Stabilization of stochastic McKean-Vlasov equations with feedback control based on discrete-time state observation

Hao Wu[a], Junhao Hu[a], Shuaibin Gao[b], Chenggui Yuan[c]

[a]School of Mathematics and Statistics, South-Central University For Nationalities

Wuhan, Hubei 430000, P.R.China

Email: wuhaomoonsky@163.com, junhaohu74@163.com

[b] Department of Mathematics, Shanghai Normal University, Shanghai, 200234, P.R.China

Email: shuaibingao@163.com

[c]Department of Mathematics, Swansea University, Bay campus, SA1 8EN, UK

Email: C.Yuan@Swansea.ac.uk

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Abstract

In this paper, the stability of solutions of stochastic McKean-Vlasov equations (SMVEs) via feedback control based on discrete-time state observation is studied, which includes the H_∞ stability, asymptotic stability and exponential stability in mean square for the solution of the controlled systems. Since the distribution of solution is difficult to be observed, the corresponding particle system which can be observed is investigated. we show that the exponential stability of control system is equivalent to the the exponential stability of the corresponding particle system. Finally, an example is provided to illustrate the theory.

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1   Introduction

Stochastic differential equations (SDEs) are widely used to model stochastic systems in different branches of science and industry. The form of SDEs reads as follows:

$$\mathrm{d}x(t) = f(x(t), t)\mathrm{d}t + g(x(t), t)\mathrm{d}B(t), \, t \geq 0.$$

One of the popular applications for SDEs is the feedback control of stochastic systems. We refer the readers to [5, 14, 19, 24] and references therein. Since, some real SDEs are often unstable, an interesting problem of the automatic control field is that for some given unstable SDEs, how can one design an effective control function for the system to make the corresponding system be stable? Among them, the feedback control based on a continuous-time state observation is an efficient one, which has been used in establishing the mean-square exponential stabilization for a class of SDEs, see e.g. [1, 4, 9] and references therein. Since the method of continuous-time state observation is usually too expensive and not realistic in real lives, [16] proposed a more effective state feedback control which is based on discrete-time state observation and is now widely studied. It is obvious that the state feedback control based on continuous-time observation requires one to observe the system all the time, while the state feedback control based on discrete-time state observation only requires one to observe the system in some discrete time. There are many results on this problem in the previous literatures (e.g. [15, 21]). In particular, for an unstable stochastic system, it is very meaningful and important to design a feedback control with the form $u(\lfloor \frac{t}{\delta} \rfloor \delta)$ embedded into the drift part, where δ is the discrete-time observation gap.

On the other hand, recently, many researchers are interested in studying the following equations, which are called stochastic Mckean-Vlasov equations (SMVEs):

$$\begin{cases} x(t) = x_0 + \int_0^t f(x(s), \mu_s) \mathrm{d}s + \int_0^t g(x(s), \mu_s) \mathrm{d}B(s), t \in [t_0, \infty), \\ \mu_t = \mathscr{L}(x(t)) := \text{the probability distribution of } x(t). \end{cases}$$

Obviously, the coefficients involved depend not only on the state process but also on its distribution. With contrast to the classical SDEs, SMVEs enjoy some essential features. The work on SMVEs was initiated by McKean [20], who was inspired by Kac's Programme in Kinetic Theory [13]. Sznitman [22] investigated the existence and uniqueness of the results under a global Lipschitz condition. Wang [23] studied the existence of invariant probability measures for SMVEs. Govindan and Ahmed [10] studied the exponential stability of the solutions for a semilinear SMVEs under the Lipschitz condition and linear growth condition. Ding and Qiao [7, 8] derived the existence and uniqueness of the solution with non-Lipschitz condition and analyzed the stability of the solutions for SMVEs, respectively. Furthermore, in addition to the theoretical values, this kind of equations also has a lot of applied values in social science, economics, engineering, etc. (see e.g. [3]).

To the best our knowledge, there is little study on the stabilization of SMVEs with feedback control based on discrete-time state observation. It is clear that the controlled Mckean-Vlasov system includes discrete-time state observations as well as its distribution observations while feedback control systems independent of distribution only need to observe state of the systems. In this paper, we shall study the stabilization problem by using the feedback control with a discrete-time version: for an unstable McKean-Vlasov system, we aim to make the Mckean-Vlasov system stable by designing a discrete-state and its distribution feedback control on this system. Our main contributions are as follows:

- We are the first to study feedback control problem for SMVEs based on discrete-time state observation.

- The Lyapunov functions used in this article not only contain state of the solution but also the distribution of the solution, while the previous Lyapunov functions used in the state feedback control system only contain state of the solution. This is an essential feature.

- We study the asymptotic stability and exponential stability in mean square of the solution for SMVEs based on discrete-time state observation.

- The distribution of analytical solution $x(t)$ is difficult to be observed while the empirical distribution can be observed more easily. Thus, we further study the corresponding particle system. We show that the exponential stability of control system is equivalent to the the exponential stability of the corresponding particle system.

We close this part by giving our organization in this article. In Section 2, we introduce some necessary notations, research objects and necessary assumptions. In Section 3, we aim to study the stability of solutions to SMVEs via feedback control based on discrete-time state observation. Then, an example is presented to illustrate the theories in Section 4.

2   Preliminaries

2.1   Notations

Throughout this paper, let $(\Omega, \mathscr{F}, \mathbb{F}, P)$ be a complete probability space with filtration $\mathbb{F} = \{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions(i.e., it is increasing and right continuous, $\mathscr{F}_0$ contains all P-null sets) taking along a standard m-Brownian motion $B(t)$. If $x, y \in \mathbb{R}^d$, we use $|x|$ to denote the Euclidean norm of x, and use $\langle x, y \rangle$ or xy to denote the Euclidean inner product. If A is a matrix, A^T is the transpose of A, and $|A|$ represents $\sqrt{\mathrm{Tr}(AA^T)}$. Moreover, let $\lfloor a \rfloor$ be the integer parts of a. For $\delta > 0$, set $\sigma_t = \lfloor \frac{t}{\delta} \rfloor \delta$, where δ is the discrete-time observation gap. Let $\mathscr{B}(\mathbb{R}^d)$ be the Borel $\sigma-$algebra on $\mathbb{R}^d$, $C(\mathbb{R}^d)$ denotes all continuous functions on $\mathbb{R}^d$ and $C^k(\mathbb{R}^d)$ denotes all continuous functions on $\mathbb{R}^d$ with continuous partial derivations of order up to k. Let $\mathscr{P}(\mathbb{R}^d)$ be the space of all probability measures, and $\mathscr{P}_p(\mathbb{R}^d)$ denotes the space of all probability measures defined on $\mathscr{B}(\mathbb{R}^d)$ with finite pth moment:

$$W_p(\mu) := \left(\int_{\mathbb{R}^d} |x|^p \mu(\mathrm{d}x) \right)^{\frac{1}{p}} < \infty.$$

For $\mu, \nu \in \mathscr{P}_p(\mathbb{R}^d)$, we define the Wasserstein distance for $p \geq 1$ as follows:

$$W_p(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \pi(\mathrm{d}x, \mathrm{d}y) \right\}^{\frac{1}{p}},$$

where $\Pi(\mu, \nu)$ is the family of all couplings for μ, ν.

Set $\mathscr{M}_{\lambda^2}^s(\mathbb{R}^d) = \{m : m \text{ is a signed measure on } \mathbb{R}^d \text{ satisfying } \|m\|_{\lambda^2}^2 = \int_{\mathbb{R}^d}(1+|x|)^2|m|(\mathrm{d}x) < \infty$, where $|m|$ is the total variation measure of $m\}$, and $\mathscr{M}_{\lambda^2}(\mathbb{R}^d) = \mathscr{M}_{\lambda^2}^s(\mathbb{R}^d) \bigcap \mathscr{P}(\mathbb{R}^d)$. We put on $\mathscr{M}_{\lambda^2}(\mathbb{R}^d)$ a topology induced by the Wasserstein distance $W_2(\cdot, \cdot)$.

2.2 Lions Derivatives

In this subsection, we will give the definition of Loins derivative for $b : \mathscr{M}_{\lambda^2}(\mathbb{R}^d) \to \mathbb{R}$ with respect to a probability measure as introduced in [8].

