



# Well-Posedness for Stochastic Fractional Navier–Stokes Equation in the Critical Fourier–Besov Space

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## Abstract

The well-posedness of stochastic Navier–Stokes equations with various noises is a hot topic in the area of stochastic partial differential equations. Recently, the consideration of stochastic Navier–Stokes equations involving fractional Laplacian has received more and more attention. Due to the scaling-invariant property of the fractional stochastic equations concerned, it is natural and also very important to study the well-posedness of stochastic fractional Navier–Stokes equations in the associated critical Fourier–Besov spaces. In this paper, we are concerned with the three-dimensional stochastic fractional Navier–Stokes equation driven by multiplicative noise. We aim to establish the well-posedness of solutions of the concerned equation. To this end, by utilising the Fourier localisation technique, we first establish the local existence and uniqueness of the solutions in the critical Fourier–Besov space  $\dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}}$ . Then, under the condition that the initial data is sufficiently small, we show the global existence of the solutions in the probabilistic sense.

**Keywords** Stochastic fractional Navier–Stokes equation · Fourier–Besov spaces · Strong solutions

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## 1 Introduction

In this paper, we are concerned with the following three-dimensional stochastic incompressible fractional Navier–Stokes equation

$$\begin{cases} du + [(-\Delta)^\alpha u + u \cdot \nabla u + \nabla \pi] dt = \sum_{k \geq 1} g_k(t, u) dB_k, & t > 0 \\ \operatorname{div} u = 0, \\ u(0) = u_0, \end{cases} \quad (1.1)$$

for unknown random field  $u = (u_1, u_2, u_3) \in \mathbb{R}^3$  representing the velocity of a fluid, where  $\pi$  stands for the pressure and the fractional Laplace operator  $(-\Delta)^\alpha$ ,  $\alpha \in (0, 1]$  is the Fourier multiplier with symbol  $|\xi|^{2\alpha}$ ,  $g_k, k \geq 1$  are jointly measurable coefficients,  $\{B_k, k \geq 1\}$  is a sequence of one-dimensional independent Brownian motions defined in a given completed filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  (see e.g. [9]). When  $\alpha = 1$ , the equation (1.1) becomes the well-known stochastic Navier–Stokes equation (SNS in short). The SNS has been intensively studied due to its feature simulation for fluid flow dynamics. Especially, Holz and Ziane [8] obtained the local well-posedness of the strong solution for the multiplicative SNS in bounded domains when the initial data are in  $H^1$ . Sritharan and Sundar [13] established Wentzell-Freidlin-type large deviation principle for the two-dimensional SNS with multiplicative Gaussian noise. Caraballo, Langa and Taniguchi [2] proved that the weak solutions for the two-dimensional SNS converge exponentially in the mean square and almost surely exponentially to the stationary solutions. Xu and Zhang [18] discussed the small time asymptotics of two-dimensional SNS in the state space  $C([0, T], H)$ . Recently, the study of well-posedness for the SNS in Besov spaces has attracted the interest of many scholars. In particular, Du and Zhang [6] obtained local and global existence of strong solutions for the SNS in the critical Besov space  $\dot{B}_{p,r}^{\frac{d}{p}-1}$ . Chang and Yang [3] studied the initial-boundary value problem of the SNS in the half space.

If  $g := \{g_k, k \geq 1\} \equiv 0$ , the system (1.1) reduces to incompressible fractional Navier–Stokes equations (FNS in short). It is scaling invariant under certain changes of spatial and temporal variables. To be more precise, one has the following

$$u_\lambda(t, x) = \lambda^{2\alpha-1} u(\lambda^{2\alpha} t, \lambda x), \quad \pi_\lambda(t, x) = \lambda^{4\alpha-2} \pi(\lambda x, \lambda^{2\alpha} t).$$

This scaling invariant property naturally leads to the definition of the critical space for the equation (1.1). Recall that a functional space  $X$  endowed with norm  $\|\cdot\|_X$  is critical for the equation (1.1) if it satisfies  $\|\varphi_\lambda\|_X = \|\varphi\|_X$ , where  $\varphi_\lambda(x) = \lambda^{2\alpha-1} \varphi(x)$ . To date, the well-posedness for FNS has been studied by many scholars in different critical spaces. For example, Wu [17] studied the well-posedness for FNS in the Besov space  $\dot{B}_{p,r}^{1-2\alpha+\frac{3}{p}}$ , Wang and Wu [15] proved the global well-posedness of mild solution and Gevrey class regularity for FNS in the Lei Lin space  $\chi^{1-2\alpha}$ , Ru and Abidin [12] discussed the global well-posedness for FNS in the variable exponent Fourier–Besov

spaces  $\dot{B}_{p(\cdot),r}^{4-2\alpha-\frac{3}{p(\cdot)}}$ , just mention a few. More discussions on the global well-posedness can be found in [4,10] (and references therein).

Motivated by the above investigations, in this paper, we want to study the well-posedness of the equation (1.1) in the critical Fourier–Besov space  $\dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}}$ , aiming to extend the well-posedness results of [12,16]. To this end, we first derive the local well-posedness for the equation (1.1) in the space  $\dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}}$ . Then for sufficiently small initial data, we show that the solution is global in the probabilistic sense.

The rest of the paper is organised as follows. In Section 2, we briefly recall some harmonic analysis tools including the Littlewood–Paley theory and the definition of the Fourier–Besov space. Section 3 is devoted to establishing our main well-posedness results.

## 2 Preliminaries

In this section, we recall the homogeneous Littlewood–Paley decomposition and the definition of Fourier–Besov spaces. For more details, the reader is referred to [1].

Let  $\mathcal{S}(\mathbb{R}^d)$ ,  $d \geq 1$ , be the Schwarz space of all smooth functions that are rapidly decreasing infinite functions along with all partial derivatives and  $\mathcal{S}'(\mathbb{R}^d)$  be the space of tempered distributions. Let the dual pairing between  $\mathcal{S}(\mathbb{R}^d)$  and  $\mathcal{S}'(\mathbb{R}^d)$  be denoted by  $\langle \cdot, \cdot \rangle$ . For any  $f \in \mathcal{S}(\mathbb{R}^d)$ , the Fourier transform and the inverse Fourier transform of  $f$  are defined, respectively, by

$$\begin{aligned} \mathcal{F}(f)(\xi) &:= \hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(x) dx, \\ \mathcal{F}^{-1}(f)(\xi) &:= \check{f}(x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(\xi) d\xi. \end{aligned}$$

Furthermore, for any  $\phi \in \mathcal{S}'(\mathbb{R}^d)$ , we define its Fourier transform and inverse Fourier transform as

$$\langle \hat{\phi}, f \rangle := \langle \phi, \hat{f} \rangle, \quad \langle \check{\phi}, f \rangle := \langle \phi, \check{f} \rangle, \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

Let  $\mathcal{C} := \{ \xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \}$  and  $\mathcal{D}(\mathcal{C})$  be the space of all test functions on  $\mathcal{C}$ , that is, the totality of all smooth functions on  $\mathcal{C}$  that have compact support. Then there exists a non-negative radial function  $\varphi \in \mathcal{D}(\mathcal{C})$ , such that

$$\sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1, \quad \forall \xi \in \mathbb{R}^d \setminus \{0\},$$

where  $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ . The homogeneous dyadic blocks is defined in the following manner

$$\dot{\Delta}_j u := \check{\varphi}_j * u, \quad j \in \mathbb{Z}.$$

We further set the quotient space  $\mathcal{S}'_h(\mathbb{R}^d) := \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}$ , where  $\mathcal{P}$  is the space of all polynomials. For any  $u \in \mathcal{S}'_h(\mathbb{R}^d)$ , we define

$$\dot{S}_j u := \sum_{j' \leq j-1} \dot{\Delta}_{j'} u.$$

We have

**Definition 1** Let  $1 \leq p, r \leq \infty, s \in \mathbb{R}$ , the homogeneous Fourier–Besov space is defined as

$$\dot{B}_{p,r}^s = \left\{ u \in \mathcal{S}'_h(\mathbb{R}^d) : \|u\|_{\dot{B}_{p,r}^s} := \left\{ 2^{js} \|\widehat{\dot{\Delta}_j u}\|_{L_\xi^p} \right\}_{l_j^r} < \infty \right\}.$$

Taking the time variable and random variables into account, we also need the following definition of Chemin–Lerner-type spaces, see, [1, 16].

