On non-negative solutions to space-time partial differential equations of higher order

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Abstract. We discuss when certain higher order partial differential operators in space and time admit non-negative solutions which have a semigroup representation as well as a representation by some associated Markov process.

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Dedicated to Masatoshi Fukushima, Scholar, Mentor, and Friend

Introduction

Modelling the dependencies of a process with the help of space-time partial differential equations shall lead to solutions which capture typical observed phenomena, e.g. the propagation of singularities, preservation of positivity, etc. The heat or diffusion equation is an example of an equation the solutions of which preserve the positivity (more correctly, the non-negativity) of initial data. It is also an example of an equation whose solution operator exhibits strong smoothing effects, e.g. continuous initial data are turned into C^∞-solutions. In addition, we encounter the semigroup property. These analytic properties do have a probabilistic companion. With the heat equation we can associate a Brownian motion and we can use Brownian motion to represent solutions to the heat equation. Indeed, the Gaussian semigroup $(T_t^G)_{t\geqslant 0}$ which gives the solution to the initial value problem to the heat equation admits a representation using Brownian motion $(B_t)_{t\geqslant 0}$ by

$$u(t,x) = (T_t^G g)(x) = E^x(g(B_t)). \tag{1}$$

Since we may construct Brownian motion with the help of the fundamental solution to the heat equation, formula (1) looks rather natural. An obvious question is to find those space-time partial differential operators which allow an analogous treatment. It is well known that this is limited to second order partial differential operators with suitable coefficients of the type

$$\frac{\partial}{\partial t} - \sum_{k,l=1}^{n} a_{kl} \frac{\partial^2}{\partial x_k \partial x_l} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} + c, \tag{2}$$

where $(a_{kl})_{k,l=1,\ldots,n}$ is a non-negative definite (symmetric) matrix. In our paper we are not interested in minimal smoothness assumptions for coefficients, but we are stimulated by the fact that certain higher order (in space and/or in time) partial differential operators still admit certain positive solutions, some of which can even be represented with the help of Markov processes, not necessarily Brownian motion. The simplest and best known example is the Laplace operator $\frac{\partial^2}{\partial t^2} + \Delta_n$ in the half-space $\mathbb{R}_+ \times \mathbb{R}^n$ which is not of the type (2), but which has solutions we can represent with the help of the Cauchy process. Indeed, the Dirichlet problem

$$\frac{\partial^2}{\partial t^2} u(t,x) + \Delta_n u(t,x) = 0 \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^n, \tag{3}$$

$$\lim_{t \to 0} u(t,x) = g(x) \tag{4}$$

has for a suitable g a (unique) solution which is given by the Poisson integral, i.e.

$$u(t,x) = \int_{\mathbb{R}^n} P_n(t, x-y) g(y)\,\mathrm{d}y. \tag{5}$$

However, this classical Poisson formula in the half-space is clearly related to the Cauchy process $(C_t)_{t \geqslant 0}$ and the Cauchy semigroup $(T_t^C)_{t \geqslant 0}$, namely by

$$u(t,x) = (T_t^C g)(x) = E^x(g(C_t)). \tag{6}$$

Note that equation (3) is of second order in t, not of first order. Taking in (3) the (partial) Fourier transform with respect to x we arrive at the ordinary differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \hat{u}(t,\xi) - |\xi|^2 \hat{u}(t,\xi) = 0 \tag{7}$$

and the "initial" condition

$$\hat{u}(0,\xi) = \hat{g}(\xi). \tag{8}$$

Note that we have only one "initial" condition for the second order equation. The Ansatz $\hat{u}(t,\xi) = e^{-\lambda t}$, $\lambda = \lambda(\xi)$, leads to the characteristic equation

$$\lambda^2 - |\xi|^2 = 0 \tag{9}$$

with the two solutions $\lambda_{1,2} = \lambda_{1,2}(\xi) = \pm|\xi|$. The solution $\lambda_1(\xi) = |\xi|$ gives

$$u(t,x) = F^{-1}_{\xi \mapsto x}(e^{-t|\cdot|} \hat{g})(x) = (T_t^C g)(x). \tag{10}$$

We may factorise (9) according to

$$(\lambda^2 - |\xi|^2) = (\lambda - |\xi|)(\lambda + |\xi|) \tag{11}$$

and the solution $\lambda(\xi) = |\xi|$ is the one of interest. It is a continuous negative definite function, hence it is associated with a convolution semigroup and therefore with a Lévy process. Guided by this well known example, see [9], we want to discuss the following problem: Let the partial differential equation with constant coefficients

$$\frac{\partial^N}{\partial t^N} u(t,x) - \sum_{j=0}^{N-1} \sum_{|\alpha| \leqslant m} a_{j\alpha} \frac{\partial^j}{\partial t^j} \left(-i \frac{\partial}{\partial x} \right)^{\alpha} u(t,x) = 0 \tag{12}$$

subject to the initial condition

$$u^{(l-1)}(0,x) = h_l(x), \quad l = 0, \dots, N-1. \tag{13}$$

Is it possible to obtain solutions to (12)/(13) of the type

$$u(t,x) = \sum_{j=1}^{L} (\gamma_j T_t^{(j)} g_j)(x) = \sum_{j=1}^{L} \gamma_j E^x (g_j(X_t^{(j)})), \quad L \leqslant N, \tag{14}$$

where $(T_t^{(j)})_{t \geqslant 0}$ is a positivity preserving semigroup acting on functions defined on $\mathbb{R}^n$ and which is associated with a Markov process $(X_t^{(j)})_{t \geqslant 0}$? Clearly, there are quite a few problems such as regularity or domain questions. To handle such question we choose to work in the Hilbert space setting, i.e. we use $L^2(\mathbb{R}^n)$ as underlying space, a restriction which is not as restrictive as it seems, other settings e.g. the Feller setting working with $C_\infty(\mathbb{R}^n)$, the space of all continuous functions vanishing at infinity, is in principle possible. In addition, we are searching only for holomorphic semigroups. Another problem is that we need to associate with (12) a total of N independent "initial" conditions, not necessarily of the form (13), but in (14) we have only $L \leqslant N$ conditions.

In Section 1 we look at an abstract version of our problem and push it formally to a stage such that we can derive conditions to solve (12) with the help of (14). We then turn to equations of the form (12) and for this we need to introduce pseudo-differential operators with constant coefficients, but rather general ξ-dependence of their symbols, see Section 2. In Section 3 we discuss in more detail the case $N = 2$ in order to understand how to transfer our problem to questions posed on the involved symbols. Maybe the most important insight of this section is that our programme to find solutions of the type (14) works in principle well, however only case by case studies will allow us to cope with initial data. The final section is devoted to various classes of examples, by no means covering the full scope of our programme. Indeed, in some sense this paper is more about a programme to obtain positivite solutions of higher order space-time partial differential equations which allow representations with the help of some Markov processes.

Our notions and notation are standard and we refer to [4]. The Fourier transform is given by

$$\hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\,\mathrm{d}x$$

which entails that the constant in Plancherel's theorem is 1 and that in the convolution theorem it is $(2\pi)^{-\frac{n}{2}}$, i.e. $(u \cdot v)\hat{} = (2\pi)^{-\frac{n}{2}}(\hat{u} * \hat{v})$. Sometimes we write Fu for $\hat{u}$ and the inverse Fourier transform is denoted by F^{-1}. Note that we mainly use the partial Fourier transform with respect to x, i.e. for $u = u(t,x)$ we denote by Fu or $\hat{u}$ the Fourier transform with respect to x only. We write $L^2_+(\mathbb{R}^n)$ or L^2_+ for the cone $\{u \in L^2(\mathbb{R}^n) | u \geqslant 0 \text{ a.e.}\}$ and $u \geqslant 0$ in the sense of $L^2(\mathbb{R}^n)$ means $u \geqslant 0$ a.e. The term $\left(-i\frac{\partial}{\partial x}\right)^\alpha$ means $(-i)^{|\alpha|}\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$. A continuous function $\psi : \mathbb{R}^n \to \mathbb{C}$ is called a continuous negative definite function if $\psi(0) \geqslant 0$ and for all $t > 0$ the function $\xi \mapsto e^{-t\psi(\xi)}$ is positive definite in the sense of Bochner. Equivalently, a function is continuous and negative definite if it admits a Lévy-Khinchin representation. A Bernstein function $f : (0,\infty) \to \mathbb{R}_+$ is a C^∞-function satisfying $(-1)^k f^{(k)}(s) \leqslant 0$, $k \in \mathbb{N}$. The most important result for us is that if f is a Bernstein function and ψ a continuous negative definite function, then $f \circ \psi$ is a continuous negative definite function too. The standard reference for Bernstein functions is [7]

