# Wasserstein Convergence for Empirical Measures of Subordinated Diffusions on Riemannian Manifolds *

**Feng-Yu Wang**[a),b)] **Bingyao Wu**[a)]

[a)] Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

[b)] Department of Mathematics, Swansea University, Bay Campus, Swansea, SA1 8EN, United Kingdom

wangfy@tju.edu.cn, F.-Y.Wang@swansea.ac.uk; bingyaowu@163.com

February 17, 2022

**Abstract**

Let $M$ be a connected compact Riemannian manifold possibly with a boundary $\partial M$, let $V \in C^2(M)$ such that $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure, where $\mathrm{d}x$ is the volume measure, and let $L = \Delta + \nabla V$. As a continuation to [14] where convergence in the quadratic Wasserstein distance $\mathbb{W}_2$ is studied for the empirical measures of the $L$-diffusion process (with reflecting boundary if $\partial M \neq \emptyset$), this paper presents the exact convergence rate for the subordinated process. In particular, letting $(\mu_t^\alpha)_{t>0}$ $(\alpha \in (0,1))$ be the empirical measures of the Markov process generated by $L^\alpha := -(-L)^\alpha$, when $\partial M$ is empty or convex we have

$$\lim_{t\to\infty} \big\{t\mathbb{E}^x[\mathbb{W}_2(\mu_t^\alpha, \mu)^2]\big\} = \sum_{i=1}^\infty \frac{2}{\lambda_i^{1+\alpha}} \text{ uniformly in } x \in M,$$

where $\mathbb{E}^x$ is the expectation for the process starting at point $x$, $\{\lambda_i\}_{i\ge1}$ are non-trivial (Neumann) eigenvalues of $-L$. In general,

$$\mathbb{E}^x[\mathbb{W}_2(\mu_t^\alpha, \mu)^2] \begin{cases} \asymp t^{-1}, & \text{if } d < 2(1+\alpha),\\ \asymp t^{-\frac{2}{d-2\alpha}}, & \text{if } d > 2(1+\alpha),\\ \preceq t^{-1}\log(1+t), & \text{if } d = 2(1+\alpha), \text{i.e. } \alpha = \frac{1}{2}, d = 3 \end{cases}$$

holds uniformly in $x \in M$, where in the last case $\mathbb{E}^x[\mathbb{W}_1(\mu_t^\alpha, \mu)^2] \succeq t^{-1}\log(1+t)$ holds for $M = \mathbb{T}^3$ and $V = 0$.

AMS subject Classification: 60D05, 58J65.
Keywords: Empirical measure, subordinated diffusion process, Riemannian manifold, Wasserstein distance, eigenvalues.

*Supported in part by NNSFC (11831014, 11921001) and the National Key R&D Program of China (No. 2020YFA0712900).

## 1 Introduction

Recently, sharp convergence rate in the Wasserstein distance has been derived in [14] for empirical measures of symmetric diffusion processes on compact Riemannian manifolds, see [10, 11, 12, 13] for further study of Dirichlet diffusion processes and SDEs/SPDEs, and see [1, 2, 5] and references within for earlier results on i.i.d. random variables and discrete time Markov chains. In this paper, we aim to extend the main results of [14] to jump processes, for which a natural model is the subordination of diffusion processes.

Let $M$ be a $d$-dimensional connected compact Riemannian manifold possibly with a smooth boundary $\partial M$. Let $V \in C^2(M)$ such that $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure on $M$, where $\mathrm{d}x$ is the Riemannian volume measure on $M$. Then the (reflecting, if $\partial M \neq \emptyset$) diffusion process $X_t$ generated by $L := \Delta + \nabla V$ on $M$ is reversible; i.e. the associated diffusion semigroup $\{P_t\}_{t\geq 0}$ is symmetric in $L^2(\mu)$, where

$$P_t f(x) := \mathbb{E}^x f(X_t), \quad t \geq 0, f \in \mathscr{B}_b(M).$$

Here, $\mathbb{E}^x$ is the expectation taken for the diffusion process $\{X_t\}_{t\geq 0}$ with $X_0 = x$, and we will use $\mathbb{P}^x$ to denote the associated probability measure. In general, for $\nu \in \mathscr{P}$ (the set of all probability measures on $M$), let $\mathbb{E}^\nu$ and $\mathbb{P}^\nu$ be the expectation and probability taken for the diffusion process with initial distribution $\nu$. For any $\nu \in \mathscr{P}$ and $t \geq 0$, $\nu P_t := \mathbb{P}^\nu(X_t \in \cdot)$ is the distribution of $X_t$ with initial distribution $\nu$.

A function $B \in C^\infty((0,\infty);[0,\infty)) \cap C([0,\infty);[0,\infty))$ is called a Bernstein function if

$$(-1)^{n-1}\frac{\mathrm{d}^n}{\mathrm{d}r^n}B(r) \geq 0, \quad n \in \mathbb{N}, r > 0.$$

We will use the following classes of Bernstein functions:

$$\mathbf{B} := \big\{B : B \text{ is a Bernstein function with } B(0) = 0, B'(0) > 0\big\},$$
$$\mathbb{B} := \left\{B \in \mathbf{B} : \int_1^\infty r^{\frac{d}{2}-1}\mathrm{e}^{-tB(r)}\mathrm{d}r < \infty \text{ for } t > 0\right\}.$$

For each $B \in \mathbf{B}$, there exists a unique stable process $S_t^B$ on $[0,\infty)$ with Laplace transform

$$\mathbb{E}\mathrm{e}^{-\lambda S_t^B} = \mathrm{e}^{-tB(\lambda)}, \quad t, \lambda \geq 0. \tag{1.1}$$

Moreover, for any $\alpha \in [0,1]$, let

$$\mathbb{B}^\alpha := \left\{B \in \mathbb{B} : \liminf_{\lambda\to\infty} \lambda^{-\alpha}B(\lambda) > 0\right\}, \quad \mathbb{B}_\alpha := \left\{B \in \mathbb{B} : \limsup_{\lambda\to\infty} \lambda^{-\alpha}B(\lambda) < \infty\right\}.$$

For any $B \in \mathbf{B}$, let $X_t^B$ be the Markov process on $M$ generated by $B(L) := -B(-L)$, which can be constructed as the time change (subordination) of $X_t$:

$$X_t^B = X_{S_t^B}, \quad t \geq 0,$$

where $(S_t^B)_{t\geq 0}$ is the stable process satisfying (1.1) independent of $(X_t)_{t\geq 0}$. We consider the empirical measure

$$\mu_t^B := \frac{1}{t}\int_0^t \delta_{X_s^B}\mathrm{d}s, \quad t > 0.$$

Let $\rho$ be the Riemiannian distance (i.e. the length of shortest curve linking two points) on $M$. For any $p>0$, the $L^p$-Wasserstein distance $\mathbb{W}_p$ is defined by

$$\mathbb{W}_p(\mu_1,\mu_2) := \inf_{\pi\in\mathscr{C}(\mu_1,\mu_2)} \bigg(\int_{M\times M} \rho(x,y)^p \pi(\mathrm{d}x,\mathrm{d}y)\bigg)^{\frac{1}{p\vee 1}}, \quad \mu_1,\mu_2\in\mathscr{P},$$

where $\mathscr{C}(\mu_1,\mu_2)$ is the set of all probability measures on $M\times M$ with marginal distributions $\mu_1$ and $\mu_2$. A measure $\pi\in\mathscr{C}(\mu_1,\mu_2)$ is called a coupling of $\mu_1$ and $\mu_2$.

Since $M$ is connected and compact, $L$ has discrete spectrum and all eigenvalues $\{\lambda_i\}_{i\ge 0}$ of $-L$ listed in the increasing order counting multiplicities satisfy (see for instance [3])

$$\kappa^{-1} i^{\frac{2}{d}} \le \lambda_i \le \kappa i^{\frac{2}{d}}, \quad i\ge 0 \tag{1.2}$$

for some constant $\kappa>1$. Our main results are stated as follows, which cover the corresponding assertions derived in [14] for $B(\lambda)=\lambda$.

**Theorem 1.1** (Lower bound estimates)**.** *Let $B\in\mathbb{B}$.*

(1) *There exists a constant $c\in(0,1]$ with $c=1$ when $\partial M$ is empty or convex, such that*

$$\liminf_{t\to\infty}\inf_{x\in M}\Big\{t\mathbb{E}^x[\mathbb{W}_2(\mu_t^B,\mu)^2]\Big\} \ge c\sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i)}.$$

(2) *Let $B\in\mathbb{B}_\alpha$ for some $\alpha\in[0,1]$. If $d>2(1+\alpha)$, then for any $p>0$,*

$$\liminf_{t\to\infty}\inf_{x\in M}\Big\{t^{\frac{2}{d-2\alpha}}\big(\mathbb{E}^x[\mathbb{W}_p(\mu_t^B,\mu)]\big)^{\frac{2}{p\wedge 1}}\Big\} > 0.$$

(3) *Let $B(\lambda)=\lambda^\alpha$ for some $\alpha\in[0,1]$. If $d=2(1+\alpha)$ (i.e. $\alpha=1$ and $d=4$, or $\alpha=\frac{1}{2}$ and $d=3$), $M=\mathbb{T}^d$ and $V=0$, then*

$$\liminf_{t\to\infty}\inf_{x\in M}\Big\{\frac{t}{\log t}\big(\mathbb{E}^x[\mathbb{W}_1(\mu_t^B,\mu)]\big)^2\Big\} > 0.$$

**Theorem 1.2** (Upper bound estimates)**.** *Let $B\in\mathbb{B}^\alpha$ for some $\alpha\in[0,1]$.*

(1) *If $d<2(1+\alpha)$, then*

$$\limsup_{t\to\infty}\sup_{x\in M}\Big\{t\mathbb{E}^x[\mathbb{W}_2(\mu_t^B,\mu)^2]\Big\} \le \sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i)} < \infty.$$

(2) *If $d>2(1+\alpha)$, then*

$$\limsup_{t\to\infty}\sup_{x\in M}\Big\{t^{\frac{2}{d-2\alpha}}\mathbb{E}^x[\mathbb{W}_2(\mu_t^B,\mu)^2]\Big\} < \infty.$$

(3) *If* $d = 2(1+\alpha)$, *i.e. either* $\alpha = 1$ *and* $d = 4$, *or* $\alpha = \frac{1}{2}$ *and* $d = 3$, *then*

$$\limsup_{t\to\infty} \sup_{x\in M} \Big\{ \frac{t}{\log t} \mathbb{E}^x[\mathbb{W}_2(\mu_t^B, \mu)^2] \Big\} < \infty.$$

The following is a straightforward consequence of Theorems 1.1 and 1.2.

**Corollary 1.3.** *Let* $B \in \mathbb{B}^\alpha \cap \mathbb{B}_\alpha$ *for some* $\alpha \in [0,1]$.

(1) *If* $\partial M$ *is empty or convex, then*

$$\lim_{t\to\infty} \Big\{ t\mathbb{E}^x[\mathbb{W}_2(\mu_t^B, \mu)^2] \Big\} = \sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i)} \tag{1.3}$$

*uniformly in* $x \in M$, *where the limit is finite if and only if* $d < 2(1+\alpha)$. *In general, there exists a constant* $c \in (0,1]$ *such that*

$$\begin{aligned} &c\sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i)} \le \liminf_{t\to\infty} \inf_{x\in M} \Big\{ t\mathbb{E}^x[\mathbb{W}_2(\mu_t^B, \mu)^2] \Big\} \\ &\le \limsup_{t\to\infty} \sup_{x\in M} \Big\{ t\mathbb{E}^x[\mathbb{W}_2(\mu_t^B, \mu)^2] \Big\} \le \sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i)}. \end{aligned} \tag{1.4}$$

(2) *If* $d > 2(1+\alpha)$, *then for any* $\varepsilon \in (0, \alpha)$ *there exist constants* $c > c(\varepsilon) > 0$ *such that*

$$\begin{aligned} &c(\varepsilon) t^{-\frac{2}{d-2\alpha}} \le \inf_{x\in M} \big( \mathbb{E}^x[\mathbb{W}_\varepsilon(\mu_t^B, \mu)] \big)^{\frac{2}{\varepsilon}} \\ &\le \inf_{x\in M} \mathbb{E}^x[\mathbb{W}_2(\mu_t^B, \mu)^2] \le \sup_{x\in M} \mathbb{E}^x[\mathbb{W}_2(\mu_t^B, \mu)^2] \le c t^{-\frac{2}{d-2\alpha}}, \quad t \ge 1. \end{aligned}$$

(3) *Let* $d = 2(1+\alpha)$, *i.e. either* $d = 3$ *and* $\alpha = \frac{1}{2}$, *or* $\alpha = 1$ *and* $d = 4$. *Then there exists a constant* $c > 0$ *such that*

$$\sup_{x\in M} \mathbb{E}^x[\mathbb{W}_2(\mu_t^B, \mu)^2] \le c t^{-1} \log t, \quad t \ge 2.$$

*On the other hand, when* $B(\lambda) = \lambda^\alpha$, $M = \mathbb{T}^d$ *and* $V = 0$, *then there exists a constant* $c' > 0$ *such that*

$$\inf_{x\in M} \mathbb{E}^x[\mathbb{W}_1(\mu_t^B, \mu)^2] \ge c' t^{-1} \log t, \quad t \ge 2.$$

Finally, we have the following result on the weak convergence of $t\mathbb{W}_2(\mu_t^B, \mu)^2$.