**Definition 2** Let  $1 \leq p, q, r, \sigma \leq \infty, s \in \mathbb{R}, T > 0$ , we define

$$\begin{aligned} \mathcal{L}_T^q \dot{B}_{p,r}^s &:= \left\{ u : \forall a.e. t \in [0, T], u(t, \cdot) \in \mathcal{S}'_h(\mathbb{R}^d), \right. \\ &\quad \left. \|u\|_{\mathcal{L}_T^q \dot{B}_{p,r}^s} := \left\{ 2^{js} \|\widehat{\dot{\Delta}_j u}\|_{L_T^q L_\xi^p} \right\}_{l_j^r} < \infty \right\}; \\ \mathcal{L}_T^q \mathcal{L}_\omega^\sigma \dot{B}_{p,r}^s &:= \left\{ u : \forall a.e. (\omega, t) \in \Omega \times [0, T], u(\omega, t, \cdot) \in \mathcal{S}'_h(\mathbb{R}^d), \right. \\ &\quad \left. \|u\|_{\mathcal{L}_T^q \mathcal{L}_\omega^\sigma \dot{B}_{p,r}^s} := \left\{ 2^{js} \|\widehat{\dot{\Delta}_j u}\|_{L_T^q L_\omega^\sigma L_\xi^p} \right\}_{l_j^r} < \infty \right\}; \\ \mathcal{L}_T^q \mathcal{L}_\omega^\sigma \dot{B}_{p,r}^s &:= \left\{ f = (f_k, k \geq 1) : \forall k \geq 1, f_k \in \mathcal{L}_T^q \mathcal{L}_\omega^\sigma \dot{B}_{p,r}^s, \right. \\ &\quad \left. \|f\|_{\mathcal{L}_T^q \mathcal{L}_\omega^\sigma \dot{B}_{p,r}^s} := \left\{ 2^{js} \|\widehat{\dot{\Delta}_j f_k}\|_{L_T^q L_\omega^\sigma L_k^2 L_\xi^p} \right\}_{l_j^r} < \infty \right\}. \end{aligned}$$

We list here some of the properties that will be used in the sequel. Let  $u, v \in \mathcal{S}'_h(\mathbb{R}^d)$ , then

- $|k - j| \geq 2 \Rightarrow \dot{\Delta}_k \dot{\Delta}_j u = 0$ ;
- $|k - j| \geq 5 \Rightarrow \dot{\Delta}_j (\dot{S}_{k-1} u \dot{\Delta}_k v) = 0$ .
- For  $1 \leq p, r \leq \infty, s < 0$ , it holds that  $u \in \dot{B}_{p,r}^s$  if and only if  $\{2^{js} \|\widehat{\dot{S}_j u}\|_{L_\xi^p}\}_{l_j^r} < +\infty$ .

Finally, we introduce the homogeneous Bony decomposition. For more details, we refer readers to [1, 11] and the references therein. Let  $u, v \in \mathcal{S}'_h(\mathbb{R}^d)$ , then the homogeneous Bony decomposition of  $uv$  is defined by

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v),$$

where  $\dot{T}_u v$  and  $\dot{T}_v u$  denote the homogenous paraproduct of  $v$  by  $u$  and  $u$  by  $v$ , respectively

$$\dot{T}_u v := \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v,$$

$$\dot{T}_v u := \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} v \dot{\Delta}_j u,$$

and  $\dot{R}(u, v)$  denotes the homogeneous remainder of  $u$  and  $v$ :

$$\begin{aligned} \dot{R}(u, v) &:= \sum_{|j-k| \leq 1} \dot{\Delta}_j u \dot{\Delta}_k v \\ &= \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \tilde{\Delta}_j v \end{aligned}$$

with  $\tilde{\Delta}_k := \dot{\Delta}_{k-1} + \dot{\Delta}_k + \dot{\Delta}_{k+1}$ .

### 3 Main Results

Acting the Leray-Hopf operator  $\mathbf{P} := I + \nabla(-\Delta)^{-1} \operatorname{div}$  (see [1, 11]) on equation (1.1), we then have

$$\begin{cases} du + [(-\Delta)^\alpha u + \mathbf{P} \operatorname{div}(u \otimes u)] dt = \sum_{k \geq 1} \mathbf{P} g_k(t, u) dB_k, \\ u(0) = u_0, \end{cases} \tag{3.1}$$

where  $\alpha \in (0, 1]$ . Following [6], we introduce the definition of local and global strong solutions for the equation (1.1).

**Definition 3** Let  $2 \leq p, r \leq \infty$  and the initial data  $u_0 \in \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}}$  be  $\mathcal{F}_0$ -measurable.

- (1)  $(u, \tau_R)$  is called a local strong solution for the equation (1.1), if
- (i)  $u$  is a progressively measurable process and for any  $0 < T < \infty$ ,

$$\begin{aligned} u &\in L^r_\omega \mathcal{L}^\infty_T \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}} \cap L^r_\omega \mathcal{L}^2_T \dot{B}_{p,r}^{4-\alpha-\frac{3}{p}}, \\ \tau_R(\omega) &= \inf \left\{ t \geq 0; \|u\|_{\mathcal{L}^2_T \dot{B}_{p,q}^{4-\alpha-\frac{3}{p}}} \geq R \right\}, \end{aligned}$$

where  $R$  is a positive constant.

- (ii) For almost all  $\omega \in \Omega$ ,  $u(t, x) \in C([0, \tau_R(\omega)); \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}})$ , and the following equality

$$u(t \wedge \tau_R) = u_0 - \int_0^{t \wedge \tau_R} [(-\Delta)^\alpha u + \mathbf{P} \operatorname{div}(u \otimes u)] ds + \sum_{k \geq 1} \int_0^{t \wedge \tau_R} \mathbf{P} g_k(s, u) dB_k$$

holds  $\mathbb{P}$ -a.s. in  $\mathcal{S}'(\mathbb{R}^d)$ .

- (2) We say that the local strong solution is unique, if  $(\tilde{u}, \tilde{\tau}_R)$  is another strong solution, then

$$\mathbb{P}(\{\omega \in \Omega : u = \tilde{u}, \forall 0 \leq t \leq \tau_R \wedge \tilde{\tau}_R\}) = 1.$$

**Definition 4** Let  $2 \leq p, r \leq \infty$  and  $u_0 \in \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}}$  be  $\mathcal{F}_0$ -measurable. We say that  $u$  is a global strong solution for the equation (1.1), if the following two conditions are fulfilled

(1)  $u$  is a progressively measurable process and for any  $0 < T < \infty$ , we have

$$u \in L^r_\omega \mathcal{L}^\infty_T \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}} \cap L^r_\omega \mathcal{L}^2_T \dot{B}_{p,r}^{4-\alpha-\frac{3}{p}}.$$

(2) For almost all  $\omega \in \Omega$ ,  $u(t, x) \in C([0, \infty); \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}})$ , and for all  $0 < t < \infty$ , the following equality

$$u(t) = u_0 - \int_0^t [(-\Delta)^\alpha u + \mathbf{P} \operatorname{div}(u \otimes u)] ds + \sum_{k \geq 1} \int_0^t \mathbf{P} g_k(s, u) dB_k$$

holds  $\mathbb{P}$ -a.s. in  $\mathcal{S}'(\mathbb{R}^d)$ .

We are now in the position to state our two main results of this paper.

**Theorem 1** Let  $2 \leq p, r \leq \infty$  and  $u_0 \in \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}}$  be  $\mathcal{F}_0$ -measurable. Assume that

$$\begin{aligned} g(t, u) &: [0, T] \times (\mathcal{L}^2_T \mathcal{L}^r_\omega \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}} \cap L^r_\omega \mathcal{L}^2_T \dot{B}_{p,r}^{4-\alpha-\frac{3}{p}}) \\ &\rightarrow \mathcal{L}^2_T \mathcal{L}^r_\omega \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}} \end{aligned}$$

fulfils

$$\begin{aligned} \|\mathbf{P}g(t, u)\|_{\mathcal{L}^2_T \mathcal{L}^r_\omega \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}}} &\leq L_1 \|u\|_{\mathcal{L}^2_T \mathcal{L}^r_\omega \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}}} \\ &\quad + L_2 \|u\|_{L^r_\omega \mathcal{L}^2_T \dot{B}_{p,r}^{4-\alpha-\frac{3}{p}}}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \|\mathbf{P}g(t, u) - \mathbf{P}g(t, v)\|_{\mathcal{L}^2_T \mathcal{L}^r_\omega \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}}} &\leq L_1 \|u - v\|_{\mathcal{L}^2_T \mathcal{L}^r_\omega \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}}} \\ &\quad + L_2 \|u - v\|_{L^r_\omega \mathcal{L}^2_T \dot{B}_{p,r}^{4-\alpha-\frac{3}{p}}}, \end{aligned} \tag{3.3}$$

where  $L_1, L_2 > 0$  and  $L_2$  is small enough. Then, there is a constant  $R > 0$  such that there exists a unique local solution  $(u, \tau_R)$  to Eq. (1.1) with

$$\mathbb{P}(\tau_R > 0) = 1.$$

**Theorem 2** With the same preamble as in Theorem 1. If further for sufficiently small  $L_3 > 0$ ,

$$\|\mathbf{P}g(t, u)\|_{\mathcal{L}^2_T \mathcal{L}^r_\omega \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}}} \leq L_3 \|u\|_{L^r_\omega \mathcal{L}^2_T \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}}}$$

$$+L_2\|u\|_{L^r_\omega \mathcal{L}^2_T \dot{\mathcal{B}}^{4-\alpha-\frac{3}{p}}_{p,r}}, \tag{3.4}$$

and for any  $\varepsilon > 0$ , there exists a constant  $\gamma = \gamma(\varepsilon) > 0$  such that

$$\|u_0\|_{L^r_\omega \dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p}}_{p,r}} < \gamma. \tag{3.5}$$

