{E}$-Models

In the same spirit of Section 3.2.2, the two dual Poisson-Lie models can be conveniently uplifted to a single action on a Drinfel'd double $\mathbb{D} = \exp(\mathfrak{d})$, $\dim \mathbb{D} = 2D$, that comprises of both G and $\widetilde{G}$. The result is called an $\mathcal{E}$-model.

The $\mathcal{E}$-model [14, 21, 89] is a theory of currents $\mathscr{J} = T_{\mathbb{A}}\mathscr{J}^{\mathbb{A}}(\sigma) = g^{-1}\partial_\sigma g$ valued in the loop algebra of $\mathfrak{d}$ which originate from the embedding map $g : \Sigma \to \mathbb{D}$ of the string worldsheet into the Drinfel'd double. The dynamics is generated by the Hamiltonian

$$\mathrm{Ham}_{\mathcal{E}} = \frac{1}{4\pi}\oint \mathrm{d}\sigma \langle\!\langle \mathscr{J}, \mathcal{E}\mathscr{J} \rangle\!\rangle\,, \tag{3.74}$$

in which the eponymous operator $\mathcal{E} : \mathfrak{d} \to \mathfrak{d}$ is an involution, $\mathcal{E}^2 = 1$, that is self-adjoint with respect to the inner product on $\mathfrak{d}$, $\langle\!\langle \cdot, \mathcal{E}\cdot \rangle\!\rangle = \langle\!\langle \mathcal{E}\cdot, \cdot \rangle\!\rangle$. This involution can be specified by D^2 parameters which are associated to those of the Poisson-Lie σ-model. If we split E_0 into a symmetric and skew-symmetric part, $E_0 = g_0 + b_0$, the mapping is

$$\mathcal{E}_{\mathbb{A}}{}^{\mathbb{B}} = (\mathcal{H}\eta^{-1})_{\mathbb{A}}{}^{\mathbb{B}}\,, \qquad \mathcal{H}_{\mathbb{A}\mathbb{B}} = \begin{pmatrix} g_0^{-1} & g_0^{-1}b_0 \\ -b_0 g_0^{-1} & g_0 - b_0 g_0^{-1} b_0 \end{pmatrix}_{\mathbb{A}\mathbb{B}}\,, \qquad \eta_{\mathbb{A}\mathbb{B}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\mathbb{A}\mathbb{B}}\,, \tag{3.75}$$

with $\mathcal{H}_{\mathbb{A}\mathbb{B}}$ the associated generalised metric. The Poisson structure of the theory is defined to be a current algebra

$$\{\mathscr{J}_{\mathbb{A}}(\sigma), \mathscr{J}_{\mathbb{B}}(\sigma')\} = 2\pi F_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}}\mathscr{J}_{\mathbb{C}}(\sigma)\delta(\sigma - \sigma') + 2\pi\eta_{\mathbb{A}\mathbb{B}}\delta'(\sigma - \sigma')\,, \tag{3.76}$$

in which $F_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}}$ are the structure constants on $\mathfrak{d}$ and $\{\cdot, \cdot\}$ denotes the *equal time* Poisson brackets. Accordingly, we find the equations of motion

$$\partial_\tau \mathscr{J} = \{\mathscr{J}, \mathrm{Ham}_{\mathcal{E}}\} = \partial_\sigma \mathcal{E}\mathscr{J} + [\mathcal{E}\mathscr{J}, \mathscr{J}]\,. \tag{3.77}$$

Taking into account that $\mathcal{E}$ is an involution, we could also decompose the currents into chiral and anti-chiral parts

$$\mathscr{J}_\pm = \frac{1}{2}(1 \pm \mathcal{E})\mathscr{J}\,, \tag{3.78}$$

and rewrite (3.77) as

$$\partial_- \mathscr{J}_+ + \partial_+ \mathscr{J}_- + [\mathscr{J}_-, \mathscr{J}_+] = 0\,. \tag{3.79}$$

Since $\mathcal{E}\mathscr{J}_\pm = \pm\mathscr{J}_\pm$, only half of the components of $\mathscr{J}_+$ and $\mathscr{J}_-$ are independent. Therefore, although the $\mathcal{E}$-model apparently depends on $2D$ degrees of freedom contained in g, the first order equations of motion allow half of them (those associated to the subalgebra $\widetilde{\mathfrak{g}}$) to be eliminated on-shell to yield second order equations for the rest (those associated to $\mathfrak{g}$). In this way, the dynamics of the theory specified by the σ-model (3.71) on the target space $M = \mathbb{D}/\widetilde{G}$ is recovered.

The Hamiltonian for a non-linear σ-model can be expressed in terms of a generalised metric $\mathcal{H}_{\mathbb{I}\mathbb{J}}$ as

$$\mathrm{Ham}_{\mathcal{H}} = \frac{1}{4\pi}\oint \mathrm{d}\sigma \begin{pmatrix} 2\pi p & \partial_\sigma x \end{pmatrix}^{\mathbb{I}} \mathcal{H}_{\mathbb{I}\mathbb{J}} \begin{pmatrix} 2\pi p \\ \partial_\sigma x \end{pmatrix}^{\mathbb{J}} \quad \text{with} \quad \mathcal{H}_{\mathbb{I}\mathbb{J}} = \begin{pmatrix} tg^{-1} & g^{-1}b \\ -bg^{-1} & t^{-1}(g - bg^{-1}b) \end{pmatrix}_{\mathbb{I}\mathbb{J}} \tag{3.80}$$

in which the canonical momentum is given by $p_{\mathbb{i}} = (g_{\mathbb{i}\mathbb{j}}\partial_\tau x^{j} - b_{\mathbb{i}\mathbb{j}}\partial_\sigma x^{\mathbb{j}})/2\pi t$ and where $(g_{\mathbb{i}\mathbb{j}}, b_{\mathbb{i}\mathbb{j}})$ is the geometric data entering the σ-model. Here the indices $\mathbb{I}$ denote that the fields act on the generalised tangent space of M and one has the usual canonical Poisson brackets $\{x^{\mathbb{i}}(\sigma), p_{\mathbb{j}}(\sigma')\} = \delta^{\mathbb{i}}_{\mathbb{j}}\delta(\sigma - \sigma')$. Given the structure of $\mathrm{Ham}_{\mathcal{H}}$, it is natural to introduce some generalised currents $\mathscr{J}_{\mathbb{I}} = \left(\partial_\sigma x^{\mathbb{i}},\, 2\pi p_{\mathbb{i}}\right)$ taking values in the generalised tangent space of the target space M. By virtue of the canonical Poisson brackets for (x, p), they can be shown to obey

$$\{\mathscr{J}_{\mathbb{I}}(\sigma), \mathscr{J}_{\mathbb{J}}(\sigma')\} = 2\pi\eta_{\mathbb{I}\mathbb{J}}\delta'(\sigma - \sigma')\,, \tag{3.81}$$

where η is again the $O(D, D)$-invariant metric. The same choice of letter for these currents is not accidental as they can be related to the $\mathcal{E}$-model currents $\mathscr{J}_{\mathbb{A}}$ by introducing generalised frame fields $E_{\mathbb{A}}{}^{\mathbb{I}}$ such that

$$\mathscr{J}_{\mathbb{A}} = E_{\mathbb{A}}{}^{\mathbb{I}}\mathscr{J}_{\mathbb{I}}\,, \quad E_{\mathbb{A}}{}^{\mathbb{I}}\eta_{\mathbb{I}\mathbb{J}}E_{\mathbb{B}}{}^{\mathbb{J}} = \eta_{\mathbb{A}\mathbb{B}}\,, \quad E_{\mathbb{A}}{}^{\mathbb{I}}\mathcal{H}_{\mathbb{I}\mathbb{J}}E_{\mathbb{B}}{}^{\mathbb{J}} = \mathcal{H}_{\mathbb{A}\mathbb{B}}\,. \tag{3.82}$$

The generalised frame field $E^{\mathbb{A}}{}_{\mathbb{I}}$ is the Generalised Geometry analogous of the left invariant Maurer-Cartan form $e^{\mathbb{a}}{}_{\mathbb{i}}$ on the target space group manifold $M = G$. For each value of the algebra index $\mathbb{A}$, $E_{\mathbb{A}}{}^{\mathbb{I}}$ defines a generalised vector comprising of a vector field and a one-form (whose components in a coordinate basis are $E_{\mathbb{A}}{}^{\mathbb{i}}$ and $E_{\mathbb{A}\mathbb{i}}$, respectively). Armed with such a generalised frame field, the elegant results of Alekseev and Strobl [90] show that one can indeed, starting from the canonical Poisson-brackets for $p_{\mathbb{i}}$ and $x_{\mathbb{i}}$ appearing in (3.80), derive the Poisson brackets for the currents $\mathscr{J}_{\mathbb{A}}$ as

$$\{\mathscr{J}_{\mathbb{A}}(\sigma), \mathscr{J}_{\mathbb{B}}(\sigma')\} = 2\pi\mathscr{L}_{E_{\mathbb{A}}}E_{\mathbb{B}}{}^{\mathbb{I}}E^{\mathbb{C}}{}_{\mathbb{I}}\mathscr{J}_{\mathbb{C}}\delta(\sigma - \sigma') + 2\pi\eta_{\mathbb{A}\mathbb{B}}\delta'(\sigma - \sigma')\,. \tag{3.83}$$

Evidently, these are coherent with (3.76) provided $\mathscr{L}_{E_{\mathbb{A}}} E_{\mathbb{B}}{}^{\mathbb{I}} = F_{\mathbb{A}\mathbb{B}}{}^{\mathbb{D}} E_{\mathbb{D}}{}^{\mathbb{I}}$. We will discuss this constraint at length in Chapter 4.

3.3 Integrable Deformations

The name "integrable deformation" is used to refer to a two-dimensional *integrable* σ-model obtained as a smooth deformation of another integrable system. Whilst this definition is quite broad, we shall restrict to deformations of the PCM, either on group manifolds or coset spaces. Depending on the precise implementation, they acquire different names and enjoy different physical as well as mathematical properties.

First off, the relevance of the PCM is to be attributed to it being a solvable toy-model that nevertheless encompasses a number of non-trivial features of QCD, such as confinement, dynamically generated mass gap and asymptotic freedom. In this sense, it is tempting to try and retain as much simplicity as possible, while extending its capabilities to more complicated scenarios.

A first attempt in this direction can be found in the work of Cherednik [91], where a PCM on $SU(2) \cong S^3$ is deformed away from the round three-sphere. In detail, the action with radius k

$$S = \frac{k^2}{\pi t} \int \mathrm{d}^2\sigma \left(\mathrm{Tr}(J_+ J_-) + C J_+^3 J_-^3 \right) \tag{3.84}$$

was considered. The currents J appearing here are the left-invariant one-forms $J = g^{-1}\mathrm{d}g$ with, in particular, $J^3 = \mathrm{Tr}(T_3 J)$ being T_3 the third $\mathfrak{su}(2)$ generator. C is a real, constant, parameter. The original $SU(2)_L \times SU(2)_R$ symmetry of the PCM is apparently broken down to $SU(2)_L \times U(1)_R$ in the deformed case. Despite this loss, the model is still classically integrable. A more sophisticated analysis [92] reveals that this naive symmetry-breaking pattern is not quite right: the Lie algebra $\mathfrak{su}(2)_R$ becomes in fact a quantum enveloping algebra $U_q(\mathfrak{su}(2))$, with parameter q[13] given by

$$q = \exp\left(\frac{\sqrt{C}}{1+C} \right) . \tag{3.85}$$

The fact that the initial symmetry is modified to a quantum group is not an accident of the squashed sphere, but rather a generic feature of integrable deformations (see e.g. [93]).

Reconnecting with more general aspects of String Theory, integrable deformations might provide new insights into the AdS/CFT correspondence. It just so happens that, possibly

[13] The parameter q describes, in general, the deviation from the canonical $\mathfrak{su}(2)$ commutation relations of the Hopf algebra $U_q(\mathfrak{su}(2))$.

in some limit, superstrings moving on a $\mathrm{AdS}_n \times M_{10-n}$ background (for some choice of $n \in \mathbb{N}$ and $(10-n)$-dimensional manifold M_{10-n}) are classically integrable; on the other side of the duality, we find a (classically) integrable CFT. The most famous example is the classical limit of type IIB superstring on $\mathrm{AdS}_5 \times S^5$ dual to the planar limit of $\mathcal{N} = 4$ super Yang-Mills theory [26]. Whereas the interpretation of a deformation on the gravity side is clear (superstring moving on a deformed geometry perhaps with non-vanishing fluxes), less so is its field-theoretical counterpart: the possibility of a deformed gauge theory is still under current investigation [94].

In this thesis, we shall restrict to two types of integrable deformations, named Yang-Baxter model and λ-deformation. They furnish instances of Poisson-Lie T-dual pairs and have been successfully applied as part of the study of the Green-Schwarz superstring on deformed backgrounds [95–100].

3.3.1 (Bi-)Yang-Baxter Model

The Yang-Baxter model, or η-deformation, [16] on a group manifold G owes its name to the parameter η governing the depth of the deformation as per the action

$$\begin{aligned} S_\eta &= \frac{1}{\pi t} \int \mathrm{d}^2\sigma \, \langle g^{-1}\partial_+ g, \frac{1}{1-\eta R_g} g^{-1}\partial_- g \rangle \\ &= \frac{1}{\pi t} \int \mathrm{d}^2\sigma \, \langle \partial_+ g g^{-1}, \frac{1}{1-\eta R} \partial_- g g^{-1} \rangle \, . \end{aligned} \tag{3.86}$$

If $g \in G$, the operator R_g is defined by $R_g = \mathrm{Ad}_g^{-1} \cdot R \cdot \mathrm{Ad}_g$, being R an R-matrix obeying the mCYBE. The prefactor t indicates the string tension. Note that we indeed have a smooth $\eta \to 0$ limit, whence we recover the PCM. The Yang-Baxter model has been proved weakly and strongly integrable in a number of works [93, 101], mostly stemming from the original paper [17]. Technicalities about the actual proof are omitted, but a much more involved set-up (double Yang-Baxter-like deformation of coset spaces) will be studied in great detail in a later chapter. Let us just mention here that R obeying the mCYBE is a fundamental ingredient of the proof.

From an algebraic point of view, the most striking consequence of the deformation is that the original global $G_L \times G_R$ symmetry of the PCM is broken down to G_R only. This is immediately evident from the second expression of S_η in (3.86), formulated in terms of right-invariant forms[14]. In terms of geometry, recall that the PCM action can be recast in the form of a Polyakov action with metric only. The identification of the dressed Killing form κ with the metric G of the Lie group, $G_{\mathbb{i}\mathbb{j}} = e_{\mathbb{i}}{}^{\mathbb{a}} \kappa_{\mathbb{a}\mathbb{b}} e^{\mathbb{b}}{}_{\mathbb{j}}$, where $g^{-1}\mathrm{d}g = e^{\mathbb{a}}{}_{\mathbb{i}} \mathrm{d}x^{\mathbb{i}} T_{\mathbb{a}}$,

[14] In fact, the story is much richer: the relation between conserved changes (coming from the integrable structure) and quantum groups unveils more subtle patterns in the symmetry-breaking process. However, this aspect will not be part of this thesis.

provides the map. The modified action (3.86) entails a different geometric structure: the metric G of the original PCM gets η-deformed to some new g and a (usually) non-trivial B-field b is turned on, $H = \mathrm{d}b \neq 0$.

The Yang-Baxter model can be tweaked so as to include a second deformation parameter, ζ. This new type of deformation, called bi-Yang-Baxter model [17], is encoded in the action

$$S = \frac{1}{\pi t} \int \mathrm{d}^2\sigma \, \langle g^{-1}\partial_+ g, \frac{1}{1 - \eta R_g - \zeta R} g^{-1}\partial_- g \rangle \, . \tag{3.87}$$

Its weak and strong integrability were only proved quite recently, in [102] and [103], respectively. The additional deformation makes it impossible to preserve any global symmetry for generic values of η and ζ.

3.3.2 λ-Deformation

The λ-deformation is a second type of integrable model obtained from a modification of the PCM. It was first introduced by Sfetsos [18] and later proven Poisson-Lie T-dual (modulo an analytic continuation of generators) to the Yang-Baxter model [21]. The construction of the action is somewhat involved and we shall briefly review it. Starting from a PCM at level $\tilde{k}$ on a Lie group $G \ni \tilde{g}$,

$$S_{\mathrm{PCM}}[\tilde{g}] = -\frac{\tilde{k}^2}{\pi} \int \mathrm{d}^2\sigma \langle \tilde{g}^{-1}\partial_+\tilde{g}, \tilde{g}^{-1}\partial_-\tilde{g} \rangle \, , \tag{3.88}$$

we gauge the G_L global symmetry, acting as $\tilde{g} \to h^{-1}\tilde{g}$, by promoting partial to covariant derivatives $\partial_\pm \to D_\pm \equiv \partial_\pm + A_\pm$ for a $\mathfrak{g}$-valued connection one-form A. We indicate the resulting action as $S_{\mathrm{gPCM}}[\tilde{g}, A]$. Additionally, the diagonal symmetry $G_{\mathrm{diag}} : g \to h^{-1}gh$ of the Wess-Zumino-Witten action[15] at level k for a new element $g \in G$,

$$S_{\mathrm{WZW}}[g] = -\frac{k}{2\pi} \int_\Sigma \mathrm{d}^2\sigma \langle g^{-1}\partial_+ g, g^{-1}\partial_- g \rangle - \frac{k}{24\pi} \int_{\mathcal{M}_3} \langle g^{-1}\mathrm{d}g, [g^{-1}\mathrm{d}g, g^{-1}\mathrm{d}g] \rangle \, , \tag{3.89}$$

can be gauged to obtain a gauged WZW action

$$\begin{aligned} S_{\mathrm{gWZW}}[g, A] = S_{\mathrm{WZW}}[g] + \frac{k}{\pi} \int \mathrm{d}^2\sigma \Big(& \langle A_-, \partial_+ g g^{-1} \rangle - \langle A_+, g^{-1}\partial_- g \rangle \\ & + \langle A_-, g A_+ g^{-1} \rangle - \langle A_-, A_+ \rangle \Big) \, . \end{aligned} \tag{3.90}$$

[15]The Wess-Zumino-Witten (WZW) model, which we have not formally introduced so far, is an extension of the PCM that includes a three-dimensional Wess-Zumino term, see (3.89). It was first considered in the context of non-Abelian bosonization by Witten [19]. It is a prime example of conformal field theory with symmetry described by an affine Lie algebra and where standard CFT techniques (e.g. the Sugawara construction) find powerful application. For an extensive treatment the reader is referred to the classic book [104].

The combination of the two actions, $S_\lambda[g,\tilde{g},A] = S_{\text{gWZW}}[g,A] + S_{\text{gPCM}}[\tilde{g},A]$, can be gauge-fixed to $\tilde{g}=1$; besides, the connection A behaves as a Lagrange multiplier and is integrated out. As a consequence, we realise that $S_\lambda[g]$ is a deformation of the WZW model at level k given by

$$S_\lambda = S_{\text{WZW}}[g] - \frac{k\lambda}{\pi}\int \mathrm{d}^2\sigma \langle O_{g^{-1}} g^{-1}\partial_- g, \partial_+ g g^{-1}\rangle\,, \tag{3.91}$$

where $O_g = (1-\lambda\mathrm{Ad}_g)^{-1}$. The effective parameter λ, whence the name of the model, amounts to a combination of the two original levels,

$$\lambda = \frac{k}{k+\tilde{k}^2}\,. \tag{3.92}$$

The λ-deformation has been rigorously proven integrable, in both weak and strong sense, in [18, 105–107]. Since then, a number of papers have addressed its properties. In particular, much like in the bi-Yang-Baxter case and even more so, multi-parameter integrable deformations are allowed [108].

3.3.3 Yang-Baxter Deformation as an $\mathcal{E}$-Model

To reconcile the presentation of Poisson-Lie T-duality with that of integrable deformations, we will show how the Yang-Baxter deformation is an instance of $\mathcal{E}$-model. This fact will be used throughout the thesis a number of times.

The construction of the $\mathcal{E}$-model starts from a comparison of the Poisson-Lie and Yang-Baxter actions, which we recall are[16]

$$S_{\text{PL}} = \frac{1}{\pi}\int \mathrm{d}^2\sigma\, e_+^{\mathbb{a}}(E_0^{-1}+\pi)^{-1}_{\mathbb{a}\mathbb{b}}e_-^{\mathbb{b}}\,, \qquad S_\eta = \frac{1}{\pi t}\int \mathrm{d}^2\sigma\, e_+^{\mathbb{a}}(\kappa^{-1}+\eta R_g)^{-1}_{\mathbb{a}\mathbb{b}}e_-^{\mathbb{b}}\,. \tag{3.93}$$

Owing to the fact that, on the semi-simple group G, $\pi = \eta\kappa^{-1}(R_g - R)$[17] we immediately identify $E_0^{-1} = t\kappa^{-1}(1+\eta R)$. The latter needs to be inverted and split into symmetric and antisymmetric parts so as to build the generalised metric in (3.75). Performing the inversion, with $E_0 = g_0 + b_0$, we find

$$g_0 = t^{-1}\frac{1}{1-\eta^2R^2}\kappa\,, \qquad \text{and} \qquad b_0 = t^{-1}\frac{1}{1-\eta^2R^2}R\kappa\,, \tag{3.94}$$

from which

$$\mathcal{H}_{\mathbb{A}\mathbb{B}} = \begin{pmatrix} t\kappa^{-1}(1-\eta^2R^2) & \eta R\kappa \\ -\eta\kappa R\kappa^{-1} & t^{-1}\kappa \end{pmatrix}_{\mathbb{A}\mathbb{B}}\,. \tag{3.95}$$

[16] Notice that, when using a notation with explicit indices, S_η acquires a slightly different form: in particular the Killing form κ enters explicitly and an additional minus sign due to the transposition of R_g appears.

[17] The presence of η in the definition of the Poisson structure is a useful convention we shall adopt.

3.4 Double Field Theory

The final ingredient of this brief review is Double Field Theory (DFT)[18]. It is well known that the low-energy limit (i.e. $\alpha' \to 0$) of Superstring Theory is Supegravity. In fact, type II superstrings of different chirality (type IIA or IIB) will reduce to supergravities with different field content. Nevertheless, at the SUGRA level they share a common Bosonic action in string frame

$$S = \int \mathrm{d}^D x \, \sqrt{g} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right] , \tag{3.96}$$

where x^μ with $\mu = 1, \ldots, D$ are local coordinates, R is the Ricci scalar of the metric g, $H = \mathrm{d}b$ the flux and ϕ the dilaton. As written, however, this shows no sign of T-duality whatsoever. The idea lying at the core of DFT is precisely to make the $O(D, D)$ T-duality group of String Theory manifest at the Supergravity level.

From a conceptual point of view, an $O(D, D)$ formulation of String Theory compactified on T^D makes perfect sense: the fundamental representation of the group is $2D$-dimensional and we identify D components with momenta and other D components with windings modes of the string on T^D. However, Supergravity is, by definition, the point particle limit of String Theory, and no such thing as winding exists for zero-dimensional objects. For example, coordinates would need to sit in some $O(D, D)$ representation, but we clearly only have D of those, x^μ. Hence, we shall introduce some dual $\tilde{x}_\mu$ so that $\mathbb{X}^I = (\tilde{x}_\mu, x^\mu)$ does transform as an $O(D, D)$ vector. We might then wonder what the new D degrees of freedom are. These are just a mere artefact of our description, introduced to ease the appearance of a manifest T-duality. Of course, such an extension results in an unwanted redundancy. Therefore, a constraint to eliminate those extra fictitious elements is most definitely needed. This is known as the *section condition* for DFT and reads[19]

$$\partial^I \partial_I (\ldots) = 0 \,, \tag{3.97}$$

where the dots indicate any field transforming in some $O(D, D)$ representation. The section condition is then equivalent to $\widetilde{\partial}^\mu \partial_\mu$ which can be solved by imposing $\widetilde{\partial}^\mu (\ldots) = 0$, i.e. removing dependence on the extraneous degrees of freedom. This choice is sometimes called the *supergravity frame*.

[18] Here we shall only provide a streamlined introduction to DFT so as to keep digressions at a minimum. Amongst the many good references, we single out [109] for a more comprehensive review of standard material on the subject.

[19] An equivalent formulation [110] is $Y^I{}_M{}^J{}_N \, \partial_I \partial_J (\ldots) = 0$ with tensor $Y^I{}_M{}^J{}_N = \eta^{IJ} \eta_{MN}$. This is to be preferred when generalising the present discussion to Exceptional Field Theory. The latter is not part of the thesis and we shall not comment on it further.

With this premise, we can start to repackage the objects in (3.96) in $O(D,D)$ multiplets. The lesson of Generalised Geometry suggests to reunite metric and B-field in a generalised metric $\mathcal{H}$ to be accompanied with an $O(D,D)$ pairing η defined as

$$\mathcal{H}_{IJ} = \begin{pmatrix} g^{-1} & g^{-1}b \\ -bg^{-1} & g - bg^{-1}b \end{pmatrix}, \qquad \eta_{IJ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.98}$$

The dilaton requires some further manipulation, though, as ϕ alone is not T-duality invariant. Actually, the combination $\sqrt{g}e^{-2\phi}$ is indeed invariant and, accordingly, we shall call generalised dilaton d the expression

$$d = \phi - \frac{1}{4}\log\det g\,. \tag{3.99}$$

In a similar fashion, diffeomorphisms and gauge transformations parametrised respectively by a vector λ^μ and a one-form $\tilde{\lambda}_\mu$ do combine in generalised diffeomorphisms. They are described by the generalised Lie derivative $\mathscr{L}$ with respect to $\xi^I = (\tilde{\lambda}_\mu, \lambda^\mu)$ acting on a tensorial density V^I of weight ω as

$$\mathscr{L}_\xi V^I = \xi^J \partial_J V^I + (\partial^I \xi_J - \partial_J \xi^I)V^J + \omega \partial_J \xi^J V^I\,. \tag{3.100}$$

With these ingredients it is possible [46] to formulate an action for DFT which reads (up to total derivatives)

$$S = \int \mathrm{d}X\, e^{-2d}\left(\frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\partial_L\mathcal{H}_{MK} + 2\mathcal{H}^{MN}\partial_M\partial_N d\right), \tag{3.101}$$

where $\mathrm{d}X \equiv \mathrm{d}^D x\,\mathrm{d}^D\tilde{x}$. To reconnect with the Einstein-Hilbert action, it is possible to introduce a generalised Ricci scalar $\mathcal{R}$ defined by

$$\begin{aligned} \mathcal{R} = {} & 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M d\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d \\ & + \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL}\,, \end{aligned} \tag{3.102}$$

so that (3.101) becomes $S = \int \mathrm{d}X e^{-2d}\mathcal{R}$. Upon imposing the section condition and parametrising $\mathcal{H}$, η and d according to (3.98) and (3.99) it is possible to recover (3.96).

3.4.1 Flux Formulation of DFT

In the same spirit of the tetrad approach to General Relativity, we could shift our focus from the (generalised) metric to the frame fields to convey a transition between (doubled) *curved* and *flat* indices. The result is known as the flux formulation of DFT [111–114]. Pivotal are generalised frame fields ${E_A}^I$ which are required to turn the generalised metric

$\mathcal{H}_{IJ}$ into a constant object $\mathcal{H}_{AB}$ via $\mathcal{H}_{IJ} = E_I{}^A \mathcal{H}_{AB} E^B{}_J$ *and* preserve the $O(D,D)$ pairing $\eta_{IJ} = E_I{}^A \eta_{AB} E^B{}_J$. A canonical choice is to adopt the parametrisation

$$E^A{}_I = \begin{pmatrix} e_a{}^\mu & e_a{}^\nu b_{\nu\mu} \\ 0 & e^a{}_\mu \end{pmatrix}, \qquad \mathcal{H}_{AB} = \begin{pmatrix} g^{ab} & 0 \\ 0 & g_{ab} \end{pmatrix}, \tag{3.103}$$

where b is the B-field and g_{ab} is the Riemannian metric in flat indices, $g_{\mu\nu} = e_\mu{}^a g_{ab} e^b{}_\nu$. Assuming Euclidean signature, g_{ab} exhibits invariance under local Lorentz group $O(D)$, and the same holds true for its inverse g^{ab}. Hence, $\mathcal{H}_{AB}$ will display a local *double Lorentz* invariance under the group $O(D) \times O(D)$, which coincides with the maximal compact subgroup of $O(D,D)$. Therefore, the generalised metric $\mathcal{H}$ is really an element of the coset space $O(D,D)/(O(D) \times O(D))$.

Now, the degrees of freedom can be encapsulated either in $\mathcal{H}$ or in E plus, in both cases, the generalised dilaton d. When opting for the frame field formulation, it is somewhat useful to introduce additional objects known as *generalised fluxes*, indicated with $\mathcal{F}_{ABC}$ and $\mathcal{F}_A$, defined in terms of the generalised dilaton and frame field as

$$\mathcal{F}_{ABC} = E_{CI} \mathscr{L}_{E_A} E_B{}^I = 3\Omega_{[ABC]}\,, \tag{3.104}$$

$$\mathcal{F}_A = -e^{-2d} \mathscr{L}_{E_A} e^{-2d} = \Omega^B{}_{BA} + 2E_A{}^I \partial_I d\,. \tag{3.105}$$

The object Ω appearing here is called the Weitzenböck connection and reads

$$\Omega_{ABC} = E_A{}^I \partial_I E_B{}^J E_{JC} = -\Omega_{ACB}\,. \tag{3.106}$$

The generalised fluxes are by construction $O(D,D)$ covariant and so will be any combination thereof. They entail another neat "Einstein-Hilbert" formulation of (3.96) in a doubled formalism

$$S = \int \mathrm{d}X\, e^{-2d} \mathcal{R}\,, \tag{3.107}$$

where, with $P = \frac{1}{2}(\eta + \mathcal{H})$ and $\overline{P} = \frac{1}{2}(\eta - \mathcal{H})$, the generalised Ricci scalar $\mathcal{R}$ is given by

$$\mathcal{R} = -\overline{P}^{AD} \overline{P}^{BE} \left(P^{CF} + \frac{1}{3} \overline{P}^{CF} \right) \mathcal{F}_{ABC} \mathcal{F}_{DEF} + 2\overline{P}^{AB} \mathcal{F}_A \mathcal{F}_B\,. \tag{3.108}$$

Chapter 4

Generalised Cosets

Abstract

Recent work has shown that two-dimensional non-linear σ-models on group manifolds with Poisson-Lie symmetry can be understood within Generalised Geometry as exemplars of generalised parallelisable spaces. Here we review and extend this idea to target spaces constructed as double cosets $M = \widetilde{G}\backslash\mathbb{D}/H$. Mirroring conventional coset geometries, we show that on M one can construct a generalised frame field and a H-valued generalised spin connection that together furnish an algebra under the generalised Lie derivative.

4.1 Introduction

The ability to construct generalised frame fields has different advantages depending on the preferred point of view and final objective.

When interested in Supergravity, one often starts from a maximal 10- or 11-dimensional theory and retains a "consistent truncation". By that we mean a lower-dimensional theory whose solutions to the equations of motion also solve the higher-dimensional ones. A consistent truncation is not always feasible, though, let alone easy to find. A possible constructive approach is the Scherk-Schwarz reduction [115]: in a nutshell, if the compactification manifold M is a unimodular Lie group G[1] it is possible to truncate the parent theory. The proof essentially relies on G being parallelisable, i.e. endowed with a set of globally defined vector fields $k_{\mathbb{a}}$ dual to the left-invariant Maurer-Cartan one-forms and obeying $L_{k_{\mathbb{a}}} k_{\mathbb{b}} = f_{\mathbb{a}\mathbb{b}}{}^{\mathbb{c}} k_{\mathbb{c}}$. Now, this approach has a major shortcoming in

[1] This condition can be slightly relaxed to M being the quotient of G by a discrete subgroup Γ acting on the left [116]. Also, recall that a Lie group is unimodular if the structure constants of its Lie algebra obey $f_{\mathbb{a}\mathbb{b}}{}^{\mathbb{b}} = 0$. This is the case for compact and/or semi-simple groups [117].

that one is often interested in "coset reductions" – where M is indeed assumed to be a coset space [117] – a scenario that escapes the Scherk-Schwarz construction. To remedy that, it was first observed in [116] and later refined in [118] that it is possible to extend the core principles of the Scherk-Schwarz reduction to Generalised Geometry requiring the presence of a global generalised frame $E_{\mathbb{A}}$ that obeys

$$\mathscr{L}_{E_{\mathbb{A}}} E_{\mathbb{B}} = \mathcal{F}_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}} E_{\mathbb{C}} \tag{4.1}$$

for some constants $\mathcal{F}$. Then, M (which is necessarily a coset [43]) is said to be generalised parallelisable and gives rise to consistent truncations provided some conditions are met [118]. The remaining challenge is to find at least one tuple $(M, E_{\mathbb{A}})$ such that (4.1) holds for a given constant *generalised torsion* $\mathcal{F}_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}}$.

When focussing on the geometric properties of a non-linear σ-model, instead, generalised frame fields have a somewhat different appeal. Let us take, for instance, the case of Poisson-Lie models introduced in the previous chapter, our main sources of interest. As they are deeply intertwined with Poisson-Lie T-duality, it is convenient to adopt a doubled formalism – the $\mathcal{E}$-model – to facilitate the analysis. If we had a canonical construction for $E_{\mathbb{A}}$ at our disposal, we could immediately retrieve the associated metric and B-field by dressing (3.75) with the generalised frame fields. Moreover, as a rule of thumb, models having a group-theoretic origin are more conveniently described with the aid of algebraic objects. In other words, it is much preferable to stick to tensors having flat generalised indices for as long as possible, reverting to curved ones only when strictly necessary. Once again, the mapping is provided by the frames, and will prove invaluable in many calculations carried out in the thesis. As an analogy, consider the Riemannian geometry of a semi-simple Lie group G: while the metric and Riemann tensor could be arbitrarily complicated when using explicit coordinate expressions, their form in flat indices is completely fixed by the structure constants and Killing form.

Although a few examples were known [116, 119, 120], the systematics for $E_{\mathbb{A}}$ were missing until recently. In Generalised Geometry, a complete construction of generalised parallelisable spaces was worked out in a series of papers [121, 122] in which the right coset $\widetilde{G}\backslash\mathbb{D}$ is identified with the internal manifold M, and $\mathbb{D}$ is a Lie group that admits a non-degenerate, invariant pairing of split signature for which the subgroup $\widetilde{G} \subset \mathbb{D}$ is maximally isotropic. The constant generalised torsion is given by the structure coefficients of $\mathfrak{d}$, the Lie algebra of $\mathbb{D}$. In the first section of this chapter we shall revisit this construction is a more mathematically oriented way with respect to the presentation in [121].

However, this is rather a special case of the dressing coset construction [123] with $M = \widetilde{G}\backslash\mathbb{D}/H$ for a trivial H. Hence, motivated by the interplay between Generalised Geometry and Poisson-Lie T-duality, we will show that dressing cosets give rise to a class of new generalised geometries which are relevant for the construction of consistent truncations. They are named generalised cosets in force of the analogy with conventional coset spaces. We shall detail them in a second section.

4.2 The Descent From $\mathbb{D}$

Let us begin with some general reasoning as to the form of the generalised frame fields before moving to their actual construction.

Consider a $2D$-dimensional Lie group $\mathbb{D}$. At the level of the algebra, $\mathfrak{d} = \mathrm{Lie}(\mathbb{D})$ is generated by a set of $T_{\mathbb{A}}$'s, $\mathbb{A} = 1, \ldots, D$, that give rise to the commutation relation $[T_{\mathbb{A}}, T_{\mathbb{B}}] = F_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}} T_{\mathbb{C}}$. Also, $\mathfrak{d}$ is equipped with an ad-invariant, non-degenerate, bi-linear, symmetric pairing of split signature $\langle\langle \cdot, \cdot \rangle\rangle$ from which we define

$$\langle\langle T_{\mathbb{A}}, T_{\mathbb{B}} \rangle\rangle = \eta_{\mathbb{A}\mathbb{B}} \,. \tag{4.2}$$

Take a subgroup $\widetilde{G} \subset \mathbb{D}$, $\dim \widetilde{G} = D$, maximally isotropic with respect to $\langle\langle \cdot, \cdot \rangle\rangle$. The Lie algebra $\widetilde{\mathfrak{g}} = \mathrm{Lie}(\widetilde{G})$ is generated by $T^{\mathbb{a}}$, $\mathbb{a} = 1, \ldots, D$, obeying $[T^{\mathbb{a}}, T^{\mathbb{b}}] = F^{\mathbb{a}\mathbb{b}}{}_{\mathbb{c}} T^{\mathbb{c}}$. In geometric terms, $\mathbb{D}$ can be thought of as the principal bundle $\widetilde{G} \hookrightarrow \mathbb{D} \to \widetilde{G}\backslash\mathbb{D}$, making $\widetilde{G}$ the fibre over the base manifold $M = \widetilde{G}\backslash\mathbb{D}$. Other than a projection map $p : \mathbb{D} \to \widetilde{G}\backslash\mathbb{D}$ that comes along with this construction, we shall eventually consider a section σ, i.e. a map from the base onto the total space, $\sigma : \widetilde{G}\backslash\mathbb{D} \to \mathbb{D}$. Generalised parallelisable spaces are those for which we can construct on M a set of $O(D, D)$-valued generalised frame fields $E_{\mathbb{A}}$ for which the generalised torsion is constant and identified with the structure constants of $\mathfrak{d}$, i.e.

$$\mathscr{L}_{E_{\mathbb{A}}} E_{\mathbb{B}} = F_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}} E_{\mathbb{C}} \qquad \text{and} \qquad \langle E_{\mathbb{A}}, E_{\mathbb{B}} \rangle = \langle\langle T_{\mathbb{A}}, T_{\mathbb{B}} \rangle\rangle = \eta_{\mathbb{A}\mathbb{B}}, \tag{4.3}$$

where we recall that for two generalised vectors $U = u + \mu$ and $V = v + \nu$ the pairing is $\langle U, V \rangle = \iota_u \nu + \iota_v \mu$. As a generalised vector, $E_{\mathbb{A}}$ will be split in a vector $k_{\mathbb{A}}$ and a one-form $\varphi_{\mathbb{A}}$, $E_{\mathbb{A}} = k_{\mathbb{A}} + \varphi_{\mathbb{A}}$. Assuming the presence of a *closed* three-form flux H, the requirements (4.3) imply that the vector part obeys

$$[k_{\mathbb{A}}, k_{\mathbb{B}}] = F_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}} k_{\mathbb{C}} \,, \tag{4.4}$$

and the one-form part should satisfy $L_{k_{\mathbb{A}}} \varphi_{\mathbb{B}} - \iota_{k_{\mathbb{B}}}(\mathrm{d}\varphi_{\mathbb{A}} - \iota_{k_{\mathbb{A}}} H) = F_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}} \varphi_{\mathbb{C}}$. While it is tempting to impose the vanishing of the expression in brackets, this cannot be done consistently. Indeed, we have not placed any demand of $\mathbb{D}$-invariance on the H-flux, whereas this would imply $L_{k_{\mathbb{A}}} H = 0$. The best we can do is assume that a set of two-forms $\vartheta_{\mathbb{A}}$ encode the failure of $\mathbb{D}$-invariance, that is $L_{k_{\mathbb{A}}}H = -\mathrm{d}\vartheta_{\mathbb{A}}$. With this choice, the one-form part of the frame algebra is satisfied provided

$$\mathrm{d}\varphi_{\mathbb{A}} = \iota_{k_{\mathbb{A}}}H + \vartheta_{\mathbb{A}}\,, \qquad L_{k_{\mathbb{A}}}\varphi_{\mathbb{B}} = F_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}}\varphi_{\mathbb{C}} + \iota_{k_{\mathbb{B}}}\vartheta_{\mathbb{A}}\,. \tag{4.5}$$

Having clarified the general approach, let us now dig deeper into its actual implementation. In simple terms, the construction of a generalised frame field on M can be achieved through some appropriate reduction of objects defined on $\mathbb{D}$. Indeed, being a group manifold, the latter has canonical frame fields given by either the left- or right-invariant Maurer-Cartan one-forms. Out of these (and their dual vector fields) we should try and carve a generalised frame field $E_{\mathbb{A}}$ which does *not* depend on $\widetilde{G}$ data. To this end, choose local coordinates $\tilde{x}_{\mathbb{i}}$ on $\widetilde{G}$ and $x^{\mathbb{i}}$ on M and let $m(x^{\mathbb{i}})$ be a representative of the coset space $M = \widetilde{G}\backslash\mathbb{D}$ such that, for $\tilde{g}(\tilde{x}_{\mathbb{i}}) \in \widetilde{G}$, the element $\mathbb{g} \in \mathbb{D}$ is parametrised by $\mathbb{g}(\tilde{x}_{\mathbb{i}}, x^{\mathbb{i}}) = \tilde{g}(\tilde{x}_{\mathbb{i}})m(x^{\mathbb{i}})$. Introduce the natural right-invariant form on $\widetilde{G}$, namely

$$A = \mathrm{d}\tilde{g}\tilde{g}^{-1}\,. \tag{4.6}$$

The structure equation $\mathrm{d}A - \frac{1}{2}[A,A] = 0$ implies that A is, in fact, a *flat* $\widetilde{\mathfrak{g}}$-valued connection on the bundle. As such, it splits the tangent bundle of $\mathbb{D}$ into a vertical and a horizontal part, locally $T_{\mathbb{g}}\mathbb{D} = V_{\mathbb{g}}\mathbb{D} \oplus H_{\mathbb{g}}\mathbb{D}$ for any $\mathbb{g} \in \mathbb{D}$. In particular, the horizontal bundle is the kernel of the connection one-form. As a group manifold, $\mathbb{D}$ is equipped with two natural sets of vector fields, $\widehat{k}_{\mathbb{A}}$ and $\widehat{v}_{\mathbb{A}}$, corresponding to the dual of the left- and right-invariant Maurer-Cartan one-forms, respectively. That is, for a group element $\mathbb{g} \in \mathbb{D}$,

$$\iota_{\widehat{k}_{\mathbb{A}}}\mathbb{g}^{-1}\mathrm{d}\mathbb{g} = T_{\mathbb{A}}\,, \qquad \iota_{\widehat{v}_{\mathbb{A}}}\mathrm{d}\mathbb{g}\mathbb{g}^{-1} = T_{\mathbb{A}}\,. \tag{4.7}$$

These vectors separately furnish two representations of the Lie algebra $\mathfrak{d}$,

$$[\widehat{k}_{\mathbb{A}}, \widehat{k}_{\mathbb{B}}] = F_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}}\,\widehat{k}_{\mathbb{C}}\,, \qquad [\widehat{v}_{\mathbb{A}}, \widehat{v}_{\mathbb{B}}] = -F_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}}\,\widehat{v}_{\mathbb{C}}\,, \qquad [\widehat{k}_{\mathbb{A}}, \widehat{v}_{\mathbb{B}}] = 0\,. \tag{4.8}$$

Even though $\widehat{k}_{\mathbb{A}}$'s do reproduce the algebra (4.4), they can not be identified with $k_{\mathbb{A}}$'s for two reasons: first, they contain a non-trivial vertical part, i.e. they retain information about $\widetilde{G}$; second, they are still defined on $\mathbb{D}$, as opposed to M. Addressing these issues is relatively straightforward: we restrict $\widehat{k}$ to its *horizontal* part $\overline{k}$[2] and then push it forward to M via the projection map. That is, we identify $k_{\mathbb{A}} = p_*\overline{k}_{\mathbb{A}}$. The only thing to be checked is that $\overline{k}$'s obey the correct algebra. By definition, the horizontal component is obtained via

$$\overline{k}_{\mathbb{A}} = \widehat{k}_{\mathbb{A}} - \iota_{\widehat{k}_{\mathbb{A}}}A_{\mathbb{b}}\widehat{v}^{\mathbb{b}}\,. \tag{4.9}$$

[2] Here and henceforth, an overbar will always indicate horizontal quantities.

From $\mathrm{d}\mathrm{g}\mathrm{g}^{-1} = A + \mathrm{Ad}_{\tilde{g}}\mathrm{d}mm^{-1}$, it follows immediately that the vector field $\widehat{v}^{\mathrm{a}}$ is dual to the connection one-form, $\iota_{\widehat{v}^{\mathrm{a}}} A_{\mathbb{b}} = \delta_{\mathbb{b}}{}^{\mathrm{a}}$, and that $\iota_{\widehat{v}_{\mathrm{a}}} A_{\mathbb{b}} = 0$. The flatness of A implies $L_{\widehat{v}^{\mathrm{a}}} A_{\mathbb{b}} = A_{\mathbb{c}} F^{\mathrm{a}\mathbb{c}}{}_{\mathbb{b}}$ which in turn can be used to reach the desired result

$$[\overline{k}_{\mathbb{A}}, \overline{k}_{\mathbb{B}}] = F_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}} \overline{k}_{\mathbb{C}} \,. \tag{4.10}$$

As the vector part is settled, let us move on to the one-form part. As a group manifold, $\mathbb{D}$ naturally comes with a bi-invariant three-form

$$\begin{aligned} \widehat{H} &= -\frac{1}{6}\langle\langle \mathrm{d}\mathrm{g}\mathrm{g}^{-1}, \mathrm{d}\mathrm{g}\mathrm{g}^{-1} \wedge \mathrm{d}\mathrm{g}\mathrm{g}^{-1} \rangle\rangle \\ &= -\frac{1}{6}\langle\langle \mathrm{d}mm^{-1}, \mathrm{d}mm^{-1} \wedge \mathrm{d}mm^{-1} \rangle\rangle + \frac{1}{2}\mathrm{d}\langle\langle \tilde{g}^{-1}\mathrm{d}\tilde{g}, \mathrm{d}mm^{-1} \rangle\rangle \,, \end{aligned} \tag{4.11}$$

which is closed thanks to the Jacobi identity. Since $\mathrm{d}mm^{-1}$ is horizontal with respect to $\widehat{v}^{a}$, i.e. $\iota_{\widehat{v}^{a}} \mathrm{d}mm^{-1} = 0$, the first term on the second line, dubbed $\overline{H}$, is horizontal. Accordingly, we let an overbar denote it. The second term is exact, and the closure of $\widehat{H}$ implies the closure of $\overline{H}$. On $\mathfrak{d}$ we can introduce an endomorphism $\mathcal{K} : \mathfrak{d} \to \mathfrak{d}$ which is an involution, $\mathcal{K}^2 = 1$, compatible with the pairing, $\langle\langle T_{\mathbb{A}}, \mathcal{K} T_{\mathbb{B}} \rangle\rangle = -\langle\langle \mathcal{K} T_{\mathbb{A}}, T_{\mathbb{B}} \rangle\rangle$, and such that its $+1$ eigenspace is identified with $\tilde{\mathfrak{g}}$. This places a para-Hermitian structure on $\mathfrak{d}$ [122]. Explicitly, $\mathcal{K}$ can be constructed as follows: suppose the $\mathfrak{d}$ generators are split according to $T_{\mathbb{A}} = (T^{\mathrm{a}}, T_{\mathbb{a}})$, where, as before, T^{a} generate $\tilde{\mathfrak{g}}$ and $T_{\mathbb{a}}$ indicate collectively the rest[3]. A matrix representation for $\mathcal{K}$ that obeys all the requirements is

$$\mathcal{K}_{\mathbb{A}}{}^{\mathbb{B}} = \begin{pmatrix} \delta_{\mathbb{a}}{}^{\mathbb{b}} & -\langle\langle T_{\mathbb{a}}, T_{\mathbb{b}} \rangle\rangle \\ 0 & -\delta^{\mathrm{a}}{}_{\mathbb{b}} \end{pmatrix} \,. \tag{4.12}$$

For $\mathfrak{d}$ a Drinfel'd double, the north-east block would vanish making the $\mathfrak{g}$ algebra coincide with the -1 $\mathcal{K}$-eigenspace, obtaining the para-Hermitian structure associated to a Manin triple, see e.g. [124]. Thanks to $\mathcal{K}$, we can define a two-form

$$\widehat{\omega} = \frac{1}{2}\langle\langle \mathrm{d}\mathrm{g}\mathrm{g}^{-1}, \mathcal{K}\, \mathrm{d}\mathrm{g}\mathrm{g}^{-1} \rangle\rangle = -\langle\langle \tilde{g}^{-1}\mathrm{d}\tilde{g}, \mathrm{d}mm^{-1} \rangle\rangle + 2\overline{\varpi} \,, \tag{4.13}$$

where we have singled out a particular *horizontal* quantity $\overline{\varpi} = \frac{1}{4}\langle\langle \mathrm{d}mm^{-1}, \mathcal{K}\mathrm{d}mm^{-1} \rangle\rangle$. Notice that, by construction, $\widehat{\omega}$ is a right-invariant two-form, i.e. it obeys $L_{\widehat{k}_{\mathbb{A}}} \widehat{\omega} = 0$. Looking back at (4.11), it is possible to employ $\widehat{\omega}$ and $\overline{\varpi}$ to write

$$\widehat{H} = \overline{H} + \mathrm{d}\overline{\varpi} - \frac{1}{2}\mathrm{d}\widehat{\omega} \,. \tag{4.14}$$

These two-forms are useful as they allow us to eventually retrieve the desired properties of the one-form part of the frame field (which we have not discussed yet). To see how, we

[3] In the particular case of $\mathfrak{d}$ being a Drinfel'd (which we don't necessarily assume to be true here), $T_{\mathbb{a}}$ would generate the dual group G.

first use the definition $\mathrm{g}^{-1}\mathrm{dg} = \mathrm{g}^{-1}A\mathrm{g} + m^{-1}\mathrm{d}m$ together with (4.7), to arrive, through a chain of equalities, to the identity

$$\frac{1}{2}\langle\langle T_{\mathbb{A}}, \mathrm{g}^{-1}\mathrm{dg}\rangle\rangle = \frac{1}{2}\iota_{\widehat{k}_{\mathbb{A}}}(\widehat{\omega} - 2\overline{\varpi}) + \langle\langle T_{\mathbb{A}}, m^{-1}\mathrm{d}m\rangle\rangle - \frac{1}{2}\langle\langle \iota_{\widehat{k}_{\mathbb{A}}} m^{-1}\mathrm{d}m, m^{-1}\mathrm{d}m\rangle\rangle\,. \tag{4.15}$$

Now, one of the requirements for the one-form part $\varphi_{\mathbb{A}}$ is that it reproduces the pairing constraint in (4.3) which, when expressed in terms of k and φ, reads $\eta_{\mathbb{AB}} = \iota_{k_{\mathbb{A}}}\varphi_{\mathbb{B}} + \iota_{k_{\mathbb{B}}}\varphi_{\mathbb{A}}$. If we contract (4.15) with $\widehat{k}_{\mathbb{B}}$ we recover on the left-hand side the pairing $\eta_{\mathbb{AB}}$ up to a constant. It is then sensible to define the horizontal one-form $\overline{\varphi}_{\mathbb{A}}$ as[4]

$$\overline{\varphi}_{\mathbb{A}} = \langle\langle T_{\mathbb{A}}, m^{-1}\mathrm{d}m\rangle\rangle - \frac{1}{2}\langle\langle \iota_{\overline{k}_{\mathbb{A}}}\mathrm{d}m m^{-1}, \mathrm{d}m m^{-1}\rangle\rangle\,, \tag{4.16}$$

$$\iota_{\overline{k}_{\mathbb{A}}}\overline{\varphi}_{\mathbb{B}} = \frac{1}{2}\langle\langle T_{\mathbb{A}}, T_{\mathbb{B}}\rangle\rangle - \frac{1}{2}\iota_{\widehat{k}_{\mathbb{A}}}\iota_{\widehat{k}_{\mathbb{B}}}(\widehat{\omega} - 2\overline{\varpi})\,. \tag{4.17}$$

Notice how the symmetrisation of (4.17) precisely returns the desired pairing (up to a push-forward/pull-back). To corroborate our finding for the one-form we shall check that the constraints in (4.5) are actually satisfied. Taking the exterior derivative of (4.15), using the fact that $L_{\overline{k}_{\mathbb{A}}}\widehat{\omega} = 0$, and finally comparing with (4.14) we obtain

$$\mathrm{d}\overline{\varphi}_{\mathbb{A}} = \iota_{\overline{k}_{\mathbb{A}}}\overline{H} + L_{\overline{k}_{\mathbb{A}}}\overline{\varpi}\,, \tag{4.18}$$

prompting for the identification $\overline{\vartheta}_{\mathbb{A}} = L_{\overline{k}_{\mathbb{A}}}\overline{\varpi}$. As for the Lie derivative, a direct calculation following from (4.18) and (4.17) leads to

$$L_{\overline{k}_{\mathbb{A}}}\overline{\varphi}_{\mathbb{B}} = F_{\mathbb{AB}}{}^{\mathbb{C}}\overline{\varphi}_{\mathbb{C}} + \iota_{\overline{k}_{\mathbb{B}}}L_{\overline{k}_{\mathbb{A}}}\overline{\varpi}\,. \tag{4.19}$$

It is actually possible to simplify this expression a little further. Owing to the fact that, for two vectors v, w, the general property $[L_v, \iota_w] = \iota_{[v,w]}$ holds, we can choose to recast the last term so as to obtain

$$L_{\overline{k}_{\mathbb{A}}}\overline{\varphi}'_{\mathbb{B}} = F_{\mathbb{AB}}{}^{\mathbb{C}}\overline{\varphi}'_{\mathbb{C}}\,, \tag{4.20}$$

where we have re-defined $\overline{\varphi}'_{\mathbb{A}} = \overline{\varphi}_{\mathbb{A}} - \iota_{\overline{k}_{\mathbb{A}}}\overline{\varpi}$. Similarly to what happened to the vectorial part of the frame field, we shall project $\overline{\varphi}_{\mathbb{A}}$ (and all the other differential forms involved in the construction) down to $M = \widetilde{G}\backslash\mathbb{D}$. To this end, consider the section $\sigma : \widetilde{G}\backslash\mathbb{D} \to \mathbb{D}$: its pull-back is the required map, from which we can define

$$\varphi_{\mathbb{A}} = \sigma^*\overline{\varphi}_{\mathbb{A}}\,, \qquad H = \sigma^*\overline{H}\,, \qquad \varpi = \sigma^*\overline{\varpi}\,. \tag{4.21}$$

[4]Since $\iota_{\widehat{k}_{\mathbb{A}}} m^{-1}\mathrm{d}m = \iota_{\overline{k}_{\mathbb{A}}} m^{-1}\mathrm{d}m$, we have changed the definition accordingly.

4.2.1 Application: Drinfel'd Double

As an application of this construction, let us consider the case where $\mathfrak{d}$ is a Drinfel'd double. This will provide us with the frame field for a number of models, including the η-deformation on a group manifold G.

Let us assume $\mathbb{D}$ is a Drinfel'd double, thereby assuring that the coset space is in fact the group G dual to $\widetilde{G}$, $M = \widetilde{G}\backslash\mathbb{D} \cong G$. As a consequence, the coset representative m is just an element of G. By construction, the algebrae of the two groups are both maximally isotropic. Indicating the generators according to $\mathfrak{g} = \mathrm{span}(T_{\mathbb{a}})$, $\widetilde{\mathfrak{g}} = \mathrm{span}(T^{\mathbb{a}})$, we then have that $\langle\!\langle T^{\mathbb{a}}, T^{\mathbb{b}}\rangle\!\rangle = \langle\!\langle T_{\mathbb{a}}, T_{\mathbb{b}}\rangle\!\rangle = 0$ and $\langle\!\langle T_{\mathbb{a}}, T^{\mathbb{b}}\rangle\!\rangle = \delta_{\mathbb{a}}{}^{\mathbb{b}}$.

Let us choose coordinates $x^{\mathbb{i}}$ on G and $\tilde{x}_{\mathbb{i}}$ on $\widetilde{G}$, so that the Drinfel'd double element $\mathrm{g}(x^{\mathbb{i}}, \tilde{x}_{\mathbb{i}})$ is split as $\mathrm{g}(x^{\mathbb{i}}, \tilde{x}_{\mathbb{i}}) = \tilde{g}(\tilde{x}_{\mathbb{i}})g(x^{\mathbb{i}})$, for $g, \tilde{g} \in G, \widetilde{G}$. The $\mathfrak{d}$-valued left-invariant form is accordingly divided as

$$\mathrm{g}^{-1}\mathrm{d}\mathrm{g} = e^{\mathbb{a}}{}_{\mathbb{i}}\mathrm{d}x^{\mathbb{i}}T_{\mathbb{a}} + \tilde{e}_{\mathbb{b}}{}^{\mathbb{i}}\mathrm{d}\tilde{x}_{\mathbb{i}}(\mathrm{Ad}_{g^{-1}})^{\mathbb{b}\mathbb{A}}T_{\mathbb{A}}\,, \tag{4.22}$$

where e and $\tilde{e}$ are frame fields on G and $\widetilde{G}$, respectively. It is now easy to find the dual vector

$$\widehat{k}_{\mathbb{a}} = e_{\mathbb{a}}\,, \qquad \widehat{k}^{\mathbb{a}} = \tilde{e}^{\mathbb{b}}(\mathrm{Ad}_g^{-1})_{\mathbb{b}}{}^{\mathbb{a}} + \pi^{\mathbb{a}\mathbb{b}}e_{\mathbb{b}}\,. \tag{4.23}$$

Here we have adopted a shorthand for displaying purposes: the fundamental vector fields on G and $\widetilde{G}$ have been indicated with $e_{\mathbb{a}} = e^{\mathbb{i}}{}_{\mathbb{a}}\frac{\partial}{\partial x^{\mathbb{i}}}$ and $\tilde{e}^{\mathbb{b}} = \tilde{e}_{\mathbb{i}}{}^{\mathbb{b}}\frac{\partial}{\partial \tilde{x}_{\mathbb{i}}}$, respectively, while the Poisson structure is obtained from the product of adjoint actions $\pi^{\mathbb{a}\mathbb{b}} = (\mathrm{Ad}_g)^{\mathbb{a}\mathbb{c}}(\mathrm{Ad}_g)^{\mathbb{b}}{}_{\mathbb{c}}$. As a consequence, $\iota_{\widehat{k}_{\mathbb{a}}}A_{\mathbb{b}} = 0$ and $\iota_{\widehat{k}^{\mathbb{a}}}A_{\mathbb{b}} = (\mathrm{Ad}_g^{-1})_{\mathbb{c}}{}^{\mathbb{a}}(\mathrm{Ad}_{\tilde{g}})^{\mathbb{c}}{}_{\mathbb{b}}$. Using the adjoint action to derive $\widehat{v}$ from $\widehat{k}$, we can obtain the vertical part of $\widehat{k}$, namely $\iota_{\widehat{k}^{\mathbb{a}}}A_{\mathbb{b}}\widehat{v}^{\mathbb{b}} = \tilde{e}^{\mathbb{b}}(\mathrm{Ad}_g^{-1})_{\mathbb{b}}{}^{\mathbb{a}}$. This is actually the first term in the second equation of (4.23): correctly, the vertical part is directed along the fibre, i.e. along the directions identified by the $\tilde{x}_{\mathbb{i}}$ coordinates. The final expression for $k_{\mathbb{A}}$ is then easily gathered as

$$k_{\mathbb{a}} = e_{\mathbb{a}}\,, \qquad k^{\mathbb{a}} = \pi^{\mathbb{a}\mathbb{b}}e_{\mathbb{b}}\,. \tag{4.24}$$

The one-form part is even easier: isotropy of $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ guarantees $\overline{H} = \overline{\varpi} = 0$ and the definition of $\overline{\varphi}_{\mathbb{A}}$ (4.16) boils down to

$$\varphi^{\mathbb{a}} = e^{\mathbb{a}}\,, \qquad \varphi_{\mathbb{a}} = 0\,, \tag{4.25}$$

where $e^{\mathbb{a}}$ is simply the vielbein for G, $e^{\mathbb{a}} = e^{\mathbb{a}}{}_{\mathbb{i}}\mathrm{d}x^{\mathbb{i}}$.

