Hafnian point processes and quasi-free states on the CCR algebra

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Abstract

Let X be a locally compact Polish space and σ a nonatomic reference measure on X (typically $X=\mathbb{R}^d$ and σ is the Lebesgue measure). Let $X^2\ni(x,y)\mapsto\mathbb{K}(x,y)\in\mathbb{C}^{2\times2}$ be a 2×2-matrix-valued kernel that satisfies $\mathbb{K}^T(x,y)=\mathbb{K}(y,x)$. We say that a point process μ in X is hafnian with correlation kernel $\mathbb{K}(x,y)$ if, for each $n\in\mathbb{N}$, the nth correlation function of μ (with respect to $\sigma^{\otimes n}$) exists and is given by $k^{(n)}(x_1,\dots,x_n)=\operatorname{haf}\left[\mathbb{K}(x_i,x_j)\right]_{i,j=1,\dots,n}$. Here $\operatorname{haf}(C)$ denotes the hafnian of a symmetric matrix C. Hafnian point processes include permanental and 2-permanental point processes as special cases. A Cox process Π_R is a Poisson point process in X with random intensity $R(x)$. Let $G(x)$ be a complex Gaussian field on X satisfying $\int_\Delta\mathbb{E}(|G(x)|^2)\sigma(dx)<\infty$ for each compact $\Delta\subset X$. Then the Cox process Π_R with $R(x)=|G(x)|^2$ is a hafnian point process. The main result of the paper is that each such process Π_R is the joint spectral measure of a rigorously defined particle density of a representation of the canonical commutation relations (CCR), in a symmetric Fock space, for which the corresponding vacuum state on the CCR algebra is quasi-free.

Keywords: Hafnian point process, Cox process, permanental point process; quasi-free state on CCR algebra

Mathematics Subject Classification (2020): Primary 60G55; 46L30 Secondary 60G15

1 Introduction

1.1 Hafnian point processes

Let X be a locally compact Polish space, let $\mathcal{B}(X)$ denote the Borel σ-algebra on X, and let $\mathcal{B}_0(X)$ denote the algebra of all pre-compact sets from $\mathcal{B}(X)$. Let σ be a reference measure on $(X,\mathcal{B}(X))$ which is non-atomic (i.e., $\sigma(\{x\})=0$ for all $x\in X$) and Radon (i.e., $\sigma(\Delta)<\infty$ for all $\Delta\in\mathcal{B}_0(X)$). For applications, the most important example is $X=\mathbb{R}^d$ $\sigma(dx)=dx$ is the Lebesgue measure.

A *(simple) configuration* γ in X is a Radon measure on X of the form $\gamma=\sum_i\delta_{x_i}$, where δ_{x_i} denotes the Dirac measure with mass at x_i and $x_i\ne x_j$ if $i\ne j$. Note that, since γ is a Radon measure, it has a finite number of atoms in each compact set in X. Let $\Gamma(X)$ denote the set of all configurations γ in X. Let $\mathcal{C}(\Gamma(X))$ denote the minimal σ-algebra on $\Gamma(X)$ such that, for each $\Delta\in\mathcal{B}_0(X)$, the mapping $\Gamma(X)\ni\gamma\mapsto\gamma(\Delta)$ is measurable. A *(simple) point process in* X is a probability measure on $(\Gamma(X),\mathcal{C}(\Gamma(X)))$.

Denote $X^{(n)} := \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}$. A measure on $X^{(n)}$ is called symmetric if it remains invariant under the natural action of the symmetric group $\mathfrak{S}_n$ on $X^{(n)}$. For each $\gamma = \sum_i \delta_{x_i} \in \Gamma(X)$, the *spatial falling factorial* $(\gamma)_n$ is the symmetric measure on $X^{(n)}$ of the form

$$(\gamma)_n := \sum_{i_1} \sum_{i_2 \neq i_1} \cdots \sum_{i_n \neq i_1, \dots, i_n \neq i_{n-1}} \delta_{(x_{i_1}, x_{i_2}, \dots, x_{i_n})}. \tag{1}$$