Then,

$$\mathbb{P}(\tau_R = \infty) \geq 1 - \varepsilon.$$

**Example 1** Here, we give an example to show that the noise coefficient in the above theorems is non-empty. Motivated by [6], let  $M > 0$  be arbitrary, we take  $g_k(t, u) = \frac{1}{\sqrt{2k}} e^{-M(1+t)} u$ ,  $k \geq 1$ . Then

$$\|\mathbf{P}g(t, u) - \mathbf{P}g(t, v)\|_{\mathcal{L}^2_T \mathcal{L}^r_\omega \dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p}}_{p,r}} \leq e^{-M} \|u - v\|_{\mathcal{L}^2_T \mathcal{L}^r_\omega \dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p}}_{p,r}},$$

Thus, conditions (3.2) and (3.3) are satisfied. Moreover, It is easy to verify that condition (3.4) also holds.

In order to prove the main results, we need the following lemmas.

**Lemma 1** ([14]) Assume  $0 < T \leq \infty$ ,  $\frac{1}{2} < \alpha \leq 1$ ,  $1 \leq p, q, \rho \leq \infty$ ,  $s \in \mathbb{R}$ . Let  $u$  be a solution of

$$\begin{cases} \partial_t u + (-\Delta)^\alpha u = f(t, x), & (t, x) \in [0, T) \times \mathbb{R}^3, \\ u(0) = u_0, \end{cases}$$

where  $f \in \mathcal{L}^\rho_T \dot{\mathcal{B}}^{s+\frac{2\alpha}{\rho}-2\alpha}_{p,q}$ . Then,  $u \in \mathcal{L}^\infty_T \dot{\mathcal{B}}^s_{p,q} \cap \mathcal{L}^\rho_T \dot{\mathcal{B}}^{s+\frac{2\alpha}{\rho}-2\alpha}_{p,q}$ . Moreover, for any  $\rho \leq \rho_1 \leq \infty$ , the following inequality

$$\|u\|_{\mathcal{L}^{\rho_1}_T \dot{\mathcal{B}}^{s+\frac{2\alpha}{\rho_1}}_{p,q}} \leq C \left( \|u_0\|_{\dot{\mathcal{B}}^s_{p,q}} + \|f\|_{\mathcal{L}^\rho_T \dot{\mathcal{B}}^{s+\frac{2\alpha}{\rho}-2\alpha}_{p,q}} \right)$$

holds for some positive constant  $C$ .

**Lemma 2** Let  $0 < T \leq \infty$ ,  $\frac{1}{2} < \alpha \leq 1$ ,  $2 \leq p < \infty$ ,  $2 \leq q \leq r < \infty$  and  $s \in \mathbb{R}$ . Then, the following stochastic fractional heat equation

$$\begin{cases} du + (-\Delta)^\alpha u dt = \sum_{k \geq 1} g_k dW_k(t), \\ u(0) = 0, \end{cases} \tag{3.6}$$

has a unique solution  $u \in L^r(\Omega; \mathcal{L}^q_T \dot{\mathcal{B}}^s_{p,r} \cap C_T \dot{\mathcal{B}}^{s-\frac{2\alpha}{q}}_{p,r})$ , where  $g = \{g_k, k \geq 1\} \in \mathcal{L}^q_T \mathcal{L}^r_\omega \dot{\mathcal{B}}^{s-\alpha}_{p,r}$  is progressively measurable. Moreover, there exists a constant  $C > 0$ ,

such that for any  $q \leq q_1 \leq \infty$ ,

$$\|u\|_{L^r_\omega \mathcal{L}^{q_1}_T \dot{\mathbf{B}}^{s+\frac{2\alpha}{q_1}-\frac{2\alpha}{q}}_{p,r}} \leq C \|g\|_{\mathcal{L}^q_T \mathcal{L}^r_\omega \dot{\mathbf{B}}^{s-\alpha}_{p,r}}. \quad (3.7)$$

**Proof** Taking the Fourier transform on both sides of (3.6), we conclude that the unique solution  $u$  satisfies

$$\hat{u}(t, \xi) = \sum_{k \geq 1} \int_0^t e^{-(t-t')|\xi|^{2\alpha}} \hat{g}_k(t', \xi) dW_k(t').$$

Therefore, by using Minkowski's inequality, Young's inequality and [6, Lemma 2.5], we get

$$\begin{aligned} \|\varphi_j(\xi) \hat{u}(t, \xi)\|_{L^r_\omega L^q_T L^p_\xi} &\leq \|\varphi_j(\xi) \hat{u}(t, \xi)\|_{L^q_T L^r_\omega L^p_\xi} \\ &= \left\| \sum_{k \geq 1} \int_0^t e^{-(t-t')|\xi|^{2\alpha}} \varphi_j(\xi) \hat{g}_k(t', \xi) dW_k(t') \right\|_{L^q_T L^r_\omega L^p_\xi} \\ &\leq C \left\| e^{-(t-t')|\xi|^{2\alpha}} \varphi_j(\xi) \hat{g}_k(t', \xi) \right\|_{L^q_T L^r_\omega L^2_{l'_k} L^p_\xi} \\ &\leq C \left\| e^{-c(t-t')2^{2\alpha j}} \varphi_j(\xi) \hat{g}_k(t', \xi) \right\|_{L^q_T L^2_{l'_k} L^r_\omega L^p_\xi} \\ &\leq C \left\| \int_0^t e^{-2c(t-t')2^{2\alpha j}} \|\varphi_j(\xi) \hat{g}_k(t', \xi)\|_{L^r_\omega l^2_{l'_k} L^p_\xi} dt' \right\|_{L^q_T}^{\frac{1}{2}} \\ &\leq C \left\| e^{-2c2^{2\alpha j} t} \right\|_{L^1_T}^{1/2} \|\varphi_j(\xi) \hat{g}_k(t', \xi)\|_{L^q_T L^r_\omega l^2_{l'_k} L^p_\xi} \\ &\leq C 2^{-\alpha j} \|\varphi_j(\xi) \hat{g}_k(t', \xi)\|_{L^q_T L^r_\omega l^2_{l'_k} L^p_\xi}. \end{aligned}$$

Consequently,

$$\begin{aligned} \|u\|_{L^r_\omega \mathcal{L}^q_T \dot{\mathbf{B}}^{s+\frac{2\alpha}{q}}_{p,r}} &= \left\| 2^{j(s+\frac{2\alpha}{q})} \|\varphi_j(\xi) \hat{u}(t, \xi)\|_{L^r_\omega L^q_T L^p_\xi} \right\|_{l'_j} \\ &\leq C \left\| 2^{j(s+\frac{2\alpha}{q}-\alpha)} \|\varphi_j(\xi) \hat{g}_k(t', \xi)\|_{L^q_T L^r_\omega l^2_{l'_k} L^p_\xi} \right\|_{l'_j} \\ &= C \|g\|_{\mathcal{L}^q_T \mathcal{L}^r_\omega \dot{\mathbf{B}}^{s+\frac{2\alpha}{q}-\alpha}_{p,r}}. \end{aligned} \quad (3.8)$$