**Theorem 1.4.** *Let* $B \in \mathbb{B}^\alpha$ *for some* $\alpha \in [0,1]$, *and let* $\partial M$ *be empty or convex. If* $d < 2(1+\alpha)$, *then*

$$\lim_{t\to\infty} \sup_{x\in M} |\mathbb{P}^x(t\mathbb{W}_2(\mu_t^B, \mu)^2 < a) - F(a)| = 0, \quad a \ge 0,$$

*where* $F(a) := \mathbb{P}(\xi < a)$ *for*

$$\xi := \sum_{i=1}^\infty \frac{2\xi_i^2}{\lambda_i B(\lambda_i)}$$

*and i.i.d. random variables* $\{\xi_i\}$ *with the standard normal distribution* $N(0,1)$.

Following the line of [14], we will first study the modified empirical measure $\mu_{t,r}^B := \mu_t^B P_r$ for $r > 0$ in Section 2, present some lemmas in Section 3, and finally prove Theorems 1.1, 1.2 and 1.4 in Sections 4, 5 and 6 respectively.

# 2 Modified empirical measures

In this part, we allow $M$ to be non-compact, but assume that the (Neumann) semigroup $P_t$ generated by $L$ is ultracontractive, i.e.

$$\|P_t\|_{1\to\infty} := \sup_{\mu(|f|)\le 1} \|P_t f\|_\infty < \infty, \quad t > 0. \tag{2.1}$$

Consequently, $-L$ has discrete spectrum and the heat kernel $p_t(x,y)$ of $P_t$ with respect to $\mu$ satisfies

$$p_t(x,y) = 1 + \sum_{i=1}^\infty \mathrm{e}^{-t\lambda_i}\phi_i(x)\phi_i(y) \le \|P_t\|_{1\to\infty} < \infty, \quad t > 0, x, y \in M, \tag{2.2}$$

where $\{\lambda_i\}_{i\ge 0}$ are all eigenvalues of $-L$ and $\{\phi_i\}_{i\ge 0}$ is the eigenbasis, i.e. $\phi_0 \equiv 1$ and $\{\phi_i\}_{i\ge 0}$ is an orthonormal basis of $L^2(\mu)$ with $L\phi_i = -\lambda_i\phi_i$.

For any $p \ge 1$ and $f \in L^p(\mu)$, let $\|f\|_p := \{\mu(|f|^p)\}^{\frac{1}{p}}$ be the $L^p(\mu)$-norm of $f$. Then there exists a constant $c > 0$ such that

$$\|P_t f\|_p \le c\mathrm{e}^{-\lambda_1 t}\|f\|_p, \quad t \ge 0, p \in [1,\infty], f \in L_0^p(\mu),$$

where $L_0^p(\mu) := \{f \in L^p(\mu) : \mu(f) := \int_M f \mathrm{d}\mu = 0\}$. Consequently, for any $B \in \mathbb{B}$,

$$\|P_t^B f\|_p = \|\mathbb{E}P_{S_t^B} f\|_p \le c\|f\|_p \mathbb{E}\mathrm{e}^{-\lambda_1 S_t^B} = c\mathrm{e}^{-B(\lambda_1)t}\|f\|_p, \quad t \ge 0, p \in [1,\infty], f \in L_0^p(\mu). \tag{2.3}$$

As in [14], we consider the modified empirical measure

$$\mu_{t,r}^B := \mu_t^B P_r, \quad r, t > 0.$$

By (2.2), we have

$$f_{t,r} := \frac{\mathrm{d}\mu_{t,r}^B}{\mathrm{d}\mu} = 1 + \frac{1}{\sqrt{t}}\sum_{i=1}^\infty \mathrm{e}^{-r\lambda_i}\psi_i(t)\phi_i, \quad \psi_i(t) := \frac{1}{\sqrt{t}}\int_0^t \phi_i(X_s^B)\mathrm{d}s, \quad r, t > 0. \tag{2.4}$$

The main result in this section is the following.

**Theorem 2.1.** *Let $B \in \mathbf{B}$, $M$ be a $d$-dimensional connected complete Riemannian manifold possibly with a boundary such that (2.1) holds.*

(1) *For any $r > 0$,*

$$\lim_{t\to\infty}\sup_{x\in M}\left| t\mathbb{E}^x[\mathbb{W}_2(\mu_{t,r}^B, \mu)^2] - \sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i)\mathrm{e}^{2r\lambda_i}}\right| = 0.$$

(2) *For any $C>0$, let*

$$\mathscr{P}(C) := \{\nu \in \mathscr{P}:\ \nu = h_\nu \mu, \|h_\nu\|_\infty \le C\}.$$

*Then for any $C>1$,*

$$\lim_{t\to\infty} \sup_{\nu\in\mathscr{P}(C)} \left|\mathbb{P}^\nu(t\mathbb{W}_2(\mu_{t,r}^B,\mu)^2 < a) - F_r(a)\right| = 0, \quad a \in \mathbb{R},$$

*where for i.i.d. random variables $\xi_i$ with distribution $N(0,1)$, $F_r := \mathbb{P}(\xi_r < \cdot)$ is the distribution function of*

$$\xi_r := \sum_{i=1}^\infty \frac{2\xi_i^2}{\lambda_i B(\lambda_i) e^{2\lambda_i r}}, \quad r>0.$$

To prove this result, we first present some lemmas, where the first follows from [14, Lemma 2.3], which goes back to [1, Proposition 2.3].

**Lemma 2.2.** *Let $B \in \mathbf{B}$, $\mathscr{M}(a,b) := \frac{a-b}{\log a - \log b} 1_{\{a\wedge b>0\}}$. Then*

$$\mathbb{W}_2(\mu_{t,r}^B,\mu)^2 \le \int_M \frac{|\nabla L^{-1}(f_{t,r}-1)|^2}{\mathscr{M}(f_{t,r},1)} d\mu, \quad t,r>0.$$

By the ergodicity we have $\lim_{t\to\infty} \mathscr{M}(f_{t,r},1) = 1$ (see Lemma 2.4 below), so that this lemma implies that $t\mathbb{W}_2(\mu_{t,r}^B,\mu)^2$ is asymptotically bounded above by

$$\Xi_r(t) := t\mu\big(|\nabla L^{-1}(f_{t,r}-1)|^2\big), \qquad t,r>0, \tag{2.5}$$

where $\mu(f) := \int_M f d\mu$ for $f \in L^1(\mu)$. Thus, we first estimate $\Xi_r(t)$.

**Lemma 2.3.** *Let $B \in \mathbf{B}$. There exists a constant $c>0$ such that*

$$\left|\mathbb{E}^\nu \Xi_r(t) - \sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i) e^{2r\lambda_i}}\right| \le \frac{c\|h_\nu\|_\infty}{t} \sum_{i=1}^\infty \frac{1}{\lambda_i B(\lambda_i) e^{2r\lambda_i}}, \quad t\ge 1, r>0, \tag{2.6}$$

*holds for any probability measure $\nu = h_\nu \mu$. Consequently,*

$$\sup_{x\in M}\left|\mathbb{E}^x \Xi_r(t) - \sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i) e^{2r\lambda_i}}\right| \le \frac{c\|P_{\frac{r}{2}}\|_{2\to\infty}^2}{t} \sum_{i=1}^\infty \frac{1}{\lambda_i B(\lambda_i) e^{r\lambda_i}}, \quad t\ge 1, r>0. \tag{2.7}$$

*Proof.* By (2.2), (2.4), (2.5), $L\phi_i = -\lambda_i \phi_i$ and $\mu(\phi_i\phi_j) = 1_{\{i=j\}}$ for $i,j\ge 0$, we obtain

$$\Xi_r(t) = \sum_{i=1}^\infty \frac{|\psi_i(t)|^2}{\lambda_i e^{2r\lambda_i}}, \quad t,r>0. \tag{2.8}$$

Since $P_t^B$ is the Markov semigroup of $X_t^B$, the Markov property implies

$$\mathbb{E}^\nu(\phi_i(X_t^B)|X_s^B) = P_{t-s}^B \phi_i(X_s^B) = e^{-B(\lambda_i)(t-s)} \phi_i(X_s^B), \quad i\ge 0, t\ge s\ge 0.$$

So, $\psi_i(t) := \frac{1}{\sqrt{t}}\int_0^t \phi_i(X_s^B)\mathrm{d}s$ satisfies

$$\begin{aligned}\mathbb{E}^\nu|\psi_i(t)|^2 &= \frac{1}{t}\mathbb{E}^\nu\left|\int_0^t \phi_i(X_s^B)\,\mathrm{d}s\right|^2 = \frac{2}{t}\int_0^t \mathrm{d}s_1\int_{s_1}^t \mathbb{E}^\nu[\phi_i(X_{s_1}^B)\phi_i(X_{s_2}^B)]\,\mathrm{d}s_2\\ &= \frac{2}{t}\int_0^t \mathbb{E}^\nu|\phi_i(X_{s_1}^B)|^2\,\mathrm{d}s_1\int_{s_1}^t \mathrm{e}^{-B(\lambda_i)(s_2-s_1)}\,\mathrm{d}s_2\\ &= \frac{2}{B(\lambda_i)t}\int_0^t \nu(P_s^B\phi_i^2)(1-\mathrm{e}^{-B(\lambda_i)(t-s)})\,\mathrm{d}s,\quad t>0.\end{aligned}\tag{2.9}$$

This together with (2.8) imply

$$\mathbb{E}^\nu\Xi_r(t) = \frac{2}{t}\sum_{i=1}^\infty \frac{1}{\lambda_i B(\lambda_i)\mathrm{e}^{2r\lambda_i}}\int_0^t \nu(P_s^B\phi_i^2)(1-\mathrm{e}^{-B(\lambda_i)(t-s)})\,\mathrm{d}s =: I_1+I_2,\tag{2.10}$$

where

$$I_1 := \frac{2}{t}\sum_{i=1}^\infty\int_0^t \frac{1-\mathrm{e}^{-(t-s)B(\lambda_i)}}{\lambda_i B(\lambda_i)\mathrm{e}^{2r\lambda_i}}\,\mathrm{d}s = \sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i)\mathrm{e}^{2r\lambda_i}} - \frac{2}{t}\sum_{i=1}^\infty \frac{1-\mathrm{e}^{-B(\lambda_i)t}}{\lambda_i B(\lambda_i)^2\mathrm{e}^{2r\lambda_i}},\tag{2.11}$$

and due to $\nu(P_s^B\phi_i^2) = \mu(h_\nu P_s^B\phi_i^2) = \mu(\phi_i^2 P_s^B h_\nu)$,

$$I_2 := \mathbb{E}^\nu\Xi_r(t) - I_1 = \frac{2}{t}\sum_{i=1}^\infty\int_0^t \frac{1-\mathrm{e}^{-(t-s)B(\lambda_i)}}{\lambda_i B(\lambda_i)\mathrm{e}^{2r\lambda_i}}\mu(\phi_i^2 P_s^B h_\nu - 1)\,\mathrm{d}s.$$

Since $\mu(\phi_i^2)=1$, by (2.3), there exists a constant $c_0>0$ such that

$$|\mu(\phi_i^2 P_s^B h_\nu - 1)| = |\mu((P_s^B h_\nu - 1)\phi_i^2)| \le \|P_s^B(h_\nu - 1)\|_\infty \le c_0\mathrm{e}^{-B(\lambda_1)s}\|h_\nu\|_\infty,\quad s\ge 0.$$

Therefore, we find a constant $c_1>0$ such that

$$|I_2| \le \frac{c_1}{t}\|h_\nu\|_\infty\sum_{i=1}^\infty \frac{1}{\lambda_i B(\lambda_i)\mathrm{e}^{2r\lambda_i}} < \infty.\tag{2.12}$$

Combining (2.10), (2.11) and (2.12), we find a constant $c_2>0$ such that

$$\left|\mathbb{E}^\nu\Xi_r(t) - \sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i)\mathrm{e}^{2r\lambda_i}}\right| \le \frac{c_2\|h_\nu\|_\infty}{t}\sum_{i=1}^\infty \frac{1}{\lambda_i B(\lambda_i)\mathrm{e}^{2r\lambda_i}}.$$

When $\nu=\delta_x$, (2.10) becomes

$$\mathbb{E}^x\Xi_r(t) \le I_1 + I_2(x),\tag{2.13}$$

where $I_1$ is in (2.11) and

$$I_2(x) := \frac{2}{t}\sum_{i=1}^\infty\int_0^t \frac{1-\mathrm{e}^{-B(\lambda_i)(t-s)}}{\lambda_i B(\lambda_i)\mathrm{e}^{2r\lambda_i}}P_s^B\{\phi_i^2-1\}(x)\,\mathrm{d}s.$$

Since $\mu(\phi_i^2)=1$, (2.3) implies $\|P_s^B\phi_i^2-1\|_\infty\le c\mathrm{e}^{-B(\lambda_1)s}\|\phi_i\|_\infty^2$. Combining this with

$$\|\phi_i\|_\infty^2=\mathrm{e}^{r\lambda_i}\|P_{\frac{r}{2}}\phi_i\|_\infty^2\le\mathrm{e}^{\lambda_i r}\|P_{\frac{r}{2}}\|_{2\to\infty}^2,$$

we find a constant $c_3>0$ such that

$$I_2(x)\le\frac{2}{t}\sum_{i=1}^\infty\int_0^t\frac{c}{\lambda_iB(\lambda_i)\mathrm{e}^{r\lambda_i}}\mathrm{e}^{-B(\lambda_1)s}\|P_{\frac{r}{2}}\|_{2\to\infty}^2\mathrm{d}s\le\frac{c_3\|P_{\frac{r}{2}}\|_{2\to\infty}^2}{t}\sum_{i=1}^\infty\frac{1}{\lambda_iB(\lambda_i)\mathrm{e}^{r\lambda_i}}.$$

This together with (2.11) and (2.13) implies (2.7). □

The following Lemma shows that $\lim_{t\to\infty}\mathscr{M}(f_{t,r},1)=1, r>0$.