Definition 2.1. *We say that $U : \mathscr{M}_{\lambda^2}(\mathbb{R}^d) \to \mathbb{R}$ is differential at $\mu \in \mathscr{M}_{\lambda^2}(\mathbb{R}^d)$, if there exists some $X \in L^2(\Omega, \mathscr{F}, P; \mathbb{R}^d)$ such that $\mu = \mathscr{L}(X)$ and the function $\tilde{U} : L^2(\Omega, \mathscr{F}, P; \mathbb{R}^d) \to \mathbb{R}$ given by $\tilde{U}(X) := U(\mathscr{L}(X))$ is Fréchet differentiable at X.*

We recall that $\tilde{U}$ is Fréchet differentiable at X means that there exists a continuous mapping $D\tilde{U}(X) : L^2(\Omega, \mathscr{F}, P; \mathbb{R}^d) \to \mathbb{R}$ such that for any $Y \in L^2(\Omega, \mathscr{F}, P; \mathbb{R}^d)$

$$\tilde{U}(X + Y) - \tilde{U}(X) = D\tilde{U}(X)(Y) + o(|Y|_{L^2}), \text{ as } |Y|_{L^2} \to 0.$$

Due to $D\tilde{U}(X) \in L(L^2(\Omega, \mathscr{F}, P; \mathbb{R}^d); \mathbb{R})$, by Riesz representation theorem, there exists a P-a.s. unique variable $Z \in L^2(\Omega, \mathscr{F}, P; \mathbb{R}^d)$ such that for any $Y \in L^2(\Omega, \mathscr{F}, P; \mathbb{R}^d)$

$$D\tilde{U}(X)(Y) = \langle Y, Z \rangle_{L^2} = \mathbb{E}[YZ].$$

Cardaliaguet [2] showed that there exists a Borel measurable function $h : \mathbb{R}^d \to \mathbb{R}^d$ which only depends on the distribution $\mathscr{L}(X)$ rather that X itself such that $Z = h(X)$. Thus, for $X \in L^2(\Omega, \mathscr{F}, P; \mathbb{R}^d)$,

$$U(\mathscr{L}(Y)) - U(\mathscr{L}(X)) = \mathbb{E}[h(X)(Y - X)] + o(|Y - X|_{L^2}).$$

We call $\partial_\mu U(\mathscr{L}(X))(y) := h(y), y \in \mathbb{R}^d$ the L-derivative of U at $\mathscr{L}(X), X \in L^2(\Omega, \mathscr{F}, P; \mathbb{R}^d)$.

Let $C^1(\mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ denote all functions $U : \mathscr{M}_{\lambda^2}(\mathbb{R}^d) \to \mathbb{R}$ such that $\partial_\mu U : \mathscr{M}_{\lambda^2}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous. Let $C_b^{1;1}(\mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ be all functions $U \in C^1(\mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ such that $\partial_\mu U$ is bounded and Lipschitz continuous, i.e., there exists a positive constant C such that

(i) $|\partial_\mu U(\mu)(x)| \leq C$ for any $\mu \in \mathscr{M}_{\lambda^2}(\mathbb{R}^d), x \in \mathbb{R}^d$.

(ii) $|\partial_\mu U(\mu)(x) - \partial_\mu U(\nu)(y)|^2 \leq C(W_2^2(\mu, \nu) + |x - y|^2), \mu, \nu \in \mathscr{M}_{\lambda^2}(\mathbb{R}^d), x, y \in \mathbb{R}^d$.

We need more definitions:

(1) The function U is said to be in $C^2(\mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ if for any $\mu \in \mathscr{M}_{\lambda^2}(\mathbb{R}^d), U \in C^1(\mathscr{M}_{\lambda^2}(\mathbb{R}^d))$, $\partial_\mu U(\mu)(\cdot)$ is differentiable and its derivative $\partial_y \partial_\mu U : \mathscr{M}_{\lambda^2}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is continuous.

(2) The function U is said to be in $C_b^{2;1}(\mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ if $U \in C^2(\mathscr{M}_{\lambda^2}(\mathbb{R}^d)) \bigcap C_b^{1;1}(\mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ and its derivative $\partial_y \partial_\mu U$ is bounded and Lipschitz continuous.

(3) The function Ψ is said to be in $C_b^{2,2}(\mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ if for any $x \in \mathbb{R}^d, \Psi(x, \cdot) \in C_b^{2,1}(\mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ and for any $\mu \in \mathscr{M}_{\lambda^2}(\mathbb{R}^d), \Psi(\cdot, \mu) \in C^2(\mathbb{R}^d)$.

(4) The function Ψ is said to be in $\mathscr{C}(\mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ if $\Psi \in C_b^{2,2}(\mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ and for any compact set $\mathcal{K} \subset \mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d)$,

$$\sup_{(x,\mu) \in \mathcal{K}} \int_{\mathbb{R}^d} (|\partial_y \partial_\mu \Psi(x,\mu)(y)|^2 + |\partial_\mu \Psi(x,\mu)(y)|^2) \mu(\mathrm{d}y) < \infty.$$

(5) $\Psi \in C_b^{2,2;1}(\mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ means that

    (i) Ψ is bicontinuous in (x,μ).

    (ii) For any $x \in \mathbb{R}^d$, $\Psi(x,\cdot) \in C_b^{2;1}(\mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ and for any $\mu \in \mathscr{M}_{\lambda^2}(\mathbb{R}^d)$, $\Psi(\cdot,\mu) \in C^2(\mathbb{R}^d)$.

    (iii) For any $\mu \in \mathscr{M}_{\lambda^2}(\mathbb{R}^d)$, $\partial_{x_i} \Psi(\cdot,\mu)$ is bounded.

(6) Let $\mathscr{C}_+(\mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ be the set of all functions $\Psi \in \mathscr{C}(\mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ such that $\Psi > 0$. $C_{b,+}^{2,2;1}(\mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ denotes all functions $\Psi \in C_b^{2,2;1}(\mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ with $\Psi \geq 0$.

2.3   The Itô formula

Consider the following equations:

$$\begin{cases} \mathrm{d}y(t) = (f(y(t),\rho_t) + u(y(t),\rho_t))\mathrm{d}t + g(y(t),\rho_t)\mathrm{d}B(t), t \in [0,\infty), \\ y(0) = x_0, \end{cases} \quad (2.1)$$

and

$$\begin{cases} \mathrm{d}x(t) = (f(x(t),\mu_t) + u(x(\sigma_t),\mu_{\sigma_t}))\mathrm{d}t + g(x(t),\mu_t)\mathrm{d}B(t), t \in [0,\infty), \\ x(0) = x_0, \end{cases} \quad (2.2)$$

where $x_0 \in \mathbb{R}^d$, and ρ_t and μ_t are the distributions of $y(t)$ and $x(t)$, respectively. Moreover, $f, u : \mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d) \to \mathbb{R}^d, g : \mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d) \to \mathbb{R}^{d \times m}$. In Eq.(2.2), one can see that the control function $u(x(\sigma_t),\mu_{\sigma_t})$ only depends on the state at discrete times $0, \delta, 2\delta, \cdots$. Moreover, we assume that $f(\mathbf{0},\delta_{\mathbf{0}}) = 0, g(\mathbf{0},\delta_{\mathbf{0}}) = 0, \mu(\mathbf{0},\delta_{\mathbf{0}}) = 0$, where $\mathbf{0}$ is a $d-$dimensinal zero vector and $\delta_{\mathbf{0}}$ is the Dirac measure at $\mathbf{0}$. For the existence and uniqueness of solutions of Eq.(2.1) and Eq.(2.2), we assume that:

(H1) Suppose that f, g, u satisfy the following Lipschitz condition, i.e., there exist positive constants $L_i, i = 1, 2, 3$ such that

$$|f(x,\mu) - f(y,\nu)|^2 \leq L_1(|x-y|^2 + W_2^2(\mu,\nu)),$$

$$|g(x,\mu) - g(y,\nu)|^2 \leq L_2(|x-y|^2 + W_2^2(\mu,\nu)),$$

$$|u(x,\mu) - u(y,\nu)|^2 \leq L_3(|x-y|^2 + W_2^2(\mu,\nu)),$$

for all $x, y \in \mathbb{R}^d, \mu, \nu \in \mathscr{M}_{\lambda^2}(\mathbb{R}^d)$.

By Theorem 3.1 in [12], under the assumption (H1), Eq.(2.1) and Eq.(2.2) have unique solutions, respectively. We now introduce the following operators.

Definition 2.2. *For $V \in \mathscr{C}(\mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d))$, the operator $LV : \mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d) \to \mathbb{R}$ for Eq.(2.1) is defined by*

$$
LV(x,\rho) = \partial_x V(x,\rho)f(x,\rho) + \partial_x V(x,\rho)u(x,\rho)
$$
$$
+ \frac{1}{2}\mathrm{tr}(g^T(x,\rho)V_{xx}(x,\rho)g(x,\rho)) + \int_{\mathbb{R}^d}\partial_\rho V(x,\rho)(y)f(y,\rho)\rho(\mathrm{d}y)
$$
$$
+ \int_{\mathbb{R}^d}\partial_\rho V(x,\rho)(y)u(y,\rho)\rho(\mathrm{d}y) + \frac{1}{2}\int_{\mathbb{R}^d}\mathrm{tr}(g^T(y,\rho)\partial_y\partial_\rho V(x,\rho)(y)g(y,\rho))\rho(\mathrm{d}y). \qquad (2.3)
$$

Definition 2.3. *Let ξ and η be two random variables whose distributions are μ and ν, respectively. Let the joint distribution of (ξ,η) be $F_{\xi,\eta}(z,\bar{z})$. For $V \in \mathscr{C}(\mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d))$, the operator $\mathbb{L}V : \mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d) \times \mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d) \to \mathbb{R}$ for Eq.(2.2) is defined by*

$$
\mathbb{L}V(x,\mu,y,\nu) = \partial_x V(x,\mu)f(x,\mu) + \partial_x V(x,\mu)u(y,\nu)
$$
$$
+ \frac{1}{2}\mathrm{tr}(g^T(x,\mu)V_{xx}(x,\mu)g(x,\mu)) + \int_{\mathbb{R}^d}\partial_\mu V(x,\mu)(z)f(z,\mu)\mu(\mathrm{d}z)
$$
$$
+ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\partial_\mu V(x,\mu)(z)u(\bar{z},\nu)F_{\xi,\eta}(\mathrm{d}z,\mathrm{d}\bar{z})
$$
$$
+ \frac{1}{2}\int_{\mathbb{R}^d}\mathrm{tr}(g^T(z,\mu)\partial_z\partial_\mu V(x,\mu)(z)g(z,\mu))\mu(\mathrm{d}z).
$$

The Itô formula has been established in [8, 11] for Eq. (2.1), we cite it as the following lemma.