Let μ be a point process in X. The *n-th correlation measure of* μ is the symmetric measure $\theta^{(n)}$ on $X^{(n)}$ defined by

$$\theta^{(n)}(dx_1 \cdots dx_n) := \frac{1}{n!} \int_{\Gamma(X)} (\gamma)_n(dx_1 \cdots dx_n)\, \mu(d\gamma). \tag{2}$$

If each measure $\theta^{(n)}$ is absolutely continuous with respect to $\sigma^{\otimes n}$, then the symmetric functions $k^{(n)} : X^{(n)} \to [0, \infty)$ satisfying

$$d\theta^{(n)} = \frac{1}{n!}\, k^{(n)} d\sigma^{\otimes n} \tag{3}$$

are called the *correlation functions of the point process* μ. Under a very weak assumption, the correlations functions (or correlation measures) uniquely identify a point process, see [24].

Let $C = [c_{ij}]_{i,j=1,\dots,2n}$ be a symmetric $2n \times 2n$-matrix. The *hafnian of* C is defined by

$$\operatorname{haf}(C) := \frac{1}{n!\, 2^n} \sum_{\pi \in \mathfrak{S}_{2n}} \prod_{i=1}^{n} c_{\pi(2i-1)\pi(i)},$$

see e.g. [7, Section 4.1]. (Note the the value of the hafnian of C does not depend on the diagonal elements of the matrix C.) The hafnian can also be written as

$$\operatorname{haf}(C) = \sum c_{i_1 j_1} \cdots c_{i_n j_n}, \tag{4}$$

where the summation is over all (unordered) partitions $\{i_1, j_1\}, \dots, \{i_n, j_n\}$ of $\{1, \dots, 2n\}$.

Hafnians were introduced by physicist Edoardo Caianiello in the 1950's, while visiting Niels Bohr's group in Copenhagen (whose latin name is Hafnia), as a Boson analogue of the formula expressing the correlations of a quasi-free Fermi state.[1]

By analogy with the definition of a pfaffian point process (see e.g. [13, Section 10] and the references therein), we now define a hafnian point process. Let $X^2 \ni (x, y) \mapsto \mathbb{K}(x, y) \in \mathbb{C}^{2\times 2}$ be a 2×2-matrix-valued kernel that satisfies $\mathbb{K}^T(x, y) = \mathbb{K}(y, x)$. We will say that a point process μ is *hafnian with correlation kernel* $\mathbb{K}(x, y)$ if, for each $n \in \mathbb{N}$, the nth correlation function of μ exists and is given by

$$k^{(n)}(x_1, \dots, x_n) = \operatorname{haf}\left[\mathbb{K}(x_i, x_j)\right]_{i,j=1,\dots,n}. \tag{5}$$

[1] We are grateful to the referee for sharing with us this historical fact.

Note that the matrix

$$\big[\mathbb{K}(x_i,x_j)\big]_{i,j=1,\dots,n}=\begin{bmatrix}\mathbb{K}(x_1,x_1) & \mathbb{K}(x_1,x_2) & \cdots & \mathbb{K}(x_1,x_n)\\ \mathbb{K}(x_2,x_1) & \mathbb{K}(x_2,x_2) & \cdots & \mathbb{K}(x_2,x_n)\\ \vdots & \vdots & \vdots & \vdots\\ \mathbb{K}(x_n,x_1) & \mathbb{K}(x_n,x_2) & \cdots & \mathbb{K}(x_n,x_n)\end{bmatrix}$$

is built upon 2×2-blocks $\mathbb{K}(x_i,x_j)$, hence it has dimension $2n\times 2n$. Furthermore, the condition $\mathbb{K}^T(x,y)=\mathbb{K}(y,x)$ ensures that the matrix $\big[\mathbb{K}(x_i,x_j)\big]_{i,j=1,\dots,n}$ is symmetric, and so its hafnian is a well-defined number.

