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# ON THE CAMPANATO AND HÖLDER REGULARITY OF LOCAL AND NONLOCAL STOCHASTIC DIFFUSION EQUATIONS

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Abstract. In this paper, we are concerned with regularity of nonlocal stochastic partial differential equations of parabolic type. By using Campanato estimates and Sobolev embedding theorem, we first show the Hölder continuity (locally in the whole state space $\mathbb{R}^d$) for mild solutions of stochastic nonlocal diffusion equations in the sense that the solutions $u$ belong to the space $C^\gamma(D_T; L^p(\Omega))$ with the optimal Hölder continuity index $\gamma$ (which is given explicitly), where $D_T := [0,T] \times D$ for $T > 0$, and $D \subset \mathbb{R}^d$ being a bounded domain. Then, by utilising tail estimates, we are able to obtain the estimates of mild solutions in $L^p(\Omega; C^{\gamma^*}(D_T))$. What's more, we give an explicit formula between the two indexes $\gamma$ and $\gamma^*$. Moreover, we prove Hölder continuity for mild solutions on bounded domains. Finally, we present a new criterion to justify Hölder continuity for the solutions on bounded domains. The novelty of this paper is that our method is suitable to the case of space-time white noise.

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1. **Introduction.** Given $T>0$ and $D\subset\mathbb{R}^d$, let $D_T:=[0,T]\times D$. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a given filtered probability space. In papers [13, 24], the authors obtained regularity of singular stochastic integrals in the following space (see section for the definition)

$$\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))$$

for $p>1,\theta>0,\delta>0$. Further, by virtue of the celebrated Sobolev embedding theorem $\mathscr{L}^{p,\theta}(D;\delta)\hookrightarrow C^\gamma(\bar{D};\delta)$ for $\theta>1$, we succeeded in obtaining estimates of solutions in the Hölder space

$$C^\gamma(D_T;L^p(\Omega)),$$

where $\gamma=\frac{(d+2)(\theta-1)}{p}$. In the present paper, we aim to obtain the estimates of solutions in the space

$$L^p(\Omega;C^\gamma(D_T)).$$

The fundamental difficulty is the fact that usually

$$\mathbb{E}\sup_{t,x}\leq\sup_{t,x}\mathbb{E}.$$

Therefore, the space $L^p(\Omega;C^\gamma(D_T))$ is a subspace of $C^\gamma(D_T;L^p(\Omega))$. Comparing with the result given in [13, 24], we shall establish a new regularity of nonlocal diffusion equations. In order to overcome the fundamental difficulty, we are going to use the tail estimates and the equivalence between the Holder space and Campanato Space to overcome the above mentioned difficulty. The idea is fairly easy to explicate. In fact, note that

$$\begin{aligned}
\mathbb{E}(|X|^p) &= \int_\Omega |X|^p d\mathbb{P}(\omega)\\
&= p\int_0^\infty \mathbb{P}\{|X|>a\}a^{p-1}da\\
&= p\int_0^M \mathbb{P}\{|X|>a\}a^{p-1}da+p\int_M^\infty \mathbb{P}\{|X|>a\}a^{p-1}da\\
&\leq M^p+p\int_M^\infty \mathbb{P}\{|X|>a\}a^{p-1}da
\end{aligned}$$

for any arbitrarily fixed constant $M>0$. In order to obtain the $L^p$-boundedness, by the above inequality, we only need to show that the second integral is bounded. Further, by utilising Chebyshev's inequality, one can derive the desired results by means of the estimates in $\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))$.

Let us recall some regularity results about stochastic partial differential equations (SPDEs). The earliest results about the $L_p$-theory of SPDEs appeared in the works of Krylov [20, 21]. Recently, Kim-Kim [14] considered the $L_p$-theory for SPDEs driven by Lévy processes, also see [7, 15, 17, 23]. In papers [29, 28], van Neerven et al. obtained the $L_p$-theory of SPDEs by means of the semigroup approach, also see [18]. There are many papers about the regularity of SPDEs on non-smooth domains, see [5, 22]. Zhang [32] obtained the $L_p$-theory of semi-linear SPDEs on general measure spaces. Let us also mention Zhang [33] where very interestingly $L_p$-maximal regularity of (deterministic) nonlocal parabolic PDEs and Krylov estimate for SDEs driven by Cauchy processes are proved.

The Hölder estimate of SPDEs has been studied by many authors. Let us mention a few. Hsu-Wang-Wang [10] established the stochastic De Giorgi iteration and regularity of semilinear SPDEs and in our paper [25], we generalized the results of [10] by using similar method. Du-Liu [8] obtained the Schauder estimate for SPDEs. Combining the deterministic theory and convolution properties, Debussche-de Moor-Hofmanová [6] established the regularity result for quasilinear SPDEs of parabolic type. Kuksin-Nadirashvili-Piatnitski [19] obtained Hölder estimates for solutions of parabolic SPDEs on bounded domains. Most recently, Tian-Ding-Wei [27] derived the local Hölder estimates of mild solutions of stochastic nonlocal diffusion equations by using tail estimates [19]. Wang [30] generalized the results of [24] to nonlocal diffusion equation. The results on Hölder estimate of PDEs with space-time white noise are few. Fortunately, our method is suitable for the space-time white noise case.

There are two methods to deal with the Schauder estimate for SPDEs. One is using the smoothing property of the kernel, the other is using the iteration technique. In this paper, we use the Morrey-Campanato estimates and tail estimates to obtain the desired results. The advantage of Morrey-Campanato estimates is to use the properties of kernel function and Sobolev embedding theorem. Comparing with other methods to obtain the Hölder estimate, it is clear that this method is relatively simple.

The rest of this paper is organized as follows. Section 2 presents some preliminaries. In section 3, we state and prove our main results on Hölder estimate over the whole spatial space. Section 4 is concerned with Hölder estimate on bounded domains. Section 5 is devoted to some applications of our main results.

## 2. **Preliminaries.**

Set, for $X = (t, x) \in \mathbb{R} \times \mathbb{R}^d$ and $Y = (s, y) \in \mathbb{R} \times \mathbb{R}^d$, the following

$$\delta(X, Y) := \max\left\{|x - y|,\, |t - s|^{\frac{1}{2\alpha}}\right\}.$$

Let $Q_c(X)$ be the ball centered in $X = (t, x)$ with radius $c > 0$, i.e.,

$$Q_c(X) := \{Y = (s, y) \in \mathbb{R} \times \mathbb{R}^d : \delta(X, Y) < R\} = (t - c^{2\alpha}, t + c^{2\alpha}) \times B_c(x).$$

Fix $T \in (0, \infty)$ arbitrarily. Denote

$$\mathcal{O}_T := (0, T) \times \mathbb{R}^d.$$

For a bounded domain $D \subset \mathbb{R}^d$, we denote $D_T := [0, T] \times D$. For a point $X \in D_T, D(X, r) := D_T \cap Q_r(X)$ and $d(D) := diam(D)$ (that is, the diameter of $D$). The following definitions are introduced in [3]. Let us first give the definition of Campanato space.

**Definition 2.1.** *(Campanato Space) Let $p \geq 1$ and $\theta \geq 0$. A function $u$ belongs to the Campanato space $\mathscr{L}^{p,\theta}(D; \delta)$, which is a subspace of $L^p(D)$, if $u$ satisfies the following condition*

$$[u]_{\mathscr{L}^{p,\theta}(D_T;\delta)} := \left(\sup_{X \in D_T, d(D) \geq \rho > 0} \frac{1}{|D(X, \rho)|^\theta} \int_{D(X,\rho)} |u(Y) - u_{X,\rho}|^p dY\right)^{1/p} < \infty,\ u \in L^p(D_T)$$

*where $|D(X, \rho)|$ stands for the Lebesgue measure of the Borel set $D(X, \rho)$ and*

$$u_{X,\rho} := \frac{1}{|D(X, \rho)|} \int_{D(X,\rho)} u(Y) dY.$$

*For $u \in \mathscr{L}^{p,\theta}(D_T;\delta)$, we define*

$$\|u\|_{\mathscr{L}^{p,\theta}(D_T;\delta)} := \Big(\|u\|^p_{L^p(D_T)} + [u]^p_{\mathscr{L}^{p,\theta}(D_T;\delta)}\Big)^{1/p}.$$

Next, we recall the definition of Hölder space.

**Definition 2.2.** *(Hölder Space) Let $0 < \gamma \le 1$. A function $u$ belongs to the Hölder space $C^\gamma(\bar{D}_T;\delta)$ if $u$ satisfies the following condition*

$$[u]_{C^\gamma(\bar{D}_T;\delta)} := \sup_{X\in D_T, d(D)\ge\rho>0} \frac{|u(X)-u(Y)|}{\delta(X,Y)^\gamma} < \infty.$$

*For $u \in C^\gamma(\bar{D}_T;\delta)$, we define*

$$\|u\|_{C^\gamma(\bar{D}_T;\delta)} := \sup_{D_T}|u| + [u]_{C^\gamma(\bar{D}_T;\delta)}.$$

**Definition 2.3.** *Let $D_T \subset \mathbb{R}^{d+1}$ be a domain. We call the domain $D_T$ an A-type domain if there exists a constant $A > 0$ such that $\forall X \in D_T$ and $\forall\, 0 < \rho \le d(D)$, it holds that*

$$|D_T(X,\rho)| = |D_T \cap Q_\rho(X)| \ge A|Q_\rho(X)|.$$

Recall that given two sets $B_1$ and $B_2$, the relation $B_1 \cong B_2$ means that both $B_1 \subseteq B_2$ and $B_2 \subseteq B_1$ hold. The notation $f(x) \approx g(x)$ means that there is a number $0 < C < \infty$ independent of $x$, i.e. a constant, such that for every $x$ we have $C^{-1}f(x) \le g(x) \le Cf(x)$. We have then the following relation of the comparison of the two spaces defined above

**Proposition 2.1.** *Assume that $D_T$ is an A-type bounded domain. Then, for $p \ge 1$ and for $1 < \theta \le 1 + \frac{p}{d+2\alpha}$ (Recall that $d$ is the dimension of the space),*

$$\mathscr{L}^{p,\theta}(D_T;\delta) \cong C^\gamma(\bar{D}_T;\delta)$$

*with*

$$\gamma = \frac{(d+2\alpha)(\theta-1)}{p}.$$

By using [3, Lemmas 2.2 and 2.3], Chen [3] proved the Proposition 2.1 with $\alpha = 1$. It is easy to generalize the results to the case of $0 < \alpha < 1$. We only note that $|Q_c(X)| \approx c^{d+2\alpha}$ and let the details to readers.