Lemma 2.1. *Assume* (H1) *and $V \in C_b^{2,2,1}(\mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d))$. Then it holds that*

$$
V(y(t),\rho_t) - V(y(0),\rho_0) = \int_0^t LV(y(s),\rho_s)\mathrm{d}s + \int_0^t V_x(y(s),\rho_s)g(y(s),\rho_s)\mathrm{d}B(s).
$$

Since the feedback control in Eq.(2.2) depends on the discrete time, we need to develop an Itô's formula for this equation.

We need more notations to formulate the Itô formula. Assume that $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{F}} = \{\tilde{\mathscr{F}}_t\}, \tilde{P})$ is another probability space taking along a $m-$dimensional Brownian motion $\{\tilde{B}(t)\}_{t\geq 0}$. Consider the following equation:

$$
\begin{cases}
\mathrm{d}\tilde{x}(t) = (f(\tilde{x}(t),\tilde{\mu}_t) + u(\tilde{x}(\sigma_t),\tilde{\mu}_{\sigma_t}))\mathrm{d}t + \tilde{g}(\tilde{x}(t),\tilde{\mu}_t)\mathrm{d}\tilde{B}(t), t \in [0,\infty), \\
\tilde{x}(0) = x_0,
\end{cases} \qquad (2.4)
$$

where $x_0 \in \mathbb{R}^d$, and $\tilde{\mu}_t$ denotes the distribution of $\tilde{x}(t)$. By the weak uniqueness, it holds that $\{x(t)\}_{t\geq 0}$ and $\{\tilde{x}(t)\}_{t\geq 0}$ are identical in probability law. Furthermore, denote by $\tilde{\mathbb{E}}[\cdot]$ the expectation under $\tilde{P}$.

We now present the Itô formula for (2.2), which is an extension of proposition 2.9 in [8].

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Lemma 2.2. *Let $V \in \mathscr{C}(\mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ and the assumption* (H1) *hold. Then one has that*

$$V(x(t), \mu_t) = V(x(0), \mu_0) + \int_0^t \mathbb{L}V(x(s), \mu_s, x(\sigma_s), \mu_{\sigma_s})\mathrm{d}s$$

$$+ \int_0^t \partial_x V(x(s), \mu_s)g(x(s), \mu_s)\mathrm{d}B(s). \tag{2.5}$$

Proof. Let $x(t)$ be the unique solution of (2.2). By the Hölder inequality and the BDG inequality, it holds that for any $T > 0$ and $0 \le t \le T$,

$$\mathbb{E} \sup_{0 \le s \le t} |x(s)|^2 \le 3\mathbb{E}|x_0|^2 + 3\mathbb{E} \sup_{0 \le s \le t} \left| \int_0^s f(x(r), \mu_r) + u(x(\sigma_r), \mu_{\sigma_r})\mathrm{d}r \right|^2$$

$$+ 3\mathbb{E} \sup_{0 \le s \le t} \left| \int_0^s g(x(r), \mu_r)\mathrm{d}B(r) \right|^2$$

$$\le 3\mathbb{E}|x_0|^2 + 6T\mathbb{E} \int_0^t |f(x(s), \mu_s)|^2\mathrm{d}s + 6T\mathbb{E} \int_0^t |u(x(\sigma_s), \mu_{\sigma_s})|^2\mathrm{d}s$$

$$+ 12\mathbb{E} \int_0^t |g(x(s), \mu_s)|^2\mathrm{d}s$$

$$\le 3\mathbb{E}|x_0|^2 + \mathbb{E} \int_0^t [(6TL_1 + 12L_2)|x(s)|^2 + 6TL_3|x(\sigma_s)|^2$$

$$+ (6TL_1 + 12L_2)W_2^2(\mu_s, \delta_0) + 6TL_3 W_2^2(\mu_{\sigma_s}, \delta_0)]\mathrm{d}s.$$

This, together with $W_2^2(\mu_s, \delta_0) \le \mathbb{E}|x(s)|^2$, yields that

$$\mathbb{E} \sup_{0 \le s \le t} |x(s)|^2 \le 3\mathbb{E}|x_0|^2 + (12TL_1 + 24L_2 + 12TL_3)\mathbb{E} \int_0^t \sup_{0 \le u \le s} |x(u)|^2\mathrm{d}s.$$

From Gronwall's formula, we get

$$\mathbb{E} \sup_{0 \le s \le T} |x(s)|^2 \le 3\mathbb{E}|\xi(0)|^2 e^{6(2TL_1 + 2TL_3 + 4L_2)T}. \tag{2.6}$$

Using similar method, one can derive that

$$\mathbb{E} \sup_{0 \le s \le T} |x(s)|^4 \le C, \tag{2.7}$$

where C is a constant depending on L_1, L_2, L_3, T. By (2.6), (2.7) and (H1), we have

$$\mathbb{E} \int_0^T (|f(x(s), \mu_s)|^2 + |u(x(\sigma_s), \mu_{\sigma_s})|^2 + |g(x(s), \mu_s)|^4)\mathrm{d}s$$

$$\le (L_1 + L_3)\mathbb{E} \int_0^T (|x(s)|^2 + |x(\sigma_s)|^2 + W_2^2(\mu_s, \delta_0) + W_2^2(\mu_{\sigma_s}, \delta_0))\mathrm{d}s$$

$$+ 2L_2^2 \mathbb{E} \int_0^T (|x(s)|^4 + W_2^4(\mu_s, \delta_0))\mathrm{d}s < \infty. \tag{2.8}$$

As in [11, Proposition A.8], we introduce another probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$, a filtration $\{\tilde{\mathscr{F}}_t\}_{t \geq 0}$, a Brownian motion $\tilde{B}$ on this probability space and processes $\tilde{B}, \tilde{x}$ on this probability space such that they have the same laws as B, x. Then, we have

$$d\tilde{x}(t) = (f(\tilde{x}(t), \tilde{\mu}_t) + u(\tilde{x}(\sigma_t), \tilde{\mu}_{\sigma_t}))\mathrm{d}t + g(\tilde{x}(t), \tilde{\mu}_t)\mathrm{d}\tilde{B}(t).$$

Let

$$\tilde{b}_t = (f(\tilde{x}(t), \tilde{\mu}_t) + u(\tilde{x}(\sigma_t), \tilde{\mu}_{\sigma_t})), \quad \tilde{\sigma}_t = g(\tilde{x}(t), \tilde{\mu}_t).$$

Then

$$d\tilde{x}(t) = \tilde{b}_t\mathrm{d}t + \tilde{\sigma}_t\mathrm{d}\tilde{B}(t).$$

Since $x(t)$ and $\tilde{x}(t)$ have the same distribution, from (2.8) one can see that

$$\tilde{\mathbb{E}} \int_0^t [|\tilde{b}_s|^2 + |\tilde{\sigma}_s|^4]\mathrm{d}t < \infty.$$

Fix x and set $h(\mu) = V(x, \mu)$. It follows from [11, Proposition A.6] that

$$h(\tilde{\mu}_t) - h(\tilde{\mu}_0) = \int_0^t \tilde{\mathbb{E}}\left[\tilde{b}_s \partial_\mu V(x, \tilde{\mu}_s)(\tilde{x}(s)) + \frac{1}{2}\mathrm{tr}[\tilde{\sigma}_s^* \partial_x \partial_\mu V(x, \tilde{\mu}_s))(\tilde{x}(s))\tilde{\sigma}_s]\right]\mathrm{d}s,$$

$$= \int_0^t \tilde{\mathbb{E}}\left[\left(f(\tilde{x}(s), \tilde{\mu}_s) + u(\tilde{x}(\sigma_s), \tilde{\mu}_{\sigma_s})\right)\partial_\mu V(x, \tilde{\mu}_s)(\tilde{x}(s))\right.$$

$$\left. + \frac{1}{2}\mathrm{tr}\left(g^*(\tilde{x}(s), \tilde{\mu}_s)\partial_x \partial_\mu V(x, \tilde{\mu}_s))(\tilde{x}(s))g(\tilde{x}(s), \tilde{\mu}_s)\right)\right]\mathrm{d}s$$

$$=: \int_0^t M(x, \tilde{\mu}_s)\mathrm{d}s.$$

Now, set $\overline{V}(x, t) = V(x, \tilde{\mu}_t)$. Thus, we have

$$\partial_t \overline{V}(x, t) = M(x, \tilde{\mu}_t).$$

Applying Itô's formula [11, Proposition A.8] to $V(x(t), \tilde{\mu}_t)$ and noting that $\mu_t = \tilde{\mu}_t$, we derive that

$$V(x(t), \tilde{\mu}_t) - V(x(0), \tilde{\mu}_0) = \overline{V}(x(t), t) - \overline{V}(x(0), 0)$$

$$= \int_0^t \left[V_x(x(s), \mu_s)(x(s))\left[f(x(s), \mu_s) + u(x(\sigma_s), \mu_{\sigma_s})\right]\right.$$

$$+ \frac{1}{2}\mathrm{trace}\left[g^T(x(s), \mu_s)V_{xx}(x(s), \mu_s)(z)g(x(s), \mu_s)\right] + M(x(s), \tilde{\mu}_s)\right]\mathrm{d}s$$

$$+ \int_0^t \partial_x V(x(s), \mu_s)g(x(s), \mu_s)\mathrm{d}B(s).$$

The desired assertion (2.5) holds. $\qquad\qquad\square$

3 Asymptotic stability and exponential stability in mean square

In order to study the asymptotic stability and the exponential stability in mean square, we impose the following assumption:

(H2) Assume there exist $V \in \mathscr{C}(\mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d))$, and four constants $\lambda_1 > 0, \lambda_2 > 0, \gamma_1 > 0, \gamma_2 \geq 0$ such that

$$\int_{\mathbb{R}^d} LV(x,\mu)\mu(\mathrm{d}x) + \lambda_1 \int_{\mathbb{R}^d} |V_x(x,\mu)|^2\mu(\mathrm{d}x) + \lambda_2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\partial_\mu V(x,\mu)(y)|^2\mu(\mathrm{d}y)\mu(\mathrm{d}x)$$
$$\leq -\gamma_1 \int_{\mathbb{R}^d} V(x,\mu)\mu(\mathrm{d}x) + \gamma_2.$$

The following two results are about the asymptotic stability of the solutions for Eq. (2.2).

Lemma 3.1. *Let* (H1) − (H2) *hold and assume further that there exists a positive constant* c_1 *such that* $c_1 \int_{\mathbb{R}^d} |x|^2\mu(\mathrm{d}x) \leq \int_{\mathbb{R}^d} V(x,\mu)\mu(\mathrm{d}x)$. *If* $\delta > 0$ *is sufficiently small such that*

$$\gamma_1 c_1 - 8L_1\theta\delta^2 - 8L_3\theta\delta^2 - 2L_2\theta\delta > 0, \delta < \sqrt{\frac{1}{8L_3}},$$

then the control system (2.2) *is* $H_\infty-$*stable, i.e.,*

$$\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t |x(s)|^2\mathrm{d}s \leq \gamma_2, \tag{3.1}$$

for any initial data $x_0 \in \mathbb{R}^n$. *Moreover, if* $\gamma_2 = 0$, *we have*

$$\mathbb{E}\int_0^\infty |x(t)|^2\mathrm{d}t < \infty. \tag{3.2}$$

Proof. We divide the proof into two parts.
   (i) *We construct the following Lyapunov functional which depends on the segment process* $x_t := \{x(t+r); -\delta \leq r \leq 0\}$ *with* $x(r) = x_0 \in \mathbb{R}^d, -\delta \leq r \leq 0$. *That is: Let*

$$\widetilde{V}(x_t,\mu_t) = V(x(t),\mu_t) + \theta \int_{t-\delta}^t \int_r^t [\delta|f(x(s),\mu_s)$$
$$+ u(x(\sigma_s),\mu_{\sigma_s})|^2 + |g(x(s),\mu_s)|^2]\mathrm{d}s\mathrm{d}r, \ t \geq 0, \tag{3.3}$$

where θ *is a positive constant to be determined later. Applying Itô's formula to* $\widetilde{V}(x_t,\mu_t)$ *and noting that*

$$\mathrm{d}\left(\theta\int_{t-\delta}^t \int_r^t [\delta|f(x(u),\mu_u) + u(x(\sigma_u),\mu_{\sigma_u})|^2 + |g(x(u),\mu_u)|^2]\mathrm{d}u\mathrm{d}r\right)$$

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$$= \left[\theta\delta[\delta|f(x(t),\mu_t) + u(x(\sigma_t),\mu_{\sigma_t})|^2 + |g(x(t),\mu_t)|^2] \right.$$
$$\left. - \theta \int_{t-\delta}^{t} [\delta|f(x(r),\mu_r) + u(x(\sigma_r),\mu_{\sigma_r})|^2 + |g(x(r),\mu_r)|^2]dr \right]dt,$$

we get

$$d\widetilde{V}(x_t,\mu_t) = \mathscr{L}V(x_t,\mu_t)dt + dM(t), \tag{3.4}$$

where $M(t)$ is a martingale and

$$\mathscr{L}V(x_t,\mu_t) = \mathbb{L}V(x(t),\mu_t,x(\sigma_t),\mu_{\sigma_t}) + \theta\delta[\delta|f(x(t),\mu_t) + u(x(\sigma_t),\mu_{\sigma_t})|^2 + |g(x(t),\mu_t)|^2]$$
$$- \theta \int_{t-\delta}^{t} [\delta|f(x(r),\mu_r) + u(x(\sigma_r),\mu_{\sigma_r})|^2 + |g(x(r),\mu_r)|^2]dr. \tag{3.5}$$

(ii) *We are going to prove* (3.1) *and* (3.2). From (2.3) and (3.5), we get

$$\mathscr{L}V(x_t,\mu_t) = LV(x(t),\mu_t) - \partial_x V(x(t),\mu_t)(u(x(t),\mu_t) - u(x(\sigma_t),\mu_{\sigma_t}))$$
$$- \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_\mu V(x(t),\mu_t)(y)(u(y,\mu_t) - u(\bar{y},\mu_{\sigma_t}))F_{x(t),x(\sigma(t))}(dy,d\bar{y})$$
$$+ \theta\delta[\delta|f(x(t),\mu_t) + u(x(\sigma_t),\mu_{\sigma_t})|^2 + |g(x(t),\mu_t)|^2]$$
$$- \theta \int_{t-\delta}^{t} [\delta|f(x(r),\mu_r) + u(x(\sigma_r),\mu_{\sigma_r})|^2 + |g(x(r),\mu_r)|^2]dr. \tag{3.6}$$

By Young's inequality, we have

$$-\partial_x V(x(t),\mu_t)(u(x(t),\mu_t) - u(x(\sigma_t),\mu_{\sigma_t}))$$
$$\leq \lambda_1|\partial_x V(x(t),\mu_t)|^2 + \frac{L_3}{4\lambda_1}|x(t) - x(\sigma_t)|^2 + \frac{L_3}{4\lambda_1}W_2^2(\mu_t,\mu_{\sigma_t}), \tag{3.7}$$

and

$$-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_\mu V(x(t),\mu_t)(y)(u(y,\mu_t) - u(\bar{y},\mu_{\sigma_t}))F_{x(t),x(\sigma(t))}(dy,d\bar{y})$$
$$\leq \lambda_2 \int_{\mathbb{R}^d} |\partial_\mu V(x(t),\mu_t)(y)|^2\mu_t(dy) + \frac{L_3}{4\lambda_2}\mathbb{E}[|x(t) - x(\sigma_t)|^2] + \frac{L_3}{4\lambda_2}W_2^2(\mu_t,\mu_{\sigma_t}). \tag{3.8}$$

Since $W_2^2(\mu_t,\delta_0) \leq \mathbb{E}|x(t)|^2$, it follows from (3.6), (3.7), (3.8) and (H1) that

$$\mathscr{L}V(x_t,\mu_t) \leq LV(x(t),\mu_t) + \lambda_1|\partial_x V(x(t),\mu_t)|^2 + \lambda_2 \int_{\mathbb{R}^d} |\partial_\mu V(x(t),\mu_t)(y)|^2\mu_t(dy)$$
$$+ (4L_1\theta\delta^2 + 4L_3\theta\delta^2 + L_2\theta\delta)|x(t)|^2 + (4L_1\theta\delta^2 + 4L_3\theta\delta^2 + L_2\theta\delta)W_2^2(\mu_t,\delta_0)$$
$$+ (\frac{L_3}{4\lambda_1} + 2L_3\theta\delta^2 + \frac{L_3}{4\lambda_2})|x(t) - x(\sigma_t)|^2 + (\frac{L_3}{4\lambda_1} + 2C_3\theta\delta^2 + \frac{L_3}{4\lambda_2})W_2^2(\mu(t),\mu(\sigma_t))$$

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$$- \theta \int_{t-\delta}^{t} [\delta|f(x(r), \mu_r) + u(x(\sigma_r), \mu_{\sigma_r})|^2 + |g(x(r), \mu_r)|^2] \mathrm{d}r. \qquad (3.9)$$