**Lemma 2.4.** *Let $\|f_{t,r}-1\|_\infty=\sup_{y\in M}|f_{t,r}(y)-1|$. Then there exists a function $c:\mathbb{N}\times(0,\infty)\to(0,\infty)$ such that*

$$\sup_{x\in M}\mathbb{E}^x[\|f_{t,r}-1\|_\infty^{2k}]\le c(k,r)t^{-k},\quad t\ge1,r>0,k\in\mathbb{N}.$$

*Proof.* For fixed $r>0$ and $y\in M$, let $f=p_r(\cdot,y)-1$. For any $k\in\mathbb{N}$, we consider

$$I_k(s):=\mathbb{E}^x\left|\int_0^s f(X_t^B)\,dt\right|^{2k}=(2k)!\mathbb{E}^x\int_{\Delta_k(s)}f(X_{s_1}^B)\cdots f(X_{s_{2k}}^B)\,\mathrm{d}s_1\cdots\mathrm{d}s_{2k},$$

where $\Delta_k(s):=\{(s_1,\cdots,s_{2k})\in[0,s]:0\le s_1\le\cdots\le s_{2k}\le s\}$.

By the proof of [14, Lemma 2.5] with $X_t^B$ replacing $X_t$, we obtain

$$I_k(t)\le\sup_{s\in[0,t]}I_k(s)\le\{2k(2k-1)\}^k\left(\int_{\Delta_1(t)}(\mathbb{E}^x|g(r_1,r_2)|^k)^{\frac{1}{k}}\,\mathrm{d}r_1\,\mathrm{d}r_2\right)^k,\tag{2.14}$$

where $g(r_1,r_2)=(fP_{r_2-r_1}^Bf)(X_{r_1}^B), r_2\ge r_1\ge0$.

By (2.1) we have

$$\|f\|_\infty=\|p_r(\cdot,y)-1\|_\infty\le2\|P_r\|_{1\to\infty}<\infty,$$

which together with (2.3) implies

$$|g(r_1,r_2)|^k\le\|fP_{r_2-r_1}^Bf\|_\infty^k\le c\mathrm{e}^{-B(\lambda_1)(r_2-r_1)k}\|f\|_\infty^{2k}\le c_1\|P_r\|_{1\to\infty}^{2k}\mathrm{e}^{-B(\lambda_1)(r_2-r_1)k}$$

for some constant $c_1>0$. Thus, there exists a constant $c_2>0$ such that

$$\int_{\Delta_1(t)}(\mathbb{E}^x|g(r_1,r_2)|^k)^{\frac{1}{k}}\,\mathrm{d}r_1\,\mathrm{d}r_2\le\int_0^t\mathrm{d}r_1\int_{r_1}^t c_1\|P_r\|_{1\to\infty}^2\mathrm{e}^{-B(\lambda_1)(r_2-r_1)}\,\mathrm{d}r_2\le c_2\|P_r\|_{1\to\infty}^2t.$$

Combining this with (2.14), we find a constant $c_3>0$ such that

$$\sup_{x,y\in M}\mathbb{E}^x[|f_{t,r}(y)-1|^{2k}]=t^{-2k}I_k(t)\le c_3\|P_r\|_{1\to\infty}^{2k}t^{-k},\quad t\ge1,r>0.$$

Noting that $f_{t,r}-1=P_{\frac{r}{2}}(f_{t,\frac{r}{2}}-1)$, this implies that for some constant $c>0$

$$\begin{aligned}&\sup_{x\in M}\mathbb{E}^x[\|f_{t,r}-1\|_\infty^{2k}]=\sup_{x\in M}\mathbb{E}^x[\|P_{\frac{r}{2}}(f_{t,\frac{r}{2}}-1)\|_\infty^{2k}]\\&\le\|P_{\frac{r}{2}}\|_{2k\to\infty}^{2k}\sup_{x\in M}\mathbb{E}^x[\mu(|f_{t,\frac{r}{2}}-1|^{2k})]\le c\|P_{\frac{r}{2}}\|_{1\to\infty}^{4k}t^{-k}.\end{aligned}$$

□

*Proof of Theorem 2.1.* (1) It suffices to verify the following estimates for any $r > 0$:

$$\liminf_{t\to\infty}\inf_{x\in M}\{t\mathbb{E}^x[\mathbb{W}_2(\mu_{t,r}^B,\mu)^2]\} \geq \sum_{i=1}^{\infty}\frac{2}{\lambda_i B(\lambda_i)\mathrm{e}^{2r\lambda_i}}, \tag{2.15}$$

$$\limsup_{t\to\infty}\sup_{x\in M}\{t\mathbb{E}^x[\mathbb{W}_2(\mu_{t,r}^B,\mu)^2]\} \leq \sum_{i=1}^{\infty}\frac{2}{\lambda_i B(\lambda_i)\mathrm{e}^{2r\lambda_i}}. \tag{2.16}$$

Let $B_\sigma := \{\|f_{t,r}-1\|_\infty \leq \sigma^{\frac{2}{3}}\}$ for $\sigma > 0$. By the proofs of [14, (2.53) and (2.54)] for $X_t^B$ replacing $X_t$, there exists a constant $c > 0$ such that

$$t\mathbb{W}_2(\mu_{t,r}^B,\mu)^2 \geq 1_{B_\sigma}\big\{\Xi_r(t) - ct\sigma^{\frac{5}{3}}\big\},\quad r,t,\sigma > 0. \tag{2.17}$$

Taking $\sigma = t^{-\gamma}$ for some $\gamma \in (\frac{3}{5},\frac{3}{4})$, we have $t\sigma^{\frac{5}{3}} \downarrow 0$ as $t \uparrow \infty$, and according to Lemma 2.4,

$$\lim_{t\to\infty}\sup_{x\in M}\mathbb{P}^x(B_\sigma^c) \leq \lim_{t\to\infty}\sup_{x\in M} t^{\frac{4\gamma}{3}}\mathbb{E}^x[\|f_{t,r}-1\|_\infty^2] = 0,$$

so that by (2.7)

$$\limsup_{t\to\infty}\sup_{x\in M}\mathbb{E}^x[1_{B_\sigma^c}\Xi_r(t)] \leq c(r)\limsup_{t\to\infty}\sup_{x\in M}\mathbb{P}^x(B_\sigma^c) = 0,$$

where $c(r) := \sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i)\mathrm{e}^{2\lambda_i r}} < \infty$. Thus, (2.17) yields

$$\liminf_{t\to\infty}\inf_{x\in M}\mathbb{E}^x\big[t\mathbb{W}_2(\mu_{t,r}^B,\mu)^2\big] \geq \liminf_{t\to\infty}\inf_{x\in M}\mathbb{E}^x\big[\Xi_r(t)\big],$$

which together with (2.7) implies (2.15).

Since $\mu(\phi_i^2) = 1$ and $\lambda_1 > 0$, by taking $x = y$ in (2.2) and integrating with respect to $\mu(\mathrm{d}x)$, we obtain

$$\sum_{i=1}^\infty \frac{1}{\lambda_i B(\lambda_i)\mathrm{e}^{2r\lambda_i}} \leq \frac{1}{\lambda_1 B(\lambda_1)}\sum_{i=1}^\infty \mathrm{e}^{-2r\lambda_i} < \infty.$$

For any $\eta \in (0,1)$, let

$$A_\eta = \{\|f_{t,r}-1\|_\infty \leq \eta\}.$$

Noting that $f_{t,r}(y) \geq 1-\eta$ implies

$$\mathscr{M}(1, f_{t,r}(y)) \geq \sqrt{f_{t,r}(y)} \geq \sqrt{1-\eta},$$

by Lemma 2.2 and (2.7), we find a constant $c(r) > 0$ such that

$$\begin{aligned} t\sup_{x\in M}\mathbb{E}^x[1_{A_\eta}\mathbb{W}_2(\mu_{t,r}^B,\mu)^2] &\leq \sup_{x\in M}\mathbb{E}^x\left\{\frac{\Xi_r(t)}{\sqrt{1-\eta}}\right\} \\ &\leq \frac{1}{\sqrt{1-\eta}}\sum_{i=1}^\infty\frac{2}{\lambda_i B(\lambda_i)\mathrm{e}^{2r\lambda_i}}\left(1+\frac{c(r)}{t}\right),\quad t>0,\eta\in(0,1). \end{aligned}$$

Thus,

$$
\begin{aligned}
&t \sup_{x\in M} \mathbb{E}^x[\mathbb{W}_2(\mu_{t,r}^B,\mu)^2] \\
&\le \frac{1}{\sqrt{1-\eta}} \sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i) \mathrm{e}^{2r\lambda_i}} \left(1+\frac{c(r)}{t}\right) + t\sup_{x\in M} \mathbb{E}^x[1_{A_\eta^c} \mathbb{W}_2(\mu_{t,r}^B,\mu)^2] \\
&\le \frac{1+c(r)t^{-1}}{\sqrt{1-\eta}} \sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i)\mathrm{e}^{2r\lambda_i}} + t\sup_{x\in M} \sqrt{\mathbb{P}^x(A_\eta^c)\mathbb{E}^x[\mathbb{W}_2(\mu_{t,r}^B,\mu)^4]}.
\end{aligned}
\tag{2.18}
$$

As shown in the proof of [14, Proposition 2.6], we have

$$
\mathbb{E}^x \mathbb{W}_2(\mu_{t,r}^B,\mu)^4 \le \|P_r\|_{1\to\infty} (\mu\times\mu)(\rho^4) < \infty. \tag{2.19}
$$

Moreover, Lemma 2.4 implies that for some constant $c(k,r)>0$

$$
\sup_{x\in M} \mathbb{P}^x(A_\eta^c) \le \eta^{-2k} c(k,r) t^{-k}.
$$

By taking $k=4$ and applying (2.18) and (2.19), we conclude that

$$
\limsup_{t\to\infty} \left\{ t\sup_{x\in M} \mathbb{E}^x[\mathbb{W}_2(\mu_{t,r}^B,\mu)^2] \right\} \le \frac{1}{\sqrt{1-\eta}} \sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i)\mathrm{e}^{2r\lambda_i}}.
$$

Then (2.16) follows by letting $\eta\to 0$.