We want to use the tail estimate to derive the following boundedness results

$$\mathbb{E}\|u\|^p_{C^\gamma([0,T]\times D)} \le C, \quad \forall p \ge 1$$

for solutions $u$ of SPDEs. To this end, we need the following proposition.

**Proposition 2.2.** [27, Lemma 2.1] *Let $u_0 \in L^p(\mathbb{R}^d \times \Omega)$. Consider the Cauchy problem*

$$\partial_t u(t,x) = \Delta^\alpha u(t,x), \quad t > 0,\ x \in \mathbb{R}^d; \quad u(0,x) = u_0(x). \tag{2.1}$$

*Then, for any $0 < \beta < 1$, the following estimates for the unique mild solution of (2.1)*

$$\|u(t,\cdot)\|_{C^\beta(\mathbb{R}^d)} \le Ct^{-\frac{\beta}{2\alpha}-\frac{d}{2p\alpha}}\|u_0\|_{L^p(\mathbb{R}^d)}, \quad \mathbb{P}-a.s.\ \omega \in \Omega, \tag{2.2}$$

*and for $\tau > 0$*

$$|u(t+\tau,x) - u(t,x)| \le Ct^{-\beta-\frac{d}{2p\alpha}}\tau^\beta\|u_0\|_{L^p(\mathbb{R}^d)}, \quad \mathbb{P}-a.s.\ \omega \in \Omega. \tag{2.3}$$

We end this section with the following properties of kernel function $K$ satisfying $K_t = \Delta^\alpha K$ (the reader is referred to [1, 2, 4, 11] for more details)

- for any $t > 0$,
$$\|K(t,\cdot)\|_{L^1(\mathbb{R}^d)} = 1 \text{ for all } t > 0.$$
- $K(t,x,y)$ is $C^\infty$ on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ for each $t > 0$;
- for $t > 0$, $x, y \in \mathbb{R}^d$, $x \neq y$, the sharp estimate of $K(t,x)$ is
$$K(t,x,y) \approx \min\left(\frac{t}{|x-y|^{d+2\alpha}}, t^{-d/(2\alpha)}\right);$$
- for $t > 0$, $x, y \in \mathbb{R}^d$, $x \neq y$, the estimate of the first order derivative of $K(t,x)$ is
$$|\nabla_x K(t,x,y)| \approx |y-x| \min\left\{\frac{t}{|y-x|^{d+2+2\alpha}}, t^{-\frac{d+2}{2\alpha}}\right\}. \tag{2.4}$$

The estimate (2.4) for the first order derivative of $K(t,x)$ was derived in [1, Lemma 5]. Xie et al. [31] obtained the estimate of the $m$-th order derivative of $p(t,x)$ by induction.

**Proposition 2.3.** [24, Proposition 5.2] *For any $m \geq 0$, we have*
$$\partial_x^m K(t,x) = \sum_{n=0}^{n=\lfloor \frac{m}{2} \rfloor} C_n |x|^{m-2n} \min\left\{\frac{t}{|x|^{d+2\alpha+2(m-n)}}, t^{-\frac{d+2(m-n)}{2\alpha}}\right\},$$
*where $\lfloor \frac{m}{2} \rfloor$ means the largest integer that is less than $\frac{m}{2}$.*

## 3. Hölder estimate in the whole space.

In this section, we establish the Morrey-Campanato estimates under different assumptions on stochastic term. Set
$$\mathcal{K}g(t,x) := \int_0^t \int_{\mathbb{R}^d} K(t-r,y) g(r,x-y)\,dy\,dW(r),$$
which is a mild solution of (3.1). The first result is similar to the deterministic case. We consider the following equation
$$du_t = \Delta^\alpha u dt + g(t,x)dW_t, \quad u|_{t=0} = 0, \tag{3.1}$$
where $\Delta^\alpha = -(-\Delta)^\alpha$ and $W_t$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

**Theorem 3.1.** *Let $D$ be an $A$-type bounded domain in $\mathbb{R}^{d+1}$ such that $\bar{D} \subset \mathcal{O}_T$. Suppose that $g \in L^\infty_{loc}(\mathbb{R}_+; L^p(\Omega\times\mathbb{R}^d))$ for $p > d/\alpha$ is $\mathcal{F}_t$-adapted process, and that $0 < \beta < \alpha$ satisfies $(\alpha-\beta)p - d \geq 0$. Then, there is a mild solution $u$ of (3.1) and $u \in \mathscr{L}^{p,\theta}((D_T;\delta); L^p(\Omega)) \cap L^p(\Omega; C^\beta(D_T))$. Moreover, it holds that*
$$\|u\|_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} \leq C\|g\|_{L^\infty([0,T];L^p(\Omega\times\mathbb{R}^d))}, \tag{3.2}$$
$$\|u\|_{C^\beta(D_T;L^p(\Omega))} \leq C\|g\|_{L^\infty([0,T];L^p(\Omega\times\mathbb{R}^d))}, \tag{3.3}$$
*where $\theta = 1 + \frac{\beta p}{d+2\alpha}$. Moreover, taking $0 < \epsilon < \beta p/2$ and $q > (d+2\alpha)/\epsilon$, we have for $1 < r < q$*
$$\|u\|_{L^r(\Omega;C^{\beta^*}(D_T))} \leq C\|g\|_{L^\infty([0,T];L^{pq}(\Omega\times\mathbb{R}^d))}, \tag{3.4}$$
*where $\beta^* = \beta - 2\epsilon/p$.*

**Proof.** The existence of mild solution of (3.1) is a classical result under the above assumptions. Now we prove the inequality (3.2). Due to the definition of Companato space, it suffices to show that

$$[u]_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))}<\infty.$$

Direct calculus shows that

$$\begin{aligned}
[u]^p_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} &\leq \sup_{D(X,c),X\in D_T,0<c\leq d(D)}\frac{1}{|D(X,c)|^{1+\theta}}\\
&\quad\times\mathbb{E}\int_{D(X,c)}\int_{D(X,c)}|u(t,x)-u(s,y)|^p dtdxdsdy\\
&\leq \sup_{D(X,c),X\in D_T,0<c\leq d(D)}\frac{1}{|D(X,c)|^{1+\theta}}\\
&\quad\times\mathbb{E}\int_{D(X,c)}\int_{D(X,c)}\Big|\int_0^t\int_{\mathbb{R}^d}K(t-r,x-z)g(r,z)dzdW(r)\\
&\quad-\int_0^s\int_{\mathbb{R}^d}K(s-r,y-z)g(r,z)dzdW(r)\Big|^p\\
&:= \sup_{D(X,c),X\in D_T,0<c\leq d(D)}\frac{1}{|D(X,c)|^{1+\theta}}\int_{D(X,c)}\int_{D(X,c)}\mathbb{E}\Upsilon dtdxdsdy.
\end{aligned}$$

Set $t\geq s$. We have the following estimates

$$\begin{aligned}
\mathbb{E}\Upsilon &\leq C\mathbb{E}\Big|\int_0^s\int_{\mathbb{R}^d}(K(t-r,x-z)-K(s-r,y-z))g(r,z)dzdW(r)\Big|^p\\
&\quad+C\mathbb{E}\Big|\int_s^t\int_{\mathbb{R}^d}K(t-r,x-z)g(r,z)dzdW(r)\Big|^p\\
&\leq C\mathbb{E}\Big|\int_0^s\Big(\int_{\mathbb{R}^d}(K(t-r,x-z)-K(s-r,y-z))g(r,z)dz\Big)^2dr\Big|^{\frac{p}{2}}\\
&\quad+C\mathbb{E}\Big|\int_s^t\Big(\int_{\mathbb{R}^d}K(t-r,x-z)g(r,z)dz\Big)^2dr\Big|^{\frac{p}{2}}\\
&:= C(H_1+H_2).
\end{aligned}$$

**Estimate of $H_1$.**
Take $\beta>0$ satisfying $(\alpha-\beta)p-d\geq 0$. We first recall the following fractional mean value formula (see (4.4) of [12])

$$f(x+h)=f(x)+\Gamma^{-1}(1+\beta)h^\beta f^{(\beta)}(x+\theta h),$$

where $0<\beta<1$ and $0\leq\theta\leq 1$ depends on $h$ satisfying

$$\lim_{h\downarrow 0}\theta^\beta=\frac{\Gamma^2(1+\beta)}{\Gamma(1+2\beta)},$$

By using the Propositions 2.2 and 2.3, the above fractional mean value formula and Hölder inequality, we have