4.3 Dressing Cosets

We shall now consider a dressing coset $M = \widetilde{G}\backslash\mathbb{D}/H$, where a second *isotropic* subgroup H is to be modded out. We start with an inspection of the geometric properties of generalised cosets with the aim of working out the appropriate frame algebra. In essence, the latter is found by requiring that both the torsion and curvature associated to a natural connection we will introduce are specified by the structure constants of $\mathfrak{d}$. Once this is achieved, we will move to the determination of an algorithm for the construction of frame fields similar to the one we detailed for $\widetilde{G}\backslash\mathbb{D}$.

4.3.1 Geometry of Generalised Cosets

Let us briefly review conventions. Generators of $\mathfrak{d}$ are, as before, $T_{\mathbb{A}}$ with indices $\mathbb{A} = 1,\dots,2D$. The Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$ is spanned by T_α, $\alpha = 1,\dots,\dim\mathfrak{h}$. As for the remainder, it is more convenient to consider $\mathfrak{k}$ such that $\mathfrak{d} = \mathfrak{h} + \mathfrak{k}$ in place of $\widetilde{\mathfrak{g}}$. In particular, $\mathfrak{k}$ is further split in two parts, $\mathfrak{k} = \mathfrak{p} + \mathfrak{q}$ such that the pairing $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ is *non-degenerate* on $\mathfrak{q}$. At the level of the generators, we take $\mathfrak{p} = \mathrm{Span}(T^\alpha)$, $\alpha = 1,\dots,\dim\mathfrak{h}$, and $\mathfrak{q} = \mathrm{Span}(T_A)$, $A = 1,\dots,2N$ for some integer N. Notice, $2D = 2N + 2\dim\mathfrak{h}$. The requirements on the pairing can be made more explicit in the form

$$\begin{gathered}\langle\!\langle T_A, T_B\rangle\!\rangle = \eta_{AB}\,, \qquad \langle\!\langle T_\alpha, T_\beta\rangle\!\rangle = 0\,, \qquad \langle\!\langle T_\alpha, T^\beta\rangle\!\rangle = \delta_\alpha^\beta\,, \\ \langle\!\langle T_\alpha, T_B\rangle\!\rangle = 0\,, \qquad \langle\!\langle T^\alpha, T_B\rangle\!\rangle = 0\,.\end{gathered} \tag{4.26}$$

We shall place further requests on the coset. Not only we want it to be reductive, in the sense that $[\mathfrak{h},\mathfrak{k}] \subset \mathfrak{k}$, but also we want $\mathfrak{p}$ and $\mathfrak{q}$ to form independent representations, meaning that

$$[\mathfrak{h},\mathfrak{p}] \subset \mathfrak{p}\,, \qquad [\mathfrak{h},\mathfrak{q}] \subset \mathfrak{q}\,. \tag{4.27}$$

We shall call a generalised coset having these properties *generalised reductive.*

Consider now a generalised frame field E_A on the generalised coset M, $\dim M = 2N$. As we are quotienting by H, there should also exist a non-vanishing $\mathfrak{h}$-valued spin connection $\Omega = \Omega^\alpha T_\alpha$. Out of it, we define two additional objects, namely $\Omega^\alpha{}_B = \langle\Omega^\alpha, E_B\rangle$ and $\Omega_{ABC} = \Omega^\alpha{}_A F_{\alpha BC}$. As before, we impose that E_A preserves the pairing

$$\langle E_A, E_B\rangle = \eta_{AB} \equiv \langle\!\langle T_A, T_B\rangle\!\rangle\,, \tag{4.28}$$

ensuring that the frames are indeed $O(N,N)$ elements. Concerning the connection, we can try and mimic the structure of this latter requirement but we are necessarily forced to introduce a *non-constant* $\Omega^{\alpha\beta}$,

$$\langle \Omega^\alpha, \Omega^\beta \rangle = \langle\!\langle T^\alpha, T^\beta \rangle\!\rangle + 2\Omega^{(\alpha\beta)}\,, \tag{4.29}$$

for, a priori, we can't guarantee the pairing to be constant as we did for E_A.

In a given patch pick coordinates x^i, $i = 1, \ldots, N$, and decompose E_A accordingly, i.e. $E_A = E_{Ai}\mathrm{d}x^i + E_A{}^i\partial_i$. It is useful to give the frame a matrix form via $E_A{}^I = (E_{Ai}, E_A{}^i)$, where indices $I, J, \ldots$ will be interpreted as *curved* indices on the generalised bundle of M. In a similar fashion, we incorporate the section condition in the "doubled" derivative $\partial_I = (0, \partial_i)$. We can use the latter to introduce a covariant derivative ∇_I acting on a frame according to

$$\nabla_I E_A{}^J = \partial_I E_A{}^J - \Omega_{IA}{}^B E_B{}^J + \Gamma^J{}_{IK} E_A{}^K\,, \tag{4.30}$$

where the spin connection is assumed to be $\Omega_{IA}{}^B = E_I{}^C \Omega_{CA}{}^B$, and $\Gamma^J{}_{IK}$ indicate some generalised Christoffel symbols. These, unless the connection is torsionless, are not symmetric in I, K. Nevertheless , if we require ∇_I to be compatible with the pairing η_{IJ}, i.e. $\nabla_K \eta_{IJ} = 0$, Γ's are constrained to obey

$$\Gamma_{KIJ} + \Gamma_{JIK} = 0\,. \tag{4.31}$$

In fact, as customary, we would like to impose the vielbein postulate $\nabla_I E_A{}^J = 0$ which, after some algebra, equivalently reads $\Gamma_{KIJ} = \Omega_{IJK} - \partial_I E_{AK} E^A{}_J$. Upon acting on the latter with frame fields and anti-symmetrising the result we obtain

$$3\Gamma_{[ABC]} = 3\Omega_{[ABC]} - 3E_{[A}{}^I \partial_{|I|} E_B{}^J E_{C]J}\,. \tag{4.32}$$

Thanks to (4.31), the left-hand side boils down to $\Gamma_{ABC} + \Gamma_{BCA} + \Gamma_{CAB}$ and, in virtue of the results in [47], coincides with the generalised torsion, T_{ABC}. As for the right-hand side, the doubled derivative ∂_I enables us to recast the (untwisted) generalised Lie derivative in the form $\mathscr{L}_{E_A} E_B{}^I = E_A{}^J \partial_J E_B{}^I - E_B{}^J \partial_J E_A{}^I + E_B{}^J \partial^I E_{AJ}$. Acting on the latter with E_{CI} on both sides we obtain $\mathscr{L}_{E_A} E_B{}^I E_{CI} = 3E_{[A}{}^J \partial_{|J|} E_B{}^I E_{|I|C]}$. Now, our goal is to obtain a constant torsion but the spin connection in (4.32) prevents us from doing so *unless* the frame algebra is tuned so as to precisely cancel off against the non-constant contribution. Therefore, we require

$$\mathscr{L}_{E_A} E_B = (F_{ABC} + 3\Omega_{[ABC]}) E^C \equiv \mathcal{F}_{ABC} E^C\,. \tag{4.33}$$

As for the curvature of ∇_I, the generalised Riemann tensor [47] in curved indices is

$$R_{IJKL} = 2\partial_{[I}\Gamma_{|L|J]K} + 2\Gamma_{L[I|M}\Gamma^M{}_{J]K} + \frac{1}{2}\Gamma_{JMI}\Gamma_L{}^M{}_K + (IJ) \leftrightarrow (KL)\,. \tag{4.34}$$

Observe how, in the third term on the right-hand side, anti-symmetry in I and J in ensured by (4.31). Its flat version, using (4.32) and (4.33), evaluates to

$$\begin{aligned} R_{ABCD} = {} & 2\partial_{[A}\Omega_{B]CD} - 2\Omega_{[A|DE}\Omega_{B]C}{}^E - (F_{ABE} + 2\Omega_{[AB]E})\Omega^E{}_{CD} \\ & - \frac{1}{2}\Omega_{EAB}\Omega^E{}_{CD} + (AB) \leftrightarrow (CD)\,. \end{aligned} \tag{4.35}$$

At this point it is not clear if, similarly to the torsion, this object is constant and completely specified by the structure constants of $\mathfrak{d}$. We begin by observing that using the definition of generalised Lie derivative, as well as the algebra (4.33),

$$E_{AI}F_{\alpha CD}\mathscr{L}_{\Omega^\alpha}E_B{}^I = 2\partial_{[A}\Omega_{B]CD} - \mathcal{F}_{EAB}\Omega^E{}_{CD}\,. \tag{4.36}$$

Also, thanks to the generalised coset being reductive by construction, we can use the Jacobi identity to prove

$$F_{\delta\epsilon}{}^\alpha F_{\alpha CD}\Omega^\delta{}_A\Omega^\epsilon{}_B = -2\Omega_{[A|DE}\Omega_{B]C}{}^E\,. \tag{4.37}$$

Finally, the pairing (4.29) ensures that

$$\frac{1}{2}\Omega_{EAB}\Omega^E{}_{CD} = \frac{1}{2}\Omega^\alpha{}_I\Omega^{\beta I}F_{\beta AB}F_{\alpha CD} = \frac{1}{2}\langle\langle T^\alpha, T^\beta\rangle\rangle F_{\beta AB}F_{\alpha CD} + \Omega^{(\alpha\beta)}F_{\beta AB}F_{\alpha CD}\,. \tag{4.38}$$

Exploiting (4.36), (4.37) and (4.38) it is possible to make the generalised Riemann tensor (4.35) constant by adding the request

$$\mathscr{L}_{\Omega^\alpha}E_B = \Omega^{\beta\alpha}F_{\beta BC}E^C + F^\alpha{}_{BC}E^C + F_{\beta\gamma}{}^\alpha\Omega^\beta{}_B\Omega^\gamma \tag{4.39}$$

to the frame algebra. Whilst the first and third term on the right-hand side serve the purpose of cancelling against non-constant contributions (pretty much as we did for the torsion), the second term seems unnecessary at a first glance. With this choice, in fact, R_{ABCD} evaluates to

$$R_{ABCD} = -F_{AB}{}^\gamma F_{\gamma CD} - F_{AB\gamma}F^\gamma{}_{CD} - F_{AB\gamma}F_{CD\delta}\,\langle\langle T^\gamma, T^\delta\rangle\rangle\,. \tag{4.40}$$

However, the generalised Riemann tensor should obey the Bianchi identities

$$3R_{[ABCD]} = 4\nabla_{[A}T_{BCD]} + 3T_{[AB}{}^E T_{CD]E}\,, \qquad \nabla_A R_{BCDE} = 0\,, \tag{4.41}$$

where $\nabla_A = E_A{}^I \nabla_I$. For the first, we notice that $\nabla_A T_{BCD} = 0$ since $\partial_A F_{BCD} = 0$ by definition and the connection term vanishes by the Jacobi identity. The first two terms in (4.40) (i.e. the ones induced by the funny contribution in (4.39)) then precisely guarantee that, again via the Jacobi identity, the rest of Bianchi identity is satisfied. Once this is settled, the second Bianchi identity follows.

Hence we conclude that, with our choice of frame algebra, both the generalised torsion and curvature of a generalised coset are completely fixed by the structure coefficients of the underlying Lie algebra $\mathfrak{d}$ and are, in particular, *constant*. This result might seem surprising, for one crucial difference between geometry and Generalised Geometry is that, for a torsion-free connection, the generalised Riemann tensor can not be completely fixed using the metric, B-field and dilaton. There remain undetermined components which however do not affect the generalised Ricci tensor and scalar. The reason for this feature is that the metric, B-field and dilaton only fix an $O(D,D)$ frame up to a local double Lorentz transformation valued in $O(D) \times O(D)$. The construction we present, however, also singles out a particular double Lorentz frame and thus determines the connection and curvature completely.

Finally, let us mention that it is possible to include the generalised dilaton in this description. In the context of DFT, the generalised dilaton d is related to the ordinary String Theory dilaton ϕ via $d = \phi - 1/4 \log(\det g)$, being g the metric. In particular, recall that the quantity e^{-2d} is used as an invariant integration measure, cf. (3.107). In General Relativity, an analogous role is played by $\sqrt{\det g}$ and it is such that, for some vector field v and metric-compatible connection ∇, $\int \mathrm{d}^D x \sqrt{\det g} \nabla_\mu v^\mu = 0$. Something similar holds in the case at hand, provided one imposes some constraints on the generalised Christoffel symbols. For a generalised vector V^I, we would like $0 = \int \mathrm{d}X \, e^{-2d} \nabla_I V^I$. Exploiting $e^{-2d} \nabla_I V^I = \nabla_I (e^{-2d} V^I) - \nabla_I (e^{-2d}) V^I$ and neglecting the boundary term $\partial_I (e^{-2d} V^I)$, we find that our request is met provided

$$2\partial_I d = -\Gamma^I{}_{IJ} \, . \tag{4.42}$$

This fixes the exterior derivative of the dilaton completely. Alternatively, one can encode the dilaton in

$$\mathcal{F}_A = E^{BI} \partial_I E_B{}^J E_{AJ} + 2 E_A{}^I \partial_I d = \Omega^B{}_{BA} \, . \tag{4.43}$$

Together with $\mathcal{F}_{ABC}$, they form the natural objects in the flux formulation of Double Field Theory.

4.3.2 Frame Field Construction

We can now proceed to a detailed account of the way such generalised frame fields are constructed on a dressing coset. First, we shall check that we can further reduce the frame field $E_{\mathbb{A}}$ inherited from $\widetilde{G}\backslash\mathbb{D}$ so as to reproduce (4.33). After that, we shall ensure that the second constraint (4.39) is met, too.

With a new subgroup to mod-out, we can interpret $\widetilde{G}\backslash\mathbb{D}$ as the principal bundle $H \hookrightarrow \widetilde{G}\backslash\mathbb{D} \to \widetilde{G}\backslash\mathbb{D}/H$. As we are considering the *right* H-coset, we can introduce an $\mathfrak{h}$-valued *left*-invariant one-form $\mathscr{A} = h^{-1}\mathrm{d}h$, for $h \in H$, acting as a connection on this space. More precisely, this would require us to be able to choose (locally perhaps) the group element as $\mathrm{g} = \tilde{g}nh$, with $\tilde{g} \in \widetilde{G}$, $h \in H$ and n parametrising M . Addressing the vector part of the generalised frame field is fairly easy and requires us to subtract the vertical part once again. In practice, the vector field $k_{\mathbb{A}}$ inherited from $\widetilde{G}\backslash\mathbb{D}$ can be restricted to the dressing coset via

$$\overline{k}_{\mathbb{A}} = k_{\mathbb{A}} - \iota_{k_{\mathbb{A}}}\mathscr{A}^{\beta}k_{\beta}\,. \tag{4.44}$$

By construction $\iota_{k_\alpha}\mathscr{A}^{\beta} = \delta_{\alpha}{}^{\beta}$, so that $\overline{k}_{\mathbb{A}}$ is horizontal, which is to say $\iota_{\overline{k}_{\mathbb{A}}}\mathscr{A}^{\beta} = 0$. This also implies that one component of the horizontal fields vanishes, namely $\overline{k}_{\alpha} = 0$. Being a left-invariant form, $\mathscr{A}$ obeys the Maurer-Cartan identity $\mathrm{d}\mathscr{A} = -\mathscr{A}\wedge\mathscr{A}$ which in turn implies immediately $L_{k_\alpha}\mathscr{A}^{\beta} = -F_{\alpha\gamma}{}^{\beta}\mathscr{A}^{\gamma}$. The Lie bracket of the horizontal vector fields evaluates, after some simple algebra, to

$$[\overline{k}_{\mathbb{A}}, \overline{k}_{\mathbb{B}}] = F_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}}\overline{k}_{\mathbb{C}} - 2\iota_{k_{[\mathbb{A}}}\mathscr{A}^{\delta}F_{\delta|\mathbb{B}]}{}^{\mathbb{C}}\overline{k}_{\mathbb{C}}\,. \tag{4.45}$$

If we appropriately restrict the indices, recalling that $\overline{k}_{\alpha}$ vanishes and that H has an action on the generalised coset, we get

$$[\overline{k}_{A}, \overline{k}_{B}] = F_{AB}{}^{C}\overline{k}_{C} + F_{AB\gamma}\overline{k}^{\gamma} - 2\iota_{k_{[A}}\mathscr{A}^{\delta}F_{\delta|B]}{}^{C}\overline{k}_{C}\,. \tag{4.46}$$

Therefore, we see that (4.46) closely resembles the algebra (4.33) we are looking for, upon making the identifications

$$E_A|_{\mathrm{vect}} = \overline{k}_A\,, \qquad \Omega^{\alpha}{}_{B} = -\iota_{k_B}\mathscr{A}^{\alpha}\,, \qquad \Omega^{\alpha}|_{\mathrm{vect}} = \overline{k}^{\alpha}\,, \qquad \Omega^{\alpha\beta} = -\iota_{k^{\beta}}\mathscr{A}^{\alpha}\,, \tag{4.47}$$

where $|_{\mathrm{vect}}$ stands for restriction to the purely vectorial part.

We now need to take care of the one-form component of the frame. As before, we take as starting point the one-forms $\varphi_{\mathbb{A}}$ on $\widetilde{G}\backslash\mathbb{D}$. Unfortunately, their horizontal projection is *not* sufficient to furnish the algebra we are looking for. Some extra modifications are in order. To begin with, let us introduce what we shall call a B-field, i.e. the two-form

$$\mathscr{B} = \mathscr{A}^\beta \wedge \varphi_\beta + \frac{1}{2}\iota_{k_\beta}\varphi_\gamma \mathscr{A}^\beta \wedge \mathscr{A}^\gamma \,. \tag{4.48}$$

Let us inspect its properties. First of all, a contraction with k_α returns

$$\iota_{k_\alpha}\mathscr{B} = \varphi_\alpha - \mathscr{A}^\beta \iota_{k_{(\alpha}}\varphi_{\beta)} = \varphi_\alpha \,. \tag{4.49}$$

The last equality follows from observing that we had (4.17), with the symmetric part corresponding to the actual inner product. However, by coset construction, $\langle\!\langle T_\alpha, T_\beta\rangle\!\rangle = 0$, hence our result. The Maurer-Cartan relation $\mathrm{d}\mathscr{A}^\beta = -\frac{1}{2}\mathscr{A}^\gamma \wedge \mathscr{A}^\delta F_{\gamma\delta}{}^\beta$ can be used, together with the isotropy condition, to check that the two-form is also invariant

$$L_{k_\alpha}\mathscr{B} = 0 \,. \tag{4.50}$$

The purpose of the two-form $\mathscr{B}$ is to tweak the naive one-form $\varphi_\mathbb{A}$ into the one appropriate for the coset construction, which we shall call $\phi_\mathbb{A}$, explicitly given by

$$\phi_\mathbb{A} = \varphi_\mathbb{A} - \iota_{k_\mathbb{A}}\mathscr{B} \,. \tag{4.51}$$

Because of (4.49), it follows that $\phi_\alpha = 0$. Moreover, one can use (4.17) and (4.49) to check that $\iota_{k_\alpha}\phi_\mathbb{A} = \langle\!\langle T_\alpha, T_\mathbb{A}\rangle\!\rangle$. This relation shows its importance when constructing the horizontal part of ϕ; by the usual procedure, we define it to be $\overline{\phi}_\mathbb{A} = \phi_\mathbb{A} - \iota_{k_\beta}\phi_\mathbb{A}\mathscr{A}^\beta$ but, in fact, one immediately checks that

$$\overline{\phi}_\alpha = 0 \,, \qquad \overline{\phi}^\alpha = \phi^\alpha - \mathscr{A}^\alpha \,, \qquad \overline{\phi}_A = \phi_A \,. \tag{4.52}$$

Therefore, both $\overline{k}_\alpha$ and $\overline{\phi}_\alpha$ vanish. We shall now prove that $\overline{\phi}$ completely specifies the one-form part of E_A and Ω^α via

$$E_A|_{\mathrm{one-form}} = \overline{\phi}_A \,, \qquad \Omega^\alpha|_{\mathrm{one-form}} = \overline{\phi}^\alpha \,. \tag{4.53}$$

Remarkably, with this choice we immediately obtain the correct pairings (4.28) and (4.29). Consider now the frame algebra. The check here is more involved as, by definition of the generalised Lie derivative, differential forms enter in a number of places. In particular, we shall prove that $L_{\overline{k}_A}\overline{\phi}^B - \iota_{\overline{k}_B}\mathrm{d}\overline{\phi}^A - \iota_{\overline{k}_A\overline{k}_B}\mathscr{H}$ obey the same algebra as (4.46), for some choice of three-form $\mathscr{H}$. Start from the bit involving the ordinary Lie derivative: exploiting (4.49), (4.50), $L_{k_\mathbb{A}}\varphi_\mathbb{B} = F_{\mathbb{AB}}{}^\mathbb{C}\varphi_\mathbb{C}$ and $\iota_{k_\beta}\phi_C = 0$, after a lengthy yet straightforward calculation we arrive at

$$L_{\overline{k}_A}\overline{\phi}_B = F_{AB}{}^C\overline{\phi}_C + F_{AB\gamma}\overline{\phi}^\gamma + F_{AB\gamma}\mathscr{A}^\gamma + \Omega^\gamma{}_A F_{\gamma B}{}^C\overline{\phi}_C - \iota_{k_B}L_{k_A}\mathscr{B} \,. \tag{4.54}$$

As for the three-form, we surely inherit H from the $\widetilde{G}\backslash\mathbb{D}$ construction: assuming there we had re-absorbed $\overline{\varpi}$ inside the definition of $\overline{\varphi}$ (which is always possible, as explained), we have, thanks to (4.18), $\iota_{\overline{k}_A}H = \mathrm{d}\varphi_A + \Omega^\beta{}_A\mathrm{d}\varphi_\beta$. However, H can be augmented with the exact three-form $\mathrm{d}\mathscr{B}$: properties of $\mathscr{B}$ then imply that $\iota_{\overline{k}_A}\mathrm{d}\mathscr{B} = L_{k_A}\mathscr{B} - \mathrm{d}\iota_{k_A}\mathscr{B} - \Omega^\beta{}_A\mathrm{d}\varphi_\beta$. Defining $\mathscr{H} = H + \mathrm{d}\mathscr{B}$, we find

$$\iota_{\overline{k}_A}\mathscr{H} = \mathrm{d}\phi_A + L_{k_A}\mathscr{B}\,. \tag{4.55}$$

Summing up the various contributions, and cleaning up the result with a symmetrised version of (4.54), we recover

$$L_{\overline{k}_A}\overline{\phi}_B - \iota_{\overline{k}_B}\mathrm{d}\overline{\phi}_A - \iota_{\overline{k}_A\overline{k}_B}\mathscr{H} = F_{AB}{}^C\overline{\phi}_C + F_{AB\gamma}\overline{\phi}^\gamma + F_{AB\gamma}\mathscr{A}^\gamma + 2\Omega^\gamma{}_{[A}F_{|\gamma|B]}{}^C\overline{\phi}_C\,. \tag{4.56}$$

This is almost the desired algebra, were not for the $\mathscr{A}$-dependent term. Actually, we are missing an ingredient: all of the objects here are still defined on $\widetilde{G}\backslash\mathbb{D}$, but we would like to project them down to the dressing coset. This is attained with a pull-back/push-forward of forms/vectors via the section $\sigma : M \to \widetilde{G}\backslash\mathbb{D}$ or the projection $p : \widetilde{G}\backslash\mathbb{D} \to M$. Since $\mathscr{A}$ is a flat connection, a section such that $\sigma^*\mathscr{A} = 0$ has to exists, at least patchwise. With this choice, the generalised frame field and Ω

$$E_A = p_*\overline{k}_A + \sigma^*\overline{\phi}_A\,, \qquad \text{and} \qquad \Omega^\alpha = p_*\overline{k}^\alpha + \sigma^*\overline{\phi}^\alpha \tag{4.57}$$

obey the frame algebra (4.33). To complete the picture, let us check the validity of (4.39). First, the following identity holds,

$$2\iota_{\overline{k}_{[A}}\mathrm{d}\Omega^\gamma{}_{B]} = -F_{\delta\epsilon}{}^\gamma\Omega^\delta{}_A\Omega^\epsilon{}_B + F_{AB\delta}\Omega^{\gamma\delta} - F_{AB}{}^\gamma + F_{AB}{}^E\Omega^\gamma{}_E + 2F_{[A|\delta}{}^E\Omega^\delta{}_{B]}\Omega^\gamma{}_E\,, \tag{4.58}$$

as can be proven with a simple calculation. Noticing that

$$\mathscr{L}_{\Omega^\alpha}E_B = \left(\Omega^\alpha{}_D\mathcal{F}^D{}_{BC} - 2\iota_{\overline{k}_{[B}}\mathrm{d}\Omega^\alpha{}_{C]}\right)E^C \tag{4.59}$$

we arrive, after some careful rearrangement of terms, at precisely (4.39).

4.3.3 Application: Drinfel'd Double and Coisotropic Subgroups

Paving the way to some calculations which will be carried out in the next chapter, let us study in detail the case where $\mathbb{D}$ is a Drinfel'd double, locally the product of two Poisson-Lie groups $\widetilde{G}$ and G, and H is a coisotropic subgroup of G. Recall that, when $H \subset G$ is coisotropic, the Poisson structure π on G descends to the Poisson-Bruhat structure $\pi_{\rm B}$ on G/H: morally, we expect a result for the frame field akin to that of Section 4.2.1, upon replacing π with $\pi_{\rm B}$ and restricting the vielbein to the coset space.

As $\mathfrak{g} = \text{Lie}(G)$ and $\widetilde{\mathfrak{g}} = \text{Lie}(\widetilde{G}) \cong \mathfrak{g}^*$ are dual algebrae, the generators T_α of $\mathfrak{h} \subset \mathfrak{g}$ will have corresponding generators T^α in $\widetilde{\mathfrak{g}}$ such that $\langle\!\langle T_\alpha, T^\beta \rangle\!\rangle = \delta^\beta_\alpha$. In this sense, we identify $\mathfrak{p} \cong \mathfrak{h}^*$ and $\mathfrak{q} \cong \mathfrak{m} + \mathfrak{m}^*$, where $\exp \mathfrak{m} \cong G/H$.

Even though we build on the previous example, we shall slightly change the notation so as to accommodate the additional level of detail entailed by the dressing coset construction. Taken an element $g \in G$, we indicate the left-invariant form with $g^{-1}\mathrm{d}g = e^{\mathbb{a}}{}_{\mathbb{i}}(g)\mathrm{d}x^{\mathbb{i}} T_{\mathbb{a}}$. Suppose we split coordinates on G according to $x^{\mathbb{i}} = (x^i, x^\mu)$, where x^i refer to G/H and x^μ to H alone. Accordingly, we pick a group element g such that $g(x^{\mathbb{i}}) = m(x^i)h(x^\mu)$. With this choice, and exploiting the defining property of a reductive coset,

$$g^{-1}\mathrm{d}g \equiv e(g)^{\mathbb{a}} T_{\mathbb{a}} = (\mathrm{Ad}_{h^{-1}})_b{}^a e^b{}_i(m)\mathrm{d}x^i T_a + (\mathrm{Ad}_{h^{-1}})_\beta{}^\alpha e^\beta{}_i(m)\mathrm{d}x^i T_\alpha + \mathscr{A}^\alpha{}_\mu \mathrm{d}x^\mu T_\alpha \,. \tag{4.60}$$

To ease the notation in the following steps, we will equivalently denote the connection one-form as $\mathscr{A}^\alpha{}_\mu = e^\alpha{}_\mu(h)$, so that its inverse reads $e^\mu{}_\alpha(h)$. The vector dual to the Maurer-Cartan one-form on G is easily found to be

$$\begin{gathered} k_a{}^i = (\mathrm{Ad}_h)_a{}^b e^i{}_b(m)\,, \qquad k_\alpha{}^\mu = e^\mu{}_\alpha(h)\,, \qquad k_\alpha{}^i = 0\,, \\ k_a{}^\mu = -e^\mu{}_\alpha(h)(\mathrm{Ad}_h^{-1})_\beta{}^\alpha e^\beta{}_i(m) e^i{}_b(m)(\mathrm{Ad}_h)_a{}^b\,. \end{gathered} \tag{4.61}$$

From the previous example, we know that $k_{\mathbb{a}} = e_{\mathbb{a}}(g)$, $k^{\mathbb{a}} = \pi^{\mathbb{a}\mathbb{b}} e_{\mathbb{b}}(g)$; therefore, we have

$$\overline{k}_a = (\mathrm{Ad}_h)_a{}^b e_b(m)\,, \qquad \overline{k}_\alpha = 0\,, \qquad \overline{k}^a = \pi^{ab}(\mathrm{Ad}_h)_b{}^c e_c(m)\,, \qquad \overline{k}^\alpha = \pi^{\alpha b} e_b(m)\,. \tag{4.62}$$

It is also possible to compute the explicit form for $\Omega^\alpha{}_B$ which, in components, reads

$$\Omega^\alpha{}_b = (\mathrm{Ad}_h^{-1})_\beta{}^\alpha e^\beta{}_i(m) e^i{}_c(m)(\mathrm{Ad}_h)_b{}^c\,, \qquad \Omega^{\alpha b} = \pi^{bc}\Omega^\alpha{}_c - \pi^{b\alpha}\,. \tag{4.63}$$

Also, we have

$$\Omega^{\alpha\beta} = \pi^{\alpha c}\Omega^\beta{}_c - \pi^{\alpha\beta}\,, \qquad \Omega^\alpha|_{\text{vect}} = \pi^{\alpha b} e_b(m)\,. \tag{4.64}$$

Let us move to the one-form part. From the construction of frame fields on $\widetilde{G}\backslash\mathbb{D}$ we recover $\varphi^{\mathbb{a}} = e^{\mathbb{a}}$ and $\varphi_{\mathbb{a}} = 0$. Dividing up the non-vanishing contribution into the different components we get

$$\varphi^\alpha = (\mathrm{Ad}_{h^{-1}})_\beta{}^\alpha e^\beta{}_i(m)\mathrm{d}x^i + e^\alpha{}_\mu(h)\mathrm{d}x^\mu\,, \qquad \varphi^a = (\mathrm{Ad}_{h^{-1}})_b{}^a e^b{}_i(m)\mathrm{d}x^i\,. \tag{4.65}$$

Since $\varphi_{\mathbb{a}} = 0$, we necessarily have no $\mathscr{B}$, thereby making ϕ coincide with φ. Hence

$$\overline{\phi}_\alpha = 0\,, \qquad \overline{\phi}_a = 0\,, \qquad \overline{\phi}^\alpha = (\mathrm{Ad}_{h^{-1}})_\beta{}^\alpha e^\beta{}_i(m)\mathrm{d}x^i\,, \qquad \overline{\phi}^a = (\mathrm{Ad}_{h^{-1}})_b{}^a e^b{}_i(m)\mathrm{d}x^i\,. \tag{4.66}$$

To obtain the final result we need to push-forward/pull-back the vectors/forms via the section/projection previously introduced. In particular, we wish to consider a section σ such that $\sigma^*\mathscr{A} = 0$. For the sake of simplicity, we will choose it so that it is equivalent to gauge-fixing the group element $h = e$, being e the identity element of H. In doing so, recall that the Poisson structure $\pi^{\mathbb{a}\mathbb{b}}$ depended on the choice of $g = mh$; therefore, upon adopting such section, every instance of $\pi^{\mathbb{a}\mathbb{b}}(mh)$ will be in fact reduced to $\pi^{\mathbb{a}\mathbb{b}}(m)$. In particular, if R is an R-matrix, we will have $\pi^{\mathbb{a}\mathbb{b}}(m) = (R_m - R)^{\mathbb{a}\mathbb{b}}$. Once restricted to coset indices, this is the Poisson-Bruhat structure π_{B}^{ab}. Hence:

$$\begin{gathered} E^a = e^a(m) + \pi^{ab}(m)e_b(m)\,, \qquad E_a = e_a(m)\,, \qquad \Omega^\alpha = \pi^{\alpha b}(m)e_b(m) + e^\alpha(m)\,, \\ \Omega^\alpha{}_b = e^\alpha{}_i(m)e^i{}_b(m)\,, \qquad \Omega^{\alpha b} = \pi^{bc}(m)\Omega^\alpha{}_c - \pi^{b\alpha}(m)\,, \qquad \Omega^{\alpha\beta} = \pi^{\alpha c}(m)\Omega^\beta{}_c - \pi^{\alpha\beta}(m)\,. \end{gathered} \tag{4.67}$$

4.4 Conclusions

Let us briefly recap the key results of this chapter. On $\widetilde{G}\backslash\mathbb{D}$ we have reviewed the construction of a set of generalised frame fields $E_{\mathbb{A}}$ that realise the algebra $\mathfrak{d}$ of $\mathbb{D}$ via the generalised Lie derivative. With respect to [121] we have improved the presentation with the adoption of a more direct, mathematical oriented and index-free approach. To show how this works in practice, the case of $\mathbb{D}$ being a Drinfel'd double was analysed in detail. We have then performed a reduction to the dressing coset $\widetilde{G}\backslash\mathbb{D}/H$ in which a second isotropic subgroup H was quotiented out, provided suitable generalised reductiveness conditions apply. In this setting we have constructed a generalised frame field obeying the vielbein postulate $\nabla E = 0$ for some covariant derivative ∇. The generalised torsion and curvature associated to ∇, in flat indices, were completely determined by a selection of the structure constants of $\mathfrak{d}$. To complement the discussion and to furnish a cornerstone of the chapter that will follow, we have finally worked out the generalised frame field for a dressing coset where $\mathbb{D}$ is a Drinfel'd double, locally $\mathbb{D} = \widetilde{G}G$, and $H \subset G$.

Chapter 5

Integrable Deformation of $\mathbb{CP}^n$ and Generalised Kähler Geometry

Abstract

We build on the results of Chapter 4 for generalised frame fields on generalised cosets and study integrable deformations for $\mathbb{CP}^n$. We elucidate how the deformed target space can be seen as an instance of generalised Kähler, or equivalently bi-Hermitian, geometry. In this respect, we find the generic form of the pure spinors for $\mathbb{CP}^n$ and the explicit expression for the generalised Kähler potential for $n = 1, 2$. In addition, we show how a two-parameter deformation can be introduced in principle. The second parameter can however be removed via a diffeomorphism, which we construct explicitly, in accordance with the results stemming from a thorough integrability analysis we carry out. We complete the discussion providing explicit expressions for the η-deformed metric and B-field of S^n and AdS_n.

5.1 Introduction

The generalised frame fields of Chapter 4 provide a privileged view point for the exploration of the geometry underpinning the target manifold M of a non-linear σ-model. For integrable deformations the sole Riemannian geometry is unable to capture the entirety of the features, as a non-vanishing B-field is most commonly part of the picture. To include it, one has then to resort to Generalised Geometry and the frame fields, among other things, precisely bridge the gap between the two-dimensional σ-model and its description in Generalised Geometry.

For holographic applications coset spaces are especially relevant as they include, among others, spheres and complex projective spaces that appear in a number of backgrounds, most notably in the original AdS/CFT formulation of type IIB on $\text{AdS}_5 \times S^5$ [26], the M-theoretic $\text{AdS}_4 \times S^7$ or the type IIA $\text{AdS}_4 \times \mathbb{CP}^3$. Generalising the AdS/CFT correspondence to spaces obtained as integrable deformations of the aforementioned cases necessarily requires a good understanding of their geometries. With this goal in mind, the aim of this chapter is to enlighten a few properties of deformed projective spaces and, tangentially, of spheres. Remarkably, as AdS_n can be thought of as the analytic continuation of the sphere S^n, we will be able to extend the analysis to anti-de Sitter spaces, thereby furnishing a more thorough description of the NS-NS sector of these backgrounds.

At this point it is fair to remind that knowing the generalised metric alone, possibly through the construction of frame fields, is far from being exhaustive: the NS-NS sector of String Theory most certainly needs the dilaton to be completed, and the RR sector has been entirely omitted thus far. Whilst algebraic approaches, based on supergroups, are a viable option for their extraction (explored, for instance, in [125]), a path much less travelled is that based on Generalised Geometry. Here, the program would be to recover the dilaton and fluxes which solve the Supergravity equations from the knowledge of a few objects that appear in Generalised Geometry such as, for instance, the pure spinors.

The major criticality of the latter approach is that, for (deformations of) String Theory backgrounds of the form $\text{AdS}_n \times X_{10-n}$ for some $(10-n)$-dimensional manifold X, whilst the overall metric and B-field are expected to factorise into those of the two manifolds, the dilaton will usually mix up, in a completely non-trivial way, AdS and X contributions. This has been proven for $X_{10-n} \cong S^{10-n}$ in [98], but we obviously do not anticipate any improvement in more convoluted cases. Hence, the completion of the NS-NS sector necessarily requires an approach where the interplay between the internal and external manifolds is manifest. A similar story holds true for the RR sector as well. Even though it might be possible to (geometrically) address these issues in the context of Generalised Supergeometry [126], we shall not comment on them any further and restrict ourselves to the generalised metric.

More specifically, we will be focusing on Yang-Baxter deformations of complex projective spaces, where the entire machinery of generalised cosets established in Chapter 4 applies. After proving that they constitute an example of Generalised Kähler Geometry, we shall resolve a little puzzle that dates back to [20]: whilst in principle these spaces allow for a *double* deformation – akin to a bi-Yang-Baxter model in the case of group manifolds –, the second deformation parameter can be smuggled away through an appropriate diffeomorphism involving the string tension, which we provide.

5.1.1 Riemannian Geometry of $\mathbb{CP}^n$

To orient the reader, we will close this introductory part with a review of well-known facts concerning the Riemannian geometry of (undeformed) $\mathbb{CP}^n$ spaces. These will serve as building blocks for a number of geometric results obtained in later sections. The references [40, 42, 127, 128] can be consulted for additional details.

Complex projective spaces can be characterised in a number of ways, most commonly as coset spaces, Kähler manifolds or flag manifolds. As a quotient space, $\mathbb{CP}^n$ represents a particular instance of complex Grassmannians

$$\mathbb{CP}^n = \frac{SU(n+1)}{S(U(1) \times U(n))} \,. \tag{5.1}$$

As a Kähler manifold, it is equipped with a Riemannian metric G, a complex structure J and a Kähler form ω related by $\omega = JG$. Being a Grassmannian, it is a compact Hermitian symmetric space and, moreover, a generalised flag manifold − a particular form of algebraic variety − describing a coadjoint orbit of $SU(n+1)$. In terms of Poisson geometry, the Poisson structure π on $SU(n+1)$ descends to a Poisson-Bruhat structure π_{B} on $\mathbb{CP}^n$; also, the Kähler form can be inverted, producing a new Poisson structure ω^{-1}. On any compact Hermitian symmetric space the two are compatible, $[\pi_{\mathrm{B}}, \omega^{-1}]_s = 0$, and can thus be linearly combined into a third object, the *Poisson pencil*, $\pi_\tau = \pi_{\mathrm{B}} - \tau\omega^{-1}$ for some choice of parameter $\tau \in \mathbb{R}$.

These facts may also be interpreted at the level of the algebra, mostly building on the results of Example 2.1.1 to which we refer for details and notation. Since $\mathfrak{g} = \mathfrak{su}(n+1)$ is semi-simple, the Killing form κ is non-degenerate. If T_{a} generate $\mathfrak{g}$ and, in the symmetric space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, T_α and T_a generate $\mathfrak{h}$ and $\mathfrak{m}$ (respectively), the restriction $\kappa_{ab} = \kappa(T_a, T_b)$ uplifts to the metric G of $\mathbb{CP}^n$. Similarly, the Drinfel'd-Jimbo R-matrix on $\mathfrak{g}$ respects the hypothesis of Koszul theorem, meaning that its restriction ${R_a}^b$ to $\mathfrak{m}$ uplifts to J. Finally, lowering its index with the Killing metric, $R_{ab} = {R_a}^c \kappa_{cb}$, we obtain the flat version of the Kähler form ω. The Poisson-Bruhat structure, instead, reads $\pi_{\mathrm{B}}^{ab} = (R_m - R)^{ab}$, where m is a coset representative obtained from exponentiating elements of $\mathfrak{m}$.

As a manifold, $\mathbb{CP}^n$ can be covered with $n+1$ patches. An alternative description is as the quotient $\mathbb{CP}^n \cong (\mathbb{C}^{n+1} - \{0\})/\sim$, where the equivalence relation $\sim$ identifies the $(Z_0, \ldots, Z_n)$ coordinates of $\mathbb{C}^{n+1} - \{0\}$ with their rescaling $(\lambda Z_0, \ldots, \lambda Z_n)$ by a constant parameter $\lambda \in \mathbb{C} - \{0\}$. With the identification implemented, the coordinates indicated with $Z \equiv [Z_0 : \cdots : Z_n]$ are called *homogeneous*. In practice we can choose $n+1$ patches $U_i = \{Z | Z_i \neq 0\}$: in the i-th patch we can introduce local coordinates $z_j := Z_j / Z_i$, $j \neq i$.

Following standard nomenclature, we shall call z_j's Fubini-Study (FS) coordinates and U_0 the *largest Bruhat cell.*

In FS coordinates (obtained from every U_i patch), the metric of $\mathbb{CP}^n$ has the well-known Fubini-Study form

$$\mathrm{d}s^2 = \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{1+|z|^2} - \frac{z_i \bar{z}_j \mathrm{d}\bar{z}_i \mathrm{d}z_j}{(1+|z|^2)^2}\,, \tag{5.2}$$

with $|z|^2 = z_i \bar{z}_i$, the corresponding Kähler potential being $K_{\mathrm{FS}} = \log(1+|z|^2)$. The complex structure J is diagonalised according to Darboux theorem, $J = i\mathrm{d}z_i \otimes \partial_{z_i} - i\mathrm{d}\bar{z}_i \otimes \partial_{\bar{z}_i}$, and ω follows accordingly. Unlike these objects, the choice of the patch U_i determines different local forms of the Poisson-Bruhat structure. In fact, FS coordinates do lead to a somewhat involved expression for π_{B} which will not be shown here.

In contrast, we can opt for other types of local coordinates − built out of FS coordinates − that result in a much simpler form for π_{B} and, anticipating some future discussion, for the metric after the deformation. Whilst these can be defined in every U_i patch we shall henceforth stick to the largest Bruhat cell for simplicity. Here, the new coordinates (x_i, ϕ_i) are related to FS coordinates by

$$z_i = \left(\frac{x_i}{1-X}\right)^{1/2} e^{i\phi_i}\,, \qquad \text{with} \quad 0 \leq x_i < 1 - \sum_{k=1}^{i-1} x_k \quad \text{and} \quad 0 \leq \phi_i < 2\pi\,, \tag{5.3}$$

where $X = \sum_i x_i$. With this choice the Poisson-Bruhat structure has a remarkably simple expression,

$$\pi_{\mathrm{B}} = \sum_i \left(-1 + \sum_{k=1}^{i} x_k\right) \partial_{x_i} \wedge \partial_{\phi_i} + \sum_{i>j} x_i \partial_{x_i} \wedge \partial_{\phi_j}\,. \tag{5.4}$$

5.2 Generalised Kähler Geometry of $\mathbb{CP}^n_\eta$

Having described in some detail the geometry underpinning $\mathbb{CP}^n$, we are now ready to tackle the problem of its η-deformation $\mathbb{CP}^n_\eta$. We introduce projectors $\mathcal{P}_i$ onto $\mathfrak{g}^{(i)}$ in the symmetric space decomposition and let $\kappa_{ab} = \langle T_a, T_b \rangle$. Denoting with $e_\pm = m^{-1}\partial_\pm m$ the left-invariant form associated to a coset representative m, the corresponding σ-model with tension t is

$$S = \frac{1}{\pi t}\int \mathrm{d}^2\sigma \, \langle \mathcal{P}_1 e_+, \frac{1}{1-\eta R_m \mathcal{P}_1} \mathcal{P}_1 e_- \rangle\,. \tag{5.5}$$

Projectors $\mathcal{P}_1$ onto $\mathfrak{g}^{(1)} \cong \mathfrak{m}$ ensure that, in the decomposition $e_\pm = e^a_\pm T_a + e^\alpha_\pm T_\alpha$, only the $\mathfrak{m}$-directed components $e^a_\pm$ contribute to the action. Notice how convenient it is to adopt the Poisson pencil: with obvious manipulations $R^{ab}_m = (R_m - R)^{ab} + R^{ab} = \pi^{ab}_\tau$ for $\tau = 1$. From this observation it is clear that the deformation entailed in the action

(5.5) is geometrically induced by the Poisson pecil. To make this more precise, one can extract a conventional metric g and NS two-form b either from directly evaluating the σ-model action or equivalently by dressing the generalised metric with the appropriate generalised frame field. In fact, the latter has already been discussed in Example 4.3.3,

$$E_A = \begin{pmatrix} e^a & \eta t \pi_{\mathrm{B}}^{ab} e_b^{-t} \\ 0 & e_a^{-t} \end{pmatrix} , \tag{5.6}$$

where we have additionally introduced the convenient factor ηt.

To construct the generalised metric, we would need to compare (5.5) with the action for a Poisson-Lie model, so as to extract E_0 out of which $\mathcal{H}_{AB}$ is built. However, so far we have only considered Poisson-Lie σ-models on group manifolds. On the (reductive, but not necessarily symmetric) coset G/H, a natural suggestion is to restrict the indices in (3.71) to run over $\mathfrak{m}$, and so we consider

$$S_{G/H} = \frac{1}{\pi} \int_\Sigma \mathrm{d}^2\sigma \, e_+^a \left(E_0^{-1} + \pi_g\right)^{-1}_{ab} e_-^b \, . \tag{5.7}$$

However, *a priori*, the degrees of freedom entering this action still contain those corresponding to the subgroup H and thus, without imposing further constraints, (5.7) does not provide a valid description of the coset G/H. This is remedied by demanding that the action develops a local gauge symmetry under the action of H from the right which serves to eliminate the unwanted degrees of freedom. A short calculation shows that under an infinitesimal transformation this is the case provided that [129]

$$0 = \tilde{f}^{ab}{}_\gamma + E_0^{-1\,ad} f_{\gamma d}{}^b + f_{\gamma d}{}^a E_0^{-1\,db} \, . \tag{5.8}$$

This result is general, and does not require H to be coisotropic, as in the case of $\mathbb{CP}^n$. Also, in a doubled fashion, it can be equivalently rewritten as[1]

$$0 = F_{\alpha A}{}^C \mathcal{H}_{CB} + F_{\alpha B}{}^C \mathcal{H}_{CA} \, . \tag{5.9}$$

When coisotropy is imposed, $\tilde{f}^{ab}{}_\gamma = 0$ and the expression above simplifies. Specifically, if E_0^{-1ab} is any linear combination of κ^{ab} and R^{ab}, the constraint is obeyed and the coset model well-defined. For $\mathbb{CP}^n_\eta$, it results in $E_0^{-1ab} = t(\kappa^{ab} + \eta R^{ab})$, giving rise to the generalised metric in flat indices

$$\mathcal{H}_{AB} = \begin{pmatrix} 1 + \eta^2 t \kappa^{-1} & \eta R^t \\ \eta R & t^{-1}\kappa \end{pmatrix}_{AB} . \tag{5.10}$$

[1]We will indicate with F the structure constants of the corresponding Drinfel'd double. For η-deformations of a Lie group $G = \exp(\mathfrak{g})$, the double is just $\mathfrak{d} \cong \mathfrak{g}_{\mathbb{C}}$.

After some manipulations one finds that the deformed geometry encapsulated by g and b can be expressed in terms of G and π_τ as

$$g^{-1} = G^{-1} - \eta^2 \pi_\tau G \pi_\tau \,, \qquad bg^{-1} = -\eta \, G\pi_\tau \,. \tag{5.11}$$

Let us emphasise that, despite the elegant form of (5.11), in terms of explicit coordinate expressions these become rather intractable. We can now observe that the generalised metric with flat indices given in (5.10) admits the decomposition $\mathcal{E} \equiv \mathcal{H}\eta^{-1} = \mathcal{J}_1\mathcal{J}_2$ with

$$\mathcal{J}_{1A}{}^B = \begin{pmatrix} R^t & 0 \\ 0 & -R \end{pmatrix}_A{}^B \quad \text{and} \quad \mathcal{J}_{2A}{}^B = \begin{pmatrix} \eta & (1+\eta^2)t\kappa^{-1}R \\ t^{-1}R\kappa & -\eta \end{pmatrix}_A{}^B , \tag{5.12}$$

such that, for $i = 1, 2$,

$$\mathcal{J}_i^2 = -1 \,, \quad [\mathcal{J}_1, \mathcal{J}_2] = 0 \,. \tag{5.13}$$

Thus, upon dressing these flat space quantities with the generalised frame fields (5.6), we see that the target space geometry is indeed generalised Kähler with

$$\mathcal{J}_1 = \begin{pmatrix} J^t & t\eta(J^t\pi_{\rm B} + \pi_{\rm B}J) \\ 0 & -J \end{pmatrix} , \quad \mathcal{J}_2 = \begin{pmatrix} -\eta\pi_\tau\omega & -t(\omega^{-1} + \eta^2\pi_\tau\omega\pi_\tau) \\ t^{-1}\omega & \eta\omega\pi_\tau \end{pmatrix} . \tag{5.14}$$

Using the parametrisation of generalised complex structures (2.30), we know that the North-East blocks of $\mathcal{J}_{1,2}$ are Poisson structures. This can rendered explicit recalling that Generalised Kähler Geometry is equivalent to bi-Hermitian geometry, the map between the two being given by (2.40). It is now easy to show that, if we introduce the quantities $Q_\pm = 1 \pm bg^{-1} = 1 \mp \eta G\pi_\tau$, the objects appearing in (2.40) are given by

$$J_\pm = Q_\pm^{-1} J Q_\pm \,, \qquad \omega_\mp^{-1} = Q_\pm^t \omega^{-1} Q_\pm \,, \qquad g = Q_\pm^{-1} G Q_\pm^{-t} \,. \tag{5.15}$$

Out of these, one can construct three Poisson structures[2] [130], namely $\sigma = g^{-1}[J_+, J_-]$ and $\pi_\pm = \pm 1/2 \left(\omega_+^{-1} \pm \omega_-^{-1}\right)$, specifically resulting in

$$\pi_+ = (1 + \eta^2\pi_\tau\omega\pi_\tau\omega)\omega^{-1} \,, \qquad \pi_- = \eta(J^t\pi_{\rm B} + \pi_{\rm B}J) \,, \qquad \sigma = \omega_-^{-1}J_+ - \omega_+^{-1}J_- \,. \tag{5.16}$$

As we can see, $\pi_\pm$ correspond to the North-East blocks of $\mathcal{J}_{1,2}$, as anticipated. Generalised Kähler Geometry is related to bi-Hermitian geometry and consequently to $\mathcal{N} = (2, 2)$ supersymmetry. In this sense, the types of supersymmetric multiplets required to furnish an $\mathcal{N} = (2, 2)$ action can be extracted from the three Poisson structures we have

[2] We choose to introduce an extra factor of $\pm 1/2$ with respect to the standard definitions so as to get rid of some numerical factors which will not affect the subsequent analysis.

just singled out. Chiral superfields parametrise

$$\text{Ker}(J_+ - J_-) = \text{Ker}\,\pi_- \,, \tag{5.17}$$

whereas twisted chirals are needed to parametrise

$$\text{Ker}(J_+ + J_-) = \text{Ker}\,\pi_+ \,. \tag{5.18}$$

The remaining directions, i.e. $(\text{Ker}[J_+, J_-])^\perp$, corresponding to the symplectic leaves of σ are to be parametrised by semi-chiral superfields [130].

Let us briefly study these superfields in our case. For π_+, its kernel is isomorphic to the kernel of $1 + (\eta\pi_\tau\omega)^2$ which, since $(\eta\pi_\tau\omega)^2$ is positive definite, is trivial. Hence, no twisted chiral multiplets are present. The kernel for π_- is better studied in Fubini-Study coordinates of the largest Bruhat cell. Here the complex structure is diagonal and the expression (5.16) for π_- amounts to selecting the diagonal blocks of π_{B} which, in this patch, turn out the complex conjugates of one another. Each one of these blocks is a $n \times n$ dimensional matrix and, therefore, has vanishing determinant for odd n. In particular, each block has a null space parametrised by one single vector so that, upon linearly combining them, we have a total of two vectors generating the null space. In the even case, it turns out that the determinant is non-vanishing, implying a trivial kernel. In summary, when n is odd we have two vectors generating the kernel of π_- and, thus, a single chiral superfield. We therefore end up with $(n-1)/2$ semi-chiral multiplets plus a single chiral multiplet in the odd case and $n/2$ semi-chiral multiplets in the even case.

5.2.1 Pure Spinors

Having identified $\mathbb{CP}^n_\eta$ as an instance of generalised Kähler geometry, we can proceed with an analysis of the associated pure spinors. These can be studied without fixing a specific (complex) dimension. We will follow the standard procedure, namely we will first compute a basis $V^j_{1,2}$ for the $+i$-eigenspace of each complex structure and then we will impose that the same basis annihilates the associated pure spinor.

Let us start from $\mathcal{J}_1$. It is most easily analysed in Fubini-Study coordinates: J is diagonal and n $+i$-eigenvectors for $\mathcal{J}_1$ are immediately found to be $V^j_1 = \partial_{z_j}$, $j = 1, \ldots, n$. On the other hand, π_- in these coordinates reads

$$\pi_- = 2\eta \sum_{j>i} (z_i z_j \partial_{z_i} \wedge \partial_{z_j} + \text{c.c.}) \,, \tag{5.19}$$

making it easy to see that the remaining eigenvectors are

$$V_1^{n+j} = \mathrm{d}\bar{z}_j + i\eta t\, \bar{z}_j \left(\sum_{i>j} \bar{z}_i \partial_{\bar{z}_i} - \sum_{i<j} \bar{z}_i \partial_{\bar{z}_i} \right) , \qquad j = 1, \ldots, n \, . \tag{5.20}$$

As proved by Gualtieri [25], the general form of a non-degenerate complex pure spinor is $\widehat{\Psi} = \Xi \wedge e^{\rho}$, where ρ is a complex two-form, Ξ a decomposable k-form and k the type of the spinor. For odd n, we proved that $\dim \mathrm{Ker}\pi_- = 2$ and the spinor will be of type 1 (that is, Ξ will be a one-form). On the contrary, for even n the Poisson structure π_- has trivial kernel: the spinor will have type 0 and we can consistently set $\Xi = 1$ since the spinor is defined up to an overall function.

Now, the requirement $V_1^j \cdot \widehat{\Psi}_1 = 0$ for $j = 1, \ldots, n$ implies that the spinor is made up of anti-holomorphic forms only. The constraints arising from $V_1^{n+j} \cdot \widehat{\Psi}_1 = 0$ with $j = 1, \ldots, n$ are equivalent to

$$0 = \xi_1^{n+j} + \iota_{v_1^{n+j}} \rho \qquad \text{for even } n \, , \tag{5.21}$$

$$0 = \iota_{v_1^{n+j}} \Xi - \Xi \wedge \iota_{v_1^{n+j}} \rho + \xi_1^{n+j} \wedge \Xi \qquad \text{for odd } n \, , \tag{5.22}$$

where v_1^{n+j} and ξ_1^{n+j} are, respectively, the vector and form part of the generalised vectors (5.20). Observe that (5.22) can be in fact split into two separate equations, corresponding to degree zero and two. In this sense, the degree zero requirement is the same as saying that the interior product of Ξ with v_1^{n+j} vanishes for all $j = 1, \ldots, n$. As one can explicitly check, all of the equations are satisfied with

$$\rho = \frac{i}{\eta t} \sum_{k>i} (-1)^{i+k} \frac{\mathrm{d}\bar{z}_i \wedge \mathrm{d}\bar{z}_k}{\bar{z}_i \bar{z}_k} \qquad \text{and} \qquad \Xi = \begin{cases} 1 & \text{even } n \\ i\eta t \sum_k (-1)^{k+1} \frac{\mathrm{d}\bar{z}_k}{\bar{z}_k} & \text{odd } n \end{cases} . \tag{5.23}$$

With this normalisation we remark that for vanishing η the pure spinor is well defined and coincides (after an appropriate rescaling) with the decomposable anti-holomorphic form $\overline{\Omega} = \mathrm{d}\bar{z}_1 \wedge \ldots \wedge \mathrm{d}\bar{z}_n$.

As for $\mathcal{J}_2$, it is sufficient to notice that its explicit form (5.14) implies that each and every generalised eigenvector V_2^j with $+i$ eigenvalue will be given by

$$V_2^j = it(\omega^{-1} + i\eta\pi_\tau)\xi_2^j + \xi_2^j \qquad j = 1, \ldots, 2n, \tag{5.24}$$

being ξ_2^j a set of $2n$ independent one-forms. The second pure spinor then results in

$$\widehat{\Psi}_2 = \exp\left[-it^{-1}(\omega^{-1} + i\eta\pi_\tau)^{-1}\right] . \tag{5.25}$$

In particular, notice that the $\eta \to 0$ limit correctly yields the exponential of the Kähler form, as it should for a Kähler manifold.

Finally, notice that, for every value of n, $\mathrm{d}\rho = \mathrm{d}\Xi = 0$; also, $\mathrm{d}(\omega^{-1} + i\eta\pi_\tau)^{-1} = 0$ follows from the compatibility of π_{B} and ω^{-1}. Thus, $\mathrm{d}\widehat{\Psi}_{1,2} = 0$. Actually, this is a consequence of our choice of normalisation for the spinors; for instance, we have set the zero-form component of $\widehat{\Psi}_2$ to one. Instead, we could impose a different normalisation using the Mukai pairing $||\widehat{\Psi}_i||^2 = \widehat{\Psi}_i \wedge \sigma(\bar{\widehat{\Psi}}_i)|_{\mathrm{top}}$. Should we scale the pure spinors such that they have equal normalisation, then they would no longer be closed. The geometry is hence not generalised Calabi-Yau[3].

5.2.2 Generalised Kähler Potential

Finding the generalised Kähler potential $\mathcal{K}$ for a deformation of $\mathbb{CP}^n$ is complicated, at least for generic n. We thus devote the present paragraph to some general reasoning.

A key challenge in establishing the generalised Kähler potential is to find appropriate coordinates. It is a trivial matter to check that $J_\pm^t \sigma J_\pm = -\sigma$, i.e. that σ splits into $\sigma = \sigma^{(2,0)} + \overline{\sigma}^{(0,2)}$ with respect to either complex structures. Invertibility, however, is not necessarily guaranteed. It is well known (see e.g. [39] for a comprehensive treatment) that each Poisson structure π defines a foliation. Specifically, although π might not be globally invertible, when restricted to one of its leaves Σ, the two-form $(\pi|_\Sigma)^{-1}$ is well-defined. It has been first proven in [57] that for $\pi = \sigma$, the leaves have real dimension $4m$, for some $m \in \mathbb{N}$. In the $\mathbb{CP}^n$ case, the integer m is related to the complex dimension of the projective space via $m = [\frac{n}{2}]$.

Suppose we now restrict to one leaf Σ, $\dim \Sigma = 4m$, where σ^{-1} is well defined[4]. Because σ is a Poisson structure $\mathrm{d}\sigma^{-1} = 0$ has to hold. In general σ^{-1} will inherit from σ the decomposition $\sigma^{-1} = \sigma^{-1\,(2,0)} + \overline{\sigma}^{-1\,(0,2)}$ and the holomorphic coordinates we look for should be such that it is brought to the canonical form

$$\sigma^{-1} = \sum_{i=1}^{m} \mathrm{d}q^i \wedge \mathrm{d}p_i + \text{c.c.} = \sum_{i=1}^{m} \mathrm{d}Q^i \wedge \mathrm{d}P_i + \text{c.c.} \tag{5.26}$$

(q,p) and (Q,P) can be thought of as the complex coordinates diagonalising, respectively, J_+ and J_- restricted to Σ (where they do not commute, so that they cannot be simultaneously diagonalised). In the language of supersymmetry, one can also look at $(q^i, \bar{q}^i, P_i, \bar{P}_i)$, for a fixed value of i, as part of a semi-chiral superfield [132]. The crucial

[3] A generalised Kähler geometry is generalised Calabi-Yau when the pure spinors associated to the generalised complex structures are nowhere-vanishing, closed when choosing their relative norm with respect to the Mukai-pairing to be a constant [25, 131].