Noting that $W_2^2(\mu_t, \mu_{\sigma_t}) \le \mathbb{E}|x(t) - x(\sigma_t)|^2$, (3.9) and (H2), we obtain

$$\mathbb{E}\mathscr{L}V(x_t, \mu_t) = \int_{\mathbb{R}^d} \mathscr{L}V(x, \mu_t)\mu_t(\mathrm{d}x) \le -\lambda_4 \mathbb{E}|x(t)|^2 + \gamma_2$$

$$+ (\frac{L_3}{4\lambda_1} + 2L_3\theta\delta^2 + \frac{L_3}{4\lambda_2})\mathbb{E}|x(t) - x(\sigma_t)|^2 + (\frac{L_3}{4\lambda_1} + 2L_3\theta\delta^2 + \frac{L_3}{4\lambda_2})\mathbb{E}|x(t) - x(\sigma_t)|^2$$

$$- \theta\mathbb{E}\int_{t-\delta}^{t} [\delta|f(x(r), \mu_r) + u(x(\sigma_r), \mu_{\sigma_r})|^2 + |g(x(r), \mu_r)|^2]\mathrm{d}r$$

$$\le -\lambda_4 \mathbb{E}|x(t)|^2 + \gamma_2$$

$$+ 2(\frac{L_3}{4\lambda_1} + 2L_3\theta\delta^2 + \frac{L_3}{4\lambda_2})\mathbb{E}|x(t) - x(\sigma_t)|^2$$

$$- \theta\mathbb{E}\int_{t-\delta}^{t} [\delta|f(x(r), \mu_r) + u(x(\sigma_r), \mu_{\sigma_r})|^2 + |g(x(r), \mu_r)|^2]\mathrm{d}r. \qquad (3.10)$$

where $\lambda_4 = \gamma_1 c_1 - 8L_1\theta\delta^2 - 8L_3\theta\delta^2 - 2L_2\theta\delta$. Noting that

$$x(t) - x(\sigma_t) = \int_{\sigma_t}^{t} (f(x(s), \mu_s) + u(x(\sigma_s), \mu_{\sigma_s}))\mathrm{d}s + \int_{\sigma_t}^{t} g(x(s), \mu_s)\mathrm{d}B(s),$$

and using the Hölder inequality and the BDG inequality, we have

$$\mathbb{E}|x(t) - x(\sigma_t)|^2 \le 2(t - \sigma_t)\mathbb{E}\int_{\sigma_t}^{t} (|f(x(r), \mu_r) + u(x(\sigma_r), \mu_{\sigma_r})|^2 + |g(x(r), \mu_r)|^2)\mathrm{d}r.$$

Since $t - \sigma_t \le \delta$, we obtain

$$\mathbb{E}|x(t) - x(\sigma_t)|^2 \le 2\mathbb{E}\int_{t-\delta}^{t} (\delta|f(x(r), \mu_r) + u(x(\sigma_r), \mu_{\sigma_r})|^2 + |g(x(r), \mu_r)|^2)\mathrm{d}r.$$

Choosing $\theta \ge \frac{(\frac{1}{\lambda_1} + \frac{1}{\lambda_2})L_3}{1 - 8L_3\delta^2}$, this together with (3.10) yields that

$$\mathbb{E}\mathscr{L}V(x_t, \mu_t) \le -\lambda_4 \mathbb{E}|x(t)|^2 + \gamma_2. \qquad (3.11)$$

From (3.4), we get

$$0 \le \mathbb{E}\widetilde{V}(x_t, \mu_t) \le L_4 - \lambda_4 \mathbb{E}\int_0^t |x(s)|^2 \mathrm{d}s + \gamma_2 t.$$

where $L_4 = V(x_0, \mu_0) + \theta \int_{-\delta}^0 \int_r^0 [\delta|f(x(u), \mu_u) + u(x(\sigma_u), \mu_{\sigma_u})|^2 + |g(x(u), \mu_u)|^2]\mathrm{d}u\mathrm{d}r$.
This leads to

$$\limsup_{t \to \infty} \frac{1}{t}\mathbb{E}\int_0^t |x(s)|^2 \mathrm{d}s \le \gamma_2.$$

If $\gamma_2 = 0$, the second assertion follows. $\qquad \square$

Lemma 3.2. *Assume that* (H1) *and* (H2) *hold. Let* $\delta > 0$ *be sufficiently small such that* $H(\delta, p) := (3^{p-1}\delta^2 2^{L_1+\frac{p}{2}} + 3^{p-1}c_p\delta^2 2^{L_2+\frac{p}{2}} + 3^{p-1}\delta^2 2^{L_2+\frac{p}{2}})e^{3^{p-1}\delta 2^{L_1+\frac{p}{2}} + 3^{p-1}\delta c_p 2^{L_2+\frac{p}{2}}} < \frac{1}{2^p}$, *where* c_p *is the constant in BDG's inequality. Then the solution* $x(t)$ *of Eq.(2.2) satisfies the following inequality for* $p \geq 2$:

$$\mathbb{E}|x(t) - x(\sigma_t)|^p \leq \frac{2^{p-1}H(\delta, p)}{1 - 2^{p-1}H(\delta, p)}\mathbb{E}|x(t)|^p. \tag{3.12}$$

Proof. Fix any integer $l \geq 0$. For $t \in [l\delta, (l+1)\delta)$, we have $\sigma_t = \lfloor \frac{t}{\delta} \rfloor \delta = l\delta$. From (2.2), we obtain

$$|x(t) - x(\sigma_t)|^p = |x(t) - x(l\delta)|^p$$
$$= \left| \int_{l\delta}^{t}(f(x(s), \mu_s) + u(x(\sigma_s), \mu_{\sigma_s}))\mathrm{d}s + \int_{l\delta}^{t}g(x(s), \mu_s)\mathrm{d}B(s) \right|^p$$
$$\leq 3^{p-1}\left| \int_{l\delta}^{t}f(x(s), \mu_s)\mathrm{d}s \right|^p + 3^{p-1}\left| \int_{l\delta}^{t}u(x(\sigma_s), \mu_{\sigma_s})\mathrm{d}s \right|^p$$
$$+ 3^{p-1}\left| \int_{l\delta}^{t}g(x(s), \mu_s)\mathrm{d}B(s) \right|^p.$$

This, together with the Lipschitz condition (H1), implies

$$\mathbb{E}|x(t) - x(\sigma_t)|^p$$
$$\leq 3^{p-1}\mathbb{E}\left| \int_{l\delta}^{t}f(x(s), \mu_s)\mathrm{d}s \right|^p + 3^{p-1}\mathbb{E}\left| \int_{l\delta}^{t}u(x(\sigma_s), \mu_{\sigma_s})\mathrm{d}s \right|^p$$
$$+ 3^{p-1}\mathbb{E}\left| \int_{l\delta}^{t}g(x(s), \mu_s)\mathrm{d}B(s) \right|^p$$
$$\leq (3^{p-1}\delta 2^{L_1+\frac{p}{2}} + 3^{p-1}c_p\delta 2^{L_2+\frac{p}{2}})\mathbb{E}\int_{l\delta}^{t}|x(s) - x(l\delta)|^2\mathrm{d}s$$
$$+ (3^{p-1}\delta^2 2^{L_1+\frac{p}{2}} + 3^{p-1}\delta^2 2^{L_2+\frac{p}{2}} + 3^{p-1}c_p\delta^2 2^{L_2+\frac{p}{2}})\mathbb{E}|x(l\delta)|^p.$$

It follows from Gronwall's inequality that

$$\mathbb{E}|x(t) - x(\sigma_t)|^p \leq H(\delta, p)\mathbb{E}|x(\sigma_t)|^p.$$

Hence, the required assertion follows from $2^{p-1}H(\delta, p) < 1$ and

$$\mathbb{E}|x(t) - x(\sigma_t)|^p \leq 2^{p-1}H(\delta, p)(\mathbb{E}|x(t)|^p + \mathbb{E}|x(t) - x(\sigma_t)|^p).$$

$\square$

The following theorem states the asymptotic stability in mean square of the solution of Eq. (2.2).

Theorem 3.3. *Assume* (H1) *and* (H2) *hold with* $\gamma_2 = 0$. *If* $\delta > 0$ *is sufficiently small such that* $H(\delta) := (12\delta^2 L_1 + 6\delta^2 L_3 + 12\delta L_2)e^{(12\delta L_1 + 12L_2)\delta} < \frac{1}{2}$, *then the solution of controlled system* (2.2) *is stable in mean square, i.e.,*

$$\lim_{t \to \infty} \mathbb{E}|x(t)|^2 = 0.$$

Proof. For $0 \le t_1 < t_2 < \infty$, we have

$$\mathbb{E}|x(t_2) - x(t_1)|^2$$

$$\le 2|t_2 - t_1|\mathbb{E} \int_{t_1}^{t_2} |f(x(s), \mu_s) + u(x(\sigma_s), \mu_{\sigma_s})|^2 \mathrm{d}s + 2\mathbb{E} \int_{t_1}^{t_2} |g(x(s), \mu_s)|^2 \mathrm{d}s$$

$$\le 4|t_2 - t_1|\mathbb{E} \int_{t_1}^{t_2} (|f(x(s), \mu_s)|^2 + |u(x(\sigma_s), \mu_{\sigma_s})|^2)\mathrm{d}s + 2\mathbb{E} \int_{t_1}^{t_2} |g(x(s), \mu_s)|^2 \mathrm{d}s$$

$$\le 16(L_1 + 2L_3)|t_2 - t_1| \int_{t_1}^{t_2} \mathbb{E}|x(s)|^2 \mathrm{d}s + 8L_2 \int_{t_1}^{t_2} \mathbb{E}|x(s)|^2 \mathrm{d}s$$

$$+ 32L_3|t_2 - t_1| \int_{t_1}^{t_2} \mathbb{E}|x(s) - x(\sigma_s)|^2 \mathrm{d}s$$

$$\le 16\left(L_1 + 2L_3 + 2L_3 \frac{H(\delta)}{1 - 2H(\delta)}\right)|t_2 - t_1| \int_{t_1}^{t_2} \mathbb{E}|x(s)|^2 \mathrm{d}s + 8L_2 \int_{t_1}^{t_2} \mathbb{E}|x(s)|^2 \mathrm{d}s.$$

By (3.2), one can see that $\int_0^t \mathbb{E}|x(s)|^2 \mathrm{d}s$ is uniformly continuous in t on $\mathbb{R}^+$. Therefore, $\mathbb{E}|x(t)|^2$ is uniformly continuous in t. This together with (3.2) implies the assertion. $\qquad\square$

Next, we will present the exponential stability in mean square.