$$
\begin{aligned}
H_1 &= \mathbb{E}\Big|\int_0^s\Big(\int_{\mathbb{R}^d}(K(t-r,x-z)-K(s-r,y-z))g(r,z)dz\Big)^2dr\Big|^{\frac{p}{2}}\\
&\le C\mathbb{E}\Big|\int_0^s\Big(\int_{\mathbb{R}^d}|K(t-r,x-z)-K(s-r,x-z)|\cdot|g(r,z)|dz\Big)^2dr\Big|^{\frac{p}{2}}\\
&\quad +C\mathbb{E}\Big|\int_0^s\Big(\int_{\mathbb{R}^d}(K(s-r,x-z)-K(s-r,y-z))\cdot g(r,z)dz\Big)^2dr\Big|^{\frac{p}{2}}\\
&\le C(t-s)^{\frac{\beta p}{2}}\mathbb{E}\Big|\int_0^s\left(\int_{\mathbb{R}^d}|\int_0^1\frac{\partial^{\frac{\beta}{2}}K}{\partial t^{\frac{\beta}{2}}}(\xi-r,x-z)d\theta|^qdz\right)^{\frac{2}{q}}\|g(r)\|^2_{L^p(\mathbb{R}^d)}dr\Big|^{\frac{p}{2}}\\
&\quad +C|x-y|^{\beta p}\mathbb{E}\Big|\int_0^s(s-r)^{-\frac{\beta}{\alpha}-\frac{d}{p\alpha}}\|g(r)\|^2_{L^p(\mathbb{R}^d)}dr\Big|^{\frac{\delta}{2}}\\
&\le C(t-s)^{\frac{\beta p}{2}}\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}\left[\int_0^s\left(\int_{\mathbb{R}^d}|\int_0^1\frac{\partial^{\frac{\beta}{2}}K}{\partial t^{\frac{\beta}{2}}}(\xi-r,x-z)d\theta|^qdz\right)^{\frac{2}{q}}dr\right]^{\frac{p}{2}}\\
&\quad +C|x-y|^{\beta p}\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}\left[\int_0^s(s-r)^{-\frac{\beta}{\alpha}-\frac{d}{p\alpha}}dr\right]^{\frac{p}{2}}\\
&\le C((t-s)^{\frac{\beta p}{2}}+|x-y|^{\beta p}),
\end{aligned}
$$

where $q=p/(p-1)$, $\xi=\theta t+(1-\theta)s$, and we used the following fact

$$
\begin{aligned}
&\int_0^s\left(\int_{\mathbb{R}^d}|\int_0^1\frac{\partial^{\frac{\beta}{2}}K}{\partial t^{\frac{\beta}{2}}}(\xi-r,x-z)d\theta|^qdz\right)^{\frac{2}{q}}dr\\
\le\ & C\int_0^s\left(\int_0^{(\xi-r)^{\frac{1}{2\alpha}}}(\xi-r)^{-\frac{dq+2q\alpha\beta}{2\alpha}}|z|^{q\alpha\beta+d-1}d|z|\right.\\
&\left.+\int_{(\xi-r)^{\frac{1}{2\alpha}}}^\infty(\xi-r)^q|z|^{-(qd+2q\alpha+2q\alpha\beta)}|z|^{q\alpha\beta+d-1}d|z|\right)^{\frac{2}{q}}dr\\
\le\ & C\left[(\theta(t-s))^{\frac{d-dq+q\alpha(1-\beta)}{q\alpha}}+\xi^{\frac{d-dq+q\alpha(1-\beta)}{q\alpha}}\right]\\
\le\ & C
\end{aligned}
$$

because using $q=p/(p-1)$, we have

$$d-dq+q\alpha(1-\beta)>0\Leftrightarrow p(\alpha-\alpha\beta)>d\Leftarrow p(\alpha-\beta)>d.$$

Similarly, we have

$$\int_0^s(s-r)^{-\frac{\beta}{\alpha}-\frac{d}{p\alpha}}dr=\frac{p\alpha}{(\alpha-\beta)p-d}s^{\frac{(\alpha-\beta)p-d}{p\alpha}}\le C$$

provided that $(\alpha-\beta)p-d\ge 0$.

**Estimate of** $H_2$.

Similar to the estimate of $H_1$, we have

$$
\begin{aligned}
H_2 &= \mathbb{E}\Big|\int_s^t \Big(\int_{\mathbb{R}^d} K(t-r,x-z)g(r,z)dz\Big)^2 dr\Big|^{\frac{p}{2}} \\
&\leq \|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}\left[\int_s^t \Big(\int_{\mathbb{R}^d} |K(t-r,x-z)|^q dz\Big)^{\frac{2}{q}} dr\right]^{\frac{p}{2}} \\
&\leq C\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}(t-s)^{\frac{q\alpha-(q-1)d}{q\alpha}\times\frac{p}{2}}
\end{aligned}
$$

provided that $\alpha p > d$. Indeed, by using $1/p + 1/q = 1$, we have

$$
q\alpha - (q-1)d > 0 \iff \alpha p > d.
$$

Combining the assumption of $p$, we have

$$
H_2 \leq C(t-s)^{\frac{p\alpha-d}{2\alpha}}.
$$

Assume that $D(X,c) = D_T \cap Q_c$ and $Q_c = Q_c(t_0,x_0)$. Noting that $(t,x) \in Q_c(t_0,x_0)$ and $(s,y) \in Q_c(t_0,x_0)$, we have

$$
0 \leq t-s \leq 2c^{2\alpha} \quad \text{and} \quad |x-y| \leq |x-x_0| + |y-x_0| \leq 2c.
$$

By using the definition of $A$-type bounded domain, we have

$$
\begin{aligned}
[u]_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} &\leq \sup_{D(X,c),X\in D_T,0<c\leq d(D)} \frac{1}{|D(X,c)|^{1+\theta}}\mathbb{E}\int_{D(X,c)}\int_{D(X,c)} \mathbb{E}\Upsilon dtdxdsdy \\
&\leq C\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))},
\end{aligned}
$$

where $\theta = 1 + \frac{\beta p}{d+2\alpha}$. This yields the inequality (3.2). Applying Proposition 2.1, one can obtain the inequality (3.3).

Next, we prove the inequality (3.4). In order to use the technique of tail estimates, we first consider the following estimates. Let $(t_0,x_0) \in D_T \subset \mathcal{O}_T$ and

$$
Q_c(t_0,x_0) = (t_0 - c^{2\alpha}, t_0 + c^{2\alpha}) \times B_c(x_0).
$$

Then we have $\bar{D}_T \subset Q_{d(D)}(t_0,x_0)$. Set $(t_1,x_1),(t_2,x_2) \in D_T$ , $Q_i := D_T \cap Q_{c_i}(t_i,x_i)$, $i = 1,2$ and

$$
\begin{aligned}
F(t_i,x_i,c_i) &= \frac{1}{|Q_i|^{1+\theta}}\int_{Q_i}\int_{Q_i} |u(t,x)-u(s,y)|^p dtdxdsdy \\
&= \frac{1}{|Q_i|^{1+\theta}}\int_{Q_i}\int_{Q_i} |\mathcal{K}g(t,x)-\mathcal{K}g(s,y)|^p dtdxdsdy.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
F(t_1,x_1,c_1) - F(t_2,x_2,c_2) &= [F(t_1,x_1,c_1) - F(t_2,x_1,c_1)] \\
&\quad + [F(t_2,x_1,c_1) - F(t_2,x_2,c_1)] \\
&\quad + [F(t_2,x_2,c_1) - F(t_2,x_2,c_2)] \\
&:= I_1 + I_2 + I_3.
\end{aligned}
$$

Estimate of $I_1$:

$$\begin{aligned}
I_1 &= F(t_1,x_1,c_1)-F(t_2,x_1,c_1)\\
&= \frac{1}{|Q_1|^{1+\theta}}\int_{Q_1}\int_{Q_1}|\mathcal{K}g(t,x)-\mathcal{K}g(s,y)|^p dtdxdsdy\\
&\quad -\frac{1}{|Q_{12}|^{1+\theta}}\int_{Q_{12}}\int_{Q_{12}}|\mathcal{K}g(t,x)-\mathcal{K}g(s,y)|^p dtdxdsdy\\
&= \frac{1}{|Q_1|^{1+\theta}}\left\{\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}}|\mathcal{K}g(t,x)-\mathcal{K}g(s,y)|^p dtdxdsdy\right.\\
&\quad \left.+\int_{Q_{12}\setminus Q_1}\int_{Q_{12}\setminus Q_1}|\mathcal{K}g(t,x)-\mathcal{K}g(s,y)|^p dtdxdsdy\right\}\\
&\quad +\left[\frac{1}{|Q_1|^{1+\theta}}-\frac{1}{|Q_{12}|^{1+\theta}}\right]\int_{Q_{12}}\int_{Q_{12}}|\mathcal{K}g(t,x)-\mathcal{K}g(s,y)|^p dtdxdsdy\\
&:= I_{11}+I_{12},
\end{aligned}$$

where $Q_{12}=D_T\cap Q_{c_1}(t_2,x_1)$. For simplicity, we assume that $|Q_1|\geq|Q_{12}|$. Otherwise, we can chance the place of $Q_1$ and $Q_{12}$. And thus $I_{12}\leq 0$ almost surely. Now, we consider the term $I_{11}$. Before giving the estimates of $I_{11}$, we first recall our aim. In order to apply the tail estimate, we want to obtain the estimates of $I_{11}$ like the followings:

$$\mathbb{E}I_{11}\leq C(t_1-t_2)^{\epsilon}\quad \text{for some}\quad \epsilon>0.$$

It is easy to see that

$$|Q_1\setminus Q_{12}|\leq C(t_1-t_2)c_1^d \quad\text{and}\quad |Q_1|\approx c_1^{d+2\alpha}.$$

So we must put some assumption on $g$ in order to get some help from it.