[4] Here and henceforth σ^{-1} should be understood as $\sigma^{-1} \equiv (\sigma|_\Sigma)^{-1}$.

point is that the transformation between $(p, \bar{p}, Q, \bar{Q})$ and $(P, \bar{P}, q, \bar{q})$ is canonical with a (real) generating function $\mathcal{K}(P, \bar{P}, q, \bar{q})$ such that

$$p_i = \frac{\partial \mathcal{K}}{\partial q^i} , \quad Q^i = \frac{\partial \mathcal{K}}{\partial P_i} . \tag{5.27}$$

It is this generating functional that becomes identified with the generalised Kähler potential. Thus, extracting $\mathcal{K}$ can in general be hard and cumbersome: first one has to obtain p_i and Q^i and then integrate the above equations to determine $\mathcal{K}$.

The discussion above completely determines the Kähler potential when semi-chiral fields parametrise the whole geometry, i.e. for the η-deformation of $\mathbb{CP}^{2m}$ (cf. the discussion around (5.17)). For $\mathbb{CP}^{2m+1}$ we need to augment the semi-chiral multiplets with a single chiral multiplet. When chiral and/or twisted chiral multiplets are required, the algorithm for determining the Kähler potential is slightly more involved but has been detailed in the literature [132, 133]. In essence, one simply repeats the above construction on each symplectic leaf; however, the resulting expressions are somewhat more complicated [132]. Here, however, we will content ourselves with considering explicitly the Kähler potential for the case of $\mathbb{CP}^1$ and $\mathbb{CP}^2$.

To find the (p, q) and (P, Q) coordinates explicitly we exploit the fact that there are n Killing vectors which leave σ^{-1} invariant (considering for simplicity here the case relevant to $\mathbb{CP}^{even}$ for which σ is invertible). The coordinates (x_m, ϕ_m) introduced in (5.3) are adapted to this such that the Killing vectors are simply given by the ∂_{ϕ_m}. We can select the holomorphic (with respect to $J_\pm$) part of σ^{-1} by acting with a projector

$$\sigma_\pm^{-1} = \frac{1}{2i}(i + J_\pm)\sigma^{-1} , \tag{5.28}$$

such that

$$\mathrm{d}q \wedge \mathrm{d}p + \mathrm{d}Q \wedge \mathrm{d}P = \sigma_+^{-1} + \sigma_-^{-1} . \tag{5.29}$$

Because both $\sigma_\pm^{-1}$ are invariant under the action of the Killing vectors ∂_{ϕ^m}

$$L_{\partial_{\phi^m}} \sigma_\pm^{-1} = 0 = \mathrm{d}(\iota_{\partial_{\phi^m}} \sigma_\pm^{-1}) , \tag{5.30}$$

we obtain the momentum maps

$$\mathrm{d}\mu_m^\pm = \iota_{\partial_{\phi^m}} \sigma_\pm^{-1} , \tag{5.31}$$

which, together with the one-forms $\mathrm{d}\phi^m$ dual to the isometries, form a basis of one-forms. A symplectic form σ^{-1} which satisfies (5.31) has to have the form

$$\sigma_\pm^{-1} = \frac{1}{2}(a + aba)_{mn} \mathrm{d}\phi^m \wedge \mathrm{d}\phi^n + (1 + ab)_m{}^n \mathrm{d}\phi^m \wedge \mathrm{d}\mu_n^\pm + \frac{1}{2} b^{mn} \mathrm{d}\mu_m^\pm \wedge \mathrm{d}\mu_n^\pm , \tag{5.32}$$

where

$$a_{mn} = \iota_{\partial_{\phi^m}} \mathrm{d}\mu_n \quad \text{with} \quad \mathrm{d}a_{mn} = 0\,. \tag{5.33}$$

Furthermore, σ^{-1} has to be closed. This implies that the only free parameter b^{mn} has to be constant like a_{mn}. To fix b^{mn}, we just have to match the left and right hand side. As result, we find that

$$\sigma_+^{-1} = \mathrm{d}q^m \wedge \mathrm{d}p_m \quad \text{and} \quad \sigma_-^{-1} = \mathrm{d}Q^m \wedge \mathrm{d}P_m\,, \tag{5.34}$$

where $\mathrm{d}q^m$ and $\mathrm{d}p_m$ are linear combinations (with constrained coefficients) of $\mathrm{d}\phi^m$ and $\mathrm{d}\mu_m^+$. The same holds for $\mathrm{d}Q^m$ and $\mathrm{d}P_m$ but with respect now to the linear combination built from $\mathrm{d}\phi^m$ and $\mathrm{d}\mu_m^-$. So the procedure is simple in principle: first integrate the moment map to find the μ_m and take appropriate linear combinations μ and ϕ to define the canonical coordinates. Then find the generating function $\mathcal{K}$ by integrating the canonical transformation of (5.27).

5.2.2.1 $\mathbb{CP}^1$

It is a well-known fact that every two-dimensional complex manifold is Kähler [134]; as such, the deformed $\mathbb{CP}^1$ geometry is completely determined by the standard (i.e. non generalised) Kähler potential. In fact, one can further notice that, given the dimensionality, the B-field is always pure gauge and thus negligible. As for the patch, we put ourselves in the largest Bruhat cell where the homogeneous coordinate $Z_0 \neq 0$ and introduce the holomorphic coordinate $z \equiv Z_1/Z_0$. The Kähler potential is[5]

$$K = -\frac{1}{2\eta} \mathrm{Im}\,\mathrm{Li}_2\left(\frac{\eta - i}{\eta + i}|z|^2\right)\,, \tag{5.35}$$

where we notice that the $\eta \to 0$ limit is non-singular and yields K_{FS}, i.e. the undeformed Fubini-Study Kähler potential for $\mathbb{CP}^1$, $K_{\mathrm{FS}} = \log(1+|z|^2)$. As $\mathbb{CP}^1$ is Kähler, $J_+ = J_-$, and π_- vanishes. In turn, there is a single set of complex coordinates diagonalising $J_\pm$ expressed by

$$q = -2\mu \log(z) = \mu\left(\log\left(\frac{1-x}{x}\right) - 2i\phi\right) = \mu\left(\log\left(\sin(\beta+\chi)\csc(\beta-\chi)\right) - 2i\phi\right) \tag{5.36}$$

and its conjugate, and[6]

$$x = \frac{1}{2\eta}(\eta - \tan\chi)\,, \qquad \mu = \frac{i-1}{8\sqrt{\eta\, t}}\,, \qquad \eta = \tan\beta\,. \tag{5.37}$$

[5] We use "Im" to indicate the imaginary part of the dilogarithm.

[6] Strictly speaking, for $\mathbb{CP}^1$ the precise form for μ is undetermined. We nevertheless choose it so as to match the higher dimensional cases, see next section.

5.2.2.2 $\mathbb{CP}^2$

$\mathbb{CP}^2$ is the first case where we can study a non-trivial generalised Kähler potential and give a rather nice explicit presentation thereof.

A first step in computing it is to find the holomorphic coordinates of $J_\pm$, that is, to identify $p(z,\bar{z}), q(z,\bar{z}), P(z,\bar{z}), Q(z,\bar{z})$, such that

$$\begin{aligned} J_+ &= i\mathrm{d}p \otimes \partial_p - i\mathrm{d}\bar{p} \otimes \partial_{\bar{p}} + i\mathrm{d}q \otimes \partial_q - i\mathrm{d}\bar{q} \otimes \partial_{\bar{q}}\,, \\ J_- &= i\mathrm{d}P \otimes \partial_P - i\mathrm{d}\bar{P} \otimes \partial_{\bar{P}} + i\mathrm{d}Q \otimes \partial_Q - i\mathrm{d}\bar{Q} \otimes \partial_{\bar{Q}}\,, \\ \sigma^{-1} &= \mathrm{d}p \wedge \mathrm{d}q + \mathrm{d}\bar{p} \wedge \mathrm{d}\bar{q} = \mathrm{d}P \wedge \mathrm{d}Q + \mathrm{d}\bar{P} \wedge \mathrm{d}\bar{Q}\,. \end{aligned} \tag{5.38}$$

Using the symplectic moment map associated to $U(1)$ actions as described previously one finds for p,q (with $\bar{p},\bar{q}$ given by standard complex conjugation)

$$\begin{aligned} q =& \mu\left(\log\left(-e^{-i\chi_2}\sin\left(\beta+\chi_1-\chi_2\right)\csc\left(\beta-\chi_1\right)\right) - 2i\phi_1\right)\,, \\ p =& \mu\left(\log\left(-ie^{-i\chi_1}\sec(\beta)\csc\left(\chi_2\right)\sin\left(\beta+\chi_1-\chi_2\right)\right) - 2i\phi_2\right)\,, \end{aligned} \tag{5.39}$$

where the angles $\chi_{1,2}$ are a generalisation of the one previously introduced

$$x_1 = \frac{1}{2} - \frac{1}{2\eta}\tan(\chi_1)\,, \quad x_2 = \frac{1}{2\eta}\sec(\chi_1)\sec(\chi_1-\chi_2)\sin(\chi_2)\,, \tag{5.40}$$

and μ and β follow the definition in (5.37). In particular, μ is a coefficient needed to ensure that σ^{-1} has the correct form (5.38). The relations (5.39) in Fubini-Study coordinates are

$$q = \mu\left[-2\log\left(\frac{\sqrt{z_1}}{\sqrt{\bar{z}_1}}\right) + \log\left(-\frac{\eta(1-|z_1|^2+|z_2|^2)+i(1+|z|^2)}{|z_1|^2(\eta(1-|z|^2)+i(1+|z|^2))}\right)\right]\,, \tag{5.41}$$

$$p = \mu\left[-2\log\left(\frac{\sqrt{z_2}}{\sqrt{\bar{z}_2}}\right) + \log\left(-\frac{\eta(1-|z_1|^2+|z_2|^2)+i(1+|z|^2)}{|z_2|^2(1+|z|^2)}\right)\right]\,, \tag{5.42}$$

where we recall $|z|^2 \equiv |z_1|^2+|z_2|^2$. Instead, one can use the angles $\chi_{1,2}$ to show that a simple relation between p,q and P,Q exists, namely

$$p + P = -2i\mu\chi_1\,, \qquad q + Q = -2i\mu\chi_2\,. \tag{5.43}$$

Letting the generating function

$$\mathcal{K}(P,\bar{P},q,\bar{q}) = -(Pq + \bar{P}\bar{q}) + \mathcal{K}_1(P,\bar{P},q,\bar{q})\,, \tag{5.44}$$

we require, in accordance with (5.27) and (5.43), that

$$\partial_q\mathcal{K}_1 = -2i\mu\chi_1\,, \quad \partial_P\mathcal{K}_1 = -2i\mu\chi_2\,. \tag{5.45}$$

A closed form for the potential in terms of the angles χ_i can be given in terms of the parametric integral

$$\mathcal{I}_\alpha(y) = \int y \cot\left(\frac{y+\alpha}{2}\right) \mathrm{d}y = 2\left(y \log\left(1 - e^{i(\alpha+y)}\right) - i\mathrm{Li}_2\left(e^{i(y+\alpha)}\right)\right) - \frac{iy^2}{2}\,, \quad (5.46)$$

such that

$$\mathcal{K}_1(\chi_1,\chi_2) = \frac{1}{32t\eta}\left(\mathcal{I}_{-2\beta}(2\chi_1) - \mathcal{I}_{2\beta}(2\chi_1 - 2\chi_2) - \mathcal{I}_0(2\chi_2)\right)\,. \quad (5.47)$$

To complete the specification of the potential one needs to express the χ_i in terms of $(P,\bar{P},q,\bar{q})$ which can be done implicitly via the relations

$$\begin{aligned} e^{|p|/|\mu|} &= \sec(\beta)\csc(\chi_2)\sin(\beta+\chi_1-\chi_2)\,,\\ e^{|q|/|\mu|} &= \csc(\beta-\chi_1)\sin(\beta+\chi_1-\chi_2)\,. \end{aligned} \quad (5.48)$$

5.2.3 T-Dual Geometry

Following the publication of [2], the authors of [135] have been able to prove that, after applying abelian T-duality on all n angular coordinates ϕ_i of $\mathbb{CP}^n_\eta$, the resulting geometry is enhanced to Kähler (in the strictly Riemannian sense). Whilst not an original work of ours, we nonetheless feel that this aspect ought to be reported so as to complement the presentation. With respect to [135], however, we shall add a brief discussion on how the dualisation is implemented on the generalised complex structures.

The η-deformed $\mathbb{CP}^n$ preserves a $U(1)^n$ toric action from the undeformed case. This is most easily seen in the (x,ϕ) coordinates, where the torus is identified with the angular coordinates ϕ_i: the n vectors ∂_{ϕ_i} are all Killing. This fact can be used to T-dualise the deformed projective space a number of times between 1 and n. For our purpose, we shall consider a duality along all angular coordinates.

Even though we could simply apply the Buscher rules, implementing T-duality in a generalised-geometric fashion best suits our description of $\mathbb{CP}^n_\eta$ as a Generalised Kähler geometry. Suppose T, with $T^2 = 1$ and $T^t = T$, is the $O(n,n)$ element implementing T-duality on the generalised metric $\mathcal{E} = \mathcal{H}\eta^{-1}$ as $\check{\mathcal{E}} = T\mathcal{E}T$[7]. Recalling the form of the metric in terms of generalised complex structures, it is immediate to see that in fact $\check{\mathcal{E}} = \check{\mathcal{J}}_1\check{\mathcal{J}}_2$, where $\check{\mathcal{J}}_1 = T\mathcal{J}_2T$ and $\check{\mathcal{J}}_2 = T\mathcal{J}_1T$. The reason for swapping the indices is that, as usual, T-duality interchanges "chiralities"[8]. It is straightforward to check that the dualised structures commute and square to minus one, making the T-dual of a

[7]For any object X we shall use $\check{X}$ to denote its T-dual.

[8]For instance, if in the $\eta \to 0$ limit $\mathcal{J}_1$ and $\mathcal{J}_2$ respectively comprised of J and ω only, the opposite is true after the transformation, hence the relabelling.

Generalised Kähler Geometry again Generalised Kähler. We now want to prove that, in our case, it is simply Kähler.

To this end, we adopt the approach reviewed in Section 3.2.3. The vector fields we consider are $v_i = \partial_{\phi_i}$. Being Killing, they obviously satisfy $L_{v_i} g = 0$ and $L_{v_i} H = 0$. An explicit computation shows that in fact also $L_{v_i} b = 0$ holds true so that the generalised vector can then be taken to be simply $V_i = \partial_{\phi_i} + \mathrm{d}\phi_i$. Finally, we compose the various dualities into a single operation $T = \prod_{i=1}^n T_i$. We can now apply T to either $\mathcal{E}$ or the generalised complex structures. In the former case, we find that $\check{b} = 0$, so that it is actually possible to have a legitimate Kähler manifold. On the complex structures, for instance, we have that

$$\check{\mathcal{J}}_1 := T\mathcal{J}_2 T = \begin{pmatrix} \check{J}^t & 0 \\ 0 & -\check{J} \end{pmatrix} \tag{5.49}$$

where we can check that in fact $\check{J}$ is a complex structure for the dual metric.

The T-dual geometry can be given explicit coordinate expression as follows [135]. We first introduce the $(n+1) \times (n+1)$ matrix $\mathcal{R}$ which is totally skew-symmetric and with upper triangular block made up of $+i$ only. We can employ $\mathcal{R}$ to construct two other quantities

$$\mathcal{N}_{jk} = i x_k \mathcal{R}_{jk}\,, \qquad \mathcal{M}_j = \sum_{k=1}^{n+1} (\mathcal{N}_{jk} \mathrm{d}\phi_k - \mathcal{N}_{kj} \mathrm{d}\phi_j)\,. \tag{5.50}$$

In turn, these do enter the definitions of the Riemannian metric and Kähler form as

$$\mathrm{d}s^2 = \sum_{j=1}^{n+1} \frac{\mathrm{d}\phi_j^2 + (\mathrm{d}x_j - \eta \mathcal{M}_j)^2}{2x_j}\,, \qquad \omega = \sum_{j=1}^{n+1} \frac{\mathrm{d}\phi_j \wedge \mathrm{d}x_j}{2x_j} - \sum_{j,k=1}^{n+1} \frac{i\eta}{2} \mathcal{R}_{jk} \mathrm{d}\phi_j \wedge \mathrm{d}\phi_k\,, \tag{5.51}$$

upon further imposing the constraints $x_{n+1} = 1 - \sum_i^n x_i$ and $\phi_{n+1} = -\sum_i^n \phi_i$ needed to remove the extra degrees of freedom we have introduced. Now, being the manifold Kähler we should be able to re-express the two quantities above in terms of a single Kähler potential. As shown in [135] this is achieved with the introduction of new coordinates (and their complex conjugates)

$$z_k = \frac{i}{2\eta} \log\left[1 + i\eta \left(1 - 2\sum_{j=1}^{k} x_j\right)\right] - i \sum_{j=1}^{k} \phi_j\,, \tag{5.52}$$

to be supplemented with the constraints

$$z_0 = i\frac{\log(1+i\eta)}{2\eta}\,, \qquad z_{n+1} = i\frac{\log(1-i\eta)}{2\eta}\,. \tag{5.53}$$

With this choice, the Kähler potential reads

$$K = i\eta \sum_{j=2}^{n} (z_j \bar{z}_{j-1} - \bar{z}_j z_{j-1}) + \frac{1}{2\eta} \sum_{j=1}^{n+1} \mathcal{P}(2\eta(z_j + \bar{z}_j - z_{j-1} - \bar{z}_{j-1}))\,, \tag{5.54}$$

where the function $\mathcal{P}$ is defined by

$$\mathcal{P}(t) = i\left(\mathrm{Li}_2(e^{it}) + \frac{t(2\pi - t)}{4}\right)\,. \tag{5.55}$$

5.3 The Double Deformation Puzzle

When discussing gauge-invariance for Poisson-Lie coset models, the constraint (5.8) was understood as a necessary ingredient for proving that H degrees of freedom can be consistently removed from the picture. As we noticed, in E_0^{-1ab} any linear combination of the $\mathfrak{m}$-restriction of the Killing form and the R-matrix, κ^{ab} and R^{ab}, would solve the equation, as $\tilde{f}^{ab}{}_{\gamma} = 0$ by coisotropy. Given this fact, one could try and mimic the way the bi-Yang-Baxter model is defined on group manifolds, namely with the introduction of a second deformation parameter ζ entering the action as

$$S = \frac{1}{\pi t} \int_{\Sigma} \mathrm{d}^2\sigma \langle \mathcal{P}_1 e_+, \frac{1}{1 - \eta R_m \mathcal{P}_1 - \zeta R \mathcal{P}_1} \mathcal{P}_1 e_- \rangle\,. \tag{5.56}$$

In [20] it was shown that, for $\mathbb{CP}^1 \cong S^2$, the ζ parameter could be reabsorbed away via a diffeomorphism, changing both the overall tension of the model and the η parameter. $\mathbb{CP}^1_\eta$, however, is a dramatically simplified setting when compared to deformed higher-dimensional projective spaces, where explicit expressions for the NS-NS fields immediately become intractable. One might then ask if the reabsorption of ζ is an accident $-$ most likely due to low dimensionality of $\mathbb{CP}^1$ $-$ or if, instead, it is a genuine feature of all these models.

Proving it in general is quite a cumbersome task, as it involves a number of non-trivial steps. First of all, it is not necessarily granted that the redefinitions $\tilde{t}, \tilde{\eta}$ of t, η induced by the diffeomorphism explicitly found for $\mathbb{CP}^1$ do generalise to higher dimensions. Ascertaining that this is case would require a dimension-independent approach: to this end we shall carry out a detailed analysis of the integrability of the system described by (5.56). Even so, albeit armed with a strong indication towards the triviality of the ζ-parameter, we would need an explicit form for the diffeomorphism. We shall provide a putative all-n expression for it, even though it was only possible to check its validity explicitly up to $n = 6$. Nevertheless, this limitation is sufficient to encompass holographic backgrounds relevant for either String or M-theory.

5.3.1 Integrability

The first objective of our analysis is to prove that the model described by (5.56) is classically integrable, both in the weak and strong sense. This has a two-fold purpose: even if the ζ parameter could not be reabsorbed, we would still put the doubly deformed model in the very special class of integrable models on coset spaces; if ζ could be reabsorbed, as we shall prove to be the case, a strong integrability analysis should suggest the new "effective" parameters $\tilde{t}$ and $\tilde{\eta}$ we mentioned earlier. In fact, they should be deducible from the comparison between the twist functions associated to this model and the one with a single deformation parameter.

5.3.1.1 Weak Integrability

Proving that a model is (classically) integrable in the weak sense amounts to find a Lax connection whose flatness implies the equations of motion and Bianchi identity. To this end we introduce the currents[9]

$$B_\pm = \frac{1}{1 \pm \eta R_m \mathcal{P}_1 \pm \zeta R \mathcal{P}_1} e_\pm(m)\,. \tag{5.57}$$

On a symmetric space, we can additionally define the projections of these currents onto $\mathfrak{g}^{(0)}$ or $\mathfrak{g}^{(1)}$ through the appropriate projectors, $B_\pm^{(i)} = \mathcal{P}_i B_\pm$, with $i = 0, 1$. After noticing the closed form for the operator appearing in (5.57)[10], we make use of the coisotropy condition to obtain the equations of motion in the form:

$$\left(\partial_- B_+^{(1)} + [B_-^{(0)}, B_+^{(1)}]\right) + \left(\partial_+ B_-^{(1)} + [B_+^{(0)}, B_-^{(1)}]\right) = 0\,. \tag{5.58}$$

The Bianchi identity for $e_\pm(m)$ coincides with the Maurer-Cartan equation $\partial_+ e_- - \partial_- e_+ - [e_-, e_+] = 0$. On symmetric spaces, however, we can employ the $\mathcal{P}_i$'s we have at hand to project the Bianchi identity down to either the $\mathfrak{g}^{(0)}$ or $\mathfrak{g}^{(1)}$ subspace. Along $\mathfrak{g}^{(0)}$ (or equivalently, the subalgebra of the group we are modding out) we find

$$\begin{aligned} \partial_+ B_-^{(0)} - \partial_- B_+^{(0)} + [B_+^{(0)}, B_-^{(0)}] + (1 + \eta^2 - \zeta^2)[B_+^{(1)}, B_-^{(1)}] \\ + \zeta[\mathcal{P}_1 R B_+^{(1)}, B_-^{(1)}] - \zeta[B_+^{(1)}, \mathcal{P}_1 R B_-^{(1)}] = 0\,. \end{aligned} \tag{5.59}$$

[9]To make contact with the literature we adopt the notation in [66].

[10]One can easily show that the inversion is given by $\frac{1}{1 \pm \eta R_m \mathcal{P}_1 \pm \zeta R \mathcal{P}_1} = \mathcal{P}_0 + (1 \mp \eta \mathcal{P}_0 R_m \mp \zeta \mathcal{P}_0 R) \frac{1}{1 \pm \eta \mathcal{P}_1 R_m \pm \zeta \mathcal{P}_1 R}$. This result is an extension of the one appearing in [93].

On $\mathfrak{g}^{(1)}$, the projection of the Bianchi identity can be mixed with the equation of motion (5.58) so as to obtain the simpler equations

$$\partial_\pm B_\mp^{(1)} + [B_\pm^{(0)}, B_\mp^{(1)}] = 0 \,. \tag{5.60}$$

It is remarkable that, in order to achieve these results, we have used multiple times the three defining hypothesis of the Koszul theorem for interpreting R as a complex structure, as this is the case for $\mathbb{CP}^n$. In particular, we have exploited $\mathcal{P}_0 R(\bullet) = 0$, where $\bullet$ denotes any operator R is acting upon. Introducing the currents

$$j_\pm = k B_\pm^{(1)} \pm \frac{\zeta}{k} \mathcal{P}_1 R B_\pm^{(1)} \,, \tag{5.61}$$

where k is the combination of parameters

$$k = \left(\frac{1 + \eta^2 - \zeta^2 + \sqrt{(1+\eta^2-\zeta^2)^2 + 4\zeta^2}}{2} \right)^{1/2} \,, \tag{5.62}$$

one can recast (5.59) and (5.60) in the canonical form

$$0 = \partial_\pm j_\mp + [B_\pm^{(0)}, j_\mp] \,, \tag{5.63}$$

$$0 = \partial_+ B_-^{(0)} - \partial_- B_+^{(0)} + [B_+^{(0)}, B_-^{(0)}] + [j_+, j_-] \,, \tag{5.64}$$

which is well known [136] to be equivalent to the existence of the Lax connection

$$\mathcal{L}_\pm = B_\pm^{(0)} + z^{\pm 1} j_\pm \,. \tag{5.65}$$

The doubly deformed model is thus weakly integrable. As a cross-check, we notice that upon taking the limit $\zeta \to 0$, k correctly reduces to $\sqrt{1+\eta^2}$ and the second term in the definition of $j_\pm$ vanishes, thus matching the result in [93].

5.3.1.2 Strong Integrability

Strong integrability requires us to pass to the Hamiltonian formalism of the system. It is convenient to introduce the Lie algebra valued quantities

$$X = e^{\mathbb{a}\mathbb{i}}(g) p_{\mathbb{i}} T_{\mathbb{a}} \,, \qquad Y = \frac{1}{2\pi t} e_\sigma^{\mathbb{a}}(g) T_{\mathbb{a}} - X^{\mathbb{b}} (\eta R_g + \zeta R)_{\mathbb{b}}{}^{\mathbb{a}} T_{\mathbb{a}} \,, \tag{5.66}$$

where we take $g = mh$ and p the canonical momentum. The advantage of these definitions is that the Lax matrix can be written as a function of X and Y,

$$L(z) = 2\pi Y^{(0)} + \left(z + \frac{1}{z}\right) \left(\pi k Y^{(1)} + \pi \frac{\zeta}{k} (RX)^{(1)}\right) + \left(z - \frac{1}{z}\right) \left(\pi k X^{(1)} + \pi \frac{\zeta}{k} (RY)^{(1)}\right)$$

$$+ f(z)X^{(0)} + g(z)(RX)^{(0)} . \tag{5.67}$$

On the second line, we have added two so far unspecified function of the spectral parameter, $f(z)$ and $g(z)$, that weigh the primary constraints $X^{(0)} \approx 0$ and $(RX)^{(0)} \approx 0$. As previously reviewed, these are needed to ensure the closure of the Maillet algebra, once a precise form for f, g is picked. X and Y are convenient as they can be embedded into a generalised vector via $\mathscr{X}_{\mathbb{A}} = 2\pi(tY^{\mathrm{a}}, X_{\mathrm{a}})$. The crucial observation is that the latter is related to the currents $\mathscr{J}_{\mathbb{A}}$ for the $\mathcal{E}$-model introduced in Section 3.2.6 by a constant $O(d,d)$ transformation β

$$\mathscr{X}_{\mathbb{A}} = \beta_{\mathbb{A}}{}^{\mathbb{B}} \mathscr{J}_{\mathbb{B}} \qquad \text{with} \qquad \beta = \begin{pmatrix} 1 & t(\eta+\zeta)\kappa^{-1}R \\ 0 & 1 \end{pmatrix} . \tag{5.68}$$

As a result, the Poisson brackets for $\mathscr{X}$, and thus the ones for $L(z)$, can be inferred from the ones for $\mathscr{J}$ already given in (3.76). Eventually, the Poisson brackets for the Lax matrix can be used to check if the Maillet algebra is satisfied: the function g turns out not to depend at all on the spectral parameter, $g = 2\pi\zeta$, whereas

$$f(z) = \pi \left(-2 + k^2(1+z^2) + \frac{\zeta^2(z^2-1)}{k^2} \right) . \tag{5.69}$$

Our model fits into the class of models with twist function, where we find

$$\varphi(z) = \frac{k^2}{t\pi(k^2+\zeta^2)} \frac{z}{(z^2-1)^2 + \frac{k^2(k^2-1)}{k^2+\zeta^2}(z^2+1)^2} . \tag{5.70}$$

Upon identifying new deformation parameter and tension according to

$$\tilde{\eta}^2 = \frac{k^2(k^2-1)}{k^2+\zeta^2} , \qquad \tilde{t} = t\frac{k^2+\zeta^2}{k^2} , \tag{5.71}$$

one can see that the twist function above coincides with the one for the single Yang-Baxter deformation already present in the literature [93]. It is tempting to infer, judging from this analysis, that the second parameter ζ is not affecting the model. Nevertheless, given the existence of transformations not affecting the twist function [137] and yet yielding to a non-trivially deformed target space, a more detailed geometric check is needed. We will explicitly construct the diffeomorphism removing ζ from the deformed metric in the next section.

5.3.1.3 Explicit Diffeomorphism

Before providing the explicit form for the diffeomorphism removing the deformation induced by ζ, let us briefly strengthen the indications of the previous section with a renormalisation group analysis. The one-loop RG flow for this model was first obtained in [20]; building on those results, we single out two RG-invariant combinations namely $t\eta$ and $\frac{1+\eta^2+\zeta^2}{\eta\zeta}$. The remaining non-trivial flow equation is

$$\frac{\mathrm{d}\eta}{\mathrm{d}\log\mu} \propto \eta t(1-\zeta^2+\eta^2)\,. \tag{5.72}$$

Now, given the effective parameters (5.71), we notice that algebraically $\eta t = \tilde{\eta}\tilde{t}$. Therefore, it holds true that $\tilde{\eta}\tilde{t}$ is RG-invariant, too. Even more generally, it is easy to prove that $\tilde{\eta}$ and $\tilde{t}$ obey the flow equation (5.72), upon setting $\zeta = 0$. Thus, it seems that our findings are consistent with one-loop quantum corrections.

To show the diffeomorphism, we put ourselves in the largest Bruhat cell and adopt (x_i, ϕ_i) coordinates; other than the parameters identified so far, it is particularly useful to introduce a new combination α defined by $\alpha^2 = k^2 - 1 - \tilde{\eta}^2$. In terms of the latter, while leaving the angles ϕ_i untouched, we redefine the x_i-coordinates through

$$\tilde{x}_i = \frac{(k^2-\alpha^2)x_i}{[k+\alpha(2\sum_{j<i}x_j - 1)][k+\alpha(2\sum_{j\leq i}x_j - 1)]}\,, \quad i = 1,\ldots,n\,. \tag{5.73}$$

This is our conjectured diffeomorphism. Indeed, although we do not dispose of a general proof, we have checked up to $n = 6$ that this removes the deformation induced on metric and 3-form $H = \mathrm{d}b$ by ζ. On dimensional grounds, this is enough for any application of deformed projective spaces to either String or M-theory. Moreover, given the extremely complex geometries this diffeomorphism has been successfully applied to, we believe we can safely conjecture its correctness for all n.

5.4 Variation on a Theme: Spheres and AdS Spaces

Yang-Baxter deformations of spheres and de Sitter spaces have been extensively considered in the literature, mainly as part of of backgrounds relevant for extensions of AdS/CFT. Oddly enough, they have only been studied on a case-by-case basis and a more general treatment is currently missing, at least to the best of our knowledge. Aim of this section is to address this point, building on some of the mathematical and physical technology developed in this chapter. As the geometry of anti-de Sitter spaces can be recovered via an appropriate analytic continuation of that of spheres, we shall mostly concentrate on the latter. We will comment on the precise implementation of the transformation in a later section. The results that follow arise from a set of unpublished notes.

5.4.1 Spheres

In analysing Yang-Baxter deformations of $\mathbb{CP}^n$, we purposefully avoided detailing the explicit Lie algebra implementation of the deformation: for instance, we refrained from picking an explicit basis for $\mathfrak{su}(n+1)$ or a coset representative. Rather, our findings were based upon a number of well-known results for the geometry of complex projective spaces, enabling us to overstep the algebraic viewpoint in favour of the more elegant geometric approach we have pursued. Reversing the logic, we shall now take the Lie algebra implementation as our starting point and construct the deformations of spheres from scratches.

The n-dimensional sphere S^n is diffeomorphic to the homogenous space $SO(n{+}1)/SO(n)$. A convenient choice for the anti-Hermitian $\frac{1}{2}n(n+1)$ generators of $\mathfrak{so}(n+1)$ is

$$(T_{a,b})_i{}^j = \delta_{ai}\delta_b{}^j - \delta_a{}^j\delta_{bi}\,, \tag{5.74}$$

where $a, b = 1, \dots, n+1$ label the matrix and $i, j = 1, \dots, n+1$ indicate its components. We choose as Killing form $\kappa(x, y) = \frac{1}{2}\,\mathrm{Tr}(xy)$, for two $x, y \in \mathfrak{g}$, which is of course non-degenerate. Following [138], we define the projector $\mathcal{P}$ onto the coset directions acting as

$$\mathcal{P}(x) = -\sum_{i=2}^{n+1} \kappa(x, T_{1,i})T_{1,i}\,, \tag{5.75}$$

for any $x \in \mathfrak{so}(n+1)$. Accordingly, we identify the n matrices $T_{1,2}, \dots, T_{1,n+1}$ as the coset generators, and the remainder as the $\mathfrak{so}(n)$ algebra we will mod out.

As a warm-up, and to set some notation, let us compute the metric for the undeformed S^n by algebraic means. To this end, let us pick a coset representative m which in spherical coordinates ϕ_i reads

$$m = \prod_{a=1}^{n} e^{(\phi_{n+1-a}-\frac{\pi}{2})T_{1,a+1}}\,. \tag{5.76}$$

Upon dressing the matrix form of κ with the vielbein for S^n constructed through the projection $\mathcal{P}(m^{-1}\mathrm{d}m)$, one obtains the well-known metric for the round n-sphere. Notation-wise, let us denote it with $\mathrm{d}s^2 = \mathrm{d}\Omega_n(\phi)$ and, more generally, indicate with $\mathrm{d}\Omega_m(\phi)$ the volume element for the m-dimensional sphere in spherical coordinates. The downside of spherical coordinates is that they make both the introduction of a deformation and the analytic continuation to AdS cumbersome, due to the presence of trigonometric functions in the metric. To circumvent this issue, we shall opt for a new set of coordinates.

To get a grasp on the simplification they entail, consider the versors $\hat{x}$ for an n-dimensional sphere defined by

$$\begin{gathered}\hat{x}_{1,n} = \cos\varphi_1 \,, \quad \hat{x}_{2,n} = \sin\varphi_1\cos\varphi_2 \ \ \ldots \ \ \hat{x}_{n-1,n} = \sin\varphi_1 \ldots \cos\varphi_{n-1} \,, \\ \hat{x}_{n,n} = \sin\varphi_1 \ldots \sin\varphi_{n-1} \,, \end{gathered} \tag{5.77}$$

and use them to construct new coordinates $(r, \varphi, \varphi_1, \ldots, \varphi_{n-1})$[11] according to

$$\begin{cases} \phi_i = \arccos\left[r \frac{\hat{x}_{i,n-1}}{(1-r^2\sum_{k=1}^{i-1}\hat{x}^2_{k,n-1})^{1/2}}\right] & i = 1, \ldots, n-1 \\ \phi_n = \varphi \end{cases} . \tag{5.78}$$

With this choice, the metric for the n-sphere becomes

$$\mathrm{d}s^2 = (1-r^2)\mathrm{d}\varphi^2 + \frac{\mathrm{d}r^2}{1-r^2} + r^2\mathrm{d}\Omega_{n-2}(\varphi) \,. \tag{5.79}$$

Notice how here the new φ's behave as spherical coordinates for a S^{n-2}. It should then be clear that in performing this change we have not really avoided the issue of having trigonometric functions, as they have been simply hidden in $\mathrm{d}\Omega_{n-2}(\varphi)$. Nevertheless, the same observation precisely suggests a way forward: define a transformation $\mathscr{G}_m$ of order $m \in \mathbb{N}$ as the set of changes

$$\mathscr{G}_m : \begin{cases} \phi_i = \arccos\left[r_{(m)} \frac{\hat{x}_{i,m-1}}{(1-r_{(m)}{}^2\sum_{k=1}^{i-1}\hat{x}^2_{k,m-1})^{1/2}}\right] \,, & i = 1, \ldots, m-1 \\ \phi_m = \varphi_{(m)} \end{cases} . \tag{5.80}$$

Suppose we start with the round metric on a n-dimensional sphere $\mathrm{d}\Omega_n(\phi)$. A single transformation $\mathscr{G}_n$ (that would correspond to (5.78)) is not sufficient for completely removing the dependence on trigonometric functions, as we have seen. However, we could apply $\mathscr{G}_{n-2}$ on the coordinates appearing in $\mathrm{d}\Omega_{n-2}(\varphi)$, thereby obtaining something analogous to (5.79). Iterating this line of thought, the composition $\mathscr{G}_{2,3} \circ \ldots \circ \mathscr{G}_{n-4} \circ \mathscr{G}_{n-2} \circ \mathscr{G}_n$ should exactly fulfil our needs. Observe how in fact the last transformation on the left is "ambiguous" as it depends on having an odd or even dimensional sphere in the first place[12]. Following this procedure we find a simple form for the S^n metric in (r_i, φ_i) coordinates (and identifying $r_0 = 1$)

$$\mathrm{d}s^2 = \sum_{i=1}^{\left[\frac{n}{2}\right]} \left(\prod_{j=1}^{i} r^2_{j-1}\right)\left((1-r_i^2)\mathrm{d}\varphi_i^2 + \frac{\mathrm{d}r_i^2}{1-r_i^2}\right) + \delta_{n,\mathrm{odd}}\left(\prod_{j=1}^{\left[\frac{n}{2}\right]} r_j^2\right)\mathrm{d}\varphi^2_{\frac{n+1}{2}} \,. \tag{5.81}$$

[11] Albeit somewhat confusing, in this standard notation r indicates an angle, and not the radius of the sphere.

[12] In either case, $\mathscr{G}_{0,1}$ are just the identity and we neglect them.

Now, suppose we want to η-deform this space. As a first step, it is necessary to construct an R-matrix, and once again we resort to the Drinfel'd-Jimbo prescription for that. Recall that this procedure requires us to single out the Cartan's subalgebra as well as the positive and negative roots of $\mathfrak{so}(n+1)$. To this end, one needs to distinguish between the even and odd case, i.e. $\mathfrak{so}(2q)$ and $\mathfrak{so}(2q+1)$ for integer q. In either case, the Cartan subalgebra $\mathfrak{c}$ has dimension q and is spanned by $H_j = iT_{2j-1,2j}$. In the even case, $\mathfrak{so}(2q)$, we pick positive and negative roots (denoted with $+$ and $-$, respectively)

$$E^+_{i,j} = \frac{1}{2\sqrt{2}}(T_{2i-1,2j+1} + iT_{2i,2j+1} + i(T_{2i-1,2j+2} + iT_{2i,2j+2}))\,, \tag{5.82}$$

$$\widetilde{E}^+_{i,j} = \frac{1}{2\sqrt{2}}(T_{2i-1,2j+1} + iT_{2i,2j+1} - i(T_{2i-1,2j+2} + iT_{2i,2j+2}))\,, \tag{5.83}$$

$$E^-_{i,j} = \frac{i}{2\sqrt{2}}(T_{2i-1,2j+1} - iT_{2i,2j+1} - i(T_{2i-1,2j+2} - iT_{2i,2j+2}))\,, \tag{5.84}$$

$$\widetilde{E}^-_{i,j} = \frac{i}{2\sqrt{2}}(T_{2i-1,2j+1} - iT_{2i,2j+1} + i(T_{2i-1,2j+2} - iT_{2i,2j+2}))\,, \tag{5.85}$$

with $i = 1, \ldots, q-1$ and $j = i, \ldots, q-1$. For $\mathfrak{so}(2q+1)$, we supplement this basis with additional $2q$ generators

$$\widehat{E}^+_j = \frac{1}{2}(T_{2j-1,2q+1} + iT_{2j,2q+1})\,, \tag{5.86}$$

$$\widehat{E}^-_j = \frac{i}{2}(T_{2j-1,2q+1} - iT_{2j,2q+1})\,, \tag{5.87}$$

with $j = 1, \ldots, q$. Gathering together the positive and negative roots as E_λ and $E_{-\lambda}$ for $\lambda \in \Delta^+$, we employ the usual formula $R = \frac{1}{2}\sum_{\lambda\in\Delta^+} E_\lambda \wedge E_{-\lambda}$ for constructing the R-matrix.

Now that we have at our disposal the Yang-Baxter matrix we are able to draw various conclusions. First and foremost, we can explicitly check that the restriction of R to coset directions, i.e. ${R_a}^b$, *vanishes*. Nonetheless, this does *not* imply that the η-deformation of S^n is trivial, as the restriction of the dressed Yang-Baxter matrix, ${(R_m)_a}^b$, is not zero[13]. Also, one can check that the coisotropy condition (2.9) is met with our embedding of $\mathfrak{so}(n)$ in $\mathfrak{so}(n+1)$: in turn, this implies that the Poisson bracket π of $SO(n+1)$ will descend to a Poisson-Bruhat bracket π_{B} for S^n. Moreover, as ${R_a}^b = 0$, its expression is simplified to $\pi^{ab}_{\text{B}} = R^{ab}_m$. Upon dressing the latter with frame fields, we arrive at a very neat expression for the Poisson-Bruhat structure for S^n in (φ, r) coordinates, namely

$$\pi_{\text{B}} = \frac{1}{2}\sum_{i=1}^{[\frac{n}{2}]} r_i \text{d}\varphi_i \wedge \text{d}r_i\,. \tag{5.88}$$

[13]That is, we first dress R with adjoint actions and *then* project it.

Comparing the action for the Yang-Baxter model for a sphere (which formally coincides with (5.5), being S^n a symmetric space too) with the generic Poisson-Lie model (5.7), we recognise immediately $E_0 = \kappa$. Replicating the steps carried out in the case of $\mathbb{CP}^n$, it is easy to see that the quantities $Q_\pm = 1 \mp \eta G \pi_{\mathrm{B}}$, where G is the undeformed metric for S^n, do in fact implement the deformed metric g as $g = Q_\pm^{-1} G Q_\pm^{-t}$. As in the case of projective spaces, this is a general result that does not rely on the chart of our choice. However, owing to the simplicity of the objects appearing in this formula when in (φ, r) coordinates, it is possible to write down the line element for S^n_η:

$$\mathrm{d}s^2 = \sum_{i=1}^{[\frac{n}{2}]} \left(\prod_{j=1}^{i} r_{j-1}^2 \right) \frac{\left((1 - r_i^2)\mathrm{d}\varphi_i^2 + \frac{\mathrm{d}r_i^2}{1-r_i^2} \right)}{1 + \eta^2 r_i^2 \prod_{j=1}^{i} r_{j-1}^4} + \delta_{n,\mathrm{odd}} \left(\prod_{j=1}^{[\frac{n}{2}]} r_j^2 \right) \mathrm{d}\varphi_{\frac{n+1}{2}}^2 \, . \tag{5.89}$$

Similarly, the explicit expression for the B-field is

$$b = \eta \sum_{i=1}^{[\frac{n}{2}]} \frac{\left(\prod_{j=1}^{i} r_{j-1}^4 \right)}{1 + \eta^2 r_i^2 \prod_{j=1}^{i} r_{j-1}^4} (r_i \mathrm{d}r_i \wedge \mathrm{d}\varphi_i) \, . \tag{5.90}$$

Despite the many elements of similarity in construction with deformed projective spaces, the spheres S^n_η do not seem to fit generically into any Generalised Complex, let alone Generalised Kähler, geometry. Of course, in principle there could be specific values of n for which they do; for instance, as $n = 2$ we have the famous $S^2 \cong \mathbb{CP}^1$.

5.4.2 Anti-de Sitter

Having detailed the construction for spheres, moving to AdS spaces is just a matter of analytically continuing the previous results to Minkowskian signature. Time t is obtained through the replacement $\varphi_1 \to -it/\alpha$, where α is the AdS/S radius, and similarly we rescale $r_1 \to r_1/\alpha$. We then multiply both the metric and B-field by a factor of α^2 (neglected so far in the discussion, as we implicitly assumed the sphere to have unit radius) and finally continue the radius, $\alpha \to i\alpha$. Having done so, we can decide to switch back to the unitary AdS setting $\alpha = 1$.

5.5 Conclusions

Using the tools of Poisson-Lie non-linear σ-models on generalised coset spaces, we have described a specific but particularly striking class of examples in which G/H were Poisson Hermitian spaces. Upon constructing an integrable Yang-Baxter deformation of these, we showed that their target space is described by generalised Kähler geometry. We discussed in detail $\mathbb{CP}^n$ as a prototypical example and, for the case of $\mathbb{CP}^2$, gave an explicit formulation of the corresponding generalised Kähler potential. We filled a gap in the literature by showing that a previously conjectured two-parameter deformation for $\mathbb{CP}^n$ is indeed integrable but coinciding with the already well-known Yang-Baxter deformation of coset spaces.

A background motivation for this work was to investigate integrable deformations of $AdS_4 \times \mathbb{CP}^3$ with the aspiration of identifying quantum group deformations in the ABJM model. At first sight the corresponding geometry is rather unattractive but in this work we have elucidated many of the key features. A complete analysis would of course require furnishing the geometry with appropriate RR fields and investigating the fermionic sector. Whilst one might "boot-strap" an RR sector, an approach done for the case of Poisson-Lie models on groups in [121], ultimately it would be desirable to extend the considerations to super-cosets [97, 125, 139–141].

We only considered coset spaces for which the gauge group is coisotropic, as these naturally solve the invariance constraint, leaving the construction of other holographically relevant coset spaces open. Moreover, the explicit examples taken into account here were based on quotients in which the Drinfel'd double was $\mathfrak{d} = \mathfrak{g}_{\mathbb{C}}$. The incorporation of λ-models requires the more general case of $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}$; the general tool kit of Chapter 4 accommodates this scenario and so it would be interesting to explore if there can be some underlying generalised Kähler structures in the λ-deformations of G/H-WZW models.

Finally, we showed how it is possible, using a similar technology, to construct the Yang-Baxter deformation for spheres and anti-de Sitter spaces in every dimension. We furnished explicit expressions for the metric and B-field that can be readily used for any application involving these deformed spacetimes.

Chapter 6

Intermezzo

The discussion in Chapters 4 and 5 has mostly remained classical. The inclusion of quantum corrections, however, is essential and must certainly be addressed in String Theory, the leading candidate for a quantum description of gravity. This is in fact an old topic that dates back to the seminal work of Friedan [142], where the renormalisation of a two-dimensional non-linear σ-model was first considered. When a Riemannian metric g completely specifies the model, the famous one-loop renormalisation group equation

$$\frac{\partial g_{\mu\nu}}{\partial \log \mu} = \alpha' R_{\mu\nu} \tag{6.1}$$

was found: at first order, g flows according to its Ricci tensor. Since then, this result has been extended in all sorts of directions, adding for example B-field and dilaton contributions, as well as higher order corrections and supersymmetry [138, 143–146].

In this regard there are, at the very least, two important questions worth of investigation. The first is rather practical and is most easily formulated with an example. In Chapter 5 we have mentioned in passing that the main tensors describing the deformed geometry of $\mathbb{CP}^n$ do have extremely involved coordinates expressions. As usual, a clever diffeomorphism alleviates this issue, but even with the best coordinates we could find (the (x, ϕ) patch we discussed at length) the metric for $\mathbb{CP}^2_\eta$ could barely fit on a page. Hence, it should really be no surprise that computing the RG flow for the deformation parameter η using (6.1) and its B-field counterpart is quite a formidable task even with the help of a computer. And this is just for the first perturbative order! On the other hand, we know that a Yang-Baxter model has a quite simple algebraic description in terms of a constant Yang-Baxter matrix. This observation suggests that the brute force approach of (6.1) is just not computationally convenient at all for sufficiently complicated geometries. We then ask:

Is it possible to characterise the renormalisation group flow algebraically rather than geometrically, at least for a restricted class of models?

The second question is more conceptual but is somehow related to the technical challenge we just discussed. T-duality is usually made explicit through a repackaging of metric and B-field in a single unified object, the generalised metric $\mathcal{H}$. However, (6.1) and its Kalb-Ramond partner by construction do not "see" any $\mathcal{H}$. Hence, we could worry about the survival of classical dualities at the quantum level. In general, and in particular for more exotic types of T-duality such as non-Abelian or Poisson-Lie, this is far from obvious and needs an explicit check. In this regard, it is possible to proceed in one of two ways: we either compute the renormalisation group flow in terms of metric and B-field and then ascertain that T-duality is left intact; or we use a doubled formalism that makes T-duality covariance explicit from the outset.

At a first glance the second path seems preferable, as one is apparently dispensed from the additional (and quite cumbersome) task of "rediscovering" duality that usually requires quite some work, especially for higher loops calculation. Moreover, it does not seem conceptually satisfactory to lose sight of T-duality, only to eventually recover it. But in fact, there are at least two serious drawbacks to this approach. In terms of convenience, the first path does *not* involve performing a new full-fledged quantum calculation from scratches: even obtaining the two-loop β-functions for the NS-NS sector, as e.g. in [145], required a remarkable effort, even leading to some initial disagreement on the correct result. Avoiding this technical challenge is extremely helpful. Besides, a quantum calculation in some doubled formalism usually implies additional intricacies: in essence, adopting e.g. the Tseytlin doubled string (3.54), the price to pay for having T-duality explicit is the breaking of Lorentz invariance: this makes many techniques canonically employed in QFT ambiguous, if not inapplicable. Nonetheless, we ask:

Is it possible to perform a two-loop calculation in a doubled approach overcoming all the technical intricacies?

Now, the two paths we have outlined should eventually converge on a final answer. Or, at least, this is what we expect. It is not necessarily granted and (higher)-loop calculations need to be performed in order to assess the validity of doubled formulations beyond the leading (classical) order. Actually, one-loop calculations for the Tseytlin string/$\mathcal{E}$-model (for Abelian/Poisson-Lie T-duality, respectively) have already been carried out in [147, 148]. Still, one-loop results are not such strong indications, and possible issues at the quantum level are better probed in a two-loop approximation (at least). Going against our own words of caution, we will devote the last two chapters to try and address these points. We anticipate that the outcome of this investigation will not be completely satisfactory: for the Tseytlin string we are able to finalise the calculation in all of its technical aspects other than possible field/coupling redefinitions that should ideally reconcile it with the expected T-duality invariant answer. For the $\mathcal{E}$-model, whilst field redefinitions should play a minor role, there are further complications that arise and we are still investigating in this direction at the time of writing.

As we will discuss quantum computations at great length in the remainder, let us briefly summarise the state of the art for the first path. In the case of Abelian T-duality, it all started with the work of Meissner. In the seminal work [149], building on the result of [145], he was able to prove that the two-loop (or fourth-order in derivatives) results *does respect* the $O(n, n)$ symmetry. This was in fact proven in a highly simplified scenario − called cosmological spacetime − where objects are only allowed to depend on a single coordinate. Taking it from here, there have been significant efforts [150–155] in developing the theory of higher derivative corrections in a T-duality covariant fashion (see [156] for recent lecture notes on this topic). Much more recently, the case of Poisson-Lie T-duality on group manifolds at two-loop was remarkably addressed in [157]. Together with the author of this paper, we were trying to explore the same setting on generalised coset spaces, exploiting the technology of Chapter 4 but the results are partial and will not be discussed here.

Without further ado, let us move on to the actual computation.

Chapter 7

The Duality-Symmetric String at Two-Loop

Abstract

The Tseytlin duality symmetric string makes manifest the $O(n,n)$ T-duality symmetry on the worldsheet at the expense of manifest Lorentz invariance. Here we consider the two-loop renormalisation of this model in the context of "cosmological" spacetimes consisting of an internal n-dimensional torus fibred over a one-dimensional base manifold. The lack of manifest Lorentz symmetry introduces a range of complexities in momenta loop integrals which we approach using different methods. Whilst the results do satisfy a number of key consistency criteria, we find however that the two-loop counter-terms are incompatible with $O(n,n)$ symmetry and obstruct the renormalisability of the duality symmetric string.

7.1 Introduction

We shall now tackle the problem of including (higher-loop) quantum corrections in a T-duality covariant formalism from the worldsheet perspective. This task will be articulated in two different chapters, the present dealing with Abelian T-duality and the following with Poisson-Lie T-duality.

The study of the renormalisation group flow for two-dimensional non-linear σ-models dates back to the original work of [142], and provides a linkage between the worldsheet and spacetime points of view of String Theory. Demanding the vanishing of the Weyl anomaly associated to a closed string propagating in a spacetime geometry requires that the target space background data (i.e. metric and NS two-form in the bosonic sector) obey a set of equations [142, 158, 159]. These equations can be interpreted as field equations of an appropriate target space gravitational theory[1]. At one-loop on the worldsheet, this target space theory has a common bosonic sector shared with the type II Supergravity theories. Higher loop quantum calculations [145, 161, 162] lead in turn to modifications to the target space effective theory, organised in an expansion in derivatives[2].

This picture largely retains the conventional geometric notions associated to point particles and does not capture all the features we have come to associate with String Theory. In particular, it is now well understood that there is a rich duality symmetry structure that heavily constrains the form of the target space effective description. The $O(n,n)$ T-duality symmetry of strings on a n-dimensional torus [9, 10] indicates that fields should be repackaged into appropriate representations of this symmetry group. The target metric and two-form become unified into a combined object, the generalised metric $\mathcal{H}$ – a representative of the coset space $O(n,n)/(O(n)\times O(n))$. The target space formulation of the dynamics of this generalised metric has been well expounded [46, 85, 111, 112, 163, 164] and has become known as Double Field Theory. Starting with the seminal work of Meissner [149], there have been significant efforts [150–155] in developing the theory of higher derivative corrections in a T-duality covariant fashion (see [156] for recent lecture notes on this topic).

This progress poses a sharp question: *can one exploit worldsheet renormalisation to obtain higher derivative corrections of the target space theory in a way that maintains T-duality covariance throughout?*

7.1.1 The Doubled String

Key to answering this question is to adopt a reformulation of the string worldsheet theory in which T-duality is promoted to a manifest symmetry [11, 83, 85, 163, 165–168]. In Chapter 3 we have already introduced the action (3.53) that indeed meets our requirement of explicit T-duality covariance. Recall that, for the target space to exhibit $O(n,n)$ T-duality, we impose a $U(1)^n$ isometry group, such that the target is a torus fibration $T^n \hookrightarrow M \to B$ over a base B. On a patch, we let x^i, $i = 1 \ldots n$, be coordinates adapted to the isometry that parametrise the fibre, and y that on the one-dimensional base. To allow T-duality to be exhibited manifestly we consider a larger *doubled* space

[1]Though somewhat tricky to source, we found the contemporary lectures notes [158, 160] to be a particularly useful resource.

[2]The full description of String Theory is richer still, since this two-dimensional quantum expansion needs to be supplemented with a g_s expansion in the worldsheet genus.

$\mathbb{M}$ with fibration $T^{2n} \hookrightarrow \mathbb{M} \to B$ in which the original coordinates x^i are supplemented with additional $\tilde{x}_i$, $i = 1, \ldots, n$, that can be thought of as describing the T-dual space, $\widetilde{T}^n$, to the torus such that locally $T^{2n} = T^n \times \widetilde{T}^n$. In fact, given the difficulties entailed in a two-loop calculation without explicit Lorentz covariance, we shall restrict ourselves to an even simpler version of (3.54),

$$S = \int \mathrm{d}^2\sigma \left(-\frac{1}{2}\mathcal{H}_{IJ}\partial_1\mathbb{X}^I\partial_1\mathbb{X}^J + \frac{1}{2}\eta_{IJ}\partial_0\mathbb{X}^I\partial_1\mathbb{X}^J + \frac{1}{2}\lambda\,\partial_\mu y\partial^\mu y \right) . \tag{7.1}$$

On the base manifold, where Lorentz invariance is unbroken, we have accordingly retained the use of a vector $\sigma^\mu = (\sigma^0, \sigma^1)$. A number of simplifications have been assumed: only a single coordinate y on B is considered, such that the base geometric data is just encoded in the coupling λ which can be thought of as a partial fixing of the target space lapse variable. Furthermore, we have set to zero off-diagonal components of the metric and two-form which together provide a connection in the doubled bundle. On a general worldsheet, this action should be supplemented with a Fradkin-Tseytlin term that induces a coupling to the duality-invariant dilaton $d = \phi - \frac{1}{4}\ln\det g$ in which ϕ is the conventional dilaton field. In the present context, we will consider the theory on two-dimensional Minkowski space such that this dilaton coupling is absent[3]. Though this is a heavily simplified set-up, even here we will find sufficient complexities.

The one-loop quantum effective action resulting from the Lagrangian in (7.1) was first considered in [147, 169]. This was later expanded to a more democratic approach in which both the fibre *and* base space are doubled allowing direct contact to be made with Double Field Theory [170, 171]. The doubled action of (7.1) can be generalised to accommodate non-Abelian and Poisson-Lie generalisations of T-duality [77], and its renormalisation was considered in [148, 172]. More recently the one-loop calculation of [147, 169] has been revisited and refined in [173] with the inclusion of a connection in the doubled bundle $\mathbb{M}$.

Here we will push the techniques of [147, 169] further and apply them at two-loop order. There are several critical aspects that make this calculation extremely challenging:

- The lack of manifest worldsheet Lorentz invariance. This has two consequences, the first is that it requires treating non-covariant loop momenta integrals and the second is that we can not disregard contributions arising from the connection $\Omega = \mathcal{V}^{-1}\mathrm{d}\mathcal{V}$ associated to the (generalised) vielbein $\mathcal{V}$ that diagonalises $\mathcal{H}$.

[3] In general, the dilaton does make a contribution to the Weyl anomaly which can be established by calculating on a curved worldsheet. However a pragmatic approach is to determine the dilaton contributions through consistency requirements as was done for the Tseytlin string at one-loop in [169].

- To treat the loop integrals we consider two different methods. In Method 1 we employ dimensional regularisation and move immediately to $d = 2+\epsilon$ and evaluate the wide variety of tensor integrals encountered by reducing them in a basis of standard scalar integrals. In Method 2 we work in $d = 2$ to perform the maximal number of simplifications of loop momenta (by e.g. replacing $p_0^2 - p_1^2 = p^2$) expressing the result implicitly in terms of a set of only five independent integrals whose regularisation and evaluation is left initially implicit. We are then able to extract conclusions that are largely independent of the choice of regularisation method[4]. For concreteness, we then complete Method 2 by employing the dimensional regularisation approach of Method 1 to the few remaining loop integrals.

- The chirality nature of the theory which obstructs a straightforward regularisation of IR divergences. Whilst this may prove challenging in general, we are able to address this potential troublesome issue in a rather naive fashion which we find to be satisfactory. By shifting momenta $p^2 \to p^2 - m^2$ in the denominator of loop integrals one regulates the divergence, and all features (such as mixed IR and UV divergences at two-loops) associated to the introduction of m^2 are removed once we include a "mass" term for the background fields.

- The departure from conventional Riemannian geometry invoked by the generalised metric $\mathcal{H}$ prevents the use of known (target space) covariant background field methods [174, 175] to simplify the calculation. Instead, we resort to a non-covariant background field expansion with a linear split between the classical background and quantum fluctuation. To tackle the abundance of diagrams produced in this expansion, we complement a pen-and-paper calculation of the counter-terms for the generalised metric $\mathcal{H}$ with a `Mathematica` implementation to determine the renormalisation of the coupling λ.

7.1.2 A Few Details on Renormalisation

If the renormalisation of the action (7.1) could be successfully performed, regardless of the detailed choice of method for evaluating the diagrams, within dimensional regularisation the bare and renormalised generalised metric and base coupling are to be related by

$$\mathcal{H}_B = \mu^\epsilon \left(\mathcal{H} + \sum_{n=1}^{\infty} \epsilon^{-n} T_n(\mathcal{H}, \lambda) \right) , \quad \lambda_B = \mu^\epsilon \left(\lambda + \sum_{n=1}^{\infty} \epsilon^{-n} \widetilde{T}_n(\mathcal{H}, \lambda) \right) , \tag{7.2}$$

[4]For instance one could invoke prescriptions that do *not* require analytic continuation of the worldsheet dimensions (Pauli-Villars and cut-off regularisations among the others) which are potentially more compatible with the chiral nature of the duality-symmetric string.

with μ some mass scale and $T_n(\mathcal{H},\lambda), \widetilde{T}_n(\mathcal{H},\lambda)$ the ϵ-independent counter-terms that render the effective action finite when phrased in terms of $\mathcal{H},\lambda$. The coupling λ also provides a loop counting parameter such that we can further express $T_n(\mathcal{H},\lambda) = \sum_L T_n^{(L)}(\mathcal{H})\lambda^L$ and $\widetilde{T}_n(\mathcal{H},\lambda) = \sum_L \widetilde{T}_n^{(L)}(\mathcal{H})\lambda^{L-1}$. The $O(n,n)$ invariant metric on the other hand will not be renormalised $\eta_B = \mu^\epsilon \eta$[5].