(2) By Lemma 2.3, it suffices to prove that for any $C>1$

$$
\lim_{t\to\infty} \sup_{\nu\in\mathscr{P}(C)} |\mathbb{P}^\nu(\Xi_r(t)<a) - \mathbb{P}(\xi_r<a)| = 0, \quad a\ge 0. \tag{2.20}
$$

Recall that

$$
\Xi_r(t) = \sum_{i=1}^\infty \frac{|\psi_i(t)|^2}{\lambda_i \mathrm{e}^{2\lambda_i r}}, \quad t,r>0.
$$

Define for any $n\ge 1$,

$$
\Psi_n(t) := (\psi_1(t),\cdots,\psi_n(t)), \quad t>0.
$$

Then, for any $\vartheta\in\mathbb{R}^n$, we have

$$
\langle \Psi_n(t),\vartheta\rangle = \frac{1}{\sqrt{t}} \int_0^t (\sum_{i=1}^n \vartheta_i \phi_i(X_s^B)) ds.
$$

By [15, Theorem 2.4′], when $t\to\infty$, the law of $\langle\Psi_n(t),\vartheta\rangle$ under $\mathbb{P}^\nu$ converges weakly to the Gaussian distribution $N(0,\sigma_{n,\vartheta})$ uniformly in $\nu\in\mathscr{P}(C)$ with variance

$$
\begin{aligned}
\sigma_{n,\vartheta} &:= \lim_{t\to\infty} \mathbb{E}^\mu \langle \Psi_n(t),\vartheta\rangle^2 \\
&= \lim_{t\to\infty} \frac{2}{t} \sum_{i=1}^n \vartheta_i^2 \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{e}^{-B(\lambda_i)(s_2-s_1)} \mathrm{d}s_2 = \sum_{i=1}^n \frac{2\vartheta_i^2}{B(\lambda_i)}.
\end{aligned}
$$

Thus, for any $\vartheta \in \mathbb{R}^n$,

$$\lim_{t\to\infty} \mathbb{E}^\nu \mathrm{e}^{\mathrm{i}\langle \Psi_n(t),\vartheta\rangle} = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle x,\vartheta\rangle} \prod_{i=1}^n N(0, 2B(\lambda_i)^{-1})(\mathrm{d}x_i) \text{ uniformly in } \nu \in \mathscr{P}(C),$$

so that the distribution of $\Psi_n(t)$ under $\mathbb{P}^\nu$ converges weakly to $\prod_{i=1}^n N(0, 2B(\lambda_i)^{-1})$ as $t \to \infty$. Therefore, letting

$$\Xi_r^{(n)}(t) := \sum_{i=1}^n \frac{|\psi_i(t)|^2}{\lambda_i B(\lambda_i)\mathrm{e}^{2\lambda_i r}}, \quad \xi_r^{(n)} := \sum_{i=1}^n \frac{2\xi_i^2}{\lambda_i B(\lambda_i)\mathrm{e}^{2\lambda_i r}},$$

we have

$$\lim_{t\to\infty} \sup_{\nu\in\mathscr{P}(C)} |\mathbb{P}^\nu(\Xi_r^{(n)}(t) < a) - \mathbb{P}(\xi_r^{(n)} < a)| = 0, \quad a \ge 0. \tag{2.21}$$

On the other hand, by (2.8) and (2.9), we find some constant $C_1 > 0$ such that

$$\begin{aligned}&\sup_{\nu\in\mathscr{P}(C)} \mathbb{E}^\nu|\Xi_r(t) - \Xi_r^{(n)}(t)| \\ &= \frac{2}{t} \sup_{\nu\in\mathscr{P}(C)} \sum_{i=n+1}^\infty \frac{\mathrm{e}^{-2\lambda_i r}}{\lambda_i B(\lambda_i)} \int_0^t \nu(P_s^B \phi_i^2)(1 - \mathrm{e}^{-B(\lambda_i)(t-s)})\mathrm{d}s \le C_1\varepsilon_n,\end{aligned}$$

where $\varepsilon_n := 2\sum_{i=n+1}^\infty \frac{2}{\lambda_i B(\lambda_i)\mathrm{e}^{2\lambda_i r}} \to 0$ as $n \to \infty$. This together with (2.21) implies (2.20). □

# 3 Some lemma

From now on, we assume that $M$ is compact. For any $q \ge p \ge 1$, let $\|\cdot\|_{p\to q}$ be the operator norm from $L^p(\mu)$ to $L^q(\mu)$. When $p = q$, we simply denote $\|\cdot\|_p = \|\cdot\|_{p\to p}$. Then there exist constants $\kappa, \lambda > 0$ such that

$$\|P_t - \mu\|_{p\to q} \le \kappa(1 \wedge t)^{-\frac{d}{2}(p^{-1}-q^{-1})}\mathrm{e}^{-\lambda_1 t}, \quad t > 0, q \ge p \ge 1. \tag{3.1}$$

Next, by the triangle inequality of $\mathbb{W}_2$, we obtain

$$\mathbb{E}[\mathbb{W}_2(\mu_t^B, \mu)^2] \le (1+\varepsilon)\mathbb{E}[\mathbb{W}_2(\mu_{t,r}^B, \mu)^2] + (1+\varepsilon^{-1})\mathbb{E}[\mathbb{W}_2(\mu_t^B, \mu_{t,r}^B)^2], \quad \varepsilon > 0. \tag{3.2}$$

As shown in [14] for $B(\lambda) = \lambda$ that, to prove Theorem 1.1, we need to estimate $\mathbb{E}[\mathbb{W}_2(\mu_t^B, \mu_{t,r}^B)^2]$ and to refine the estimate on $\mathbb{E}[\mathbb{W}_2(\mu_{t,r}^B, \mu)^2]$ for compact $M$. These are included in the following lemmas.

**Lemma 3.1.** *Let $B \in \mathbb{B}$ and $\mu_{t,r,\varepsilon}^B = (1-\varepsilon)\mu_{t,r}^B + \varepsilon\mu, \varepsilon \in [0,1]$. There exists a constant $c > 0$ such that*

$$\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B, \mu_{t,r}^B)^2] \le c\|h_\nu\|_\infty r, \quad \nu = h_\nu\mu, \tag{3.3}$$

$$\mathbb{W}_2(\mu_{t,r,\varepsilon}^B, \mu_{t,r}^B)^2 \le c\varepsilon, \quad t, r \ge 0, \varepsilon \in [0,1]. \tag{3.4}$$

*Proof.* Since for $t > 0$,

$$\pi_t(\mathrm{d}x, dy) := \left(\frac{1}{t}\int_0^t p_r(x,y)\delta_{X_s^B}(\mathrm{d}x)\,\mathrm{d}s\right)\mu(\mathrm{d}y) \in \mathscr{C}(\mu_t^B, \mu_{t,r}^B),$$

we have

$$\begin{aligned}\mathbb{W}_2(\mu_{t,r}^B, \mu_t^B)^2 &\le \int_M \rho(x,y)^2\,\pi_t(\mathrm{d}x,\mathrm{d}y)\\ &= \frac{1}{t}\int_0^t \mathrm{d}s\int_M p_r(X_s^B, y)\,\rho(X_s^B, y)^2\mu(\mathrm{d}y).\end{aligned}\tag{3.5}$$

Since $\nu = h_\nu\mu$, by the $P_t^B$-invariance of $\mu$, we find a constant $c_1 > 0$ such that

$$\begin{aligned}&\mathbb{E}^\nu\int_M p_r(X_s^B, y)\rho(X_s^B, y)^2\mu(\mathrm{d}y) \le \|h_\nu\|_\infty\mu\left[P_s^B\left(\int_M p_r(x,y)\rho(\cdot,y)^2\mu(\mathrm{d}y)\right)\right]\\ &= \|h_\nu\|_\infty\mathbb{E}^\mu[\rho(X_0, X_r)^2] \le c_1\|h_\nu\|_\infty r, \quad s \ge 0,\end{aligned}\tag{3.6}$$

where the last step is due to [14, Lemma 3.1]. Substituting this into (3.5), we prove (3.3).

On the other hand, let $D$ be the diameter of $M$. Since

$$\pi(\mathrm{d}x,\mathrm{d}y) := (1-\varepsilon)\mu_{t,r}^B(\mathrm{d}x)\delta_x(\mathrm{d}y) + \varepsilon\mu(\mathrm{d}x)\mu_{t,r}^B(\mathrm{d}y) \in \mathscr{C}(\mu_{t,r,\varepsilon}^B, \mu_{t,r}^B),$$

we obtain

$$\mathbb{W}_2(\mu_{t,r,\varepsilon}^B, \mu_{t,r}^B)^2 \le \int_{M\times M}\rho(x,y)^2\pi(\mathrm{d}x,\mathrm{d}y) \le \varepsilon D^2, \quad t,r > 0, \varepsilon\in[0,1].$$

Then the proof is finished. □

**Lemma 3.2.** *Let $B \in \mathbb{B}^\alpha$ for some $\alpha \in [0,1]$, and let $d < 2(1+\alpha)$.*

(1) *For any $q \in (\frac{d}{2}\vee 1, \frac{d}{d-2\alpha})$, there exists a constant $c > 0$, such that*

$$\sup_{y\in M}\mathbb{E}^\mu[|f_{t,r}(y) - 1|^2] \le \frac{c}{tr^{\frac{d}{2q}}}, \quad t \ge 1, r \in (0,1].\tag{3.7}$$

(2) *For any $q \in (\frac{d}{2}\vee 1, \frac{d}{d-2\alpha})$ and $\gamma \in (1, \frac{2q}{d})$,*

$$\lim_{t\to\infty}\sup_{y\in M}\mathbb{E}^\mu[|\mathscr{M}((1-t^{-\gamma})f_{t,t^{-\gamma}}(y) + t^{-\gamma}, 1)^{-1} - 1|^p] = 0, \quad p > 0.\tag{3.8}$$

*Proof.* (1) For fixed $y \in M$, simply denote $f = p_r(\cdot, y) - 1$. Then

$$\mathbb{E}^\mu\big[|f_{t,r} - 1|^2\big] = \frac{2}{t^2}\mathbb{E}^\mu\int_0^t f(X_{r_1}^B)\mathrm{d}r_1\int_{r_1}^t f(X_{r_2}^B)\mathrm{d}r_2,\tag{3.9}$$

Since $P^B_t$ is invariant with respect to $\mu$, we obtain

$$\begin{aligned}(3.10)\qquad &\mathbb{E}^\mu[f(X^B_{r_1})f(X^B_{r_2})] = \mu\big(P^B_{r_1}(fP^B_{r_2-r_1}f)\big) = \mu(fP^B_{r_2-r_1}f)\\ &\le \|f\|_{\frac{q}{q-1}}\|P^B_{r_2-r_1}f\|_q \le \|f\|_{\frac{q}{q-1}}\|P^B_{\frac{r_2-r_1}{2}}\|_{1\to q}\|P^B_{\frac{r_2-r_1}{2}}f\|_1,\quad r_2>r_1\ge 0.\end{aligned}$$

By $f = p_r(\cdot,y)-1$ and (3.1), we find some constants $c_1>0$ such that

$$(3.11)\qquad \|f\|_{\frac{q}{q-1}} \le 1+\|p_r(\cdot,y)\|_{\frac{q}{q-1}} \le 1+\|P_{\frac{r}{2}}\|_{1\to\frac{q}{q-1}} \le c_1 r^{-\frac{d}{2q}},\quad r\in(0,1], q\ge 1.$$

Moreover, since $P^B_t$ is the semigroup of $X^B_t := X_{S^B_t}$, by (3.1) and noting that $B\in\mathbb{B}^\alpha$ implies

$$(3.12)\qquad B(r)\ge k_0(r\wedge r^\alpha)\ge k_1 r^\alpha - k_2,\quad r\ge 0$$

for some constants $k_0,k_1,k_2>0$, we find a constant $c_2>0$ such that

$$\begin{aligned}&\|P^B_r\|_{1\to q}\le \mathbb{E}\|P_{S^B_r}\|_{1\to q}\le c\mathbb{E}\big[(1\wedge S^B_r)^{-\frac{d(q-1)}{2q}}\big]\\ &\le c + c\mathbb{E}\big[(S^B_r)^{-\frac{d(q-1)}{2q}}\big] = c+\frac{c}{\Gamma(\frac{d(q-1)}{2q})}\int_0^\infty t^{\frac{d(q-1)}{2q}-1}\mathrm{e}^{-rB(t)}\mathrm{d}t\\ &\le c_2(r^{-\frac{d(q-1)}{2\alpha q}}+1),\quad r>0.\end{aligned}$$

Since $\frac{d(q-1)}{2\alpha q}<1$, by combining this with (2.3), (3.9) and (3.11), we find constants $c_3,c_4>0$ such that

$$\mathbb{E}^\mu[|f_{t,r}(y)-1|^2]\le \frac{c_3}{r^{\frac{d}{2q}}t^2}\int_0^t \mathrm{d}r_1\int_{r_1}^t\big((r_2-r_1)^{-\frac{d(q-1)}{2\alpha q}}+1\big)\mathrm{e}^{-\lambda_1^\alpha(r_2-r_1)}\,\mathrm{d}r_2\le \frac{c_4}{r^{\frac{d}{2q}}t},\quad t,r>0.$$

(2) Let $\theta>0$ be small enough such that $\gamma(\frac{d}{2q}+\frac{\theta p}{2})<1$. According to the proof of the [14, Lemma 3.3], there exists a map $C:(0,1)\to(0,\infty)$ such that

$$\begin{aligned}&\sup_{y\in M}\mathbb{E}^\mu[|\mathscr{M}((1-r)f_{t,r}(y)+r,1)^{-1}-1|^p]\le \delta_\eta+(1+\theta^{-1}r^{-\frac{\theta}{2}})^p\sup_{y\in M}\mathbb{P}^\mu(\{|f_{t,r}(y)-1|>\eta\})\\ &\le \delta_\eta + C(\eta)t^{-1}r^{-\frac{d}{2p}-\frac{\theta p}{2}},\quad t\ge 1, r,\eta\in(0,1),\end{aligned}$$

holds for $\delta_\eta = \left|\frac{1}{\sqrt{1-\eta}}-\frac{2}{2+\eta}\right|^q$, $\eta\in(0,1)$. This implies (3.8) by taking $r=t^{-\gamma}$ and letting first $t\to\infty$ then $\eta\to 0$. □

**Lemma 3.3.** *Let $B\in\mathbb{B}^\alpha$ for some $\alpha\in[0,1]$. For any $p\in[1,2]$, there exists a constant $c>0$ such that*

$$(3.13)\qquad \mathbb{E}^\mu[|\psi_i(t)|^{2p}]\le c\lambda_i^{\alpha(p-2)+(p-1)(\frac{d}{2}-2\alpha)},\quad i\ge 1.$$