Set $t>s$. Denote

$$\begin{aligned}
&\mathbb{E}\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}}|\mathcal{K}g(t,x)-\mathcal{K}g(s,y)|^p dtdxdsdy\\
=&\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}}\mathbb{E}\Upsilon dtdxdsdy.
\end{aligned}$$

Similar to the proof of inequality (3.2), we have

$$\mathbb{E}\Upsilon\leq Cc_1^{\beta p}.$$

Noting that $(t,x)\in Q_1$ and $(s,y)\in Q_1$, we have

$$0\leq t-s\leq 2{c_1}^{2\alpha}\quad\text{and}\quad |x-y|\leq|x-x_1|+|y-x_1|\leq 2c_1.$$

Using the above inequalities and the properties of $A$-type domain, we deduce

$$\begin{aligned}
&\mathbb{E}\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}}\mathbb{E}\Upsilon dtdxdsdy\\
\leq\ & C(p,T)c_1^{\beta p}|Q_1\setminus Q_{12}|^2\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}.
\end{aligned}$$

Since $D_T$ is a $A$-type bounded domain, we have for $2c_1\leq diamD$,

$$\begin{aligned}
&A|Q_{c_1}(t_1,x_1)|\leq|Q_1|\leq|Q_{c_1}(t_1,x_1)|\\
&A|Q_{c_1}(t_1,x_1)\setminus Q_{c_1}(t_2,x_1)|\leq|Q_1\setminus Q_{12}|\leq|Q_{c_1}(t_1,x_1)\setminus Q_{c_1}(t_2,x_1)|.
\end{aligned}$$

We remark that

$$
\begin{aligned}
&|Q_{c_1}(t_1,x_1)| \approx c_1^{d+2\alpha},\\
&|Q_{c_1}(t_1,x_1) \setminus Q_{c_1}(t_2,x_1)| \leq C c_1^d [c_1^2 \wedge (t_1-t_2)],
\end{aligned}
$$

where $C$ is a positive constant which does not depend on $c_1$. Noting that $Q_1 \setminus Q_{12} \subset Q_1$ and taking $0<\epsilon<\beta p/2$, we have

$$
\begin{aligned}
& \mathbb{E}\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} |\mathcal{K}g(t,x)-\mathcal{K}g(s,y)|^p dtdxdsdy\\
\leq\ & C(C_0,D,d,T)\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}|Q_1|^{2+\frac{\beta p-2\epsilon}{d+2\alpha}}|t_1-t_2|^\epsilon.
\end{aligned}
$$

Similarly, we can get

$$
\begin{aligned}
& \mathbb{E}\int_{Q_{12}\setminus Q_1}\int_{Q_{12}\setminus Q_1} |u(t,x)-u(s,y)|^p dtdxdsdy\\
\leq\ & C(D,d,T)\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}|Q_1|^{2+\frac{\beta p-2\epsilon}{d+2\alpha}}|t_1-t_2|^\epsilon.
\end{aligned}
$$

Due to the fact that $I_{12}\leq 0$, we have

$$
\mathbb{E}I_1 \leq C(D,d,T)\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}|t_1-t_2|^\epsilon,
$$

where $\theta = 1+\frac{\beta p-2\epsilon}{d+2\alpha}$.

Next, we estimate $I_2$. By using the fact that

$$
\Big|[D\cap Q_{c_1}(t_2,x_1)]\setminus[D\cap Q_{c_1}(t_2,x_2)]\Big| \leq C c_1^{d-1+2\alpha}|x_1-x_2|,
$$

similar to the estimates of $I_1$, we can take $0<\epsilon<\beta p/2$ such that

$$
\begin{aligned}
\mathbb{E}I_2 &= \mathbb{E}[F(t_2,x_1,c_1)-F(t_2,x_2,c_1)]\\
&\leq C(D,d,T)\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}|x_1-x_2|^\epsilon,
\end{aligned}
$$

where $\theta = 1+\frac{\beta p-\epsilon}{d+2\alpha}$.

Next, we estimate $I_3$. By using the fact that

$$
\Big|[D\cap Q_{c_1}(t_2,x_2)]\setminus[D\cap Q_{c_2}(t_2,x_2)]\Big| \leq C c_1^{d-1+2\alpha}(c_1-c_2), \quad \text{if} \quad c_1\geq c_2,
$$

similar to the estimates of $I_1$, we can estimate

$$
\begin{aligned}
\mathbb{E}I_3 &= \mathbb{E}[F(t_2,x_2,c_1)-F(t_2,x_2,c_2)]\\
&\leq C(D,d,T)\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}|c_1-c_2|^\epsilon,
\end{aligned}
$$

where $\theta = 1+\frac{\beta p-\epsilon}{d+2\alpha}$.

Therefore, we have

$$
\begin{aligned}
& \mathbb{E}|F(t_1,x_1,c_1)-F(t_2,x_2,c_2)|^q\\
\leq\ & C(D,d,T)\|g\|^{pq}_{L^{pq}(\Omega;L^\infty([0,T];L^{pq}(\mathbb{R}^d)))}(|t_1-t_2|+|x_1-x_2|+|c_1-c_2|)^{\epsilon q},
\end{aligned}
$$

where $\theta = 1+\frac{\beta p-2\epsilon}{d+2\alpha}$, $(t_i,x_i)\in D_T$ and $0<c_i\leq d(D)$, $i=1,2$.

For simplicity, we set $D_T=[0,1]^{d+1}$ and $c\in[0,2]$. One introduces a sequence of sets:

$$
\mathcal{S}_n = \{z\in\mathbb{Z}^{d+2}|\ z2^{-n}\in(0,1)^{d+1}\times(0,2)\}, \quad n\in\mathbb{N}.
$$

For an arbitrary $e=(e_1,e_2,\cdots,e_{d+2})\in\mathbb{Z}^{d+2}$ such that

$$
|e|_\infty = \max_{1\leq j\leq d+2}|e_j| = 1,
$$

and for every $z, z+e \in \mathcal{S}_n$, we define $v_z^{n,e} = |F((z+e)2^{-n}) - F(z2^{-n})|$. From the above discussion, we have

$$\mathbb{E}|v_z^{n,e}|^q \leq C(\beta, C_0, D, d, T)\|g\|^{pq}_{L^{pq}(\Omega;L^\infty([0,T];L^{pq}(\mathbb{R}^d)))}2^{-n\epsilon q} := \hat{C}2^{-n\epsilon q}.$$

For any $\tau > 0$ and $K > 0$, one sets a number of events

$$\mathcal{A}_{z,\tau}^{n,e} = \{\omega \in \Omega | v_z^{n,e} \geq K\tau^n,\ z, z+e \in \mathcal{S}_n\},$$

which yields that

$$\mathbb{P}(\mathcal{A}_{z,\tau}^{n,e}) \leq \frac{\mathbb{E}|v_z^{n,e}|^q}{K^q\tau^{qn}} \leq \frac{\hat{C}2^{-n\epsilon q}}{K^q\tau^{qn}}.$$

Noting that for each $n$, the total number of the events $\mathcal{A}_{z,\tau}^{n,e}, z, z+e \in \mathcal{S}_n$ is not larger than $2^{d+2}3^{d+2}$. Hence the probability of the union

$$\mathcal{A}_\tau^n = \cup_{z,z+e\in\mathcal{S}_n}(\cup_{\|e\|_\infty=1}\mathcal{A}_{z,\tau}^{n,e})$$

meets the estimate

$$\mathbb{P}(\mathcal{A}_\tau^n) \leq \frac{\hat{C}2^{-n\epsilon q}}{K^q\tau^{qn}}2^{(d+2)n} \leq \hat{C}K^{-q}\left(\frac{2^{d+2}}{(2^\epsilon\tau)^q}\right)^n.$$

Let $\tau = 2^{-\nu\epsilon}$, where $\nu > 0$ satisfies $(1-\nu)\epsilon q \geq d+2$. Then the probability of the event $\mathcal{A} = \cup_{n\geq 1}\mathcal{A}_\tau^n$ can be calculated that

$$\mathbb{P}(\mathcal{A}) \leq C\hat{C}K^{-q}. \tag{3.5}$$

For every point $\xi = (t, x, c) \in (0,1)^{d+1} \times (0,2)$, we have $\xi = \sum_{i=0}^\infty e_i 2^{-i}$ ($\|e_i\|_\infty \leq 1$). Denote $\xi_k = \sum_{i=0}^k e_i 2^{-i}$ and $\xi_0 = 0$. For any $\omega \notin \mathcal{A}$, we have $|F(\xi_{k+1}) - F(\xi_k)| < K\tau^{k+1}$, which implies that

$$|F(t,x,c)| \leq \sum_{k=0}^\infty |F(\xi_{k+1}) - F(\xi_k)| < K\sum_{k=1}^\infty \tau^k \leq K(2^{\nu\epsilon} - 1)^{-1}. \tag{3.6}$$

Set $v_1 = \sup_{(t,x,c)\in(0,1)^{d+1}\times(0,2)}|F(t,x,c)|$, then $v_1 = \sup_{(t,x,c)\in[0,1]^{d+1}\times[0,2]}|F(t,x,c)|$ since $F$ has a continuous version. For $1 < r < q$, we have

$$\mathbb{E}v_1^r = r\int_0^\infty a^{r-1}\mathbb{P}(v_1 \geq a)da = r\int_0^{\gamma K} a^{r-1}\mathbb{P}(v_1 \geq a)da + r\int_{\gamma K}^\infty a^{r-1}\mathbb{P}(v_1 \geq a)da. \tag{3.7}$$

If one chooses $\gamma \geq (2^{\nu\epsilon} - 1)^{-1}$, using (3.5), (3.6) and (3.7), we get

$$\begin{aligned}\mathbb{E}v_1^r &\leq (\gamma K)^r + C\hat{C}^q r\int_{\gamma K}^\infty a^{r-1-q}da \\ &\leq (\gamma K)^r + C\hat{C}r(cK)^{r-q},\end{aligned}$$

which yields that

$$\mathbb{E}v_1^r \leq C(D, d, T)\|g\|^{pr}_{L^{pq}(\Omega;L^\infty([0,T];L^{pq}(\mathbb{R}^d)))},$$

if we choose $\|g\|^p_{L^{pq}(\Omega;L^\infty([0,T];L^{pq}(\mathbb{R}^d)))}$. By using the following embed inequality

$$L^p(\Omega; \mathscr{L}^{p,\theta}(D_T;\delta)) \cong L^p(\Omega; C^\gamma(\bar{D}_T;\delta)),$$

we obtain the inequality (3.4). The proof is complete. □

**Remark 3.1.** *It follows from Theorem 3.1 that the index $\beta$ and $\beta^*$ satisfy $\beta > \beta^*$, which implies that if we want to change the places of $\mathbb{E}$ and $\sup_{t,x}$, we must pay it on the index.*

*Comparing with the earlier results of [27] (Tian et al. obtained the Hölder estimate to equation (3.1) locally in $\mathbb{R}^d$), we find the Hölder continuous index in this paper is larger than that in [27]. More precisely, we obtain the index of time variable is closed to $1/2$. Since the index of Hölder continuous of Brownian motion is $\frac{1}{2}-$, maybe the index obtained in this paper is optimal.*

Next, we consider another case. If $g$ is a Hölder continuous function, the following theorem shows that what assumptions should be put on the kernel function $K$.