Because at the classical level the lapse variable can be fixed to $\lambda = 1$ by rescaling y, one anticipates it should be irrelevant to the quantum theory. However, some care is required since the one-loop result [147] indicates that a non-constant divergent counter-term proportional to $\partial_\mu y \partial^\mu y$ is produced even when $\lambda = 1$. Such divergences at one-loop can be removed by the addition of a counter-term proportional to the equation of motion for the background fields (equivalently via a field re-definition of y). The required term depends rather implicitly on $\widetilde{T}$, and can be established exactly for particular models as in [176]. Doing so will induce a modification of the fibre counter-term. We prefer to keep λ explicit as it serves as a loop counting parameter and mirrors the way the lapse function enters in the target space description [149][6].

As the bare couplings are independent of the scale we can take the derivative of $\mathcal{H}_B, \lambda_B$ to determine the β-functions in terms of the ϵ^{-1} poles[7]:

$$\beta^{\mathcal{H}} \equiv \mu\frac{\partial\mathcal{H}}{\partial\mu}+\epsilon\mathcal{H} = -\sum_L LT_1^{(L)}(\mathcal{H},\eta)\lambda^{-L}\,, \quad \beta^{\lambda} \equiv \mu\frac{\partial\lambda}{\partial\mu}+\epsilon\lambda = -\sum_L L\widetilde{T}_1^{(L)}(\mathcal{H},\eta)\lambda^{-L+1}\,. \tag{7.3}$$

The higher order poles are not independent and instead provide a set of consistency relations known as the pole equations[8]. Specialising to the two-loop case we have in particular that on the fibre[9]

$$0 = 2T_2^{(2)} - T_1^{(1)} \circ \frac{\delta}{\delta\mathcal{H}} T_1^{(1)} + \widetilde{T}_1^{(1)} T_1^{(1)}\,. \tag{7.4}$$

[5]We will prove this statement explicitly soon. For the time being, let us just assume it to be true.

[6]In Appendix D.1 we illustrate this point with an explicit example and furthermore show that such field redefinitions do not ameliorate the difficulties encountered at two-loop order.

[7]The derivation of this result slightly deviates from that in the conventional non-linear σ-model, as the counter-terms are not homogeneous in $\mathcal{H}$ alone but are homogeneous when viewed as a function of both $\mathcal{H}$ and η. In particular $T_n^{(L)}(\mathcal{H},\eta)$ is of homogeneity degree 1. One must then keep track of the η dependence, even though all its counter-terms vanish.

[8]As we are employing a simple linear splitting of a quantum fluctuation on top of a classical background field it is sufficient to use the pole equations as presented here rather than the more general ones [177, 178] required of a geometric non-linear background field method.

[9]The symbol $\circ$ is used here to indicate both index contraction and integration, e.g. $X \circ \frac{\delta}{\delta Y} \equiv \int \mathrm{d}y X_{IJ}(y) \frac{\delta}{\delta Y_{IJ}(y)}$.

In addition to this we have a further consistency requirement that comes from demanding that $\mathcal{H}_B$ preserves η_B, which invokes not only $\eta = \mathcal{H}\eta^{-1}\mathcal{H}$ but also

$$0 = T_n\eta^{-1}\mathcal{H} + \mathcal{H}\eta^{-1}T_n + \sum_{m=1}^{n-1} T_m\eta^{-1}T_{n-m}\,. \tag{7.5}$$

Inserting the loop expansion of the counter-term one obtains the requirements relevant for two loops:

$$\begin{aligned} 0 =& T_1^{(1)}\eta^{-1}\mathcal{H} + \mathcal{H}\eta^{-1}T_1^{(1)}\,, \\ 0 =& T_1^{(2)}\eta^{-1}\mathcal{H} + \mathcal{H}\eta^{-1}T_1^{(2)}\,, \\ 0 =& T_2^{(2)}\eta^{-1}\mathcal{H} + \mathcal{H}\eta^{-1}T_2^{(2)} + T_1^{(1)}\eta^{-1}T_1^{(1)}\,. \end{aligned} \tag{7.6}$$

Our findings however show that this renormalisation programme, whilst valid at one-loop order, is unsuccessful at two-loop order.

Certain features are valid and provide strong checks on the calculation; the ϵ^{-2} pole of fibre obeys both the pole equation of (7.4) and the consistency relation of the last (7.6). The ϵ^{-2}-pole on the base $\widetilde{T}_2^{(2)}$ reflects the unbroken Lorentz covariance of the base part of the Lagrangian in (7.1) - this is despite that fact that the constituent diagrams that produce $\partial_0 y\partial_0 y$ and $\partial_1 y\partial_1 y$ are vastly different. Moreover, $\widetilde{T}_2^{(2)}$ is expressible only in terms of $\mathcal{H}$ and η which is a deeply non-trivial fact since the individual diagrams depend, in addition, on the connection $\Omega = \mathcal{V}^{-1}\mathrm{d}\mathcal{V}$.

However, the simple pole on the fibre $T_1^{(2)}$ does not, regardless of the method used to evaluate momentum integrals, satisfy the $O(n,n)$ compatibility requirement (7.6). This alone is enough to raise serious questions as to the quantum validity of the Lagrangian in (7.1) since it would seem to destroy the possibility of integrating out the T-dual coordinates and hence prevent the matching of degrees of freedom with the standard string. Moreover, this conclusion is robust and can be reached regardless of the precise prescription for evaluating loop integrals.

The situation on the base is less pleasant still. We find that the simple poles do not respect the unbroken Lorentz covariance of the base part of the Lagrangian in (7.1). That is, $\partial_0 y\partial_0 y$ and $\partial_1 y\partial_1 y$ come with different counter-terms and, moreover, a counter-term is produced for $\partial_0 y\partial_1 y$. Furthermore, unlike the one-loop result or the two-loop ϵ^{-2} pole, these counter-terms are not evidently expressible in terms of just η and $\mathcal{H}$.

The outline of the chapter is as follows. In Section 7.2, we recall the basics of the background field method and furnish an all-order expansion of the action. In Section 7.3, we summarise one-loop results as first obtained in [147, 169]. In Section 7.4 we first detail the relevant Wick contractions that produce the contributing diagrams to the two-loop calculation. We consolidate these in an appendix by then summing all contributing diagrams, reorganised in a basis of tensors of $\mathcal{H}$ and its derivatives, expressing the result in terms of un-evaluated momentum integrals[10]. Then, in Section 7.5 we tackle the evaluation of the resultant loop integrals using the two alternative methodologies described above. Multiple technical appendices detailing the calculation supplement the presentation.

7.2 Expansion

To calculate the renormalisation of the action (7.1) we adopt a background field expansion method, (Taylor) expanding around some classical saddle for the fibre and base coordinates

$$\mathbb{X}^I = \mathbb{X}^I_{\rm cl} + \xi^I \ , \quad y = y_{\rm cl} + \zeta \ . \tag{7.7}$$

The fluctuations ξ and ζ are dynamical, whilst the classical background fields $\mathbb{X}_{\rm cl}$ and $y_{\rm cl}$ are frozen (i.e. we will not integrate over them in the path-integral approach). The quantum effective action Γ is then obtained via Wick contraction of the exponential of the interacting Lagrangian $\mathcal{L}_{\rm I}$

$$e^{i\Gamma} = e^{iS_{\rm cl}} \langle \exp\left(i\int {\rm d}^2\sigma \mathcal{L}_{\rm I}\right)\rangle_{\rm 1PI} \, . \tag{7.8}$$

Some comments are in order. On the right-hand side, only one-particle-irreducible (1PI) diagrams need to be considered and the average is taken with respect to the quantum fluctuations ξ, ζ. $S_{\rm cl}$ denotes the classical action, i.e. the one comprising of the classical fields $\mathbb{X}_{\rm cl}$ and $y_{\rm cl}$ only[11]. We choose[12] the background fields to be on-shell: terms in $\mathcal{L}_{\rm I}$ linear in the fluctuations are necessarily proportional to the equations of motion and will be dropped. The effective action is effectively recovered taking logarithms, thus removing disconnected diagrams.

As this is a Taylor expansion of actual coordinate values, it is evidently not geometrically covariant. In the context of the conventional non-linear σ-model it is much more preferable to maintain geometric covariance in the calculation, and this can be achieved by means of a covariant background field expansion [175]. Here, however, the departure from conventional Riemannian geometry entailed by the introduction of the generalised metric $\mathcal{H}$ means such notion of "covariant" background field expansion is currently (at least to our knowledge) lacking. Instead, we will proceed non-covariantly adopting (7.7).

[10] This provides an intermediate result that is largely independent of any details of the regularisation procedure. It could be harnessed in further studies which might employ different regularisation schemes.

[11] To ease the notational burden we henceforward omit the subscript $\mathbb{X}_{\rm cl} \to \mathbb{X}$ on the classical background and, where useful, indicate the worldsheet point with a subscript, e.g. $\zeta(\sigma) \equiv \zeta_\sigma$.

[12] This is not mandatory, and one could decide to work off-shell for the classical background.

Our first task is to single out, for each fluctuation type, a kinetic term. While for ζ the latter has a standard form, $\frac{1}{2}\lambda\partial_\mu\zeta\partial^\mu\zeta$, fluctuations along the fibre coordinates require the introduction of a (generalised) vielbein $\mathcal{V}_I{}^A$ to remedy for the non-constant $\mathcal{H}_{IJ}(y)$ factor. The vielbein and its derivatives are defined by

$$\mathcal{H}_{IJ} = \mathcal{V}_I{}^A \mathcal{H}_{AB} \mathcal{V}_J{}^B\,, \qquad \eta_{IJ} = \mathcal{V}_I{}^A \eta_{AB} \mathcal{V}_J{}^B\,, \qquad \Omega_\mu{}^{IJ} = \mathcal{V}^I{}_A \partial_\mu \mathcal{V}^{JA}\,, \tag{7.9}$$

where Ω is known as the Weitzenböck connection. Early alphabet capital Latin indices $A, B, \ldots$ are used to indicate the flat generalised tangent bundle. Accordingly, the fluctuations on the torus can be "flattened" by defining $\xi^I = \mathcal{V}^I{}_A \xi^A$.

Both ξ^A's and ζ's have now a canonical kinetic term

$$\mathcal{L}_{\mathrm{K}} = -\frac{1}{2}\mathcal{H}_{AB}\partial_1\xi^A\partial_1\xi^B + \frac{1}{2}\eta_{AB}\partial_0\xi^A\partial_1\xi^B + \frac{1}{2}\lambda\partial_\mu\zeta\partial^\mu\zeta\,, \tag{7.10}$$

from which two-point functions are easily extracted[13] as

$$\langle \xi^A(\sigma)\xi^B(\sigma')\rangle = \mathcal{H}^{AB}\Delta(\sigma-\sigma') + \eta^{AB}\theta(\sigma-\sigma')\,, \qquad \langle\zeta(\sigma)\zeta(\sigma')\rangle = \lambda^{-1}\Delta(\sigma-\sigma')\,, \tag{7.11}$$

where

$$\Delta(\sigma) = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} e^{-ip\cdot\sigma}\frac{i}{p^2}\,, \qquad \theta(\sigma) = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} e^{-ip\cdot\sigma}\frac{i}{p^2}\frac{p_0}{p_1}\,. \tag{7.12}$$

Performing the Taylor series one finds the entire interaction Lagrangian, at any order, is given by

$$\begin{aligned}
-2\mathcal{L}_{\mathrm{I}} = &-\xi^I\xi^J\Omega_{1I}{}^K\Omega_{0JK} - \xi^I\partial_1\xi^A\Omega_{0IA} - \xi^I\partial_0\xi^A\Omega_{1IA} \\
&+ \sum_{n\geq 2}\bigg(\frac{1}{n!}\mathcal{H}^{(n)}_{IJ}\zeta^n\partial_1\mathbb{X}^I\partial_1\mathbb{X}^J + \frac{2}{(n-1)!}\mathcal{H}^{(n-1)}_{IA}\zeta^{n-1}\partial_1\xi^A\partial_1\mathbb{X}^I \\
&- \frac{2}{(n-1)!}\mathcal{H}^{(n-1)}_{IJ}\Omega_1{}^J{}_K\zeta^{n-1}\xi^K\partial_1\mathbb{X}^I + \frac{1}{(n-1)!}\mathcal{H}^{(n-1)}_{AB}\zeta^{n-1}\partial_1\xi^A\partial_1\xi^B \\
&- \frac{2}{(n-2)!}\mathcal{H}^{(n-2)}_{AJ}\Omega_1{}^J{}_I\zeta^{n-2}\xi^I\partial_1\xi^A + \frac{1}{(n-2)!}\mathcal{H}^{(n-2)}_{IJ}\Omega_1{}^I{}_K\Omega_1{}^J{}_L\zeta^{n-2}\xi^K\xi^L\bigg)\,.
\end{aligned} \tag{7.13}$$

To simplify the presentation of the tensorial structure we have introduced some notation. We indicate derivatives with respect to the base coordinate with a dot, e.g. $\dot{\mathcal{H}} \equiv \partial_y\mathcal{H}$, or in general for the n-th derivative we use $\mathcal{H}^{(n)}$ (so that $\mathcal{H}^{(0)} \equiv \mathcal{H}$). Concatenations of (matrix) products of $\mathcal{H}$'s and their derivatives (assuming η is used to raise indices

[13] There is one subtlety if we were to consider dimensional regularisation; when computing the Green function for ξ one ends up with an expression such as $p_0^2 - p_1^2$. It is not clear that one should directly trade this combination for p^2 in d-dimensions. However our general approach will be to remain strictly in $d = 2$ where $p_0^2 - p_1^2 \equiv p^2$ and only continuing to $d = 2 + \epsilon$ when evaluating integrals.

whenever needed) will be indicated with $\cdot$, e.g. $\mathcal{H}^{(i,j,k)} = \mathcal{H}^{(i)} \cdot \mathcal{H}^{(j)} \cdot \mathcal{H}^{(k)}$. Contractions of $\mathcal{H}$ with an external torus leg $\partial_1 \mathbb{X}$ will be shown with a $\bullet$, i.e. $\mathcal{H}_{I\bullet} \equiv \mathcal{H}_{IJ}\partial_1\mathbb{X}^J$. We also use as shorthand $\mathcal{H}^{(n)}_{AB} \equiv \mathcal{V}_A{}^I \mathcal{H}^{(n)}_{IJ} \mathcal{V}_B{}^J$.

By inspection of this expansion we can immediately make two general statements that are true to all orders perturbatively. First is that the $O(n,n)$ pairing η does not receive quantum corrections at any order in perturbation theory; as no $\partial_0\mathbb{X}^I$ legs appear in (7.13)[14], it is impossible to generate terms proportional to $\partial_0\mathbb{X}^I\partial_1\mathbb{X}^J$ upon expanding $\exp(iS_{\rm I})$. Secondly, we cannot generate any mixed base-fibre terms of the form $\partial_1\mathbb{X}\partial_\mu y$ in the effective action. This follows as any such term would necessarily involve an odd number of ξ fields, and thus vanish upon Wick contraction.

7.3 One-loop Recap

Before moving on to the two-loop calculation, let us recapitulate the situation at one-loop. At this order it is sufficient to consider only quadratic terms in the fluctuations and the effective action is given by (assuming only 1PI and connected diagrams are considered)

$$\Gamma = S_{\rm cl} + \langle S_{\rm I}\rangle + \frac{i}{2}\langle S_{\rm I}^2\rangle\,, \tag{7.14}$$

where $S_{\rm I} = \int {\rm d}^2\sigma \mathcal{L}_{\rm I}$. To calculate the quantum correction to $\mathcal{H}_{IJ}\partial_1\mathbb{X}^I\partial_1\mathbb{X}^J$ we can ignore the Weitzenböck connection pieces all together, as any such contribution implicitly contains derivatives of the base coordinates y, such that the relevant interaction term reads

$$\mathcal{L}_{\rm I} \supset -\frac{1}{4}\ddot{\mathcal{H}}_{IJ}\zeta_\sigma^2\partial_1\mathbb{X}^I\partial_1\mathbb{X}^J - \dot{\mathcal{H}}_{AI}\zeta_\sigma\partial_1\xi^A_\sigma\partial_1\mathbb{X}^I\,. \tag{7.15}$$

There are two contributions to consider here. The first requires only a single copy of the worldsheet and is given by

$$-\frac{1}{4}\ddot{\mathcal{H}}_{IJ}\partial_1\mathbb{X}^I\partial_1\mathbb{X}^J\langle\zeta_\sigma\zeta_\sigma\rangle = -\frac{1}{4}\mathcal{H}^{(2)}_{\bullet\bullet}\Delta(0) = -\frac{i}{4}\mathcal{H}^{(2)}_{\bullet\bullet}\mathbf{I}, \tag{7.16}$$

in which $\mathcal{H}^{(2)}_{\bullet\bullet} \equiv \ddot{\mathcal{H}}_{IJ}\partial_1\mathbb{X}^I\partial_1\mathbb{X}^J$. We will denote this contraction as a *bubble* and introduce the divergent integral

$$\mathbf{I} = -i\Delta(0) = \int \frac{{\rm d}^2 p}{(2\pi)^2}\frac{1}{p^2}\,. \tag{7.17}$$

[14]The reason why this is the case is straightforward: any $\partial_0\mathbb{X}^I$ leg must necessarily come from the background field expansion of $\eta_{IJ}\partial_0\mathbb{X}^I\partial_1\mathbb{X}^J$; however, η is constant and terms linear in the fluctuations are discarded on-shell.

The second contribution arises from $\langle S_{\rm I}^2\rangle$ and requires two copies of the worldsheet:

$$\int \mathrm{d}^2\sigma_2 \left(\frac{i}{2}\dot{\mathcal{H}}_{AI}\dot{\mathcal{H}}_{BJ}\partial_1\mathbb{X}^I\partial_1\mathbb{X}^J\langle\zeta_{\sigma_1}\zeta_{\sigma_2}\rangle\langle\partial_1\xi^A_{\sigma_1}\partial_1\xi^B_{\sigma_2}\rangle\right) = \frac{i}{2\lambda}\mathbf{L}\mathcal{H}^{(1,1,0)}_{\bullet\bullet} - \frac{i}{2\lambda}\widetilde{\mathbf{L}}\mathcal{H}^{(1,1)}_{\bullet\bullet}\,. \tag{7.18}$$

We will denote such contraction as a *loop* and introduce the integrals

$$\mathbf{L} = \int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{p_1^2}{(p^2)^2}\,,\quad \widetilde{\mathbf{L}} = \int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{p_1p_0}{(p^2)^2}\,. \tag{7.19}$$

On general grounds (e.g. integration over an odd integrand) we may assume $\widetilde{\mathbf{L}} = 0$, however the integral $\mathbf{L}$ is expected to result in a UV divergence.

At one-loop order we can simply regulate IR divergences by the replacement of $p^2 \to p^2 - m^2$ in integrals, and UV divergences can be unambiguously regulated in $d = 2+\epsilon$. The fundamental divergent integral $\mathbf{I}$ evaluates, in dimensional regularisation, to

$$\mathbf{I} = \mathbf{P} + \frac{i\bar{\gamma}}{4\pi}\,,\quad \mathbf{P} \equiv \frac{i}{2\pi\epsilon}\,, \tag{7.20}$$

where $O(\epsilon)$ contributions have been dropped and the combination $\bar{\gamma} = \gamma_E + \log(m^2/4\pi)$ introduced. In $d = 2+\epsilon$ we can invoke Lorentz invariance to relate

$$\int \frac{\mathrm{d}^dp}{(2\pi)^d}\frac{p_\mu p_\nu}{(p^2-m^2)^2} = \frac{\eta_{\mu\nu}}{d}\mathbf{I}\,. \tag{7.21}$$

A naive prescription to compute $\mathbf{L}$ is to simply set the free Lorentz indices $\mu = \nu = 1$ to give

$$\mathbf{L} \sim -\frac{1}{2}\mathbf{I}\,. \tag{7.22}$$

This is sufficient at one-loop, and in general allows an unambiguous determination of the leading divergence of any integral we encounter. However, strictly speaking the relation (7.21) is only valid in $d = 2+\epsilon$ and the process of specifying the component p_1 of a $d = 2+\epsilon$ dimensional vector is rather ambiguous. Different prescriptions for doing so will lead to different finite parts. At two-loop this ambiguity becomes more acute since whilst the ϵ^{-2} pole will be well determined, a prescription needs to be given to find the ϵ^{-1} pole.

The minimal subtraction procedure (i.e. removal of $\frac{1}{\epsilon}$ divergences only) then gives a counter-term to $\mathcal{H}_{\bullet\bullet}$ of

$$T_1^{(1)} = \frac{1}{4\pi\lambda}\left(\mathcal{H}^{(2)} + \mathcal{H}^{(1,1,0)}\right)\,. \tag{7.23}$$

Calculating the correction to the base term is a little more complicated since there are a number of possible diagrams involving the vielbein pieces. After some work, and invoking identities such as

$$\mathrm{Tr}(\mathcal{H}\Omega_0\mathcal{H}\Omega_0) = \mathrm{Tr}(\Omega_0\Omega_0) + \frac{1}{2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\partial_0 y\partial_0 y\,, \tag{7.24}$$

one finds a counter-term to λ of

$$\widetilde{T}_1^{(1)} = -\frac{1}{16\pi}\,\mathrm{Tr}\left(\mathcal{H}^{(1,1)}\right)\,. \tag{7.25}$$

Though far from obvious from the intermediate stages of the calculation, one finds that the counter-term for $\lambda(\partial_0 y)^2$ matches that of $\lambda(\partial_1 y)^2$ and that no terms in the form of $\partial_0 y\partial_1 y$ are generated, as expected. According to the general results as per (7.3), the β-functions are extracted as

$$\beta^{\mathcal{H}}_{(1)} = -\frac{1}{4\pi\lambda}(\mathcal{H}^{(2)} + \mathcal{H}^{(1,1,0)})\,, \qquad \beta^{\lambda}_{(1)} = \frac{1}{16\pi}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\,, \tag{7.26}$$

where the subscript is used to emphasise the loop order we are working at.

In the above we regulated IR divergences in an ad-hoc fashion by replacing $p^2 \to p^2 - m^2$. However, the inclusion of a mass term is a delicate matter as it is in general disruptive to the chiral nature of the fields $\mathbb{X}$. Indeed, it seems hard to find a local term that precisely recreates this prescription. We come closer by introducing

$$\mathcal{L}_{\mathrm{mF}} = -\frac{m^2}{4}\mathcal{H}_{IJ}\mathbb{X}^I\mathbb{X}^J\,, \qquad \mathcal{L}_{\mathrm{mB}} = -\frac{\lambda}{2}m^2 y^2\,, \tag{7.27}$$

as mass terms on the fibre and base, respectively. For two-loop calculations both the background field expansion of $\mathcal{L}_{\mathrm{mF}}$ mass term, and its one loop renormalisation are important. A straightforward calculation shows that the counter-term for the mass is

$$T_m = \frac{1}{4\pi\lambda}\mathcal{H}^{(2)}\,. \tag{7.28}$$

Together we end up with a one-loop counter-term Lagrangian

$$\mathcal{L}_{\mathrm{CT}} = -\frac{1}{2\epsilon}\left(T_1^{(1)}\right)_{IJ}\partial_1\mathbb{X}^I\partial_1\mathbb{X}^J + \frac{1}{2\epsilon}\widetilde{T}_1^{(1)}\partial_\mu y\partial^\mu y - \frac{m^2}{4\epsilon}\left(T_m\right)_{IJ}\mathbb{X}^I\mathbb{X}^J\,. \tag{7.29}$$

In addition, we have a freedom to add any terms that vanish as a consequence of the classical equations of motion satisfied by the background

$$\Box y + \frac{1}{2}\dot{\mathcal{H}}_{IJ}\partial_1\mathbb{X}^I\partial_1\mathbb{X}^J = 0\,. \tag{7.30}$$

Multiplying this equation by a function $f(y)$ assumed to be first order in derivatives, and integrating by parts allows us to consider the addition of

$$\mathcal{L}_{\text{on-shell}} = -\frac{1}{\epsilon}\dot{f}(y)\partial_\mu y\partial^\mu y + \frac{1}{2\epsilon}f(y)\dot{\mathcal{H}}_{IJ}\partial_1\mathbb{X}^I\partial_1\mathbb{X}^J\,. \tag{7.31}$$

Choosing $\dot{f}(y) = \frac{1}{2}\widetilde{T}$ eliminates the base divergence in

$$\mathcal{L}_{\text{CT}} + \mathcal{L}_{\text{on-shell}} = -\frac{1}{2\epsilon}\left(T_1^{(1)} - f\dot{\mathcal{H}}\right)_{IJ}\partial_1\mathbb{X}^I\partial_1\mathbb{X}^J - \frac{m^2}{4\epsilon}\left(T_m\right)_{IJ}\mathbb{X}^I\mathbb{X}^J\,. \tag{7.32}$$

The same result can be obtained by performing a field redefinition $y \to y - \frac{1}{\lambda\epsilon}f(y)$. In what follows we choose not to perform this redefinition (though we will examine its consequences for the two-loop result in Appendix D.1).

7.3.1 Renormalisation of Ω

A subtlety is the renormalisation of the Weitzenböck connection Ω. Albeit somewhat irrelevant at one-loop – it recombines so as to return only instances of $\mathcal{H}$ in the final result – it does bear a significance at two-loop, as we shall see. Notice that, since Ω is first order in derivatives by construction, $\beta^{\Omega}_{(1)}$ will necessarily be third order. Actually, Ω is not quite an independent quantity as it is obtained from the generalised frame fields that also determine $\mathcal{H}$. Thus, it should be possible to extract $\beta^{\Omega}_{(1)}$ from $\beta^{\mathcal{H}}_{(1)}$.

Using (7.9), and explicitly avoiding indices for the sake of simplicity, we could write $\mathcal{H} = \mathcal{V}\mathcal{H}_0\mathcal{V}^t$, $\mathcal{H}_0$ being the identity[15]. If t is the RG time and $r = \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{V})\mathcal{V}^{-1}$ we have from this

$$\beta^{\mathcal{H}} = r\mathcal{H} + \mathcal{H}r^t\,. \tag{7.33}$$

In particular, compatibility of $\mathcal{V}$ with η ensures that $r^t = -\eta^{-1}r\eta$. The most general form of r that satisfies these two constraints is[16]

$$r\eta = \frac{1}{16\pi\lambda}(\mathcal{H}^{(0,2)} - \mathcal{H}^{(2,0)})\,. \tag{7.34}$$

[15] In general $\mathcal{H}_0$ coincides with a constant $\mathcal{H}_{AB}$. However, it is possible that a constant generalised metric contains flowing parameters, cf. (3.95). Via an $O(n,n)$ rotation it is possible to move this dependence onto the frame fields, and we assume to have done so here.

[16] Actually, once all identities involving $\mathcal{H}$ and Ω are exploited, the most general would be $r\eta = \frac{1}{16\pi\lambda}(\mathcal{H}^{(0,2)} - \mathcal{H}^{(2,0)}) + a\{\Omega^{(1)}, \mathcal{H}^{(0)}\} + b(\Omega^{(1)} + \mathcal{H}^{(0)}\Omega^{(1)}\mathcal{H}^{(0)})$, for two unspecified parameters a, b and where $\{\cdot,\cdot\}$ denotes the anti-commutator. The coefficient of the first combination that does not involve Ω is completely fixed by $\beta^{\mathcal{H}}$. The fact that the result is ambiguous should not be surprising, for the generalised frames are not completely specified by $\mathcal{H}$. However, as the terms involving $\Omega^{(1)}$ are not created at one-loop and, besides, do not contribute to the result we will omit them.

Using the definition of Ω in terms of $\mathcal{V}$ we easily find $\beta^{\Omega} = -\eta^{-1}\partial_y r - \Omega r - r^t\Omega$, hence

$$\beta^{\Omega}_{(1)} = \frac{1}{16\pi\lambda}\left(\mathcal{H}^{(3,0)} - \mathcal{H}^{(0,3)} + \mathcal{H}^{(2,1)} - \mathcal{H}^{(1,2)} + [\mathcal{H}^{(0,2)},\Omega] - [\mathcal{H}^{(2,0)},\Omega]\right) . \tag{7.35}$$

As expected, this object is third order in derivatives and skew-symmetric.

7.4 Two-loop Expansion and Wick Contractions

The two-loop effective action is evaluated to

$$\Gamma = S_{\rm cl} + \langle S_{\rm All}\rangle + \frac{i}{2}\langle S^2_{\rm All}\rangle - \frac{1}{6}\langle S^3_{\rm All}\rangle - \frac{i}{24}\langle S^4_{\rm All}\rangle\,, \tag{7.36}$$

where the restriction to 1PI connected diagrams is understood. Here $S_{\rm All} = \int {\rm d}^2\sigma \mathcal{L}_{\rm All}$ contains the interaction Lagrangian expansion to quartic order in fluctuations *and* the background field expansion of the one-loop counter-term Lagrangian to quadratic order[17]. Because this is quite involved we will treat the fibre and the various contributions to Lorentz components $\partial_0 y\partial_0 y$, $\partial_1 y\partial_1 y$ and $\partial_0 y\partial_1 y$ separately.

7.4.1 Fibre Contributions

To renormalise the term $\mathcal{H}_{IJ}\partial_1\mathbb{X}^I\partial_1\mathbb{X}^J$ we can discard all terms in which the classical background Ω is involved. This leaves only a few contributors

$$\begin{aligned}\mathcal{L}_{\rm All} \supset \; & \left(-\frac{1}{4}\mathcal{H}^{(2)}\zeta^2 - \frac{1}{12}\mathcal{H}^{(3)}\zeta^3 - \frac{1}{48}\mathcal{H}^{(4)}\zeta^4 - \frac{1}{2}X^{(2)}\zeta^2\right)_{IJ}\partial_1\mathbb{X}^I\partial_1\mathbb{X}^J \\ &+ \left(-\mathcal{H}^{(1)}\zeta - \frac{1}{2}\mathcal{H}^{(2)}\zeta^2 - \frac{1}{6}\mathcal{H}^{(3)}\zeta^3 - X^{(1)}\zeta\right)_{IJ}\partial_1\xi^I\partial_1\mathbb{X}^J \\ &+ \left(-\frac{1}{2}\mathcal{H}^{(1)}\zeta - \frac{1}{4}\mathcal{H}^{(2)}\zeta^2 - X\right)_{AB}\partial_1\xi^A\partial_1\xi^B \\ &+ \frac{1}{2}Y\partial_\mu\zeta\partial^\mu\zeta\,. \end{aligned} \tag{7.37}$$

[17]This approach of expanding the counter term Lagrangian is something of a shortcut and one of the great virtues of the background field method; however when employing covariant background field expansions in which the quantum-classical splitting is non-linear this approach is not complete [177] and instead one should renormalise each and every vertex in $\mathcal{L}_{\rm I}$. Here however we *are* employing a linear splitting and so anticipate that the resultant Ward identity ensures we can complete the renormalisation by considering only diagrams with external classical background fields. For completeness we have calculated the full one-loop renormalisation of $\mathcal{L}_{\rm I}$ by splitting the fluctuations into a quantum-background (indicated with a tilde) and a dynamical part $\xi \to \xi + \tilde{\xi}$, $\zeta \to \zeta + \tilde{\zeta}$ performing the path-integral over the latter. As expected, doing so recovers the expansion of the one-loop counter-term Lagrangian to quadratic order.

Here $X_{IJ} = \frac{1}{\epsilon}\left(T_1^{(1)}\right)_{IJ}$ and $Y = \frac{1}{\epsilon}\tilde{T}_1^{(1)}$ arise from the expansion of the counter-term Lagrangian in the MS scheme. Schematically we group the terms here in the number of classical background fields as

$$\mathcal{L}_{\rm All} \supset \underbrace{A^{[2]}_{IJ}\partial_1\mathbb{X}^I\partial_1\mathbb{X}^J}_{\mathcal{A}^{[2]}} + \underbrace{A^{[1]}_{A}\partial_1\mathbb{X}^A}_{\mathcal{A}^{[1]}} + \mathcal{A}^{[0]}\,, \tag{7.38}$$

where we further denote by $\mathcal{A}^{[n]}_i$ the term in $\mathcal{A}^{[n]}$ that contains i derivatives of $\mathcal{H}$. As the loop expansion is organised into a derivative expansion of the generalised metric, two-loop contributions occur with fourth order in derivatives. Since $\mathcal{A}^{[0]}$ carries at least one derivative, the expansion of Γ truncates to this order with the term $\langle \mathcal{L}^4_{\rm All}\rangle$. We only require terms with exactly two occurrences of the background field $\partial_1\mathbb{X}$. This will give the following contributions to deal with

$$\begin{aligned} a_1 = \langle \mathcal{A}^{[2]}\rangle\,, \quad a_2 = i\langle \mathcal{A}^{[0]}\mathcal{A}^{[2]}\rangle\,, \quad a_3 = \frac{i}{2}\langle \mathcal{A}^{[1]}\mathcal{A}^{[1]}\rangle\,,\\ a_4 = -\frac{1}{2}\langle \mathcal{A}^{[0]}\mathcal{A}^{[0]}\mathcal{A}^{[2]}\rangle\,, \quad a_5 = -\frac{1}{2}\langle \mathcal{A}^{[0]}\mathcal{A}^{[1]}\mathcal{A}^{[1]}\rangle\,, \quad a_6 = -\frac{i}{4}\langle \mathcal{A}^{[0]}\mathcal{A}^{[0]}\mathcal{A}^{[1]}\mathcal{A}^{[1]}\rangle\,. \end{aligned} \tag{7.39}$$

The first step is to evaluate the Wick contractions to obtain expressions containing an un-evaluated momentum integral of the form

$$[[T(p_0,p_1,k_0,k_1)]]_{i,j,k} = \int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{T(p_0,p_1,k_0,k_1)}{(p^2)^i(k^2)^j[(k+p)^2]^k}\,, \tag{7.40}$$

where the $T(p_0,p_1,k_0,k_1)$ will be some specific components of momenta k and p, arising predominantly from the fibre propagator terms. The Wick contraction is standard though tedious, here we only present a_6 in detail. In a_6 there are three distinct contractions to consider. The first is

$$\begin{aligned} a_{6_a} &= -\frac{i}{4}\dot{\mathcal{H}}_{AB}\dot{\mathcal{H}}_{CD}\dot{\mathcal{H}}_{\bullet E}\dot{\mathcal{H}}_{\bullet F}\langle\zeta_{\sigma_1}\zeta_{\sigma_3}\rangle\langle\zeta_{\sigma_2}\zeta_{\sigma_4}\rangle\langle\partial_1\xi^A_{\sigma_1}\partial_1\xi^C_{\sigma_2}\rangle\langle\partial_1\xi^B_{\sigma_1}\partial_1\xi^D_{\sigma_2}\rangle\langle\partial_1\xi^E_{\sigma_3}\partial_1\xi^F_{\sigma_4}\rangle\\ &= -\frac{1}{4\lambda^2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,1,0)}_{\bullet\bullet}\times\left([[(p_1+k_1)k_1p_1^2((p_0+k_0)k_0-(p_1+k_1)k_1)]]_{3,1,1}\right)\,. \end{aligned} \tag{7.41}$$

In deriving this expression we have made use of cyclicity of trace to discard terms involving $\mathrm{Tr}(\mathcal{H}^{(1,1,0)}) = -\,\mathrm{Tr}(\mathcal{H}^{(1,1,0)})$. The diagram that gives rise to a $[[\ldots]]_{3,1,1}$ we call *square envelope* topology (see Appendix C.1 for more details). The remaining contractions here yield

$$\begin{aligned} a_{6_b} =& -\frac{i}{4}\dot{\mathcal{H}}_{AB}\dot{\mathcal{H}}_{CD}\dot{\mathcal{H}}_{\bullet E}\dot{\mathcal{H}}_{\bullet F}\langle\zeta_{\sigma_1}\zeta_{\sigma_2}\rangle\langle\zeta_{\sigma_3}\zeta_{\sigma_4}\rangle\langle\partial_1\xi^A_{\sigma_1}\partial_1\xi^C_{\sigma_2}\rangle\langle\partial_1\xi^B_{\sigma_1}\partial_1\xi^E_{\sigma_3}\rangle\langle\partial_1\xi^D_{\sigma_2}\partial_1\xi^F_{\sigma_4}\rangle\\ =&\frac{1}{2\lambda^2}\mathcal{H}^{(1,1,1,1,0)}_{\bullet\bullet}\left([[k_1^2p_0^2p_1^2]]_{3,1,1} - 2[[k_0k_1p_0p_1^3]]_{3,1,1} + [[k_1^2p_1^4]]_{3,1,1}\right)\,, \end{aligned} \tag{7.42}$$

which is another square envelope and

$$\begin{aligned} a_{6_c} &= -\frac{i}{4}\dot{\mathcal{H}}_{AB}\dot{\mathcal{H}}_{CD}\dot{\mathcal{H}}_{\bullet E}\dot{\mathcal{H}}_{\bullet F}\langle\zeta_{\sigma_1}\zeta_{\sigma_3}\rangle\langle\zeta_{\sigma_2}\zeta_{\sigma_4}\rangle\langle\partial_1\xi^A_{\sigma_1}\partial_1\xi^C_{\sigma_2}\rangle\langle\partial_1\xi^B_{\sigma_1}\partial_1\xi^F_{\sigma_4}\rangle\langle\partial_1\xi^D_{\sigma_2}\partial_1\xi^E_{\sigma_3}\rangle \\ &= \frac{1}{2\lambda^2}\mathcal{H}^{(1,1,1,1,0)}_{\bullet\bullet}\Big([[(p_1+k_1)^2p_1^2k_1^2]]_{2,2,1} - 2[[(p_1+k_1)(p_0+k_0)p_1p_0k_1^2]]_{2,2,1} \\ &\quad + [[(p_1+k_1)^2p_1p_0k_1k_0]]_{2,2,1}\Big)\,, \end{aligned} \tag{7.43}$$

where the $[[\dots]]_{2,2,1}$ integral arises from what we call *diamond sunset* topology.

The remaining contributions $a_1,\dots,a_5$ are dealt with, in an similar fashion, in Appendix B.1. Combining these contributions results in the following tensor structures with coefficients given by un-evaluated integrals:

$$\begin{aligned} \mathcal{H}^{(4)}_{\bullet\bullet}:&\quad \frac{1}{16}\mathbf{I}^2 - \frac{1}{8}\mathbf{IP} \\ \mathcal{H}^{(3,1,0)}_{\bullet\bullet}:&\quad \frac{1}{2}(\mathbf{P}-\mathbf{I})[[p_1^2]]_{2,0,0} - \frac{1}{4}\mathbf{IP} \\ \mathcal{H}^{(2,0,2)}_{\bullet\bullet}:&\quad \frac{1}{4}\mathbf{IP} + \frac{1}{4}[[p_1^2]]_{1,1,1} \\ \mathcal{H}^{(2,1,1)}_{\bullet\bullet}:&\quad \frac{1}{2}\mathbf{IP} - \frac{1}{2}\mathbf{P}[[p_1^2]]_{2,0,0} + [[p_1k_1p\cdot k]]_{2,1,1} \\ \mathcal{H}^{(1,2,1)}_{\bullet\bullet}:&\quad \frac{1}{8}\mathbf{IP} - \frac{1}{2}\mathbf{P}[[p_1^2]]_{2,0,0} - \frac{1}{2}[[p_1^2k_1^2]]_{2,2,0} - \frac{1}{4}(\mathbf{P}-\mathbf{I})[[p_1^2p^2]]_{3,0,0} \\ \mathcal{H}^{(1,1,1,1,0)}_{\bullet\bullet}:&\quad \frac{1}{2}\mathbf{P}[[p_1^2]]_{2,0,0} + [[p_1^2k_1^2]]_{2,2,0} + \frac{1}{2}\mathbf{I}[[p_1^4]]_{3,0,0} + \frac{1}{4}\mathbf{P}[[p_1^2p^2]]_{3,0,0} \\ &\quad - [[p_1^3k_1k\cdot p]]_{3,1,1} + \frac{1}{2}[[p_1^2k_1^2p^2]]_{3,1,1} - [[(p_1^2k_1^2+p_1^3k_1)k^2]]_{2,2,1} \\ \mathcal{H}^{(2)}_{\bullet\bullet}\,\mathrm{Tr}(\mathcal{H}^{(1,1)}):&\quad -\frac{1}{32}\mathbf{P}[[p^2]]_{2,0,0} + \frac{1}{8}[[(p+k)_1k_1(p\cdot k+k^2)]]_{2,1,1} \\ \mathcal{H}^{(1,1,0)}_{\bullet\bullet}\,\mathrm{Tr}(\mathcal{H}^{(1,1)}):&\quad \frac{1}{16}\mathbf{P}[[p_1^2p^2]]_{3,0,0} - \frac{1}{4}[[(p_1+k_1)p_1^2k_1(p\cdot k+k^2)]]_{3,1,1}\,. \end{aligned} \tag{7.44}$$

This result provides a determination of the contributions to the generalised metric two-loop counter term for which any subsequent regularisation scheme can be employed. Occurrences of $\mathbf{P} = \frac{i}{2\pi\epsilon}$ arise in these expressions from diagrams with one-loop counter-term insertions. The explicit evaluation of the remaining integrals is a delicate matter and will be discussed at length in Section 7.5.

In addition, if IR divergences are to be regulated by including an explicit mass term as described before, the background field expansion of the mass terms in the Lagrangian contained in (7.27) must be included as should one-loop diagrams with the mass counter-term of (7.28) insertion. These contractions are detailed in Appendix B.1.

7.4.2 Base Contributions

Contributions to the base manifold can be organised in terms of the type of external legs, $(\partial_0 y)^2$, $(\partial_1 y)^2$ or $\partial_0 y \partial_1 y$, they come with. Given the chiral nature of the action (7.1) these are better treated separately; eventually, we shall compare our findings for the various cases and comment on the (broken) Lorentz invariance of the final result.

Restricting ourselves to $(\partial_0 y)^2$, we can discard all terms in which the classical backgrounds Ω_1 and $\partial_1 \mathbb{X}$ are involved. This leaves only a few contributors

$$\begin{aligned}\mathcal{L}_{\text{All}} \supset &-\frac{1}{2}\Omega_{0AI}\partial_1\xi^A\xi^I \\ &-\frac{1}{2}\left(\mathcal{H}^{(1)}\zeta+\frac{1}{2}\mathcal{H}^{(2)}\zeta^2+X\right)_{AB}\partial_1\xi^A\partial_1\xi^B \\ &+\frac{1}{2}Y\partial_\mu\zeta\partial^\mu\zeta+Y^{(1)}\zeta\partial_\mu\zeta\partial^\mu y+\frac{1}{4}Y^{(2)}\zeta^2\partial_\mu y\partial^\mu y\,. \end{aligned} \tag{7.45}$$

Similarly to the fibre, we group terms in the number of classical background fields they come with. Recalling that Ω_0 counts as a $\partial_0 y$ insertion, we have

$$\mathcal{L}_{\text{All}} \supset \mathcal{B}^{[0]} + \underbrace{B^{[1]}\partial_0 y}_{\mathcal{B}^{[1]}} + \underbrace{B^{[2]}(\partial_0 y)^2}_{\mathcal{B}^{[2]}}\,. \tag{7.46}$$

We denote by $\mathcal{B}_i^{[n]}$ the term in $\mathcal{B}^{[n]}$ that contains i derivatives of $\mathcal{H}$ or $\mathcal{V}$ such that

$$\mathcal{B}^{[0]} = \mathcal{B}_1^{[0]} + \mathcal{B}_2^{[0]}\,, \qquad \mathcal{B}^{[1]} = \mathcal{B}_1^{[1]} + \mathcal{B}_3^{[1]}\,, \qquad \mathcal{B}^{[2]} = \mathcal{B}_4^{[2]}\,. \tag{7.47}$$

First we consider only contractions that lead to exactly two occurrences of $\partial_0 y$ and four derivatives in $\mathcal{H}$ or $\mathcal{V}$. These are given by

$$\begin{aligned} b_1 =& \langle \mathcal{B}_4^{[2]}\rangle\,, \quad b_2 = i\langle \mathcal{B}_1^{[1]}\mathcal{B}_3^{[1]}\rangle\,, \\ b_3 =& -\frac{1}{2}\langle \mathcal{B}_2^{[0]}\mathcal{B}_1^{[1]}\mathcal{B}_1^{[1]}\rangle\,, \quad b_4 = -\frac{i}{4}\langle \mathcal{B}_1^{[0]}\mathcal{B}_1^{[0]}\mathcal{B}_1^{[1]}\mathcal{B}_1^{[1]}\rangle\,. \end{aligned} \tag{7.48}$$

We detail the explicit evaluation of each of these in Appendix B.2, but note here that b_2 vanishes outright since it contains no connected diagram.

With the classical background fields viewed as *external* legs to the diagrams in $b_1 - b_4$, the divergences are extracted at zero *external* momenta (i.e the momenta associated to the Fourier transform of the background field on external legs). In addition to these contributions, we must take into account some diagrams for which the vertices contribute fewer than four derivatives of background fields, but for which the loop integrals carry divergences that are linear or quadratic in the external momenta. Fourier transforming these external momenta then produces a worldsheet derivative which acts on the background fields $\mathcal{H}, \mathcal{V}, \Omega$. The inclusion of such contributions is vital to the cancellation of terms involving Ω which could not otherwise be rewritten in terms of the generalised metric $\mathcal{H}$. If vertices contain no occurrences of $\partial_0 y$ and two derivatives of $\mathcal{H}$ or $\mathcal{V}$, the relevant contribution is

$$b_5 = \frac{i}{2}\langle \mathcal{B}_1^{[0]}\mathcal{B}_1^{[0]}\rangle\,. \tag{7.49}$$

If vertices contain $\partial_0 y$ once and three derivatives of $\mathcal{H}$ or $\mathcal{V}$, the relevant contributions are

$$b_6 = i\langle \mathcal{B}_1^{[1]}\mathcal{B}_2^{[0]}\rangle\,, \quad b_7 = -\frac{1}{2}\langle \mathcal{B}_1^{[1]}\mathcal{B}_1^{[0]}\mathcal{B}_1^{[0]}\rangle\,. \tag{7.50}$$

Note that $b_{5,6,7}$ do also source $(\partial_1 y)^2$ and $\partial_1 y \partial_0 y$ terms, which we will carry forward for inclusion in the relevant calculation later. The evaluation of these contractions is detailed in the appendix.

Terms with $\partial_0 y \partial_1 y$ legs are somewhat simple to study. Adopting our by now familiar approach, we single out in $\mathcal{L}_{\rm All}$ the relevant terms and name them $\mathcal{C}_i^{[n;\sigma^\mu]}$ where $[n;\sigma^\mu]$ denotes the number of occurrences of $\partial_\mu y$ and i the number of derivatives of background fields.The relevant combinations are

$$c_1 = -\frac{i}{2}\langle \mathcal{C}_2^{[2;\tau,\sigma]}\mathcal{C}_1^{[0]}\mathcal{C}_1^{[0]}\rangle\,, \quad c_2 = -\langle \mathcal{C}_2^{[2;\tau,\sigma]}\mathcal{C}_2^{[0]}\rangle\,, \quad c_3 = \frac{1}{2}\langle \mathcal{C}_1^{[1;\tau]}\mathcal{C}_1^{[1;\sigma]}\mathcal{C}_1^{[0]}\mathcal{C}_1^{[0]}\rangle\,, \tag{7.51}$$
$$c_4 = -i\langle \mathcal{C}_1^{[1;\tau]}\mathcal{C}_1^{[1;\sigma]}\mathcal{C}_2^{[0]}\rangle\,, \quad c_5 = -i\langle \mathcal{C}_2^{[1;\sigma]}\mathcal{C}_1^{[1;\tau]}\mathcal{C}_1^{[0]}\rangle\,, \quad c_6 = -\langle \mathcal{C}_3^{[1;\tau]}\mathcal{C}_1^{[1;\sigma]}\rangle\,. \tag{7.52}$$

These c's must be supplemented by some other contributions to get the full picture; in fact, as anticipated, integrals with external momentum insertion, such as the ones appearing in $b_{5,6,7}$, can give rise to terms with $\partial_0 y \partial_1 y$ legs. We shall deal with them explicitly in Appendix B.3.

Terms with $(\partial_1 y)^2$ legs are high in number and complexity when compared to those we just analysed. To deal with them (and the others, too) more efficiently we have created an appropriate `Mathematica` notebook[18]. We shall collect the results for all Wick contractions, including those on the fibre, in Appendix B.4. Albeit involved, they are obtained with minimal assumptions[19] and could be employed as the starting point for testing new methods for the evaluation of non-invariant integrals.

[18]We report additional details concerning the implementation in Appendix E.

[19]We have assumed in particular: momentum routing as per the figures in Appendix C.1, Taylor expansion of integrals involving external momentum insertion.

7.5 Evaluation of Integrals

We turn now to the evaluation of the momentum integrals for which we can follow two slightly different methods. Adopting dimensional regularisation, a critical decision is *when* in the calculation one assumes $d = 2+\epsilon$ or $d = 2$, and what cancellations are made before the evaluation of integrals:

- **Method 1:** We move immediately to $d = 2 + \epsilon$ and do not make any assumption on the relation between $p_0^2 - p_1^2$ and p^2 to combine integrals. Instead, we use arguments of Lorentz invariance to evaluate the myriad of tensor integrals $[[T(p_0, p_1, k_0, k_1)]]_{i,j,k}$ that are encountered.
- **Method 2:** We remain in $d = 2$ for as long as possible, and simplify combinations of integrals by replacing $p_0^2 = p^2 + p_1^2$, $k_0^2 = k^2 + k_1^2$ and $p_0 k_0 = p \cdot k + p_1 k_1$. The invariant combination are left to cancel against the denominators. Only once all such cancellations are made we continue to $d = 2 + \epsilon$. This method dramatically simplifies the situation as the calculation can be reduced to the evaluation of just four loop integrals.

7.5.1 Method 1

The first strategy we consider is to use $O(d)$ symmetry to relate the various integrals with non-scalar numerators in terms of a basis of scalar integrals, i.e. those of form $[[f(p^2, k^2, p \cdot k]]_{i,j,k}$, which can be easily evaluated in terms of the basic dimensional regularised integrals (in Minkowski space)

$$I_n = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{(p^2 - m^2)^n} \,. \tag{7.53}$$

In particular, the integrals we shall encouter are

$$I_1 = \mathbf{I} = \frac{i}{2\pi\epsilon} + \frac{i\bar{\gamma}}{4\pi} \,, \qquad m^2 I_2 = \frac{i}{4\pi} \,, \qquad m^4 I_3 = -\frac{i}{8\pi} \,, \tag{7.54}$$

where the expressions have been truncated to the relevant order. To achieve this we operate as follows: given a non-invariant integral of the form $[[p_0^{n_1} p_1^{n_2} k_0^{n_3} k_1^{n_4}]]_{i,j,k}$ for some $n_i \in \mathbb{N}$, we consider the associated integral where momenta are given d-dimensional Lorentz indices, $[[p_{\mu_1} \dots p_{\mu_{n_1+n_2}} k_{\nu_1} \dots k_{\nu_{n_3+n_4}}]]_{i,j,k}$. We then use $O(d)$ invariance to argue that the latter should equal a combination of Minkowski metrics multiplied by both a d-dependent finite factor and a scalar integral. Finally, we set the Lorentz indices so as to match our initial expression and recover the desired result. All of the relevant integrals are listed in Appendix C.1.

Albeit standard in QFT, this technique here necessarily involves explicitly the η_{11}, η_{00} and η_{01} components of the now d-dimensional worldsheet Minkowski metric. A prescription for these needs to be given and might in principle depend on ϵ. To keep track of this possibility we consider[20] $\eta_{00} = -\eta_{11} = 1 + \mathfrak{g}\epsilon$ for some $\mathfrak{g} \in \mathbb{R}$ (with higher orders in ϵ irrelevant to the two-loop calculation) and $\eta_{01} = \eta_{10} = 0$. A simple example encountered is

$$[[p_\mu p_\nu]]_{2,0,0} = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{p_\mu p_\nu}{(p^2 - m^2)^2} = \frac{\eta_{\mu\nu}}{d}(I_1 + m^2 I_2)\,. \tag{7.55}$$

Specialising the Lorentz indices this prescription gives

$$[[p_0 p_1]]_{2,0,0} = 0\,, \quad [[p_0 p_0]]_{2,0,0} = -[[p_1 p_1]]_{2,0,0} = \frac{1+\mathfrak{g}\epsilon}{2+\epsilon}(I_1 + m^2 I_2)\,. \tag{7.56}$$

One might wonder if there is some preferred value of $\mathfrak{g}$ required by consistency. A natural demand might be to set

$$[[p_0 p_0]]_{2,0,0} - [[p_1 p_1]]_{2,0,0} \equiv [[p_\mu p^\mu]]_{2,0,0} = (I_1 + m^2 I_2)\,, \tag{7.57}$$

which is achieved for $\mathfrak{g} = \frac{1}{2}$. However, consider now the "triangle" integral

$$I_\triangle = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{p_1^2(p_0^2 - p_1^2)}{(p^2 - m^2)^3} = -4\frac{(1+\mathfrak{g}\epsilon)^2}{d(d+2)}(I_1 + 2m^2 I_2 + m^4 I_3)\,. \tag{7.58}$$

In deriving this, we have prevented the numerator from cancelling against the denominator, thereby computing, in schematic form, $[[p_1^2 p_0^2]]_{3,0,0} - [[p_1^4]]_{3,0,0}$. On the other hand, if we now replace $p_0^2 - p_1^2 = p^2$ *prior* to integrating we get

$$I_\triangle = [[p_1^2]]_{2,0,0} + m^2[[p_1^2]]_{3,0,0} = -\frac{1+\mathfrak{g}\epsilon}{d}(I_1 + 2m^2 I_2 + m^4 I_3)\,. \tag{7.59}$$

These two results agree in their leading $\frac{1}{\epsilon}$ singularity but differ in the finite parts by $\frac{i(1-4\mathfrak{g})}{16\pi}$. This shows that there is no universal unambiguous choice for $\mathfrak{g}$. At one-loop this has no material impact on the β-functions, but at two-loops this ambiguity is dangerous because the $I_\triangle$ appears multiplied by a further $\frac{1}{\epsilon}$ (coming either from a counter-term insertion or from a factorised loop in a diagram). The prescription we follow in this Method 1 is to *not* combine explicit factors of $p_0^2 - p_1^2$ into p^2 prior to performing the integral[21].

[20] One can consider a more general choice where $\eta_{00} = 1 + \mathfrak{g}\epsilon$ and $\eta_{11} = -1 - \mathfrak{f}\epsilon$, however setting $\mathfrak{f} \neq \mathfrak{g}$ does not produce simplifications of the final result.

[21] With respect to the un-evaluated loop integrals reported in Appendix B.4, we should re-expand $p^2 = p_0^2 - p_1^2$ etc. and then apply the rules for Method 1.

When evaluating the counter-term contributions on the base, one additional technical difficulty is posed by integrals with non-invariant denominators, containing explicit components, e.g. p_1, in place of invariant combinations. These we tackle by means of a Schwinger parametrisation

$$\frac{1}{p_1} = \int_0^\infty \mathrm{d}u \, e^{-up_1} \, . \tag{7.60}$$

As described in detail in Appendix C.1.4, we then proceed formally by series expansion of this exponential to produce a sum of loop integrals with non-invariant numerators; each term can be recast, using the same $O(d)$ symmetry technique, as some invariant integral multiplied by a combinatorial (and d-dependent) factor. In the integrals encountered, we found that we could resum the series obtaining a hypergeometric function of the invariant combinations of momenta. The Schwinger parameter can then be integrated using standard identities for hypergeometric functions[22] to produce an expression for which the final loop momenta can then be integrated.

At two-loop order we expect divergences of the form $\frac{1}{\epsilon^2}$ and $\frac{1}{\epsilon}$. The latter contribute to the β-function whilst the former are constrained from the one-loop $\frac{1}{\epsilon}$ contribution by pole equations. Terms of $\frac{1}{\epsilon^2}$ can be sourced in one of two ways; either as a two-loop diagram giving a contribution I_1^2, or as a one-loop diagram giving a contribution I_1 with a counter-term insertion carrying a further $\frac{1}{\epsilon}$. Terms of $\frac{1}{\epsilon}$, instead, can arise in several ways:

(i) First, expanding I_1^2 and $\frac{1}{\epsilon}I_1$ produces sub-leading $\frac{1}{\epsilon}$ poles proportional to $\frac{\bar{\gamma}}{\epsilon}$. These we anticipate should cancel out, and indeed the correct counter-term Lagrangian should make this the case.

(ii) Second, we can find in a two-loop diagram a contribution proportional to either $m^2 I_1 I_2$ or $m^4 I_1 I_3$. The explicit mass that enters here as a result of the IR regulator cancels the same in the finite integrals I_2 and I_3. We anticipate that these divergences should also cancel as happens in the standard string β-function.

(iii) Finally we can have a pre-factor $f(d)I_1^2$ whose expansion results in a $\frac{1}{\epsilon}$ pole. These are the terms responsible for the β-function.

We will now collect and present the results for our calculation using Method 1. In doing so, let us recall that terms with a double ϵ-pole shall *not* depend on our prescription for computing integrals and are thus unambiguous.

[22]For instance

$$\int_0^\infty \mathrm{d}u \, u^{\alpha-1} \, {}_0\tilde{F}_1(b; -u) = \frac{\Gamma(\alpha)}{\Gamma(b-\alpha)} \, . \tag{7.61}$$

Let us start from the fibre. The ϵ^{-2} counter-term turns out to be

$$T_2^{(2)} = \frac{1}{32\pi^2\lambda^2}\left(\mathcal{H}^{(4)} + 4\mathcal{H}^{(3,1,0)} - 2\mathcal{H}^{(2,0,2)} - 6\mathcal{H}^{(2,1,1)} - 4\mathcal{H}^{(1,2,1)} + 3\mathcal{H}^{(1,1,1,1,0)}\right) + \frac{1}{128\pi^2\lambda^2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\left(\mathcal{H}^{(2)} + \mathcal{H}^{(1,1,0)}\right), \tag{7.62}$$

in exact agreement with the pole equation

$$0 = 2T_2^{(2)} - T_1^{(1)} \circ \frac{\delta}{\delta\mathcal{H}} T_1^{(1)} + \widetilde{T}_1^{(1)} T_1^{(1)}\,. \tag{7.63}$$

Regarding the single ϵ-pole, $\bar{\gamma}$'s cancel out, as they should, among different a's (equivalently: topologies). Contributions coming from IR regularisation - those involving $I_{2,3}$ - do not vanish on their own but can be removed with the addition of the appropriate mass term (7.27) to the Lagrangian. Crucially this term has a non-trivial expansion so that interaction vertices with mass insertions are produced at all orders. These are relevant, as they can be used to precisely cancel off against $I_{2,3}$-dependent terms produced in the calculation. More concretely, as we explain in Appendix B.1, four different topologies are involved: triangle envelope, square envelope, decorated loop and decorated triangle. Their contributions respectively evaluate to

$$\begin{aligned} m_1 &= \frac{m^2 I_1 I_2}{8\lambda^2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(2)}_{\bullet\bullet}\,, \qquad & m_2 &= \frac{I_1(m^2 I_2 + m^4 I_3)}{8\lambda^2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,1,0)}_{\bullet\bullet}\,, \\ m_3 &= -\frac{m^2 I_1 I_2}{16\lambda^2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(2)}_{\bullet\bullet}\,, \qquad & m_4 &= -\frac{I_1(m^2 I_2 + m^4 I_3)}{16\lambda^2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,1,0)}_{\bullet\bullet}\,. \end{aligned} \tag{7.64}$$

In summation these yield

$$\sum_{i=1}^{4} m_i = -\frac{1}{256\pi^2\epsilon\lambda^2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\left(\mathcal{H}^{(1,1,0)}_{\bullet\bullet} + 2\mathcal{H}^{(2)}_{\bullet\bullet}\right). \tag{7.65}$$

Including these, which precisely cancel all $m^2 I_1 I_2$ and $m^4 I_1 I_3$ contributions, we find the ϵ^{-1} counter-term

$$\begin{aligned} T_1^{(2)} = {} & \frac{2\mathfrak{g}-1}{32\pi^2\lambda^2}\mathcal{H}^{(2,0,2)} + \frac{4\mathfrak{g}-1}{16\pi^2\lambda^2}\mathcal{H}^{(2,1,1)} + \frac{3(8\mathfrak{g}-1)}{128\pi^2\lambda^2}\mathcal{H}^{(1,1,1,1,0)} \\ & + \frac{1-2\mathfrak{g}}{128\pi^2\lambda^2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(2)} + \frac{1-4\mathfrak{g}}{512\pi^2\lambda^2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,1,0)}\,. \end{aligned} \tag{7.66}$$

Let us consider the constraint that $T_1^{(2)}$ be compatible with the $O(n,n)$ structure. We find that the consistency condition of eq. (7.6) is not obeyed and instead:

$$T_1^{(2)}\eta^{-1}\mathcal{H} + \mathcal{H}\eta^{-1}T_1^{(2)} = \frac{4\mathfrak{g}-3}{256\pi^2\lambda^2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,1)} + \frac{13-40\mathfrak{g}}{64\pi^2\lambda^2}\mathcal{H}^{(1,1,1,1)}$$

$$+\frac{2\mathfrak{g}-1}{32\pi^2\lambda^2}\left(\mathcal{H}^{(2,0,2,0)}+\mathcal{H}^{(0,2,0,2)}\right)\,, \tag{7.67}$$

which does not vanish for any choice of $\mathfrak{g}$. On the base we find the ϵ^{-2} pole

$$\widetilde{T}_2^{(2)} = -\frac{1}{64\pi^2\lambda}\left(2\,\mathrm{Tr}(\mathcal{H}^{(3,1)})+\mathrm{Tr}(\mathcal{H}^{(2,2)})+\mathrm{Tr}(\mathcal{H}^{(1,1,1,1)})\right)\,, \tag{7.68}$$

in which it is notable that we could combine all terms containing the connection Ω to give a final answer in terms of the generalised metric alone. Moreover, we find that the ϵ^{-2} counter-term for $\partial_0 y\partial_0 y$ matches that of $\partial_1 y\partial_1 y$ (despite arising from a totally different set of diagrams and contractions) and that no $\epsilon^{-2}\partial_0 y\partial_1 y$ counter-term is produced.