1 An Abstract Problem

Let $(A_j, D(A_j))$, $1 \leqslant j \leqslant N$, be a finite family of closable operators densely defined on $L^2(\mathbb{R}^n)$, each of which extends to a generator, denoted again by A_j, of a strongly continuous contraction semigroup $(T_t^{(j)})_{t\geqslant 0}$ on $L^2(\mathbb{R}^n)$. Since

$$D(A_j \circ A_l) = \{g \in D(A_l) | A_l g \in D(A_j)\}$$

it follows that the assumption

$$[A_j, A_l] = A_j A_l - A_l A_j = 0 \quad \text{for all} \quad 1 \leqslant j, l \leqslant N \tag{15}$$

implies that for any $j_1, \ldots, j_M$, $1 \leqslant j_k \leqslant N$, the operator $A_{j_1} \circ A_{j_2} \circ \cdots \circ A_{j_M}$ is defined on

$$V := D(A_1 \circ \cdots \circ A_N) \tag{16}$$

which we assume to be dense in $L^2(\mathbb{R}^n)$ too. We find for the Yosida approximation $A_{j,\lambda}$ of A_j that $[A_{j,\lambda}, A_{l,\lambda}] = 0$ and it follows that for all $1 \leqslant j, l \leqslant N$ we have

$$[A_j, T_t^{(l)}] = 0, \quad t \geqslant 0. \tag{17}$$

As a further assumption we pose

$$T_t^{(j)} V \subset V \quad \text{for all } 1 \leqslant j \leqslant N. \tag{18}$$

Note that in later situations we will replace V in (16) and (18) with a smaller subspace of V. Clearly we have the equalities

$$\frac{\mathrm{d}}{\mathrm{d}t}T_t^{(j)}g = A_j T_j^{(j)}g, \quad g \in D(A_j), \tag{19}$$

as well as

$$\frac{\mathrm{d}}{\mathrm{d}t}T_t^{(j)}g = A_j T_t^{(j)}g \;\text{ on } V. \tag{20}$$

By (17) and (18) we have with $1 \leqslant j_1, \ldots, j_M \leqslant N$, $1 \leqslant l_1, \ldots, l_k \leqslant N$ that any permutation of the compositions $A_{j_1} \circ \cdots \circ A_{j_M} \circ T_t^{(l_1)} \circ \cdots \circ T_t^{(l_N)}$ is defined on V and these permutations are equal to each other. Consequently we have for each $1 \leqslant j \leqslant N$ and for $g \in V$ that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_1\right) \cdots \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_N\right) T_t^{(j)}g =$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_1\right) \cdots \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_{j-1}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_{j+1}\right) \cdots \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_N\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_j\right) T_j^{(j)}g = 0$$

holds. Thus

$$u_j(t,x) := (T_t^{(j)}g_j)(x) \quad (\text{in } L^2(\mathbb{R}^n)) \tag{21}$$

is a solution to the equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_1\right) \cdots \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_N\right) u_j = 0, \quad 1 \leqslant j \leqslant N. \tag{22}$$

Hence for any scalars $\gamma_j \in \mathbb{R}$ we find for $g_j \in V$ a solution to (22) by

$$v(t,x) := \sum_{j=1}^{N} (\gamma_j T_j^{(j)} g_j)(x) \quad (\text{in } L^2(\mathbb{R}^n)). \tag{23}$$

By our assumption, $(T_t^{(j)})_{t \geqslant 0}$ is a strongly continuous contraction semigroup on $L^2(\mathbb{R}^n)$ and therefore we have in $L^2(\mathbb{R}^n)$

$$\lim_{t \to 0} T_t^{(j)} g_j = g_j, \tag{24}$$

and consequently as an identity in $L^2(\mathbb{R}^n)$

$$v(0,x) = \left(\sum_{j=1}^{N} \gamma_j g_j\right)(x). \tag{25}$$

For $t > 0$ we may formally differentiate (23) k-times, $k \in \mathbb{N}$, to find

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} v(t,x) = \left(\sum_{j=1}^{N} \gamma_j A_j^k T_t^{(j)} g_j\right)(x) \quad (\text{in } L^2(\mathbb{R}^n)), \tag{26}$$

however in order to justify (26) we need to assume $T_t^{(j)} g_j \in D(A_j^k)$. Note that for a holomorphic semigroup $(T_t^{(j)})_{t\geqslant 0}$ this condition is always satisfied. Passing in (26) formally to the limit as $t \to 0$ we arrive at

$$\frac{\mathrm{d}^{\mathrm{k}}}{\mathrm{d}t^k} v(0,x) = \left(\sum_{j=1}^{N} \gamma_j A_j^k g_j \right)(x) \quad (\text{in } L^2(\mathbb{R}^n)) \tag{27}$$

and once again, if for example $(T_t^{(j)})_{t\geqslant 0}$ is for each $1 \leqslant j \leqslant N$ a holomorphic semigroup, the calculation can be justified.

Now we change our point of view and consider (22) as an ordinary operator-differential equation of order N in $L^2(\mathbb{R}^n)$, i.e. we consider

$$\left(\frac{\mathrm{d}^N}{\mathrm{d}t^N} - \left(\sum_{j=1}^{N} A_j \right) \frac{\mathrm{d}^{N-1}}{\mathrm{d}t^{N-1}} + \cdots + (-1)^N A_1 \circ \cdots \circ A_N \right) u = 0 \tag{28}$$

and for this equation we prescribe the N initial conditions

$$\left. \begin{aligned} u(0,x) &= \tilde{h}_0(x) = h_1(x) \\ \vdots \quad & \quad \vdots \qquad\quad \vdots \\ u^{(N-1)}(0,x) &= \tilde{h}_{N-1}(x) = h_N(x) \end{aligned} \right\} \tag{29}$$

The function $v(t,x) := \sum_{j=1}^{N} (\gamma_j T_t^{(j)} g_j)(x)$ is of course a special solution to (28) as are $u_j(t,x) := (T_t^{(j)} g_j)(x)$, and we shall not expect that we can always fit the initial conditions using only these solutions. However, under certain (in general, restrictive) conditions on $h_1, \ldots, h_N$ it might become possible to single out solutions to (28) and (29) having special properties, e.g. being positivity preserving.

We want to note that when considering an operator of the type

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_1 \right) \cdots \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_N \right) \sum_{j=0}^{M} \sum_{l=0}^{m} a_{jl} \frac{\mathrm{d}^j}{\mathrm{d}t^j} B_l, \quad a_{jl} \in \mathbb{R}, \tag{30}$$

where the operators B_l are densely defined on $L^2(\mathbb{R}^n)$ and satisfy certain commutator relations, then under reasonable domain conditions it is still possible to obtain solutions of the corresponding equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_1 \right) \cdots \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_N \right) \sum_{j=0}^{M} \sum_{l=0}^{m} \frac{\mathrm{d}^j}{\mathrm{d}t^j} B_l u(t,x) = 0 \tag{31}$$

with the help of the semigroups $(T_t^{(k)})_{t\geqslant 0}$, $1 \leqslant k \leqslant N$.

It is clear that, in general, no unique solution of (28) and (29) of the type $v(t,x) = \left(\sum_{j=1}^{N} \gamma_j T_t^{(j)} g_j \right)(x)$ with g depending on $h_1, \ldots, h_N$ exists. Indeed, neither the existence nor the uniqueness of such a solution can be taken for granted. In order to get some ideas we now restrict ourselves to the case $N = 2$ and we assume that $(T_t^{(1)})_{t\geqslant 0}$ is positivity preserving (or sub-Markovian) on $L^2(\mathbb{R}^n)$ whereas $(T_t^{(2)})_{t\geqslant 0}$ is not. For simplicity we add the assumption that $(T_t^{(1)})_{t\geqslant 0}$ is holomorphic, which follows for example if $(T_t)_{t\geqslant 0}$ is symmetric and conservative. Thus for $g \in L^2(\mathbb{R}^n)$, $g \geqslant 0$ in $L^2(\mathbb{R}^n)$ a non-negative solution to

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_2\right) u(t,x) = 0 \tag{32}$$

is given by $u(t,x) = (T_t^{(1)} g)(x)$. Moreover, we have

$$\lim_{t\to 0} u(t,x) = \lim_{t\to 0} T_t^{(1)} g(x) = g(x) \quad \text{ in } L^2(\mathbb{R}^n)$$

and differentiation yields

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t,x) = A_1 T_t^{(1)} g(x) = (T_t^{(1)} A_1 g)(x)$$

where for the last step we need to assume that $g \in D(A_1)$. Under this assumption we find

$$\lim_{t\to 0} \frac{\mathrm{d}u(t,x)}{\mathrm{d}t} = \lim_{t\to 0} \left(T_t^{(1)} A_1 g\right)(x) = (A_1 g)(x) \quad \text{ in } L^2(\mathbb{R}^n).$$