**Theorem 3.2.** *Let $u = \mathcal{K} * g$ and $D_T$ be an $A$-type bounded domain in $\mathbb{R}^{d+1}$ such that $\bar{D}_T \subset \mathcal{O}_T$. Suppose that $g \in C^\beta(\mathbb{R}_+ \times \mathbb{R}^d)$, $0 < \beta < 1$, is a non-random function and $g(0,0) = 0$. Assume that there exists positive constants $\gamma_i$ $(i = 1, 2)$ such that the non-random kernel function satisfies that for any $t \in (0, T]$*

$$\int_0^s \left( \int_{\mathbb{R}^d} |K(t-r,z) - K(s-r,z)|(1+|z|^\beta) dz \right)^2 dr \le C(T,\beta)(t-s)^{\gamma_1}, \tag{3.8}$$

$$\int_0^s \left( \int_{\mathbb{R}^d} |K(s-r,z)| dz \right)^2 dr \le C_0, \tag{3.9}$$

$$\int_s^t \left( \int_{\mathbb{R}^d} |K(t-r,z)|(1+|z|^\beta) dz \right)^2 dr \le C(T,\beta)(t-s)^{\gamma_2}, \tag{3.10}$$

*where $C_0$ is a positive constant. Then we have, for $p \ge 1$ and $\beta < \gamma$,*

$$\begin{aligned} &\|u\|_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} \le C\|g\|_{C^\beta(\mathbb{R}_+\times\mathbb{R}^d))}, \\ &\|u\|_{C^\beta(D_T;L^p(\Omega))} \le C\|g\|_{C^\beta(\mathbb{R}_+\times\mathbb{R}^d))}, \end{aligned} \tag{3.11}$$

*where $\theta = 1 + \frac{\gamma p}{d+2}$ and $\gamma = \min\{\gamma_1, \gamma_2, \beta\}$. Moreover, taking $0 < \delta < \gamma p/2$ and $q > (d+2)/\delta$, we have for $0 < r < q$*

$$\|u\|_{L^r(\Omega;C^{\beta^*}(D_T))} \le C\|g\|_{C^\beta(\mathbb{R}_+\times\mathbb{R}^d))}, \tag{3.12}$$

*where $\beta^* = \gamma - 2\delta/p$.*

**Proof.** The proof of the (3.11) is contained in our paper [24]. And we only focus on the proof of (3.12).

Similar to the proof of Theorem 3.1, we need to estimate $I_i$, $i=1,2,3$. Estimate of $I_1$:

$$
\begin{aligned}
I_1 &= F(t_1,x_1,c_1) - F(t_2,x_1,c_1) \\
&= \frac{1}{|Q_1|^{1+\theta}} \int_{Q_1}\int_{Q_1} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \\
&\quad - \frac{1}{|Q_{12}|^{1+\theta}} \int_{Q_{12}}\int_{Q_{12}} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \\
&= \frac{1}{|Q_1|^{1+\theta}} \left\{ \int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \right. \\
&\quad \left. + \int_{Q_{12}\setminus Q_1}\int_{Q_{12}\setminus Q_1} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \right\} \\
&\quad + \left[ \frac{1}{|Q_1|^{1+\theta}} - \frac{1}{|Q_{12}|^{1+\theta}} \right] \int_{Q_{12}}\int_{Q_{12}} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \\
&:= I_{11} + I_{12},
\end{aligned}
$$

where $Q_{12} = D \cap Q_{c_1}(t_2,x_1)$. For simplicity, we assume that $|Q_1| \geq |Q_{12}|$. Otherwise, we can chance the place of $Q_1$ and $Q_{12}$. And thus $I_{12} \leq 0$ almost surely.

It is easy to see that

$$
|Q_1 \setminus Q_{12}| \leq C(t_1 - t_2)c_1^d \quad \text{and} \quad |Q_1| \approx C c_1^{d+2}.
$$

So we must put some assumption on $g$ in order to get some help from it.

Set $t > s$. By the BDG inequality, we have

$$
\begin{aligned}
& \mathbb{E}\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \\
= \; & \mathbb{E}\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} \Big| \int_0^t \int_{\mathbb{R}^d} K(t-r,z)g(r,x-z)dzdW(r) \\
& - \int_0^s \int_{\mathbb{R}^d} K(s-r,z)g(r,y-z)dzdW(r) \Big|^p dtdxdsdy \\
\leq \; & 2^{p-1}\mathbb{E}\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} \Big| \int_0^s \int_{\mathbb{R}^d} (K(t-r,z) - K(s-r,z))g(r,x-z)dzdW(r) \Big|^p \\
& + 2^{p-1}\mathbb{E}\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} \Big| \int_0^s \int_{\mathbb{R}^d} K(s-r,z)(g(r,x-z) - g(r,y-z))dzdW(r) \Big|^p \\
& + 2^{p-1}\mathbb{E}\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} \Big| \int_s^t \int_{\mathbb{R}^d} K(t-r,z)g(r,x-z)dzdW(r) \Big|^p dtdxdsdy \\
\leq \; & C(p)\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} \left( \int_0^s | \int_{\mathbb{R}^d} |K(t-r,z) - K(s-r,z)||g(r,x-z)|dz|^2 dr \right)^{\frac{p}{2}} \\
& + C(p)\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} \left( \int_0^s | \int_{\mathbb{R}^d} |K(s-r,z)||g(r,x-z) - g(r,y-z)|dz|^2 dr \right)^{\frac{p}{2}} \\
& + C(p)\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} \left( \int_s^t | \int_{\mathbb{R}^d} K(t-r,z)g(r,x-z)dz|^2 dr \right)^{\frac{p}{2}} \\
=: \; & \int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} (J_1 + J_2 + J_3) dtdxdsdy.
\end{aligned}
$$

Estimate of $J_1$. By using the Hölder continuous of $g$, i.e.,

$$\begin{aligned}
|g(r,x-z)-g(0,0)| &\le C_g \max\left\{r^{\frac{1}{2}}, |x-z|\right\}^\beta \\
&\le C(g,\beta)(T^{\frac{\beta}{2}}+|x-x_1|^\beta+|x_1|^\beta+|z|^\beta) \\
&\le C(g,\beta)(T^{\frac{\beta}{2}}+c_1^\beta+|x_1|^\beta+|z|^\beta),
\end{aligned}$$

and (3.8), we have

$$\begin{aligned}
J_1 &= C(p)\left(\int_0^s|\int_{\mathbb{R}^d}|K(t-r,z)-K(s-r,z)||g(r,x-z)|dz|^2dr\right)^{\frac{p}{2}} \\
&\le C(p,\beta,T)\left(\int_0^s|\int_{\mathbb{R}^d}|K(t-r,z)-K(s-r,z)|(1+|z|^\beta)dz|^2dr\right)^{\frac{p}{2}} \\
&\quad +c_1^{\beta p}C(p,\beta)\left(\int_0^s\int_{\mathbb{R}^d}|K(t-r,z)-K(s-r,z)|dr\right)^{\frac{p}{2}} \\
&\le C(p,\beta,T)(1+c_1^{\beta p})(t-s)^{\frac{\gamma_1 p}{2}}.
\end{aligned}$$

Here and in the rest part of the proof, we write the constant depending on $\|g\|_{C^\beta(\mathbb{R}_+\times\mathbb{R}^d))}$ as $C(\beta)$ for simplicity. The condition (3.9) and

$$|g(r,x-z)-g(r,y-z)|\le C_g|x-y|^\beta$$

imply the following derivation

$$\begin{aligned}
J_2 &= C(p)\int_Q\int_Q\left(\int_0^s|\int_{\mathbb{R}^d}|K(s-r,z)||g(r,x-z)-g(r,y-z)|dz|^2dr\right)^{\frac{p}{2}} \\
&\le C(p,g)\int_Q\int_Q\left(\int_0^s|\int_{\mathbb{R}^d}|K(r,z)||x-y|^\beta dz|^2dr\right)^{\frac{p}{2}} \\
&\le C(N_0,p,g,\beta)|x-y|^{\beta p}.
\end{aligned}$$

Estimate of $I_3$. By using the property $g(0,0)=0$ and (3.10), we get

$$\begin{aligned}
J_3 &= C(p)\left(\int_s^t|\int_{\mathbb{R}^d}K(t-r,z)g(r,x-z)dz|^2dr\right)^{\frac{p}{2}} \\
&\le C\left(\int_s^t\left|\int_{\mathbb{R}^d}|K(r,z)|(T+|x-x_1|^\beta+|x_1|^\beta+|z|^\beta)dz\right|^2dr\right)^{\frac{p}{2}} \\
&\le C(p,T,\beta)\left(\int_s^t\left|\int_{\mathbb{R}^d}|K(t-r,z)|(1+|z|^\beta)dz\right|^2dr\right)^{\frac{p}{2}} \\
&\quad +C(p,T,\beta)|x-y|^{\beta p}\left(\int_s^t\left|\int_{\mathbb{R}^d}|K(t-r,z)|dz\right|^2dr\right)^{\frac{p}{2}} \\
&\le C(p,T,\beta)(t-s)^{\frac{\gamma_2 p}{2}}(1+|x-y|^{\beta p}).
\end{aligned}$$