Theorem 3.4. *Assume that* (H1) *and* (H2) *hold with* $\gamma_2 = 0$. *Let* δ *be sufficiently small such that* $\lambda_4 = \gamma_1 c_1 - 8L_1\theta\delta^2 - 8L_3\theta\delta^2 - 2L_2\theta\delta > 0$. *If there exist two positive constants* c_1 *and* c_2 *such that*

$$c_1 \int_{\mathbb{R}^d} |x|^2 \mu(\mathrm{d}x) \le \int_{\mathbb{R}^d} V(x, \mu)\mu(\mathrm{d}x) \le c_2 \int_{\mathbb{R}^d} |x|^2 \mu(\mathrm{d}x), \tag{3.13}$$

then the solution $x(t)$ *of Eq.* (2.2) *is exponentially stable in mean square, i.e.,*

$$\limsup_{t \to \infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^2) \le -\alpha,$$

where $\alpha > 0$ *is a constant satisfying* $\alpha K \delta e^{\alpha\delta} + \alpha c_2 - \lambda_4 < 0$ *with* $K = \frac{8\theta\delta^2 L_3 H(\delta)}{1 - 2H(\delta)} + 4\theta\delta^2 L_1 + 8\theta\delta^2 L_3 + 2\theta\delta L_2$.

Proof. Let $\widetilde{V}(x_t, \mu_t)$ be defined by (3.3). Applying Itô's formula to $e^{\alpha t}\widetilde{V}(x_t, \mu_t)$ and using (3.4), we get

$$\mathbb{E}e^{\alpha t}\widetilde{V}(x_t, \mu_t) = \mathbb{E}\widetilde{V}(x_0, \delta_{x_0}) + \mathbb{E} \int_0^t e^{\alpha s}[\alpha\widetilde{V}(x_s, \mu_s) + \mathscr{L}V(x_s, \mu_s)]\mathrm{d}s.$$

By (3.11), we obtain

$$\mathbb{E}e^{\alpha t}\widetilde{V}(x_t, \mu_t) \le \mathbb{E}\widetilde{V}(x_0, \delta_{x_0}) + \int_0^t e^{\alpha s}[\alpha\mathbb{E}\widetilde{V}(x_s, \mu_s) - \lambda_4\mathbb{E}|x(s)|^2]ds. \tag{3.14}$$

Due to (3.3) and (3.13), one can see that

$$\mathbb{E}\widetilde{V}(x_t, \mu_t) \le c_2\mathbb{E}|x(t)|^2 + \mathbb{E}(\Gamma(x_t, \mu_t)),$$

where $\Gamma(x_t, \mu_t) = \theta \int_{t-\delta}^t \int_r^t [\delta|f(x(u), \mu_u) + u(x(\sigma_u), \mu_{\sigma_u})|^2 + |g(x(u), \mu_u)|^2]dudr$. It follows from (H1) that

$$\begin{aligned}
\mathbb{E}\Gamma(x_t, \mu_t) \le\ & 4\theta\delta^2 L_3\mathbb{E}\int_{t-\delta}^t |x(s) - x(\sigma_s)|^2 ds \\
& + 4\theta\delta^2 L_3\mathbb{E}\int_{t-\delta}^t W_2^2(\mu_s, \mu_{\sigma_s})ds \\
& + (2\theta\delta^2 L_1 + 4\theta\delta^2 L_3 + \theta\delta L_2)\mathbb{E}\int_{t-\delta}^t |x(s)|^2 ds \\
& + (2\theta\delta^2 L_1 + 4\theta\delta^2 L_3 + \theta\delta L_2)\mathbb{E}\int_{t-\delta}^t W_2^2(\mu_s, \delta_0)ds \\
\le\ & 8\theta\delta^2 L_3\mathbb{E}\int_{t-\delta}^t |x(s) - x(\sigma_s)|^2 ds \\
& + (4\theta\delta^2 L_1 + 8\theta\delta^2 L_3 + 2\theta\delta L_2)\mathbb{E}\int_{t-\delta}^t |x(s)|^2 ds.
\end{aligned}$$

By Lemma 3.3, we have

$$\begin{aligned}
\mathbb{E}\Gamma(x_t, \mu_t) \le\ & \frac{8\theta\delta^2 L_3 H(\delta, 2)}{1 - 2H(\delta, 2)}\mathbb{E}\int_{t-\delta}^t |x(s)|^2 ds \\
& + (4\theta\delta^2 L_1 + 8\theta\delta^2 L_3 + 2\theta\delta L_2)\mathbb{E}\int_{t-\delta}^t |x(s)|^2 ds \\
\le\ & K\mathbb{E}\int_{t-\delta}^t |x(s)|^2 ds.
\end{aligned}$$

From (3.13) and (3.14), we derive that

$$c_1 e^{\alpha t}\mathbb{E}|x(t)|^2 \le \widetilde{V}(x_0, \delta_{x_0}) + (\alpha K\delta e^{\alpha\delta} + \alpha c_2 - \lambda_4)\int_0^t \mathbb{E}|x(s)|^2 ds.$$

This together with $\alpha K\delta e^{\alpha\delta} + \alpha c_2 - \lambda_4 < 0$ yields

$$c_1 e^{\alpha t}\mathbb{E}|x(t)|^2 \le \widetilde{V}(x_0, \delta_{x_0}), \ t > \delta.$$

The required assertion follows. $\qquad\square$

4 Interacting particle systems

Assume that $\{B^1(t)\}, \{B^2(t)\}, \cdots$, are independent $m-$dimensional Brownian motions. We now consider the following equations, for $i = 1, 2, \cdots$

$$\begin{cases} \mathrm{d}x^i(t) = (f(x^i(t), \mu_t^{x_i}) + u(x^i(\sigma_t), \mu_{\sigma_t}^{x_i}))\mathrm{d}t + g(x^i(t), \mu_t^{x_i})\mathrm{d}B^i(t), t \in [0, \infty), \\ x(0) = x_0, \end{cases} \quad (4.1)$$

where $\mu_t^{x_i}$ represents the law of $x^i(t)$. Let $\{x^i(t), i = 1, 2, \cdots\}$ be the unique solution of the above equations. We now write the corresponding interacting particle systems as follows:

$$\begin{cases} \mathrm{d}x^{i,N}(t) = (f(x^{i,N}(t), \mu_t^{x,N}) + u(x^{i,N}(\sigma_t), \mu_{\sigma_t}^{x,N}))\mathrm{d}t + g(x^{i,N}(t), \mu_t^{x,N})\mathrm{d}B^i(t), t \in [0, \infty), \\ x(0) = x_0, \end{cases}$$

$$(4.2)$$

where $\mu_t^{x,N}(\cdot) := \frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}(t)}(\cdot)$. Obviously, in real world, the distribution of $x(t)$ is difficult to be observed. However, the corresponding one of the particle system can be observed. We will prove that the exponential stability of system (2.2) is equivalent to the exponential stability of corresponding particle system (4.2).

First of all, we make the following assumption:

(H3) (i) Let ξ and η be two random variables whose distributions are μ and ν, respectively, and the joint distribution of (ξ, η) be $F_{\xi,\eta}(z, \bar{z})$. Assume that there exists a Lyapunov function $U \in \mathscr{C}(\mathbb{R}^d \times \mathscr{M}_{\lambda^2}(\mathbb{R}^d))$ such that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{L}U(x, \mu, y, \nu)F_{\xi,\eta}(x, y)$$

$$\leq -\alpha_1 \int_{\mathbb{R}^d} |x|^p \mu(\mathrm{d}x) + \alpha_2 W_2^p(\mu, \nu) + \alpha_3 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x - y|^p F_{\xi,\eta}(x, y) + \beta$$

for $p \geq 2$, where $\alpha_1, \alpha_2, \alpha_3, \beta$ are four constants satisfying $\alpha_1 > 0, \alpha_2 \geq 0, \alpha_3 \geq 0, \beta \geq 0$.
(ii) There exist two positive constants c_1 and c_2 such that

$$\bar{c}_1 \int_{\mathbb{R}^d} |x|^p \mu(\mathrm{d}x) \leq \int_{\mathbb{R}^d} U(x, \mu)\mu(\mathrm{d}x) \leq \bar{c}_2 \int_{\mathbb{R}^d} |x|^p \mu(\mathrm{d}x),$$

For the future use, we cite [3, Theorem 5.8, pp.362] as the following lemma.