If we add to (32) the initial condition

$$\left.\begin{aligned} u(0,x) &:= \lim_{t\to 0} u(t,x) = h_1(x) \\ \frac{\mathrm{d}}{\mathrm{d}t} u(0,x) &:= \lim_{t\to 0} \frac{\mathrm{d}}{\mathrm{d}t} u(t,x) = h_2(x) \end{aligned}\right\} \tag{33}$$

we arrive at the relation

$$h_1 = g \quad \text{ and } \quad h_2 = A_1 g. \tag{34}$$

Thus, for the initial value problem

$$\left.\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_2\right) u(t,x) = \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - (A_1 + A_2)\frac{\mathrm{d}}{\mathrm{d}t} + A_1 A_2\right) u(t,x) = 0 \\ u(0,x) = g \text{ and } \frac{\mathrm{d}}{\mathrm{d}t} u(0,x) = A_1 g \end{aligned}\right\} \tag{35}$$

a solution is given by $u(t,x) := (T_t^{(1)} g)(x)$ and this solution is positive in the sense that $g \geqslant 0$ in $L^2(\mathbb{R}^n)$ implies $u(t,x) \geqslant 0$. Of course, a uniqueness result for (28) and (29) (with $N = 2$) holds in our situation, but we have to note that the initial data h_1 and h_2 are not independent of each other.

We now want to study the more general case, namely to find positive solutions to (28) and (29) under the assumption that for $M \leqslant N$ the semigroup $(T_t^{(j)})_{t\geqslant 0}$,

$1 \leqslant j \leqslant M$, generated by A_j are positivity preserving in $L^2(\mathbb{R}^n)$. In this case, for $g_j \in L^2(\mathbb{R}^n)$, $g_j \geqslant 0$, $1 \leqslant j \leqslant M$, and coefficients $\gamma_j \geqslant 0$ each of the functions

$$v(t,x) := \sum_{j=1}^{M} \gamma_j T_j^{(j)} g_j(x) \quad (\text{ in } L^2(\mathbb{R}^n)) \tag{36}$$

gives a non-negative solution to (28) and we need to relate the functions g_j to the initial data $h_1, \ldots, h_N$. Under appropriate conditions on $(T_t^{(j)})_{t\geqslant 0}$, for example holomorphy, we derive using (27) the equations

$$v^{(k-1)}(0,\cdot) = h_k = \sum_{j=1}^{M} \gamma_j A_j^{k-1} g_j, 1 \leqslant k \leqslant N. \tag{37}$$

Thus, in the situation under discussion, given $\gamma_j \geqslant 0$, $1 \leqslant j \leqslant M$, and functions $g_j \in D(A_j^{N-1})$, $1 \leqslant j \leqslant M$, for h_k, $1 \leqslant j \leqslant M$, determined by (37) we have a non-negative solution to (28) and (29) by (36) provided $g_j \geqslant 0$. The more interesting question is of course whether we can determine $g_j \in D(A_j^N)$, $g_j \geqslant 0$, and $\gamma_j \geqslant 0$, $1 \leqslant j \leqslant M$, for given functions h_k, $1 \leqslant k \leqslant N$. These are N equations for (essentially) $M < N$ unknown functions, but due to the conditions $g_j \geqslant 0$, these are non-linear equations. We have to solve for the mapping

$$S : \bigtimes_{j=1}^{M} D_+(A_j^{N-1}) \qquad \to (L^2(\mathbb{R}^n))^N \tag{38}$$

$$SG = H, G = (g_1, \ldots, g_M) \mapsto (h_1, \ldots, h_N) = H \tag{39}$$

where h_k is given by (37) and $D_+(A_j^{N-1}) = \{g_j \in D(A_j^{N-1}) | g \geqslant 0\}$. Clearly $\bigtimes_{j=1}^{M} D_+(A_j^{N-1})$ is a convex set in $(L^2(\mathbb{R}^n))^M$ and S maps convex combinations onto convex combinations implying that the image of $\bigtimes_{j=1}^{M} D_+(A_j^{N-1})$ under S is a convex subset in $(L^2(\mathbb{R}^n))^N$. For $M = 1$, N fixed and $\gamma_1 = 1$ for simplicity, we have the N equations

$$h_k = A_1^{k-1} g_1, \quad 1 \leqslant k \leqslant N, \tag{40}$$

which implies of course $g_1 = h_1 = \tilde{h}_0$. Moreover, for $k = 2$ we get formally

$$g_2 = (A_1)^{-1} h_2 = (A_1)^{-1} \tilde{h}_0. \tag{41}$$

In general, we may try to interpret $(A_1)^{-1}$ as the abstract potential operator in the sense of Yosida associated with $(T_t^{(1)})_{t\geqslant 0}$. But of course we have to sort out domain problems, and similarly we may try to handle $g_k = (A_1)^{-1} \circ \cdots \circ (A_1)^{-1} h_k$ with k copies of $(A_1)^{-1}$. The case $M \geqslant 2$ is obviously much more complicated and we will pick it up in forthcoming investigations. In the next section we want to turn our attention to concrete pseudo-differential operators and by this we can reduce our consideration to the level of symbols, i.e. functions which are easier to handle than abstract operators.

2 Some translation invariant pseudo-differential operators

In order to handle operators such as (30) for concrete operators A_j and B_l we now introduce translation invariant pseudo-differential operators in a quite general manner. Note that any translation invariant operator on $\mathcal{S}'(\mathbb{R}^n)$ is indeed a convolution operator, but its kernel might be rather singular.
Let $q : \mathbb{R}^n \to \mathbb{R}$ be a continuous function of at most polynomial growth, i.e. we have for some $c \geqslant 0$ and $m \geqslant 0$ the estimate

$$|q(\xi)| \leqslant c(1+|\xi|^2)^{\frac{m}{2}} \quad \text{for all } \xi \in \mathbb{R}^n. \tag{42}$$

On $\mathcal{S}(\mathbb{R}^n)$ we can define the pseudo-differential operator

$$q(D)u(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(\xi)\hat{u}(\xi)\,\mathrm{d}\xi. \tag{43}$$

From (42) and Plancherel's theorem we deduce immediately that

$$\|q(D)u\|_s \leqslant c_{q,s}\|u\|_{s+m} \tag{44}$$

for all u belonging to $\mathcal{S}(\mathbb{R}^n)$, or for $u \in H^{s+m}(\mathbb{R}^n)$, where $H^t(\mathbb{R}^n)$, $t \in \mathbb{R}$, denotes the standard Bessel potential space (or Sobolev space of fractional order) with the norm

$$\|u\|_t^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)^t |\hat{u}(\xi)|^2\,\mathrm{d}\xi. \tag{45}$$

The operator $q(D)$ has extensions $q(D) : H^{m+s}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$, however, in general, we cannot determine the domain of the closure of $(q(D), \mathcal{S}(\mathbb{R}^n))$ in $L^2(\mathbb{R}^n)$ in terms of classical Sobolev spaces. If q_1 and q_2 are continuous symbols each satisfying (42) with c_j and m_j, then their compositions $q_1(D) \circ q_2(D)$ is given on $\mathcal{S}(\mathbb{R}^n)$ by

$$(q_1(D) \circ q_2(D)u)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q_1(\xi)q_2(\xi)\hat{u}(\xi)\,\mathrm{d}\xi \tag{46}$$

which extends to an operator on $L^2(\mathbb{R}^2)$ with domain $H^{m_1+m_2}(\mathbb{R}^n)$. Moreover, on $H^{m_1+m_2}(\mathbb{R}^n)$ we have $[q_1(D), q_2(D)] = 0$ since all translation invariant operators on $\mathcal{S}'(\mathbb{R}^n)$ commute. Note that $q_j(D)$ maps $H^{m_1+m_2}(\mathbb{R}^n)$ continuously into $H^{m_k}(\mathbb{R}^n)$, $j, k = 1, 2, j \neq k$.
Thus, if we restrict in (30) the operators A_j and B_l to be operators of the type (43), translation invariance and hence commutativity can be taken for granted and in addition we can always operate on some space $H^m(\mathbb{R}^n)$, m sufficiently large, in order to handle various compositions of the operators A_j and B_l.