Let  $p'$  be the conjugate number of  $p$ , then for any  $\beta \in (0, \frac{1}{2})$ , we have

$$\begin{aligned} \|\varphi_j \hat{u}\|_{L_\omega^r L_T^\infty L_\xi^p} &\leq C \left\| \int_0^t e^{-(t-t')|\xi|^{2\alpha}} (t-s)^{\beta-1} \times \right. \\ &\quad \left. \left( \sum_{k \geq 1} \int_0^s e^{-(s-t')|\xi|^{2\alpha}} (s-t')^{-\beta} \varphi_j \hat{g}_k dW_k \right) ds \right\|_{L_\omega^r L_T^\infty L_\xi^p} \\ &\leq C \left\| \int_0^t e^{-c(t-t')2^{2\alpha j}} (t-s)^{\beta-1} \times \right. \\ &\quad \left. \left\| \sum_{k \geq 1} \int_0^s e^{-(s-t')|\xi|^{2\alpha}} (s-t')^{-\beta} \varphi_j \hat{g}_k dW_k \right\|_{L_\xi^p} ds \right\|_{L_\omega^r L_T^\infty} \\ &\leq C \|e^{-ct2^{2\alpha j}} t^{\beta-1}\|_{L_T^{q'}} \left\| \sum_{k \geq 1} \int_0^s e^{-(s-t')|\xi|^{2\alpha}} (s-t')^{-\beta} \varphi_j \hat{g}_k dW_k \right\|_{L_\omega^r L_T^q L_\xi^p}, \end{aligned}$$

where we have used the factorisation formula (see e.g. [5])

$$\begin{aligned} &\sum_{k \geq 1} \int_0^t e^{-(t-t')|\xi|^{2\alpha}} \varphi_j(\xi) \hat{g}_k(t', \xi) dW_k(t') \\ &= \frac{\sin \beta \pi}{\pi} \int_0^t e^{-(t-t')|\xi|^{2\alpha}} (t-s)^{\beta-1} \\ &\quad \left( \sum_{k \geq 1} \int_0^s e^{-(s-t')|\xi|^{2\alpha}} (s-t')^{-\beta} \varphi_j \hat{g}_k dW_k \right) ds. \end{aligned} \tag{3.9}$$

Basic calculus then implies that

$$\|e^{-ct2^{2\alpha j}} t^{\beta-1}\|_{L_T^{q'}} \leq C 2^{2j\alpha(\frac{1}{q}-\beta)}.$$

Substituting this estimate into (3.8), we obtain

$$\begin{aligned} \|\varphi_j \hat{u}\|_{L_\omega^r L_T^\infty L_\xi^p} &\leq C 2^{2j\alpha(\frac{1}{q}-\beta)} \\ &\quad \left\| \sum_{k \geq 1} \int_0^s e^{-(s-t')|\xi|^{2\alpha}} (s-t')^{-\beta} \varphi_j \hat{g}_k dW_k \right\|_{L_T^q L_\omega^r L_\xi^p} \\ &\leq C 2^{2j\alpha(\frac{1}{q}-\beta)} \left\| e^{-(s-t')|\xi|^{2\alpha}} (s-t')^{-\beta} \varphi_j \hat{g}_k \right\|_{L_T^q L_\omega^r L_s^2 L_k^2 L_\xi^p} \\ &\leq C 2^{2j\alpha(\frac{1}{q}-\beta)} \left\| e^{-(s-t')|\xi|^{2\alpha}} (s-t')^{-\beta} \varphi_j \hat{g}_k \right\|_{L_T^q L_s^2 L_\omega^r L_k^2 L_\xi^p} \\ &\leq C 2^{2j\alpha(\frac{1}{q}-\beta)} \left\| \int_0^s e^{-2c2^{2j\alpha}(s-t')(s-t')^{-2\beta}} \|\varphi_j \hat{g}_k\|_{L_\omega^r L_k^2 L_\xi^p} dt' \right\|_{L_T^{\frac{q}{2}}}^{\frac{1}{2}} \\ &\leq C 2^{2j\alpha(\frac{1}{q}-\beta)} \|e^{-c2^{2j\alpha}t} t^{-2\beta}\|_{L_T^1}^{\frac{1}{2}} \|\varphi_j \hat{g}_k\|_{L_T^q L_\omega^r L_k^2 L_\xi^p} \\ &\leq C 2^{j\alpha(\frac{2}{q}-1)} \|\varphi_j \hat{g}_k\|_{L_T^q L_\omega^r L_k^2 L_\xi^p}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|u\|_{L^r_\omega \mathcal{L}^\infty_T \dot{B}^s_{p,r}} &= \left\| 2^{js} \|\varphi_j \hat{u}\|_{L^r_\omega L^\infty_T L^p_\xi} \right\|_{l^r_j} \\ &\leq C \left\| 2^{j(s+\frac{2\alpha}{q}-\alpha)} \|\varphi_j \hat{g}_k\|_{L^q_T L^r_\omega l^2_k L^p_\xi} \right\|_{l^r_j} \\ &\leq C \|g\|_{\mathcal{L}^q_T \mathcal{L}^r_\omega \dot{B}^{s+\frac{2\alpha}{q}-\alpha}}. \end{aligned} \tag{3.10}$$

Using the interpolation inequality for (3.8) and (3.10), we obtain (3.7). Finally,  $u \in L^r(\Omega; C_T \dot{B}^{s-\frac{2\alpha}{q}}_{p,r})$  holds by utilising factorisation formula (3.9) again (see e.g. [5]). We thus complete the proof.  $\square$

**Lemma 3** *Let  $1 < p, \rho, q \leq \infty, \frac{1}{2} < \alpha \leq 1$  with  $\frac{2}{\rho} + 1 - 2\alpha < 0$  and  $5 - 4\alpha - \frac{3}{p} + \frac{4\alpha}{\rho} > 0$ . Then, there exists a constant  $C > 0$  such that*

$$\begin{aligned} \|uv\|_{\mathcal{L}^{\frac{\rho}{2}}_T \dot{B}^{5-4\alpha-\frac{3}{p}+\frac{4\alpha}{\rho}}_{p,q}} &\leq C \|u\|_{\mathcal{L}^\rho_T \dot{B}^{4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho}}_{p,q}} \\ &\quad \|v\|_{\mathcal{L}^\rho_T \dot{B}^{4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho}}_{p,q}}. \end{aligned}$$

**Proof** According to Bony’s decomposition, we have

$$\begin{aligned} \dot{\Delta}_j(uv) &= \dot{\Delta}_j \dot{T}_u v + \dot{\Delta}_j \dot{T}_v u + \dot{\Delta}_j R(u, v) \\ &= \sum_{|k-j| \leq 4} \dot{\Delta}_j (\dot{S}_{k-1} u \dot{\Delta}_k v) + \sum_{|k-j| \leq 4} \dot{\Delta}_j (\dot{S}_{k-1} v \dot{\Delta}_k u) \\ &\quad + \sum_{k-j \geq 3} \dot{\Delta}_j \left( \sum_k \tilde{\Delta}_k u \dot{\Delta}_k v \right) \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

For the term  $I_1$ , by Young’s inequality and  $l^q \hookrightarrow l^\infty$ , we get

$$\begin{aligned} \|\widehat{I_1}\|_{L^{\frac{\rho}{2}}_T L^p_\xi} &\leq \sum_{|k-j| \leq 4} \|\varphi_j (\widehat{\dot{S}_{k-1} u \dot{\Delta}_k v})\|_{L^{\frac{\rho}{2}}_T L^p_\xi} \\ &\leq \sum_{|k-j| \leq 4} \|\widehat{\dot{S}_{k-1} u} * \widehat{\dot{\Delta}_k v}\|_{L^{\frac{\rho}{2}}_T L^p_\xi} \\ &\leq \sum_{|k-j| \leq 4} \|\varphi_k \hat{v}\|_{L^{\frac{\rho}{2}}_T L^p_\xi} \|\widehat{\dot{S}_{k-1} u}\|_{L^{\frac{\rho}{2}}_T L^1_\xi} \\ &\leq \sum_{|k-j| \leq 4} 2^{-k(1-2\alpha+\frac{2\alpha}{\rho})} \left\| 2^{k(1-2\alpha+\frac{2\alpha}{\rho})} \|\widehat{\dot{S}_{k-1} u}\|_{L^{\frac{\rho}{2}}_T L^1_\xi} \right\|_{l^\infty} \|\varphi_k \hat{v}\|_{L^{\frac{\rho}{2}}_T L^p_\xi} \\ &\leq \sum_{|k-j| \leq 4} 2^{-k(1-2\alpha+\frac{2\alpha}{\rho})} \|\varphi_k \hat{v}\|_{L^{\frac{\rho}{2}}_T L^p_\xi} \|u\|_{\mathcal{L}^\rho_T \dot{B}^{1-2\alpha+\frac{2\alpha}{\rho}}_{1,q}}. \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \left\| 2^{j(5-4\alpha-\frac{3}{p}+\frac{4\alpha}{\rho})} \|\mathbf{I}_1\|_{\mathcal{L}_T^\rho L_\xi^p} \right\|_{l_j^q} \\
 & \leq \left\| \sum_{|k-j|\leq 4} 2^{(j-k)(5-4\alpha-\frac{3}{p}+\frac{4\alpha}{\rho})} 2^{k(4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho})} \|\varphi_k \hat{v}\|_{L_T^\rho L_\xi^p} \right\|_{l_j^q} \|u\|_{\mathcal{L}_T^\rho \dot{B}_{1,q}^{1-2\alpha+\frac{2}{\rho}}} \\
 & \leq \sum_{|j|\leq 4} 2^{j(5-4\alpha-\frac{3}{p}+\frac{4\alpha}{\rho})} \left\| 2^{j(4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho})} \|\varphi_j \hat{v}\|_{L_T^\rho L_\xi^p} \right\|_{l_j^q} \|u\|_{\mathcal{L}_T^\rho \dot{B}_{1,q}^{1-2\alpha+\frac{2}{\rho}}} \\
 & \leq C \|v\|_{\mathcal{L}_T^\rho \dot{B}_{p,q}^{4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho}}} \|u\|_{\mathcal{L}_T^\rho \dot{B}_{1,q}^{1-2\alpha+\frac{2}{\rho}}} \\
 & \leq C \|v\|_{\mathcal{L}_T^\rho \dot{B}_{p,q}^{4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho}}} \|u\|_{\mathcal{L}_T^\rho \dot{B}_{p,q}^{4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho}}},
 \end{aligned}$$