Turning to the single ϵ-pole on the base, $\bar{\gamma}$'s cancel out. Contributions coming from IR regularisation cancel entirely. For the remaining ϵ^{-1} counter-terms, indicated with $\widetilde{T}_1^{(2)}|_{\mu\nu}\partial^\mu y\partial^\nu y$, we obtain

$$\widetilde{T}_1^{(2)}|_{01} = \frac{1-6\mathfrak{g}}{64\pi^2\lambda}\,\mathrm{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega) - \frac{\frac{16}{3}+4\mathfrak{g}}{64\pi^2\lambda}\,\mathrm{Tr}(\mathcal{H}^{(2)}\Omega\Omega)\,, \tag{7.69}$$

$$\begin{aligned}\widetilde{T}_1^{(2)}|_{00} &= \frac{1-8\mathfrak{g}}{128\pi^2\lambda}\,\mathrm{Tr}(\mathcal{H}^{(1,1,1,1)}) - \frac{1}{384\pi^2\lambda}(13-10\mathfrak{g})\,\mathrm{Tr}(\mathcal{H}^{(2,2)})\\ &+ \frac{35-8\mathfrak{g}}{384\pi^2\lambda}(\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega)-\mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega)) + \frac{1}{384\pi^2\lambda}(56+8\mathfrak{g})\,\mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega)\,,\end{aligned} \tag{7.70}$$

$$\begin{aligned}\widetilde{T}_1^{(2)}|_{11} &= \frac{-3+4\mathfrak{g}}{128\pi^2\lambda}\,\mathrm{Tr}(\mathcal{H}^{(1,1,1,1)}) - \frac{9-10\mathfrak{g}}{384\pi^2\lambda}\,\mathrm{Tr}(\mathcal{H}^{(2,2)})\\ &+ \frac{13+16\mathfrak{g}}{384\pi^2\lambda}\,\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega) - \frac{11-8\mathfrak{g}}{384\pi^2\lambda}\,\mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega) + \frac{5(1-\mathfrak{g})}{48\pi^2\lambda}\,\mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega)\,.\end{aligned} \tag{7.71}$$

It is clear that not only does the result depend on the connection Ω rather than $\mathcal{H}$ alone, there is no value for $\mathfrak{g}$ for which $\widetilde{T}_1^{(2)}|_{01}=0$ and $\widetilde{T}_1^{(2)}|_{00}=\widetilde{T}_1^{(2)}|_{11}$.

7.5.2 Method 2

In this method we shall perform the maximal simplifications that we can before actually evaluating any integral. We make four key assumptions:

1. Integrals are first dealt with in $d=2$. In particular, we replace factors of p_0^2 in the numerator of momentum integrals with $p^2+p_1^2$ and cancel off against factors of p^2 between numerator and the denominator[23].

[23] This step is potentially ambiguous as there can be multiple ways to implement such a simplification, e.g. in $k_0^3p_0^3$ we could extract either $(k\cdot p)^3$ or $k^2p^2k\cdot p$. However, at least at the two-loop order we are working to, no such possible ambiguity occurs.

2. After these cancellations have happened we continue the integral to $d = 2 + \epsilon$ dimensions and, in particular, we assume shift-symmetry in the momenta k and p.

3. IR regulating should be done at the end of such simplifications, and based on the experience of Method 1 when done successfully will not be important for the $\frac{1}{\epsilon}$ pole.

4. Having done these simplifications, we will assume that the integration method is such that any integrand whose numerator contains an odd number of temporal components of momenta vanish[24].

This methodology greatly assists in dealing with integrals that contain non-Lorentz invariant denominators. For instance, consider the following expression which is encountered in the computation

$$\mathcal{J} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{p_1}{k^2 p^2 (k_1 + p_1)} \,. \tag{7.72}$$

Suppose in the numerator we sum and subtract k_1:

$$\begin{aligned} \mathcal{J} &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{k_1 + p_1}{k^2 p^2 (k_1 + p_1)} - \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{k_1}{k^2 p^2 (k_1 + p_1)} \\ &= \mathbf{I}^2 - \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{k_1}{k^2 p^2 (k_1 + p_1)} = \mathbf{I}^2 - \mathcal{J} \,, \end{aligned} \tag{7.73}$$

where in the last step we used the fact that the denominator is invariant under the swapping of k and p. Hence, we see that the integral is easily solved as $\mathcal{J} = \frac{1}{2}\mathbf{I}^2$.

Further simplifications follow from the momenta shift-symmetry. Consider $[[(k_1+p_1)^2]]_{1,1,1}$; expanding the square and using the $k \leftrightarrow p$ symmetry of the integrand gives $[[(k_1 + p_1)^2]]_{1,1,1} = 2[[k_1^2]]_{1,1,1} + 2[[k_1 p_1]]_{1,1,1}$. On the other hand, shifting $k \to k - p$ followed by $p \to -p$ yields $[[(k_1 + p_1)^2]]_{1,1,1} = [[k_1^2]]_{1,1,1}$. Hence $[[k_1 p_1]]_{1,1,1} = -\frac{1}{2}[[k_1^2]]_{1,1,1}$.

With these rules implemented the entire two-loop contributions can be remarkably expressed in terms of only five (non-invariant) integrals:

$$\begin{aligned} &\mathbf{I} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2} \,, \quad \mathbf{L} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k_1^2}{(k^2)^2} \,, \qquad \mathbf{T} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k_1^4}{(k^2)^3} \,, \\ &\mathbf{S} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{k_1^2}{p^2 k^2 (k+p)^2} \,, \qquad \mathbf{TE} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{k_1 p_1^2 (k_1 + p_1)}{(p^2)^2 k^2 (k+p)^2} \,, \end{aligned} \tag{7.74}$$

corresponding respectively to the fundamental integral **I** and then integrals in the **L**oop, **T**riangle, **S**unset and **T**riangle **E**nvelope topologies. For each of these we denote by $\mathbf{X}_{(i)}$

[24] One could do away this restriction, however this will not effect the broad conclusions we reach as integrands with odd and even numbers of temporal components of momenta are associated to different tensorial combinations of the background fields. This is because a factor of $p_1 p_0$ can only arise in conjunction with an η and a factor of $p_1 p_1$ comes with an $\mathcal{H}$.

the $\frac{1}{\epsilon^i}$ contribution to the integral $\mathbf{X}$. The leading divergence and $\bar{\gamma}$-dependence of the remaining integrals is unambiguous, such that we may express

$$\begin{aligned}
&\mathbf{I} = \mathbf{P} + \frac{i\bar{\gamma}}{4\pi}\,, \qquad \mathbf{L} = -\frac{1}{2}\mathbf{P} - \frac{i\bar{\gamma}}{8\pi} + i\mathbf{L}_{(0)}\,, \qquad \mathbf{T} = \frac{3}{8}\mathbf{P} + \frac{3i\bar{\gamma}}{32\pi} + i\mathbf{T}_{(0)}\,, \\
&\mathbf{S} = -\frac{1}{2}\mathbf{P}^2 + \frac{\bar{\gamma}}{8\pi^2\epsilon} + \frac{\mathbf{S}_{(1)}}{\pi\epsilon}\,, \qquad \mathbf{TE} = \frac{1}{8}\mathbf{P}^2 - \frac{\bar{\gamma}}{32\pi^2\epsilon} + \frac{\mathbf{TE}_{(1)}}{\pi\epsilon}\,,
\end{aligned} \tag{7.75}$$

in which we recall $\mathbf{P} \equiv \frac{i}{2\pi\epsilon}$ and have introduced $\mathbf{L}_{(0)}, \mathbf{T}_{(0)}, \mathbf{S}_{(1)}$ and $\mathbf{TE}_{(1)}$ to signify the undetermined contributions from these integrals.

In general, two-loop diagrams which can be factorised into the product of one-loop diagrams do not lead to simple $\frac{1}{\epsilon}$ poles once the appropriate one-loop diagrams with counter-term insertions are subtracted off [162]. Here we see this through contributions of the form $(\mathbf{I}-\mathbf{P})\mathbf{L}$ where $\mathbf{P}$ comes from a counter-term insertion in the MS scheme; the simple pole part of $(\mathbf{I}-\mathbf{P})$ drops such that only a term proportional to $\frac{\bar{\gamma}}{\epsilon}$ is produced (such terms should cancel with an appropriate treatment of counter-terms). Using (7.75) we see similarly that the combination $(\mathbf{P}+\mathbf{L})\mathbf{L}$ has no $\bar{\gamma}$-independent $\frac{1}{\epsilon}$ contribution.

To expose the simplifications of this method one needs to organise the calculation by grouping all terms with the same tensorial structure. In general these occur from different Wick contractions and across different topologies. Let us highlight this method by examining the term $\mathcal{H}^{(1,1,1,1,0)}$ in more detail. The terms in a_6 require the most work and proceed as follows. The strategy is first remove all temporal components of momenta by replacing e.g. $k_0 p_0 = k\cdot p + k_1 p_1$, and then cancel numerators and denominators. For example

$$\begin{aligned}
a_{6_b} =& \frac{1}{2\lambda^2}\mathcal{H}^{(1,1,1,1,0)}_{\bullet\bullet}\left(-2[[k_1 p_1^3 k\cdot p]]_{3,1,1} + [[k_1^2 p_1^2 p^2]]_{3,1,1}\right) \\
=& \frac{1}{2\lambda^2}\mathcal{H}^{(1,1,1,1,0)}_{\bullet\bullet}\left(-[[k_1 p_1^3]]_{3,1,0} + [[k_1 p_1^3]]_{3,0,1} + [[k_1 p_1^3]]_{2,1,1} + [[k_1^2 p_1^2]]_{2,1,1}\right) \\
=& \frac{1}{2\lambda^2}\mathcal{H}^{(1,1,1,1,0)}_{\bullet\bullet}\left(-[[k_1 p_1^3]]_{3,1,0} + [[(k_1-p_1)p_1^3]]_{3,1,0} + [[k_1 p_1^3]]_{2,1,1} + [[k_1^2 p_1^2]]_{2,1,1}\right) \\
=& \frac{1}{2\lambda^2}\mathcal{H}^{(1,1,1,1,0)}_{\bullet\bullet}\left(\mathbf{TE} - \mathbf{I}\times\mathbf{T}\right)\,.
\end{aligned} \tag{7.76}$$

The contributions arising from the remaining Wick contractions relevant to $\mathcal{H}^{(1,1,1,1,0)}$ are summarised below.

	$\lambda^{-2}\mathcal{H}_{\bullet\bullet}^{(1,1,1,1,0)}$
a_{3_c}	$\frac{1}{2}\mathbf{P}\times\mathbf{L}$
a_{51_a}	$\mathbf{L}^2$
a_{51_b}	$\frac{1}{2}\mathbf{I}\times\mathbf{T}$
a_{51_e}	$\frac{1}{4}\mathbf{P}\times\mathbf{L}$
a_{6_b}	$\frac{1}{2}\left(\mathbf{TE}-\mathbf{I}\times\mathbf{T}\right)$
a_{6_c}	$-\mathbf{TE}$
Tot.	$\mathbf{L}^2-\frac{1}{2}\mathbf{TE}+\frac{3}{4}\mathbf{P}\times\mathbf{L}$

TABLE 7.1: Method 2 two-loop contributions to $\mathcal{H}_{\bullet\bullet}^{(1,1,1,1,0)}$.

The other tensorial structures can be treated in a similar fashion, both on the fibre and base manifold. We will refrain from detailing the discussion any further here and rather report our findings in tabular forms. The interested reader is referred to the appendices where the explicit steps for carrying out the calculation are shown.

Tensor	Result	$\frac{1}{64\pi\epsilon^2}$	$\frac{\bar{\gamma}}{4\pi\epsilon}$	$\frac{1}{4\pi\epsilon}$
$\mathcal{H}_{\bullet\bullet}^{(4)}$	$\frac{1}{16}\mathbf{I}^2-\frac{1}{8}\mathbf{IP}$	1	0	0
$\mathcal{H}_{\bullet\bullet}^{(3,1,0)}$	$-\frac{1}{2}\mathbf{IL}-\frac{1}{4}\mathbf{IP}+\frac{1}{2}\mathbf{PL}$	-4	0	0
$\mathcal{H}_{\bullet\bullet}^{(2,0,2)}$	$\frac{1}{4}\mathbf{IP}+\frac{1}{4}\mathbf{S}$	-2	0	$\mathbf{S}_1$
$\mathcal{H}_{\bullet\bullet}^{(2,1,1)}$	$\frac{1}{2}\mathbf{IP}-\frac{1}{2}\mathbf{PL}+\frac{1}{2}\mathbf{IL}+\frac{1}{4}\mathbf{S}$	-6	0	$\mathbf{S}_1$
$\mathcal{H}_{\bullet\bullet}^{(1,2,1)}$	$\frac{1}{8}\mathbf{IP}-\frac{3}{4}\mathbf{PL}-\frac{1}{2}\mathbf{L}^2+\frac{1}{4}\mathbf{IL}$	-4	0	0
$\mathcal{H}_{\bullet\bullet}^{(1,1,1,1,0)}$	$-\frac{1}{2}\mathbf{TE}+\mathbf{L}^2+\frac{3}{4}\mathbf{PL}$	3	0	$-\frac{1}{2}(-\mathbf{L_0}+4\mathbf{TE}_1)$
$\mathcal{H}_{\bullet\bullet}^{(2)}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})$	$-\frac{1}{32}\mathbf{IP}-\frac{1}{32}\mathbf{S}$	$\frac{1}{4}$	0	$-\frac{1}{8}\mathbf{S}_1$
$\mathcal{H}_{\bullet\bullet}^{(1,1,0)}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})$	$\frac{1}{16}\mathbf{PL}+\frac{1}{8}\mathbf{TE}$	$\frac{1}{4}$	0	$\frac{1}{8}(-\mathbf{L}_0+4\mathbf{TE}_1)$

TABLE 7.2: Two-loop contribution for each tensorial structure on the fibre.

Tensor	Result	$\frac{1}{64\pi\epsilon^2}$	$\frac{\bar{\gamma}}{4\pi\epsilon}$	$\frac{1}{2\pi\epsilon}$
$\mathrm{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega)$	$-\frac{1}{2}\mathbf{LP}-\frac{1}{2}\mathbf{LI}-\frac{1}{4}\mathbf{PI}-2\mathbf{TE}$	0	0	$\mathbf{L}_0-4\mathbf{TE}_1$
$\mathrm{Tr}(\mathcal{H}^{(2)}\Omega\Omega)$	$\frac{1}{4}(2\mathbf{L}-\mathbf{I})(\mathbf{I}-\mathbf{P})+4\mathbf{TI}+2\mathbf{S}-4\mathbf{TE}$	0	0	$-\frac{1}{4}\mathbf{S}_1-2\mathbf{TE}_1$

TABLE 7.3: Two-loop contribution for each tensorial structure on the base with external legs $\partial_0 y\partial_1 y$.

Tensor	Result	$\frac{1}{64\pi\epsilon^2}$	$\frac{\bar{\gamma}}{4\pi\epsilon}$	$\frac{1}{4\pi\epsilon}$
$\mathrm{Tr}(\mathcal{H}^{(3,1)})$	$\frac{3}{8}\mathbf{L}(\mathbf{I}-\mathbf{P})-\frac{1}{16}\mathbf{I}(\mathbf{P}-2\mathbf{I})$	-1	0	0
$\mathrm{Tr}(\mathcal{H}^{(2,2)})$	$-\frac{1}{16}\mathbf{S}-\frac{1}{2}\mathbf{TE}+\frac{1}{16}\mathbf{PI}$	$-\frac{1}{2}$	0	$-\frac{1}{4}\mathbf{S}_{(1)}-2\mathbf{TE}_{(1)}$
$\mathrm{Tr}(\mathcal{H}^{(1,1,1,1)})$	$\frac{1}{8}\mathbf{PI}+\frac{1}{8}\mathbf{L}(\mathbf{P}-\mathbf{L})+\frac{1}{8}\mathbf{S}+\frac{1}{2}\mathbf{TE}$	$-\frac{1}{2}$	0	$\frac{1}{2}(\mathbf{S}_{(1)}-\mathbf{L}_{(0)}+4\mathbf{TE}_{(1)})$
$\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega)$	$2\mathbf{LI}+2\mathbf{TI}+\frac{1}{4}\mathbf{I}^2$	0	0	$-4\mathbf{L}_{(0)}-4\mathbf{T}_{(0)}$
$\mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega)$	$\frac{1}{2}\mathbf{L}^2-\frac{3}{2}\mathbf{LI}-2\mathbf{TI}-\frac{1}{8}\mathbf{I}^2$	0	0	$4\mathbf{L}_{(0)}+4\mathbf{T}_{(0)}$
$\mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega)$	$\frac{5}{2}\mathbf{LI}+2\mathbf{TI}-\mathbf{LP}-\frac{1}{2}\mathbf{PI}+\frac{1}{2}\mathbf{I}^2$	0	0	$-3\mathbf{L}_{(0)}-4\mathbf{T}_{(0)}$

TABLE 7.4: Two-loop contribution for each tensorial structure on the base with external legs $\partial_1 y \partial_1 y$.

Tensor	Result	$\frac{1}{64\pi\epsilon^2}$	$\frac{\bar{\gamma}}{4\pi\epsilon}$	$\frac{1}{4\pi\epsilon}$
$\mathrm{Tr}(\mathcal{H}^{(3,1)})$	$\frac{1}{8}\mathbf{LP}-\frac{1}{8}\mathbf{LI}-\frac{1}{16}\mathbf{PI}$	1	0	0
$\mathrm{Tr}(\mathcal{H}^{(2,2)})$	$-\frac{3}{16}\mathbf{S}-\frac{1}{2}\mathbf{TE}-\frac{1}{16}\mathbf{PI}$	$\frac{1}{2}$	0	$-\frac{3}{4}\mathbf{S}_{(1)}-2\mathbf{TE}_{(1)}$
$\mathrm{Tr}(\mathcal{H}^{(1,1,1,1)})$	$\frac{1}{8}\mathbf{L}^2+\frac{1}{8}\mathbf{LP}$	$\frac{1}{2}$	0	0
$\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega)$	$\frac{1}{2}\mathbf{S}+\mathbf{LI}+2\mathbf{TI}$	0	0	$-2\mathbf{L}_{(0)}-4\mathbf{T}_{(0)}+2\mathbf{S}_{(1)}$
$\mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega)$	$-\frac{1}{2}\mathbf{L}^2-\frac{1}{2}\mathbf{LI}-2\mathbf{TI}-\frac{3}{4}\mathbf{S}+2\mathbf{TE}$	0	0	$4\mathbf{T}_{(0)}-3\mathbf{S}_{(1)}+8\mathbf{TE}_{(1)}$
$\mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega)$	$\frac{1}{2}\mathbf{LI}+2\mathbf{TI}+\mathbf{S}$	0	0	$-\mathbf{L}_{(0)}+4\mathbf{S}_{(1)}-4\mathbf{T}_{(0)}$

TABLE 7.5: Two-loop contribution for each tensorial structure on the base with external legs $\partial_0 y \partial_0 y$.

In summary we find the results for the simple ϵ-poles of counter-terms to be given by

$$\begin{aligned}\pi\lambda^2 T_1^{(2)} =& -\frac{1}{2}\mathbf{S}_{(1)}\left(\mathcal{H}^{(2,0,2)}+\mathcal{H}^{(2,1,1)}-\frac{1}{8}\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(2)}\right)\\ &+\left(\mathbf{TE}_{(1)}-\frac{1}{4}\mathbf{L}_{(0)}\right)\left(\mathcal{H}^{(1,1,1,1,0)}-\frac{1}{4}\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,1,0)}\right),\end{aligned}$$

$$\pi\lambda\widetilde{T}_1^{(2)}|_{01}= -2\left(\mathbf{TE}_{(1)}-\frac{1}{4}\mathbf{L}_{(0)}\right)\mathrm{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega)+2\left(\mathbf{S}_{(1)}-2\mathbf{TE}_{(1)}-\mathbf{T}_{(0)}\right)\mathrm{Tr}(\mathcal{H}^{(2)}\Omega\Omega),$$

$$\begin{aligned}\pi\lambda\widetilde{T}_1^{(2)}|_{11}=& -\left(\frac{1}{8}\mathbf{S}_{(1)}+\mathbf{TE}_{(1)}\right)\mathrm{Tr}(\mathcal{H}^{(2,2)})+\left(\frac{1}{4}\mathbf{S}_{(1)}+\mathbf{TE}_{(1)}-\frac{1}{4}\mathbf{L}_{(0)}\right)\mathrm{Tr}(\mathcal{H}^{(1,1,1,1)})\\ &-2\left(\mathbf{L}_{(0)}+\mathbf{T}_{(0)}\right)\left(\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega)-\mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega)\right)\\ &-\left(\frac{3}{2}\mathbf{L}_{(0)}+2\mathbf{T}_{(0)}\right)\mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega),\end{aligned}$$

$$
\begin{aligned}
\pi\lambda\widetilde{T}_1^{(2)}|_{00} =& \left(\frac{3}{8}\mathbf{S}_{(1)} + \mathbf{TE}_{(1)}\right)\mathrm{Tr}(\mathcal{H}^{(2,2)}) + \left(-\mathbf{S}_{(1)} + \mathbf{L}_{(0)} + 2\mathbf{T}_{(0)}\right)\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega) \\
&+ \left(\frac{3}{2}\mathbf{S}_{(1)} - 4\mathbf{TE}_{(1)} - 2\mathbf{T}_{(0)}\right)\mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega) \\
&+ \left(-2\mathbf{S}_{(1)} + \frac{1}{2}\mathbf{L}_{(0)} + 2\mathbf{T}_{(0)}\right)\mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega)\,.
\end{aligned}
$$

7.5.2.1 $O(n,n)$ Consistency Requirement

We can now return to question of compatibility of the fibre counter-term with the $O(n,n)$ structure. At one-loop, it follows immediately that $T_1^{(1)}$ satisfies the required condition $T_1^{(1)}\cdot\eta^{-1}\cdot\mathcal{H} + \mathcal{H}\cdot\eta^{-1}\cdot T_1^{(1)} = 0$. At two-loop order $T_2^{(2)}$, we recall, ought to obey $0 = T_2^{(2)}\eta^{-1}\mathcal{H} + \mathcal{H}\eta^{-1}T_2^{(2)} + T_1^{(1)}\eta^{-1}T_1^{(1)}$. There are four relevant independent tensors $X_a(\mathcal{H},\eta)$ with four derivatives and total homogeneity one that obey $\mathcal{H}\cdot\eta^{-1}\cdot X_a + X_a\cdot\eta^{-1}\cdot\mathcal{H} = 0$ given by

$$
\begin{aligned}
X_1 =& \mathcal{H}^{(4)} + 2(\mathcal{H}^{(3,1,0)} + \mathcal{H}^{(0,1,3)}) - 3\mathcal{H}^{(2,0,2)} + 6\mathcal{H}^{(1,1,0,1,1)}\,, \\
X_2 =& \mathcal{H}^{(2,1,1)} + \mathcal{H}^{(1,1,2)} + 2\mathcal{H}^{(1,1,0,1,1)}\,, \\
X_3 =& \mathcal{H}^{(1,2,1)} - \mathcal{H}^{(1,1,0,1,1)}\,, \\
X_4 =& \left(\mathcal{H}^{(2)} - \mathcal{H}^{(1,0,1)}\right)\mathrm{Tr}(\mathcal{H}^{(1,1)})\,.
\end{aligned}
\tag{7.77}
$$

It is useful to introduce these combinations in $T_2^{(2)}$ as they just drop when checking the compatibility condition:

$$
T_2^{(2)} = \frac{1}{128\pi^2\lambda^2}(4X_1 - 12X_2 - 16X_3 + X_4) + \frac{1}{32\pi^2\lambda^2}(\mathcal{H}^{(2,0,2)} - \mathcal{H}^{(1,1,0,1,1)})\,. \tag{7.78}
$$

The rest of the proof is easy and only involves simple identities to recast

$$
\mathcal{H}\cdot\eta^{-1}\cdot\mathcal{H}^{(2,0,2)} + \mathcal{H}^{(2,0,2)}\cdot\eta^{-1}\cdot\mathcal{H} = -2(\mathcal{H}^{(2,2)} + \mathcal{H}^{(1,1,0,2)} + \mathcal{H}^{(2,0,1,1)})\,. \tag{7.79}
$$

As for $T_1^{(2)}$, the tensor $\mathcal{H}^{(2,0,2)}$ enters into the result but there are no contributions of $\mathcal{H}^{(4)}$ and $\mathcal{H}^{(3,1,0)}$ that allow for its completion into X_1. As a result, to ensure $T_1^{(2)}$ is $O(n,n)$ compatible we are required to enforce $\mathbf{S}_{(1)} = 0$. This on its own is not an unexpected conclusion, in fact is the case if we use the Lorentz invariant regularisation scheme with $\mathfrak{g} = \frac{1}{2}$. However once $\mathbf{S}_{(1)} = 0$ is set, we are left with

$$
\pi\lambda^2 T_1^{(2)} \approx \left(\mathbf{TE}_{(1)} - \frac{1}{4}\mathbf{L}_{(0)}\right)\left(\mathcal{H}^{(1,1,1,1,0)} - \frac{1}{4}\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,1,0)}\right)\,, \tag{7.80}
$$

and again this is not $O(n,n)$ compatible unless $\mathbf{L}_{(0)} = 4\mathbf{TE}_{(1)}$. Unlike $\mathbf{S}_{(1)}$ it is impossible to tune $\mathfrak{g}$ within the Lorentz invariant regularisation scheme to make this combination vanish since $\mathbf{L}_{(0)} - 4\mathbf{TE}_{(1)} = \frac{1}{32\pi}$ is independent of $\mathfrak{g}$.

The conclusion of this analysis is that the only way the counter-term $T_1^{(2)}$ is compatible with the $O(n,n)$-structure (neglecting scheme changes, which we will discuss) is that the prescription for evaluating the integrals be such that $T_1^{(2)} \approx 0$ and hence $\beta^{\mathcal{H}}$ receives no contribution at two-loops.

7.5.2.2 Lorentz Consistency Requirement

Turning now to the base, we examine if restoration of Lorentz invariance is possible. To eliminate the mixed $\widetilde{T}_1^{(2)}|_{01}$ contribution we require again that $\mathbf{L}_{(0)} = 4\mathbf{TE}_{(1)}$ and additionally $\mathbf{T}_{(0)} = \mathbf{S}_{(1)} - 2\mathbf{TE}_{(1)}$. Notice also that $\mathrm{Tr}(\mathcal{H}^{(1,1,1,1)})$ enters in $\widetilde{T}_1^{(2)}|_{11}$ and not in $\widetilde{T}_1^{(2)}|_{00}$, so to eliminate this mismatch requires once again that $\mathbf{S}_{(1)} = 0$. Eliminating $\mathbf{S}_{(1)}$, $\mathbf{TE}_{(1)}$ and $\mathbf{T}_{(0)}$ in this way yields

$$\begin{aligned}\pi\widetilde{T}_1^{(2)}|_{11} &= -\frac{1}{4}\mathbf{L}_{(0)}\,\mathrm{Tr}(\mathcal{H}^{(2,2)}) - \mathbf{L}_{(0)}\left(\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega) - \mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega)\right) \\ &\quad - \frac{1}{2}\mathbf{L}_{(0)}\,\mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega)\,, \end{aligned} \tag{7.81}$$

$$\pi\widetilde{T}_1^{(2)}|_{00} = +\frac{1}{4}\mathbf{L}_{(0)}\,\mathrm{Tr}(\mathcal{H}^{(2,2)}) - \frac{1}{2}\mathbf{L}_{(0)}\,\mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega)\,. \tag{7.82}$$

Lorentz symmetry is restored only when also $\mathbf{L}_{(0)} = 0$ and the entire counter-term $\widetilde{T}_1^{(2)} = 0$.

7.5.2.3 Evaluation of Remaining Integrals

We can invoke the $O(d)$ Lorentz invariant integration prescription employed throughout Method 1 to now evaluate the remaining integrals to be:

$$\mathbf{TE}_{(1)} = -\frac{1}{6}\mathbf{T}_{(0)} = \frac{3-8\mathfrak{g}}{128\pi}\,, \quad \mathbf{S}_{(1)} = -\frac{1}{2}\mathbf{L}_{(0)} = \frac{2\mathfrak{g}-1}{16\pi}\,. \tag{7.83}$$

Using these values we can eliminate $\mathbf{TE}_{(1)}$ and $\mathbf{S}_{(1)}$ to give:

$$\begin{aligned}\pi\lambda^2 T_1^{(2)} &= \frac{1}{4}\mathbf{L}_{(0)}\left(\mathcal{H}^{(2,0,2)} + \mathcal{H}^{(2,1,1)} - \frac{1}{8}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(2)}\right) \\ &\quad - \left(\frac{1}{6}\mathbf{T}_{(0)} + \frac{1}{4}\mathbf{L}_{(0)}\right)\left(\mathcal{H}^{(1,1,1,1,0)} - \frac{1}{4}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,1,0)}\right)\,. \end{aligned} \tag{7.84}$$

On the base we find

$$\pi\widetilde{T}_1^{(2)}|_{01} = \left(\frac{1}{3}\mathbf{T}_{(0)} + \frac{1}{2}\mathbf{L}_{(0)}\right)\mathrm{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega) - \left(\mathbf{L}_{(0)} + \frac{4}{3}\mathbf{T}_{(0)}\right)\mathrm{Tr}(\mathcal{H}^{(2)}\Omega\Omega)\,, \quad (7.85)$$

$$\begin{aligned}\pi\widetilde{T}_1^{(2)}|_{11} = &\left(\frac{1}{16}\mathbf{L}_{(0)} + \frac{1}{6}\mathbf{T}_{(0)}\right)\mathrm{Tr}(\mathcal{H}^{(2,2)}) - \left(\frac{3}{8}\mathbf{L}_{(0)} + \frac{1}{6}\mathbf{T}_{(0)}\right)\mathrm{Tr}(\mathcal{H}^{(1,1,1,1)})\\ &- 2\left(\mathbf{L}_{(0)} + \mathbf{T}_{(0)}\right)\left(\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega) - \mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega)\right)\\ &- \left(\frac{3}{2}\mathbf{L}_{(0)} + 2\mathbf{T}_{(0)}\right)\mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega)\,, \end{aligned} \quad (7.86)$$

$$\begin{aligned}\pi\widetilde{T}_1^{(2)}|_{00} = &-\left(\frac{3}{16}\mathbf{L}_{(0)} + \frac{1}{6}\mathbf{T}_{(0)}\right)\mathrm{Tr}(\mathcal{H}^{(2,2)}) + \left(\frac{3}{2}\mathbf{L}_{(0)} + 2\mathbf{T}_{(0)}\right)\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega)\\ &- \left(\frac{3}{4}\mathbf{L}_{(0)} + \frac{4}{3}\mathbf{T}_{(0)}\right)\mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega)\\ &+ \left(\frac{3}{2}\mathbf{L}_{(0)} + 2\mathbf{T}_{(0)}\right)\mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega)\,. \end{aligned} \quad (7.87)$$

The $\mathfrak{g} = \frac{1}{2}$ prescription could now be adopted to further simplify the results by setting $\mathbf{L}_{(0)} = 0$. As a final simplification let us assume further that the background is such that the one-loop counter-terms all vanish, in which case

$$T_1^{(2)} \to -\frac{1}{6}\mathbf{T}_{(0)}\mathcal{H}^{(1,1,1,1,0)}\,, \quad (7.88)$$

$$\pi\widetilde{T}_1^{(2)}|_{01} \to \frac{5}{3}\mathbf{T}_{(0)}\,\mathrm{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega)\,, \quad (7.89)$$

$$\pi(\widetilde{T}_1^{(2)}|_{11} - \widetilde{T}_1^{(2)}|_{00}) \to \mathbf{T}_{(0)}\left(\frac{1}{6}\,\mathrm{Tr}(\mathcal{H}^{(1,1,1,1)}) - 4\,\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega) + \frac{10}{3}\,\mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega)\right)\,. \quad (7.90)$$

These final expressions demonstrate that even with such additional assumptions, the fibre counter-term remains incompatible with $O(n, n)$ structure and Lorentz invariance of the base counter-terms can not be enforced without placing constraints.

7.6 Couplings Reparametrisation

We have so far neglected the possibility of scheme changes and redefinitions that could possibly remedy the broken Lorentz-invariance and/or the $O(n, n)$ compatibility. We shall now expand on the discussion in the pre-print [3] and discuss this point more thoroughly.

Suppose a theory depends on some set of couplings φ^i, $i = 1, \ldots, N$ such that the β-function can be viewed as a vector $\beta = \beta^i \frac{\partial}{\partial\varphi^i}$ on the space with coordinates parametrised by φ's. Up to now we have worked in a minimal subtraction (MS) renormalisation scheme:

that is, we have only removed divergent parts and ignored finite pieces when renormalising the action. There are other possibilities, of course. Changing the scheme means adding *finite* local counter-terms and is equivalent to a redefinition of the couplings φ, sometimes called "coupling reparametrisation" (CR) [145]. For a multi-coupling theory, the β-functions are affected by a CR starting at two loops. Under $\varphi^i \to \varphi^i + \delta\varphi^i$, the change in the β-function β^i for the i-th φ is given by the Lie derivative

$$\delta\beta^i = \delta\varphi^j \frac{\partial}{\partial\varphi^j}\beta^i - \beta^j \frac{\partial}{\partial\varphi^j}\delta\varphi^i \,. \tag{7.91}$$

Momentarily reinstating the string tension, at two-loop the reparametrisation $\delta\varphi^i$ is order α' so that, according to (7.91), a change $\delta\beta^i_{(2)}$ in the two-loop β-function $\beta^i_{(2)}$ for the i-th coupling will be induced by the one-loop results $\beta^j_{(1)}$.

Now, the counter-terms of the previous sections indicate that, at two-loop, the $(\partial_0 y)^2$ and $(\partial_1 y)^2$ legs do *not* renormalise in the same way. This clashes with the fact that they share a common coupling λ: as a result, we see broken Lorentz invariance on the base upon renormalisation. This observation is additionally strengthened by the appearance of the non-vanishing $\widetilde{T}^{(2)}_1|_{01}$ counter-term. Let us then re-start from scratches and give up on Lorentz-covariance on the base from the outset by considering

$$\mathcal{L}_G = -\frac{1}{2}\partial_1\mathbb{X}^I\mathcal{H}_{IJ}\partial_1\mathbb{X}^J + \frac{1}{2}\partial_0\mathbb{X}^I\eta_{IJ}\partial_1\mathbb{X}^J + \frac{\lambda_{00}}{2}(\partial_0 y)^2 + \frac{\lambda_{01}}{2}\partial_0 y\partial_1 y - \frac{\lambda_{11}}{2}(\partial_1 y)^2 \,. \tag{7.92}$$

In $\mathcal{L}_G$, we can consider the three base-couplings as distinct objects and treat their renormalisations separately. If the three RG flows are eventually consistent, i.e. λ_{00} and λ_{11} receive the same quantum correction and λ_{01} receives none, we do take the limit $\lambda_{00} \to \lambda$, $\lambda_{11} \to \lambda$, $\lambda_{01} \to 0$. When no confusion can arise, we shall employ the shorthand

$$\mathbf{lim}_\lambda \equiv \lim_{\lambda_{00}\to\lambda}\lim_{\lambda_{11}\to\lambda}\lim_{\lambda_{01}\to 0} \qquad \text{so that e.g.} \qquad \mathcal{L} = \mathbf{lim}_\lambda \mathcal{L}_G \,. \tag{7.93}$$

With this choice we can fully leverage CR/finite counter-terms: within $\mathcal{L}$ we could only add a finite counter-term for $\mathcal{H}$ and/or λ. Now we can re-parametrise *separately* $\mathcal{H}$, λ_{00}, λ_{11} and λ_{01}: that is, we have twice as many degrees of freedom! Equivalently: for each coupling we can add different finite counter-terms that need not be related to one another.

The price to pay for this generalisation is the re-computation of one-loop β-functions for all couplings. For instance, the one-loop result (7.26) is really the $\mathbf{lim}_\lambda$ of the one we would obtain from the use of $\mathcal{L}_G$. The limiting procedure indeed washes away the precise dependence of, say, $\beta^{\mathcal{H}}_{(1)}$ on λ_{00}, λ_{11} and λ_{01}. However, this is crucial as we eventually need to calculate

$$\delta\beta^{\mathcal{H}}_{(2)} = \delta\mathcal{H}\frac{\partial}{\partial\mathcal{H}}\beta^{\mathcal{H}}_{(1)} + \delta\lambda_{00}\frac{\partial}{\partial\lambda_{00}}\beta^{\mathcal{H}}_{(1)} + \delta\lambda_{11}\frac{\partial}{\partial\lambda_{11}}\beta^{\mathcal{H}}_{(1)} + \delta\lambda_{01}\frac{\partial}{\partial\lambda_{01}}\beta^{\mathcal{H}}_{(1)} \\ - \beta^{\mathcal{H}}_{(1)}\frac{\partial}{\partial\mathcal{H}}\delta\mathcal{H} - \beta^{\lambda_{00}}_{(1)}\frac{\partial}{\partial\lambda_{00}}\delta\mathcal{H} - \beta^{\lambda_{11}}_{(1)}\frac{\partial}{\partial\lambda_{11}}\delta\mathcal{H} - \beta^{\lambda_{01}}_{(1)}\frac{\partial}{\partial\lambda_{01}}\delta\mathcal{H}\,. \tag{7.94}$$

In fact, for a two-loop calculation, the use of $\mathcal{L}_G$ (as opposed to the simpler $\mathcal{L}$) can be limited to the computation of one-loop β-functions. The reason is that, eventually, we will be interested in

$$\hat{\beta}^i_{(2)} = \mathbf{lim}_\lambda\left(\beta^i_{(2)} + \delta\beta^i_{(2)}\right)\,. \tag{7.95}$$

The derivatives in $\delta\beta^i$ as per (7.91) imply that we cannot exchange limit and reparametrisation. However, the polynomial base couplings and the particular limit we are considering imply that it is safe to directly use $\mathcal{L}$ to compute $\beta^i_{(2)}$.

7.6.1 One-Loop, Again

Singling out the precise dependence of the one-loop β-functions on the couplings is vital for determining the effects of scheme changes on the two-loop β-functions. Therefore, we need to re-perform the one-loop calculation, this time with $\mathcal{L}_G$, though. As all λ's are assumed constant, the effect of trading $\mathcal{L}$ for $\mathcal{L}_G$ is only to modify the propagator for the base fluctuation ζ. As no new interaction terms are created, the one-loop computation will be identical to the one performed with $\mathcal{L}$, at the price of changing some of the loop integrals. Most importantly, these modifications will *not* affect the renormalisation of the λ's. At one loop, the only possible source for ∂y legs of either types is the Weitzenböck connection Ω. Exploiting this fact, it is easy to see that only fibre fluctuations ξ can contribute to the production of two ∂y legs in the effective action. Thus, $\mathcal{L}_G$ will only modify the renormalisation of $\mathcal{H}$ (η is still protected by the general all-loop arguments that applied to $\mathcal{L}$). More concretely, we introduce a new notation for the ζ-propagator

$$\langle\zeta(\sigma_1)\zeta(\sigma_2)\rangle = \frac{1}{\lambda_{00}}\Xi(\sigma_1-\sigma_2)\,, \tag{7.96}$$

$$\Xi(\sigma_1-\sigma_2) = \int\frac{\mathrm{d}^2k}{(2\pi)^2}\,e^{-ik(\sigma_1-\sigma_2)}\,\frac{i}{k^2}\,\frac{1}{1+\frac{\Delta\lambda}{\lambda_{00}}\frac{k_1^2}{k^2}+\frac{\lambda_{01}}{\lambda_{00}}\frac{k_0k_1}{k^2}}\,, \tag{7.97}$$

where $\Delta\lambda = \lambda_{00}-\lambda_{11}$ and obviously $\mathbf{lim}_\lambda\Xi = \Delta$. The λ_{01}-dependent term in $\mathcal{L}_G$ implies that structures with different parity than the ones encountered thus far can appear in the RG flow. In fact, in Fourier space, λ_{01} adds to Ξ an explicit dependence on k_0 that was previously absent. To guarantee that this is paired up with another *odd* power of k_0 (so as to produce a non-vanishing loop integral) combinations of $\mathcal{H}$ and η other than the ones considered in the $\lambda_{01}=0$ case have to be considered.

Evidently, Ξ will make the evaluation of certain loop integrals much nastier, mostly due to the presence of λ_{01}. In fact, we can generically solve integrals coming from this propagator by Taylor-expanding the second fraction in Ξ (that depending on λ's), evaluating each integral with Method 1[25] and then re-summing the series. As we shall see in a moment, this is easily done when $\lambda_{01} = 0$. For non-vanishing λ_{01} we are unfortunately unable to obtain a closed expression for the re-summation. Nevertheless, we are only interested in the linear dependence of $\beta^{\mathcal{H}}_{(1)}$ on λ_{01}: this is the only order for which the combination of $\mathbf{lim}_\lambda$ and $\partial/\partial\lambda_{01}$ can produce a non-vanishing result. On top of that, when restricting to integrals with linear dependence on λ_{01}, we can in fact set $\lambda_{00} = \lambda_{11} = \lambda$ from the outset: possible deviations will be killed by $\mathbf{lim}_\lambda$.

Upon re-performing the entire calculation, it turns out that there are just two integrals that need evaluation. Following the guidelines of Method 1 we find

$$J_1 = \int \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{k_1^2}{(k^2)^2} \frac{1}{1 + \frac{\Delta\lambda}{\lambda_{00}} \frac{k_1^2}{k^2}} = \frac{1 - e^{\frac{\Delta\lambda}{2\lambda_{00}}}}{\frac{\Delta\lambda}{\lambda_{00}}} \mathbf{P} \,, \tag{7.98}$$

$$J_2 = \int \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{1}{k^2} \frac{1}{1 + \frac{\Delta\lambda}{\lambda_{00}} \frac{k_1^2}{k^2}} = e^{\frac{\Delta\lambda}{2\lambda_{00}}} \mathbf{P} \,. \tag{7.99}$$

Notice that, as it should, $\mathbf{lim}_\lambda J_1 = -\frac{1}{2}\mathbf{P}$ and $\mathbf{lim}_\lambda J_2 = \mathbf{P}$. We can now report the results for the one-loop calculation using $\mathcal{L}_G$ (where $O(\lambda_{01}^2)$ terms are being suppressed):

$$\begin{aligned} \beta^{\mathcal{H}}_{(1)} &= -\frac{e^{\frac{\Delta\lambda}{2\lambda_{00}}}}{4\pi\lambda_{00}} \mathcal{H}^{(2)} + \frac{1 - e^{\frac{\Delta\lambda}{2\lambda_{00}}}}{2\pi\Delta\lambda} \mathcal{H}^{(1,0,1)} - \frac{\lambda_{01}}{16\pi\lambda^2} \mathcal{H}^{(1,1)} + O(\lambda_{01}^2) \,, \\ \beta^{\lambda_{00}}_{(1)} &= \frac{1}{16\pi} \operatorname{Tr}(\mathcal{H}^{(1,1)}) = \beta^{\lambda_{11}}_{(1)} \,, \qquad \beta^{\lambda_{01}}_{(1)} = 0 \,. \end{aligned} \tag{7.100}$$

The RG flows of λ_{00} and λ_{11} are equal and unaffected by the more general action. This could be anticipated from the fact that (7.26) did not depend on λ in the first place. The new coupling λ_{01}, in particular, does not flow, making our $\mathbf{lim}_\lambda$ consistent at one-loop. However, the β-function for $\mathcal{H}$ depends on the different λ's (in the case of λ_{01} we have only kept the linear contribution, as explained before).

7.6.2 Scheme Choices

We are now in the position to explore the effect of CR on the two-loop β-functions. Let us start by considering $\delta\lambda_{00} = c_{00} \operatorname{Tr}(\mathcal{H}^{(1,1)})$ and $\delta\lambda_{11} = c_{11} \operatorname{Tr}(\mathcal{H}^{(1,1)})$. Plugging them

[25] Recall that, at one-loop, Method 1 is completely unambiguous and affords us with the possibility of using combinatorial arguments to easily evaluate complicated integrals.

into (7.94), using (7.100) and finally taking the limit we find

$$\mathbf{lim}_\lambda \delta\beta^{\mathcal{H}}_{(2)} = 2\frac{c_{00}+c_{11}}{16\pi\lambda^2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(2)} + \frac{3c_{00}+c_{11}}{16\pi\lambda^2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,0,1)}\,. \tag{7.101}$$

Recall that one of the issues with $O(n,n)$ compatibility was that terms with traces in $\beta^{\mathcal{H}}_{(2)}$, such as the ones above, did not have the same coefficient. Since in (7.101) we have two independent parameters we can always choose c_{00} and c_{11} so that in $\hat{\beta}^{\mathcal{H}}_{(2)} = \beta^{\mathcal{H}}_{(2)} + \delta\beta^{\mathcal{H}}_{(2)}$ the terms $\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(2)}$ and $\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,0,1)}$ share the same pre-factor. Notice how this is made possible by the different finite counter-terms for λ_{00} and λ_{11}: if they were equal, (7.101) would only depend on an effective parameter $c = c_{00} + c_{11}$ and we would not have enough freedom to fix both tensorial structures. Similarly, consider the $O(n,n)$-violating reparametrisation $\delta\mathcal{H} = A_1(\lambda_{00},\lambda_{11},\lambda_{01})\mathcal{H}^{(2)} + A_2(\lambda_{00},\lambda_{11},\lambda_{01})\mathcal{H}^{(1,0,1)}$, for two functions $A_{1,2}$ ot the λ-couplings. Given the one-loop result, these should be such that $\mathbf{lim}_\lambda A_{1,2} = \frac{a_{1,2}}{\lambda}$, for some *constant* coefficients $a_{1,2}$. we arrive at

$$\begin{aligned}\mathbf{lim}_\lambda \delta\beta^{\mathcal{H}}_{(2)} = &\frac{a_1-a_2}{2\pi\lambda^2}\left(\mathcal{H}^{(2,0,2)} + \mathcal{H}^{(2,1,1)} + \mathcal{H}^{(1,1,2)}\right)\\ &-\frac{1}{16\pi}\left(\mathbf{lim}_\lambda\frac{\partial A_1}{\partial\lambda_{00}} + \mathbf{lim}_\lambda\frac{\partial A_1}{\partial\lambda_{11}}\right)\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(2)}\\ &-\frac{1}{16\pi}\left(\mathbf{lim}_\lambda\frac{\partial A_2}{\partial\lambda_{00}} + \mathbf{lim}_\lambda\frac{\partial A_2}{\partial\lambda_{11}}\right)\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,0,1)}\,.\end{aligned} \tag{7.102}$$

The $\mathcal{H}^{(2,0,2)}$ structure was already identified as problematic for $O(n,n)$ compatibility. With this addition it is always possible to make its coefficient in $\hat{\beta}^{\mathcal{H}}_{(2)}$ vanish. Once again, observe how this is afforded by a reparametrisation that explicitly violates one of the symmetries we are eventually trying to restore. Also, λ_{01} does not enter here, as $\beta^{\lambda_{01}}_{(1)} = 0$.

A problem arises with the reparametrisation of λ_{01}, though. Given that $\widetilde{T}_1{}^{(2)}|_{01}$ explicitly depends on Ω, we are forced to assume that $\delta\lambda_{01}$ does too. The only term that might satisfy our needs is $\delta\lambda_{01} = c_{01}\,\mathrm{Tr}(\Omega\mathcal{H}^{(0)}\Omega)$. Indeed, this would result in

$$\mathbf{lim}_\lambda \delta\beta^{\lambda_{01}}_{(2)} = \frac{c_{01}}{2\pi\lambda}\left(\mathrm{Tr}(\mathcal{H}^{(2)}\Omega\Omega) - \mathrm{Tr}(\mathcal{H}^{(1,0,1)}\Omega\Omega)\right)\,. \tag{7.103}$$

Hence, *if* the integrals in $\widetilde{T}_1{}^{(2)}|_{01}$, upon evaluation, were to precisely give opposite coefficient to $\mathrm{Tr}(\mathcal{H}^{(2)}\Omega\Omega)$ and $\mathrm{Tr}(\mathcal{H}^{(1,0,1)}\Omega\Omega)$, we could always make $\widetilde{T}_1{}^{(2)}|_{01}$ vanish. However, a nasty consequence of this form for $\delta\lambda_{01}$ is that it will affect $\delta\beta^{\mathcal{H}}_{(2)}$, too. Concretely, this would yield, because of the form of $\beta^{\mathcal{H}}_{(1)}$,

$$\mathbf{lim}_\lambda \delta\beta^{\mathcal{H}}_{(2)} \supset \frac{c_{01}}{16\pi\lambda^2}\,\mathrm{Tr}(\Omega\mathcal{H}^{(0)}\Omega)\mathcal{H}^{(1,1)}\,. \tag{7.104}$$

Notice that i) any *non-vanishing* $\delta\lambda_{01}$ implies a contribution to $\delta\beta^{H}_{(2)}$ proportional to

$\mathcal{H}^{(1,1)}$: given the "parity" of $\mathcal{H}^{(1,1)}$, it can't be removed with other reparametrisations; ii) if we picked a precise evaluation of the integrals, we could make $\widetilde{T}_1{}^{(2)}|_{01}$ vanish from the outset, sparing us from the introduction of $\delta\lambda_{01}$.

7.7 Summary and Conclusions

In this chapter we have computed the two-loop effective action for the T-duality symmetric bosonic string.

As a first step we provided a complete calculation of all contributions to the effective action arising from Wick contraction keeping loop integrals unevaluated. We then employed two methods to simplify and evaluate these (non-Lorentz covariant) loop integrals. The two methods agree with each other for the ϵ^{-2} divergences, but are subtly different when it comes to the sub-leading ϵ^{-1} divergences that contribute to the β-functions at two-loop order.

Of the two approaches, Method 2, in which the maximal number of simplifications is performed in $d = 2$, results in compelling simplifications such that the results can be phrased in terms of a basis of just five independent integrals. We are then able to analyse the results in a way that keeps the choice of regularisation method implicit giving general conclusions that would hold with any choice of regularisation (dimensional or otherwise). For concreteness, here we completed Method 2 by employing continuation to $d = 2 + \epsilon$ after all simplifications have been made to evaluate the remaining integrals.

Both methods pass a number of important consistency checks:

- The ϵ^{-2} contributions are in exact accordance with the expectations from the pole equation on the doubled fibre in which the T-duality acts.
- The ϵ^{-2} contributions on the base are consistent with Lorentz invariance. This is to say, no $\partial_0 y \partial_1 y$ legs are produced (even though they do appear in intermediate steps) and the counter-terms for $\partial_0 y \partial_0 y$ and $\partial_1 y \partial_1 y$ coincide (even though they come from totally different sets of diagrams).
- For ϵ^{-2} poles, all occurrences of the Weitzenböck connection Ω combine in a fashion to be expressible in terms of $\mathcal{H}$ alone.
- After regularising IR divergence as described, mixed IR/UV divergences of the form $\frac{\log(m)}{\epsilon}$ are removed in the cancellation of $\bar{\gamma}$ terms.

- On the doubled fibre, possible contributions to the $\frac{1}{\epsilon}$ pole due to the IR mass regulator giving divergences of the form $m^2 I_1 I_2$ are cancelled with the introduction of an appropriate mass term and its background field expansion.

Notwithstanding the dramatic simplifications afforded by Method 2 compared to Method 1, the results for the ϵ^{-1} pole that contribute to the two-loop β-function present some puzzles:

- On the fibre, the counter-term does not have the right structure to allow the $O(n,n)$ constraint $\mathcal{H}\eta\mathcal{H} = \eta$ to be preserved by RG flow.
- The ϵ^{-1} pole on the base manifold is *not* Lorentz invariant. A new interaction vertex proportional to $\partial_0 y \partial_1 y$ is created, and the counter-terms for the legs $\partial_0 y \partial_0 y$ and $\partial_1 y \partial_1 y$ have differences.
- The ϵ^{-1} pole on the base manifold involves the connection Ω in a way that can not be combined into something expressible in terms of the generalised metric $\mathcal{H}$ alone.

Method 2 does afford one possible avenue to resolve these puzzles. Namely the possibility that an integration prescription can be invoked such that the entire two-loop ϵ^{-1} counter-terms vanish. This is the case if the undetermined subleading part of the integrals $\mathbf{L}, \mathbf{T}, \mathbf{S}, \mathbf{TE}$ of (7.75) vanish. Such a result would be equally surprising as it would be in contradiction to that of the conventional non-linear σ-model for the bosonic string.

More likely, appropriate finite Lorentz- and $O(n,n)$-violating one-loop counter-terms could be added so as to cure the pathological behaviours. To this end, it is important to enlarge the set of admissible finite counter-terms by considering a more general Lagrangian $\mathcal{L}_G$. However, their determination appears very difficult in practice and might also involve an explicit evaluation of the integrals to be carried out successfully.

The calculation involved in arriving at these results is of considerable complexity (especially with regards to the counter-terms on the base) and so we can't rule out that these issues pointed out here may be resolvable. As methods other than ours might in principle be considered, we have collected the relevant loop integrals, prior to any evaluation, in a way which is suitable to further investigation. Even though it is possible that different prescriptions might result in a non-vanishing β-function compatible with both $O(n,n)$- and Lorentz-symmetry, it seems likely that the resolution would be highly non-trivial[26]

[26] One might contend that an anomaly in double Lorentz transformations for the duality-symmetric string, [179], could play a role here. The Green-Schwarz mechanism required to cancel this would doubtless be important in the most general setting, however in the present "cosmological" set-up, there is no such anomaly to contend with, as the base manifold has only one dimension.

and would need to give a compelling non-ambiguous proposal for regulating the loop integrals involved.

As it stands, however, the results obtained cast some doubt as to the full validity of the doubled action in the form of (7.1) at the quantum level. At the very least one can say that the power of invoking manifest T-duality on the worldsheet is far outweighed by the added complexities that the chiral nature of this formalism entails at the quantum level.

Chapter 8

Towards Poisson-Lie T-Duality at Two-Loop

Abstract

We begin a study of the two-loop renormalisation of the action that makes Poisson-Lie T-duality manifest. Similarly to the Tseytlin string, the breakdown of explicit Lorentz covariance entails a number of technical difficulties which we try to address. We report partial results, as the project is ongoing at the time of writing.

8.1 Introduction

Klimčík and Ševera proposed in [14] a (worldsheet) action that describes two Poisson-Lie T-dual models on a Drinfel'd double $\mathbb{D}$. Similarly to the doubled string, the price for explicit T-duality covariance is the breakdown of Lorentz invariance. More concretely, their proposal is to extend the *chiral* WZW action by an $S_{\mathcal{H}}$ term, $S = S_{\text{WZW}} + S_{\mathcal{H}}$, being $\mathcal{H}$ the generalised metric in flat indices and

$$S_{\text{WZW}} = \frac{1}{2}\int_{\Sigma} \mathrm{d}^2\sigma \, \langle L_1|L_0\rangle + \frac{1}{12}\int_{\mathcal{M}_3} \langle L|[L,L]\rangle \,, \qquad S_{\mathcal{H}} = -\frac{1}{2}\int_{\Sigma} \mathrm{d}^2\sigma \, \langle L_1|\mathcal{H}|L_1\rangle \,. \quad (8.1)$$

Here Σ indicates the two-dimensional worldsheet with coordinates $\sigma^\mu = (\tau, \sigma)$, and $\mathcal{M}_3$ a three-dimensional manifold such that $\partial\mathcal{M}_3 \cong \Sigma$. L is used to indicate the Maurer-Cartan left-invariant one-form $L = g^{-1}\mathrm{d}g$, for $g \in \mathbb{D}$[1]. For generators $T_{\mathbb{A}}$ of $\mathfrak{d} = \text{Lie}(\mathbb{D})$, the pairing $\langle\cdot|\cdot\rangle$ is $\langle T_{\mathbb{A}}|T_{\mathbb{B}}\rangle = \eta_{\mathbb{A}\mathbb{B}}$.

[1] Accordingly, the right-invariant one-form is denoted with $R = \mathrm{d}g g^{-1}$.

This action offers a privileged standpoint when exploring the quantum corrections to models it encompasses. Indeed, we only have a single coupling $\mathcal{H}_{\mathbb{A}\mathbb{B}}$ to renormalise[2] and, furthermore, this is a constant object. In fact, the power of this formalism is that, whatever the result might eventually be, it needs to be a function of only $\mathcal{H}_{\mathbb{A}\mathbb{B}}$, $\eta_{\mathbb{A}\mathbb{B}}$ and the structure constants $F_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}}$ of $\mathfrak{d}$. The geometric intricacies due to e.g. a very complicated metric (think of $\mathbb{CP}^n_\eta$, for instance) are effectively stripped out of the picture, and one is left with a much neater renormalisation group flow in terms of *constant* and *algebraic* objects.

The one-loop renormalisation of (8.1) was first addressed in [148]. There it was shown that the β-function for $\mathcal{H}$ was given by

$$\beta^{\mathcal{H}}_{(1)} = \frac{1}{8\pi}(\eta^{\mathbb{A}\mathbb{C}}\eta^{\mathbb{B}\mathbb{D}} - \mathcal{H}^{\mathbb{A}\mathbb{C}}\mathcal{H}^{\mathbb{B}\mathbb{D}})(\eta^{\mathbb{E}\mathbb{F}}\eta^{\mathbb{G}\mathbb{H}} - \mathcal{H}^{\mathbb{E}\mathbb{F}}\mathcal{H}^{\mathbb{G}\mathbb{H}})F_{\mathbb{C}\mathbb{E}\mathbb{G}}F_{\mathbb{D}\mathbb{F}\mathbb{H}}\,. \tag{8.2}$$

From the target space perspective, i.e. without re-performing a quantum computation as explained in Chapter 6, this result was recovered in [157] and further extended to two loops. Aim of this chapter is to carry out a two-loop calculation using the action in (8.1) to obtain $\beta^{\mathcal{H}}_{(2)}$: ideally, that should also agree with the one found in [157].