We are interested in the case where some of the operators A_j are generators of translation invariant sub-Markovian semigroups and in this case we know much more.
Let $(\mu_t)_{t\geqslant 0}$ be a convolution semigroup on $\mathbb{R}^n$ and $\psi : \mathbb{R}^n \to \mathbb{C}$ its associated continuous negative definite function, i.e. we have

$$\hat{\mu}_t(\xi) = (2\pi)^{-\frac{n}{2}} e^{-t\psi(\xi)}, t > 0 \text{ and } \xi \in \mathbb{R}^n. \tag{47}$$

We can associate with $(\mu_t)_{t\geqslant 0}$ an L^2-sub-Markovian semigroup

$$(T_t^\psi g)(x) = (\mu_t * g)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t\psi(\xi)} \hat{g}(\xi)\,\mathrm{d}\xi, \tag{48}$$

i.e. $(T_t^\psi)_{t\geqslant 0}$ is a strongly continuous contraction semigroup on $L^2(\mathbb{R}^n)$ satisfying $0 \leqslant g \leqslant 1$ in $L^2(\mathbb{R}^n)$, i.e. $\lambda^{(n)}$-almost everywhere, implies $0 \leqslant T_t^\psi g \leqslant 1$ in $L^2(\mathbb{R}^n)$. Moreover, if ψ is real-valued then $(T_t^\psi)_{t\geqslant 0}$ is symmetric, i.e. $(T_t^\psi g, h)_0 = (g, T_t^\psi h)_0$, and if in addition $\psi(0) = 0$, then $(T_t^\psi)_{t\geqslant 0}$ is conservative, hence Markovian, which means that its extension to $L^\infty(\mathbb{R}^n)$ has the property that $T_t^\psi 1 = 1$ $\lambda^{(n)}$-almost everywhere.
By a theorem of E. M. Stein [8] such a semigroup has a holomorphic extension $z \mapsto T_z^\psi$ for z in a certain sector of $\mathbb{C}$.
For every continuous negative definite function ψ the function $\xi \mapsto \psi(\xi) - \psi(0)$ is again a continuous negative definite function and if ψ is real-valued then $(T_t^{\psi-\psi(0)})_{t\geqslant 0}$ is a symmetric Markovian strongly continuous contraction semigroup on $L^2(\mathbb{R}^n)$, hence it has a holomorphic extension. However we have

$$\begin{aligned} T_t^\psi g &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t\psi(\xi)} \hat{g}(\xi)\,\mathrm{d}\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t\psi(0)} e^{-t(\psi(\xi)-\psi(0))} \hat{g}(\xi)\,\mathrm{d}\xi \\ &= e^{-t\psi(0)} T_t^{\psi-\psi(0)} g \end{aligned}$$

implying that for every real-valued continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ we can consider $(T_t^\psi)_{t\geqslant 0}$ as a holomorphic semigroup. We also note that on $L^2(\mathbb{R}^n) \cap C_\infty(\mathbb{R}^n)$ this semigroup admits the representation

$$(T_t^\psi g)(x) = \int_{\mathbb{R}^n} g(x-y)\mu_t(\mathrm{d}y), \quad g \in L^2(\mathbb{R}^2) \cap C_\infty(\mathbb{R}^n) \tag{49}$$

which is pointwisely defined and which admits a pointwise extension to $C_b(\mathbb{R}^n)$.
Let $(T_t^\psi)_{t\geqslant 0}$ be the symmetric L^2-semigroup associated by (48) with ψ. The L^2-generator of $(T_t^\psi)_{t\geqslant 0}$ is the operator $(A^\psi, H^{\psi,2}(\mathbb{R}^n))$ where

$$H^{\psi,s}(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) | \|u\|_{\psi,s} < \infty \right\}, s \geqslant 0, \tag{50}$$

and

$$\|u\|_{\psi,s}^2 := \int_{\mathbb{R}^n} (1+\psi(\xi))^s |\hat{u}(\xi)|^2 \mathrm{d}\xi = \|(1+\psi(D)^{\frac{s}{2}} u\|_{L^2}^2, \tag{51}$$

where we denote by $\psi(D)$ and $(1+\psi(D))^{\frac{s}{2}}$ the pseudo-differential operators

$$\psi(D)u(x) = F^{-1}(\psi\hat{u})(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \psi(\xi) \hat{u}(\xi)\,\mathrm{d}\xi \tag{52}$$

and

$$(1+\psi(D))^{\frac{s}{2}} u(x) = F^{-1}((1+\psi(\cdot))^{\frac{s}{2}} \hat{u})(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} (1+\psi(\xi))^{\frac{s}{2}} \hat{u}(\xi)\,\mathrm{d}\xi, \tag{53}$$

respectively. These operators are considered as extensions from $\mathcal{S}(\mathbb{R}^n)$ to their natural L^2-domains, i.e. $H^{\psi,2}(\mathbb{R}^n)$ and $H^{\psi,s}(\mathbb{R}^n)$, respectively. An easy calculation shows now that

$$A^{\psi} = -\psi(D), \quad D(A^{\psi}) = H^{\psi,2}(\mathbb{R}^n). \tag{54}$$

(For details we refer to [4] or [2].) In order to cover interesting examples we want to emphasise that if $\varphi : \mathbb{R}^m \to \mathbb{R} = \varphi(\xi_1, \dots, \xi_m)$, $m \leqslant n$, is a continuous negative definite function on $\mathbb{R}^m$ then $\psi : \mathbb{R}^n \to \mathbb{R}$, $\psi(\xi_1, \dots, \xi_n) := \varphi(\xi_1, \dots, \xi_m)$, is a continuous negative definite function on $\mathbb{R}^n$. Moreover, the sum ψ of finitely many continuous negative definite functions $\psi_j : \mathbb{R}^n \to \mathbb{R}$, $1 \leqslant j \leqslant N$, i.e. the function $\psi = \psi_1 + \cdots + \psi_N$, is again a continuous negative definite function as is $\lambda\psi$, $\lambda > 0$, for ψ continuous negative definite. Finally, we note that for every continuous negative definite function we have the estimate

$$|\psi(\xi)| \leqslant c_{\psi}(1 + |\xi|^2) \tag{55}$$

which implies that $H^s(\mathbb{R}^n) \subset H^{\psi,s}(\mathbb{R}^n)$ for all $s \geqslant 0$.

We now suggest to first study problem (22), (29) (or (28), (29)) in the context of generators of the type A^{ψ_j}, $j = 1, 2$, then to investigate the case where $N = 2$ but only A_1 is of the type A^{ψ}. The aim is to come towards an understanding of constraints needed to arrive at certain families of positivity preserving solutions.

3 Some discussions on the case $N = 2$

With $N = 2$ we choose $A_1 = -\psi_1(D)$ and $A_2 = -\psi_2(D)$, where $\psi_1, \psi_2 : \mathbb{R}^n \to \mathbb{R}$ are continuous negative definite functions with corresponding operator semigroups $(T_t^{(1)})_{t\geqslant 0}$ and $(T_t^{(2)})_{t\geqslant 0}$. For the moment we pretend that all data belong to $\mathcal{S}(\mathbb{R}^n)$, later we will take care on precise domains. The equation we want to solve is

$$\left(\frac{\partial}{\partial t} + \psi_1(D)\right)\left(\frac{\partial}{\partial t} + \psi_2(D)\right) u(t,x) = 0 \tag{56}$$

under the initial conditions

$$\left.\begin{aligned} u(0,x) &= h_1(x) \\ \frac{\partial u}{\partial t}(0,x) &= h_2(x) \end{aligned}\right\} \tag{57}$$

We are looking for solutions of the form

$$v(t,x) = \gamma_1 T_t^{(1)} g_1(x) + \gamma_2 T_t^{(2)} g_2(x). \tag{58}$$

If $\gamma_1, \gamma_2 \geqslant 0$ and $g_1, g_2 \geqslant 0$ then v is a non-negative solution to (56). Thus the problem is to find g_1, g_2 (non-negative) for given h_1, h_2. From (57) and the holomorphy of $(T_t^{(1)})_{t\geqslant 0}$ and $(T_t^{(2)})_{t\geqslant 0}$ we deduce

$$\left.\begin{aligned} \gamma_1 g_1(x) + \gamma_2 g_2(x) &= h_1(x) \\ (-\psi_1(D)\gamma_1 g_1)(x) + (-\psi_2(D)\gamma_2 g_2)(x) &= h_2(x). \end{aligned}\right\} \tag{59}$$

Using the Fourier transform we arrive at

$$\gamma_1\hat{g}_1(\xi) + \gamma_2\hat{g}_2(\xi) = \hat{h}_1(\xi) \tag{60}$$

and

$$\gamma_1\psi_1(\xi)\hat{g}_1(\xi) + \gamma_2\psi_2(\xi)\hat{g}_2(\xi) = -\hat{h}_2(\xi). \tag{61}$$