where we have used the fact that  $\dot{B}_{p_1,q}^{s_1} \hookrightarrow \dot{B}_{p_2,q}^{s_2}$ ,  $p_2 \leq p_1$ ,  $s_1 + \frac{d}{p_1} = s_2 + \frac{d}{p_2}$ . Similarly,

$$\left\| 2^{j(5-4\alpha-\frac{3}{p}+\frac{4\alpha}{\rho})} \|\mathbf{I}_2\|_{\mathcal{L}_T^\rho L_\xi^p} \right\|_{l_j^q} \leq C \|v\|_{\mathcal{L}_T^\rho \dot{B}_{p,q}^{4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho}}} \|u\|_{\mathcal{L}_T^\rho \dot{B}_{p,q}^{4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho}}}.$$

For the term  $I_3$ , utilising Bernstein inequality, we get

$$\begin{aligned}
 & 2^{j(5-4\alpha-\frac{3}{p}+\frac{4\alpha}{\rho})} \|\widehat{\mathbf{I}}_3\|_{\mathcal{L}_T^\rho L_\xi^p} \\
 & \leq 2^{j(5-4\alpha-\frac{3}{p}+\frac{4\alpha}{\rho})} \sum_{k-j\geq 3} \left\| \varphi_j \widehat{\tilde{\Delta}_k u \dot{\Delta}_k v} \right\|_{\mathcal{L}_T^\rho L_\xi^p} \\
 & \leq 2^{j(5-4\alpha-\frac{3}{p}+\frac{4\alpha}{\rho})} \sum_{k-j\geq 3} \left\| \widehat{\tilde{\Delta}_k u} * \widehat{\dot{\Delta}_k v} \right\|_{\mathcal{L}_T^\rho L_\xi^p} \\
 & \leq 2^{j(5-4\alpha-\frac{3}{p}+\frac{4\alpha}{\rho})} \sum_{k-j\geq 3} \left\| \widehat{\tilde{\Delta}_k u} \right\|_{L_T^\rho L_\xi^p} \left\| \widehat{\dot{\Delta}_k v} \right\|_{L_T^\rho L_\xi^1} \\
 & \leq 2^{j(5-4\alpha-\frac{3}{p}+\frac{4\alpha}{\rho})} \sum_{k-j\geq 3} 2^{k(3-\frac{3}{p})} \left\| \widehat{\tilde{\Delta}_k u} \right\|_{L_T^\rho L_\xi^p} \left\| \widehat{\dot{\Delta}_k v} \right\|_{L_T^\rho L_\xi^p} \\
 & \leq \|v\|_{\mathcal{L}_T^\rho \dot{B}_{p,\infty}^{4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho}}} \sum_{k-j\geq 3} 2^{(j-k)(5-4\alpha-\frac{3}{p}+\frac{4\alpha}{\rho})} 2^{k(4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho})} \left\| \widehat{\tilde{\Delta}_k u} \right\|_{L_T^\rho L_\xi^p}.
 \end{aligned}$$

Thus,

$$\begin{aligned} & \left\| 2^{j(5-4\alpha-\frac{3}{p}+\frac{4\alpha}{\rho})} \|\widehat{\mathbf{I}}_3\|_{L_T^{\frac{\rho}{2}} L_{\xi}^{\rho} L_j^q} \right\| \\ & \leq \|v\|_{\mathcal{L}_T^{\rho} \dot{\mathcal{B}}_{p,\infty}^{4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho}}} \sum_{l \leq -3} 2^{l(5-4\alpha-\frac{3}{p}+\frac{4\alpha}{\rho})} \|u\|_{\mathcal{L}_T^{\rho} \dot{\mathcal{B}}_{p,q}^{4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho}}} \\ & \leq C \|v\|_{\mathcal{L}_T^{\rho} \dot{\mathcal{B}}_{p,\infty}^{4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho}}} \|u\|_{\mathcal{L}_T^{\rho} \dot{\mathcal{B}}_{p,q}^{4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho}}} \\ & \leq C \|v\|_{\mathcal{L}_T^{\rho} \dot{\mathcal{B}}_{p,q}^{4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho}}} \|u\|_{\mathcal{L}_T^{\rho} \dot{\mathcal{B}}_{p,q}^{4-2\alpha-\frac{3}{p}+\frac{2\alpha}{\rho}}}. \end{aligned}$$

The proof is completed. □

Let  $0 < R \leq 1$ , which will be determined later. We introduce a continuous decreasing function  $\vartheta : [0, \infty) \rightarrow [0, 1]$  defined via

$$\vartheta(x) := \{0 \vee (2 - R^{-1}x)\} \wedge 1.$$

We consider the following modified system of the equation (1.1)

$$\begin{cases} du + [(-\Delta)^\alpha u + \mathbf{P} \operatorname{div}(\chi_u u \otimes u)] dt = \sum_{k \geq 1} \mathbf{P} g_k(t, u) dB_k, \\ u(0) = u_0, \end{cases} \tag{3.11}$$

where  $\chi_u(t) = \vartheta(\|u\|_{\mathcal{L}_T^{\rho} \dot{\mathcal{B}}_{p,r}^{4-\alpha-\frac{3}{p}}})$ .

**Proposition 1** *Under the assumptions of Theorem 1. Equation (3.11) has a global strong solution.*

**Proof** Equation (3.11) can be rewritten as

$$u(t) = e^{-t(-\Delta)^\alpha} u_0 + S(\chi_u u \otimes u) + K(t, u),$$

where  $S$  and  $K$  are, respectively, the solutions to the fractional heat equation

$$\begin{cases} \partial_t S(A) + (-\Delta)^\alpha S(A) = -\mathbf{P} \operatorname{div} A, \\ S(A)|_{t=0} = 0, \end{cases}$$

and stochastic fractional heat equation

$$\begin{cases} dK(t, u) + (-\Delta)^\alpha K(t, u) dt = \sum_{k \geq 1} g_k dW_k(t), \\ K(t, u)|_{t=0} = 0. \end{cases}$$

Define

$$\Psi(u) := e^{-t(-\Delta)^\alpha} u_0 + S(\chi_u u \otimes u) + K(t, u).$$

Let

$$X_T := \{w : w \text{ is progressively measurable, } \|w\|_{X_T} < +\infty\}$$

with

$$\|w\|_{X_T} := \|u\|_{L^r_\omega \mathcal{L}^\infty_T \dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p}}_{p,r}} + \|u\|_{L^r_\omega \mathcal{L}^2_T \dot{\mathcal{B}}^{4-\alpha-\frac{3}{p}}_{p,r}}.$$

For any  $p_1 \geq 2$ , we obtain

$$\begin{aligned} & \|e^{-t(-\Delta)^\alpha} u_0\|_{L^r_\omega \mathcal{L}^{q_1}_T \dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p}+\frac{2\alpha}{q_1}}_{p,r}} \\ &= \|2^{j(4-2\alpha-\frac{3}{p}+\frac{2\alpha}{q_1})} \|\varphi_j e^{-t|\xi|^{2\alpha}} \hat{u}_0\|_{L^{q_1}_T L^p_\xi} \|L^r_\omega J^r_j\| \\ &\leq \|2^{j(4-2\alpha-\frac{3}{p}+\frac{2\alpha}{q_1})} \|\varphi_j e^{-ct2^{2j\alpha}} \hat{u}_0\|_{L^{q_1}_T L^p_\xi} \|L^r_\omega J^r_j\| \\ &\leq C \|2^{j(4-2\alpha-\frac{3}{p})} \|\varphi_j \hat{u}_0\|_{L^p_\xi} \|L^r_\omega J^r_j\| \\ &= C \|u_0\|_{L^r_\omega \dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p}}_{p,r}}. \end{aligned}$$