Lemma 4.1. *Assume that $\{x_n\}_{n\geq1}$ is a sequence of independent identically distributed (i.i.d. for short) random variables in $\mathbb{R}^d$ with common distribution $\mu \in \mathscr{P}(\mathbb{R}^d)$. For any $N \in \mathbb{N}$, we define the empirical measure $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$. If $\mu \in \mathscr{P}_q(\mathbb{R}^d)$ with $q > 4$, then there exists a constant $C = C(d, q, W_q(\mu))$ such that for any $N \geq 2$,*

$$\mathbb{E}[W_2^2(\mu^N, \mu)] \leq \tau(N) = C \begin{cases} N^{-\frac{1}{2}}, & 1 \leq d < 4, \\ N^{-\frac{1}{2}}\ln(N), & d = 4, \\ N^{-\frac{2}{d}}, & 4 < d. \end{cases}$$

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The constant C in the lemma above depends on the qth moment $W_q(\mu)$ of i.i.d. random variables. In order to apply this result to the solution $x(t)$ of Eq. (2.2), we give the following moment estimate of $x(t)$.

Lemma 4.2. *Assume* (H1) *and* (H3). *Then it holds that*

$$\sup_{t \geq 0} \mathbb{E}|x(t)|^p \leq C_{x_0,p}, \quad p \geq 2.$$

where $C_{x_0,p}$ only depends on x_0, p.

Proof. Let α, δ be two positive constants sufficiently small such that

$$H(\delta, p) < \frac{1}{2} \text{ and } \alpha \bar{c}_2 - \alpha_1 + \alpha_2 \frac{H(\delta, p)}{1 - 2H(\delta, p)} + \alpha_3 \frac{H(\delta, p)}{1 - 2H(\delta, p)} < 0.$$

From Itô's formula, (H3) and (3.12), we have

$$\mathbb{E}[e^{\alpha t} U(x(t), \mu_t)] = U(x_0, \mu_0) + \mathbb{E} \int_0^t e^{\alpha s} (\alpha U(x(s), \mu_s) + \mathbb{L}U(x(s), \mu_s, x(\sigma_s), \mu_{\sigma_s})) \mathrm{d}s$$

$$\leq U(x_0, \mu_0) + \mathbb{E} \int_0^t e^{\alpha s} (\alpha \bar{c}_2 |x(s)|^p - \alpha_1 |x(s)|^p + \alpha_2 W_p^p(\mu_s, \mu_{\sigma_s}) + \alpha_3 |x(s) - x(\sigma_s)|^p + \beta) \mathrm{d}s$$

$$\leq U(x_0, \mu_0) + \mathbb{E} \int_0^t e^{\alpha s} \left[\left(\alpha \bar{c}_2 - \alpha_1 + \alpha_2 \frac{H(\delta, p)}{1 - 2H(\delta, p)} + \alpha_3 \frac{H(\delta, p)}{1 - 2H(\delta, p)} \right) |x(s)|^p + \beta) \right] \mathrm{d}s.$$

Due to the assumption in the theorem, we have

$$\mathbb{E}[|x(t)|^p] \leq e^{-\alpha t} U(x_0, \mu_0) + \frac{\beta}{\alpha}(1 - e^{-\alpha t}).$$

We obtain the required results from the above inequality. $\qquad \square$

From the above lemma, set $W_q(\mu_t) = W_q(\mu_t, \delta_0)$, one can see that $\sup_{t \geq 0} W_q(\mu_t, \delta_0) \leq C_q$. Thus we have the following theorem.

Theorem 4.3. *Assume* (H1) $-$ (H3), *and $p > 4$. Then, we have the following two results:*

(1)

$$\lim_{N \to \infty} \mathbb{E}[|x^i(t) - x^{i,N}(t)|^2] = 0.$$

(2) The solution of Eq.(2.2) is exponentially stable in mean square, i.e., there exists a positive constant ℓ_1 such that

$$\limsup_{t \to \infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^2) \leq -\ell_1, \tag{4.3}$$

if and only if the solution of (4.2) is exponentially stable in mean square, i.e., there exists a positive constant ℓ_2 and for any i such that

$$\limsup_{t \to \infty} \lim_{N \to \infty} \frac{1}{t} \log(\mathbb{E}|x^{i,N}(t)|^2) \leq -\ell_2. \tag{4.4}$$

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Proof. Firstly, we prove the first result. By Itô's formula and Assumption (H1), we derive

$$\mathbb{E}|x^i(t) - x^{i,N}(t)|^2 \leq \mathbb{E}\Bigg\{ \int_0^t \Big[2\langle x^i(s) - x^{i,N}(s), f(x^i(s), \mu_s^{x^i}) - f(x^{i,N}(s), \mu_s^{x,N})\rangle$$

$$+ 2\langle x^i(s) - x^{i,N}(s), u(x^i(\sigma_s), \mu_{\sigma_s}^{x^i}) - u(x^{i,N}(\sigma_s), \mu_{\sigma_s}^{x,N})\rangle$$

$$+ |g(x^i(s), \mu_s^{x^i}) - g(x^{i,N}(s), \mu_s^{x,N})|^2 \Big] \mathrm{d}s \Bigg\}$$

$$\leq L \int_0^t \mathbb{E}|x^i(s) - x^{i,N}(s)|^2 \mathrm{d}s + L \int_0^t \mathbb{E}[W_2^2(\mu_s^{x^i}, \mu_s^{x,N})]\mathrm{d}s$$

$$+ L \int_0^t \mathbb{E}|x^i(\sigma_s) - x^{i,N}(\sigma_s)|^2 \mathrm{d}s + L \int_0^t \mathbb{E}[W_2^2(\mu_{\sigma_s}^{x^i}, \mu_{\sigma_s}^{x,N})]\mathrm{d}s,$$

where L is a constant independent of t, whose value may vary from one place to another. From the above inequality, we derive

$$\sup_{0\leq s\leq t} \mathbb{E}|x^i(s) - x^{i,N}(s)|^2 \leq L \int_0^t \sup_{0\leq s\leq r} \mathbb{E}|x^i(s) - x^{i,N}(s)|^2 \mathrm{d}r + L \int_0^t \mathbb{E}[W_2^2(\mu_s^{x^i}, \mu_s^{x,N})]\mathrm{d}s$$

$$+ L \int_0^t \mathbb{E}[W_2^2(\mu_{\sigma_s}^{x^i}, \mu_{\sigma_s}^{x,N})]\mathrm{d}s, \tag{4.5}$$

We construct another empirical measure which comes from (4.1) as follows:

$$\mu_s^N(\mathrm{d}x) = \frac{1}{N} \sum_{j=1}^N \delta_{x^j(s)}(\mathrm{d}x).$$

Note that

$$W_2^2(\mu_s^{x^i}, \mu_s^{x,N}) \leq 2(W_2^2(\mu_s^{x^i}, \mu_s^N) + W_2^2(\mu_s^N, \mu_s^{x,N}))$$

$$= 2W_2^2(\mu_s^{x^i}, \mu_s^N) + \frac{2}{N} \sum_{j=1}^N |x^j(s) - x^{j,N}(s)|^2,$$

and

$$\frac{1}{N} \sum_{j=1}^N \mathbb{E}|x^j(s) - x^{j,N}(s)|^2 = \mathbb{E}|x^j(s) - x^{j,N}(s)|^2.$$

Similarly,

$$W_2^2(\mu_{\sigma_s}^{x^i}, \mu_{\sigma_s}^{x,N}) \leq 2(W_2^2(\mu_{\sigma_s}^{x^i}, \mu_{\sigma_s}^N) + W_2^2(\mu_{\sigma_s}^N, \mu_{\sigma_s}^{x,N}))$$

$$= 2W_2^2(\mu_{\sigma_s}^{x^i}, \mu_{\sigma_s}^N) + \frac{2}{N} \sum_{j=1}^N |x^j(\sigma_s) - x^{j,N}(\sigma_s)|^2,$$

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and

$$\frac{1}{N} \sum_{j=1}^{N} \mathbb{E}|x^j(\sigma_s) - x^{j,N}(\sigma_s)|^2 = \mathbb{E}|x^j(\sigma_s) - x^{j,N}(\sigma_s)|^2.$$

Thus, we have

$$\int_0^t \mathbb{E}[W_2^2(\mu_s^{x^i}, \mu_s^{x,N})]\mathrm{d}s \leq 2 \int_0^t \mathbb{E}[W_2^2(\mu_s^{x^i}, \mu_s^N)]\mathrm{d}s + 2 \int_0^t \mathbb{E}|x^j(s) - x^{j,N}(s)|^2\mathrm{d}s,$$

and

$$\int_0^t \mathbb{E}[W_2^2(\mu_{\sigma_s}^{x^i}, \mu_{\sigma_s}^{x,N})]\mathrm{d}s \leq 2 \int_0^t \mathbb{E}[W_2^2(\mu_{\sigma_s}^{x^i}, \mu_{\sigma_s}^N)]\mathrm{d}s + 2 \int_0^t \mathbb{E}|x^j(\sigma_s) - x^{j,N}(\sigma_s)|^2\mathrm{d}s.$$