Under the assumption $\gamma_1\gamma_2(\psi_2(\xi) - \psi_1(\xi)) \neq 0$, i.e. $\gamma_1 \neq 0$, $\gamma_2 \neq 0$ and $\psi_2(\xi) \neq \psi_1(\xi)$ we obtain

$$\hat{g}_1(\xi) = \frac{-\hat{h}_1(\xi)\psi_2(\xi) - \hat{h}_2(\xi)}{\gamma_1(\psi_1(\xi) - \psi_2(\xi))}, \quad \hat{g}_2(\xi) = \frac{\hat{h}_1(\xi)\psi_1(\xi) + \hat{h}_2(\xi)}{\gamma_2(\psi_1(\xi) - \psi_2(\xi))} \tag{62}$$

In order to find g_1 and g_2 we now need some conditions. Even with h_j in $\mathcal{S}(\mathbb{R}^n)$ we cannot expect $\hat{g}_1$ or $\hat{g}_2$ to belong to $\mathcal{S}(\mathbb{R}^n)$, however, $\hat{g}_1$ and $\hat{g}_2$ need only to be in $L^2(\mathbb{R}^n)$ in order to find g_1 and g_2 in $L^2(\mathbb{R}^n)$ too. The holomorphy of $(T_t^{(j)})_{t\geqslant 0}$ then implies that $T_t^{(j)}g_j \in \bigcap_{k\in\mathbb{N}} D([\psi_j(D)]^k) = \bigcap_{k\in\mathbb{N}} H^{\psi_j,2k}(\mathbb{R}^n)$, hence we can achieve sufficient regularity to obtain a solution of (56). What becomes obvious is that a trade-off between the behaviour of the zeroes of $\psi_1-\psi_2$ and the zeroes of $\hat{h}_1\psi_j \mp \hat{h}_2$ is now needed to determine g_1 and g_1 uniquely. This shall not surprise us, in general we shall not expect (56) and (57) to have a unique (non-negative) solution of the type (58). Given our initial question, it is natural to change the point of view and to start with $\gamma_1, \gamma_2 \geqslant 0$ as well as with $g_1, g_2 \geqslant 0$ and to use (57) to determine conditions for h_1 and h_2. In this case, h_1 is already determined by (59) as is h_2 determined by (60). We introduce the mapping

$$\begin{aligned} S : H^{\psi_1,2}(\mathbb{R}^n) \times H^{\psi_2,2}(\mathbb{R}^n) &\to L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \\ (g_1, g_2) \mapsto S(g_1, g_2) &= (\gamma_1 g_1 + \gamma_2 g_2, -\gamma_1\psi_1(D)g_1 - \gamma_2\psi_2(D)g_2) \\ &= (F^{-1}(\gamma_1\hat{g}_1 + \gamma_2\hat{g}_2), F^{-1}(-\gamma_1\psi_1\hat{g}_1 - \gamma_2\psi_2\hat{g}_2)) \end{aligned} \tag{63}$$

and by construction, essentially the range of $S|_{H^{\psi_1,2}\times H^{\psi_2,2}\cap L^2_+\times L^2_+}$ will consist of exactly those elements (h_1, h_2) for which we can find (g_1, g_2) such that $u(t,x) = \gamma_1 T_t^{(1)}g_1(x) + \gamma_2 T_t^{(2)}g_2(x)$ is a non-negative solution to (56). Since S is linear and $H^{\psi_1,2} \times H^{\psi_2,2} \cap L^2_+ \times L^2_+$ is convex the range of $S|_{H^{\psi_1,2}\times H^{\psi_2,2}\cap L^2_+\times L^2_+}$ is convex too and it always contains the zero function. The range of $\tilde{S} := S|_{H^{\psi_1,2}\times H^{\psi_2,2}\cap L^2_+\times L^2_+}$ can be characterised in more detail. Since by assumption $g_j \geqslant 0$ and $g_j \in L^2(\mathbb{R}^n)$ its Fourier transform $\hat{g}_j$ must be a positive definite distribution belonging to $L^2(\mathbb{R}^n)$. Thus we have

$$\begin{aligned} R(\tilde{S}) = \{(F^{-1}(\gamma_1 w_1 + \gamma_2 w_2), F^{-1}(-\gamma_1\psi_1 w_1 - \gamma_2\psi_2 w_2)) | \gamma_1, \gamma_2 \geqslant 0, \\ w_1, w_2 \in L^2(\mathbb{R}^n) \text{ positive definite}, \psi_j w_j \in L^2(\mathbb{R}^n)\}. \end{aligned} \tag{64}$$

Thus we have

Proposition 1. *For $(h_1, h_2) \in R(\tilde{S})$ there exists $(g_1, g_2) \in (H^{\psi_1,2} \times H^{\psi_2,2}) \cap (L^2_+ \times L^2_+)$ such that $v(t,x) = (T_t^{(1)}g_1)(x) + (T_t^{(2)}g_2)(x) \geqslant 0$. If in addition $v(t,\cdot), t > 0$ belongs to $\{u \in L^2(\mathbb{R}^n) | (1 + \psi_1(D))(1 + \psi_2(D))u \in L^2(\mathbb{R}^n)\}$ then v is a non-negative solution to* (56) *and* (57).

Remark 1. We may introduce the space $H^{\psi_1,\psi_2,s}(\mathbb{R}^n)$ as the space of all elements in $L^2(\mathbb{R}^n)$ such that

$$\|u\|^2_{\psi_1,\psi_2,s} = \int_{\mathbb{R}^n} (1+\psi_1(\xi))^s (1+\psi_2(\xi))^s |\hat{u}(\xi)|^2 \,\mathrm{d}\xi < \infty$$

and replace in Proposition 1 the condition $(g_1, g_2) \in (H^{\psi_1,2} \times H^{\psi_2,2}) \cap (L^2_+ \times L^2_+)$ by $(g_1, g_2) \in (H^{\psi_1,\psi_2,2} \times H^{\psi_1,\psi_2,2}) \cap (L^2_+ \times L^2_+)$. A more practical, but less sharp condition would be $(g_1, g_2) \in (H^4 \times H^4) \cap (L^2_+ \times L^2_+)$, and in the case where ψ_j satisfies $|\psi_j(\xi)| \leqslant c_{\psi_j}(1+|\xi|^2)^{m_j}$, $0 < m_j < 1$, instead of the estimate $|\psi_j(\xi)| \leqslant c_{\psi_j}(1+|\xi|^2)$ we may require $(g_1, g_2) \in (H^{2(m_1+m_2)} \times H^{2(m_1+m_2)}) \cap (L^2_+ \times L^2_+)$.

We next want to look at the case where $A_1 = -\psi(D)$ is a generator of a symmetric sub-Markovian semigroup, but A_2 is not. We assume that A_2 is of the type $q(D)$ with q satisfying (42). A positive solution to (56) with $\psi_2(D)$ being replaced by $-A_2$ is now sought in the form

$$v(t,x) = T^{(1)}_t g_1(x), \tag{65}$$

since $\gamma_1 \neq 0$ is needed we now may chose $\gamma_1 = 1$, hence we put in (58) $\gamma_2 = 0$ and $\gamma_1 = 1$. This leads to

$$\hat{g}_1(\xi) = \hat{h}_1(\xi) \quad \text{and} \quad \psi_1(\xi)\hat{g}_1(\xi) = -\hat{h}_2(\xi) \tag{66}$$

or $\hat{h}_2(\xi) = -\psi_1(\xi)\hat{h}_1(\xi)$. Thus we may obtain positive solutions to (56) and (57) if $h_1 \in L^2(\mathbb{R}^2)$ is a positive definite distribution such that $\psi_1 \hat{h}_1 \in L^2(\mathbb{R}^n)$ and if in addition we have $\hat{h}_2 = -\psi_1 \hat{g}_1$, i.e. $h_2 = -\psi_1(D) g_1$. However, as an additional condition we need $\psi_1(D) T^{(1)}_t g \in D(A_2)$, for which $g \in H^{m+2}(\mathbb{R}^n)$ is a sufficient condition.