Noting that $(t,x)\in Q_1$ and $(s,y)\in Q_1$, we have

$$0\le t-s\le 2{c_1}^2 \quad \text{and} \quad |x-y|\le|x-x_1|+|y-x_1|\le 2c_1.$$

Using the above inequality and the properties of $A$-type domain, we deduce

$$\begin{aligned}
\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} J_1 dtdxdsdy &\leq C(p,T,\beta)(1+c_1^{\beta p}){c_1}^{\gamma_1 p}|Q_1\setminus Q_{12}|^2;\\
\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} J_2 dtdxdsdy &\leq C(C_0,p,g,\beta){c_1}^{\beta p}|Q_1\setminus Q_{12}|^2;\\
\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} J_3 dtdxdsdy &\leq C(p,T,\beta)|Q_1\setminus Q_{12}|^2{c_1}^{\gamma_2 p}(1+{c_1}^{\beta p}).
\end{aligned}$$

Combining the estimates of $J_1, J_2$ and $J_3$, we get

$$\begin{aligned}
&\mathbb{E}\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}}|u(t,x)-u(s,y)|^p dtdxdsdy\\
\leq\ & C(\beta,C_0,T,p)|Q_1\setminus Q_{12}|^2(c_1^{\beta p}+1)(c_1^{\beta p}+c_1^{\gamma_1 p}+c_1^{\gamma_2 p}).
\end{aligned}$$

Since $D$ is a $A$-type bounded domain, we have for $2c_1\leq diamD$,

$$\begin{aligned}
&A|Q_{c_1}(t_1,x_1)|\leq|Q_1|\leq|Q_{c_1}(t_1,x_1)|\\
&A|Q_{c_1}(t_1,x_1)\setminus Q_{c_1}(t_2,x_1)|\leq|Q_1\setminus Q_{12}|\leq|Q_{c_1}(t_1,x_1)\setminus Q_{c_1}(t_2,x_1)|.
\end{aligned}$$

We remark that

$$\begin{aligned}
&|Q_{c_1}(t_1,x_1)|\approx Cc_1^{d+2},\\
&|Q_{c_1}(t_1,x_1)\setminus Q_{c_1}(t_2,x_1)|\leq Cc_1^d[c_1^2\wedge(t_1-t_2)],
\end{aligned}$$

where $C$ is a positive constant which does not depend on $c_1$. Noting that $Q_1\setminus Q_{12}\subset Q_1$ and taking $0<\delta<1$, we have

$$\begin{aligned}
&\mathbb{E}\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}}|\mathcal{K}g(t,x)-\mathcal{K}g(s,y)|^p dtdxdsdy\\
\leq\ & C(\beta,C_0,D,d,T)|Q_1|^{2+\frac{\gamma p-2\delta}{d+2}}|t_1-t_2|^\delta,
\end{aligned}$$

where $\gamma=\min\{\gamma_1,\gamma_2,\beta\}$.

Similarly, we can get

$$\begin{aligned}
&\mathbb{E}\int_{Q_{12}\setminus Q_1}\int_{Q_{12}\setminus Q_1}|u(t,x)-u(s,y)|^p dtdxdsdy\\
\leq\ & C(\beta,C_0,D,d,T)|Q_1|^{2+\frac{\gamma p-2\delta}{d+2}}|t_1-t_2|^\delta.
\end{aligned}$$

Due to the fact that $I_{12}\leq 0$, we have

$$\mathbb{E}I_1\leq C(\beta,C_0,D,d,T)|t_1-t_2|^\delta,$$

where $\theta=1+\frac{\gamma p-2\delta}{d+2}$.

Next, similar to the proof of Theorem 3.1, one can estimate $I_2$ and $I_3$ as followings

$$\begin{aligned}
\mathbb{E}I_2&=\mathbb{E}[F(t_2,x_1,c_1)-F(t_2,x_2,c_1)]\leq C(\beta,C_0,D,d,T)|x_1-x_2|^\delta,\\
\mathbb{E}I_3&=\mathbb{E}[F(t_2,x_2,c_1)-F(t_2,x_2,c_2)]\leq C(\beta,C_0,D,d,T)|c_1-c_2|^\delta,
\end{aligned}$$

where $\theta=1+\frac{\gamma p-\delta}{d+2}$.

Therefore, we have

$$\begin{aligned}
&\mathbb{E}|F(t_1,x_1,c_1)-F(t_2,x_2,c_2)|^q\\
\leq\ & C(C_0,D,d,T)\|g\|^q_{C^\beta(\mathbb{R}_+\times\mathbb{R}^d))}(|t_1-t_2|+|x_1-x_2|+|c_1-c_2|)^{\delta q},
\end{aligned}$$

where $\theta = 1 + \frac{\beta p - 2\delta}{d+2}$, $(t_i, x_i) \in D_T$ and $0 < c_i \leq d(D)$, $i = 1, 2$. The rest proof of this theorem is exactly similar to that of Theorem 3.1 and we omit it here. The proof of Theorem 3.2 is complete. □

Next, we consider the following equation

$$\frac{\partial}{\partial t} u(t,x) = \Delta^\alpha u(t,x) + g(t,x)\dot{W}(t,x), \quad u|_{t=0} = 0, \tag{3.13}$$

where $\Delta^\alpha = -(-\Delta)^\alpha$ and $W(t,x)$ is a standard space-time white noise.

**Theorem 3.3.** *Let $D$ be an $A$-type bounded domain in $\mathbb{R}^{d+1}$ such that $\bar{D} \subset \mathcal{O}_T$. Suppose that $g \in L^\infty_{loc}(\mathbb{R}_+; L^p(\Omega \times \mathbb{R}^d))$ is $\mathcal{F}_t$-adapted process. Set $d = 1$. Assume that $\frac{1}{2} < \alpha \leq 1$, $p > \frac{2}{2\alpha - 1}$. Let $\beta > 0$ be sufficiently small such that $p(2\alpha - 2\beta - 1) > 2$. Then, there is a mild solution $u$ of (3.13) and $u \in \mathscr{L}^{p,\theta}((D_T;\delta); L^p(\Omega)) \cap L^p(\Omega; C^\beta(D_T))$. Moreover, it holds that*

$$\|u\|_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} \leq C\|g\|_{L^\infty([0,T];L^p(\Omega\times\mathbb{R}))}, \tag{3.14}$$
$$\|u\|_{C^\beta(D_T;L^p(\Omega))} \leq C\|g\|_{L^\infty([0,T];L^p(\Omega\times\mathbb{R}))}, \tag{3.15}$$

*where $\theta = 1 + \frac{\beta p}{3}$. Moreover, taking $0 < \epsilon < \beta p/2$ and $q > 3/\epsilon$, we have for $1 < r < q$*

$$\|u\|_{L^r(\Omega;C^{\beta^*}(D_T))} \leq C\|g\|_{L^\infty([0,T];L^{pq}(\Omega\times\mathbb{R}))}, \tag{3.16}$$

*where $\beta^* = \beta - 2\epsilon/p$.*

**Proof.** The existence of mild solution of (3.13) is a classical result under the above assumptions. Now we prove the inequality (3.14). Due to the definition of Companato space, it suffices to show that

$$[u]_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} < \infty.$$

Direct calculus shows that

$$\begin{aligned}
[u]^p_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} &\leq \sup_{D(X,c),X\in D_T,0<c\leq d(D)} \frac{1}{|D(X,c)|^{1+\theta}} \\
&\quad \times \mathbb{E}\int_{D(X,c)}\int_{D(X,c)} |u(t,x) - u(s,y)|^p dtdxdsdy \\
&\leq \sup_{D(X,c),X\in D_T,0<c\leq d(D)} \frac{1}{|D(X,c)|^{1+\theta}} \\
&\quad \times \mathbb{E}\int_{D(X,c)}\int_{D(X,c)} \Big| \int_0^t \int_{\mathbb{R}} K(t-r,x-z)g(r,z)W(dr,dz) \\
&\quad - \int_0^s \int_{\mathbb{R}} K(s-r,y-z)g(r,z)W(dr,dz)\Big|^p \\
&:= \sup_{D(X,c),X\in D_T,0<c\leq d(D)} \frac{1}{|D(X,c)|^{1+\theta}} \int_{D(X,c)}\int_{D(X,c)} \mathbb{E}\Upsilon dtdxdsdy.
\end{aligned}$$

Set $t \geq s$. We have the following estimates

$$
\begin{aligned}
\mathbb{E}\Upsilon &\leq C\mathbb{E}\Big|\int_0^s \int_{\mathbb{R}} (K(t-r,x-z)-K(s-r,y-z))g(r,z)W(dr,dz)\Big|^p \\
&\quad +C\mathbb{E}\Big|\int_s^t \int_{\mathbb{R}} K(t-r,x-z)g(r,z)W(dr,dz)\Big|^p \\
&\leq C\mathbb{E}\Big|\int_0^s \int_{\mathbb{R}} (K(t-r,x-z)-K(s-r,y-z))^2 g^2(r,z)dzdr\Big|^{\frac{p}{2}} \\
&\quad +C\mathbb{E}\Big|\int_s^t \int_{\mathbb{R}} K^2(t-r,x-z)g^2(r,z)dzdr\Big|^{\frac{p}{2}} \\
&=: C(H_1+H_2).
\end{aligned}
$$

**Estimate of $H_1$.**

Take $\beta > 0$ satisfying $(2\alpha - 2\beta - 1)p - 2 \geq 0$. By using the Proposition 2.3, and Hölder inequality, we have

$$
\begin{aligned}
H_1 &= \mathbb{E}\Big|\int_0^s \int_{\mathbb{R}} (K(t-r,x-z)-K(s-r,y-z))^2 g^2(r,z)dzdr\Big|^{\frac{p}{2}} \\
&\leq C\mathbb{E}\Big|\int_0^s \int_{\mathbb{R}} |K(t-r,x-z)-K(s-r,x-z)|^2 \cdot |g^2(r,z)|dzdr\Big|^{\frac{p}{2}} \\
&\quad +C\mathbb{E}\Big|\int_0^s \int_{\mathbb{R}} (K(s-r,x-z)-K(s-r,y-z))^2 \cdot g^2(r,z)dzdr\Big|^{\frac{p}{2}} \\
&=: H_{11}+H_{12}.
\end{aligned}
$$