*Proof.* Since $P^B_t\phi_i = \mathrm{e}^{-B(\lambda_i)t}\phi_i$, we have

$$(3.14)\qquad g(r_1,r_2) := (\phi_i P^B_{r_2-r_1}\phi_i)(X^B_{r_1}) = \mathrm{e}^{-(r_2-r_1)B(\lambda_i)}\phi_i(X^B_{r_1})^2.$$

By (2.14), $\mu(P^B_{r_1}\phi_i^2)=\mu(\phi_i^2)=1$ and (3.12), we find a constant $c_1>0$ such that

$$\begin{aligned} t\mathbb{E}^\mu[|\psi_i(t)|^2] &\le c_1\int_0^t \mathrm{d}r_1\int_{r_1}^t \mathbb{E}^\mu[g(r_1,r_2)]\,\mathrm{d}r_2 \\ &= c_1\int_0^t \mathrm{d}r_1\int_{r_1}^t \mathrm{e}^{-(r_2-r_1)B(\lambda_i)}\mu(P^B_{r_1}\phi_i^2)\,\mathrm{d}r_2 \le \frac{c_1 t}{\lambda_i^\alpha},\quad t\ge 1, i\in\mathbb{N}. \end{aligned}\tag{3.15}$$

On the other hand, by (2.14), (3.12) and (3.14), we find a constant $c_2>0$ such that

$$\begin{aligned} t^2\mathbb{E}^\mu[|\psi_i(t)|^4] &\le c_2\left(\int_0^t \mathrm{d}r_1\int_{r_1}^t (\mathbb{E}^\mu[|g(r_1,r_2)|^2])^{\frac{1}{2}}\,\mathrm{d}r_2\right)^2 \\ &\le c_2\left(\int_0^t \mathrm{d}r_1\int_{r_1}^t \mathrm{e}^{-(r_2-r_1)\lambda_i^\alpha}\sqrt{\mu(P^B_{r_1}\phi_i^4)}\,\mathrm{d}r_2\right)^2. \end{aligned}\tag{3.16}$$

Moreover, (3.1) and $P_t\phi_i=\mathrm{e}^{-\lambda_i t}\phi_i$ yield

$$\|\phi_i\|_\infty=\inf_{t>0}\{\mathrm{e}^{\lambda_i t}\|P_t\phi_i\|_\infty\}\le\inf_{t>0}\{\mathrm{e}^{\lambda_i t}\|P_t\|_{2\to\infty}\}\le c_3\lambda_i^{\frac{d}{4}},\quad i\ge 1$$

for some constant $c_3>0$, so that

$$\sqrt{\mu(P^B_r\phi_i^4)}=\sqrt{\mu(\phi_i^4)}\le\sqrt{\|\phi_i\|_\infty^2\mu(\phi_i^2)}\le c_3\lambda_i^{\frac{d}{4}},\quad i\ge 1.$$

This together with (3.16) implies that for some constant $c_4>0$

$$\mathbb{E}^\mu[|\psi_i(t)|^4]\le c_4\lambda_i^{\frac{d}{2}-2\alpha},\quad i\ge 1.$$

Combining this with (3.15) and using Hölder's inequality, we find a constant $c_5>0$ such that

$$\begin{aligned} &\mathbb{E}^\mu[|\psi_i(t)|^{2p}]=\mathbb{E}^\mu[|\psi_i(t)|^{4-2p}|\psi_i(t)|^{4(p-1)}] \\ &\le(\mathbb{E}^\mu[|\psi_i(t)|^2])^{2-p}(\mathbb{E}^\nu[|\psi_i(t)|^4])^{p-1}\le c_5\lambda_i^{\alpha(p-2)+(p-1)(\frac{d}{2}-2\alpha)}. \end{aligned}$$

□

**Lemma 3.4.** *Let $B\in\mathbb{B}^\alpha$ for some $\alpha\in[0,1]$. If $d<2(1+\alpha)$, then there exists a constant $p>1$ such that*

$$\limsup_{t\to\infty}\sup_{r>0}\left\{t^p\mathbb{E}^\mu\int_M|\nabla L^{-1}(f_{t,r}-1)|^{2p}\,d\mu\right\}<\infty.$$

*Proof.* According to the proof of [14, Lemma 3.5], for any $p>1$ and $\varepsilon>p-1$, there exists a constant $c_1(p,\varepsilon)>0$ such that

$$t^p\mathbb{E}^\mu\int_M|\nabla L^{-1}(f_{t,r}-1)|^{2p}\,d\mu\le c_1(p,\varepsilon)\sum_{i=1}^\infty i^\varepsilon\lambda_i^{\frac{d(p-1)}{2}-1}\mathbb{E}^\mu[|\psi_i(t)|^{2p}].$$

Combining this with Lemma 3.3 and (1.2), we find a constant $c_2(p,\varepsilon) > 0$ such that

$$\mathbb{E}^\mu \int_M |\nabla L^{-1}(f_{t,r}-1)|^{2p}\, d\mu \le c_2(p,\varepsilon) t^{-p} \sum_{i=1}^\infty i^{\delta_{p,\varepsilon}} \tag{3.17}$$

holds for

$$\delta_{p,\varepsilon} := \varepsilon + \frac{2}{d}\left\{(p-1)(d-2\alpha) + \alpha p - (2\alpha+1)\right\}.$$

So, it remains to show that $\delta_{p,\varepsilon} < -1$ holds for some constants $p > 1$ and $\varepsilon > p-1$. This follows from the fact that for $\varepsilon > 0$ and $p_\varepsilon := 1 + \frac{\varepsilon}{2}$ we have $\varepsilon > p_\varepsilon - 1$ and

$$\lim_{\varepsilon \downarrow 0} \delta_{p_\varepsilon,\varepsilon} = -\frac{2(1+\alpha)}{d} < -1.$$

□

Finally, the following lemma reduces arbitrary initial values to initial distributions with bounded density.

**Lemma 3.5.** *Let $B \in \mathbb{B}$ and $p \in (0,2]$. Then for any $\varepsilon > 0$,*

$$\begin{aligned}
&\alpha_\varepsilon := \|P^B_{\varepsilon^2}\|_{1\to\infty} < \infty,\\
&\sup_{x\in M} \mathbb{E}^x\left[\mathbb{W}_p(\mu^B_t,\mu)^{1\vee p}\right] \le (1+\varepsilon) \sup_{\nu\in\mathscr{P}(\alpha_\varepsilon)} \mathbb{E}^\nu\left[\mathbb{W}_p(\mu^B_t,\mu)^{1\vee p}\right] + \frac{\varepsilon(1+\varepsilon)D^p}{t},\\
&\inf_{\nu\in\mathscr{P}(\alpha_\varepsilon)} \mathbb{E}^\nu\left[\mathbb{W}_p(\mu^B_t,\mu)^{1\vee p}\right] \le (1+\varepsilon) \inf_{x\in M} \mathbb{E}^x\left[\mathbb{W}_p(\mu^B_t,\mu)^{1\vee p}\right] + \frac{\varepsilon(1+\varepsilon)D^p}{t}, \quad t > \varepsilon^2,
\end{aligned}$$

*where $D$ is the diameter of $M$.*

*Proof.* There exists a constant $c > 0$ such that

$$\|P_t\|_{1\to\infty} \le c(1 + t^{-\frac{d}{2}}), \quad t > 0.$$

This together with (1.1) and $\int_1^\infty r^{\frac{d}{2}-1} e^{-tB(r)} dr < \infty$ implies

$$\begin{aligned}
\|P^B_t\|_{1\to\infty} &= \sup_{\mu(|f|)\le 1} \sup_{x\in M} |\mathbb{E}^x f(X_{S^B_t})| \le \mathbb{E}\|P_{S^B_t}\|_{1\to\infty} \le c + c\mathbb{E}(S^B_t)^{-\frac{d}{2}}\\
&= c + \frac{c}{\Gamma(\frac{d}{2})} \int_0^\infty r^{\frac{d}{2}-1} e^{-tB(r)} dr < \infty, \quad t > 0.
\end{aligned}$$

Next, for any $x \in M$ and $\varepsilon > 0$, let $\nu_{x,\varepsilon}$ be the distribution of $X^B_{\varepsilon^2}$. Then

$$\left\|\frac{d\nu_{x,\varepsilon}}{d\mu}\right\|_\infty = \sup_{\mu(|f|)\le 1} |P^B_{\varepsilon^2} f(x)| \le \|P^B_{\varepsilon^2}\|_{1\to\infty} = \alpha_\varepsilon,$$

so that

$$\nu_{x,\varepsilon} \in \mathscr{P}(\alpha_\varepsilon), \quad x \in M, \varepsilon > 0. \tag{3.18}$$

Let

$$\tilde{\mu}_{\varepsilon,t}^B := \frac{1}{t}\int_{\varepsilon^2}^{\varepsilon^2+t} \delta_{X_s^B}\mathrm{d}s, \quad t > 0.$$

By Markov property, we have

$$\mathbb{E}^x\big[\mathbb{W}_2(\tilde{\mu}_{\varepsilon,t}^B, \mu)^2\big] = \mathbb{E}^{\nu_{x,\varepsilon}}\big[\mathbb{W}_2(\mu_t^B, \mu)^2\big], \quad x \in M, t, \varepsilon > 0. \tag{3.19}$$

Moreover, it is easy to see that for any $t > \varepsilon^2 > 0$,

$$\pi := \frac{1}{t}\int_{\varepsilon^2}^{t} \delta_{(X_s^B, X_s^B)}\mathrm{d}s + \frac{1}{t}\int_0^{\varepsilon^2} \delta_{(X_s^B, X_{t+s}^B)}\mathrm{d}s \in \mathscr{C}(\mu_t^B, \tilde{\mu}_{\varepsilon,t}^B),$$

so that

$$|\mathbb{W}_p(\mu_t, \mu) - \mathbb{W}_p(\tilde{\mu}_{\varepsilon,t}^B, \mu)|^{1\vee p} \le \big\{\mathbb{W}_p(\mu_t, \tilde{\mu}_{\varepsilon,t}^B)\big\}^{1\vee p} \le \int_{M\times M} \rho^p \mathrm{d}\pi \le \frac{\varepsilon^2 D^p}{t}.$$

Combining this with (3.18) and (3.19), we obtain

$$\begin{aligned}
&\sup_{x\in M} \mathbb{E}^x\big[\mathbb{W}_p(\mu_t, \mu)^{1\vee p}\big] \le (1+\varepsilon)\sup_{x\in M} \mathbb{E}^x\big[\mathbb{W}_p(\tilde{\mu}_{\varepsilon,t}^B, \mu)^{1\vee p}\big] + (1+\varepsilon^{-1})\frac{\varepsilon^2 D^p}{t} \\
&\le (1+\varepsilon) \sup_{\nu\in\mathscr{P}(\alpha_\varepsilon)} \mathbb{E}^\nu\big[\mathbb{W}_p(\mu_t^B, \mu)^{1\vee p}\big] + \frac{\varepsilon(1+\varepsilon)D^p}{t}.
\end{aligned}$$

Similarly, the last estimate also holds.

□

# 4 Proof of Theorem 1.1

## 4.1 Proof of Theorem 1.1(1)

Since $M$ is compact and $V \in C^2(M)$, there exists a constant $K > 0$ such that

$$\mathrm{Ric}_V := \mathrm{Ric} - \mathrm{Hess}_V \ge -K,$$

where Ric is the Ricci curvature.

When $\partial M$ is either convex or empty, then

$$\mathbb{W}_p(\mu, \nu P_r)^2 \le \mathrm{e}^{2Kr}\mathbb{W}_p(\mu, \nu)^2, \quad r > 0, p \ge 1, \tag{4.1}$$

see for instance [8, 9]. Since $\mu_{t,r}^B = \mu_t^B P_r$, this and (2.15) imply

$$\begin{aligned}
&\mathrm{e}^{2Kr} \liminf_{t\to\infty} \Big\{t \inf_{x\in M} \mathbb{E}^x[\mathbb{W}_2(\mu, \mu_t^B)^2]\Big\} \\
&\ge \liminf_{t\to\infty} \Big\{t \inf_{x\in M} \mathbb{E}^x[\mathbb{W}_2(\mu, \mu_{t,r}^B)^2]\Big\} \ge \sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i)\mathrm{e}^{2r\lambda_i}}, \quad r > 0.
\end{aligned}$$

By letting $r \to 0$, we prove the desired estimate for $c = 1$.

When $\partial M$ is non-convex, the desired inequality follows by using the following estimate due to [4, Theorem 2.7] replacing (4.1): there exist constants $c, \lambda > 0$ such that

$$c\mathbb{W}_2(\nu P_r, \mu) \le \mathrm{e}^{\lambda r}\mathbb{W}_2(\nu, \mu), \quad \nu \in \mathscr{P}, r > 0.$$

## 4.2 Proof of Theorem 1.1(2)

It suffices to prove for $p \in (0,\alpha)$. The proof is modified from that of the proof of [14, Theorem 1.1], the only difference is that we have to use $\mathbb{W}_p$ for $p \in (0,\alpha)$ replacing $\mathbb{W}_1$, since in this case we have $\mathbb{E}[(S_t^B)^p] < \infty$.