Since

$$X^2=\{(x,x)\mid x\in X\}\sqcup X^{(2)},$$

for the definition of a hafnian point process, it is sufficient to assume that $\mathbb{K}(x,x)$ is defined for σ-a.a. $x\in X$, and the restriction of $\mathbb{K}(x,y)$ to $X^{(2)}$ is defined for $\sigma^{\otimes 2}$-a.a. $(x,y)\in X^{(2)}$.

Note that, for the hafnian point process μ, the correlation kernel $\mathbb{K}(x,y)$ is not uniquely determined by μ. Indeed, since the hafnian of a matrix does not depend on its diagonal elements, formula (5) implies that the correlation functions $k^{(n)}(x_1,\dots,x_n)$ do not depend on the diagonal elements of the 2×2-matrices $\mathbb{K}(x,x)$ for $x\in X$. Hence, these elements can be chosen arbitrarily.

Let $\alpha\in\mathbb{R}$ and let $B=[b_{ij}]_{i,j=1,\dots,n}$ be an $n\times n$ matrix. The *α-determinant of B* is defined by

$$\det{}_\alpha(B):=\sum_{\pi\in\mathfrak{S}_n}\prod_{i=1}^n\alpha^{n-\nu(\pi)}\,b_{i\,\pi(i)},\tag{6}$$

see [33, 35]. In formula (6), for $\pi\in\mathfrak{S}_n$, $\nu(\pi)$ denotes the number of cycles in the permutation π. In particular, for $\alpha=1$, $\det_1(B)$ is the usual permanent of B.

A point process μ is called *α-permanental (or α-determinantal) with correlation kernel* $K:X^2\to\mathbb{C}$ if, for each $n\in\mathbb{N}$, the nth correlation function of μ exists and is given by

$$k^{(n)}(x_1,\dots,x_n)=\det{}_\alpha[K(x_i,x_j)]_{i,j=1,\dots,n},$$

[33], see also [17]. For $\alpha=1$, one calls μ a *permanental point process.*

As easily follows from [7, Section 4.1] a permanental point process with correlation kernel $K(x,y)$ is hafnian with correlation kernel

$$\mathbb{K}(x,y)=\begin{bmatrix}0 & K(x,y)\\ K(y,x) & 0\end{bmatrix}.$$

Furthermore, similarly to [20, Proposition 1.1], we see that a 2-permanental point process with a symmetric correlation kernel $K(x,y)=K(y,x)$ is hafnian with the correlation kernel

$$\mathbb{K}(x,y)=\begin{bmatrix}K(x,y) & K(x,y)\\ K(x,y) & K(x,y)\end{bmatrix}.$$

For studies of permanental, and more generally α-permanental point processes, we refer to [9, 22, 27, 28, 33].

Recall that a *Cox process* Π_R is a Poisson point process with a random intensity $R(x)$. Here $R(x)$ is a random field defined for σ-a.a $x \in X$ and taking a.s. non-negative values. The correlation functions of the Cox process Π_R are given by

$$k^{(n)}(x_1, \dots, x_n) = \mathbb{E}\big(R(x_1) \cdots R(x_n)\big). \tag{7}$$

Let $G(x)$ be a mean-zero, complex Gaussian field defined for σ-a.a. $x \in X$. Assume additionally that $\int_\Delta \mathbb{E}(|G(x)|^2)\sigma(dx) < \infty$ for each $\Delta \in \mathcal{B}_0(X)$. Let $R(x) := |G(x)|^2 = G(x)\overline{G(x)}$. Comparing the classical moment formula for Gaussian random variables with formula (4), we immediately see that

$$\mathbb{E}\big(R(x_1) \cdots R(x_n)\big) = \operatorname{haf}\big[\mathbb{K}(x_i, x_j)\big]_{i,j=1,\dots,n}, \tag{8}$$

where

$$\mathbb{K}(x, y) = \begin{bmatrix} \mathbb{E}(G(X)G(y)) & \mathbb{E}(G(x)\overline{G(y)}) \\ \mathbb{E}(\overline{G(x)}G(y)) & \mathbb{E}(\overline{G(x)G(y)}) \end{bmatrix} = \begin{bmatrix} \mathcal{K}_2(x, y) & \mathcal{K}_1(x, y) \\ \overline{\mathcal{K}_1(x, y)} & \overline{\mathcal{K}_2(x, y)} \end{bmatrix}. \tag{9}$$