Therefore,

$$\|e^{-t(-\Delta)^\alpha} u_0\|_{X_T} \leq C \|u_0\|_{L^r_\omega \dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p}}_{p,r}}. \tag{3.12}$$

By Lemmas 1 and 3, we have

$$\begin{aligned} \|S(\chi_u u \otimes u)\|_{X_T} &\leq C \|\chi_u u \otimes u\|_{L^r_\omega \mathcal{L}^1_T \dot{\mathcal{B}}^{5-2\alpha-\frac{3}{p}}_{p,q}} \\ &\leq C \left\| \|\chi_u u\|_{L^2_T \dot{\mathcal{B}}^{4-\alpha-\frac{3}{p}}_{p,q}} \|u\|_{L^2_T \dot{\mathcal{B}}^{4-\alpha-\frac{3}{p}}_{p,q}} \right\|_{L^r_\omega} \\ &\leq CR \|u\|_{L^r_\omega \mathcal{L}^2_T \dot{\mathcal{B}}^{4-\alpha-\frac{3}{p}}_{p,q}}. \end{aligned} \tag{3.13}$$

For the stochastic term, by Lemma 2, we get

$$\begin{aligned} \|K(t, u)\|_{X_T} &\leq C \|\mathbf{P}f(t, u)\|_{L^2_T \mathcal{L}^r_\omega \dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p}}_{p,r}} \\ &\leq C \left( L_1 T^{\frac{1}{2}} \|u\|_{L^r_\omega \mathcal{L}^\infty_T \dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p}}_{p,r}} + L_2 \|u\|_{L^r_\omega \mathcal{L}^2_T \dot{\mathcal{B}}^{4-\alpha-\frac{3}{p}}_{p,r}} \right). \end{aligned} \tag{3.14}$$

Combining (3.12), (3.13) and (3.14), we obtain

$$\begin{aligned} \|\Psi(u)\|_{X_T} &\leq C_1 \left[ \|u_0\|_{L^r_\omega \dot{B}^{4-2\alpha-\frac{3}{p}}_{p,r}} + R \|u\|_{L^r_\omega \mathcal{L}^2_T \dot{B}^{4-\alpha-\frac{3}{p}}_{p,q}} \right. \\ &\quad \left. + L_1 T^{\frac{1}{2}} \|u\|_{L^r_\omega \mathcal{L}^\infty_T \dot{B}^{4-2\alpha-\frac{3}{p}}_{p,r}} + L_2 \|u\|_{L^r_\omega \mathcal{L}^2_T \dot{B}^{4-\alpha-\frac{3}{p}}_{p,r}} \right] \\ &\leq C_1 \left[ \|u_0\|_{L^r_\omega \dot{B}^{4-2\alpha-\frac{3}{p}}_{p,r}} + (R + L_1 T^{\frac{1}{2}} + L_2) \|u\|_{S_T} \right]. \end{aligned} \tag{3.15}$$

Next, we estimate the term  $\Psi(u) - \Psi(v)$  and we have

$$\begin{aligned} \Psi(u) - \Psi(v) &= S(\chi_u u \otimes u - \chi_v v \otimes v) + [K(t, u) - K(t, v)] \\ &=: \Pi_1 + \Pi_2. \end{aligned}$$

In order to estimate  $\Pi_1$ . Firstly, similar to [3], one can get that

$$|\chi_u - \chi_v|_{L^\infty_T} \leq R^{-1} \|u - v\|_{\mathcal{L}^2_T \dot{B}^{4-\alpha-\frac{3}{p}}_{p,r}}. \tag{3.16}$$

We now divide  $\Pi_1$  into the three following cases.

(1) If  $\chi_u > 0, \chi_v > 0$ , then

$$\begin{aligned} |\chi_u u \otimes u - \chi_v v \otimes v| &\leq |(\chi_u - \chi_v)u \otimes u| + |\chi_v(u - v) \otimes u| \\ &\quad + |\chi_v v \otimes (u - v)|. \end{aligned}$$

Furthermore, by Lemma 3 and (3.16), we then have

$$\begin{aligned} \|(\chi_u - \chi_v)u \otimes u\|_{\mathcal{L}^1_T \dot{B}^{5-2\alpha-\frac{3}{p}}_{p,r}} &\leq |\chi_u - \chi_v|_{L^\infty_T} \|u \otimes u\|_{\mathcal{L}^1_T \dot{B}^{5-2\alpha-\frac{3}{p}}_{p,r}} \\ &\leq |\chi_u - \chi_v|_{L^\infty_T} \|u\|^2_{\mathcal{L}^2_T \dot{B}^{4-\alpha-\frac{3}{p}}_{p,r}} \\ &\leq 4R \|u - v\|_{\mathcal{L}^2_T \dot{B}^{4-\alpha-\frac{3}{p}}_{p,r}}, \end{aligned}$$

and

$$\begin{aligned} \|\chi_u(u - v) \otimes u\|_{\mathcal{L}^1_T \dot{B}^{5-2\alpha-\frac{3}{p}}_{p,r}} &+ \|\chi_v v \otimes (u - v)\|_{\mathcal{L}^1_T \dot{B}^{5-2\alpha-\frac{3}{p}}_{p,r}} \\ &\leq \|u - v\|_{\mathcal{L}^2_T \dot{B}^{4-\alpha-\frac{3}{p}}_{p,r}} \left( \|u\|_{\mathcal{L}^2_T \dot{B}^{4-\alpha-\frac{3}{p}}_{p,r}} + \|v\|_{\mathcal{L}^2_T \dot{B}^{4-\alpha-\frac{3}{p}}_{p,r}} \right) \\ &\leq 4R \|u - v\|_{\mathcal{L}^2_T \dot{B}^{4-\alpha-\frac{3}{p}}_{p,r}}. \end{aligned}$$

(2) If  $\chi_u > 0, \chi_v = 0$ , then

$$|\chi_u u \otimes u - \chi_v v \otimes v| \leq |(\chi_u - \chi_v)u \otimes u|,$$

Therefore,

$$\|\chi_u u \otimes u - \chi_v v \otimes v\|_{\mathcal{L}_T^1 \mathcal{B}_{p,r}^{5-2\alpha-\frac{3}{p}}} \leq 4R \|u - v\|_{\mathcal{L}_T^2 \mathcal{B}_{p,r}^{4-\alpha-\frac{3}{p}}}.$$

(3) If  $\chi_u = 0, \chi_v > 0$ , then by the similar argument, we have

$$\|\chi_u u \otimes u - \chi_v v \otimes v\|_{\mathcal{L}_T^1 \mathcal{B}_{p,r}^{5-2\alpha-\frac{3}{p}}} \leq 4R \|u - v\|_{\mathcal{L}_T^2 \mathcal{B}_{p,r}^{4-\alpha-\frac{3}{p}}}.$$

Combing all the above estimates, we obtain

$$\begin{aligned} \|\Pi_1\|_{X_T} &\leq C \|\chi_u u \otimes u - \chi_v v \otimes v\|_{L_\omega^r \mathcal{L}_T^1 \mathcal{B}_{p,r}^{5-2\alpha-\frac{3}{p}}} \\ &\leq CR \|u - v\|_{L_\omega^r \mathcal{L}_T^2 \mathcal{B}_{p,r}^{4-\alpha-\frac{3}{p}}}. \end{aligned}$$

To estimate the term  $\Pi_2$ , we observe that Lemma 3 implies

$$\begin{aligned} \|\Pi_2\|_{X_T} &\leq C \|\mathbf{P}f(t, u) - \mathbf{P}f(t, v)\|_{\mathcal{L}_T^2 \mathcal{L}_\omega^r \mathbf{B}_{p,r}^{4-2\alpha-\frac{3}{p}}} \\ &\leq C \left( L_1 \|u - v\|_{\mathcal{L}_T^2 \mathcal{L}_\omega^r \mathcal{B}_{p,r}^{4-2\alpha-\frac{3}{p}}} + L_2 \|u - v\|_{L_\omega^r \mathcal{L}_T^2 \mathcal{B}_{p,r}^{4-\alpha-\frac{3}{p}}} \right) \\ &\leq C \left( L_1 T^{\frac{1}{2}} \|u - v\|_{L_\omega^r \mathcal{L}_T^\infty \mathcal{B}_{p,r}^{4-2\alpha-\frac{3}{p}}} + L_2 \|u - v\|_{L_\omega^r \mathcal{L}_T^2 \mathcal{B}_{p,r}^{4-\alpha-\frac{3}{p}}} \right). \end{aligned}$$

Consequently,

$$\|\Psi(u) - \Psi(v)\|_{X_T} \leq C_2(R + L_1 T^{\frac{1}{2}} + L_2) \|u - v\|_{X_T}. \tag{3.17}$$

Define

$$M := 2C_1 \|u_0\|_{L_\omega^r \mathcal{B}_{p,r}^{4-2\alpha-\frac{3}{p}}}, \tag{3.18}$$

$$R := \min\{1, (8 \min\{C_1, C_2\})^{-1}\}, \tag{3.19}$$

$$X_{T,M} := \{w : w \in X_T, \|w\|_{X_T} \leq M\}, \tag{3.20}$$

$$\tilde{T} := \min\left\{1, \left(\frac{1}{4L_1 \max\{C_1, C_2\}}\right)^2\right\}. \tag{3.21}$$

If  $L_2$  is sufficiently small such that

$$\max\{C_1, C_2\}L_2 < \frac{1}{8}, \tag{3.22}$$

then the mapping  $\Psi$  becomes a contracting mapping on  $X_{\tilde{T},M}$ . Therefore, the equation (3.11) has a unique strong solution  $u$  on the interval  $[0, \tilde{T}]$  by Banach’s fixed point

theorem (see [1, Lemma5.5]). Repeating this procedure, we obtain a global strong solution  $u$  for the equation (3.11).  $\square$

With all these in hand, we proceed to show our two main theorems.