For any $t \geq 1$ and $N \in \mathbb{N}$, we consider $\mu_N^B := \frac{1}{N}\sum_{i=1}^N \delta_{X_{t_i}^B}$, where $t_i := \frac{(i-1)t}{N}, 1 \leq i \leq N$. By taking the Wasserstein coupling

$$\frac{1}{t}\sum_{i=1}^N \int_{t_i}^{t_{i+1}} \delta_{X_s^B}(\mathrm{d}x)\delta_{X_{t_i}^B}(\mathrm{d}y)\,\mathrm{d}s \in \mathscr{C}(\mu_t^B, \mu_N^B),$$

we obtain

$$\mathbb{W}_p(\mu_t^B, \mu_N^B) \leq \frac{1}{t}\sum_{i=1}^N \int_{t_i}^{t_{i+1}} \rho(X_s^B, X_{t_i}^B)^p\,\mathrm{d}s.$$

By [14, (3.6)] that

$$\sup_{x\in M} \mathbb{E}^x \rho(X_0, X_t)^2 \leq ct, \quad t \geq 0$$

holds for some constant $c > 0$. So, by Jensen's inequality, for any $p \in (0,\alpha)$, there exists a constant $c_1 > 0$ such that

$$\sup_{x\in M} \mathbb{E}^x[\rho(X_0^B, X_r^B)^p] = \sup_{x\in M} \mathbb{E}^x[\rho(X_0, X_{S_r^B})^p] \leq c^{p/2}\mathbb{E}\big[(S_r^B)^{\frac{p}{2}}\big] \leq c_1 r^{\frac{p}{2\alpha}}, \quad r \in [0,1],$$

where the last step follows from (1.1) and $B \in \mathbb{B}_\alpha$ from which we find constants $c_2, c_3 > 0$ such that for $\varepsilon := \frac{p}{2}$,

$$\begin{aligned}
&\mathbb{E}\big[(S_r^B)^\varepsilon\big] = \frac{\varepsilon}{\Gamma(1-\varepsilon)}\int_0^\infty (1-\mathrm{e}^{-rB(t)})t^{-\varepsilon-1}\mathrm{d}t \\
&\leq c_2\int_0^\infty (1-\mathrm{e}^{-c_2 r - c_2 r t^\alpha})t^{-\varepsilon-1}\mathrm{d}t \leq c_2 \mathrm{e}^{c_2 r}\int_0^\infty (1-\mathrm{e}^{-c_2 r t^\alpha})t^{-\varepsilon-1}\mathrm{d}t \leq c_3 r^{\frac{\varepsilon}{\alpha}}, \quad r \in [0,1].
\end{aligned}$$

Therefore, there exists a constant $c_4 > 0$ such that

$$\sup_{x\in M} \mathbb{E}^x\big[\mathbb{W}_p(\mu_t^B, \mu_N^B)\big] \leq c_4(tN^{-1})^{\frac{p}{2\alpha}}, \quad t \geq 1, N \in \mathbb{N}. \tag{4.2}$$

On the other hand, since $M$ is compact, there exists a constant $c_5 > 0$ such that

$$\mu(\{\rho(x,\cdot)^p \leq r\}) \leq c_5 r^{\frac{d}{p}}, \quad r > 0, x \in M.$$

By [6, Proposition 4.2], this implies

$$\mathbb{W}_p(\mu_N^B, \mu) \geq c_6 N^{-\frac{p}{d}}, \quad N \in \mathbb{N}, t \geq 1$$

for some constant $c_6 > 0$. This and (4.2) yield

$$\inf_{x\in M} \mathbb{E}^x[\mathbb{W}_p(\mu, \mu_t^B)] \geq \inf_{x\in M} \mathbb{E}^x[\mathbb{W}_p(\mu, \mu_N^B)] - \sup_{x\in M} \mathbb{E}^x[\mathbb{W}_p(\mu_t^B, \mu_N^B)]$$

$$\geq c_6 N^{-\frac{p}{d}} - c_4 (tN^{-1})^{\frac{p}{2\alpha}}, \quad t \geq 1, N \in \mathbb{N}.$$

By taking $N := \inf\{n \in \mathbb{N} : n \geq \delta t^{\frac{d}{d-2\alpha}}\}$ for small $\delta > 0$, find a constant $c_7 > 0$ such that for large enough $t > 1$,

$$\inf_{x \in M} \mathbb{E}^x[\mathbb{W}_p(\mu, \mu_t^B)] \geq c_7 t^{\frac{p}{d-2\alpha}}.$$

Hence, the desired estimate holds.

## 4.3 Proof of Theorem 1.1(3)

We only consider the case that $\alpha = \frac{1}{2}, d = 3$, since the proof for $\alpha = 1$ and $d = 4$ has been presented in [14]. In this case, the assertion is implied by the following two lemmas which essentially due to [14] for $\alpha = 1$.

**Lemma 4.1.** *Let $B(\lambda) = \lambda^{\frac{1}{2}}$ and $d = 3$. If for any constant $C > 1$ there exist constants $\gamma, \varepsilon, t_0 > 0$, such that*

$$\{\mathbb{E}^\nu \mathbb{W}_1(\mu^B_{t,t^{-\gamma}}, \mu)\}^2 \geq \varepsilon \mathbb{E}^\nu \mu(|\nabla(-L)^{-1}(f_{t,t^{-\gamma}} - 1)|^2), \quad \nu \in \mathscr{P}(C), t > t_0, \tag{4.3}$$

*then the estimate in Theorem* 1.1(3) *holds.*

*Proof.* By Lemma 3.5 for $p = 1$, it suffices to prove that for any constant $C > 1$,

$$\liminf_{t \to \infty} t(\log t)^{-1} \inf_{\nu \in \mathscr{P}(C)} \{\mathbb{E}^\nu \mathbb{W}_1(\mu_t^B, \mu)\}^2 > 0. \tag{4.4}$$

By (2.6) and (4.3), there exists a constant $c_1, t_1 > 0$ such that

$$\inf_{\nu \in \mathscr{P}(C)} \{\mathbb{E}^\nu \mathbb{W}_1(\mu^B_{t,t^{-\gamma}}, \mu)\}^2 \geq \frac{c_1}{t} \sum_{i=1}^\infty \frac{1}{\lambda_i^{\frac{3}{2}} \mathrm{e}^{2t^{-\gamma}\lambda_i}}, \quad t > t_1.$$

Since $d = 3$, (1.2) implies $\lambda_i \leq c i^{\frac{2}{3}}$ for some constant $c > 0$, so that we find constants $c_2, c_3 > 0$ such that

$$\inf_{\nu \in \mathscr{P}(C)} \{\mathbb{E}^\nu \mathbb{W}_1(\mu^B_{t,t^{-\gamma}}, \mu)\}^2 \geq \frac{1}{c_2 t} \int_1^\infty \frac{\mathrm{d}s}{s \mathrm{e}^{c_2 t^{-\gamma} s^{\frac{2}{3}}}} \geq \frac{c_3 \log t}{t}, \quad t > t_1.$$

Combining this with (4.1), we find a constant $c_4 > 0$ such that

$$\inf_{\nu \in \mathscr{P}(C)} \{\mathbb{E}^\nu \mathbb{W}_1(\mu_t^B, \mu)\}^2 \geq \frac{c_4 \mathrm{e}^{-2Kt^{-\gamma}} \log t}{t}, \quad t > t_1.$$

This implies (4.4). □

**Lemma 4.2.** *Let $M = \mathbb{T}^3, V = 0$ and $B(\lambda) = \lambda^{\frac{1}{2}}$. Then for any $\gamma \in (0, \frac{2}{5})$ there exist constants $\varepsilon, t_0 > 0$ such that*

$$\{\mathbb{E}^\nu \mathbb{W}_1(\mu^B_{t,t^{-\gamma}}, \mu)\}^2 \geq \varepsilon \mathbb{E}^\nu \mu(|\nabla(-\Delta)^{-1}(f_{t,t^{-\gamma}} - 1)|^2), \quad \nu \in \mathscr{P}, t > t_0. \tag{4.5}$$

*Proof.* The proof is similar to that of [14, Proposition 5.3] with $X_t^B$ replacing $X_t$.

Let $f_t = (-\Delta)^{-1}(f_{t,t^{-\gamma}} - 1)$. It is shown in the proof of [14, Proposition 5.3] that

$$\mathbb{W}_1(\mu^B_{t,t^{-\gamma}}, \mu) \geq \beta^{-1}\mu(|\nabla f_t|^2) - K_1\beta^{-3}\mu(|\nabla f_t|^4), \quad \beta > 0$$

holds for some constant $K_1 > 0$. If there exist a constant $K_2 > 0$ such that

$$\mathbb{E}^\nu \mu(|\nabla f_t|^4) \leq K_2[\mathbb{E}^\nu \mu(|\nabla f_t|^2)]^2, \quad t \geq 2, \tag{4.6}$$

then

$$\mathbb{E}^\nu \mathbb{W}_1(\mu^B_{t,t^{-\gamma}}, \mu) \geq \beta^{-1}\mathbb{E}^\nu\mu(|\nabla f_t|^2) - \beta^{-3}K_1K_2[\mathbb{E}^\nu\mu(|\nabla f_t|^2)]^2, \quad \beta > 0.$$

Taking $\beta = N\mathbb{E}^\nu[\mu(|\nabla f_t|^2)^{\frac{1}{2}}]$ for large enough $N > 1$, we prove (4.5) for some constant $c > 0$. So, it remains to prove (4.6).

We identify $\mathbb{T}$ with $[0, 2\pi)$ by the one-to-one map

$$[0, 2\pi) \ni s \mapsto \mathrm{e}^{\mathrm{i}s},$$

where i is the imaginary unit. In this way, a point in $\mathbb{T}^3$ is regarded as a point in $[0, 2\pi)^3$, so that $\{\mathrm{e}^{\mathrm{i}\langle m,\cdot\rangle}\}_{m\in\mathbb{Z}^3}$ consist of an eigenbasis of $\Delta$ in the complex $L^2$-space of $\mu$, where $\mu$ is the normalized volume measure on $\mathbb{T}^3$. Since $X_t^B$ is generated by $-(-\Delta)^{\frac{1}{2}}$, we have

$$\mathbb{E}^x \mathrm{e}^{\mathrm{i}\langle m, X_t^B\rangle} = \mathrm{e}^{-|m|t}\mathrm{e}^{\mathrm{i}\langle m,x\rangle}, \quad t \geq 0, x \in \mathbb{T}^3, m \in \mathbb{Z}^3. \tag{4.7}$$

Moreover,

$$f_t := (-\Delta)^{-1}(f_{t,t^{-\gamma}} - 1) = \sum_{m\in\mathbb{Z}^3\setminus\{0\}} b_m \mathrm{e}^{-\mathrm{i}\langle m,\cdot\rangle},$$

where

$$b_m := \frac{\mathrm{e}^{-|m|^2t^{-\gamma}}}{|m|^2 t}\int_0^t \mathrm{e}^{\mathrm{i}\langle m, X_s^B\rangle}\,\mathrm{d}s, \quad m \in \mathbb{Z}^3. \tag{4.8}$$

Then

$$|\nabla f_t(x)|^2 = -\sum_{m_1,m_2\in\mathbb{Z}^3\setminus\{0\}} \langle m_1, m_2\rangle b_{m_1}b_{m_2}\mathrm{e}^{-\mathrm{i}\langle m_1+m_2,x\rangle},$$

$$|\nabla f_t(x)|^4 = \sum_{m_1,\cdots,m_4\in\mathbb{Z}^3\setminus\{0\}} \langle m_1, m_2\rangle\langle m_3, m_4\rangle b_{m_1}b_{m_2}b_{m_3}b_{m_4}\mathrm{e}^{-\mathrm{i}\langle m_1+m_2+m_3+m_4,x\rangle}.$$

Noting that, $\mu(\mathrm{e}^{-\mathrm{i}\langle m,\cdot\rangle}) = 0$ when $m \neq 0$, we get

$$\mathbb{E}^\nu\mu(|\nabla f_t|^2) = \sum_{m\in\mathbb{Z}^3\setminus\{0\}} |m|^2\mathbb{E}^\nu[b_m b_{-m}], \tag{4.9}$$

$$\mathbb{E}^\nu\mu(|\nabla f_t|^4) = \sum_{(m_1,m_2,m_3,m_4)\in\mathbb{S}} \langle m_1, m_2\rangle\langle m_3, m_4\rangle\mathbb{E}^\nu[b_{m_1}b_{m_2}b_{m_3}b_{m_4}], \tag{4.10}$$

where $\mathbb{S} := \{(m_1, m_2, m_3, m_4) \in \mathbb{Z}^3\backslash\{0\} : m_1 + m_2 + m_3 + m_4 = 0\}$.