Here $\mathcal{K}_1(x, y) := \mathbb{E}(G(x)\overline{G(y)})$ is the *covariance* of the Gaussian field and $\mathcal{K}_2(x, y) := \mathbb{E}(G(x)G(y))$ is the *pseudo-covariance* of the Gaussian field. By (7)–(9), the corresponding Cox process Π_R is hafnian with the correlation kernel (9).

In the case where the Gaussian field $G(x)$ is real-valued, the moments of $R(x)$ are given by the 2-determinants built upon the kernel $K(x, y) := \mathcal{K}_1(x, y) = \mathcal{K}_2(x, y)$, hence $R(x)$ is a 2-permanental process. For studies of α-permanental processes, we refer e.g. to [16–19, 22, 30–32] and the references therein. Obviously, in this case, Π_R is a 2-permanental point process with the correlation kernel $K(x, y)$, compare with [33, Subsection 6.4].

A Gaussian random field is called *proper* if $\mathcal{K}_2(x, y) = 0$ for all x and y. Since the moments of the random field $R(x)$ are given by permanents built upon the kernel $K(x, y) := \mathcal{K}_1(x, y)$, $R(x)$ is a permanental process, compare with [9, 27, 28]. We note, however, that the available studies of α-permanental processes usually discuss only the case where the kernel is real-valued. In the case of $R(x)$, the correlation kernel is, of course, complex-valued.

1.2 Aim of the paper

Quasi-free states play a central role in studies of operator algebras related to quantum statistical mechanics, see e.g. [1–4, 14, 15, 29].

Let $\mathcal{H} = L^2(X, \sigma)$ be the L^2-space of σ-square-integrable functions $h : X \to \mathbb{C}$. Let $\mathfrak{F}$ be a separable complex Hilbert spaces. Let $A^+(h)$, $A^-(h)$ ($h \in \mathcal{H}$) be linear operators in $\mathfrak{F}$ that satisfy the following assumptions:

(i) $A^+(h)$ and $A^-(h)$ depend linearly on $h \in \mathcal{H}$;

(ii) for each $h \in \mathcal{H}$, $A^-(\overline{h})$ is (the restriction of) the adjoint operator of $A^+(h)$, where $\bar{h}$ is the complex conjugate of h;

(iii) the operators $A^+(h)$, $A^-(h)$ satisfy the canonical commutation relations (CCR).

See Section 3 for details.

Let $\mathbb{A}$ be the unital $*$-algebra generated by the operators $A^+(h)$, $A^-(h)$. If we additionally assume that $\mathfrak{F}$ is a certain symmetric Fock space, then we can define the vacuum state τ on $\mathbb{A}$. If τ appears to be a quasi-free state, one says that the operators $A^+(h)$ and $A^-(h)$ form a *quasi-free representation of the CCR*.

We define operator-valued distributions $A^+(x)$ and $A^-(x)$ ($x \in X$) through the equalities

$$A^+(h) = \int_X h(x)A^+(x)\sigma(dx), \quad A^-(h) = \int_X h(x)A^-(x)\sigma(dx), \tag{10}$$

holding for all $h \in \mathcal{H}$.

Then the *particle density* $\rho(x)$ is formally defined as

$$\rho(x) := A^+(x)A^-(x), \quad x \in X.$$

We called this definition *formal* since it requires to take product of two operator-valued distributions, and *a priori* it is not clear if this product indeed makes sense. Nevertheless, in all the examples below, we will be able to rigorously define $\rho(x)$ as an operator-valued distribution.