**Proof of Theorem 1.** Let  $u$  be the solution for the equation (3.11). Define the stopping time

$$\tau_R(\omega) := \inf \left\{ t \geq 0; \|u\|_{\mathcal{L}_T^2 \dot{\mathcal{B}}_{p,q}^{4-\alpha-\frac{3}{p}}} \geq R \right\}$$

with the convention  $\inf \emptyset = \infty$ . Then for almost all  $\omega \in \Omega$ , it holds that

$$u \in C([0, \tau_R(\omega)); \dot{\mathcal{B}}_{p,q}^{4-\alpha-\frac{3}{p}}).$$

If  $\tau_R(\omega) < +\infty$ , we set  $u(t) := 0, \forall t \geq \tau_R(\omega)$ . Then,  $(u, \tau_R)$  is the unique local strong solution to (3.1). According to (3.15), we can show that

$$\begin{aligned} \|u\|_{L_\omega^r \mathcal{L}_1^\infty \frac{1}{k} \wedge \tau_R \dot{\mathcal{B}}_{p,r}^{4-2\alpha-\frac{3}{p}}} &\leq C_1 \left[ \|u_0\|_{L_\omega^r \dot{\mathcal{B}}_{p,r}^{4-2\alpha-\frac{3}{p}}} + R \|u\|_{L_\omega^r \mathcal{L}_1^2 \frac{1}{k} \wedge \tau_R \dot{\mathcal{B}}_{p,q}^{4-\alpha-\frac{3}{p}}} \right. \\ &\quad \left. + \frac{L_1}{k^{\frac{1}{2}}} \|u\|_{L_\omega^r \mathcal{L}_1^\infty \frac{1}{k} \wedge \tau_R \dot{\mathcal{B}}_{p,r}^{4-2\alpha-\frac{3}{p}}} + L_2 \|u\|_{L_\omega^r \mathcal{L}_1^2 \frac{1}{k} \wedge \tau_R \dot{\mathcal{B}}_{p,r}^{4-\alpha-\frac{3}{p}}} \right] \\ &\leq C_1 \|u_0\|_{L_\omega^r \dot{\mathcal{B}}_{p,r}^{4-2\alpha-\frac{3}{p}}} + \frac{1}{2} \|u\|_{L_\omega^r \mathcal{L}_1^\infty \frac{1}{k} \wedge \tau_R \dot{\mathcal{B}}_{p,r}^{4-2\alpha-\frac{3}{p}}} \\ &\quad + \frac{1}{4} \|u\|_{L_\omega^r \mathcal{L}_1^2 \frac{1}{k} \wedge \tau_R \dot{\mathcal{B}}_{p,r}^{4-\alpha-\frac{3}{p}}} \end{aligned}$$

holds for all  $k > (2C_1 L_1)^2$ . Consequently,

$$\|u\|_{L_\omega^r \mathcal{L}_1^\infty \frac{1}{k} \wedge \tau_R \dot{\mathcal{B}}_{p,r}^{4-2\alpha-\frac{3}{p}}} \leq 2C_1 \|u_0\|_{L_\omega^r \dot{\mathcal{B}}_{p,r}^{4-2\alpha-\frac{3}{p}}} + \frac{1}{2} \|u\|_{L_\omega^r \mathcal{L}_1^2 \frac{1}{k} \wedge \tau_R \dot{\mathcal{B}}_{p,r}^{4-\alpha-\frac{3}{p}}}.$$



Similarly, we get

$$\begin{aligned} \|u\|_{L^r_\omega \mathcal{L}^2_{\frac{1}{k} \wedge \tau_R} \dot{B}^{4-\alpha-\frac{3}{p}}} &\leq \|e^{-t(-\Delta)^\alpha} u_0\|_{L^r_\omega \mathcal{L}^2_{\frac{1}{k}} \dot{B}^{4-\alpha-\frac{3}{p}}} + \frac{C_1 L_1}{k^{\frac{1}{2}}} \|u\|_{L^r_\omega \mathcal{L}^\infty_{\frac{1}{k} \wedge \tau_R} \dot{B}^{4-2\alpha-\frac{3}{p}}} \\ &\quad + \frac{1}{4} \|u\|_{L^r_\omega \mathcal{L}^2_{\frac{1}{k} \wedge \tau_R} \dot{B}^{4-\alpha-\frac{3}{p}}} \\ &\leq \|e^{-t(-\Delta)^\alpha} u_0\|_{L^r_\omega \mathcal{L}^2_{\frac{1}{k}} \dot{B}^{4-\alpha-\frac{3}{p}}} + \frac{2C_1^2 L_1}{k^{\frac{1}{2}}} \|u_0\|_{L^r_\omega \dot{B}^{4-2\alpha-\frac{3}{p}}} \\ &\quad + \frac{1}{2} \|u\|_{L^r_\omega \mathcal{L}^2_{\frac{1}{k} \wedge \tau_R} \dot{B}^{4-\alpha-\frac{3}{p}}}. \end{aligned}$$

Therefore,

$$\|u\|_{L^r_\omega \mathcal{L}^2_{\frac{1}{k} \wedge \tau_R} \dot{B}^{4-\alpha-\frac{3}{p}}} \leq 2 \|e^{-t(-\Delta)^\alpha} u_0\|_{L^r_\omega \mathcal{L}^2_{\frac{1}{k}} \dot{B}^{4-\alpha-\frac{3}{p}}} + \frac{4C_1^2 L_1}{k^{\frac{1}{2}}} \|u_0\|_{L^r_\omega \dot{B}^{4-2\alpha-\frac{3}{p}}}.$$

It follows from Chebyshev’s inequality that

$$\begin{aligned} \mathbb{P}\left(\tau_R \leq \frac{1}{k}\right) &\leq \mathbb{P}\left(\|u\|_{\mathcal{L}^2_{\tau_R \wedge \frac{1}{k}} \dot{B}^{4-\alpha-\frac{3}{p}}} \geq R\right) \\ &\leq \frac{1}{R} \|u\|_{L^1_\omega \mathcal{L}^2_{\tau_R \wedge \frac{1}{k}} \dot{B}^{4-\alpha-\frac{3}{p}}} \\ &\leq \frac{1}{R} \|u\|_{L^r_\omega \mathcal{L}^2_{\tau_R \wedge \frac{1}{k}} \dot{B}^{4-\alpha-\frac{3}{p}}} \\ &\leq \frac{1}{R} \left(2 \|e^{-t(-\Delta)^\alpha} u_0\|_{L^r_\omega \mathcal{L}^2_{\frac{1}{k}} \dot{B}^{4-\alpha-\frac{3}{p}}} + \frac{4C_1^2 L_1}{k^{\frac{1}{2}}} \|u_0\|_{L^r_\omega \dot{B}^{4-2\alpha-\frac{3}{p}}}\right). \end{aligned}$$

For any  $0 < \varepsilon < 1$ , there exists  $\varrho_\varepsilon > 0$ , such that

$$\left[ \mathbb{E} \sum_{j \geq \varrho_\varepsilon} 2^{jr(4-2\alpha-\frac{3}{p})} \|\varphi_j \widehat{u}_0\|_{L^r_{L^\xi}} \right]^{\frac{1}{r}} \leq \frac{R\varepsilon}{8}.$$