These together with (4.5) imply

$$\sup_{0 \leq s \leq t} \mathbb{E}|x^i(s) - x^{i,N}(s)|^2 \leq L \int_0^t \sup_{0 \leq s \leq r} \mathbb{E}|x^i(s) - x^{i,N}(s)|^2\mathrm{d}r + L \int_0^t \mathbb{E}[W_2^2(\mu_s^{x^i}, \mu_s^N)]\mathrm{d}s$$
$$+ L \int_0^t \mathbb{E}[W_2^2(\mu_{\sigma_s}^{x^i}, \mu_{\sigma_s}^N)]\mathrm{d}s \tag{4.6}$$

By Lemma 4.1, one has

$$\mathbb{E}[W_2^2(\mu_s^{x^i}, \mu_s^N)] + \mathbb{E}[W_2^2(\mu_{\sigma_s}^{x^i}, \mu_{\sigma_s}^N)] \leq \sup_{s \geq 0} C(d, q, W_q(\mu_s^i)) \begin{cases} N^{-\frac{1}{2}}, & 1 \leq d < 4, \\ N^{-\frac{1}{2}} \ln(N), & d = 4, \\ N^{-\frac{2}{d}}, & 4 < d. \end{cases}$$

Additionally, Lemma 4.2 implies that

$$\sup_{s \geq 0} C(d, q, W_q(\mu_s^i)) < \infty.$$

Therefore, the first result holds by Gronwall's inequality and (4.6).

Now, we prove the second result. One may complete the proof by the following two inequalities:

$$\mathbb{E}|x^{i,N}(t)|^2 \leq 2\mathbb{E}|x^i(t) - x^{i,N}(t)|^2 + 2\mathbb{E}|x^i(t)|^2,$$
$$\mathbb{E}|x^i(t)|^2 \leq 2\mathbb{E}|x^i(t) - x^{i,N}(t)|^2 + 2\mathbb{E}|x^{i,N}(t)|^2.$$

$\square$

4.1 Example

In this section, we give an example to illustrate the theory.

Example 4.4. *Consider the following equation:*

$$\begin{cases} \mathrm{d}y(t) = \left(2y(t) + \int_{\mathbb{R}} z\mu_t(\mathrm{d}z)\right)\mathrm{d}t + y(t)\mathrm{d}B(t), \\ y(0) = y_0, \end{cases} \tag{4.7}$$

where y_0 is a positive constant. Set

$$V(x,\mu) = |x|^2 + \int_{\mathbb{R}} |z|^2 \mu(\mathrm{d}z).$$

By the fact of $\partial_\mu\left(\int_{\mathbb{R}} |z|^2 \mu(\mathrm{d}z)\right)(y) = 2y$, computing the operator $LV(x,\mu)$ of Eq.(4.7) acting on $V(x,\mu)$, we have

$$LV(x,\mu) = (2x + \int_{\mathbb{R}} z\mu(\mathrm{d}z))2x + |x|^2 + \int_{\mathbb{R}}\left(2y + \int_{\mathbb{R}} z\mu(\mathrm{d}z)\right)2y\mu(\mathrm{d}y) + \int_{\mathbb{R}} y^2 \mu(\mathrm{d}y)$$

$$\geq 4|x|^2 + 5\int_{\mathbb{R}} |z|^2 \mu(\mathrm{d}z) + \left|\int_{\mathbb{R}} z\mu(\mathrm{d}z)\right|^2,$$

Thus, from Itô's formula, we can know that the solution of Eq.(4.7) is unstable in the sense of mean square expectation.

We now consider the following equation with discrete time feedback control:

$$\mathrm{d}x(t) = [2x(t) + \int_{\mathbb{R}} z\mu_t(\mathrm{d}z) - k_1 x(\sigma_t) - k_2 \int_{\mathbb{R}} z\mu_{\sigma_t}(\mathrm{d}z)]\mathrm{d}t + x(t)\mathrm{d}B(t), \tag{4.8}$$

where k_1 and k_2 are constants. The corresponding equation of (4.8) without delay (discrete time feedback control) is as follows:

$$\mathrm{d}\bar{x}(t) = [2\bar{x}(t) + \int_{\mathbb{R}} z\rho_t(\mathrm{d}z) - k_1\bar{x}(t) - k_2 \int_{\mathbb{R}} z\rho_t(\mathrm{d}z)]\mathrm{d}t + \bar{x}(t)\mathrm{d}B(t), \tag{4.9}$$

where ρ_t is the distribution of $\bar{x}(t)$. Computing the operator $LV(x,\mu)$ of Eq.(4.9) acting on $V(x,\mu)$, one can see that

$$LV(x,\mu) = \left(2x + \int_{\mathbb{R}} z\mu(\mathrm{d}z) - k_1 x - k_2 \int_{\mathbb{R}} z\mu(\mathrm{d}z)\right)2x + |x|^2$$

$$+ \int_{\mathbb{R}}\left(2y + \int_{\mathbb{R}} z\mu(\mathrm{d}z) - k_1 y - k_2 \int_{\mathbb{R}} z\mu(\mathrm{d}z)\right)2y\mu(\mathrm{d}y) + \int_{\mathbb{R}} y^2 \mu(\mathrm{d}y)$$

$$\leq (6 - 2k_1 + k_2)|x|^2 + (5 - 2k_1)\int_{\mathbb{R}} |z|^2 \mu(\mathrm{d}z) + (3 - k_2)\left(\int_{\mathbb{R}} z\mu(\mathrm{d}z)\right)^2.$$

and

$$\lambda_1 \int_{\mathbb{R}} |V_x(x,\mu)|^2 \mu(\mathrm{d}x) = 4\lambda_1 \int_{\mathbb{R}} |x|^2 \mu(\mathrm{d}x), \lambda_2 \int_{\mathbb{R}} |\partial_\mu V_x(x,\mu)|^2 \mu(\mathrm{d}x) \leq 4\lambda_2 \int_{\mathbb{R}} |x|^2 \mu(\mathrm{d}x).$$

Choosing $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{2}, k_1 = 8, k_2 = 3$, we have

$$\int_{\mathbb{R}} LV(x,\mu)\mu(\mathrm{d}x) + \lambda_1 \int_{\mathbb{R}} |V_x(x,\mu)|^2 \mu(\mathrm{d}x) + \lambda_2 \int_{\mathbb{R}} |\partial_\mu V_x(x,\mu)|^2 \mu(\mathrm{d}x)$$
$$\leq -5 \int_{\mathbb{R}} x^2 \mu(\mathrm{d}x). \tag{4.10}$$

Obviously, (H1) holds, and (H2) holds with $\gamma_1 = -5, \gamma_2 = 0$. Moreover, $\int_{\mathbb{R}} |x|^2 \mu(\mathrm{d}x) \leq \int_{\mathbb{R}} V(x,\mu)\mu(\mathrm{d}x) \leq 2\int_{\mathbb{R}} |x|^2 \mu(\mathrm{d}x)$. This means that the conditions of Theorem 3.5 hold. Therefore, we conclude that the solution of Eq.(4.8) is exponentially stable in mean square. Set $U(x,\mu) = |x|^6 + \int_{\mathbb{R}} |z|^6 \mu(\mathrm{d}z)$. Furthermore, from (3.6) and Lemma 3.2, we know that

$$\mathbb{E}[\mathbb{L}U(x(t),\mu_t,x(\sigma_t),\mu_{\sigma_t})]$$
$$= \mathbb{E}[LU(x(t),\mu_t) - \partial_x U(x(t),\mu_t)(u(x(t),\mu_t) - u(x(\sigma_t),\mu_{\sigma_t}))]$$
$$- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_\mu U(x(t),\mu_t)(y)(u(y,\mu_t) - u(\bar{y},\mu_{\sigma_t}))F_{x(t),x(\sigma(t))}(\mathrm{d}y,\mathrm{d}\bar{y})$$
$$\leq (52 - 12k_1 + 12k_2 + \frac{5}{3})\mathbb{E}[|x(t)|^6] + 6^5 2L_3^3 \mathbb{E}[|x(t) - x(\sigma_t)|^6]$$
$$\leq (52 - 12k_1 + 12k_2 + \frac{5}{3})\mathbb{E}[|x(t)|^6] + 6^5 2L_3^3 \frac{32H(\delta,6)}{1 - 32H(\delta,6)}\mathbb{E}[|x(t)|^6]$$
$$\leq -\frac{13}{3}\mathbb{E}[|x(t)|^6] + 6^5 2L_3^3 \frac{32H(\delta,6)}{1 - 32H(\delta,6)}\mathbb{E}[|x(t)|^6]. \tag{4.11}$$

Letting δ be small enough such that $6^5 2L_3^3 \frac{32H(\delta,6)}{1-32H(\delta,6)} < 1$, we infer that the conditions of Lemma 4.2 hold. Thus, the corresponding interacting particle system is exponentially stable in mean square.

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