Superficially, the computation resembles the one for the doubled string detailed in Chapter 7 but in fact there are a few differences worth highlighting. Since for $\mathbb{D}$ we have no fibration, we do not require to distinguish between base and fibre any more; this brings several advantages:

1. We only have one type of fluctuation, called ξ. As a consequence, there is no need to distinguish between base and fibre propagators.
2. There is no ambiguity on whether to treat ∂_μ as a two- or d-dimensional quantity when regularising.
3. There are no Lorentz-invariant combinations to reconstruct. Possibly the main issue of the doubled string calculation is avoided altogether from the outset.

Despite these remarkable perks, the absence of a base/fibre split has a major drawback: recall that, for the doubled string, the two-point function for mixed fluctuations ξ and ζ vanished, $\langle \xi\, \zeta \rangle = 0$. This fact, together with the structure of the expanded action, implied that a 10-point function[3] (schematically) factorised $\langle \xi^8\, \zeta^2 \rangle = \langle \xi^8 \rangle \langle \zeta^2 \rangle$, resulting in roughly order 1,500 terms. Conversely, for the $\mathcal{E}$-model action (8.1) we easily find terms of the form $\langle \xi^{10} \rangle$, contributing some 30,000 terms each. This added extra factor

[2] That is, assuming that $\eta_{\mathbb{A}\mathbb{B}}$ does not flow, as we expect.

[3] At two-loop order the 10-point function is, on dimensional grounds, the one containing maximum number of fluctuations. The absence of e.g. $\langle \xi^{10} \rangle$ depends on the details of the calculation.

of 20 makes the calculation much more computationally intensive with respect to that in Chapter 7.

8.2 Symmetries

Before delving into its details, let us take some time to discuss a few peculiarities of the calculation as, with respect to the doubled string, we have now at our disposal a much more rigid group structure due to the Drinfel'd double.

8.2.1 Constraints on Results

Even though (8.2) is by itself pretty elegant, its form can be made much more compelling, as first noticed by Klimčík in [180]. To this end introduce the projectors $\mathcal{P} = \frac{1}{2}(\eta + \mathcal{H})$ and $\overline{\mathcal{P}} = \frac{1}{2}(\eta - \mathcal{H})$ and view them as elements of the symmetric product $S^2\mathfrak{d}$ thanks to $\mathcal{P} = \mathcal{P}^{\mathbb{AB}}T_{\mathbb{A}} \otimes T_{\mathbb{B}}$ and similarly for $\overline{\mathcal{P}}$. It is possible to introduce a double bracket $[\![\cdot,\cdot]\!] : S^2\mathfrak{d} \times S^2\mathfrak{d} \to S^2\mathfrak{d}$ defined through

$$[\![T_{\mathbb{A}} \otimes T_{\mathbb{B}}, T_{\mathbb{C}} \otimes T_{\mathbb{D}}]\!] = [T_{\mathbb{A}}, T_{\mathbb{C}}] \otimes [T_{\mathbb{B}}, T_{\mathbb{D}}]\,, \tag{8.3}$$

where $[\cdot,\cdot]$ is the usual Lie bracket on $\mathfrak{d}$. Using this notation, the one-loop β-function reads

$$\beta^{\mathcal{H}}_{(1)} = \frac{1}{\pi}\Big(\mathcal{P}[\![\mathcal{P},\overline{\mathcal{P}}]\!]\overline{\mathcal{P}} + \overline{\mathcal{P}}[\![\mathcal{P},\overline{\mathcal{P}}]\!]\mathcal{P}\Big)\,. \tag{8.4}$$

This rewriting makes the underlying structure much more transparent, but also advises us against the use of η and $\mathcal{H}$ in formatting the final result. In fact, the adoption of projectors enables us to draw some conclusion as to the general shape of the (all-loop) answer.

Projectors need to obey their defining relations, namely $\mathcal{P}^2 = \mathcal{P}$, $\overline{\mathcal{P}}^2 = \overline{\mathcal{P}}$ and $\mathcal{P}\overline{\mathcal{P}} = 0$. Defining for the sake of simplicity the RG time $t = \log\mu$, taking a t-derivative of the first constraint yields

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t}\mathcal{P} + \mathcal{P}\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t}\,. \tag{8.5}$$

If we multiply the latter by $\mathcal{P}$ or $\overline{\mathcal{P}}$ on the left *and* right, we easily arrive at respectively

$$\mathcal{P}\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t}\mathcal{P} = 0\,, \qquad \overline{\mathcal{P}}\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t}\overline{\mathcal{P}} = 0\,. \tag{8.6}$$

Inspired by the one-loop result (8.4), it seems reasonable to assume that generically

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t} = \mathcal{P}F_1\overline{\mathcal{P}} + \overline{\mathcal{P}}F_2\mathcal{P}\,, \tag{8.7}$$

for some unknown tensors $F_{1,2}$. This would indeed guarantee that (8.6) be automatically satisfied. An identical result holds true for the renormalisation of $\overline{\mathcal{P}}$, for some possibly different tensors $F_{3,4}$. However, as $\eta = \mathcal{P} + \overline{\mathcal{P}}$ should not be renormalised, these new objects are in fact constrained by $\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t} = -\frac{\mathrm{d}\overline{\mathcal{P}}}{\mathrm{d}t}$. Finally, since the projectors are symmetric, we find $F_2^t = F_1$ and so

$$\frac{1}{2}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \mathcal{P}F_1\overline{\mathcal{P}} + \overline{\mathcal{P}}F_1^t\mathcal{P}\,. \tag{8.8}$$

Now, for two objects $\mathcal{O}_{1,2}$ it is easily proven that $[\![\mathcal{O}_1, \mathcal{O}_2]\!]^t = [\![\mathcal{O}_1^t, \mathcal{O}_2^t]\!]$: hence, we see that if these two coincide with projectors (which are symmetric by construction), the identification $F_1 = \frac{1}{2\pi}[\![\mathcal{P}, \overline{\mathcal{P}}]\!]$ indeed perfectly fits the discussion.

8.2.2 Graphical Representation

The fact that the one-loop β-function is completely specified by the double bracket $[\![\cdot,\cdot]\!]$ should not be too surprising after all. At one loop, we expect the result to depend on two structure constants[4] and, given (8.8), there are four indices left to be paired: this makes $[\![\cdot,\cdot]\!]$ the unique candidate. At higher orders, things start getting more convoluted as there are potentially more "pairings" than just $[\![\cdot,\cdot]\!]$ the β-function could be made up of. For instance, at two loops where four structure constants are needed, we could i) concatenate two double brackets $[\![\cdot,\cdot]\!][\![\cdot,\cdot]\!]$, or ii) compose them $[\![\cdot,[\![\cdot,\cdot]\!]]\!]$ or iii) create a completely new pairing which we will denote with $\{\cdot,\cdot,\cdot\}$.

This added complexity can be facilitated by the use of a graphic representation of the result, mostly following that introduced in [157] with few minor changes. A word of caution: even though these graphs look like proper Feynman diagrams, i.e. pictorial representations of a perturbative series, they have nothing to do with an actual quantum computation other than shedding some light into the structure of the perturbatively computed β-function. In this sense, we will see that proper Feynman diagrams will still be described, in our jargon, as "sunsets", "square envelopes", etc. Also, to further minimise any source of confusion, we will reserve the word "graphs" to indicate this pictorial description of the β-function.

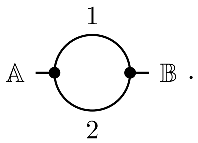
The idea is to associate to every tensor a graphical element. For instance, we shall adopt a solid line to represent $\mathcal{H}$ and a wiggly line to indicate η, treating $\mathcal{H}$ and η as "propagators", the extrema being labelled by their indices. Conversely, a structure constant is described by a three-vertex interaction. Graphically:

$F_{\mathbb{ABC}} =$, $\mathcal{H}^{\mathbb{AB}} =$, $\eta^{\mathbb{AB}} =$. (8.9)

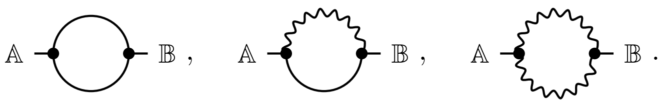
[4] In this algebraic setting, structure constants morally play the role of derivatives in the geometric approach. That is, the n-loop result should contain $2n$ structure constants/derivatives.

Even though we advocated for the usage of projectors as opposed to η and $\mathcal{H}$, we found this exceptionally convenient as a final polishing step. For example, we only used $\mathcal{P}$ and $\overline{\mathcal{P}}$ for rearranging our final expression (8.2) and not before. At higher loops there is even more compelling evidence for adopting this approach: contractions of structure constants through η are constrained by Jacobi identities that allow reshuffling and, possibly, simplifications. If we move too soon to projectors, we could simply overlook such identities, being them now expressed in a more convoluted and less transparent way.

Let us warm up exploring the one-loop renormalisation. Using a placeholder M for either η or $\mathcal{H}$, we consider the double bracket $[\![M_1, M_2]\!]_{\mathbb{AB}}$. According to our rules, its representation is

$$[\![M_1, M_2]\!]_{\mathbb{AB}} = F_{\mathbb{ACD}} F_{\mathbb{BEF}}\, M_1^{\mathbb{CE}} M_2^{\mathbb{DF}} =$$  . (8.10)

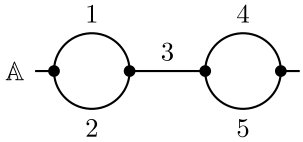
Since $[\![\cdot,\cdot]\!]$ is symmetric under the exchange of its arguments, three graphs will exhaust all possibilities corresponding to $M_1, M_2 \in \{\eta, \mathcal{H}\}$, namely

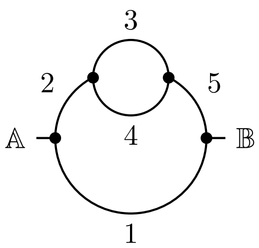
 (8.11)

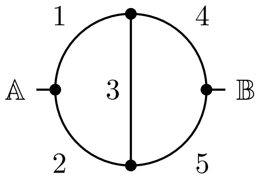
Were we to precisely reproduce $\beta^{\mathcal{H}}_{(1)}$, these pictures ought to be supplemented with the appropriate projectors attached to either the $\mathbb{A}$ or $\mathbb{B}$ endpoint. As anticipated, at two loops more possibilities have to be taken into account. The generic contribution to $\beta^{\mathcal{H}}_{(2)}$ at this order is schematically given by

$$L_{\mathbb{C}} M^{\mathbb{CA}} \left(F^4 M^5\right)_{\mathbb{AB}} M^{\mathbb{BD}} L_{\mathbb{D}}\,, \tag{8.12}$$

where once again $M \in \{\eta, \mathcal{H}\}$. There are only three possible connected graphs, corresponding to

 $$= [\![M_1, M_2]\!] M_3 [\![M_4, M_5]\!]_{\mathbb{AB}}\,, \tag{8.13a}$$

 $$= \left[\!\!\left[M_1, M_2 [\![M_3, M_4]\!] M_5\right]\!\!\right]_{\mathbb{AB}}\,, \tag{8.13b}$$

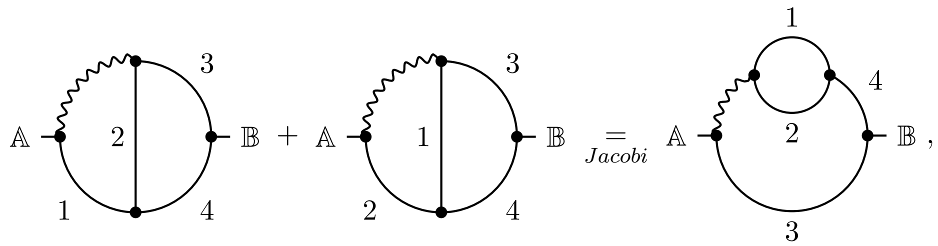


$$= \Big\{ M_1, M_2; M_3; M_4, M_5 \Big\}_{\mathbb{A}\mathbb{B}} \, . \tag{8.13c}$$

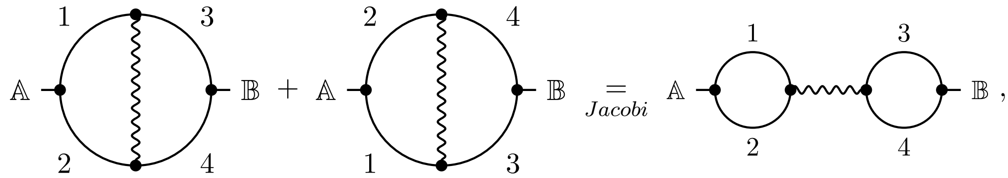
The tensorial expression associated to the last one reads

$$\begin{aligned} \Big\{ M_1, M_2; M_3; M_4, M_5 \Big\}_{\mathbb{A}\mathbb{B}} &= F_{\mathbb{A}\mathbb{A}_2\mathbb{A}_3} F_{\mathbb{B}\mathbb{B}_2\mathbb{B}_3} F_{\mathbb{C}_1\mathbb{C}_2\mathbb{C}_3} F_{\mathbb{D}_1\mathbb{D}_2\mathbb{D}_3} \\ &\times M_1^{\mathbb{A}_2\mathbb{C}_2} M_2^{\mathbb{A}_3\mathbb{D}_3} M_3^{\mathbb{C}_1\mathbb{D}_1} M_4^{\mathbb{B}_3\mathbb{C}_3} M_5^{\mathbb{B}_2\mathbb{D}_2} \, , \end{aligned} \tag{8.14}$$

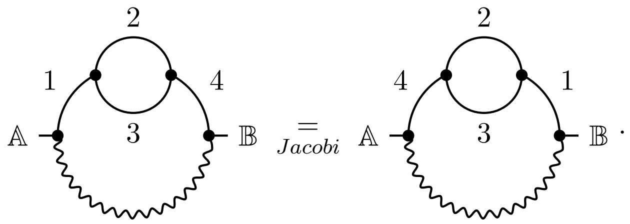
where we have added subscripts on indices for displaying purposes. Actually, these graphs are not independent as they can be related via the Jacobi identity, which is why we kept η and $\mathcal{H}$ (as opposed to projectors) in the first place. Specifically, the relations are



(8.15a)

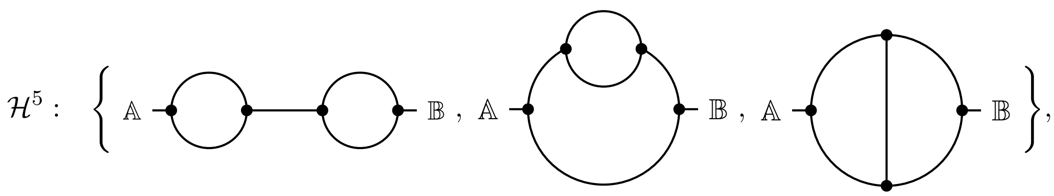


(8.15b)

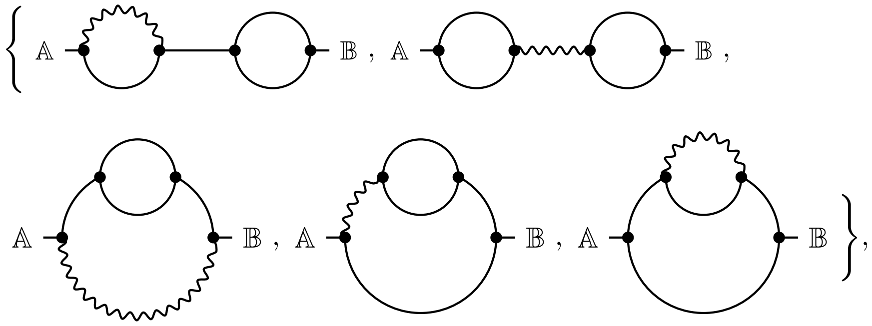


(8.15c)

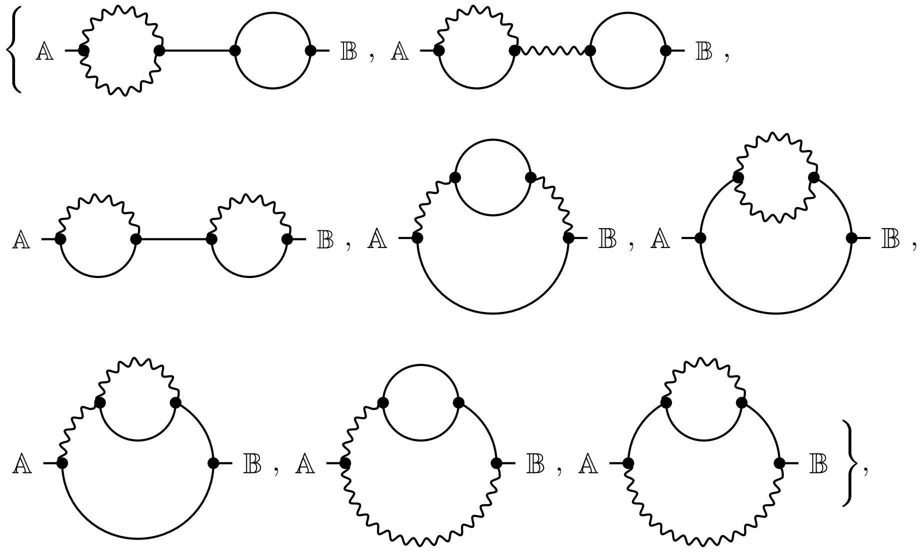
Using these identities, we can build a basis of all independent couplings of the form $\big(F^4 M^5\big)_{\mathbb{A}\mathbb{B}}$,



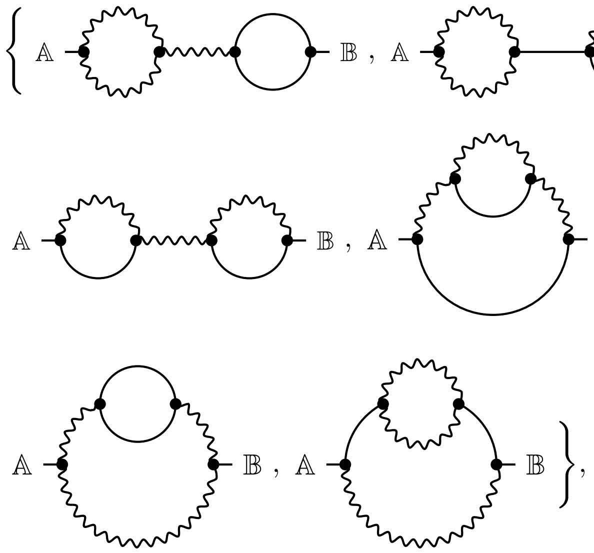
$\mathcal{H}^4\eta$:



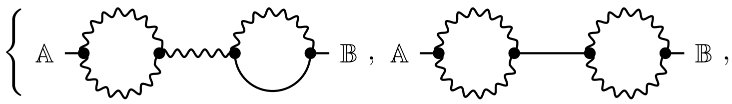
$\mathcal{H}^3\eta^2$:

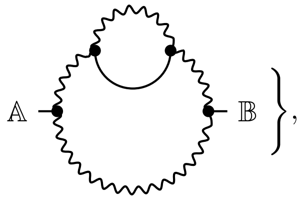


$\mathcal{H}^2\eta^3$:

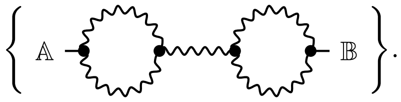


$\mathcal{H}\eta^4$:





η^5 :



Hence, we have a total of 26 possible contributions to $\beta^{\mathcal{H}}_{(2)}$. Notice, in particular, how the third topology in (8.13) only enters once, in the $\mathcal{H}^5$ coupling. In fact, it is possible to show that, although the result in [157] naively depends on this topology, the correct application of the identities (8.15) makes $\beta^{\mathcal{H}}_{(2)}$ boil down to an expression involving only (8.13a) and (8.13b), i.e. $[\![\cdot,\cdot]\!]$.

8.2.3 Diagrams with External Momentum Insertion

When renormalising the doubled string, a crucial aspect was not to miss out on Feynman diagrams involving the insertion of external momenta. Let us briefly recapitulate this point for the reader's convenience and then show how it does *not* apply to the case at hand.

The inverse string tension α' is a convenient loop counting parameter around which to organise the perturbative renormalisation of the (generalised) metric. Having dimensions of an area, the n-th loop contribution to β should comprise of $2n$ derivatives on purely dimensional grounds. For the two-loop renormalisation of the doubled string, we realised that if the required number of derivatives (that is, four) was already saturated by the tensorial expression associated to a particular loop integral, we could simply set the "external" momenta (i.e. those on which the tensors depend upon, in Fourier space) in the integral to zero. Hence, the resulting "zero-momentum" integral was much easier to compute depending only on two "internal" momenta (which we called p and k). Conversely, if the number of derivatives was *not* saturated (e.g. it was three) we had to retain some power of the external momentum in the integral (linear q, for three derivatives in the tensorial structure) and later convert it into a derivative via a Fourier transformation.

In principle, the same reasoning applies to the Poisson-Lie case. Here, the only objects derivatives can non-trivially act upon are left-invariant one-forms L, as the rest ($\mathcal{H}_{\mathbb{AB}}$, $\eta_{\mathbb{AB}}$ and $F_{\mathbb{ABC}}$) are all constants. Now, the β-function for $\mathcal{H}_{\mathbb{AB}}$ can be extrapolated from terms that, after the evaluation of integrals (possibly with external momentum insertion), have two occurrences of L_1. If, *prior* to integration, a term had no L's in its tensorial part, it would definitely be impossible to create two L_1's for the derivatives would act on constant objects. Conversely, if we had two or more left-invariant forms, integrals with momentum insertion would play no role either. Instead, expressions involving a single L_1 could benefit from momentum insertion as the Maurer-Cartan equation would allow for the creation of the missing L through $\mathrm{d}L - L \wedge L = 0$. Nevertheless, even if this was the case, the term would necessarily look like $\int \mathrm{d}^2\sigma(\partial L) \times \text{const}$, which is a total derivative. We conclude that integrals with momentum insertion are irrelevant for the renormalisation of $\mathcal{H}_{\mathbb{A}\mathbb{B}}$.

8.3 Expansion

Having elucidated a number of aspects that relied on the generic features of the theory rather than on the specifics of the calculation, let us now venture into more technical details. We shall start with the background field expansion of (8.1). To better leverage the group structure underpinning this action, we base our expansion on the decomposition

$$g = g_{\text{cl}}\,\Xi\,, \tag{8.16}$$

for a "classical" group element g_{cl}, and a quantum fluctuation $\Xi \equiv e^{\xi}$ for $\xi \in \mathfrak{d} = \text{Lie}(\mathbb{D})$. At the level of Maurer-Cartan forms, this amounts to

$$L = \text{Ad}_{\Xi}^{-1} L_{\text{cl}} + \mathscr{L}\,, \qquad \text{with} \qquad \mathscr{L} = \Xi^{-1}\mathrm{d}\Xi\,. \tag{8.17}$$

A similar notation is employed for right-invariant forms, e.g. $\mathscr{R} = \text{Ad}_{\Xi}\mathscr{L}$. The latter can be given an all-loop expansion in powers of the fluctuation ξ by means of an argument that dates back to Schur [181]. Let $\xi = \xi(\sigma^\mu)$ be a differentiable curve, $t \in \mathbb{R}$ a real parameter and $\Xi(t) = e^{t\xi}$ a one-parameter family of group elements. Defining the object

$$\mathscr{R}(t) := \mathrm{d}\Xi(t)\,\Xi(t)^{-1}\,, \qquad \text{with} \qquad \partial_t\mathscr{R}(t) = \text{Ad}_{\Xi(t)}\,\mathrm{d}\xi = e^{\text{ad}_{t\xi}}\,\mathrm{d}\xi\,, \tag{8.18}$$

it is obvious that we can write

$$\mathscr{R} = \mathscr{R}(1) = \int_0^1 \mathrm{d}t\,\partial_t\mathscr{R}(t) = \int_0^1 \mathrm{d}t\,e^{\text{ad}_{t\xi}}\,\mathrm{d}\xi = \int_0^1 \mathrm{d}t\,e^{t\text{ad}_{\xi}}\,\mathrm{d}\xi\,. \tag{8.19}$$

If we expand the exponential factor in the previous expression in t and perform the integration we get

$$\mathscr{R} = \sum_{j=0}^{\infty} \frac{1}{(j+1)!}\text{ad}_{\xi}^{j}\,\mathrm{d}\xi = \sum_{j=0}^{\infty} \frac{1}{(j+1)!}\mathrm{d}\xi^{\mathbb{A}}(\rho^j)_{A}{}^{\mathbb{B}}T_{\mathbb{B}}\,, \tag{8.20}$$

where, in the last step, we have introduced the matrix $\rho_A{}^B \equiv \xi^C F_{CA}{}^B$ to render the adjoint action explicit, $\mathrm{ad}_\xi T_{\mathbb{A}} = \rho_{\mathbb{A}}{}^{\mathbb{C}} T_{\mathbb{C}}$. A completely identical argument reveals that the expansion for the left-invariant form is given by

$$\mathscr{L} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)!} \mathrm{ad}_\xi^j \, \mathrm{d}\xi = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)!} \mathrm{d}\xi^{\mathbb{A}} (\rho^j)_{\mathbb{A}}{}^{\mathbb{B}} T_{\mathbb{B}} \,. \tag{8.21}$$

The expansion of the WZW term is best treated recalling the Polyakov-Wiegmann identity which, in the chiral case, reads

$$S_{\mathrm{WZW}}[g_{\mathrm{cl}}\, \Xi] = S_{\mathrm{WZW}}[g_{\mathrm{cl}}] + S_{\mathrm{WZW}}[\Xi] + \int_\Sigma \mathrm{d}^2\sigma \, \langle L_{\mathrm{cl}\,1} | \mathscr{R}_0 \rangle \,. \tag{8.22}$$

The first term is just the classical contribution while the last is immediately dealt with adopting (8.20). With this technique, and using the Cauchy product for series, the full expansion gives

$$\begin{aligned} S = \frac{1}{2} \int_\Sigma \mathrm{d}^2\sigma \Bigg[& L_{\mathrm{cl}\,1}^{\mathbb{A}} L_{\mathrm{cl}\,0\,\mathbb{A}} + 2\, L_{\mathrm{cl}\,1}^{\mathbb{A}} \partial_0 \xi^{\mathbb{B}} \sum_{k\geq 0} \frac{(-1)^k}{(k+1)!} \rho^k_{\mathbb{A}\mathbb{B}} + 2\, \partial_1 \xi^{\mathbb{A}} \partial_0 \xi^{\mathbb{B}} \sum_{k \geq 0} \frac{1}{(2k+2)!} \rho^{2k}_{\mathbb{A}\mathbb{B}} \\ & - L_{\mathrm{cl}\,1}^{\mathbb{A}} L_{\mathrm{cl}\,1}^{\mathbb{B}} \mathcal{H}^{\mathbb{C}\mathbb{D}} \sum_{k\geq 0} \sum_{m=0}^{k} \frac{(-1)^m}{m!\,(k-m)!} \rho^m_{\mathbb{A}\mathbb{C}}\, \rho^{k-m}_{\mathbb{D}\mathbb{B}} \\ & - 2\, L_{\mathrm{cl}\,1}^{\mathbb{A}} \partial_1 \xi^{\mathbb{B}} \mathcal{H}^{\mathbb{C}\mathbb{D}} \sum_{k\geq 0} \sum_{m=0}^{k} \frac{(-1)^m}{m!\,(k+1-m)!} \rho^m_{\mathbb{A}\mathbb{C}}\, \rho^{k-m}_{\mathbb{D}\mathbb{B}} \\ & - \partial_1 \xi^{\mathbb{A}} \partial_1 \xi^{\mathbb{B}} \mathcal{H}^{\mathbb{C}\mathbb{D}} \sum_{k\geq 0} \sum_{m=0}^{k} \frac{(-1)^m}{(m+1)!\,(k+1-m)!} \rho^m_{\mathbb{A}\mathbb{C}}\, \rho^{k-m}_{\mathbb{D}\mathbb{B}} \Bigg] \\ + \frac{1}{12} \int_{\mathcal{M}_3} \Bigg[& F_{\mathbb{A}\mathbb{B}\mathbb{C}}\, \mathscr{L}^{\mathbb{A}} \wedge \mathscr{L}^{\mathbb{B}} \wedge \mathscr{L}^{\mathbb{C}} + F_{\mathbb{A}\mathbb{B}\mathbb{C}}\, L_{\mathrm{cl}}{}^{\mathbb{A}} \wedge L_{\mathrm{cl}}{}^{\mathbb{B}} \wedge L_{\mathrm{cl}}{}^{\mathbb{C}} \Bigg]. \end{aligned} \tag{8.23}$$

Some comments are in order. Unlike the doubled string, there is no need for the introduction of (generalised) frame fields as the kinetic term turns out to be canonical. Specifically, we easily see that the two-point function evaluates to

$$\langle \xi^{\mathbb{A}}(\sigma) \xi^{\mathbb{B}}(\sigma') \rangle = \mathcal{H}^{\mathbb{A}\mathbb{B}} \Delta(\sigma - \sigma') + \eta^{\mathbb{A}\mathbb{B}} \theta(\sigma - \sigma') \tag{8.24}$$

where Δ and θ do coincide with those introduced in (7.12). Also, the expansion (8.23) still contains terms defined on the three-manifold $\mathcal{M}_3$ but, order by order in the fluctuations number, one can prove that Stokes theorem applies.

Notice how it is possible to conclude, in full generality, that $\eta_{\mathbb{A}\mathbb{B}}$ does not flow: the Polyakov-Wiegmann identity (8.22), together with the expansion of $S_{\mathcal{H}}$, simply forbid the presence of $L_{\mathrm{cl}0}$ external legs in the interacting Lagrangian.

8.4 Current Status

Using (8.23), the one-loop β-function (8.2) is easily recovered. As in the case of the doubled string of Chapter 7, the result is unambiguous, as we are only determining the leading divergence of the loop integrals involved in the computation. Also, were we to adopt Method 2, the integrals **I**, **L** and **T** would completely specify the outcome.

At higher order the situation worsens significantly. The explicit calculation shows that, within Method 2, numerous new integrals are created, at least in the intermediate steps of the calculation. Many of them eventually simplifies once all terms are taken into account and summed, but a few seem to survive. These are:

$$\begin{gathered}\mathbf{X} = -\frac{p_1^3}{p^4 k^2 (k_1 + p_1)}\,, \qquad \mathbf{X}_2 = \frac{p_1^4}{p^4 k^2 (k_1+p_1)^2}\,, \\ \mathbf{Y} = -\frac{p_1^5}{p^6 k^2 (k_1+p_1)}\,, \qquad \mathbf{DL}_1 = \frac{k_1^2}{p^2 k^2 (k_1+p_1)^2}\,, \\ \mathbf{TE}_2 = \frac{k_1 p_1^3}{p^4 k^2 (k+p)^2}\,. \end{gathered} \tag{8.25}$$

Actually, $\mathbf{TE}_2$ is not genuinely new as it is part of what we used to call **TE**. However, it is missing its partner that would made up **TE**. Let us briefly sketch how to obtain the leading ϵ^{-2} order of these.

Let us start from $\mathbf{TE}_2$. As it contains no spurious p_1 in the denominator, it can be immediately evaluated thanks to the rules in Appendix C. Dropping sub-leading parts here and henceforth, we find

$$\mathbf{TE}_2 = \frac{3}{64\pi^2\epsilon^2}\,. \tag{8.26}$$

The evaluation of the other integrals is much more subtle. It is based on the Schwinger trick

$$\frac{1}{A^n} = \frac{1}{(n-1)!}\int_0^\infty \mathrm{d}u\, u^{n-1} e^{-uA}\,. \tag{8.27}$$

Let us explain in detail what happens in the case of **X**: the others are completely identical in spirit. First notice that, up to the (k_1+p_1) term (to which we will apply the Schwinger trick) the integral is completely factorised between k and p integrals. Now there are two ways to proceed:

1. Newton's binomial

$$\begin{aligned}\frac{1}{k_1+p_1} &= \int_0^\infty \mathrm{d}u\, e^{-u(k_1+p_1)} = \int_0^\infty \mathrm{d}u \sum_{n=0}^\infty \frac{1}{n!} u^n (k_1+p_1)^n \\ &= \int_0^\infty \mathrm{d}u \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{1}{n!} u^n \binom{n}{m} k_1^m p_1^{n-m}\,. \end{aligned} \tag{8.28}$$

2. Separation of k and p in the exponential

$$\frac{1}{k_1+p_1} = \int_0^\infty \mathrm{d}u\, e^{-u(k_1+p_1)} = \int_0^\infty \mathrm{d}u\, e^{-uk_1} e^{-up_1}$$
$$= \int_0^\infty \mathrm{d}u \left(\sum_{n=0}^{\infty} \frac{1}{n!} u^n k_1^n\right)\left(\sum_{m=0}^{\infty} \frac{1}{m!} u^m p_1^m\right) . \tag{8.29}$$

In general, the two options are reconciled using Cauchy product formula. However, option 1 turns out more complicated in the practical goal of resumming series. Reinserting the integrals over momentum we see that

$$\mathbf{X} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d p}{(2\pi)^d} \int_0^\infty \mathrm{d}u \left(\frac{1}{k^2} \sum_{n=0}^{\infty} \frac{u^{2n} k_1^{2n}}{(2n)!}\right)\left(\frac{1}{p^4} \sum_{m=0}^{\infty} \frac{u^{2m+1} p_1^{2m+4}}{(2m+1)!}\right) . \tag{8.30}$$

We have used the fact the we only care about even powers of either k and p, and odd powers have been dropped accordingly. Now we employ the usual trick of Method 1: for each momentum integral we replace every combination of momenta with explicit indices with a scalar object multiplied by a combinatorial factor. As we will only deal with a bunch of either k_1 or p_1, we can simplify this approach down to

$$k_1^{2n} = \frac{(2n-1)!!}{2^{n-1} d \left(1+\frac{d}{2}\right)_{n-1}} (\eta_{11})^n (k^2)^n . \tag{8.31}$$

Doing so we find

$$\mathbf{X} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d p}{(2\pi)^d} \int_0^\infty \mathrm{d}u \left(\sum_{n=0}^{\infty} \frac{(2n-1)!!\,(\eta_{11})^n}{(2n)!\; 2^{n-1} d \left(1+\frac{d}{2}\right)_{n-1}} (k^2)^{n-1}\right)$$
$$\times \left(\sum_{m=0}^{\infty} \frac{(2m+3)!!\,(\eta_{11})^{m+2}}{(2m+1)!\; 2^{m+1} d \left(1+\frac{d}{2}\right)_{m+1}} (p^2)^m\right) . \tag{8.32}$$

At this point our usual approach is to resum the series and obtain hypergeometric functions. This works in this case, too.

$$\mathbf{X} = \frac{3\psi^2 \Gamma(d/2)^2}{2^{2+2d}\pi^{2d}} \int \mathrm{d}^d k\, \mathrm{d}^d p\, \frac{1}{k^2} \int_0^\infty \mathrm{d}u\, u\, {}_0\widetilde{F}_1\left(;2+\frac{d}{2};-\frac{1}{4}p^2u^2\psi\right) {}_0\widetilde{F}_1\left(;\frac{d}{2};-\frac{1}{4}k^2u^2\psi\right)$$
$$+ \frac{\psi^3 \Gamma(d/2)^2}{2^{3+2d}\pi^{2d}} \int \mathrm{d}^d k\, \mathrm{d}^d p\, \frac{p^2}{k^2} \int_0^\infty \mathrm{d}u\, u^3\, {}_0\widetilde{F}_1\left(;3+\frac{d}{2};-\frac{1}{4}p^2u^2\psi\right) {}_0\widetilde{F}_1\left(;\frac{d}{2};-\frac{1}{4}k^2u^2\psi\right) . \tag{8.33}$$

The problem with this result is that we can't really integrate over u now: there are no rules, to the best of our knowledge, for performing the integral of the product of two hypergeometric functions. Actually, there are identities for turning the product of two into a single hypergeometric, but that requires a particular fine tuning between the parameters and variables which we don't have here. Hence, we adopt a different approach. First, as everything depends on either p^2 or k^2, we move to polar coordinates $\mathrm{d}^d p \to \Omega_d \mathrm{d}p p^{d-1}$, where Ω_d is the volume of the d-dimensional sphere. Then we redefine the Schwinger parameter $v = u^2$ and introduce new momentum variables

$$K = \frac{1}{4}k^2 v\psi \,, \qquad P = \frac{1}{4}p^2 v\psi \,. \tag{8.34}$$

We end up with

$$\begin{aligned}\mathbf{X} = &\frac{\Gamma(d/2)^2\Omega_d^2\psi^{3-d}}{64\pi^{2d}}\int_0^\infty \mathrm{d}v\, v^{1-d}\int_0^\infty \mathrm{d}K\, K^{d/2-2}{}_0\widetilde{F}_1\left(;\frac{d}{2};-K\right)\\ &\times\left[\frac{3}{2}\int_0^\infty \mathrm{d}P\, P^{d/2-1}{}_0\widetilde{F}_1\left(;2+\frac{d}{2};-P\right)-\int_0^\infty \mathrm{d}P\, P^{d/2}{}_0\widetilde{F}_1\left(;3+\frac{d}{2};-P\right)\right] \,.\end{aligned} \tag{8.35}$$

With respect to the doubled string, we now want to integrate over K and P first and over v last. Integrals are carried out with the usual identities (C.34) and

$$\Omega_d = \frac{\pi^{d/2}}{\Gamma(1+d/2)} \,, \tag{8.36}$$

and we end up with

$$\mathbf{X} = \frac{(3-d)(d-2)\psi^{3-d}}{64 d^2 \pi^d}\Gamma\left(\frac{d}{2}-1\right)^2\int_0^\infty \mathrm{d}v\, v^{1-d} \,. \tag{8.37}$$

Notice that the gamma function would produce a double ϵ pole, but it is reduced to a single by the $(d-2)$ factor. The missing ϵ-pole sits in the v-integral. The problem is that, if we are integrating from $v = 0$ we necessarily incur in an additional divergence which can *not* be regulated with ϵ. To cope with this, suppose our integration ranges from some small $a > 0$ to infinity. Then

$$\int_a^\infty \mathrm{d}v\, v^{1-d} = \frac{a^{2-d}}{d-2} = \frac{1}{\epsilon} - \log(a) + O(\epsilon) \,. \tag{8.38}$$

This has precisely the form of an IR regularised integral, and we shall interpret a as a sort of mass. In any case, the leading pole is insensitive to this choice. Hence we find

$$\mathbf{X} = \frac{1}{64\pi^2\epsilon^2} \,. \tag{8.39}$$

Replicating this construction for all cases we find the following results at leading order

$$\begin{gathered}\mathbf{X} = \frac{1}{64\pi^2\epsilon^2} \,, \qquad \mathbf{X}_2 = -\frac{1}{64\pi^2\epsilon^2} \,,\\ \mathbf{Y} = -\frac{3}{256\pi^2\epsilon^2} \,, \qquad \mathbf{DL}_1 = \frac{1}{32\pi^2\epsilon^2} \,,\\ \mathbf{TE}_2 = \frac{3}{64\pi^2\epsilon^2} \,.\end{gathered} \tag{8.40}$$

These can be used to extract the $T_2^{(2)}$ counter-term for $\mathcal{H}$. From the pole equation, it is possible to check that the expected answer is

$$\begin{aligned} T^{(2)}_{2\,\mathbb{A}\mathbb{B}} = \frac{1}{2\pi^2}\Big(& -\mathcal{P}_{\mathbb{A}}{}^{\mathbb{C}}\mathcal{P}_{\mathbb{B}}{}^{\mathbb{C}}\,[\![\mathcal{P},\overline{\mathcal{P}}]\!]\overline{\mathcal{P}}[\![\mathcal{P},\overline{\mathcal{P}}]\!]_{\mathbb{C}\mathbb{D}} + \overline{\mathcal{P}}_{\mathbb{A}}{}^{\mathbb{C}}\overline{\mathcal{P}}_{\mathbb{B}}{}^{\mathbb{C}}\,[\![\mathcal{P},\overline{\mathcal{P}}]\!]\mathcal{P}[\![\mathcal{P},\overline{\mathcal{P}}]\!]_{\mathbb{C}\mathbb{D}} \\ & + \mathcal{P}_{\mathbb{A}}{}^{\mathbb{C}}\overline{\mathcal{P}}_{\mathbb{B}}{}^{\mathbb{C}}[\![\mathcal{P},\overline{\mathcal{P}}]\!]\overline{\mathcal{P}}[\![\mathcal{P},\overline{\mathcal{P}}]\!]_{\mathbb{C}\mathbb{D}} - \mathcal{P}_{\mathbb{A}}{}^{\mathbb{C}}\overline{\mathcal{P}}_{\mathbb{B}}{}^{\mathbb{C}}[\![\mathcal{P},\overline{\mathcal{P}}]\!]\mathcal{P}[\![\mathcal{P},\overline{\mathcal{P}}]\!]_{\mathbb{C}\mathbb{D}} \\ & - \mathcal{P}_{\mathbb{A}}{}^{\mathbb{C}}\overline{\mathcal{P}}_{\mathbb{B}}{}^{\mathbb{C}}\Big[\!\Big[\mathcal{P},\mathcal{P}[\![\mathcal{P},\overline{\mathcal{P}}]\!]\overline{\mathcal{P}}\Big]\!\Big]_{\mathbb{C}\mathbb{D}} - \mathcal{P}_{\mathbb{A}}{}^{\mathbb{C}}\overline{\mathcal{P}}_{\mathbb{B}}{}^{\mathbb{C}}\Big[\!\Big[\mathcal{P}-\overline{\mathcal{P}},\overline{\mathcal{P}}[\![\mathcal{P},\overline{\mathcal{P}}]\!]\mathcal{P}\Big]\!\Big]_{\mathbb{C}\mathbb{D}} \\ & + \mathcal{P}_{\mathbb{A}}{}^{\mathbb{C}}\overline{\mathcal{P}}_{\mathbb{B}}{}^{\mathbb{C}}\Big[\!\Big[\overline{\mathcal{P}},\mathcal{P}[\![\mathcal{P},\overline{\mathcal{P}}]\!]\overline{\mathcal{P}}\Big]\!\Big]_{\mathbb{C}\mathbb{D}} + \mathcal{P}_{\mathbb{A}}{}^{\mathbb{C}}\overline{\mathcal{P}}_{\mathbb{B}}{}^{\mathbb{C}}\Big[\!\Big[\overline{\mathcal{P}},\overline{\mathcal{P}}[\![\mathcal{P},\overline{\mathcal{P}}]\!]\mathcal{P}\Big]\!\Big]_{\mathbb{C}\mathbb{D}}\Big)\,. \end{aligned} \tag{8.41}$$

We still have not managed to reproduce it but, given the complexity of the calculation, this counter-term should represent a very strong indication towards the robustness of our implementation.

8.5 Conclusions

In this chapter we have carried out a preliminary analysis of two-loop quantum corrections to Poisson-Lie T-duality in the form of an $\mathcal{E}$-model. This project is currently under investigation together with C. Eloy and D.C. Thompson, and the results are necessarily partial and inconclusive at this stage. Nevertheless, it is possible to draw some conclusions from the experience we have gained so far.

In this scenario, the absence of a fibration similar to that of Chapter 7 is a double-edged sword: if some theoretical issues are more easily addressed (e.g. no ambiguity in deciding how to treat covariant Lorentz indices and no loop integrals with momentum insertion), the computation is rendered much more intensive due to an abundance of terms. Nonetheless, we should be able to overcome this additional difficulty through the use of a sufficiently powerful computer and the optimisation of the `Mathematica` code used for the doubled string.

The one-loop result can be recovered with little effort and the procedure is unambiguous and conceptually identical to that of Chapter 7. At two-loop, however, when adopting Method 2 we find that new integrals other than the ones previously encountered need to be considered. Some of these, whilst appearing in intermediate steps of the calculation, do end up summing to zero in the final result. This could be an indication that, once again, a limited subset of integrals shall determine the two-loop contributions.

Chapter 9

Epilogue

As we are on the verge of a wrap up, it seems appropriate to loosen it up a little and adopt first person singular here and henceforth. Conclusions should not just be a summary of results − each chapter has a dedicated section for that − but a broader discussion as to the current status of the field, what we have and have not achieved and which questions I would like to know the answer to.

Nunc Fluens Facit Tempus, Nunc Stans Facit Aeternitatem

People working at the interface of Duality and Integrability have gone a long way during the years of my Ph.D. course. It seems reasonable to start from the trilogy of Costello, Yamazaki and Witten [182–184], offering a radically new perspective on integrable models via four-dimensional Chern-Simons theory. Building on this, Delduc, Lacroix, Magro and Vicedo [185, 186] were able to embed many integrable σ-models − including (but not limited to) Yang-Baxter and λ-deformations − as well as re-interpret Poisson-Lie T-duality in this framework. Additionally, some of these authors, together with collaborators, pointed out the relevance of affine Gaudin models for our comprehension of two-dimensional integrability [66, 187, 188]. Broadly speaking, the twist function has emerged as a central actor out of which new models can be easily built and quantum corrections explored [67, 68].

In fact, (higher) loop calculations for integrable deformations have seen an impressive surge of interest. The first project I was assigned as a Ph.D. student was what appears here as Chapter 8. Back then only a handful of people (if any) were working on quantum corrections. As years passed, numerous papers have appeared. Hoare, Levine and Tseytlin have studied to great depth the interplay of integrability and RG flow [189–193]; Hassler and Rochais addressed the α'-corrections to Poisson-Lie T-duality from the DFT perspective [157, 194]; Pulman, Ševera and Youmans explored the one-loop renormalisation of $\mathcal{E}$-models using Chern-Simons theory [195], just to name a few.

On the duality side, a lot of effort has gone into finding an extension of U-duality similar to Poisson-Lie T-duality. This line of enquiry resulted in what is now called an exceptional Drinfel'd algebra [196–199]. Since then, various articles have stemmed to provide a mathematical formulation in terms of algebroids[1] [202, 203] and applications in Supergravity [204, 205]. As this field is extremely recent, I expect many more interesting works to appear soon, possibly merging Poisson-Lie U- and T-duality in a unified framework, along the lines of [202].

It's All About Geometry

Having sketched the evolution the field has gone through in the last four years, I will now take the chance to analyse what I would hope to see achieved in the future. The exposition will clearly suffer from my own personal bias, and should not be regarded, by any means, as a list of topics sorted by relevance.

In Chapter 5 we made some progress towards a deeper understanding of the geometry underpinning Yang-Baxter deformations. Poisson structure(s), as expected, do play a pivotal role in determining the metric and B-field of the deformed manifold. And yet, I feel like we are missing the bigger picture. Supergravity solutions for the NS-NS sector do require additional information such as dilaton and fluxes which, to the best of our knowledge, simply cannot be extrapolated from a Poisson structure, but demand an algebraic (and cumbersome) approach to be flashed out. While this definitely works, I find it unsatisfactory.

My suggestion is that it should be possible to extract the RR sector (which is somehow related to fermions) from a Poisson superstructure Π of a projectable Poisson supermanifold [206]. In this case, Π would contain the "bosonic" Poisson structure π as well as other supertensors that likely describe the missing part of the geometry. With additional effort, it would be extremely interesting to inspect how Supergeometry interacts with Generalised Geometry, along the lines of [126]. This would have a twofold purpose: first, to elucidate the general structure of SUGRA solutions involving Yang-Baxter deformed spacetimes highlighting what the algebraic approach overshadows; second, to provide a handle for generalisations of the AdS/CFT correspondence. Unless we have the gravitational (i.e. geometrical) side fully under control, I find it unlikely that we will

[1] In fact, Poisson-Lie T-duality was given a very rigorous formulation in terms of exact Courant algebroids since its very early days [200, 201].

be able to probe the dual gauge theory to a great extent. Unfortunately, I did not have the time to explore this topic adequately.

Speaking of Supersymmetry, it is intriguing to understand if $\mathbb{CP}^n_\eta$ can be re-obtained as a gauged linear σ-model (GLSM). The theory of toric varieties as GSLMs has been well expounded in [207] in the context of Mirror Symmetry. It is in general possible to recover a NLSM with target manifold M as the low-energy limit of a GLSM with moduli space described by M. In particular, it is possible to apply this procedure to "squashed" toric manifolds [208, 209]: for example, even though the squashed $\mathbb{CP}^1$ does naively resemble $\mathbb{CP}^1_\eta$, they are definitely not the same. Finding the right superpotential necessary for eventually obtaining $\mathbb{CP}^n_\eta$ seems in general quite hard. Perhaps, it would be preferable to attack the problem using gauged Gross-Neveu models, as recently proposed in [210, 211]. Either ways, one final goal is to exploit supersymmetric localisation techniques to obtain exact results, e.g. the partition function of a deformed model.

To close off the discussion on possible research avenues in the geometry realm, let me also point out the geometric quantisation of $\mathcal{E}$-models, as well as the study of Poisson models. The former shall put $\mathcal{E}$-models on solid mathematical foundations, and its exploration lead to a deeper understanding of Poisson-Lie T-duality as a whole. In the light of [212] it should also be possible to use the power of QP manifolds and the Batalin-Vilkovisky formalism to further address additional quantum properties. Finally, one could try and use all of the above, together with the AKSZ construction of topological field theories (see [213] for a recent review), to study Poisson σ-models and their interplay with Poisson-Lie groups, as initiated in [214].

Silicon Loops

It should be clear from the discussion in Chapters 7 and 8 that the determination of higher-loop quantum corrections in a T-duality covariant formalism is extremely involved. Loop integrals are in general ambiguous or, at least, there is no settled method for computing them beyond the leading ϵ-divergence; similarly, it is not clear which renormalisation scheme is to be preferred. Practically speaking, these are very serious drawbacks as they pose a particularly hard challenge for those who want to probe the quantum structure of (generalised) T-dualities. Unless more robust procedures are found, it seems implausible that three, or even higher, loop calculations can be carried out without gigantic effort. Nonetheless, I am somewhat confident that the renormalisation group flow for Poisson-Lie T-duality on Drinfel'd doubles can be given, at least in the form of a conjecture, an all-loop expression. The constraints imposed by the underlying structure are quite stringent, and point towards the use of the double bracket $[\![\cdot,\cdot]\!]$ as the elementary building block.

Irrespectively of the success of this research, I fear the power afforded by computers is not being fully leveraged or, at least, not in a homogeneous way. If probably every theoretician has some degree of familiarity with `Mathematica`, the absence of a shared and multi-purpose toolkit − perhaps spun out of a collective project − is quite remarkable. There are certainly various (unofficial) packages for performing General Relativity or Standard Model calculations, but a single framework that can handle both the diversity of mathematical notions (tensors, differential forms, spinors, etc.) as well as the physical theories is missing. This necessity became evident to me while developing the notebook for the computations in Chapter 7: most of the tools were already present, but scattered and not capable of performing quantum computations right off the bat. With minor modifications, the notebook can correctly compute the two-loop β-function for $\lambda\phi^4$ theory and also deliver results for the $\mathcal{E}$-model of Chapter 8. These are completely different-looking theories, signalling that a generalisation of the code to encompass most of the commonly studied theories should be within reach. This putative software would then take care of most heavy-lifting duties in the common practice of Theoretical Physics and, besides, provide a common ground where to test calculations and reproduce results.

Appendix A

Conventions

A.1 Indices

Even though we try to minimise the displaying of indices, there will still be an abundance of those. Somewhat unconventionally, we adopt small double-stroked letters to indicate elements of a generic Lie algebra $\mathfrak{g}$ as well as coordinates on the corresponding group G. This choice serves the purpose of reserving greek and latin letters for subgroups and coset space, respectively. As long as algebraic objects are involved we will stick to the following conventions[1]:

	$\mathfrak{g}$	$\tilde{\mathfrak{g}}$	$\mathfrak{h}$	$\mathfrak{m}$	$\mathfrak{d}$	$\mathfrak{p}$	$\mathfrak{q}$
Generators	$T_{\mathbb{a}}$	$T^{\mathbb{a}}$	T_α	T_a	$T_{\mathbb{A}}$	T^α	T_A
Structure Constants	$f_{\mathbb{a}\mathbb{b}}{}^{\mathbb{c}}$	$\tilde{f}^{\mathbb{a}\mathbb{b}}{}_{\mathbb{c}}$	$f_{\alpha\beta}{}^\gamma$	$f_{ab}{}^{\mathbb{c}}$	$F_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}}$	$F^{\alpha\beta}{}_{\mathbb{C}}$	$F_{AB}{}^{\mathbb{C}}$
Exponentiation	G	$\widetilde{G}$	H	G/H	$\mathbb{D}$	P	Q

TABLE A.1: Conventions for algebraic objects.

On the (co)tangent bundle of groups and manifolds, we will employ again a similar idea, this time with letters from the second part of the alphabet. As the local frame from a group coincides with its algebra we opt for a notation similar to that used before:

	G	$\widetilde{G}$	$\mathbb{D}$	E	T^d	$\widetilde{T}^d$	B	$\mathbb{E}$	T^{2d}
Coordinates	$x^{\mathbb{i}}$	$\tilde{x}_{\mathbb{i}}$	$\mathbb{X}^{\mathbb{I}}$	x^μ	$x^{\mathbb{i}}$	$\tilde{x}_{\mathbb{i}}$	y	$\mathbb{X}^I$	$\mathbb{X}^{\mathbb{I}}$
Local (Flat) Frame	$v^{\mathbb{a}}$	$v_{\mathbb{a}}$	$V^{\mathbb{A}}$					V^A	$V^{\mathbb{A}}$

TABLE A.2: Conventions for geometric objects.

[1]Some of the entities indicated here will be given proper meaning during the relevant chapter.