For $H_{11}$, we have

$$
\begin{aligned}
H_{11} &\leq C(t-s)^{\frac{\beta p}{2}}\mathbb{E}\Big|\int_0^s \left(\int_{\mathbb{R}} |\frac{\partial^{\frac{\beta}{2}} K}{\partial t^{\frac{\beta}{2}}}(\xi-r,x-z)|^q dz\right)^{\frac{2}{q}} \|g(r)\|^2_{L^p(\mathbb{R})}dr\Big|^{\frac{p}{2}} \\
&\leq C(t-s)^{\frac{\beta p}{2}}\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R})))}\left[\int_0^s \left(\int_{\mathbb{R}} |\frac{\partial^{\frac{\beta}{2}} K}{\partial t^{\frac{\beta}{2}}}(\xi-r,x-z)|^q dz\right)^{\frac{2}{q}} dr\right]^{\frac{p}{2}},
\end{aligned}
$$

where $q = 2p/(p-2)$, $\xi = \theta t + (1-\theta)s$, $0 < \theta < 1$ and we used the following fact

$$
\begin{aligned}
&\int_0^s \left(\int_{\mathbb{R}} |\frac{\partial^{\frac{\beta}{2}} K}{\partial t^{\frac{\beta}{2}}}(\xi-r,x-z)|^q dz\right)^{\frac{2}{q}} dr \\
&\leq C\int_0^s \Bigg(\int_0^{(\xi-r)^{\frac{1}{2\alpha}}} (\xi-r)^{-\frac{q+2q\alpha\beta}{2\alpha}}|z|^{q\alpha\beta}d|z| \\
&\quad + \int_{(\xi-r)^{\frac{1}{2\alpha}}}^\infty (\xi-r)^q|z|^{-(q+2q\alpha+2q\alpha\beta)}|z|^{q\alpha\beta}d|z|\Bigg)^{\frac{2}{q}} dr \\
&\leq C\left[(\theta(t-s))^{\frac{1-q+q\alpha(1-\beta)}{q\alpha}} + \xi^{\frac{1-q+q\alpha(1-\beta)}{q\alpha}}\right] \\
&\leq C
\end{aligned}
$$

because using $q = 2p/(p-2)$, we have

$$
1-q+q\alpha(1-\beta) > 0 \Leftrightarrow p(2\alpha-2\alpha\beta-1) > 2 \Leftarrow p(2\alpha-2\beta-1) > 2.
$$

For $H_{12}$, by using the fractional mean value formula again, we have

$$\begin{aligned}
H_{12} &\leq C|x-y|^{\beta p}\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R})))}\Big|\int_0^s \Big(\int_{\mathbb{R}}[K^{(\beta)}(s-r,\xi-z)]^q dzdr\Big)^{\frac{2}{q}} dr\Big|^{\frac{p}{2}} \\
&\leq C|x-y|^{\beta p}\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R})))}\Big[\int_0^s (s-r)^{-\frac{d(q-1)+\beta q}{q\alpha}}dr\Big]^{\frac{p}{2}} \\
&\leq C|x-y|^{\beta p},
\end{aligned}$$

where $q = 2p/(p-2)$, $\xi = \theta x + (1-\theta)y$ and we used the following inequality

$$\int_0^s (s-r)^{-\frac{d(q-1)+\beta q}{q\alpha}}dr = \frac{q\alpha}{q(\alpha-\beta)-(q-1)}s^{\frac{q(\alpha-\beta)-(q-1)}{q\alpha}} \leq C$$

provided that $(2\alpha-2\beta-1)p-2 \geq 0$.

**Estimate of $H_2$.**

Similar to the estimate of $H_1$, we have

$$\begin{aligned}
H_2 &= \mathbb{E}\Big|\int_s^t\int_{\mathbb{R}^d} K^2(t-r,x-z)g^2(r,z)dzdr\Big|^{\frac{p}{2}} \\
&\leq \|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}\Big[\int_s^t\Big(\int_{\mathbb{R}^n}|K(t-r,x-z)|^q dz\Big)^{\frac{2}{q}}dr\Big]^{\frac{p}{2}} \\
&\leq C\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}(t-s)^{\frac{q\alpha-(q-1)d}{q\alpha}\times\frac{p}{2}}
\end{aligned}$$

provided that $p(2\alpha-1) > 2$. Indeed, by using $q = \frac{2p}{p-2}$, we have

$$q\alpha-(q-1) > 0 \iff p(2\alpha-1) > 2.$$

Combining the assumption of $p$, we have

$$H_2 \leq C(t-s)^{\frac{p(2\alpha-1)-2}{2\alpha}}.$$

The rest proof is similar to that of 3.1 and we omit it here. □

4. **Hölder estimate on a bounded domain.** In this section, we consider the SPDEs of the following form

$$\begin{cases} du = Audt + g(t,x)dW_t, & (t,x)\in(0,\infty)\times D, \\ u|_{\partial D} = 0, \\ u_{t=0} = 0, \end{cases} \tag{4.1}$$

where $D$ is a smooth bounded domain in $\mathbb{R}^d$, $W_t$ is standard one-dimensional Brownian motion, and $g$ is progressively measurable $L^\infty$- or $L^p$-function.

Throughout this section, we assume that $A$ is a uniformly elliptic second-order differential operator of the form

$$A = a_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

with smooth coefficients. Furthermore, we assume that at least one of the following two assumptions holds:

$$\begin{aligned}
B^\infty: &\quad \|g\|_{L^\infty([0,T];L^p(\Omega;L^\infty(D)))} < \infty, \\
B^p: &\quad \|g\|_{L^\infty([0,T];L^p(\Omega\times D)))} < \infty.
\end{aligned}$$

In order to obtain the Hölder estimate, we need the following Lemma. Consider the following initial-boundary problem:

$$\frac{\partial v}{\partial t} - Av = 0, \quad v|_{t=0} = F(x), \quad v|_{\partial D} = 0, \tag{4.2}$$

and denote by $S_t$ the corresponding semigroup:

$$v(t,\cdot) = (S_t F)(\cdot), \quad F = F(\cdot).$$

**Lemma 4.1.** [19, Lemma 1] *Let $|F(x)| < M$. Then, for any $\theta < 1$, the following estimates hold with $c > 0$ and $\tau > 0$:*

$$\|v(t,\cdot)\|_{C^\theta(D)} \le c(\theta) M t^{-\theta/2} \exp(-ct),$$
$$|v(t+\tau, x) - v(t,x)| \le c(\theta) M t^{-\theta} \tau^\theta \exp(-ct).$$

*Moreover, if $\|F\|_{L^p(D)} \le M$ and $p > 1$, then for $\tau > 0$*

$$\|v(t,\cdot)\|_{C^\theta(D)} \le c(\theta) M t^{-\theta/2 - d/(2p)} \exp(-ct),$$
$$|v(t+\tau, x) - v(t,x)| \le c(\theta) M t^{-\theta - d/(2p)} \tau^\theta \exp(-ct).$$

**Theorem 4.1.** *Let $D_T$ be an $A$-type bounded domain in $\mathbb{R}^{d+1}$.*

*(i) Suppose that $B^p$ holds for $p > d$ and that $0 < \beta < 1$ satisfies $(1-\beta)p - d \ge 0$. Then, there is a mild solution $u$ of (4.1) and $u \in \mathscr{L}^{p,\theta}((D_T;\delta); L^p(\Omega)) \cap L^p(\Omega; C^\beta(D_T))$. Moreover, it holds that*

$$\|u\|_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} \le C\|g\|_{L^\infty([0,T];L^p(\Omega\times D))},$$
$$\|u\|_{C^\beta(D_T;L^p(\Omega))} \le C\|g\|_{L^\infty([0,T];L^p(\Omega\times D))},$$

*where $\theta = 1 + \frac{\beta p}{d+2}$. Moreover, taking $0 < \epsilon < \beta p/2$ and $q > (d+2)/\epsilon$, we have for $1 < r < q$*

$$\|u\|_{L^r(\Omega;C^{\beta^*}(D_T))} \le C\|g\|_{L^\infty([0,T];L^{pq}(\Omega\times\mathbb{R}^d))},$$

*where $\beta^* = \beta - 2\epsilon/p$.*

*(ii) Suppose that $B^\infty$ holds for $p > 1$. Then, there is a mild solution $u$ of (4.1) and $u \in \mathscr{L}^{p,\theta}((D_T;\delta); L^p(\Omega)) \cap L^p(\Omega; C^\beta(D_T))$. Moreover, it holds that*

$$\|u\|_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} \le C\|g\|_{L^\infty([0,T];L^p(\Omega\times D))},$$
$$\|u\|_{C^\beta(D_T;L^p(\Omega))} \le C\|g\|_{L^\infty([0,T];L^p(\Omega\times D))},$$

*where $\theta = 1 + \frac{p}{d+2}$. Moreover, taking $0 < \epsilon < \beta p/2$ and $q > (d+2)/\epsilon$, we have for $1 < r < q$*

$$\|u\|_{L^r(\Omega;C^{\beta^*}(D_T))} \le C\|g\|_{L^\infty([0,T];L^{pq}(\Omega\times\mathbb{R}^d))},$$

*where $\beta^* = 1 - 2\epsilon/p$.*

**Proof.** The proof of this Theorem is exactly similar to that of Theorem 3.1 by using Lemma 4.1. We omit it to the readers. We only emphasize that the distance will be used here

$$\delta(X,Y) := \max\left\{|x-y|, \, |t-s|^{\frac{1}{2}}\right\},$$

where is different from that in Section 2. The proof is complete. □

**Remark 4.1.** *Theorem 4.1 does not hold for the nonlocal operator because we did not have the similar properties of kernel function on bounded domain.*

*Comparing Theorem 4.1 with* [19, Theorems 1 and 2], *we find the index of* [19] *is $\beta < \frac{1}{2} - \frac{d}{2p}$ for the case $B^p$ and the index in this paper is larger than that of* [19].