Thus,

$$\begin{aligned} \|e^{-t(-\Delta)^\alpha} u_0\|_{L_\omega^r \mathcal{L}_1^2 \dot{\mathcal{B}}_{p,r}^{4-\alpha-\frac{3}{p}}} &= \left( \mathbb{E} \sum_{j \in \mathbb{Z}} 2^{jr(4-\alpha-\frac{3}{p})} \|e^{-t|\xi|^{2\alpha}} \varphi_j \widehat{u}_0\|_{L_{\frac{1}{k}}^2 L_\xi^p}^r \right)^{\frac{1}{r}} \\ &\leq \frac{R\varepsilon}{8} + \left( \mathbb{E} \sum_{j \leq J_\varepsilon} 2^{jr(4-\alpha-\frac{3}{p})} \|e^{-t|\xi|^{2\alpha}} \varphi_j \widehat{u}_0\|_{L_{\frac{1}{k}}^2 L_\xi^p}^r \right)^{\frac{1}{r}} \\ &\leq \frac{R\varepsilon}{8} + C_3 k^{-\frac{1}{2}} \|u_0\|_{L_\omega^r \dot{\mathcal{B}}_{p,r}^{4-2\alpha-\frac{3}{p}}}. \end{aligned}$$

If we choose

$$k \geq \max \left\{ (2C_1 L_1)^2, 16R^{-2} \varepsilon^{-2} (2C_3 + 4C_1 L_1)^2 \|u_0\|_{L_\omega^r \dot{\mathcal{B}}_{p,r}^{4-2\alpha-\frac{3}{p}}}^2 \right\},$$

then we have

$$\mathbb{P} \left( \tau_R \leq \frac{1}{k} \right) \leq \varepsilon.$$

Thereby,

$$\begin{aligned} \mathbb{P}(\tau_R > 0) &= 1 - \mathbb{P}(\tau_R = 0) \\ &\geq 1 - \lim_{k \rightarrow \infty} \mathbb{P} \left( \tau_R \leq \frac{1}{k} \right) \\ &\geq 1 - \varepsilon. \end{aligned}$$

Since  $\varepsilon$  is arbitrary small, we get  $\mathbb{P}(\tau_R > 0) = 1$ . The proof is thus completed.  $\square$

**Proof of Theorem 2.** Similar to (3.15), for any  $0 < t < \infty$ , we have

$$\begin{aligned} \|u\|_{X_{t \wedge \tau_R}} &\leq C_1 \left[ \|u_0\|_{L_\omega^r \dot{\mathcal{B}}_{p,r}^{4-2\alpha-\frac{3}{p}}} + L_3 \|u\|_{L_\omega^r \mathcal{L}_{t \wedge \tau_R}^\infty \dot{\mathcal{B}}_{p,r}^{4-2\alpha-\frac{3}{p}}} \right. \\ &\quad \left. + (R + L_2) \|u\|_{L_\omega^r \mathcal{L}_T^2 \dot{\mathcal{B}}_{p,r}^{4-\alpha-\frac{3}{p}}} \right]. \end{aligned} \tag{3.23}$$

Choose  $L_3 > 0$  such that  $C_1 L_3 < 1$ . Utilising (3.19) and (3.22), we obtain

$$\|u\|_{L_\omega^r \mathcal{L}_{t \wedge \tau_R}^2 \dot{\mathcal{B}}_{p,r}^{4-\alpha-\frac{3}{p}}} \leq 2C_1 \|u_0\|_{L_\omega^r \dot{\mathcal{B}}_{p,r}^{4-2\alpha-\frac{3}{p}}}.$$

By Fatou’s lemma (see [7, Theorem 1.5.4]), we have

$$\begin{aligned}
 R\mathbb{P}(\tau_R < \infty) &= \mathbb{E}\left(1_{(\tau_R < \infty)} \lim_{t \rightarrow \infty} \|u\|_{\mathcal{L}_{t \wedge \tau_R}^2 \dot{B}_{p,r}^{4-\alpha-\frac{3}{p}}}\right) \\
 &\leq \liminf_{t \rightarrow \infty} \mathbb{E}\left(1_{(\tau_R < \infty)} \|u\|_{\mathcal{L}_{t \wedge \tau_R}^2 \dot{B}_{p,r}^{4-\alpha-\frac{3}{p}}}\right) \\
 &\leq 2C_1 \|u_0\|_{L_\omega^r \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}}}.
 \end{aligned}$$

Hence,

$$\mathbb{P}(\tau_R < \infty) \leq \frac{2C_1}{R} \|u_0\|_{L_\omega^r \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}}}.$$

Thus,

$$\mathbb{P}(\tau_R = \infty) \geq 1 - \frac{2C_1}{R} \|u_0\|_{L_\omega^r \dot{B}_{p,r}^{4-2\alpha-\frac{3}{p}}}.$$

It follows that for any  $\varepsilon > 0$ , choosing  $\gamma = \frac{\varepsilon R}{2C_1}$ , we obtain

$$\mathbb{P}(\tau_R = \infty) \geq 1 - \varepsilon.$$

We thus complete the proof of Theorem 2. □

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## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

**Availability of Data and Material (data transparency)** Data and material sharing not applicable to this article as no datasets and material were generated or analysed during the current (theoretical) study.

**Data Availability Statement** No data were generated or analysed in this article.

**Code Availability (software application or custom code)** Code sharing is not applicable to this article as no code was generated during the current (theoretical) study.

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## References

1. Bahouri, H., Chemin, J.Y., Danchin, R.: *Fourier Analysis and Nonlinear Partial Differential Equations*, vol. 343. Springer, Berlin (2011)
2. Caraballo, T., Langa, J., Taniguchi, T.: The exponential behavior and stabilizability of stochastic 2D-Navier-Stokes equations. *J. Differ. Equ.* **179**(2), 714–737 (2002)
3. Chang, T., Yang, M.: On initial-boundary value problem of the stochastic Navier-Stokes equations in the half space, (2020), [arXiv:2012.01730](https://arxiv.org/abs/2012.01730)
4. Chen, J., Song, C. M.: Global stability for the fractional Navier-Stokes equations in the Fourier-Herz space. *Math. Methods Appl. Sci.* **41**, 2696–3717 (2018)
5. Da Prato, G., Zabczyk, J.: *Stochastic equations in infinite dimensions*, 2nd edn. *Encyclopedia of Mathematics and its Applications*, vol. 152, Cambridge, (2014)
6. Du, L., Zhang, T.: Local and global strong solutions to the stochastic incompressible Navier-Stokes equations in critical Besov space. *J. Math. Anal. Appl.* **481**, 123472 (2020)
7. Durrett, R.: *Probability: Theory and Examples*, vol. 49. Cambridge University Press, Cambridge (2019)
8. Glatt-Holtz, N., Ziane, M.: Strong pathwise solutions of the stochastic Navier-Stokes system. *Adv. Differ. Equ.* **14**, 567–600 (2009)
9. Karatzas, I., Shreve, S.E.: *Brownian Motion and Stochastic Calculus*, vol. 113, 2nd edn. Springer, Berlin (1991)
10. Liu, Q., Zhao, J.H.: Global well-posedness for the generalized magneto-hydrodynamic equations in the critical Fourier-Herz spaces. *J. Math. Anal. Appl.* **420**, 1301–1315 (2014)
11. Miao, C.X., Wu, J.H., Zhang, Z.F.: *Littlewood-Paley Theory and its Application in Hydrodynamic Equations*. *Fundamental Series in Mordn Mathematics*, vol. 126. Science Press, Beijing (2012)
12. Ru, S. L., Abidin, M. Z.: Global well-posedness of the incompressible fractional Navier-Stokes equations in Fourier-Besov spaces with variable exponents. *Comput. Math. Appl.* **77**, 1082–1090 (2019)
13. Sritharan, S., Sundar, P.: Large deviations for the two-dimensional Navier-Stokes equations with multiplicative noise. *Stoch. Process. Appl.* **116**, 1636–1659 (2006)
14. Wang, W.H.: Global well-posedness and analyticity for the 3D fractional magnetohydrodynamics equations in variable Fourier-Besov spaces. *Z. Angew. Math. Phys.* **70**, 163 (2019)
15. Wang, W.H., Wu, G.: Global mild solution of the generalized Navier-Stokes equations with the Coriolis force. *Appl. Math. Lett.* **76**, 181–186 (2018)
16. Wang, W.H., Wu, G.: Global mild solution of stochastic generalized Navier-Stokes. *Acta Math. Sin. Engl. Ser.* **34**(11), 1635–1647 (2018)
17. Wu, J.: Lower bounds for an integral involving fractional Laplacians and the generalized Navier-Stokes equations in Besov spaces. *Commun. Math. Phys.* **263**(3), 803–831 (2006)
18. Xu, T., Zhang, T.S.: On the small time asymptotics of the two-dimensional stochastic Navier-Stokes equations. *Ann. Inst. Henri Poincaré Probab. Stat.* **45**(4), 1002–1019 (2009)

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