By (4.8), we have

$$\mathbb{E}^\nu[b_m b_{-m}] = \frac{\mathrm{e}^{-2|m|^2 t^{-\gamma}}}{|m|^4 t^2} \int_{[0,t]^2} \mathbb{E}^\nu \mathrm{e}^{\mathrm{i}\langle m, X^B_{s_2} - X^B_{s_1}\rangle}\, \mathrm{d}s_1 \mathrm{d}s_2.$$

The Markov property and (4.7) yield

$$\mathbb{E}^\nu(\mathrm{e}^{\mathrm{i}\langle m, X^B_{s_2} - X^B_{s_1}\rangle} | \mathscr{F}_{s_1 \wedge s_2}) = \mathrm{e}^{-|m||s_1 - s_2|}, \quad s_1, s_2 \geq 0. \tag{4.11}$$

Then we find a constant $\kappa > 0$ such that

$$\mathbb{E}^\nu[b_m b_{-m}] = \frac{\mathrm{e}^{-2|m|^2 t^{-\gamma}}}{|m|^4 t^2} \int_{[0,t]^2} \mathrm{e}^{-|m||s_1 - s_2|}\, \mathrm{d}s_1 \mathrm{d}s_2 \geq \frac{\kappa \mathrm{e}^{-2|m|^2 t^{-\gamma}}}{|m|^5 t}, \quad t \geq 2.$$

Using this and (4.9), we get that

$$\begin{aligned} \mathbb{E}^\nu \mu(|\nabla f_t|^2) &\geq \sum_{m \in \mathbb{Z}^3\backslash\{0\}} \frac{\kappa \mathrm{e}^{-2|m|^2 t^{-\gamma}}}{|m|^3 t} \geq \frac{\kappa_1}{t} \int_1^\infty \frac{\mathrm{e}^{-2s^2 t^{-\gamma}}}{s}\, \mathrm{d}s \\ &\geq \frac{\kappa_1}{t\mathrm{e}^2} \int_1^{t^{\frac{\gamma}{2}}} s^{-1}\, \mathrm{d}s = \frac{\kappa_1 \gamma}{2\mathrm{e}^2}(t^{-1} \log t), \quad t \geq 2. \end{aligned} \tag{4.12}$$

Let $\mathbf{S}$ be the set of all the permutations of $\{1, 2, 3, 4\}$, $D(t) = \{(s_1, s_2, s_3, s_4) \in [0, t]^4 : 0 \leq s_1 \leq s_2 \leq s_3 \leq s_4 \leq t\}$. We have

$$\begin{aligned} &\mathbb{E}^\nu[b_{m_1} b_{m_2} b_{m_3} b_{m_4}] \\ &= \frac{\mathrm{e}^{-\sum_{p=1}^4 |m_p|^2 t^{-\gamma}}}{t^4 \prod_{p=1}^4 |m_p|^2} \int_{[0,t]^4} \mathbb{E}^\nu[\mathrm{e}^{\mathrm{i}\langle m_1, X^B_{s_1}\rangle} \mathrm{e}^{\mathrm{i}\langle m_2, X^B_{s_2}\rangle} \mathrm{e}^{\mathrm{i}\langle m_3, X^B_{s_3}\rangle} \mathrm{e}^{\mathrm{i}\langle m_4, X^B_{s_4}\rangle}]\, \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \mathrm{d}s_4 \\ &= \frac{\mathrm{e}^{-\sum_{p=1}^4 |m_p|^2 t^{-\gamma}}}{t^4 \prod_{p=1}^4 |m_p|^2} \sum_{(i,j,k,l) \in \mathbf{S}} \int_{D(t)} \mathbb{E}^\nu[\mathrm{e}^{\mathrm{i}\langle m_i, X^B_{s_1}\rangle} \mathrm{e}^{\mathrm{i}\langle m_j, X^B_{s_2}\rangle} \mathrm{e}^{\mathrm{i}\langle m_k, X^B_{s_3}\rangle} \mathrm{e}^{\mathrm{i}\langle m_l, X^B_{s_4}\rangle}]\, \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \mathrm{d}s_4 \end{aligned}$$

Since $m_1 + m_2 + m_3 + m_4 = 0$, by (4.7) and the Markov property we obtain

$$\mathbb{E}^\nu[\mathrm{e}^{\mathrm{i}\langle m_i, X^B_{s_1}\rangle} \mathrm{e}^{\mathrm{i}\langle m_j, X^B_{s_2}\rangle} \mathrm{e}^{\mathrm{i}\langle m_k, X^B_{s_3}\rangle} \mathrm{e}^{\mathrm{i}\langle m_l, X^B_{s_4}\rangle}] = \mathrm{e}^{-|m_l|(s_4 - s_3) - |m_l + m_k|(s_3 - s_2) - |m_i|(s_2 - s_1)}.$$

Thus,

$$\begin{aligned} &\frac{t^4 \prod_{p=1}^4 |m_p|^2}{\mathrm{e}^{-\sum_{p=1}^4 |m_p|^2 t^{-\gamma}}} \mathbb{E}^\nu[b_{m_1} b_{m_2} b_{m_3} b_{m_4}] \\ &= \sum_{(i,j,k,l) \in \mathbf{S}} \int_{D(t)} \mathrm{e}^{-|m_l|(s_4 - s_3) - |m_l + m_k|(s_3 - s_2) - |m_i|(s_2 - s_1)}\, \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \mathrm{d}s_4. \end{aligned} \tag{4.13}$$

If $m_l + m_k = 0$, then

$$\begin{aligned}
&\int_{D(t)} \mathrm{e}^{-|m_l|(s_4-s_3)-|m_l+m_k|(s_3-s_2)-|m_i|(s_2-s_1)}\,\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}s_4\\
&= \int_0^t\int_{s_1}^t\int_{s_2}^t\int_{s_3}^t \mathrm{e}^{-|m_l|(s_4-s_3)}\mathrm{e}^{-|m_i|(s_2-s_1)}\,\mathrm{d}s_4\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1 \le \frac{t^2}{|m_i||m_l|}.
\end{aligned}$$

If $m_l + m_k \neq 0$, then

$$\begin{aligned}
&\int_{D(t)} \mathrm{e}^{-|m_l|(s_4-s_3)-|m_l+m_k|(s_3-s_2)-|m_i|(s_2-s_1)}\,\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}s_4\\
&= \int_0^t\int_{s_1}^t\int_{s_2}^t\int_{s_3}^t \mathrm{e}^{-|m_l|(s_4-s_3)}\mathrm{e}^{-|m_l+m_k|(s_3-s_2)}\mathrm{e}^{-|m_i|(s_2-s_1)}\,\mathrm{d}s_4\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1\\
&\le \frac{t}{|m_i||m_l+m_k||m_l|}.
\end{aligned}$$

Combining these with (4.13) leads to

$$\mathbb{E}^\nu[b_{m_1}b_{m_2}b_{m_3}b_{m_4}] \le \frac{\mathrm{e}^{-\sum_{p=1}^4 |m_p|^2 t^{-\gamma}}}{\prod_{p=1}^4 |m_p|^2} \sum_{(i,j,k,l)\in\mathbf{S}} \left\{ \frac{t^{-2}1_{\{m_l+m_k=0\}}}{|m_i||m_l|} + \frac{t^{-3}1_{\{m_l+m_k\neq 0\}}}{|m_i||m_l+m_k||m_l|} \right\}.$$

Therefore, by (4.10), we find a constant $c > 0$ such that

$$\mathbb{E}^\nu \mu(|\nabla f_t|^4) \le c(I_1 + I_2), \quad t \ge 2, \tag{4.14}$$

holds for

$$I_1 := \frac{1}{t^2} \sum_{a,b\in\mathbb{Z}^3\setminus\{0\}} \frac{1}{|a|^3|b|^3}\mathrm{e}^{-2(|a|^2+|b|^2)t^{-\gamma}},$$

$$I_2 := \frac{1}{t^3} \sum_{\substack{m_1,m_2,m_3,m_4\in\mathbb{Z}^3\setminus\{0\}\\ m_3+m_4\neq 0}} \frac{\mathrm{e}^{-\sum_{p=1}^4 |m_p|^2 t^{-\gamma}}}{|m_1|^2|m_2||m_3||m_3+m_4||m_4|^2}.$$

It is easy to see that there exists constants $c_1, c_2 > 0$, such that

$$I_1 \le \frac{c_1}{t^2}\left(\int_1^\infty \frac{\mathrm{e}^{-2s^2t^{-\gamma}}}{s}\,\mathrm{d}s\right)^2 \le c_2(t^{-1}\log t)^2, \quad t \ge 2, \tag{4.15}$$

and similarly

$$\sum_{m\in\mathbb{Z}^3\setminus\{0\}} \frac{\mathrm{e}^{-|m|^2t^{-\gamma}}}{|m|^2} \le c_2 t^{\frac{\gamma}{2}}, \quad \sum_{m\in\mathbb{Z}^3\setminus\{0\}} \frac{\mathrm{e}^{-|m|^2t^{-\gamma}}}{|m|} \le c_2 t^{\gamma}, \quad t \ge 2,$$

Then by reformulating $I_2$ as

$$I_2 = \frac{1}{t^3}\left(\sum_{m_1\in\mathbb{Z}^3\setminus\{0\}} \frac{\mathrm{e}^{-|m_1|^2t^{-\gamma}}}{|m_1|^2}\right)\left(\sum_{m_2\in\mathbb{Z}^3\setminus\{0\}} \frac{\mathrm{e}^{-|m_2|^2t^{-\gamma}}}{|m_2|}\right) \sum_{\substack{m_3,m_4\in\mathbb{Z}^3\setminus\{0\}\\ m_3+m_4\neq 0}} \frac{\mathrm{e}^{-(|m_3|^2+|m_4|^2)t^{-\gamma}}}{|m_3||m_3+m_4||m_4|^2},$$

we find a constant $c_3 > 0$ such that

$$I_2 \leq c_3^2 t^{\frac{3\gamma}{2}-3} \sum_{m_4 \in \mathbb{Z}^3\backslash\{0\}} \frac{\mathrm{e}^{-|m_4|^2 t^{-\gamma}}}{|m_4|^2} \sum_{m_3 \in \mathbb{Z}^3\backslash\{0,-m_4\}} \frac{\mathrm{e}^{-|m_3|^2 t^{-\gamma}}}{|m_3||m_3+m_4|}. \tag{4.16}$$

Write

$$\sum_{m_3 \in \mathbb{Z}^3\backslash\{0,-m_4\}} \frac{\mathrm{e}^{-|m_3|^2 t^{-\gamma}}}{|m_3||m_3+m_4|} =: J_1 + J_2 + J_3 \tag{4.17}$$

for

$$J_1 := \sum_{\substack{m_3 \in \mathbb{Z}^3\backslash\{0,-m_4\} \\ |m_3| \leq \frac{|m_4|}{2}}} \frac{\mathrm{e}^{-|m_3|^2 t^{-\gamma}}}{|m_3||m_3+m_4|},$$

$$J_2 := \sum_{\substack{m_3 \in \mathbb{Z}^3\backslash\{0,-m_4\} \\ \frac{|m_4|}{2} < |m_3| \leq 2|m_4|}} \frac{\mathrm{e}^{-|m_3|^2 t^{-\gamma}}}{|m_3||m_3+m_4|},$$

$$J_3 := \sum_{\substack{m_3 \in \mathbb{Z}^3\backslash\{0,-m_4\} \\ |m_3| > 2|m_4|}} \frac{\mathrm{e}^{-|m_3|^2 t^{-\gamma}}}{|m_3||m_3+m_4|}.$$

On the region $\{m_3 \in \mathbb{Z}^3\backslash\{0,-m_4\} : |m_3| \leq \frac{|m_4|}{2}\}$ we find a constant $c_4 > 0$ such that

$$J_1 \leq \frac{2}{|m_4|} \sum_{m_3 \in \mathbb{Z}^3\backslash\{0\}} \frac{\mathrm{e}^{-|m_3|^2 t^{-\gamma}}}{|m_3|} \leq \frac{c_4 t^{\gamma}}{|m_4|}, \quad t \geq 2. \tag{4.18}$$

Next, on the region $\{m_3 \in \mathbb{Z}^3\backslash\{0,-m_4\} : |m_3| > 2|m_4|\}$, we have $|m_3 + m_4| \sim |m_3|$ and $|m_3|^2 \geq \frac{|m_3|^2}{2} + 2|m_4|^2$, so we find a constant $c_5 > 0$ such that

$$J_3 \leq 4 \sum_{\substack{m_3 \in \mathbb{Z}^3\backslash\{0\} \\ |m_3| > 2|m_4|}} \frac{\mathrm{e}^{-|m_3|^2 t^{-\gamma}}}{|m_3|^2} \leq 4\mathrm{e}^{-2|m_4|^2 t^{-\gamma}} \sum_{m_3 \in \mathbb{Z}^3\backslash\{0\}} \frac{\mathrm{e}^{-\frac{|m_3|^2 t^{-\gamma}}{2}}}{|m_3|^2} \leq c_5 t^{\frac{\gamma}{2}} \mathrm{e}^{-2|m_4|^2 t^{-\gamma}}.$$

This together with $\mathrm{e}^{-s} \leq s^{-\frac{1}{2}}$ gives

$$J_3 \leq \frac{c_5 t^{\gamma}}{|m_4|}, \quad t \geq 2. \tag{4.19}$$

Finally, on the region $\{m_3 \in \mathbb{Z}^3\backslash\{0,-m_4\} : \frac{|m_4|}{2} < |m_3| \leq 2|m_4|\}$, we have $|m_3| \sim |m_4|$ and $1 \leq |m_3 + m_4| \leq 3|m_4|$, so that there for a constant $c_6 > 0$

$$J_2 \leq \frac{2\mathrm{e}^{-\frac{|m_4|^2 t^{-\gamma}}{4}}}{|m_4|} \sum_{1 \leq |m_3+m_4| \leq 3|m_4|} \frac{1}{|m_3+m_4|} \leq c_6 |m_4| \mathrm{e}^{-\frac{|m_4|^2 t^{-\gamma}}{4}}.$$

By $e^{-s} \le s^{-1}$, we get the upper estimate of $J_2$,

$$J_2 \le \frac{c_7 t^\gamma}{|m_4|}, \quad t \ge 2.$$

Combining this with (4.16),(4.17),(4.18) and (4.19), we find a constant $c_8 > 0$ such that

$$I_2 \le c_8 t^{\frac{5}{2}\gamma - 3} \log t, \quad t \ge 2.$$

Substituting this and (4.15) into (4.14), and combining with (4.12), we prove (4.6). The proof is finished.