5. **Applications and further discussions.** We first give an example for Theorem 3.2. Consider the equation (3.1). In our paper [24], by using Proposition 2.3, we got the following result, where $p$ is the heat kernel.

**Lemma 5.1.** *Let $0 \le \epsilon < \alpha$. The following estimates hold.*

$$\int_0^s \left( \int_{\mathbb{R}^d} |\nabla^\epsilon p(t-r,z) - \nabla^\epsilon p(s-r,z)|(1+|z|^\beta)dz \right)^2 dr \le N(T,\beta)(t-s)^\gamma,$$

$$\int_0^s \left( \int_{\mathbb{R}^d} |\nabla^\epsilon p(s-r,z)|dz \right)^2 dr \le N_0,$$

$$\int_s^t \left( \int_{\mathbb{R}^d} |\nabla^\epsilon p(t-r,z)|(1+|z|^\beta)dz \right)^2 dr \le N(T,\beta)(t-s)^\gamma,$$

*where $\gamma = \frac{\alpha-\epsilon}{\alpha}$.*

Then applying Theorem 3.2 with $u = \mathcal{K} * g$ and $\mathcal{K} = \nabla^\epsilon p$, we have the following result.

**Theorem 5.1.** *Let $0 \le \epsilon < \alpha$ and $D_T$ be an A-type bounded domain in $\mathbb{R}^{d+1}$ such that $\bar{D}_T \subset \mathcal{O}_T$. Suppose that $g \in C^\beta(\mathbb{R}_+ \times \mathbb{R}^d)$, $0 < \beta < 1$, is a non-random function and $g(0,0) = 0$. Then we have, for $p \ge 1$ and $\beta < \gamma$,*

$$\|\nabla^\epsilon u\|_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} \le C\|g\|_{C^\beta(\mathbb{R}_+\times\mathbb{R}^d))},$$
$$\|\nabla^\epsilon u\|_{C^\beta(D_T;L^p(\Omega))} \le C\|g\|_{C^\beta(\mathbb{R}_+\times\mathbb{R}^d))},$$

*where $\theta = 1 + \frac{\gamma p}{d+2}$ and $\gamma = \frac{\alpha-\epsilon}{\alpha}$. Moreover, taking $0 < \delta < \gamma p/2$ and $q > (d+2)/\delta$, we have for $0 < r < q$*

$$\|\nabla^\epsilon u\|_{L^r(\Omega;C^{\beta^*}(D_T))} \le C\|g\|_{C^\beta(\mathbb{R}_+\times\mathbb{R}^d))},$$

*where $\beta^* = \gamma - 2\delta/p$.*

In fact, one can use the factorization method to obtain the Hölder estimates of solutions to the following equation

$$du_t = [\Delta^\alpha u + f(t,x,u)]dt + g(t,x)dW_t, \quad u|_{t=0} = u_0(x),$$

where $\Delta^\alpha = -(-\Delta)^\alpha$, $\alpha \in (0,1]$ and $W_t$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. About the factorization method, see [6].

In addition, one can use the Kunita's first inequality to deal with a general case. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space such that $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration on $\Omega$ containing all $P$-null subsets of $\Omega$ and $\mathbb{F}$ be the predictable $\sigma$-algebra associated with the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. We are given a $\sigma$-finite measure space $(Z, \mathcal{Z}, \nu)$ and a Poisson random measure $\mu$ on $[0,T] \times Z$, defined on the stochastic basis. The compensator of $\mu$ is Leb$\otimes\nu$, and the compensated martingale measure $\tilde{N} := \mu - Leb \otimes \nu$. The method used here is also suitable to the case that

$$\begin{aligned} \mathcal{G}g(t,x) &= \int_0^t \int_Z K(t,s,\cdot) * g(s,\cdot,z)(x)\tilde{N}(dz,ds) \\ &= \int_0^t \int_Z \int_{\mathbb{R}^d} K(t-s,x-y)g(s,y,z)dy\tilde{N}(dz,ds) \qquad (5.1) \end{aligned}$$

for $\mathbb{F}$-predictable processes $g : [0,T] \times \mathbb{R}^d \times Z \times \Omega \to \mathbb{R}$.

In the end of this section, we give a new criteria based on the following Proposition.

**Proposition 5.1.** [26, Theorem 2.1] *Let $\{X_t, t \in [0,1]\}$ be a Banach-valued stochastic field for which there exist three strictly positive constants $\gamma, c, \varepsilon$ such that*

$$\mathbb{E}[\sup_{0\le t\le 1} |X_t(x) - X_t(y)|^\gamma] \le c|x-y|^{d+\varepsilon},$$

*then there is a modification $\tilde{X}$ of $X$ such that*

$$\mathbb{E}\Big[\Big(\sup_{s\neq t} \frac{|\tilde{X}_t - \tilde{X}_s|}{|t-s|^\alpha}\Big)^\gamma\Big] < \infty$$

*for every $\alpha \in [0, \varepsilon/\gamma)$. In particular, the paths of $\tilde{X}$ are Hölder continuous in $x$ of order $\alpha$.*

For applications, we need prove the Kolmogorov criterion with the following form.

**Theorem 5.2.** *Let $\{X_t(x), x \in [0,1]^d, t \in [0,1]\}$ be a Banach-valued stochastic field for which there exist three strictly positive constants $\gamma, c, \varepsilon$ such that*

$$\mathbb{E}[\sup_{0\le t\le 1} |X_t(x) - X_t(y)|^\gamma] \le c|x-y|^{d+\varepsilon},$$

*then there is a modification $\tilde{X}$ of $X$ such that*

$$\mathbb{E}\Big[\sup_{0\le t\le 1}\Big(\sup_{x\neq y} \frac{|\tilde{X}_t(x) - \tilde{X}_t(y)|}{|x-y|^\alpha}\Big)^\gamma\Big] < \infty$$

*for every $\alpha \in [0, \varepsilon/\gamma)$. In particular, the paths of $\tilde{X}$ are Hölder continuous in $x$ of order $\alpha$.*

**Proof.** Let $D_m$ be the set of points in $[0,1]^d$ whose components are equal to $2^{-m}i$ for some integral $i \in [0, 2^m]$. The set $D = \cup_m D_m$ is the set of dyadic numbers. Let further $\Delta_m$ be the set of pairs $(x, y)$ in $D_m$ such that $|x-y| = 2^{-m}$. There are $2^{(m+1)d}$ such pairs in $\Delta_m$.

Let us finally set $K_i(t) = \sup_{(x,y)\in\Delta_i} |X_t(x) - X_t(y)|$. The hypothesis entails that for a constant $J$,

$$\mathbb{E}[\sup_{0\le t\le 1} K_i(t)^\gamma] \le \sum_{(x,y)\in\Delta_i} \mathbb{E}[\sup_{0\le t\le 1} |X_t(x) - X_t(y)|^\gamma] \le c2^{(i+1)d}2^{-i(d+\varepsilon)} = J2^{-i\varepsilon}.$$

For a point $x$ (resp. $y$) in $D$, there is an increasing sequences $\{x_m\}$ (resp. $\{y_m\}$) of points in $D$ such that $x_m$ (resp. $y_m$) is in $D_m$ for each $m$, $x_m \le x$ ($y_m \le y$) and $x_m = x$ ($y_m = y$) from some $m$ on. If $|x-y| \le 2^{-m}$, then either $x_m = y_m$ or $(x_m, y_m) \in \Delta_m$ and in any case

$$X_t(x) - X_t(y) = \sum_{i=m}^{\infty}(X_t(x_{i+1}) - X_t(x_i)) + X_t(x_m) - X_t(y_m) - \sum_{i=m}^{\infty}(X_t(y_{i+1}) - X_t(y_i)),$$

where the series are actually finite sums. It follows that

$$|X_t(x) - X_t(y)| \le K_m + 2\sum_{i=m+1}^{\infty} K_i(t) \le 2\sum_{i=m}^{\infty} K_i(t).$$

As a result, setting $M_\alpha(t) = \sup\{|X_t(x) - X_t(y)|/|x-y|^\alpha,\ x, y \in D,\ x \neq y\}$, we have

$$M_\alpha(t) \quad\le\quad \sup_{m\in N}\Big\{2^{m\alpha} \sup_{|x-y|\le 2^{-m}} |X_t(x) - X_t(y)|,\ x, y \in D,\ x \neq y\Big\}$$

$$\begin{aligned}&\leq \sup_{m\in N}\left\{2^{m\alpha+1}\sum_{i=m}^{\infty}K_i(t)\right\}\\&\leq 2\sum_{i=0}^{\infty}2^{i\alpha}K_i(t).\end{aligned}$$

For $\gamma \geq 1$ and $\alpha < \varepsilon/\gamma$, we get with $J' = 2J$,

$$[\mathbb{E}\sup_{0\leq t\leq 1}M_\alpha(t)^\gamma]^{1/\gamma} \leq 2\sum_{i=0}^{\infty}2^{i\alpha}[\mathbb{E}\sup_{0\leq t\leq 1}K_i(t)^\gamma]^{1/\gamma} \leq J'\sum_{i=0}^{\infty}2^{i(\alpha-\varepsilon/\gamma)} < \infty.$$

For $\gamma < 1$, the same reasoning applies to $[\mathbb{E}\sup_{0\leq t\leq 1}M_\alpha(t)^\gamma]$ instead of $[\mathbb{E}\sup_{0\leq t\leq 1}M_\alpha(t)^\gamma]^{1/\gamma}$.

It follows in particular that for almost every $\omega$, $X_t(\cdot)$ is uniformly continuous on $D$ and it is uniformly in $t$, so it make sense to set

$$\tilde{X}_t(x,\omega) = \lim_{y\in D, y\to x}X_t(y,\omega).$$

By Fatou's lemma and the hypothesis, $\tilde{X}_t(x) = X_t(x)$ a.s. and $\tilde{X}$ is clearly the desired modification. □

It is easy to see that one can use Theorem 5.2 to consider the equation (3.1) and (5.1)

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