Eventually we want to switch from (56) and (57) to

$$\left(\frac{\partial}{\partial t} + \psi_1(D)\right)\left(\frac{\partial}{\partial t} + \psi_2(D)\right) Bu(t,x) = 0 \tag{67}$$

or

$$\left(\frac{\partial}{\partial t} + \psi_1(D)\right)\left(\frac{\partial}{\partial t} - A_2\right) Bu(t,x) = 0 \tag{68}$$

under the initial conditions

$$\left.\begin{aligned} u(0,x) &= h_1(x) \\ \frac{\partial u}{\partial t}(0,x) &= h_2(x). \end{aligned}\right\} \tag{69}$$

Here $B = q(D)$ is a pesudo-differential operator with symbol $q(\xi)$ satisfying (42). For $g_1, g_2 \in H^{m+4}(\mathbb{R}^n)$ the operators $\psi_1(D), \psi_2(D)$ and $q(D)$ mutually commute and hence we may search for the solutions of the type

$$v(t,x) = \gamma_1 T^{(1)}_t g_1(x) + \gamma_2 T^{(2)}_t g_2(x) \tag{70}$$

or

$$v(t,x) = T_t^{(1)} g_1(x), \tag{71}$$

respectively. This implies that all of our previous considerations carry over to the new case, however we need to add additional assumptions, i.e. domain conditions. For the case of equation (67) the precise condition is of course

$$\gamma_1 \psi_1(D) T_t^{(1)} g_1 + \gamma_2 \psi_2(D) T_t^{(2)} g_2 \in D(B), \tag{72}$$

and only if $D(B)$ is better known, say as an anisotropic Bessel potential space, we can say more. In the best case we would expect $D(B) = H^{q,2}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) | q(D) \in L^2(\mathbb{R}^n)\}$ and then we can give more detailed conditions.

We now consider operators of the type (30) where we assume that for some $L \leqslant N$ the operators A_j, $1 \leqslant j \leqslant L$, have an extension from $\mathcal{S}(\mathbb{R}^n)$ to a generator of a holomorphic sub-Markovian semigroup $(T_t)_{t \geqslant 0}$ on $L^2(\mathbb{R}^n)$. Our aim is to find solutions to (31) of the type

$$v(t,x) = \sum_{j=1}^{L} (\gamma_j T_t^{(j)} g_j)(x), \quad \gamma_j \geqslant 0 \text{ and } g_j \geqslant 0 \text{ in } L^2(\mathbb{R}^n). \tag{73}$$

In addition, we add the initial conditions (29). It is clear that in this generality we cannot obtain existence or uniqueness results. Most of all we need to consider carefully domains of suitable extensions of the operators A_j, $L < j \leqslant N$, and B_l, $1 \leqslant l \leqslant M$, and further, on some suitable common domain we need the commutator relations $[A_j, A_l] = 0$ and $[A_j, B_l]$ to hold. We do not want to follow the general abstract case, but we want to assume that all operators involved are translation invariant pseudo-differential operators of the type (43). More precisely, for $1 \leqslant j \leqslant L$ we assume that $A_j = -\psi_j(D)$ where $\psi_j : \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function and for $L < j \leqslant N$, as well as for $1 \leqslant l \leqslant M$, we assume that the symbols of the operators A_j and B_l satisfy (42) for some growth exponent depending on m_j and $\tilde{m}_l$ respectively. We put

$$m := 2L + \sum_{j=L+1}^{N} m_j + \sum_{l=1}^{M} \tilde{m}_j, \tag{74}$$

and we consider all operators on $H^m(\mathbb{R}^n)$. It follows that on $H^m(\mathbb{R}^n)$ any composition of operators $A_{j_1} \circ \cdots \circ A_{j_k} \circ B_{l_1} \circ \cdots \circ B_{l_j}$, $1 \leqslant j_\alpha \leqslant N$, $1 \leqslant l_\beta \leqslant M$, maps $H^m(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$, and any of such a composition for $L < j_\alpha \leqslant N$ maps $H^m(\mathbb{R}^n)$ into $H^{2L}(\mathbb{R}^n)$. Moreover we have $H^m(\mathbb{R}^n) \subset H^{2L}(\mathbb{R}^n)$ and the compositions do not depend on the ordering of the operators. Since by assumption $(T_t^{(j)})_{t \geqslant 0}$, $1 \leqslant j \leqslant L$, extends to a holomorphic semigroup we have for every $g \in L^2(\mathbb{R}^n)$ that $T_t^{(j)} g \in \bigcap_{k \in \mathbb{N}} H^{2k,\psi_j}(\mathbb{R}^n)$, $t > 0$. In order to guarantee that $T_t^{(j)} g \in H^m(\mathbb{R}^n)$, $t > 0$, and hence that all operators $A_{j_1} \circ \cdots \circ A_{j_k} \circ B_{l_1} \circ \cdots \circ B_{l_i}$ commute with $T_t^{(j)}$, $t > 0$, we add the assumption

$$(1 + \psi_j(\xi)) \geqslant \kappa_0 (1 + |\xi|^2)^{\frac{m_j'}{2}}, \quad \kappa_0 > 0, m_j' > 0, j = 1, \ldots, L. \tag{75}$$

Now it follows that for every collection $g_j \in L^2(\mathbb{R}^n)$, $1 \leqslant j \leqslant L$, a solution to (31) is given by

$$v(t,x) := \sum_{j=1}^{L} (\gamma_j T_t^{(j)} g_j)(x), \quad t > 0, \tag{76}$$

and for $\gamma_j \geqslant 0$, $g_j \geqslant 0$ in $L^2(\mathbb{R}^n)$ this solution is non-negative.

Next we want to adjust the initial conditions. For $g_j \in H^{2L}(\mathbb{R}^n)$, or even $g_j \in H^{2(m'_1+\cdots+m'_L)}(\mathbb{R}^n)$, due to the holomorphy of the semigroups $(T_t^{(j)})_{t\geqslant 0}$ we find for $1 \leqslant k \leqslant N$

$$v^{(k)}(0,x) = \sum_{j=1}^{L} (\gamma_j A_j^{k-1} g_j)(x) = h_k(x) \quad (\text{in } L^2(\mathbb{R}^n)). \tag{77}$$

Switching to the Fourier transforms we obtain the following system of N equations for the L unknown functions $\hat{g}_j$:

$$\sum_{j=1}^{L} \gamma_j (-\psi_j(\xi))^{k-1} \hat{g}_j(\xi) = \hat{h}_k(\xi), \quad 1 \leqslant k \leqslant N. \tag{78}$$

Once more, we change our point of view and we consider (78) as conditions for the initial values $h_1, \ldots, h_N$ to hold in order that (31) under (29) admits a non-negative solution.

We introduce the mappings S and $\tilde{S}$ analogously to (63) by

$$\begin{aligned} S : (H^m \times \cdots \times H^m) &\to L^2 \times \cdots \times L^2 \\ g := (g_1, \ldots, g_L) &\mapsto Sg := (h_1, \ldots, h_N), \end{aligned} \tag{79}$$

where

$$h_k = (Sg)_k := \sum_{j=1}^{L} \gamma_j (-\psi_j(D)))^{k-1} g_j, \quad 1 \leqslant k \leqslant N, \tag{80}$$

i.e.

$$\hat{h}_k = (Sg)^\wedge{}_k = \sum_{j=1}^{L} \gamma_j (-\psi_j)^{k-1} \hat{g}_j. \tag{81}$$

If by assumption $g_j \geqslant 0$, $1 \leqslant j \leqslant L$, then there exists positive definite distributions $w_j \in L^2(\mathbb{R}^n)$ such that $w_j = \hat{g}_j$ and we find

$$h_k = F^{-1}\left(\sum_{j=1}^{L} \gamma_j (-\psi_j)^{k-1} w_j\right). \tag{82}$$

For the range of $\tilde{S} := S|_{(H^m\times\cdots\times H^m)\cap(L^2_+\times\cdots\times L^2_+)}$ we derive in analogy to (64)

$$R(\tilde{S}) = \left\{ \left(F^{-1}(\sum_{j=1}^{L} \gamma_j w_j), F^{-1}(\sum_{j=1}^{L} \gamma_j (-\psi_j) w_j), \ldots, F^{-1}(\sum_{j=1}^{L} \gamma_j (-\psi_j)^{N-1} w_j) \right) \right| \\ \left. \gamma_1, \ldots, \gamma_L \geqslant 0, w_1, \ldots, w_L \text{ positive definite}, \psi_j^{k-1} w_j \in L^2(\mathbb{R}^n), 1 \leqslant k \leqslant N \right\}. \tag{83}$$

Thus we arrive at

Proposition 2. *For* $(h_1, \ldots, h_N) \in R(\tilde{S})$ *there exists* $(g_1, \ldots, g_L) \in (H^m \times \cdots \times H^m) \cap (L^2_+ \times \cdots \times L^2_+)$ *such that* $v(t,x) := \sum_{j=1}^{L} \gamma_j T_t^{(j)} g_j \geqslant 0$. *If in addition each* $(T_t^{(j)})_{t\geqslant 0}$ *is holomorphic and* (75) *is satisfied, then* v *solves* (31) *under the initial condition* (29).