□

# 5 Proof of Theorem 1.2

(1) By Lemma 3.5 for $p = 2$, it suffices to prove

$$\limsup_{t\to\infty} \sup_{\nu\in\mathscr{P}(C)} \left\{ t\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B, \mu)^2] \right\} \le \sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i)}, \quad C > 1. \tag{5.1}$$

By the triangle inequality of $\mathbb{W}_2$ and Lemma 3.1, for any $\varepsilon > 0$ there exists a constant $c(\varepsilon) > 0$ such that

$$\begin{aligned}
&\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B, \mu)^2] \\
&\le (1+\varepsilon)\mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r_t,r_t}^B, \mu)^2] + 2(1+\varepsilon^{-1})\left\{ \mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r_t}^B, \mu_{t,r_t,r_t}^B)^2] + \mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B, \mu_{t,r_t}^B)^2] \right\} \\
&\le (1+\varepsilon)\mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r_t,r_t}^B, \mu)^2] + c(\varepsilon) r_t,
\end{aligned}$$

where $r_t = t^{-\beta}, \beta \in (1, \frac{2q}{d}), q \in (\frac{d}{2} \vee 1, \frac{d}{d-2\alpha}), t \ge 1$. Since $\frac{d\mu_{t,r_t,r_t}}{d\mu} = (1-r_t) f_{t,r_t} + r_t$, by combining this with Lemma 2.2 and Hölder's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r_t,r_t}^B, \mu)^2] \le \mathbb{E}^\nu \int_M \frac{|\nabla L^{-1}(f_{t,r_t} - 1)|^2}{\mathscr{M}((1-r_t) f_{t,r_t} + r_t, 1)} \, d\mu \\
&\le \mathbb{E}^\nu \int_M \left\{ |\nabla L^{-1}(f_{t,r_t} - 1)|^2 + |\nabla L^{-1}(f_{t,r_t} - 1)|^2 |\mathscr{M}((1-r_t) f_{t,r_t} + r_t, 1)^{-1} - 1| \right\} d\mu \\
&\le \mathbb{E}^\nu \int_M |\nabla L^{-1}(f_{t,r_t} - 1)|^2 \, d\mu + \left( \mathbb{E}^\nu \int_M |\nabla L^{-1}(f_{t,r_t} - 1)|^{2p} \, d\mu \right)^{\frac{1}{p}} \\
&\times \left( \mathbb{E}^\nu \int_M |\mathscr{M}((1-r_t) f_{t,r_t} + r_t, 1)^{-1} - 1|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}.
\end{aligned}$$

Since $B \in \mathbb{B}^\alpha$, by Lemma 2.3, Lemma 3.4 and (3.8), this implies

$$\limsup_{t\to\infty} \sup_{\nu\in\mathscr{P}(C)} \left\{ t\mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r_t,r_t}^B, \mu)^2] \right\} \le \sum_{i=1}^\infty \frac{2}{\lambda_i B(\lambda_i)}.$$

Combining this with Lemma 3.1 for $\varepsilon = r = r_t := t^{-\beta}$ where $\beta > 1$, we prove (5.1).

(2) By Lemma 3.5 for $p = 2$, it suffices to prove

$$\limsup_{t\to\infty} \sup_{\nu\in\mathscr{P}(C)} \Big\{ t^{\frac{2}{d-2\alpha}} \mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B, \mu)^2] \Big\} < \infty, \quad C > 1. \tag{5.2}$$

Let $r : (1, \infty) \to (0, 1)$ to be determined. By [7], we have

$$t\mathbb{W}_2(\mu_{t,r}^B, \mu)^2 \le 4\Xi_r(t), \quad t, r > 0.$$

Combining this with Lemma 3.1 and Lemma 2.3, we find a constant $c_0 > 0$ such that

$$\begin{aligned} \mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B, \mu)^2] &\le 2\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B, \mu_{t,r_t}^B)^2] + 2\mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r_t}^B, \mu)^2] \\ &\le c_0 r_t + c_0 \frac{\|h_\nu\|_\infty}{t} \sum_{i=1}^\infty \frac{1}{\lambda_i^{1+\alpha} \mathrm{e}^{2r_t\lambda_i}}, \quad t > 1. \end{aligned} \tag{5.3}$$

By (1.2), there exists constants $c_2, c_3 > 0$ such that

$$\sum_{i=1}^\infty \frac{1}{\lambda_i^{1+\alpha} \mathrm{e}^{2r_t\lambda_i}} \le c_2 \int_1^\infty s^{-\frac{2(1+\alpha)}{d}} \mathrm{e}^{-c_3 r_t s^{\frac{2}{d}}} \,\mathrm{d}s,$$

so that (5.3) implies

$$\sup_{\nu\in\mathscr{P}(C)} \mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B, \mu)^2] \le c r_t + \frac{c}{t} \int_1^\infty s^{-\frac{2(1+\alpha)}{d}} \mathrm{e}^{-c_3 r_t s^{\frac{2}{d}}} \,\mathrm{d}s, \quad t > 1, r_t > 0 \tag{5.4}$$

for some constant $c > 0$ depending on $C$.

Since $d > 2(1 + \alpha)$, we find a constant $c_4 > 0$ such that

$$\int_1^\infty s^{-\frac{2(1+\alpha)}{d}} \mathrm{e}^{-c_3 r_t s^{\frac{2}{d}}} \,\mathrm{d}s = \int_{r_t^{\frac{d}{2}}}^\infty (r_t^{-\frac{d}{2}} u)^{-\frac{2(1+\alpha)}{d}} \mathrm{e}^{-c_3 u^{\frac{2}{d}}} r_t^{-\frac{d}{2}} \,\mathrm{d}u \le c_4 r_t^{-\frac{d-2(1+\alpha)}{2}}, \quad t > 1.$$

Combining this with (5.4) and taking

$$r_t = t^{-\frac{2}{d-2\alpha}}, \quad t > 1,$$

we prove (5.2).

(3) Since $d = 2(1 + \alpha)$, for any $c > 0$ there exists a constant $c_1 > 0$ such that there exist a constants $c_1 > 0$ such that

$$\int_1^\infty s^{-\frac{2(1+\alpha)}{d}} \mathrm{e}^{-c r_t s^{\frac{2}{d}}} \,\mathrm{d}s \le c_1 \ln(1 + r_t^{-1}), \quad t > 1,$$

so that (5.4) implies

$$\mathbb{E}^\mu[\mathbb{W}_2(\mu_t^B, \mu)^2] \le c' r_t + c' t^{-1} \log(1 + r_t^{-1}), \quad t > 1$$

for some constant $c' > 0$. Taking $r_t = t^{-1} \log(1 + t^{-1})$ for $t \ge 2$, we find a constant $c_2 > 0$ such that

$$\mathbb{E}^\mu[\mathbb{W}_2(\mu_t^B, \mu)^2] \le c_2 t^{-1} \log(1 + t), \quad t \ge 2.$$

Since $\mathbb{E}^\nu \le \|h_\nu\|_\infty \mathbb{E}^\mu$ for $\nu = h_\nu \mu$, combining this with Lemma 3.5 for $p = 2$ and $\varepsilon = 1$, we obtain

$$\limsup_{t\to\infty} \frac{t}{\log t} \sup_{x\in M} \mathbb{E}^x[\mathbb{W}_2(\mu_t^B, \mu)^2] < \infty.$$

## 6 Proof of Theorem 1.4

*Proof.* By (3.18), (3.19) and noting that the Markov property implies

$$\mathbb{P}^x(t\mathbb{W}_2(\tilde{\mu}^B_{\varepsilon,t},\mu)^2 < a) = \mathbb{P}^{\nu_{x,\varepsilon}}(t\mathbb{W}_2(\mu^B_t,\mu)^2 < a), \quad a \geq 0,$$

it suffices to prove that for any $C > 1$,

$$\liminf_{t\to\infty} \inf_{\nu\in\mathscr{P}(C)} \mathbb{P}^\nu(t\mathbb{W}_2(\mu^B_t,\mu)^2 < a) \geq F(a), \quad a \geq 0, \tag{6.1}$$

$$\limsup_{t\to\infty} \sup_{\nu\in\mathscr{P}(C)} \mathbb{P}^\nu(t\mathbb{W}_2(\mu^B_t,\mu)^2 < a) \leq F(a), \quad a \geq 0. \tag{6.2}$$

It is easy to see that (6.2) follows from Theorem 2.1(2) and (4.1).

To prove (6.1), let $\gamma > 1$ be in Lemma 3.2(2), and denote

$$\begin{aligned}
\tilde{\Xi}(t) &:= t\int_M \frac{|\nabla L^{-1}(f_{t,t^{-\gamma}}-1)|^2}{\mathscr{M}((1-t^{-\gamma})f_{t,t^{-\gamma}}+t^{-\gamma},1)}\mathrm{d}\mu,\\
\Xi(t) &:= \Xi_{t^{-\gamma}}(t) = t\mu\big(|\nabla L^{-1}(f_{t,t^{-\gamma}}-1)|^2\big), \quad t > 1.
\end{aligned}$$

Then Lemma 3.2(2) and Lemma 3.4 yield

$$\limsup_{t\to\infty} \sup_{\nu\in\mathscr{P}(C)} \mathbb{P}^\nu(|\tilde{\Xi}(t)-\Xi(t)| > \varepsilon) = 0, \quad \varepsilon > 0.$$

Combining this with Lemma 2.2, (2.6) and noting that $\sum_{i=1}^\infty \frac{1}{\lambda_i B(\lambda_i)} < \infty$, we prove (6.1). $\square$

**Data Availability Statements.** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**Acknowledgement.** The authors would like to thank the referee for corrections.

# References

[1] L. Ambrosio, F. Stra, D. Trevisan, *A PDE approach to a 2-dimensional matching problem*, Probab. Theory Relat. Fields 173(2019), 433–477.

[2] E. Boissard, T. Le Gouic, *On the mean speed of convergence of empirical and occupation measures in Wasserstein distance,* Ann. Inst. Henri Poincar Probab. Stat. 50(2014), 539–563.

[3] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, 1984.

[4] L.-J. Cheng, A. Thalmaier and J. Thompson, *Functional inequalities on manifolds with non-convex boundary*, Science China Math. 61(2018), 1421–1436.

[5] N. Fournier, A. Guillin, *On the rate of convergence in Wasserstein distance of the empirical measure,* Probab. Theory Relat. Fields 162(2015), 707–738.

[6] B. Kloeckner, *Approximation by finitely supported measures*, ESAIM Control Optim. Calc. Var. 18(2012), 343–359.

[7] M. Ledoux, *On optimal matching of Gaussian samples*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 457, Veroyatnost' i Statistika. 25(2017), 226–264.

[8] F.-Y. Wang, *Harnack inequalities on manifolds with boundary and applications,* J. Math. Pures Appl. 94(2010), 304–321.

[9] F.-Y. Wang, *Analysis for Diffusion Processes on Riemannian Manifolds,* World Scientific, Singapore, 2014.

[10] F.-Y. Wang, *Precise limit in Wasserstein distance for conditional empirical measures of Dirichlet diffusion processes*, J. Funct. Anal. 280(2021), 108998.

[11] F.-Y. Wang, *Convergence in Wasserstein Distance for empirical measures of Dirichlet diffusion processes on manifolds*, to appear in J. Eur. Math. Soc. arXiv:2005.09290.

[12] F.-Y. Wang, *Wasserstein convergence rate for empirical measures on noncompact manifolds*, Stoch. Proc. Appl. 144(2022), 271–287.

[13] F.-Y. Wang, *Convergence in Wasserstein distance for empirical measures of semilinear SPDEs*, arXiv:2102.00361.

[14] F.-Y. Wang, J.-X. Zhu, *Limit theorems in Wasserstein distance for empirical measures of diffusion processes on Riemannian manifolds*, to appear in Ann. lInst. H. Poinc. arXiv:1906.03422.

[15] L. Wu, Moderate deviations of dependent random variables related to CLT, Ann. Probab. 23(1995), 420–445.