Appendix B

Wick Contractions

B.1 Fibre Wick Contractions

We report here Wick contractions which are relevant for the two-loop computation on the fibre. We use conventions as explained in the main text, namely we set

$$a_1 = \langle \mathcal{A}^{[2]} \rangle \,, \quad a_2 = i \langle \mathcal{A}^{[0]} \mathcal{A}^{[2]} \rangle \,, \quad a_3 = \frac{i}{2} \langle \mathcal{A}^{[1]} \mathcal{A}^{[1]} \rangle \,,$$
$$a_4 = -\frac{1}{2} \langle \mathcal{A}^{[0]} \mathcal{A}^{[0]} \mathcal{A}^{[2]} \rangle \,, \quad a_5 = -\frac{1}{2} \langle \mathcal{A}^{[0]} \mathcal{A}^{[1]} \mathcal{A}^{[1]} \rangle \,, \quad a_6 = -\frac{i}{4} \langle \mathcal{A}^{[0]} \mathcal{A}^{[0]} \mathcal{A}^{[1]} \mathcal{A}^{[1]} \rangle \tag{B.1}$$

with

$$\mathcal{A}_2^{[0]} = -\frac{1}{32\pi\epsilon} \mathrm{Tr}(\mathcal{H}^{(1,1)}) \partial_\mu \zeta \partial^\mu \zeta - \frac{1}{8\pi\epsilon\lambda} (\mathcal{H}^{(2)}_{AB} - \mathcal{H}^{(1,0,1)}_{AB}) \partial_1 \xi^A \partial_1 \xi^B - \frac{1}{4} \mathcal{H}^{(2)}_{AB} \zeta^2 \partial_1 \xi^A \partial_1 \xi^B \tag{B.2}$$

$$\mathcal{A}_1^{[0]} = -\frac{1}{2} \mathcal{H}^{(1)}_{AB} \zeta \partial_1 \xi^A \partial_1 \xi^B \,, \tag{B.3}$$

$$\mathcal{A}_1^{[1]} = -\mathcal{H}^{(1)}_{A\bullet} \zeta \partial_1 \xi^A \,, \tag{B.4}$$

$$\mathcal{A}_2^{[1]} = -\frac{1}{2} \mathcal{H}^{(2)}_{A\bullet} \zeta^2 \partial_1 \xi^A \,, \tag{B.5}$$

$$\mathcal{A}_3^{[1]} = \frac{1}{4\pi\epsilon\lambda} \left(-\mathcal{H}^{(3)} + \mathcal{H}^{(1,1,1)} + \mathcal{H}^{(1,0,2)} + \mathcal{H}^{(2,0,1)} \right)_{A\bullet} \zeta \partial_1 \xi^A - \frac{1}{6} \mathcal{H}^{(3)}_{A\bullet} \zeta^3 \partial_1 \xi^A \,, \tag{B.6}$$

$$\mathcal{A}_2^{[2]} = -\frac{1}{4} \mathcal{H}^{(2)}_{\bullet\bullet} \zeta^2 \,, \tag{B.7}$$

$$\mathcal{A}_3^{[2]} = -\frac{1}{12} \mathcal{H}^{(3)}_{\bullet\bullet} \zeta^3 \,, \tag{B.8}$$

$$\mathcal{A}_4^{[2]} = \frac{1}{16\pi\epsilon\lambda} \left(-\mathcal{H}^{(4)} + \mathcal{H}^{(1,2,1)} + 4\mathcal{H}^{(1,1,2)} + 2\mathcal{H}^{(2,0,2)} + 2\mathcal{H}^{(1,0,3)} \right)_{\bullet\bullet} \zeta^2 - \frac{1}{48} \mathcal{H}^{(4)}_{\bullet\bullet} \zeta^4 \,. \tag{B.9}$$

Owing to the fact that $\mathcal{H}$ is an involution, and thus $\dot{\mathcal{H}} \cdot \mathcal{H} = -\mathcal{H} \cdot \dot{\mathcal{H}}$, at two-loop order a basis for the relevant independent tensors without traces is

$$\mathcal{H}^{(4)}_{\bullet\bullet}\,, \quad \mathcal{H}^{(3,1,0)}_{\bullet\bullet}\,, \quad \mathcal{H}^{(2,0,2)}_{\bullet\bullet}\,, \quad \mathcal{H}^{(2,1,1)}_{\bullet\bullet}\,, \quad \mathcal{H}^{(1,2,1)}_{\bullet\bullet}\,, \quad \mathcal{H}^{(1,1,1,1,0)}_{\bullet\bullet}\,, \tag{B.10}$$

which can be extended by the ones with trace

$$\mathcal{H}^{(2)}_{\bullet\bullet}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\,, \quad \mathcal{H}^{(1,1,0)}_{\bullet\bullet}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\,. \tag{B.11}$$

We first extract the coefficients of this basis in terms of the unevaluated tensorial integrals $[[f(p_0, p_1, q_0, q_1)]]_{i,j,k}$ which can be evaluated using the Method 1 rules. We then use Method 2 rules to present a final answer in the $\mathbf{I}, \mathbf{L}, \mathbf{T}, \mathbf{S}, \mathbf{TE}$ basis of integrals. When dealing with counter-term insertions we adopt the shorthands

$$X = \frac{1}{8\pi\epsilon\lambda}(\mathcal{H}^{(2)} + \mathcal{H}^{(1,1,0)})\,, \qquad Y = \frac{1}{32\pi\epsilon}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\,. \tag{B.12}$$

a_1

The contributing diagrams are either bubbles or decorated bubbles and evaluate to

$$\begin{aligned} a_1 &= \langle \mathcal{A}_4^{[2]} \rangle = -\frac{1}{48}\mathcal{H}^{(4)}_{\bullet\bullet}\langle\zeta^4\rangle - \frac{1}{2}X^{(2)}_{\bullet\bullet}\langle\zeta^2\rangle \\ &= \frac{1}{16\lambda^2}\mathcal{H}^{(4)}_{\bullet\bullet}[[1]]_{1,1,0} - \frac{i}{2\lambda}X^{(2)}_{\bullet\bullet}[[1]]_{1,0,0} = \frac{1}{16\lambda^2}\mathcal{H}^{(4)}_{\bullet\bullet}\mathbf{I}^2 - \frac{i}{2\lambda}X^{(2)}_{\bullet\bullet}\mathbf{I}\,. \end{aligned} \tag{B.13}$$

Expanding the derivatives of the counter-term insertion yields

$$a_1 = \frac{1}{16\lambda^2}\mathbf{I}^2\mathcal{H}^{(4)}_{\bullet\bullet} - \frac{1}{8\lambda^2}\mathbf{IP}\left(\mathcal{H}^{(4)} - 4\mathcal{H}^{(2,1,1)} - 2\mathcal{H}^{(2,0,2)} + 2\mathcal{H}^{(3,1,0)} - \mathcal{H}^{(1,2,1)}\right)_{\bullet\bullet} \tag{B.14}$$

a_2

After discarding non-1PI graphs we obtain

$$\begin{aligned} a_2 &= i\langle \mathcal{A}_1^{[0]}\mathcal{A}_3^{[2]} + \mathcal{A}_2^{[0]}\mathcal{A}_2^{[2]} \rangle = \frac{i}{16}\mathcal{H}^{(2)}_{AB}\mathcal{H}^{(2)}_{\bullet\bullet}\langle\zeta^2_{\sigma_1}\partial_1\xi^A_{\sigma_1}\partial_1\xi^B_{\sigma_1}\zeta^2_{\sigma_2}\rangle + \frac{i}{4}Y\mathcal{H}^{(2)}_{\bullet\bullet}\langle\partial_\mu\zeta_{\sigma_1}\partial^\mu\zeta_{\sigma_1}\zeta^2_{\sigma_2}\rangle \\ &= -\frac{1}{8\lambda^2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(2)}_{\bullet\bullet}[[k_1^2]]_{2,1,0} - \frac{i}{2\lambda^2}Y[[p^2]]_{2,0,0}\,. \end{aligned} \tag{B.15}$$

The potentially linearly divergent $[[k_1^2]]_{2,1,0}$ term cancels with the same from a_4 and can be set to zero. In Method 2 we cancel the p^2 in numerator and denominator of $[[p^2]]_{2,0,0}$ to yield

$$a_2 = -\frac{\mathbf{IP}}{32\lambda^2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(2)}_{\bullet\bullet}\,. \tag{B.16}$$

a_3

There are two contributions in a_3 without counter-term insertion, of which the first comes from

$$\begin{aligned} a_{3a} &= i\langle \mathcal{A}_1^{[1]}\mathcal{A}_3^{[1]}\rangle = \frac{i}{6}\mathcal{H}^{(1)}_{\bullet A}\mathcal{H}^{(3)}_{\bullet B}\langle \zeta_{\sigma_1}\zeta^3_{\sigma_2}\partial_1\xi^A_{\sigma_1}\partial_1\xi^B_{\sigma_2}\rangle \\ &= -\frac{1}{2\lambda^2}\mathcal{H}^{(3,1,0)}_{\bullet\bullet}[[p_1^2]]_{2,1,0} + \frac{1}{2\lambda^2}\mathcal{H}^{(3,1)}_{\bullet\bullet}[[p_1p_0]]_{2,1,0} \\ &= -\frac{\mathbf{IL}}{2\lambda^2}\mathcal{H}^{(3,1,0)}_{\bullet\bullet}\,. \end{aligned} \tag{B.17}$$

The second contribution arises from

$$\begin{aligned} a_{3b} &= \frac{i}{2}\langle \mathcal{A}_2^{[1]}\mathcal{A}_2^{[1]}\rangle = \frac{i}{8}\mathcal{H}^{(2)}_{\bullet A}\mathcal{H}^{(2)}_{\bullet B}\langle \zeta^2_{\sigma_1}\zeta^2_{\sigma_2}\partial_1\xi^A_{\sigma_1}\partial_1\xi^B_{\sigma_2}\rangle \\ &= \frac{1}{4\lambda^2}\mathcal{H}^{(2,0,2)}_{\bullet\bullet}[[p_1^2]]_{1,1,1} + \frac{1}{4\lambda^2}\mathcal{H}^{(2,2)}_{\bullet\bullet}[[p_1p_0]]_{1,1,1} \\ &= \frac{\mathbf{S}}{4\lambda^2}\mathcal{H}^{(2,0,2)}_{\bullet\bullet}\,. \end{aligned} \tag{B.18}$$

We also have a one-loop diagram with a counter-term insertion

$$\begin{aligned} a_{3c} &= i\mathcal{H}^{(1)}_{\bullet A}X^{(1)}_{B\bullet}\langle \zeta_{\sigma_1}\zeta_{\sigma_2}\partial_1\xi^A_{\sigma_1}\partial_1\xi^B_{\sigma_2}\rangle \\ &= -i\mathcal{H}^{(1)}_{\bullet A}\mathcal{H}^{AB}X^{(1)}_{B\bullet}[[p_1^2]]_{2,0,0} - i\mathcal{H}^{(1)}_{\bullet A}\eta^{AB}X^{(1)}_{B\bullet}[[p_1p_0]]_{2,0,0} \\ &= \frac{\mathbf{PL}}{2\lambda^2}\left(\mathcal{H}^{(3,1,0)} + H^{(1,1,1,1,0)} - \mathcal{H}^{(1,2,1)} - \mathcal{H}^{(1,1,2)}\right)\,. \end{aligned} \tag{B.19}$$

In the final steps we have invoked that $[[p_1p_0]]_{2,0,0} = [[p_1p_0]]_{2,1,0} = [[p_1p_0]]_{1,1,1} = 0$.

a_4

This triangle envelope topology diagram evaluates to

$$\begin{aligned} a_4 &= -\frac{1}{2}\langle \mathcal{A}_2^{[2]}\mathcal{A}_1^{[0]}\mathcal{A}_1^{[0]}\rangle = \frac{1}{32}\mathcal{H}^{(2)}_{\bullet\bullet}\mathcal{H}^{(1)}_{AB}\mathcal{H}^{(1)}_{CD}\langle \zeta^2_{\sigma_1}\zeta_{\sigma_2}\zeta_{\sigma_3}\partial_1\xi^A_{\sigma_2}\partial_1\xi^B_{\sigma_2}\partial_1\xi^C_{\sigma_3}\partial_1\xi^D_{\sigma_3}\rangle \\ &= \frac{1}{8\lambda^2}\mathcal{H}^{(2)}_{\bullet\bullet}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\left([[(p_1+k_1)(p_0+k_0)k_1k_0]]_{2,1,1} - [[(p_1+k_1)^2k_1^2]]_{2,1,1}\right) \end{aligned} \tag{B.20}$$

Under Method 2 we proceed by replacing e.g. $p_0^2 = p^2 + p_1^2$ to give

$$\begin{aligned} a_4 &= \frac{1}{8\lambda^2}\mathcal{H}^{(2)}_{\bullet\bullet}\mathrm{tr}\mathcal{H}^{(1,1)}\left(-\frac{1}{4}[[p_1^2]]_{1,1,1} + [[k_1^2]]_{2,1,0}\right) \\ &= -\frac{\mathbf{S}}{32\lambda^2}\mathcal{H}^{(2)}_{\bullet\bullet}\mathrm{tr}\mathcal{H}^{(1,1)}\,. \end{aligned} \tag{B.21}$$

In the last line we dispensed with the potentially linearly divergent contribution $[[k_1^2]]_{2,1,0}$ which in a case cancels against the same from a_2.

a_5

First we consider the part of

$$a_{5_1} = -\frac{1}{2}\langle \mathcal{A}_1^{[1]}\mathcal{A}_1^{[1]}\mathcal{A}_2^{[0]}\rangle \tag{B.22}$$

that does not involve the insertion of one-loop counter-term operators. There are three topologies involved here giving contributions $a_{5_{1a}}$, $a_{5_{1b}}$, $a_{5_{1c}}$:

$$\begin{aligned} a_{5_{1a}} &= \frac{1}{2}\mathcal{H}^{(1)}_{\bullet A}\mathcal{H}^{(2)}_{CD}\mathcal{H}^{(1)}_{B\bullet}\langle\zeta_{\sigma_1}\zeta_{\sigma_3}\rangle\langle\zeta_{\sigma_2}\zeta_{\sigma_3}\rangle\langle\partial_1\xi^A_{\sigma_1}\partial_1\xi^C_{\sigma_3}\rangle\langle\partial_1\xi^B_{\sigma_2}\partial_1\xi^D_{\sigma_3}\rangle \\ &= \frac{1}{2\lambda^2}\left(-\mathcal{H}^{(1,2,1)}+2\mathcal{H}^{(1,1,1,1,0)}\right)_{\bullet\bullet}[[p_1^2k_1^2]]_{2,2,0}+\frac{1}{2\lambda^2}\mathcal{H}^{(1,2,1)}_{\bullet\bullet}[[p_1p_0k_1k_0]]_{2,2,0} \\ &= \frac{\mathbf{L}^2}{2\lambda^2}\left(-\mathcal{H}^{(1,2,1)}+2\mathcal{H}^{(1,1,1,1,0)}\right)_{\bullet\bullet}. \end{aligned} \tag{B.23}$$

Here there is a small subtlety; in Method 2 one could have made a replacement such as $[[p_1p_0k_1k_0]]_{2,2,0} \to [[p_1k_1p\cdot k]]_{2,2,0} + [[p_1p_1k_1k_1]]_{2,2,0}$ and produced a $\frac{1}{\epsilon}$ pole; however, as this is factorised diagram, general arguments [162] imply that the counter-term contribution must cancel such a pole. Hence the correct procedure is to replace $[[p_1p_0k_1k_0]]_{2,2,0} \to ([[p_1p_0]]_{2,0,0})^2 \to 0$. The second and third parts are

$$\begin{aligned} a_{5_{1b}} &= \frac{1}{4}\mathcal{H}^{(1)}_{\bullet A}\mathcal{H}^{(2)}_{CD}\mathcal{H}^{(1)}_{B\bullet}\langle\zeta_{\sigma_3}\zeta_{\sigma_3}\rangle\langle\zeta_{\sigma_1}\zeta_{\sigma_2}\rangle\langle\partial_1\xi^A_{\sigma_1}\partial_1\xi^C_{\sigma_3}\rangle\langle\partial_1\xi^B_{\sigma_2}\partial_1\xi^D_{\sigma_3}\rangle \\ &= \frac{1}{4\lambda^2}\left(2\mathcal{H}^{(1,1,1,1,0)}_{\bullet\bullet}[[p_1^4]]_{3,1,0}+\mathcal{H}^{(1,2,1)}_{\bullet\bullet}([[p_1^2p_0^2]]_{3,1,0}-[[p_1^4]]_{3,1,0})\right) \\ &= \frac{1}{4\lambda^2}\left(2\mathbf{IT}\mathcal{H}^{(1,1,1,1,0)}_{\bullet\bullet}+\mathbf{IL}\mathcal{H}^{(1,2,1)}_{\bullet\bullet}\right), \end{aligned} \tag{B.24}$$

$$\begin{aligned} a_{5_{1c}} &= \frac{1}{4}\mathcal{H}^{(1)}_{\bullet A}\mathcal{H}^{(2)}_{CD}\mathcal{H}^{(1)}_{B\bullet}\langle\partial_1\xi^C_{\sigma_3}\partial_1\xi^D_{\sigma_3}\rangle\langle\zeta_{\sigma_1}\zeta_{\sigma_3}\rangle\langle\zeta_{\sigma_2}\zeta_{\sigma_3}\rangle\langle\partial_1\xi^A_{\sigma_1}\partial_1\xi^B_{\sigma_2}\rangle \\ &= \frac{1}{2\lambda^2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,1,0)}_{\bullet\bullet}[[p_1^2k_1^2]]_{3,1,0}\,. \end{aligned} \tag{B.25}$$

Then there are two contributions, $a_{5_{1d}}$ and $a_{5_{1e}}$, from one-loop triangles with counter-term insertions:

$$\begin{aligned} a_{5_{1d}} &= \frac{1}{2}\mathcal{H}^{(1)}_{\bullet A}\mathcal{H}^{(1)}_{B\bullet}Y\langle\zeta_{\sigma_1}\partial_1\xi^A_{\sigma_1}\zeta_{\sigma_2}\partial_1\xi^B_{\sigma_2}\partial_\mu\zeta_{\sigma_3}\partial^\mu\zeta_{\sigma_3}\rangle \\ &= -\frac{i}{2\lambda^2}\mathcal{H}^{(1,1,0)}_{\bullet\bullet}Y[[p_1^2p^2]]_{3,0,0}+\frac{i}{\lambda^2}\mathcal{H}^{(1,1)}_{\bullet\bullet}Y[[p_1p_0p^2]]_{3,0,0} \\ &= \frac{\mathbf{PL}}{16\lambda^2}\,\mathrm{Tr}\,\mathcal{H}^{(1,1)}\mathcal{H}^{(1,1,0)}_{\bullet\bullet}, \end{aligned} \tag{B.26}$$

$$\begin{aligned}
a_{51e} &= \frac{1}{2}\mathcal{H}^{(1)}_{\bullet A}\mathcal{H}^{(1)}_{B\bullet}X_{CD}\langle\zeta_{\sigma_1}\partial_1\xi^A_{\sigma_1}\zeta_{\sigma_2}\partial_1\xi^B_{\sigma_2}\partial_1\xi^C_{\sigma_3}\partial_1\xi^D_{\sigma_3}\rangle \\
&= -\frac{i}{\lambda}\Big((\mathcal{H}^{(1,0)}X\mathcal{H}^{(0,1)})_{\bullet\bullet}[[p_1^4]]_{3,0,0} + (\mathcal{H}^{(1)}X\mathcal{H}^{(1)})_{\bullet\bullet}[[p_1^2p_0^2]]_{3,0,0}\Big) \\
&= -\frac{\mathbf{PL}}{4\lambda^2}\Big(\mathcal{H}^{(1,2,1)} - \mathcal{H}^{(1,1,1,1,0)}\Big)_{\bullet\bullet}\,. \qquad \text{(B.27)}
\end{aligned}$$

Finally we have a second contraction with three vertices given by

$$\begin{aligned}
a_{5_2} &= -\langle\mathcal{A}^{[1]}_1\mathcal{A}^{[1]}_2\mathcal{A}^{[0]}_1\rangle = \frac{1}{4}\mathcal{H}^{(1)}_{\bullet A}\mathcal{H}^{(1)}_{CD}\mathcal{H}^{(2)}_{B\bullet}\langle\zeta_{\sigma_1}\zeta^2_{\sigma_2}\zeta_{\sigma_3}\partial_1\xi^A_{\sigma_1}\partial_1\xi^B_{\sigma_2}\partial_1\xi^C_{\sigma_3}\partial_1\xi^D_{\sigma_3}\rangle \\
&= \frac{1}{\lambda^2}\mathcal{H}^{(1,1,2)}_{\bullet\bullet}\big(-[[p_1^2(p+k)_1^2]]_{2,1,1} + [[p_1p_0(p+k)_1(p+k)_0]]_{2,1,1}\big) \\
&= \frac{\mathbf{S}+2\mathbf{IL}}{4\lambda^2}\mathcal{H}^{(1,1,2)}_{\bullet\bullet} - \frac{1}{4\lambda^2}\mathrm{tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,1,0)}_{\bullet\bullet}[[p_1^2k_1^2]]_{3,1,0}\,. \qquad \text{(B.28)}
\end{aligned}$$

The final $[[p_1^2k_1^2]]_{3,1,0}$ contribution (which we expect not to contain divergent terms in $\frac{1}{\epsilon}$ or $\frac{1}{\epsilon^2}$) cancel between a_{5_2} and $a_{5_{1c}}$, and a_{6a}.

a_6

For the last contraction,

$$a_6 = -\frac{i}{4}\langle\mathcal{A}^{[0]}_1\mathcal{A}^{[0]}_1\mathcal{A}^{[1]}_1\mathcal{A}^{[1]}_1\rangle\,, \qquad \text{(B.29)}$$

there are three different topologies of diagrams to consider:

$$\begin{aligned}
a_{6_a} &= -\frac{i}{4}\mathcal{H}^{(1)}_{AB}\mathcal{H}^{(1)}_{CD}\mathcal{H}^{(1)}_{\bullet E}H^{(1)}_{\bullet F}\langle\zeta_{\sigma_1}\zeta_{\sigma_3}\rangle\langle\zeta_{\sigma_2}\zeta_{\sigma_4}\rangle\langle\partial_1\xi^A_{\sigma_1}\partial_1\xi^C_{\sigma_2}\rangle\langle\partial_1\xi^B_{\sigma_1}\partial_1\xi^D_{\sigma_2}\rangle\langle\partial_1\xi^E_{\sigma_3}\partial_1\xi^F_{\sigma_4}\rangle \\
&= -\frac{1}{4\lambda^2}\mathrm{tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,1,0)}_{\bullet\bullet}\times \\
&\times\ \big(-[[(p+k)_1^2k_1^2p_1^2]]_{3,1,1} + [[(p+k)_1(p+k)_0k_1k_0p_1^2]]_{3,1,1}\big)\,,
\end{aligned}$$

$$\begin{aligned}
a_{6_b} &= -\frac{i}{4}\mathcal{H}^{(1)}_{AB}\mathcal{H}^{(1)}_{CD}\mathcal{H}^{(1)}_{\bullet E}H^{(1)}_{\bullet F}\langle\zeta_{\sigma_1}\zeta_{\sigma_2}\rangle\langle\zeta_{\sigma_3}\zeta_{\sigma_4}\rangle\langle\partial_1\xi^A_{\sigma_1}\partial_1\xi^C_{\sigma_2}\rangle\langle\partial_1\xi^B_{\sigma_1}\partial_1\xi^E_{\sigma_3}\rangle\langle\partial_1\xi^D_{\sigma_2}\partial_1\xi^F_{\sigma_4}\rangle \\
&= \frac{1}{2\lambda^2}\mathcal{H}^{(1,1,1,1,0)}_{\bullet\bullet}\times \\
&\times\ \big([[(p+k)_1^2p_1^4]]_{3,1,1} - 2[[(p+k)_1(p+k)_0p_1^3p_0]]_{3,1,1} + [[(p+k)_1^2p_1^2p_0^2]]_{3,1,1}\big)\,,
\end{aligned}$$

$$\begin{aligned}
a_{6_c} &= -\frac{i}{4}\mathcal{H}^{(1)}_{AB}\mathcal{H}^{(1)}_{CD}\mathcal{H}^{(1)}_{\bullet E}H^{(1)}_{\bullet F}\langle\zeta_{\sigma_1}\zeta_{\sigma_3}\rangle\langle\zeta_{\sigma_2}\zeta_{\sigma_4}\rangle\langle\partial_1\xi^A_{\sigma_1}\partial_1\xi^C_{\sigma_2}\rangle\langle\partial_1\xi^B_{\sigma_1}\partial_1\xi^F_{\sigma_4}\rangle\langle\partial_1\xi^D_{\sigma_2}\partial_1\xi^E_{\sigma_3}\rangle \\
&= \frac{1}{2\lambda^2}\mathcal{H}^{(1,1,1,1,0)}_{\bullet\bullet}\times \qquad \text{(B.30)} \\
&\times\ \Big([[(p+k)_1^2p_1^2k_1^2]]_{2,2,1} - 2[[(p+k)_1(p+k)_0p_1p_0k_1^2]]_{2,2,1} + [[(p+k)_1^2p_1p_0k_1k_0]]_{2,2,1}\Big)\,.
\end{aligned}$$

As detailed in the main body for the case of a_{6_b}, under Method 2 each of these can be simplified to yield

$$a_{6_a} = -\frac{1}{8\lambda^2}\left(2[[p_1^2 q_1^2]]_{3,1,0} - \mathbf{TE}\right)\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,1,0)}_{\bullet\bullet}\,, \tag{B.31}$$

$$a_{6_b} = \frac{1}{2\lambda^2}\left(\mathbf{TE} - \mathbf{IT}\right)\mathcal{H}^{(1,1,1,1,0)}_{\bullet\bullet}\,, \tag{B.32}$$

$$a_{6_c} = -\frac{1}{\lambda^2}\mathbf{TE}\,\mathcal{H}^{(1,1,1,1,0)}_{\bullet\bullet}\,. \tag{B.33}$$

B.1.1 IR Regularisation in Method 1

Let us explore here in some detail the way our IR regularisation prescription deals with the cancellation of loop integrals proportional to $I_{2,3}$ in the final result. It is fairly easy to tackle this problem explicitly once some observations are made. First, if we are interested in $I_1 I_2$ or $I_1 I_3$ contributions only, we can safely neglect the counter-term insertions, as by definition they would not give rise to any such term at this loop order. Obviously, we will also drop in $\mathcal{L}_\mathrm{I}$ any term proportional to Ω. Another important remark is that, when considering $\exp(iS_\mathrm{I})$, we can discard any term which is not proportional to m^2, as these do not originate from the expansion of the mass term and have thus been previously considered. Finally, to keep things simple, we restrict ourselves to combinations that lead eventually to the desired tensor structures, namely $\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(2)}$ and $\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,1,0)}$. While other tensors might arise in the full calculation, they eventually cancel in the final result. The first contribution belongs to what we call triangle envelope topology. It is given by

$$\begin{aligned} m_1 &= \frac{m^2}{32}\mathcal{H}^{(2)}_{\bullet\bullet}\mathcal{H}^{(1)}_{AB}\mathcal{H}^{(1)}_{CD}\langle\zeta^2_{\sigma_1}\zeta_{\sigma_2}\zeta_{\sigma_3}\rangle\langle\xi^A_{\sigma_2}\xi^B_{\sigma_2}\partial_1\xi^C_{\sigma_3}\partial_1\xi^D_{\sigma_3}\rangle \\ &= \frac{m^2}{8\lambda^2}[[k^2 + k\cdot p]]_{2,1,1}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(2)}_{\bullet\bullet}\,. \end{aligned} \tag{B.34}$$

The second contribution comes from the square envelope topology and evaluates to

$$\begin{aligned} m_2 &= -\frac{im^2}{16}\mathcal{H}^{(1)}_{AB}\mathcal{H}^{(1)}_{C\bullet}\mathcal{H}^{(1)}_{D\bullet}\mathcal{H}^{(1)}_{EF}\langle\zeta_{\sigma_1}\zeta_{\sigma_2}\zeta_{\sigma_3}\zeta_{\sigma_4}\rangle\langle\xi^A_{\sigma_4}\xi^B_{\sigma_4}\partial_1\xi^C_{\sigma_1}\partial_1\xi^D_{\sigma_2}\partial_1\xi^E_{\sigma_3}\partial_1\xi^F_{\sigma_3}\rangle \\ &= \frac{m^2}{4\lambda^2}[[p_1^2(k^2 + k\cdot p)]]_{3,1,1}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,1,0)}_{\bullet\bullet}\,. \end{aligned} \tag{B.35}$$

The third possibility is a decorated loop diagram, coming from

$$m_3 = \frac{i}{32}m^2\mathcal{H}^{(2)}_{AB}\mathcal{H}^{(2)}_{\bullet\bullet}\langle\zeta^2_{\sigma_1}\zeta^2_{\sigma_2}\rangle\langle\xi^A_{\sigma_2}\xi^B_{\sigma_2}\rangle = -\frac{m^2}{16\lambda^2}\mathbf{I}[[1]]_{2,0,0}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(2)}_{\bullet\bullet}\,. \tag{B.36}$$

Finally, we have a decorated triangle (we neglect the double loop part, as it is not relevant for the tensor structure we are interested in)

$$m_4 = \frac{m^2}{16}\mathcal{H}^{(2)}_{AB}\mathcal{H}^{(1)}_{C\bullet}\mathcal{H}^{(1)}_{D\bullet}\langle\zeta_{\sigma_1}\zeta_{\sigma_2}\zeta^2_{\sigma_3}\rangle\langle\xi^A_{\sigma_3}\xi^B_{\sigma_3}\partial_1\xi^C_1\partial_1\xi^D_2\rangle = \frac{m^2}{8\lambda^2}I_1[[p_1^2]]_{3,0,0}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\mathcal{H}^{(1,1,0)}_{\bullet\bullet}\,. \tag{B.37}$$

B.2 Base $(\partial_0 y)^2$ Wick Contractions

We are now to evaluate in detail the Wick contractions associated to the base $(\partial_0 y)^2$ term. Completing combinations which are already fourth-order in derivatives, namely

$$b_1 = \langle\mathcal{B}^{[2]}_4\rangle\,,\quad b_2 = i\langle\mathcal{B}^{[1]}_1\mathcal{B}^{[1]}_3\rangle\,,\quad b_3 = -\frac{1}{2}\langle\mathcal{B}^{[0]}_2\mathcal{B}^{[1]}_1\mathcal{B}^{[1]}_1\rangle\,,\quad b_4 = -\frac{i}{4}\langle\mathcal{B}^{[0]}_1\mathcal{B}^{[0]}_1\mathcal{B}^{[1]}_1\mathcal{B}^{[1]}_1\rangle\,,$$

we have three which are second- or third-order

$$b_5 = \frac{i}{2}\langle\mathcal{B}^{[0]}_1\mathcal{B}^{[0]}_1\rangle\,,\quad b_6 = i\langle\mathcal{B}^{[1]}_1\mathcal{B}^{[0]}_2\rangle\,,\quad b_7 = -\frac{1}{2}\langle\mathcal{B}^{[1]}_1\mathcal{B}^{[0]}_1\mathcal{B}^{[0]}_1\rangle\,. \tag{B.38}$$

We need the following identifications

$$\mathcal{B}^{[0]}_1 = -\frac{1}{2}\mathcal{H}^{(1)}_{AB}\zeta\partial_1\xi^A\partial_1\xi^B\,, \tag{B.39}$$

$$\mathcal{B}^{[1]}_1 = \frac{1}{2}\Omega_{0AB}\xi^A\partial_1\xi^B\,, \tag{B.40}$$

$$\mathcal{B}^{[1]}_3 = -\frac{1}{8\pi\epsilon}\,\mathrm{Tr}(\mathcal{H}^{(2,1)})\zeta\partial^\mu\zeta\partial_\mu y \tag{B.41}$$

$$\mathcal{B}^{[2]}_0 = -\frac{1}{32\pi\epsilon}\,\mathrm{Tr}(\mathcal{H}^{(1,1)})\partial_\mu\zeta\partial^\mu\zeta - \frac{1}{8\pi\epsilon\lambda}\left(\mathcal{H}^{(2)} - \mathcal{H}^{(1,0,1)}\right)_{AB}\partial_1\xi^A\partial_1\xi^B - \frac{1}{4}\mathcal{H}^{(2)}_{AB}\zeta^2\partial_1\xi^A\partial_1\xi^B\,, \tag{B.42}$$

$$\mathcal{B}^{[2]}_4 = -\frac{1}{32\pi\epsilon\lambda}\left(-\,\mathrm{Tr}(\mathcal{H}^{(2,2)}) + \mathrm{Tr}(\mathcal{H}^{(3,1)})\right)\zeta^2\partial_\mu y\partial^\mu y\,. \tag{B.43}$$

The combinations $b_{5,6,7}$ require us to evaluate integrals with insertion of external momenta. When looking at terms on the base manifold with legs $\partial_0 y\partial_0 y$ or $\partial_1 y\partial_1 y$, the relevant basis of tensors turns out to be

$$\mathrm{Tr}(\mathcal{H}^{(3,1)})\,,\quad \mathrm{Tr}(\mathcal{H}^{(2,2)})\,,\quad \mathrm{Tr}(\mathcal{H}^{(1,1,1,1)})\,,\quad \mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega)\,, \\ \mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega)\,,\quad \mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega)\,. \tag{B.44}$$

b_1

The first case is immediately solved as

$$b_1 = \langle \mathcal{B}_4^{[2]} \rangle = -Y^{(2)}(\partial_0 y)^2 \langle \zeta^2 \rangle = -\frac{\mathbf{IP}}{16\lambda} \left(\mathrm{Tr}(\mathcal{H}^{(2,2)}) + \mathrm{Tr}(\mathcal{H}^{(3,1)}) \right) (\partial_0 y)^2 \,. \tag{B.45}$$

b_2

This contribution consists of diagrams in which one vertex contains only ζ and the other only ξ fields and hence upon Wick contraction no relevant 1PI graphs are produced.

b_3

b_3 has a simple structure

$$\begin{aligned} b_3 &= -\frac{1}{2} \langle \mathcal{B}_2^{[0]} \mathcal{B}_1^{[1]} \mathcal{B}_1^{[1]} \rangle = \frac{1}{16} \left(4X + \frac{i}{2\lambda} \mathbf{I} \mathcal{H}^{(2)} \right)_{EF} \Omega_{0AB} \Omega_{0CD} \langle \partial_1 \xi^E_{\sigma_1} \partial_1 \xi^F_{\sigma_1} \xi^A_{\sigma_2} \partial_1 \xi^B_{\sigma_2} \xi^C_{\sigma_3} \partial_1 \xi^D_{\sigma_3} \rangle \\ &= \frac{1}{2} \left(4X + \frac{i}{2\lambda} \mathbf{I} \mathcal{H}^{(2)} \right)_{EF} \Omega_{0AB} \Omega_{0CD} \langle \partial_1 \xi^E_{\sigma-1} \partial_1 \xi^B_{\sigma_2} \rangle \langle \partial_1 \xi^F_{\sigma_1} \partial_1 \xi^D_{\sigma_3} \rangle \langle \xi^A_{\sigma_2} \xi^C_{\sigma_3} \rangle \,, \end{aligned} \tag{B.46}$$

where we have already contracted $\langle \zeta_\sigma^2 \rangle = i\lambda^{-1}\mathbf{I}$ for simplicity and used the symmetries of the tensorial part to simplify the Wick contraction. We immediately recognise a triangle diagram, possibly decorated in the case of $\mathcal{H}^{(2)}$. The contractions are easily calculated as

$$\begin{aligned} & i \int \mathrm{d}\sigma_2 \mathrm{d}\sigma_3 \langle \partial_1 \xi_{\sigma_1} \partial_1 \xi_{\sigma_2} \rangle \otimes \langle \partial_1 \xi_{\sigma_1} \partial_1 \xi_{\sigma_3} \rangle \otimes \langle \xi_{\sigma_2} \xi_{\sigma_3} \rangle \\ &= [[(p^1)^4]]_{3,0,0} \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} + [[(p^0)^2 (p^1)^2]]_{3,0,0} (\eta \otimes \eta \otimes \mathcal{H} + \eta \otimes \mathcal{H} \otimes \eta + \mathcal{H} \otimes \eta \otimes \eta) \,. \end{aligned} \tag{B.47}$$

in which we have once again omitted the vanishing $[[p^0(p^1)^3]]_{3,0,0}$. Using Method 2 we replace $[[(p^0)^2(p^1)^2]]_{3,0,0} \to \mathbf{T}+\mathbf{L}$ and $[[(p^1)^4]]_{3,0,0} \to \mathbf{T}$. This produces after simplification of the tensors

$$\lambda b_3 = \left(\frac{3}{4} \mathbf{IL} + \mathbf{IT} \right) \mathrm{Tr}(\mathcal{H}^{(1,1)} \Omega\Omega) + \frac{1}{4} \mathbf{L}(\mathbf{I} - \mathbf{P}) \,\mathrm{Tr}(\mathcal{H}^{(2,1)} \Omega) + \frac{1}{8} \left(2\mathbf{IT} + \mathbf{LP} \right) \mathrm{Tr}(\mathcal{H}^{(1,1,1,1)}) \,. \tag{B.48}$$

Two remarks are in order. Assuming that integration by parts holds, we can write

$$\mathbf{L} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{k_1^2}{(k^2)^2} = -\int \frac{\mathrm{d}^2 k}{(2\pi)^2} k_1 \partial_{k_1} \frac{k_1^2}{(k^2)^2} = -2\mathbf{L} - 4\mathbf{T} \tag{B.49}$$

which implies that the $\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega)$ term vanishes. However we will not enforce this directly but allow $3\mathbf{L}_0 + 4\mathbf{T}_0 \neq 0$ to keep track of any ambiguity. The $\mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega)$ coefficient gives rise only to a $\frac{\bar{\gamma}}{\epsilon}$ that will cancel against a counter-term insertion.

b_4

Within

$$\begin{aligned} b_4 &= -\frac{i}{4}\langle \mathcal{B}_1^{[0]}\mathcal{B}_1^{[0]}\mathcal{B}_1^{[1]}\mathcal{B}_1^{[1]}\rangle \\ &= -\frac{i}{64}\mathcal{H}^{(1)}_{EF}\mathcal{H}^{(1)}_{GH}\Omega_{0AB}\Omega_{0CD}\langle\zeta_{\sigma_1}\zeta_{\sigma_2}\rangle\langle\xi^A_{\sigma_3}\xi^C_{\sigma_4}\partial_1\xi^E_{\sigma_1}\partial_1\xi^F_{\sigma_1}\partial_1\xi^G_{\sigma_2}\partial_1\xi^H_{\sigma_2}\partial_1\xi^B_{\sigma_3}\partial_1\xi^D_{\sigma_4}\rangle\,, \end{aligned} \quad \text{(B.50)}$$

there are two different topologies of Wick contractions to consider. First is a *diamond sunset* arising from

$$\mathrm{DS} = \int \mathrm{d}\sigma_2\mathrm{d}\sigma_3\mathrm{d}\sigma_4\langle\zeta_{\sigma_1}\zeta_{\sigma_2}\rangle\langle\partial_1\xi_{\sigma_1}\xi_{\sigma_3}\rangle\otimes\langle\partial_1\xi_{\sigma_3}\partial_1\xi_{\sigma_2}\rangle\otimes\langle\partial_1\xi_{\sigma_2}\partial_1\xi_{\sigma_4}\rangle\otimes\langle\partial_1\xi_{\sigma_1}\xi_{\sigma_4}\rangle\,. \quad \text{(B.51)}$$

Setting $[[\ldots]]_{2,2,1}$ integrals with odd number of timelike or spacelike components of momenta to zero, we get the contribution to b_4 from diamond sunset diagrams

$$\begin{aligned} b_4|_{\mathrm{DS}} &= \frac{1}{4\lambda}\,\mathrm{Tr}(\Omega\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)})[[k^1p^1(k^0p^0-k^1p^1)^2]]_{2,2,1} \\ &\quad -\frac{1}{2\lambda}\,\mathrm{Tr}(\Omega\mathcal{H}^{(1,0)}\Omega\mathcal{H}^{(1,0)})[[(k^1)^2p^0p^1(k^1p^0-k^0p^1)]]_{2,2,1}\,. \end{aligned} \quad \text{(B.52)}$$

Proceeding now to Method 2 we obtain

$$\lambda b_4|_{\mathrm{DS}}=\left(\frac{1}{8}\mathbf{L}^2-\frac{1}{4}\mathbf{TE}\right)\mathrm{Tr}(\mathcal{H}^{(1,1,1,1)})+\left(-\frac{1}{2}\mathbf{L}^2+\mathbf{TE}-\frac{1}{8}\mathbf{S}\right)\mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega)\,. \quad \text{(B.53)}$$

In addition the *square envelope* topology is given by

$$\mathrm{SE} = \int \mathrm{d}\sigma_2\mathrm{d}\sigma_3\mathrm{d}\sigma_4\langle\zeta_{\sigma_1}\zeta_{\sigma_2}\rangle\langle\partial_1\xi_{\sigma_1}\partial_1\xi_{\sigma_2}\rangle\otimes\langle\partial_1\xi_{\sigma_2}\xi_{\sigma_3}\rangle\otimes\langle\partial_1\xi_{\sigma_3}\xi_{\sigma_4}\rangle\otimes\langle\partial_1\xi_{\sigma_4}\partial_1\xi_{\sigma_1}\rangle\,. \quad \text{(B.54)}$$

Once the contractions are carried out, and basic identities applied, two tensors only appear. The final result for the square envelopes evaluates to

$$\begin{aligned} \lambda b_4|_{\mathrm{SE}} &= \frac{1}{2}\,\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega)[[k_1p_1(3k_1p_1p_0^2-3k_0p_1^2p_0+k_1p_1^3-k_0p_0^3)]]_{3,1,1} \\ &\quad +\frac{1}{4}\,\mathrm{Tr}(\mathcal{H}^{(1,1,1,1)})[[k_1p_1^2(k_1p_0^2+k_1p_1^2-2k_0p_0p_1)]]_{3,1,1}\,. \end{aligned} \quad \text{(B.55)}$$

Proceeding with Method 2 we have

$$\lambda b_4|_{\rm SE} = \left(\frac{1}{4}\mathbf{TE} - \frac{1}{4}\mathbf{IT}\right) \mathrm{Tr}(\mathcal{H}^{(1,1,1,1)}) - \left(-\mathbf{TE} + \frac{1}{4}\mathbf{IL} + \frac{1}{8}\mathbf{S} + \mathbf{IT}\right) \mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega)\,. \tag{B.56}$$

In summation we obtain

$$\begin{aligned} \lambda b_4 \;&=\; \left(\frac{1}{8}\mathbf{L}^2 - \frac{1}{4}\mathbf{IT}\right) \mathrm{Tr}(\mathcal{H}^{(1,1,1,1)}) - \left(-\mathbf{TE} + \frac{1}{4}\mathbf{IL} + \frac{1}{8}\mathbf{S} + \mathbf{IT}\right) \mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega) \\ &\quad + \left(-\frac{1}{2}\mathbf{L}^2 + \mathbf{TE} - \frac{1}{8}\mathbf{S}\right) \mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega)\,. \end{aligned} \tag{B.57}$$

b_5

As it stands, b_5 contains tensors which, in total, are second-order in derivatives. We are thus prompted to extract terms quadratic in the external momentum q from the loop integral. We have

$$\begin{aligned} b_5 &= \frac{i}{2}\langle \mathcal{B}_1^{[0]}\mathcal{B}_1^{[0]}\rangle = \frac{i}{8}\int \mathrm{d}^2\sigma_2 \mathcal{H}^{(1)}_{AB}(\sigma_1)\mathcal{H}^{(1)}_{CD}(\sigma_2)\langle\zeta_{\sigma_1}\zeta_{\sigma_2}\rangle\langle\partial_1\xi^A_{\sigma_1}\partial_1\xi^B_{\sigma_1}\partial_1\xi^C_{\sigma_2}\partial_1\xi^D_{\sigma_2}\rangle \\ &= \frac{1}{4}\mathcal{H}^{(1)}_{AB}(\sigma_1)\int\frac{\mathrm{d}^2q}{(2\pi)^2}\mathcal{H}^{(1)}_{CD}(q)e^{-iq\sigma_1}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{p_1^2k_1^2\mathcal{H}^{AC}\mathcal{H}^{BD} + p_1p_0k_1k_0\eta^{AC}\eta^{BD}}{p^2k^2[q-(k+p)]^2}\,. \end{aligned} \tag{B.58}$$

Now, to second order, the denominator with external momentum insertion is expanded as

$$\frac{1}{(q-(k+p))^2} = \cdots + 4\frac{[(k+p)\cdot q]^2}{[(k+p)^2]^3} - \frac{q^2}{[(k+p)^2]^2} + \ldots\,. \tag{B.59}$$

In the previous expression, we restrict ourselves to terms involving q_0^2, as we want to concentrate on external legs $(\partial_0 y)^2$. In passing from momentum to position space, we turn q_0^2 into $-\partial_0^2$ acting upon $\mathcal{H}^{(1)}_{CD}$. However, as a second derivative would necessarily produce a derivative of Ω, integration by parts is in order. Specifically we let $\mathcal{H}^{(1)}_{AB}\partial_0^2(\mathcal{H}^{(1)}_{CD}) = -\partial_0(\mathcal{H}^{(1)}_{AB})\partial_0(\mathcal{H}^{(1)}_{CD})$. We eventually arrive at

$$\begin{aligned} b_5 = &-\frac{1}{2}\left(\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega) - \mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega) + 2\,\mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega) - \frac{1}{2}\mathrm{Tr}(\mathcal{H}^{(2,2)})\right) \\ &\times\left(-[[p_1k_1(p_0k_0-p_1k_1)]]_{1,1,2} + 4[[(k_0+p_0)^2p_1k_1(p_0k_0-p_1k_1)]]_{1,1,3}\right)(\partial_0 y)^2\,. \end{aligned} \tag{B.60}$$

With Method 2, the above expression boils down to

$$b_5 = \left(\frac{3}{4}\mathbf{S} + 2\mathbf{TE}\right)\times$$

$$\times\left(\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega) - \mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega) + 2\,\mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega) - \frac{1}{2}\,\mathrm{Tr}(\mathcal{H}^{(2,2)})\right)(\partial_0 y)^2\,. \quad \text{(B.61)}$$

b_6

Although $\mathcal{B}_2^{[0]} = -Y\partial_\mu\zeta\partial^\mu\zeta - \frac{1}{4}\mathcal{H}^{(2)}_{AB}\zeta^2\partial_1\xi^A\partial_1\xi^B - X_{AB}\partial_1\xi^A\partial_1\xi^B$ we can simplify the evaluation of $b_6 = i\langle\mathcal{B}_1^{[1]}\mathcal{B}_2^{[0]}\rangle$ by noting the term with Y will not contribute (it is a disconnected diagram) and the ζ loop on the term with $\mathcal{H}^{(2)}$ is evaluated to $i\mathbf{I}$; hence effectively we use

$$\mathcal{B}_2^{[0]} = A_{AB}\partial_1\xi^A\partial_1\xi^B\,, \quad A_{AB} = -\frac{i}{4}\mathbf{I}\mathcal{H}^{(2)}_{AB} - X_{AB}\,. \quad \text{(B.62)}$$

The contraction gives

$$\begin{aligned} b_6 =& -\frac{i}{2}\Omega_{0AB}(\sigma_1)A_{CD}(\sigma_2)\langle\partial_1\xi^A_{\sigma_1}\xi^B_{\sigma_1}\partial_1\xi^C_{\sigma_2}\partial_1\xi^D_{\sigma_2}\rangle \\ =& -\Omega_{0AB}(\sigma_1)\int\frac{\mathrm{d}^2q}{(2\pi)^2}A_{CD}(q)e^{-iq\sigma_1}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{1}{p^2(p-q)^2} \\ &\times\Big(p_1p_1(q-p)_1\mathcal{H}^{AC}\mathcal{H}^{BD} + p_0p_1(q-p)_0\eta^{AC}\eta^{BD} \\ &+ p_1p_1(q-p)_0\mathcal{H}^{AC}\eta^{BD} + p_0p_1(q-p)_1\eta^{AC}\mathcal{H}^{BD}\Big)\,. \end{aligned} \quad \text{(B.63)}$$

To proceed one simply Taylor expands to extract the linear dependence on q_0 (and q_1, even though we will omit that part here) from the integrands. In this case it is not even really necessary to use the specific rules for Method 2, for no p_0^2 appears once the dust settles. Still, in the language of Method 2 we can rephrase the result as

$$b_6 = \frac{1}{4}\mathbf{L}(\mathbf{I}-\mathbf{P})\,\mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega)\partial_0 y\partial_0 y - \frac{1}{8}\mathbf{L}(\mathbf{I}-\mathbf{P})\,\mathrm{Tr}(\mathcal{H}^{(3,1)})\partial_0 y\partial_0 y\,. \quad \text{(B.64)}$$

Note that the final contribution from this diagram to $\partial_0 y\partial_1 y$ cancels out.

b_7

This is by far the most complicated case and we shall provide the reader with additional details. The Wick contractions are easily simplified exploiting the symmetries of the tensorial structure

$$\begin{aligned} b_7 =& -\frac{1}{2}\langle\mathcal{B}_1^{[0]}\mathcal{B}_1^{[0]}\mathcal{B}_1^{[1]}\rangle \\ =& \frac{1}{16}\mathcal{H}^{(1)}_{EF}(\sigma_1)\int\mathrm{d}\sigma_2\mathrm{d}\sigma_3\,\mathcal{H}^{(1)}_{CD}(\sigma_2)\Omega_{0AB}(\sigma_3)\langle\partial_1\xi^A_{\sigma_3}\xi^B_{\sigma_3}\partial_1\xi^C_{\sigma_2}\partial_1\xi^D_{\sigma_2}\partial_1\xi^E_{\sigma_1}\partial_1\xi^F_{\sigma_1}\rangle\langle\zeta_{\sigma_1}\zeta_{\sigma_2}\rangle \end{aligned}$$

$$= \frac{1}{2}\mathcal{H}^{(1)}_{EF}(\sigma_1)\int \mathrm{d}\sigma_2\mathrm{d}\sigma_3\, \mathcal{H}^{(1)}_{CD}(\sigma_2)\Omega_{0AB}(\sigma_3)\Delta_{12}\partial_{12}\Delta^{CE}_{12}\partial_{23}\Delta^{AD}_{23}\partial_1\Delta^{BF}_{13}\,, \tag{B.65}$$

where we have shortened the expression using $\Delta_{ij} \equiv \Delta(\sigma_i - \sigma_j)$ and $\Delta^{AB}_{ij} \equiv \mathcal{H}^{AB}\Delta(\sigma_i - \sigma_j) + \eta^{AB}\theta(\sigma_i - \sigma_j)$. The momentum routing is slightly subtle. First: two external momenta, q and l (Fourier partners of σ_2 and σ_3), have to be introduced. Second: we are entitled to choose the routing that will best suite our purpose.

Killing every instance of q_1 and l_1 (they would eventually produce some $\partial_1 y$ leg) we arrive at

$$\begin{aligned}
b_7 =& -\frac{i}{2}\mathcal{H}^{(1)}_{EF}(\sigma_1)\int \frac{\mathrm{d}^2q}{(2\pi)^2}\Omega_{0AB}(q)e^{-iq\sigma_1}\int \frac{\mathrm{d}^2l}{(2\pi)^2}\mathcal{H}^{(1)}_{CD}(l)e^{-il\sigma_1}\\
&\times \int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{p_1(p_1+k_1)}{k^2(p+k)^2(p+l)^2(p+l+q)^2}\\
&\times \left((k_1+p_1)\mathcal{H}^{CE} + (p_0+k_0)\eta^{CE}\right)\left(p_1\mathcal{H}^{AD} + (p_0+l_0)\eta^{AD}\right)\\
&\times \left(p_1\mathcal{H}^{BF} + (p+l+q)_0\eta^{BF}\right)\,.
\end{aligned} \tag{B.66}$$

Judging from the tensorial structures already at our disposal, we need to extract from the loop integral linear terms in either q_0 or l_0

$$\begin{aligned}
b_7 =& -\frac{i}{2}\mathcal{H}^{(1)}_{EF}(\sigma_1)\int \frac{\mathrm{d}^2q}{(2\pi)^2}\Omega_{0AB}(q)e^{-iq\sigma_1}\int \frac{\mathrm{d}^2l}{(2\pi)^2}\mathcal{H}^{(1)}_{CD}(l)e^{-il\sigma_1}\\
&\times \int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{(p^2)^3}\frac{1}{k^2}\frac{p_1(p_1+k_1)}{(p+k)^2}\left((p_1+k_1)\mathcal{H}^{CE} + (p_0+k_0)\eta^{CE}\right)\\
&\times \Big(l_0[4p_0p_1^2\mathcal{H}^{AD}\mathcal{H}^{BF} + 2p_0(p_0^2+p_1^2)\eta^{AD}\eta^{BF}\\
&+ \left(2p_0^2p_1 + p_1(p_0^2+p_1^2)\right)\left(\mathcal{H}^{AD}\eta^{BF} + \eta^{AD}\mathcal{H}^{BF}\right)]\\
&+ q_0[2p_0p_1^2\mathcal{H}^{AD}\mathcal{H}^{BF} + p_0(p_0^2+p_1^2)\eta^{AD}\eta^{BF}\\
&+ 2p_0^2p_1\eta^{AD}\mathcal{H}^{BF} + p_1(p_0^2+p_1^2)\mathcal{H}^{AD}\eta^{BF}]\Big)\,.
\end{aligned} \tag{B.67}$$

We shall now pass to position space and turn l_0, q_0 into derivatives. While the former would hit a generalised metric, the latter would act upon Ω. As we would like to avoid that situation we integrate by parts. Massaging the expression we obtain a little further, dropping terms linear in p_0 or k_0

$$\begin{aligned}
b_7 =& \frac{1}{2}\Omega_{0AB}\left(\mathcal{H}^{(1)}_{EF}\partial_0\mathcal{H}^{(1)}_{CD} - \partial_0\mathcal{H}^{(1)}_{EF}\mathcal{H}^{(1)}_{CD}\right)\int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{(p^2)^3k^2(p+k)^2}\\
&\times \Big[p_1^2(p_1+k_1)^2\left(2p_0^2\eta^{AD}\mathcal{H}^{BF} + (p_0^2+p_1^2)\mathcal{H}^{AD}\eta^{BF}\right)\mathcal{H}^{CE}
\end{aligned}$$

$$+ p_1 p_0 (p_1 + k_1)(p_0 + k_0) \left(2p_1^2 \mathcal{H}^{AD}\mathcal{H}^{BF} + (p_0^2 + p_1^2)\eta^{AD}\eta^{BF}\right)\eta^{CE}\Big] . \qquad \text{(B.68)}$$

We can now perform the tensor contractions. The crucial observation here is that, when contracted, $\mathcal{H}_{AB}$ anti-commutes with both $\mathcal{H}^{(1)}_{AB}$ and $\partial_0\mathcal{H}^{(1)}_{AB}$. To see this, it is important to keep in mind that $\mathcal{H}^{(1)}_{AB} \equiv \mathcal{V}_A{}^I \mathcal{H}^{(1)}_{IJ}\mathcal{V}_B{}^J$. We obtain

$$b_7 = I\left(\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega) - \mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega) + \mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega)\right)\partial_0 y\partial_0 y\,, \qquad \text{(B.69)}$$

where I stands for the integral we are left with, namely

$$I = [[p_1(k_1+p_1)\left(k_0\left(p_0^3 + 3p_0p_1^2\right) - 3k_1p_0^2p_1 - p_1^3(k_1+p_1) + p_0^4\right)]]_{3,1,1}\,. \qquad \text{(B.70)}$$

Using Method 2 the latter becomes

$$I = \frac{1}{4}\mathbf{S} - 2\mathbf{TE} + \frac{1}{2}\mathbf{LI} + 2\mathbf{TI}\,. \qquad \text{(B.71)}$$

B.3 Base $\partial_0 y\partial_1 y$ Wick Contractions

In order to evaluate Wick contractions associated to the $\partial_0 y\partial_1 y$ legs we single out the following terms in the action

$$\mathcal{C}^{[0]}_1 = -\frac{1}{2}\mathcal{H}^{(1)}_{AB}\zeta\partial_1\xi^A\partial_1\xi^B\,, \qquad \text{(B.72)}$$

$$\mathcal{C}^{[0]}_2 = -\frac{1}{4}\mathcal{H}^{(2)}_{AB}\zeta^2\partial_1\xi^A\partial_1\xi^B - Y\partial_\mu\zeta\partial^\mu\zeta - X_{AB}\partial_1\xi^A\partial_1\xi^B\,, \qquad \text{(B.73)}$$

$$\mathcal{C}^{[1;\tau]}_3 = -2Y^{(1)}\zeta\partial_0\zeta\partial_0 y \qquad \text{(B.74)}$$

$$\mathcal{C}^{[1;\sigma]}_3 = -\frac{1}{2}\mathcal{H}^{(2)}_{BC}\Omega_{1A}{}^C\zeta^2\xi^A\partial_1\xi^B + 2Y^{(1)}\zeta\partial_1\zeta\partial_1 y - 2X_{BC}\Omega_{1A}{}^C\xi^A\partial_1\xi^B\,, \qquad \text{(B.75)}$$

$$\mathcal{C}^{[1;\sigma]}_2 = -\mathcal{H}^{(1)}_{BC}\Omega_{1A}{}^C\zeta\xi^A\partial_1\xi^B\,, \qquad \text{(B.76)}$$

$$\mathcal{C}^{[1;\sigma]}_1 = -\mathcal{H}_{BC}\Omega_{1A}{}^C\xi^A\partial_1\xi^B + \frac{1}{2}\Omega_{1AB}\xi^A\partial_0\xi^B\,, \qquad \text{(B.77)}$$

$$\mathcal{C}^{[1;\tau]}_1 = \frac{1}{2}\Omega_{0AB}\xi^A\partial_1\xi^B\,, \qquad \text{(B.78)}$$

$$\mathcal{C}^{[2;\tau,\sigma]}_2 = \frac{1}{2}\Omega_{1AC}\Omega_{0B}{}^C\xi^A\xi^B\,, \qquad \text{(B.79)}$$

where $\mathcal{C}^{[n;\sigma^\mu]}_p$ indicates a term with p derivatives, n external legs of type σ^μ. They can be used to be create the following combinations

$$c_1 = -\frac{i}{2}\langle\mathcal{C}^{[2;\tau,\sigma]}_2\mathcal{C}^{[0]}_1\mathcal{C}^{[0]}_1\rangle\,,\quad c_2 = -\langle\mathcal{C}^{[2;\tau,\sigma]}_2\mathcal{C}^{[0]}_2\rangle\,,\quad c_3 = \frac{1}{2}\langle\mathcal{C}^{[1;\tau]}_1\mathcal{C}^{[1;\sigma]}_1\mathcal{C}^{[0]}_1\mathcal{C}^{[0]}_1\rangle\,, \qquad \text{(B.80)}$$

$$c_4 = -i\langle\mathcal{C}^{[1;\tau]}_1\mathcal{C}^{[1;\sigma]}_1\mathcal{C}^{[0]}_2\rangle\,,\quad c_5 = -i\langle\mathcal{C}^{[1;\sigma]}_2\mathcal{C}^{[1;\tau]}_1\mathcal{C}^{[0]}_1\rangle\,,\quad c_6 = -\langle\mathcal{C}^{[1;\tau]}_3\mathcal{C}^{[1;\sigma]}_1\rangle\,. \qquad \text{(B.81)}$$

These should be supplemented with additional terms coming from integral with external momenta insertion. The relevant basis of tensors is simply

$$\mathrm{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega)\,, \quad \mathrm{Tr}(\mathcal{H}^{(2)}\Omega\Omega)\,. \tag{B.82}$$

c_1

In this case

$$\begin{aligned} c_1 &= -\frac{1}{16}\mathcal{H}^{(1)}_{CD}\mathcal{H}^{(1)}_{EF}\Omega_{1AI}\Omega_{0B}{}^{I}\langle\zeta_{\sigma_2}\zeta_{\sigma_3}\rangle\langle\xi^A_{\sigma_1}\xi^B_{\sigma_1}\partial_1\xi^C_{\sigma_2}\partial_1\xi^D_{\sigma_2}\partial_1\xi^E_{\sigma_3}\partial_1\xi^F_{\sigma_3}\rangle \\ &= -\frac{1}{2\lambda}[[k_1^2(p_0^2+p_1^2)-2k_1k_0p_1p_0]]_{2,1,1}\,\mathrm{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega)\partial_0 y\partial_1 y \\ &= \frac{1}{2\lambda}\left(\mathbf{LI}-\frac{1}{2}\mathbf{S}\right)\mathrm{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega)\partial_0 y\partial_1 y\,. \end{aligned} \tag{B.83}$$

c_2

In c_2 we discard a disconnected contraction and obtain

$$\begin{aligned} c_2 &= -\frac{i}{2}X_{CD}\Omega_{1AI}\Omega_{0B}{}^{I}\langle\xi^A_{\sigma_1}\xi^B_{\sigma_2}\partial_1\xi^C_{\sigma_2}\partial_2\xi^D_{\sigma_2}\rangle-\frac{i}{8}\mathcal{H}^{(2)}_{CD}\Omega_{1AI}\Omega_{0B}{}^{I}\langle\zeta_{\sigma_2}\zeta_{\sigma_2}\rangle\langle\xi^A_{\sigma_1}\xi^B_{\sigma_1}\partial_1\xi^C_{\sigma_2}\partial_1\xi^D_{\sigma_2}\rangle \\ &= -\frac{i}{2}\Omega_{1AI}\Omega_{0B}{}^{I}\left(X+\frac{i}{4}\mathbf{I}\mathcal{H}^{(2)}\right)_{CD}\langle\xi^A_1\xi^B_1\partial_1\xi^C_2\partial_1\xi^D_2\rangle \\ &= \frac{1}{4\lambda}(\mathbf{I}-\mathbf{P})[[p_0^2-p_1^2]]_{2,0,0}\,\mathrm{Tr}(\mathcal{H}^{(2)}\Omega\Omega)\partial_0 y\partial_1 y \\ &\quad -\frac{1}{2\lambda}\left(\mathbf{I}[[p_1^2]]_{2,0,0}+\frac{1}{2}\mathbf{P}[[p_0^2-p_1^2]]_{2,0,0}\right)\mathrm{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega)\partial_0\partial_1 y \\ &= -\frac{1}{2\lambda}\mathbf{I}\left(\mathbf{L}+\frac{\mathbf{1}}{\mathbf{2}}\mathbf{P}\right)\mathrm{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega)\partial_0 y\partial_1 y+\frac{1}{4\lambda}\mathbf{I}\left(\mathbf{I}-\mathbf{P}\right)\mathrm{Tr}(\mathcal{H}^{(2)}\Omega\Omega)\partial_0 y\partial_1 y\,. \end{aligned} \tag{B.84}$$

c_3

This combination comprises of square envelope and diamond sunset topologies, making it the hardest to compute. However, one can show that the diamond sunset topology does not ultimately contribute to the final result, as the corresponding integral always contains odd powers of 0-components.

$$\begin{aligned} c_3 = &-\frac{i}{16}\Omega_{0AB}\mathcal{H}^{(1)}_{CD}\mathcal{H}^{(1)}_{EF}\mathcal{H}_{HI}\Omega_1{}^{I}{}_{G}\langle\xi^G_{\sigma_1}\partial_1\xi^H_{\sigma_1}\xi^A_{\sigma_2}\partial_1\xi^B_{\sigma_2}\partial_1\xi^C_{\sigma_3}\partial_1\xi^D_{\sigma_3}\partial_1\xi^E_{\sigma_4}\partial_1\xi^F_{\sigma_4}\rangle\langle\zeta_{\sigma_3}\zeta_{\sigma_4}\rangle \\ &-\frac{i}{32}\mathcal{H}^{(1)}_{CD}\mathcal{H}^{(1)}_{EF}\Omega_{1GH}\Omega_{0AB}\langle\xi^G_{\sigma_1}\partial_0\xi^H_{\sigma_1}\xi^A_{\sigma_2}\partial_1\xi^B_{\sigma_2}\partial_1\xi^C_{\sigma_3}\partial_1\xi^D_{\sigma_3}\partial_1\xi^E_{\sigma_4}\partial_1\xi^F_{\sigma_4}\rangle\langle\zeta_{\sigma_3}\zeta_{\sigma_4}\rangle \end{aligned}$$

$$= \frac{1}{\lambda}[[(k_1^2(p_0^2+2p_1^2)-k_0k_1p_0p_1)(p_0^2-p_1^2)+4k_1p_0p_1^2(k_0p_1-k_1p_0)]]_{3,1,1}\operatorname{Tr}(\mathcal{H}^{(1,1,0)}\Omega_0\Omega_1)$$
$$= \frac{1}{\lambda}\left(\frac{3}{4}\mathbf{S}-2\mathbf{TE}-\frac{1}{2}\mathbf{LI}+2\mathbf{TI}\right)\operatorname{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega)\partial_0 y\partial_1 y\,. \qquad \text{(B.85)}$$

c_4

In this case the topologies involved are triangle (with counter-term insertion) and decorated triangle.