Remark 2. While $R(\tilde{S})$ is in general difficult to determine, we may of course choose some of the parameters γ_j to be 0 and then the situation becomes more transparent. For example, we may choose $\gamma_j = 0$ for all $j \neq j_0$ for a fixed $j_0 \in \{1, \ldots, L\}$ and $\gamma_{j_0} = 1$. In this case the condition (80) reduces to

$$h_1 = F^{-1} w_{j_0}, h_2 = F^{-1}(-\psi_{j_0} w_{j_0}), \ldots, h_N = F^{-1}(-\psi_{j_0}^{N-1} w_{j_0}).$$

In the following chapter we will turn to concrete partial differential operators with constant coefficients and we will try to find families of non-negative solutions for related initial value problems.

However, we first want to extend our considerations by allowing complex-valued continuous negative definite functions $\psi_j : \mathbb{R}^n \to \mathbb{C}$ as symbols of $A_j = -\psi_j(D)$. The only change in our argument is required to justify that the associated operator semigroup $(T_t^{(j)})_{t\geqslant 0}$ is holomorphic on $L^2(\mathbb{R}^n)$. As the example of the drift, which corresponds to $\psi(\xi) = -i\xi, n = 1$, shows us that we cannot expect for a general complex-valued continuous negative definite function ψ_j the semigroup $(T_t^{(j)})$ to be holomorphic. However, in the case where ψ satisfies the sector condition

$$|\mathrm{Im}\,\psi(\xi)| \leqslant \kappa_0 \mathrm{Re}\,\psi(\xi), \quad \kappa_0 > 0, |\xi| \geqslant R \geqslant 0, \tag{84}$$

it follows that $-\psi(D)$ is a sectorial operator and hence the generator of a holomorphic semigroup on $L^2(\mathbb{R}^n)$, see [6] or [10]. Moreover, since $\mathrm{Re}\,\psi$ is a continuous negative function too, we can form the spaces $H^{\mathrm{Re}\,\psi,s}(\mathbb{R}^n)$. Thus replacing in our previous considerations the real-valued continuous negative definite functions by complex-valued continuous negative definite functions each satisfying the sector condition and using the spaces $H^{\mathrm{Re}\,\psi,s}(\mathbb{R}^n)$ with $\mathrm{Re}\,\psi$ satisfying (where appropriate) additional conditions such as (84), we obtain the previous results in the more general situation. For more details we refer to [1] and [4].

4 Higher Order Partial Differential Equations Admitting Non-negative Solutions

We now turn from operator-valued differential operators to partial differential equations of the type

$$\frac{\partial^N}{\partial t^N} u(t,x) - \sum_{j=0}^{N-1} \sum_{|\alpha|\leqslant m} a_{j\alpha} \frac{\partial^j}{\partial t^j} \left(-i\frac{\partial}{\partial x}\right)^{\alpha} u(t,x) = 0, \quad a_{jl} \in \mathbb{R}, \tag{85}$$

and we ask when does such an equation admit a solution given by

$$v(t,x) = \sum_{j=1}^{L} (\gamma_j T_t^{(j)} g_j)(x), \quad \gamma_j \geqslant 0, g_j \geqslant 0, L \leqslant N, \tag{86}$$

where $g_j \in L^2(\mathbb{R}^n)$ and $(T_t^{(j)})_{t\geqslant 0}, 1 \leqslant j \leqslant L$, is an L^2-sub-Markovian semigroup. When taking in (85) the Fourier transform with respect to x we arrive at the parameter dependent ordinary differential equation

$$\frac{\mathrm{d}^N}{\mathrm{d}t^N}\hat{u}(t,\xi) - \sum_{j=0}^{N-1} \sum_{|\alpha|\leqslant m} a_{j\alpha}\xi^\alpha \frac{\mathrm{d}^j}{\mathrm{d}t^j}\hat{u}(t,\xi) = 0. \tag{87}$$

We long for solutions of (87) of the form

$$\hat{u}(t,\xi) = \hat{v}_k(t,\xi) = e^{-\psi_k(\xi)t} \tag{88}$$

where $\psi_k : \mathbb{R}^n \to \mathbb{C}$ is a continuous negative definite function satisfying the sector condition and $\operatorname{Re}\psi_k$ satisfies the growth condition (75). From (87) we arrive (with $\lambda(\xi) = \psi_k(\xi)$) at the **characteristic equation**

$$\lambda^N(\xi) + \sum_{j=0}^{N-1} \sum_{|\alpha|\leqslant m} a_{j\alpha}(-1)^{N-j-1}\xi^\alpha \lambda^j(\xi) = 0 \tag{89}$$

for which we seek solutions $\lambda_k = \lambda_k(\xi)$ which are continuous negative definite, satisfying the sector condition and the real part of which satisfies (75). Every such solution will give rise to a holomorphic sub-Markovian semigroup $(T_t^{(k)})_{t\geqslant 0}$ associated with λ_k by

$$(T_t^{(k)}u)^\wedge(\xi) = e^{-t\lambda_k(\xi)}\hat{u}(\xi) \tag{90}$$

and we may apply the considerations of the previous chapters to obtain non-negative solutions of the type (86) for the equation (85). The problem is of course to find such solutions λ to (89). Even in the cases where we can obtain solutions with the help of radicals, it is not clear which properties the function $\xi \mapsto \lambda_k(\xi)$ will have. So far we have no general answer to our problem, however the following examples show the scope of our considerations. It is clear that if we obtain solutions of the type (86) the function $\lambda_k(\xi)$ in (90) must be a continuous negative function satisfying the sector condition, provided we assume that $(T_t^{(j)})_{t\geqslant 0}$ to be holomorphic. We prefer to provide some rather concrete examples, but in each case it is obvious that we can include more general and complicated cases with similar symbol structure.

Example 1. **A.** We take in (85) the dimension $n = 1$ and the values $N = 2$, $m = 4$ and $a_{j\alpha} = \delta_{0,4}$. Then we are dealing with the equation

$$\frac{\partial^2}{\partial t^2}u(t,x) - \frac{\partial^4}{\partial x^4}u(t,x) = 0 \tag{91}$$

which yields

$$\lambda^2 - \xi^4 = 0. \tag{92}$$

Since $\lambda - \xi^4 = (\lambda - \xi^2)(\lambda + \xi^2)$ we have the continuous negative definite function $\lambda_1(\xi) = \xi^2$ as a solution which satisfies all our conditions, and for $g \in L^2(\mathbb{R})$, $g \geqslant 0$, a non-negative solution to (91) is given by $(x,t) \mapsto (T_t^G g)(x)$, where $(T_t^G)_{t\geqslant 0}$ is the Gaussian semigroup on $L^2(\mathbb{R})$.

B. Taking next $N = 4$ and $m = 2$ in (85), but again $n = 1$, and further $a_{j\alpha} = \delta_{0,2}$, we get the equation

$$\frac{\partial^4}{\partial t^4} u(t,x) + \frac{\partial^2}{\partial x^2} u(t,x) = 0 \tag{93}$$

which leads to $\lambda^4 - \xi^2 = 0$. Obviously $\lambda(\xi) = |\xi|^{\frac{1}{2}}$ is a solution of this equation and this is a continuous negative definite function which fulfills all of our requirements. The associated semigroup $(T_t^\lambda)_{t\geqslant 0}$ is the semigroup subordinate to the Gaussian semigroup with the help of the Bernstein function $f(s) = s^{\frac{1}{4}}$. The polynomial $\lambda^4 - \xi^2$ admits the factorisation $\lambda^4 - \xi^2 = (\lambda - |\xi|^{\frac{1}{2}})(\lambda + |\xi|^{\frac{1}{2}})(\lambda - i|\xi|^{\frac{1}{2}})(\lambda + i|\xi|^{\frac{1}{2}})$ and therefore only one solution of $\lambda^4 - \xi^2 = 0$ is a continuous negative definite function as sought.

C. Now we take $a_{j\alpha} = \delta_{0,2m}$ as coefficients for $n = 1$ and $2N, 2m \in \mathbb{N}$ and hence (89) becomes $\lambda^{2N} = |\xi|^{2m}$. Further, by $\lambda = |\xi|^{\frac{m}{N}}$ we always have for $\frac{m}{N} < 2$, i.e. $m \leqslant 2N$, a continuous negative definite function as a solution satisfying all of our conditions. We can phrase this differently, namely that for $n = 1$ to every α-stable process $(X_t^{(\alpha)})_{t\geqslant 0}$ with α rational we can find a partial differential equation of $\frac{\partial^{2N}}{\partial t^{2N}} u(t,x) = \frac{\partial^{2m}}{\partial x^{2m}}, u(t,x) = 0$ such that the transition function of $(X_t^{(\alpha)})_{t\geqslant 0}$ gives the solution to that equation. We refer to [11] where (fractional) differential equations being solved by transition functions of certain stable processes, i.e. densities of certain convolution semigroups, are discussed.