$$\begin{aligned}
c_4 &= \frac{1}{2}\mathcal{H}_{DI}{\Omega_1}^I{}_C X_{EF}\Omega_{0AB}\langle\xi^C_{\sigma_1}\xi^A_{\sigma_2}\partial_1\xi^D_{\sigma_1}\partial_1\xi^B_{\sigma_2}\partial_1\xi^E_{\sigma_3}\partial_1\xi^F_{\sigma_3}\rangle\\
&\quad+\frac{1}{8}\mathcal{H}^{(2)}_{EF}\mathcal{H}_{DI}{\Omega_1}^I{}_C\Omega_{0AB}\langle\zeta^2_{\sigma_3}\rangle\langle\xi^C_{\sigma_1}\xi^A_{\sigma_2}\partial_1\xi^D_{\sigma_1}\partial_1\xi^B_{\sigma_2}\partial_1\xi^E_{\sigma_3}\partial_1\xi^F_{\sigma_3}\rangle\\
&\quad+\frac{1}{4}X_{EF}\Omega_{1CD}\Omega_{0AB}\langle\xi^C_{\sigma_1}\xi^A_{\sigma_2}\partial_0\xi^D_{\sigma_1}\partial_1\xi^B_{\sigma_2}\partial_1\xi^E_{\sigma_3}\partial_1\xi^F_{\sigma_3}\rangle\\
&\quad+\frac{1}{16}\mathcal{H}^{(2)}_{EF}\Omega_{1CD}\Omega_{0AB}\langle\zeta^2_{\sigma_3}\rangle\langle\xi^C_{\sigma_1}\xi^A_{\sigma_2}\partial_0\xi^D_{\sigma_1}\partial_1\xi^B_{\sigma_2}\partial_1\xi^E_{\sigma_3}\partial_1\xi^F_{\sigma_3}\rangle\\
&= \frac{1}{\lambda}\left(\frac{1}{2}\mathbf{P}[[(p_0^2-p_1^2)^2]]_{3,0,0}-\frac{1}{2}\mathbf{I}[[p_1^2(3p_1^2+p_0^2)]]_{3,0,0}\right)\operatorname{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega)\partial_0 y\partial_1 y\\
&\quad+\frac{1}{2\lambda}(\mathbf{P}-\mathbf{I})[[(p_0^2-p_1^2)^2]]_{3,0,0}\operatorname{Tr}(\mathcal{H}^{(2)}\Omega\Omega)\partial_0 y\partial_1 y\\
&= \frac{\mathbf{I}}{2\lambda}(\mathbf{P}-\mathbf{L}-4\mathbf{T})\operatorname{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega)\partial_0 y\partial_1 y+\frac{\mathbf{I}}{2\lambda}(\mathbf{P}-\mathbf{I})\operatorname{Tr}(\mathcal{H}^{(2)}\Omega\Omega)\partial_0 y\partial_1 y\,. \qquad \text{(B.86)}
\end{aligned}$$

c_5

The only topology involved here are triangle envelopes.

$$\begin{aligned}
c_5 &= \frac{1}{4}\mathcal{H}^{(1)}_{BL}\mathcal{H}^{(1)}_{EF}{\Omega_1}^L{}_A\Omega_{0CD}\langle\zeta_{\sigma_1}\zeta_{\sigma_3}\rangle\langle\xi^A_{\sigma_1}\xi^C_{\sigma_2}\partial_1\xi^B_{\sigma_1}\partial_1\xi^F_{\sigma_2}\partial_1\xi^E_{\sigma_3}\partial_1\xi^F_{\sigma_3}\rangle\\
&= -\frac{1}{\lambda}[[k_1^2(p_0^2+p_1^2)-2k_0k_1p_0p_1]]_{2,1,1}\operatorname{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega)\partial_0 y\partial_1 y\\
&= \frac{1}{\lambda}\left(\mathbf{LI}-\frac{1}{2}\mathbf{S}\right)\operatorname{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega)\partial_0 y\partial_1 y\,. \qquad \text{(B.87)}
\end{aligned}$$

c_6

This one contains only loop and decorated loop topologies.

$$\begin{aligned}
c_6 &= iX_{DE}\Omega_{0AB}\Omega^E_1{}_C\langle\xi^C_{\sigma_1}\xi^A_{\sigma_2}\partial_1\xi^D_{\sigma_1}\partial_1\xi^B_{\sigma_2}\rangle+\frac{i}{4}\mathcal{H}^{(2)}_{DI}{\Omega_1}^I{}_C\Omega_{0AB}\langle\zeta_{\sigma_1}\zeta_{\sigma_1}\rangle\langle\xi^C_{\sigma_1}\xi^A_{\sigma_2}\partial_1\xi^D_{\sigma_1}\partial_1\xi^B_{\sigma_2}\rangle\\
&= -\frac{1}{2\lambda}\left(\mathbf{P}[[p_0^2]]_{2,0,0}+\mathbf{I}[[p_1^2]]_{2,0,0}\right)\operatorname{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega)\partial_0 y\partial_1 y\\
&\quad+\frac{1}{2\lambda}[[p_0^2]]_{2,0,0}(\mathbf{I}-\mathbf{P})\operatorname{Tr}(\mathcal{H}^{(2)}\Omega\Omega)\partial_0 y\partial_1 y
\end{aligned}$$

$$= -\frac{1}{2\lambda}\left(\mathbf{IP}+\mathbf{LI}+\mathbf{LP}\right)\mathrm{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega)\partial_0 y\partial_1 y + \frac{1}{2\lambda}(\mathbf{I}+\mathbf{L})(\mathbf{I}-\mathbf{P})\,\mathrm{Tr}(\mathcal{H}^{(2)}\Omega\Omega)\partial_0 y\partial_1 y\,. \tag{B.88}$$

Contributions From External Momentum Insertion

There are a few contributors to $\partial_0 y\partial_1 y$ legs that $c_1,\ldots,c_6$ have missed. These arise from loop integral with non-vanishing external momentum. A careful analysis reveals that in fact only the sunset topology with a linear insertion of external momentum is responsible for such contributions. Also, we shall only get a correction, call it c_7, to $\mathrm{Tr}(\mathcal{H}^{(2)}\Omega\Omega)$. We find

$$\begin{aligned} c_7 &= \frac{2}{\lambda}[[k_1p_0(k_1p_0-k_0p_1)]]_{2,1,1}\,\mathrm{Tr}(\mathcal{H}^{(2)}\Omega\Omega)\partial_0 y\partial_1 y \\ &= \frac{1}{\lambda}\left(\frac{3}{2}\mathbf{S}-\mathbf{LI}\right)\mathrm{Tr}(\mathcal{H}^{(2)}\Omega\Omega)\partial_0 y\partial_1 y\,. \end{aligned} \tag{B.89}$$

B.4 Results of Wick Contractions

Before we evaluate the loop integrals, we collect and present in this section the results of the standard Wick contractions. To slightly reduce the length of expressions, each time a numerator of a loop integrand contains an even number of τ-components of momenta, these are swapped for the invariant combination e.g. $p_0^2 \to p^2+p_1^2$, though at this stage no assumption about how this holds in $d=2+\epsilon$ is made and, in particular, no cancellations of factors of p^2 between numerator and denominator are employed up to this point. For the counter-term attached to $\lambda^{-2}\partial_1\mathbb{X}^I\partial_1\mathbb{X}^J$ we obtain

$$\begin{aligned} \mathcal{H}^{(4)}_{\bullet\bullet}:&\quad \frac{1}{16}\mathbf{I}^2-\frac{1}{8}\mathbf{IP}\\ \mathcal{H}^{(3,1,0)}_{\bullet\bullet}:&\quad \frac{1}{2}(\mathbf{P}-\mathbf{I})[[p_1^2]]_{2,0,0}-\frac{1}{4}\mathbf{IP}\\ \mathcal{H}^{(2,0,2)}_{\bullet\bullet}:&\quad \frac{1}{4}\mathbf{IP}+\frac{1}{4}[[p_1^2]]_{1,1,1}\\ \mathcal{H}^{(2,1,1)}_{\bullet\bullet}:&\quad \frac{1}{2}\mathbf{IP}-\frac{1}{2}\mathbf{P}[[p_1^2]]_{2,0,0}+[[p_1k_1p\cdot k]]_{2,1,1}\\ \mathcal{H}^{(1,2,1)}_{\bullet\bullet}:&\quad \frac{1}{8}\mathbf{IP}-\frac{1}{2}\mathbf{P}[[p_1^2]]_{2,0,0}-\frac{1}{2}[[p_1^2k_1^2]]_{2,2,0}-\frac{1}{4}(\mathbf{P}-\mathbf{I})[[p_1^2p^2]]_{3,0,0}\\ \mathcal{H}^{(1,1,1,1,0)}_{\bullet\bullet}:&\quad \frac{1}{2}\mathbf{P}[[p_1^2]]_{2,0,0}+[[p_1^2k_1^2]]_{2,2,0}+\frac{1}{2}\mathbf{I}[[p_1^4]]_{3,0,0}+\frac{1}{4}\mathbf{P}[[p_1^2p^2]]_{3,0,0}\\ &\quad -[[p_1^3k_1k\cdot p]]_{3,1,1}+\frac{1}{2}[[p_1^2k_1^2p^2]]_{3,1,1}-[[(p_1^2k_1^2+p_1^3k_1)k^2]]_{2,2,1}\\ \mathcal{H}^{(2)}_{\bullet\bullet}\,\mathrm{Tr}(\mathcal{H}^{(1,1)}):&\quad -\frac{1}{32}\mathbf{P}[[p^2]]_{2,0,0}+\frac{1}{8}[[(p+k)_1k_1(p\cdot k+k^2)]]_{2,1,1}\\ \mathcal{H}^{(1,1,0)}_{\bullet\bullet}\,\mathrm{Tr}(\mathcal{H}^{(1,1)}):&\quad \frac{1}{16}\mathbf{P}[[p_1^2p^2]]_{3,0,0}-\frac{1}{4}[[(p_1+k_1)p_1^2k_1(p\cdot k+k^2)]]_{3,1,1} \end{aligned} \tag{B.90}$$

For counter-term attached to $\partial_0 y \partial_0 y$ external legs we find:

$$
\begin{aligned}
\mathrm{Tr}(\mathcal{H}^{(2,2)}): \quad & [[k_1(k_1+p_1)(p^2+p_1^2)(k^2+k\cdot p)]]_{3,1,1} \\
& -\frac{1}{4}[[k_1(k_1+p_1)(k^2+k\cdot p)]]_{2,1,1} \\
\mathrm{Tr}(\mathcal{H}^{(3,1)}): \quad & \frac{1}{8}\left(\mathbf{P}-\mathbf{I}\right)[[p_1^2]]_{2,0,0} \\
\mathrm{Tr}(\mathcal{H}^{(1,1,1,1)}): \quad & \frac{1}{8}\mathbf{P}[[p_1^2p^2]]_{3,0,0}+\frac{1}{4}\mathbf{I}[[p_1^4]]_{3,0,0}+\frac{1}{4}[[k_1p_1^2(k_1p^2-2p_1k\cdot p)]]_{3,1,1} \\
& -\frac{1}{16}[[k_1^2p^2-4p_1k_1k\cdot p+3p_1^2k^2]]_{2,2,1} \\
\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega): \quad & \frac{1}{4}\mathbf{I}[[p_1^2(3p^2+4p_1^2)]]_{3,0,0}+\frac{1}{2}[[k_1(k\cdot p(k_1-p_1)+k^2(k_1+p_1))]]_{2,1,1} \\
& -\frac{1}{2}[[k_1(2p^2k_1p_1^2+4k^2(k_1+p_1)(p^2+p_1^2))]]_{3,1,1} \\
& -\frac{1}{2}[[k_1k\cdot p(4k_1p_1^2+p^2(4k_1+p_1)))]]_{3,1,1} \\
\mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega): \quad & -\frac{1}{2}[[k_1(k\cdot p(k_1-p_1)+k^2(k_1+p+1))]]_{2,1,1} \\
& +2[[k_1(k_1(k\cdot p+k^2)p^2+p_1k^2p^2]]_{3,1,1} \\
& +2[[k_1(k_1p_1^2(k\cdot p+k^2+p^2)+p_1^3(k^2-k\cdot p))]]_{3,1,1} \\
\mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega): \quad & \frac{1}{4}(\mathbf{P}-\mathbf{I})[[p_1^2]]_{2,0,0}-\frac{1}{4}(\mathbf{P}-\mathbf{I})[[p_1^2p^2]]_{3,0,0} \\
& +[[k_1(k^2(k_1+p_1)+k_1k\cdot p)]]_{2,1,1} \\
& -2[[k_1(k_1p_1^2p^2+2(k_1+p_1)k^2(p^2+p_1^2)]]_{3,1,1} \\
& -2[[k_1(2k_1p_1^2+2k_1p^2+p_1p^2)k\cdot p)]]_{3,1,1}\,.
\end{aligned}
\tag{B.91}
$$

For the counter-term attached to mixed derivatives of the background field $\partial_0 y \partial_1 y$ we obtain:

$$
\begin{aligned}
\mathrm{Tr}(\mathcal{H}^{(1,1,0)}\Omega\Omega): \quad & -\frac{1}{4}\mathbf{P}[[3p^2-2p_1^2]]_{2,0,0}-\mathbf{I}[[p_1^2]]_{2,0,0}+\frac{1}{2}\mathbf{I}[[(p^2)^2]]_{3,0,0} \\
& -\frac{1}{2}\mathbf{I}[[p^2p_1^2-4p_1^4]]_{3,0,0}-\frac{3}{2}[[k_1^2p^2-2k_1p_1k\cdot p]]_{2,1,1} \\
& +[[k_1(k_1(p^2)^2+4p_1^3k\cdot p-p_1p^2(k\cdot p+2k_1p_1))]]_{3,1,1} \\
\mathrm{Tr}(\mathcal{H}^{(2)}\Omega\Omega): \quad & -\frac{1}{4}(\mathbf{P}-\mathbf{I})[[3p^2+2p_1^2]]_{2,0,0}+\frac{1}{2}(\mathbf{P}-\mathbf{I})[[(p^2)^2]]_{3,0,0} \\
& +2[[k_1(k_1(p^2)^2-2k_1p_1^2p^2+4p_1^3k\cdot p)]]_{3,1,1}
\end{aligned}
\tag{B.92}
$$

For the counter-term attached to $\partial_1 y \partial_1 y$ we obtain

$$
\begin{aligned}
\mathrm{Tr}(\mathcal{H}^{(2,2)}): \quad & [[k_1(k_1+p_1)p_1^2(k^2+k\cdot p)]]_{3,1,1}+\frac{1}{4}[[k_1(k_1+p_1)(k^2+k\cdot p)]]_{2,1,1} \\
\mathrm{Tr}(\mathcal{H}^{(3,1)}): \quad & \frac{1}{8}\mathbf{I}^2-\frac{1}{8}\mathbf{P}[[p^2-p_1^2]]_{2,0,0}+\frac{3}{8}\mathbf{I}[[p_1^2]]_{2,0,0}-\frac{1}{2}\mathbf{P}[[p_1^2p^2]]_{3,0,0}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}(\mathcal{H}^{(1,1,1,1)}):\quad & \frac{1}{8}\mathbf{P}[[p^2+2p_1^2]]_{2,0,0}+\frac{1}{4}\mathbf{I}[[p_1^2]]_{2,0,0}-\frac{1}{8}\mathbf{P}[[p_1^2p^2+(p^2)^2]]_{3,0,0}\\
&+\frac{1}{4}\mathbf{I}[[p_1^4]]_{3,0,0}+\frac{1}{4}[[k_1^2+k_1p_1]]_{1,1,1}+\frac{1}{4}[[k_1^2p^2-2p_1(p_1+2k_1)k\cdot p]]_{2,1,1}\\
&-\frac{1}{4}[[k_1(k_1p^2-2p_1k\cdot p)(p^2-p_1^2)]]_{3,1,1}+\frac{1}{16}[[k_1^2p^2(k\cdot p+k_1p_1)]]_{2,2,1}\\
\mathrm{Tr}(\mathcal{H}^{(1,1)}\Omega\Omega):\quad & \frac{1}{4}\mathbf{I}[[p^2+6p_1^2]]_{2,0,0}-\frac{1}{4}\mathbf{I}[[(p^2)^2+p_1^2p^2-4p_1^4]]_{3,0,0}\\
&-\frac{1}{2}[[\frac{(k_1+p_1)(p^2+k\cdot p)}{p_1}]]_{1,1,1}\\
&-\frac{1}{2}[[\frac{k_1}{p_1}(-2p^2k\cdot p+k_1p_1(k\cdot p+k^2-3p^2)+5p_1^2(k^2+k\cdot p))]]_{2,1,1}\\
&-\frac{1}{2}[[\frac{k_1}{p_1}((p^2)^2k\cdot p+2p_1k_1(p^2)^2+p_1^2p^2k\cdot p]]_{3,1,1}\\
&-[[\frac{k_1}{p_1}(k_1p_1^3(p^2+2k\cdot p+2k^2)+2k^2p_1^4)]]_{3,1,1}\\
\mathrm{Tr}(\mathcal{H}^{(1)}\Omega\mathcal{H}^{(1)}\Omega):\quad & -[[p^2+k\cdot p]]_{1,1,1}\\
&+\frac{1}{2}[[(4p_1^2-k_1p_1+k_1^2-2p^2)k\cdot p+k_1(k^2k_1+5p_1k^2-2p_1p^2)]]_{2,1,1}\\
&+2[[k_1p_1(p^2k\cdot p)+k_1p_1(k\cdot p+k^2+p^2)+(k^2-k\cdot p)p_1^2]]_{3,1,1}\\
&-\frac{1}{8}[[k_1p_1(-2k^2p^2+k_1^2p^2+7p_1^2k^2)]]_{2,2,1}\\
&-\frac{1}{8}[[k\cdot p(k_1^2(3p^2-8p_1^2)+k^2(-2p^2+5p_1^2))]]_{2,2,1}\\
\mathrm{Tr}(\mathcal{H}^{(2,1)}\Omega):\quad & -\frac{1}{4}\mathbf{P}[[2p^2+p_1^2]]_{2,0,0}+\frac{1}{4}\mathbf{I}[[p^2+5p_1^2]]_{2,0,0}\\
&-\frac{1}{4}(\mathbf{P}-\mathbf{I})[[(p^2)^2]]_{3,0,0}-\frac{3}{4}\mathbf{P}[[p^2p_1^2]]_{3,0,0}-\frac{1}{4}\mathbf{I}[[p_1^2p^2]]_{3,0,0}\\
&+[[k_1(k^2(k_1+p_1)+k_1k\cdot p)]]_{2,1,1}\\
&-2[[k_1(k_1p_1^2p^2+2(k_1+p_1)k^2(p^2+p_1^2)]]_{3,1,1}\\
&-2[[k_1(2k_1p_1^2+2k_1p^2+p_1p^2)k\cdot p)]]_{3,1,1}\,.
\end{aligned}
\qquad \text{(B.93)}
$$

Appendix C

Loop Integrals

C.1 Loop Integrals via $O(d)$-invariance (Method 1)

For a L-loop calculation, $2L$ copies of the worldsheet are needed. We shall label each copy by a number and indicate the n-th with $\mathbf{n}$. Propagators stretching from $\mathbf{n}$ to $\mathbf{m}$ will be reported schematically as $\mathbf{n} \to \mathbf{m}$, where $\mathbf{n}$ and $\mathbf{m}$ are allowed to coincide. Accordingly, a sequence with the same extrema shall indicate a closed loop.

At one-loop order, two topologies only contribute to the β-function: having two copies of the worldsheet, propagators can either stretch from $\mathbf{1}$ to $\mathbf{2}$ or close on the same copy. All possible loops are exhausted by $\mathbf{1} \to \mathbf{1}$ and $(\mathbf{1} \to \mathbf{2})^2$. We shall name them *bubble* and *loop*, respectively.

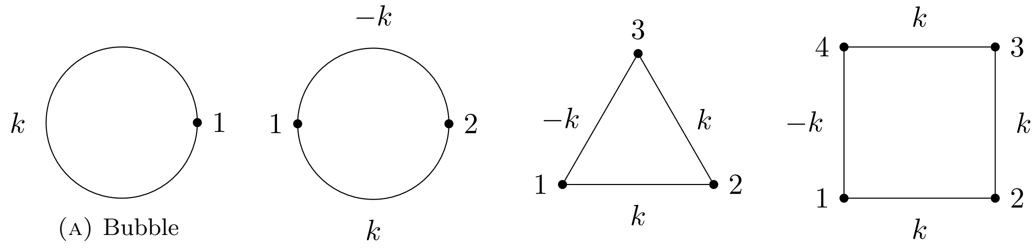
At two-loop order, divergences can originate from either i) one-loop integrals multiplied by a $1/\epsilon$ pole due to one-loop counter-term insertions; ii) products of one-loop integrals; iii) genuinely new two-loop integrals. New topologies for one-loop diagrams arise in i) and ii).

For a two-loop calculation, triangle-shaped and square-shaped loops, corresponding schematically to $\mathbf{1} \to \mathbf{2} \to \mathbf{3} \to \mathbf{1}$ and $\mathbf{1} \to \mathbf{2} \to \mathbf{3} \to \mathbf{4} \to \mathbf{1}$, are in fact allowed. For the sake of simplicity, we will refer to them as *triangle* and *square* diagrams. On top of that, $\mathbf{1} \to \mathbf{1}$ and $(\mathbf{1} \to \mathbf{2})^2$ can still appear, even though their finite $O(\epsilon^0)$ parts are now to be kept[1].

Graphs of type ii) can all be seen graphically as dressings of the previous diagrams (up to the squares) with an extra loop or bubble. When a bubble is added, we call the resulting diagram "decorated": for example, adding a bubble to a triangle results in a *decorated triangle*.

[1] Actually, square diagrams are only required in calculating one-loop diagrams with external quantum fields.

Scenario iii) is the most intricate as it allows for many different topologies. The simplest instance of genuine two-loop diagram is the *sunset* diagram, corresponding to $(\mathbf{1}\to\mathbf{2})^3$. The only way to non-trivially extend the triangle diagram to two-loop is by adding an internal line, resulting in $(\mathbf{1}\to\mathbf{2})^2\to\mathbf{3}\to\mathbf{1}$. We shall call it *triangle envelope* diagram. Squares allow for two extensions: we can either add a line joining two adjacent vertices, $(\mathbf{1}\to\mathbf{2})^2\to\mathbf{3}\to\mathbf{4}$ (*square envelope*), or let is stretch to the opposite vertex, $(\mathbf{1}\to\mathbf{2}\to\mathbf{3}\to\mathbf{4}\to\mathbf{1})(\mathbf{1}\to\mathbf{3})$ (*diamond sunset*).



(A) Bubble (B) Loop (C) Triangle (D) Square

FIGURE C.1: One-loop diagrams.

C.1.1 Combinatorics

The basic integrals in Minkowski space are given by

$$I_n = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2-m^2)^n} = \frac{(-1)^n i}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}(m^2)^{\frac{d}{2}-n}\,. \tag{C.1}$$

Our aim is to recast divergent 1-loop and 2-loop integrals as combinations of the basic integrals I_n, postponing the evaluation of their precise dependence on the dimensional-regulator ϵ. At 2-loop order we require only I_1, I_2 and I_3 which are given by

$$I_1 \approx \frac{i}{2\pi\epsilon}+\frac{i\bar\gamma}{4\pi}+\frac{i\epsilon\left(6\bar\gamma^2+\pi^2\right)}{96\pi}\,,\quad m^2 I_2 \approx +\frac{i}{4\pi}+\frac{i\epsilon\bar\gamma}{8\pi}\,,\quad m^4 I_3 \approx -\frac{i}{8\pi}-\frac{i\epsilon\,(\bar\gamma-1)}{16\pi}\,, \tag{C.2}$$

where

$$\bar\gamma = \gamma_E + \log\left(\frac{m^2}{4\pi}\right)\,. \tag{C.3}$$

Integrals with non-positive n can also be dropped: $\Gamma(n)$ has poles for $n\in\mathbb{Z}^-$, thus forcing $I_{n\le 0}\to 0$.

Prior to venturing into the explicit evaluation of loop integrals, let us pause and analyse their combinatorial structure. Fix some integer number $q\in\mathbb{N}$ and consider the one-loop integral

$$I_{\lambda_1\ldots\lambda_{2q}} = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{p_{\lambda_1}\ldots p_{\lambda_{2q}}}{D} \tag{C.4}$$

for some denominator $D = D(p^2)$. Integrals of this form are usually evaluated assuming $O(d)$ symmetry in the final result; that is, we postulate that the right-hand side can be recast in the form

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{p_{\lambda_1} \dots p_{\lambda_{2q}}}{D} = A(\eta_{\lambda_1\lambda_2} \dots \eta_{\lambda_{2q-1}\lambda_{2q}} + \text{perms.}) \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{(p^2)^q}{D}\,, \tag{C.5}$$

where the "scalar" integral is in general known, and the prefactor A is to be fixed by taking appropriate contractions of both the left- and right-hand side with the d-dimensional Minkowski metric.

Let us indicate by $P_{\lambda_1 \dots \lambda_{2q}}$ the rank $2q$ totally symmetric tensor made up of all possible permutations of tensor products of the d-dimensional Minkowski metric

$$P_{\lambda_1 \dots \lambda_{2q}} = \eta_{\lambda_1\lambda_2} \dots \eta_{\lambda_{2q-1}\lambda_{2q}} + \text{perms.} \tag{C.6}$$

As can be easily seen, a total number of $(2q-1)!!$ terms have to be accounted for in the permutations. Particularly important is the trace of this object, $P^{(2q)}$,

$$P^{(2q)} \equiv \eta^{\lambda_1\lambda_2} \dots \eta^{\lambda_{2q-1}\lambda_{2q}} P_{\lambda_1 \dots \lambda_{2q}} = \prod_{j=0}^{q-1} (d+2j) = 2^{q-1} d \left(1 + \frac{d}{2}\right)_{q-1}\,. \tag{C.7}$$

To be concrete, the constant A in (C.5) would be $1/P^{(2q)}$.

In the case of two-loop integrals, the situation is more involved as there are two distinct momenta to deal with. A prototypical example is for instance

$$\begin{aligned} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k_\mu k_\nu p_{\lambda_1} \dots p_{\lambda_{2q}}}{D} &= A\eta_{\mu\nu}(\eta_{\lambda_1\lambda_2} \dots \eta_{\lambda_{2q-1}\lambda_{2q}} + \text{perms.}) \\ &+ B(\eta_{\mu\lambda_1}\eta_{\nu\lambda_2}\eta_{\lambda_3\lambda_4} \dots \eta_{\lambda_{2q-1}\lambda_{2q}} + \text{perms.})\,, \end{aligned} \tag{C.8}$$

where A, B will now be combinations of scalar loop integrals and some combinatorial factors. Contracting both sides with the tensor $\eta^{\mu\nu}\eta^{\lambda_1\lambda_2} \dots \eta^{\lambda_{2q-1}\lambda_{2q}}$ we obtain a first equation

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^2 (p^2)^q}{D} = P^{(2q)}(dA + 2qB)\,. \tag{C.9}$$

A second equation is deduced in a similar manner by contracting with $\eta^{\mu\lambda_1}\eta^{\nu\lambda_2} \dots \eta^{\lambda_{2q-1}\lambda_{2q}}$

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(k \cdot p)^2 (p^2)^{q-1}}{D} = P^{(2q)} \left[A + (2q + d - 1)B\right]\,. \tag{C.10}$$

Solving (C.9) and (C.10) for A and B, one finds the correct rewriting of (C.8) in terms of scalar loop integrals and combinatorial factors.

In a theory with non-manifest Lorentz invariance, $P_{\lambda_1 \ldots \lambda_{2q}}$ is most often encountered with explicit values assigned to its indices (either 0 or 1). As we shall make large use of this formula, let us mention that if $P_{\lambda_1 \ldots \lambda_{2q}}$ comes with $2q_0$ 0-indices and $2q_1$ 1-indices (so that $2q = 2q_0 + 2q_1$), the tensor specifically evaluates to

$$P_{0_1 \ldots 0_{2q_0} 1_1 \ldots 1_{2q_1}} = (2q_0-1)!!\,(2q_1-1)!!\,(\eta_{00})^{q_0}(\eta_{11})^{q_1} = (2q_0-1)!!\,(2q-2q_0-1)!!\,(\eta_{00})^{q_0}(\eta_{11})^{q_1}\,. \tag{C.11}$$

C.1.2 One-loop Integrals

Bubble Diagrams

Integrals for bubble diagrams are trivial, as they simply coincide with I_1.

Loop Diagrams

Loop diagrams are only slightly more involved. In general we are interested in both the divergent and convergent (at least $O(\epsilon^0)$) parts, as the latter is important for counter-terms. The divergent diagrams we shall encounter are of the form

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k_\mu k_\nu}{(k^2 - m^2)^2} = \frac{\eta_{\mu\nu}}{d}(I_1 + m^2 I_2)\,. \tag{C.12}$$

Diagrams with an odd number of momenta in the numerator vanish by symmetry. Fully convergent integrals are necessary of the form

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)^2} = I_2\,. \tag{C.13}$$

Triangle Diagrams

Divergent integrals stemming from triangle diagrams are of the form

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2 - m^2)^3} = \frac{1}{d(d+2)}(\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})(I_1 + 2m^2 I_2 + m^4 I_3)\,, \tag{C.14}$$

where we have made use of $O(d)$ symmetry to perform the integral. By the remarks above, the right-hand side is mass independent at zero-th order in ϵ. Other finite results are

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)^3} = I_3\,, \qquad \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k_\mu k_\nu}{(k^2 - m^2)^3} = \frac{\eta_{\mu\nu}}{d}(I_2 + m^2 I_3)\,. \tag{C.15}$$

C.1.3 Two-loop Integrals

A two-loop integral with vanishing external momenta is of the form

$$[[\ldots]]_{ijk} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{(\ldots)}{(p^2 - m^2)^i (k^2 - m^2)^j [(k+p)^2 - m^2]^k} \,, \tag{C.16}$$

where the dots will be specified on a case-by-case basis. In performing manipulation we discard finite terms, for instance:

$$[[1]]_{i,j,k} = 0 \,, \quad \text{for } i, j, k \geq 1 \,. \tag{C.17}$$

Typical UV divergent integrals encountered are

$$[[1]]_{1,0,1} = [[1]]_{1,1,0} = [[1]]_{0,1,1} = I_1^2 \;, \quad [[1]]_{2,0,1} = I_1 I_2 \;. \tag{C.18}$$

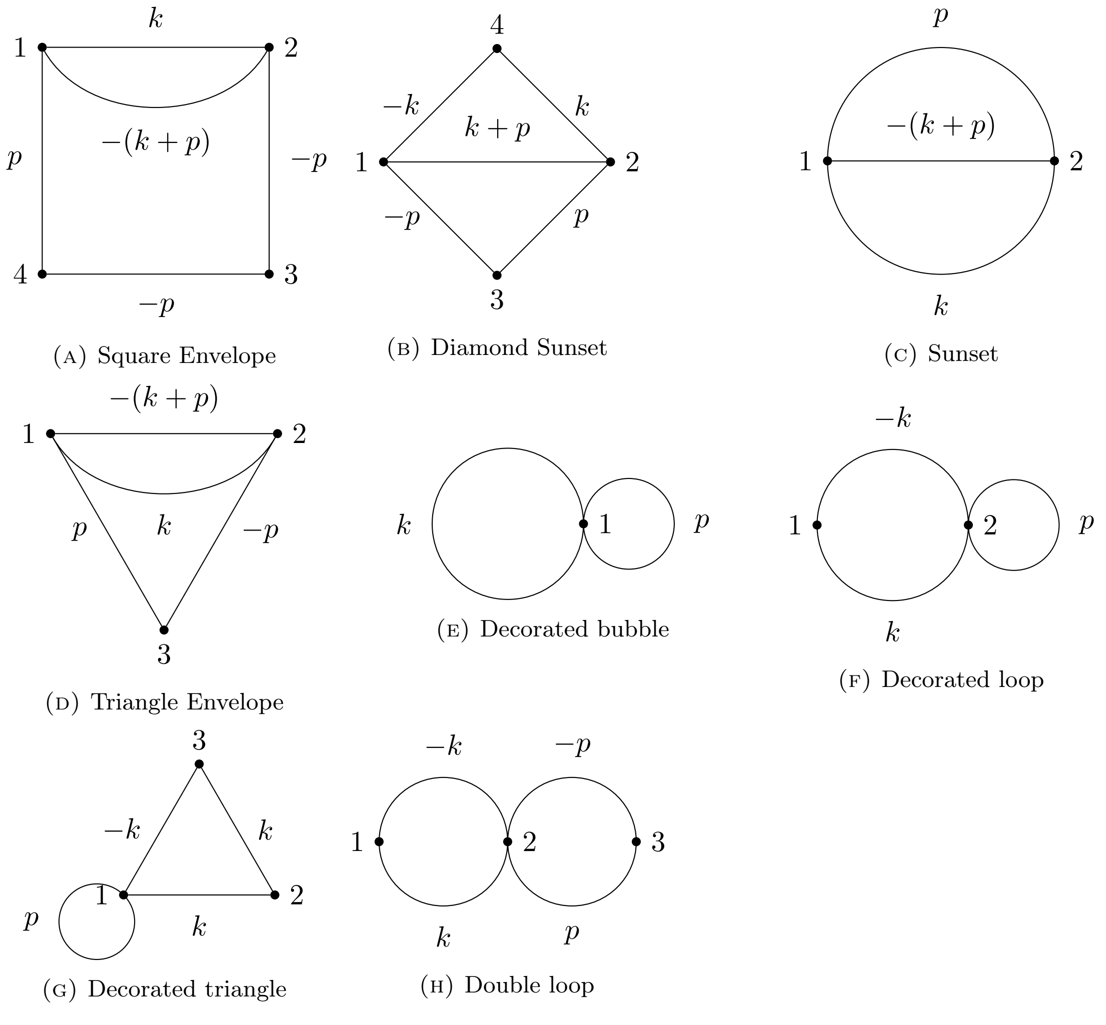


FIGURE C.2: Two-loop diagrams. Momentum flows are aligned with numerical ordering of vertices.

Sunset Diagrams

Sunset diagrams are equivalent to $[[\dots]]_{1,1,1}$ integrals. Given the exchange symmetry $k \leftrightarrow p$ in the denominator, the only relevant integrals are

$$[[p^2]]_{1,1,1} = I_1^2\,, \qquad [[(p\cdot k)]]_{1,1,1} = -\frac{1}{2}I_1^2\,. \tag{C.19}$$

It immediately follows that

$$[[p_\mu p_\nu]]_{1,1,1} = \frac{1}{d}\eta_{\mu\nu}I_1^2\,, \qquad [[p_\mu k_\nu]]_{1,1,1} = -\frac{1}{2d}\eta_{\mu\nu}I_1^2\,. \tag{C.20}$$

Diamond Sunset Diagrams

Diamond sunset diagrams correspond to $[[\dots]]_{2,2,1}$. Notice that this class is symmetric under $k \leftrightarrow p$; hence, we will omit integrals which can be deduced from symmetry arguments. Again, we begin with the scalar integrals

$$[[(p^2)^3]]_{2,2,1} = I_1^2 + 4m^2 I_1 I_2\,, \tag{C.21a}$$
$$[[(p^2)^2(p\cdot k)]]_{2,2,1} = -I_1^2 - m^2 I_1 I_2\,, \tag{C.21b}$$
$$[[(p^2)^2 k^2]]_{2,2,1} = I_1^2 + m^2 I_1 I_2\,, \tag{C.21c}$$
$$[[p^2(p\cdot k)k^2]]_{2,2,1} = -\frac{1}{2}I_1^2\,, \tag{C.21d}$$
$$[[p^2(p\cdot k)^2]]_{2,2,1} = \frac{3}{4}I_1^2 + \frac{1}{2}m^2 I_1 I_2\,, \tag{C.21e}$$
$$[[(p\cdot k)^3]]_{2,2,1} = \frac{2-3d}{4d}I_1^2 + \frac{2-d}{2d}m^2 I_1 I_2\,. \tag{C.21f}$$

Tensorial integrals can now be deduced

$$[[p_\mu p_\nu p_\rho p_\sigma p_\kappa p_\lambda]]_{2,2,1} = F_1(\eta_{\mu\nu}\eta_{\rho\sigma}\eta_{\kappa\lambda} + 14\,\text{perms})(I_1^2 + 4m^2 I_1 I_2)\,, \tag{C.22a}$$
$$[[p_\mu p_\nu p_\rho p_\sigma p_\kappa k_\lambda]]_{2,2,1} = F_1(\eta_{\mu\nu}\eta_{\rho\sigma}\eta_{\kappa\lambda} + 14\,\text{perms})\left(-I_1^2 - m^2 I_1 I_2\right)\,, \tag{C.22b}$$
$$\begin{aligned}[[p_\mu p_\nu p_\rho p_\sigma k_\kappa k_\lambda]]_{2,2,1} = {}& \frac{F_1}{d-1}(\eta_{\mu\nu}\eta_{\rho\sigma} + 2\,\text{perms})\eta_{\kappa\lambda}(dI_1^2 + (d+1)m^2 I_1 I_2)\\ &+ \frac{F_1}{4(d-1)}(\eta_{\mu\kappa}\eta_{\nu\lambda}\eta_{\rho\sigma} + 11\,\text{perms})((3d-4)I_1^2 + 2(d-2)m^2 I_1 I_2)\,,\end{aligned} \tag{C.22c}$$

$$\begin{aligned}[[p_\mu p_\nu p_\rho k_\sigma k_\kappa k_\lambda]]_{2,2,1} = {}& \frac{F_1}{4d(d-1)}(\eta_{\mu\nu}\eta_{\rho\sigma}\eta_{\kappa\lambda} + 8\,\text{perms})\\ &\times\left[-2(d^2-2d+2)I_1^2 + 4(d-2)m^2 I_1 I_2\right]\\ &+ \frac{F_1}{8d(d-1)}(\eta_{\mu\sigma}\eta_{\nu\kappa}\eta_{\rho\lambda} + 5\,\text{perms})\end{aligned}$$

$$\times \left[-2(3d^2 - 2d - 4) I_1^2 - 4(d^2 - 4) m^2 I_1 I_2 \right] . \tag{C.22d}$$

Triangle Envelope Diagrams

This class is equivalent to $[[\ldots]]_{2,1,1}$. Scalar integrals are

$$[[(p^2)^2]]_{2,1,1} = I_1^2 \, , \tag{C.23a}$$
$$[[p^2 k^2]]_{2,1,1} = I_1^2 + m^2 I_1 I_2 \, , \tag{C.23b}$$
$$[[(p \cdot k)^2]]_{2,1,1} = \frac{3}{4} I_1^2 + \frac{1}{2} m^2 I_1 I_2 \, , \tag{C.23c}$$
$$[[p^2 (p \cdot k)]]_{2,1,1} = -\frac{1}{2} I_1^2 \, , \tag{C.23d}$$
$$[[(p \cdot k) k^2]]_{2,1,1} = -I_1^2 - m^2 I_1 I_2 \, , \tag{C.23e}$$
$$[[k^4]]_{2,1,1} = I_1^2 + 3 m^2 I_1 I_2 \, . \tag{C.23f}$$

Adopting the symbol $F_2 = \frac{1}{d(d+2)}$, tensorial integrals evaluate to

$$[[p_\mu p_\nu p_\rho p_\sigma]]_{2,1,1} = F_2 (\eta_{\mu\nu} \eta_{\rho\sigma} + 2\,\text{perms}) I_1^2 \, , \tag{C.24a}$$
$$[[p_\mu p_\nu p_\rho k_\sigma]]_{2,1,1} = -\frac{1}{2} F_2 (\eta_{\mu\nu} \eta_{\rho\sigma} + 2\,\text{perms}) I_1^2 \, , \tag{C.24b}$$
$$\begin{aligned} [[p_\mu p_\nu k_\rho k_\sigma]]_{2,1,1} = & \frac{F_2}{2(d-1)} \eta_{\mu\nu} \eta_{\rho\sigma} I_1 [(2d-1) I_1 + 2 d m^2 I_2] \\ & + \frac{F_2}{4(d-1)} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) I_1 [(3d-4) I_1 + 2(d-2) m^2 I_2] \, , \end{aligned} \tag{C.24c}$$
$$[[p_\mu k_\nu k_\rho k_\sigma]]_{2,1,1} = -F_2 (\eta_{\mu\nu} \eta_{\rho\sigma} + 2\,\text{perms}) I_1 (I_1 + m^2 I_2) \, , \tag{C.24d}$$
$$[[k_\mu k_\nu k_\rho k_\sigma]]_{2,1,1} = F_2 (\eta_{\mu\nu} \eta_{\rho\sigma} + 2\,\text{perms}) (I_1^2 + 3 m^2 I_1 I_2) \, . \tag{C.24e}$$

Square Envelope Diagrams

Square envelope diagrams correspond to $[[\ldots]]_{3,1,1}$, and the scalar integrals we shall need for this topology are

$$[[(p^2)^3]]_{3,1,1} = I_1^2 \, , \tag{C.25a}$$
$$[[(p^2)^2 k^2]]_{3,1,1} = I_1 (I_1 + 2 m^2 I_2 + m^4 I_3) \, , \tag{C.25b}$$
$$[[p^2 (k^2)^2]]_{3,1,1} = I_1 (I_1 + 4 m^2 I_2 + 3 m^4 I_3) \, , \tag{C.25c}$$
$$[[(p^2)^2 (p \cdot k)]]_{3,1,1} = -\frac{1}{2} I_1^2 \, , \tag{C.25d}$$
$$[[p^2 (p \cdot k)^2]]_{3,1,1} = \frac{3}{4} I_1^2 + m^2 I_1 I_2 + \frac{m^4}{2} I_1 I_3 \, , \tag{C.25e}$$
$$[[p^2 (p \cdot k) k^2]]_{3,1,1} = -I_1^2 - 2 m^2 I_1 I_2 - m^4 I_1 I_3 \, , \tag{C.25f}$$

$$[[(p\cdot k)^2k^2]]_{3,1,1} = I_1^2 + \frac{7d+2}{2d}m^2 I_1(I_2+m^2I_3)\,, \tag{C.25g}$$

$$[[(p\cdot k)^3]]_{3,1,1} = -\frac{7}{8}I_1^2 - \frac{3}{2}m^2I_1I_2 - \frac{3}{4}m^4I_1I_3\,. \tag{C.25h}$$

Integrals with "open" Lorentz indices can be built out of these. They all share a common pre-factor $F_1 = [d(d+2)(d+4)]^{-1}$ coming from the contraction of indices. The results are:

$$[[p_\mu p_\nu p_\rho p_\sigma p_\kappa p_\lambda]]_{3,1,1} = F_1(\eta_{\mu\nu}\eta_{\rho\sigma}\eta_{\kappa\lambda} + 14\,\text{perms})I_1^2\,, \tag{C.26a}$$

$$[[p_\mu p_\nu p_\rho p_\sigma p_\kappa k_\lambda]]_{3,1,1} = -\frac{1}{2}F_1(\eta_{\mu\nu}\eta_{\rho\sigma}\eta_{\kappa\lambda} + 14\,\text{perms})I_1^2\,, \tag{C.26b}$$

$$\begin{aligned}[[p_\mu p_\nu p_\rho p_\sigma k_\kappa k_\lambda]]_{3,1,1} &= \frac{F_1}{d-1}(\eta_{\mu\nu}\eta_{\rho\sigma} + 2\,\text{perms})\eta_{\kappa\lambda}I_1\left[dI_1 + m^2(d+1)(2I_2+m^2I_3)\right]\\ &+ \frac{F_1}{4(d-1)}(\eta_{\mu\kappa}\eta_{\nu\lambda}\eta_{\rho\sigma} + 11\,\text{perms})\\ &\times I_1\left[(3d-4)\,I_1 + 2\,(d-2)\,m^2(2I_2+m^2I_3)\right]\,,\end{aligned} \tag{C.26c}$$

$$\begin{aligned}[[p_\mu p_\nu p_\rho k_\sigma k_\kappa k_\lambda]]_{3,1,1} &= \frac{F_1}{4(d-1)}(\eta_{\mu\nu}\eta_{\rho\sigma}\eta_{\kappa\lambda} + 8\,\text{perms})I_1[(3-4d)I_1\\ &+ 2(1-2d)m^2(2I_2+m^2I_3)]\\ &+ \frac{F_1}{8(d-1)}(\eta_{\mu\sigma}\eta_{\nu\kappa}\eta_{\rho\lambda} + 5\,\text{perms})I_1[(10-7d)I_1\\ &+ 6(2-d)m^2(2I_2+m^2I_3)]\,,\end{aligned} \tag{C.26d}$$

$$\begin{aligned}[[p_\mu p_\nu k_\rho k_\sigma k_\kappa k_\lambda]]_{3,1,1} &= \frac{F_1}{d(d-1)}(\eta_{\mu\nu}\eta_{\rho\sigma}\eta_{\kappa\lambda} + 2\,\text{perms})\Big[d(d-1)I_1^2\\ &+ (4d^2-2d-4)m^2I_1I_2 + (3d^2-5d-4)m^4I_1I_3\Big]\\ &+ \frac{F_1}{2(d-1)}(\eta_{\mu\rho}\eta_{\nu\sigma}\eta_{\kappa\lambda} + 11\,\text{perms})\Big[2(d-1)I_1^2 + (7d-6)m^2I_1I_2\\ &+ (7d-4)m^4I_1I_3\Big]\,.\end{aligned} \tag{C.26e}$$

C.1.4 Schwinger Parametrisation

When considering the renormalisation of the $(\partial_1 y)^2$ component of the metric, the integrals above are not sufficient since we also encounter diagrams which give rise to integrands in Fourier space wit non-scalar denominators.

The first integral we shall be concerned with arises from a particular instance of the sunset topology. Consider

$$J_1 = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d p}{(2\pi)^d}\frac{k^1(p^0)^2}{p^1}\frac{1}{(k^2-m^2)(p^2-m^2)[(k+p)^2-m^2]}\,. \tag{C.27}$$

To address the p^1 momentum in the denominator we employ a Schwinger parametrisation

$$\frac{1}{p^1} = \int_0^\infty \mathrm{d}u\, e^{-up^1}\,. \tag{C.28}$$

The Lorentz-scalar part of (C.27) and the momentum integrals are easily dealt with using the above techniques so we momentarily omit them by defining a "reduced" integral

$$\tilde{J}_1 = \int_0^\infty \mathrm{d}u\, e^{-up^1} k^1 (p^0)^2\,. \tag{C.29}$$

We now perform a series expansion of this exponential which we understand will give a series of loop integrals we can evaluate using some combinatorics and formulas in Appendix C.1. Even powers of u drop out since the numerator of the corresponding momentum integral will contain an odd number of σ components of momentum - this implies that any scalar contraction will come with a factor of η_{01} and hence will drop. So we need retain only

$$\tilde{J}_1 = -\int_0^\infty \mathrm{d}u u \sum_{n=0}^{\infty} \frac{u^{2n}}{(2n+1)!} k^1 (p^0)^2 (p^1)^{2n+1}\,. \tag{C.30}$$

Now, the precise replacement stems from the usual combinatorics in loop integrals

$$k^\mu p^{\nu_1} \dots p^{\nu_{2n+3}} \to \frac{1}{P^{(2n+4)}} P^{\mu\nu_1\dots\nu_{2n+3}} (k\cdot p)(p^2)^{n+1}\,. \tag{C.31}$$

In a scheme where $\eta^{01} = 0$, there are $(2n+1)!!$ non vanishing contributions in $P^{\mu\nu_1\dots\nu_{2n+3}}$ with a choice of indices as in (C.30). We then end up with

$$\tilde{J}_1 = \varphi(\epsilon)\psi(\epsilon)(k\cdot p)p^2 \int_0^\infty \mathrm{d}u u \sum_{n=0}^{\infty} \frac{(2n+1)!!}{(2n+1)!\, P^{(2n+4)}} (-up^2\psi(\epsilon))^n\,, \tag{C.32}$$

where we have adopted the parametrisations $\eta^{11} = -\psi(\epsilon)$ and $\eta^{00} = \varphi(\epsilon)$, with $\psi = 1 + \mathfrak{g}\epsilon$ and $\varphi = 1 + \mathfrak{g}\epsilon$ are both positive definite. Introducing a new integration variable $z = up^2\psi/4$, one can prove that the series can be resummed to yield the regularised hypergeometric function ${}_0\tilde{F}_1$

$$\tilde{J}_1 = \frac{\varphi(\epsilon)}{2}\Gamma\left(\frac{d}{2}\right)(k\cdot p)\int_0^\infty \mathrm{d}z\, {}_0\tilde{F}_1\left(2+\frac{d}{2}, -z\right)\,. \tag{C.33}$$

The remaining integral is part of a family of parametric integrals whose precise evaluation is known and equals

$$\int_0^\infty \mathrm{d}z\, z^{\alpha-1}\, {}_0\tilde{F}_1(b; -z) = \frac{\Gamma(\alpha)}{\Gamma(b-\alpha)}\,. \tag{C.34}$$

Adapting this formula to the case at hand we find

$$\tilde{J}_1 = \frac{\varphi(\epsilon)}{d}(k \cdot p)\,, \qquad J_1 = -\frac{\varphi(\epsilon)}{2d} I_1^2\,. \tag{C.35}$$

Other than J_1, we have a family of relevant non-invariant integrals depending on a integer parameter $\alpha \in \mathbb{N}$ reading

$$J_2^{(\alpha)} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{p^1} \frac{k^0 k^1 (p^0)^{2\alpha+1}}{(k^2 - m^2)(p^2 - m^2)^{\alpha+1}[(k+p)^2 - m^2]}\,. \tag{C.36}$$

Using the same technique the relevant cases evaluate to

$$J_2^{(0)} = \frac{3d-4}{4d(d-1)} \varphi(\epsilon) I_1^2\,, \tag{C.37}$$

$$J_2^{(1)} = \frac{3(3d-4)}{4(d-1)d(d+2)} \varphi(\epsilon)^2 I_1^2 + \frac{3(d-2)\varphi(\epsilon)^2}{2(d-1)d(d+2)} \varphi(\epsilon)^2 m^2 I_1 I_2\,, \tag{C.38}$$

$$J_2^{(2)} = \frac{15(3d-4)}{4(d-1)d(d+2)(d+4)} \varphi(\epsilon)^3 I_1^2 + \frac{15(d-2)m^2}{(d-1)d(d+2)(d+4)} \varphi(\epsilon)^3 m^2 I_1 \left(I_2 + \frac{m^2}{2} I_3 \right)\,. \tag{C.39}$$

Appendix D

Example

D.1 Example

We complement the main presentation with an explicit toy example. This allows for an independent computerised cross-check of the calculations done elsewhere in the project. In this toy model we are also able to explicitly examine the implication of removing the base manifold counter-term at one-loop via a field redefinition/addition of total derivative.

We begin with a model consisting of a single S^1 direction of radius $r = r(y)$ for the fibre with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_1 x \partial_0 \tilde{x} + \partial_0 x \partial_1 \tilde{x}) - \frac{1}{2}\left(r^2 \partial_1 x \partial_1 x + r^{-2} \partial_1 \tilde{x} \partial_1 \tilde{x}\right) + \frac{1}{2}\partial_\mu y \partial^\mu y \,. \tag{D.1}$$

The one-loop renormalisation yields

$$\mathcal{L}_{\mathrm{CT}} = \frac{1}{4\pi\epsilon}\frac{r\ddot{r} - \dot{r}^2}{r^4}(\partial_1 \tilde{x} \partial_1 \tilde{x} - r^4 \partial_1 x \partial_1 x) + \frac{1}{4\pi\epsilon}\frac{\dot{r}^2}{r^2}\partial_\mu y \partial^\mu y \,, \tag{D.2}$$

where over-dots denote, as usual, derivatives with respect to y. Using Method 2 followed by Method 1 for final evaluation of integrals, we find the two-loop contributions

$$\widetilde{T}^{(2)}_2|_{11} = \frac{1}{8\pi^2\epsilon^2}\frac{5(r^{(1)})^4 - 8r(r^{(1)})^2 r^{(2)} + r^2(r^{(2)})^2 + 2r^2 r^{(1)} r^{(3)}}{r^4} = \widetilde{T}^{(2)}_2|_{00} \,, \tag{D.3}$$

$$\widetilde{T}^{(2)}_2|_{01} = 0 \,, \tag{D.4}$$

$$T^{(2)}_2|_{xx} = -\frac{1}{16\pi^2\epsilon^2}\frac{2(r^{(1)})^4 - 6r(r^{(1)})^2 r^{(2)} + 4r^2 r^{(1)} r^{(3)} + r^2((r^{(2)})^2 - r^{(4)})}{r^2} \,, \tag{D.5}$$

$$T^{(2)}_2|_{\tilde{x}\tilde{x}} = \frac{1}{16\pi^2\epsilon^2}\frac{6(r^{(1)})^4 - 14r(r^{(1)})^2 r^{(2)} + 4r^2 r^{(1)} r^{(3)} + r^2(5(r^{(2)})^2 - rr^{(4)})}{r^6} \,, \tag{D.6}$$

$$\widetilde{T}^{(2)}_1|_{00} = \frac{\mathfrak{g}}{8\pi^2\epsilon}\frac{((r^{(1)})^2 - rr^{(2)})^2}{r^4}\,, \tag{D.7}$$

$$\widetilde{T}^{(2)}_1|_{11} = \frac{1}{8\pi^2\epsilon}\frac{(\mathfrak{g}-1)(r^{(1)})^4 + 2(3\mathfrak{g}-1)r(r^{(1)})^2 r^{(2)} - (3\mathfrak{g}-1)r^2(r^{(2)})^2}{r^4}\,, \tag{D.8}$$

$$\widetilde{T}^{(2)}_1|_{01} = 0\,, \tag{D.9}$$

$$T^{(2)}_1|_{xx} = \frac{1}{16\pi^2\epsilon}\frac{2(1-\mathfrak{g})(r^{(1)})^4 + (2\mathfrak{g}-1)r(r^{(1)})^2 r^{(2)} + 2(2\mathfrak{g}-1)r^2(r^{(2)})^2}{r^2}\,, \tag{D.10}$$

$$T^{(2)}_1|_{\tilde{x}\tilde{x}} = \frac{1}{16\pi^2\epsilon}\frac{2(-4+9\mathfrak{g})(r^{(1)})^4 - 9(2\mathfrak{g}-1)r(r^{(1)})^2 r^{(2)} + 2(2\mathfrak{g}-1)r^2(r^{(2)})^2}{r^6}\,, \tag{D.11}$$

where $r^{(n)}$ is an alternative notation for the n-th derivative of r with respect to y. These agree with the results of the main text upon the substitution of the generalised metric $\mathcal{H} = \mathrm{diag}(r^2, r^{-2})$. We remark that the vanishing of $\widetilde{T}^{(2)}|_{01}$ is due to the triviality of the tensorial combinations within this example.

Let us note that we can amend the one-loop counter-term through the inclusion of any piece that vanishes upon integration by parts and application of the equations of motion

$$\mathcal{L}_{\mathrm{on-shell}} = \dot{f}(y)\partial_\mu y \partial^\mu y - \frac{1}{2}f(y)\mathcal{H}^{(1)}_{\bullet\bullet} \approx 0\,, \tag{D.12}$$

for some function $f(y)$. With the choice that $\dot{f} = -\frac{1}{2}\widetilde{T}^{(1)}$ the base divergence of the one-loop counter can be removed, considering instead

$$\mathcal{L}_{\mathrm{CT}} + \mathcal{L}_{\mathrm{on-shell}} = \frac{f(y)}{4\pi\epsilon}\left(r\dot{r}\,\partial_1 x\partial_1 x - \frac{\dot{r}}{r^3}\partial_1\tilde{x}\partial_1\tilde{x}\right) + \frac{1}{4\pi\epsilon}\frac{r\ddot{r} - \dot{r}^2}{r^4}(\partial_1\tilde{x}\partial_1\tilde{x} - r^4\partial_1 x\partial_1 x)\,. \tag{D.13}$$

We may now proceed to re-calculate the two-loop divergences using this modified one-loop counter term. One might anticipate that this resolves some of the discrepancies between $\widetilde{T}^{(2)}_1|_{00}$ and $\widetilde{T}^{(2)}_1|_{11}$ seen in the above. However, an explicit calculation yields

$$\begin{aligned}\widetilde{T}^{(2)}_1|_{00} &= \frac{1}{16\pi^2\epsilon}\frac{(1-\mathfrak{g})(r^{(1)})^4 - 2\mathfrak{g}r(r^{(1)})^2 r^{(2)} + \mathfrak{g}r^2(r^{(2)})^2}{r^4}\\ &\quad + \frac{1}{16\pi^2\epsilon}\frac{(-1+2\mathfrak{g})frr^{(1)}((r^{(1)})^2 - rr^{(2)})}{r^4} + \dots\,,\end{aligned} \tag{D.14}$$

$$\begin{aligned}\widetilde{T}^{(2)}_1|_{11} &= \frac{1}{16\pi^2\epsilon}\frac{(5-14\mathfrak{g})(r^{(1)})^4 + 4(1-3\mathfrak{g})r(r^{(1)})^2 r^{(2)} + 2(1-3\mathfrak{g})r^2(r^{(2)})^2}{r^4}\\ &\quad + \frac{1}{8\pi^2\epsilon}\frac{3(-1+2\mathfrak{g})frr^{(1)}((r^{(1)})^2 - rr^{(2)})}{r^4} + \dots\,,\end{aligned} \tag{D.15}$$

where ellipsis indicate terms proportional to $\frac{\bar{\gamma}}{\epsilon}$ (which do *not* now vanish). Far from ameliorating the situation, we still have Lorentz violation from the base counter terms and, moreover, un-cancelled $\bar{\gamma}$ terms.

Appendix E

Mathematica Implementation

To perform the calculations in Chapter 7 and 8 we wrote a specific `Mathematica` notebook, capable of expanding the action to any given order, exponentiating the interacting Lagrangian, performing Wick contractions, computing loop integrals etc. As, to the best of our knowledge, no available packages comprise of these features, we shall spend some time reviewing our implementation.

E.1 Tweaking xAct

The best tool to deal with canonicalisation of tensorial expressions is the xAct suite [215] and, in particular, the xTensor and xPerm packages. However, they are not suited for a quantum computation out of the box, and we shall push them a little further to achieve this. More in detail, the first issue we encounter is the Taylor expansion of the exponentiated interacting action $\exp(i\int \mathrm{d}^2\sigma\mathcal{L}_\mathrm{I})$. Up to a prefactor, its n-th term is in fact the product of n S_I's, each one evaluated on a *different* copy of the worldsheet,

$$\exp\left(i\int \mathrm{d}^2\sigma\,\mathcal{L}_\mathrm{I}\right) = 1 + \sum_{n=1}^{\infty}\frac{i^n}{n!}\left(\int \mathrm{d}^2\sigma_1\,\mathcal{L}_\mathrm{I}(\sigma_1)\right)\times\ldots\times\left(\int \mathrm{d}^2\sigma_n\,\mathcal{L}_\mathrm{I}(\sigma_n)\right)\,. \tag{E.1}$$

Hence, tensors appearing in $\mathcal{L}_\mathrm{I}(\sigma_n)$ shall accordingly retain a dependence on the worldsheet copy even in the code. This is achieved treating the worldsheet coordinates as *parameters* (in the language of xAct) and defining $4n$ of those (i.e. $2n$ τ's and $2n$ σ's) for an n-loop calculation. Similarly, for each copy of the worldsheet, we will define a set of tensors ($\mathcal{H}$, Ω, ξ etc.) carrying explicit dependence on the relevant set of parameters. The result of Wick contractions[1] will be products of propagators of either types (Δ and θ) treated as tensors with zero indices.

[1]The Wick theorem is easily implemented with a recursive function that does not require xAct.

E.2 Loop Counting and 1PI Feynman Diagrams

As for Feynman diagrams, we have a twofold issue. First, we want to discard diagrams that do exceed the loop order we are working at. This problem is easily tackled observing that all relevant graphs are planar and hence we can use Euler formula at genus $g = 0$, i.e.

$$V - E + L = 1\,, \tag{E.2}$$

where V, E and L are respectively the number of vertices, edges and loops. Notice how we are using L, as opposed to the usual F (faces): this is due to the fact that the number of faces equals the number of loops plus one, as it also includes the outer and infinitely large region. In practice we do not need to actually draw the graph, nor apply Wick theorem, to ascertain the loop number: we can simply solve (E.2) for L, being E half the number of the propagators[2] and V the number of relevant copies of the worldsheet present in the expression at hand. Second, we shall restrict ourselves to connected, 1PI graphs. To check this property it is firstly necessary to draw the graph; `Mathematica` has a dedicated function, and we suggest the adoption of Tutte embedding for displaying purposes. Building on this, we found very useful the package IGraph, which implements a number of functions for graph analysis currently unavailable in the `Mathematica` main architecture. In mathematical jargon, a 1PI diagram is called 2-edge-connected. The function IGEdgeConnectivity, when applied to a graph, precisely returns the number of edges and we shall hence discard those which fail to meet our criterion. On top of that, IGraph also provides the handy function IGGetIsomorphism able to find the relabelling of indices between that makes two graph isomorphic. This is extremely useful, as it enables us to vastly reduce the number of total terms by graphically implementing the relabelling we implicitly perform when counting the symmetry factor associated to a diagram.

[2]If odd, the expression is discarded from the outset, thanks to Wick theorem.

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