Our next examples show that there are more than just symmetric stable semigroups which give solutions of the type (86). We still assume $n = 1$.

Example 2. **A.** Consider the differential operator $\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + a\frac{\partial}{\partial x}$, where $a \in \mathbb{R}$ is a parameter. This operator leads to the characteristic equation $\lambda^2 - \xi^2 - ia\xi = 0$ which we can factorise according to $\lambda^2 - \xi^2 - ia\xi = (\lambda - (\xi^2 + ia\xi)^{\frac{1}{2}})(\lambda + (\xi^2 + ia\xi)^{\frac{1}{2}})$. The function $\xi \mapsto \xi^2 + ia\xi$ is a continuous negative definite function for every $a \in \mathbb{R}$. Since $s \mapsto f(s) = s^{\frac{1}{2}}$ is a Bernstein function, it follows that $\xi \mapsto \lambda(\xi) = (\xi^2 + ia\xi)^{\frac{1}{2}}$ is a continuous negative definite function too. Moreover, since $\operatorname{Re}\lambda^2(\xi) = \xi^2$ and $\operatorname{Im}\lambda^2(\xi) = a\xi$, it follows that $\lambda^2(\xi)$ fulfills the sector condition as well as the growth condition (75). Hence the semigroup generated by the differential operator with symbol $\lambda^2(\xi)$ is on $L^2(\mathbb{R})$ holomorphic which is inherited by the semigroup obtained by subordination with the help of the Bernstein function f. In addition, since $|\lambda^{\frac{1}{2}}(\xi)| = (\xi^4 + a\xi^2)^{\frac{1}{2}}$ the growth condition (75) is fulfilled too. Thus for $g \geqslant 0$, $g \in L^2(\mathbb{R})$, a non-negative solution to $\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} = 0$ is given by $u(t,x) := F^{-1}_{\xi\mapsto x}(r^{-(\xi^2+ia\xi)^{\frac{1}{2}}t}\hat{g}(\xi))(x)$.

B. The wave operator $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)$ needs a more careful discussion. The characteristic equation $\lambda^2 + \xi^2 = 0$ admits the factorisation $(\lambda - i\xi)(\lambda + i\xi)$. Although $\xi \mapsto \pm i\xi$ are continuous negative definite functions, we cannot apply our considerations since these functions do not satisfy the sector condition and hence the corresponding pseudo-differential operators are not sectorial, hence do not generate a holomorphic semigroup.
C. We want to investigate the operator $\frac{\partial^2}{\partial t^2} + a\frac{\partial}{\partial t} + b\frac{\partial^2}{\partial x^2}$ with $a, b \in \mathbb{R}$. This gives the characteristic equation $\lambda^2 - a\lambda - b\xi^2 = 0$ with solutions $\lambda_{1,2} = \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 + 4b\xi^2}$. For $a > 0$, $b > 0$ the function $\lambda_1(\xi) = \frac{a}{2} + \frac{1}{2}(a^2 + 4b\xi^2)^{\frac{1}{2}}$ is a continuous negative definite function satisfying the sector as well as growth conditions.

Remark 3. It is easy to see that if a continuous negative definite solution to a one-dimensional ($n = 1$) characteristic equation depends only on ξ^2, then we can handle the $n-$dimensional case when replacing ξ^2 by $|\xi|^2$, i.e. $-\frac{\partial^2}{\partial x^2}$ by $-\Delta_n$.

Example 3. In Example 4.1.C. the operator $\frac{\partial^N}{\partial t^N} - (-1)^m \frac{\partial^{2m}}{\partial x^{2m}}$ was discussed and we want to extend our considerations to the case $\frac{\partial^N}{\partial t^N} - \frac{\partial^m}{\partial x^m}$. This entails the characteristic equation $\lambda^N - (-i\xi)^m = 0$ and we always have a solution $\lambda = (-i\xi)^{\frac{m}{N}}$. For $m \leqslant N$ this is a continuous negative definite function since $\xi \mapsto -i\xi$ is one and $s \mapsto s^{\frac{m}{N}}$, $s \geqslant 0$, $m \leqslant N$, is a Bernstein function. However, for $m, N \in \mathbb{N}$, $m < N$, we find $(-i\xi)^{\frac{m}{N}} = |\xi|^{\frac{m}{N}} e^{-i\frac{m}{N}\pi}$, or

$$(-i\xi)^{\frac{m}{N}} = |\xi|^{\frac{m}{N}}(\cos \tfrac{m}{N}\pi - i \sin \tfrac{m}{N}\pi)$$

which gives

$$|\mathrm{Im}\,(-i\xi)^{\frac{m}{N}}| = |\xi|^{\frac{m}{N}} \sin \tfrac{m}{N}\pi = |\xi|^{\frac{m}{N}} \tan(\tfrac{m}{N}\pi) \cos \tfrac{m}{N}\pi = c_{m,N} \mathrm{Re}\,(-i\xi)^{\frac{m}{N}},$$

where $c_{m,N} > 0$ for $0 < m < N$. Thus $\xi \mapsto (-i\xi)^{\frac{m}{N}}$ fulfills the sector condition as well as the growth condition and for $0 < m < N$ our previous results apply to $\frac{\partial^N}{\partial t^N} - \frac{\partial^m}{\partial x^m}$. Note that Example 4.1.C. extends by Remark 3 to the case $\frac{\partial^N}{\partial t^N} - (-\Delta_n)^m$, but an extension of the example $\frac{\partial^N}{\partial t^N} - \frac{\partial^m}{\partial x^m}$ to higher dimensions is not obvious. For more properties of the one-dimensional drift operator in relation to fractional derivatives we refer to [5].

We now have a look at the Laplace operator in the half-space $\mathbb{R}_+ \times \mathbb{R}^n, n \geqslant 1$.

Example 4. The operator is of course $\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ and we treat the variable $t \geqslant 0$ differently to the variable $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The characteristic equation becomes $\lambda^2 - |\xi|^2 = 0$ which we can factorise according to $(\lambda - |\xi|)(\lambda + |\xi|)$. The function $\xi \mapsto |\xi|$ is of course a continuous negative definite function satisfying all of our conditions. The corresponding operator semigroup is the Cauchy semigroup and the result will lead us to the Poisson formula for the Laplacian in the half-space, see [9] and our introduction.

Example 5. We may now use our previous examples to study higher order equations in several space dimensions such as $\frac{\partial^4}{\partial t^4} - \frac{\partial^4}{\partial t^2 \partial y^2} + \frac{\partial^3}{\partial t^2 \partial x} - \frac{\partial^3}{\partial y^3}$ the characteristic equation of which is

$$\lambda^4 - i\lambda^2\eta - \lambda^2|\xi|^2 - i\eta|\xi|^2 = (\lambda - (i\eta)^{\frac{1}{2}})(\lambda - |\xi|)(\lambda + (i\eta)^{\frac{1}{2}})(\lambda + |\xi|).$$

The function $\psi_1(\eta) = (i\eta)^{\frac{1}{2}}$ and $\psi_2(\xi) = |\xi|$ are continuous negative definite functions in $\mathbb{R}$, both satisfying all of our conditions on $\mathbb{R}$ and hence the corresponding semigroup $(T_t^{(j)})_{t\geqslant 0}$, $j = 1, 2$, are holomorphic sub-Markovian semigroups on $L^2(\mathbb{R})$. However we cannot expect these semigroups to be holomorphic on $L^2(\mathbb{R}^2)$ when associated with $\varphi_1(\xi,\eta) = \psi_1(\eta)$ or $\varphi_2(\xi,\eta) = \psi_2(\xi)$, respectively. Nonetheless, all of our results still apply provided the data $g_1, g_2 \in L^2(\mathbb{R}^2)$ when forming $u(t,x,y) = (T_t^{(1)} g_1)(x,y) + (T_t^{(2)} g_2)(x,y)$, where $(T_t^{(j)} g_j)^\wedge(\cdot,\cdot) = e^{-\varphi_j(\cdot)t}\hat{g}_j(\cdot,\cdot)$, provided the data g_1 and g_2 are sufficiently smooth.

These examples demonstrate the scope of our results as they show how to construct many further ones. However, the central question "How many continuous negative definite solutions does the characteristic equation admit?" is for the general case open, which of course should not be a surprise. In particular, we want to point out that in higher dimensions, i.e. $n \geqslant 2$, special combinations of terms in the characteristic equation may lead to "unexpected" solutions, similar to the cases where we treat $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ as one variable $|\xi|^2$, or where $(\xi, \eta) \mapsto (\xi^2 - i\eta)^{\frac{1}{2}}$ is treated as one variable when solving the characteristic equation.

In light of the results in [3], handling equations of the type (85) with t-dependent coefficients would be of great interest.

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