The CCR imply the commutation $[\rho(x), \rho(y)] = 0$ ($x, y \in X$), where $[\cdot,\cdot]$ denotes the commutator. For each $\Delta \in \mathcal{B}_0(X)$, we denote

$$\rho(\Delta) := \int_\Delta \rho(x)\sigma(dx) = \int_\Delta A^+(x)A^-(x)\sigma(dx), \tag{11}$$

which is a family of Hermitian commuting operators in the Fock space $\mathfrak{F}$. In view of the spectral theorem, one can expect that the operators $(\rho_\Delta)_{\Delta\in\mathcal{B}_0(X)}$ can be realized as operators of multiplication in $L^2(\Gamma_X, \mu)$, where μ is the joint spectral measure of this family of operators at the vacuum.

Let $G(x)$ be a complex-valued Gaussian field and $R(x) = |G(x)|^2$. The main aim of the paper is show that the Cox process Π_R is the joint spectral measure of a (rigorously defined) particle density $(\rho_\Delta)_{\Delta\in\mathcal{B}_0(X)}$ for a certain quasi-free representation of the CCR. As a by-product, we obtain a unitary isomorphism between a subspace of a Fock space and $L^2(\Gamma_X, \Pi_R)$.

In the special case where Π_R is a permanental point process (with a real-valued correlation kernel), such a statement was proved in [26] (see also [25]). In that case, the corresponding quasi-free state has an additional property of being gauge-invariant, so one could use the gauge-invariant quasi-free representation of the CCR by Araki and Woods [5].

We stress that, even in the case of a gauge-invariant quasi-free state, the representation of the CCR that we use in this paper has a different form as compared to the one by Araki and Woods [5]. Nevertheless, since both gauge-invariant quasi-free representations have the same n-point functions, one can show that these representations are unitarily equivalent.

We note that, in [25, 26], it was also shown that each determinantal point process ($\alpha = -1$) arises as the joint spectral measure of the particle density of a quasi-free representation of the Canonical Anticommutation Relations (CAR). In that case, the state is also gauge-invariant, so one can use the Araki–Wyss representation of the CAR from [6].

It is worth to compare our result with the main result of Koshida [23]. In the latter paper, it is proven that, when the underlying space X is *discrete*, every pfaffian point process on X arises as the particle density of a quasi-free representation of the CAR. As noted in [23], a similar statement in the case of a continuous space X is still an open problem.

1.3 Organization of the paper

The starting point of our considerations is the observation that the Poisson point process with (deterministic) intensity $|\lambda(x)|^2$ arises from the trivial (quasi-free) representation of the CCR with

$$A^+(x) = a^+(x) + \overline{\lambda(x)}, \quad A^-(x) = a^-(x) + \lambda(x), \tag{12}$$

where $a^+(x)$, $a^-(x)$ are the creation and annihilation operators at point x, acting in the symmetric Fock space $\mathcal{F}(\mathcal{H})$ over $\mathcal{H}$, compare with [21]. We then proceed as follows:

- We realize a Gaussian field $G(x)$ as a family of operators $\Phi(x)$ acting in a Fock space $\mathcal{F}(\mathcal{G})$ over a Hilbert space $\mathcal{G}$ (typically $\mathcal{G} = \mathcal{H}$ or $\mathcal{G} = \mathcal{H} \oplus \mathcal{H}$).
- We consider a quasi-free representation of the CCR with

$$A^+(x) = a^+(x) + \Phi^*(x), \quad A^-(x) = a^-(x) + \Phi(x) \tag{13}$$

  acting in the Fock space $\mathcal{F}(\mathcal{H} \otimes \mathcal{G}) = \mathcal{F}(\mathcal{H}) \otimes \mathcal{F}(\mathcal{G})$.
- We prove that the corresponding particle density $(\rho_\Delta)_{\Delta \in \mathcal{B}_0(X)}$ is well-defined and has the joint spectral measure Π_R.

The paper is organized as follows. In Section 2, we discuss complex-valued Gaussian fields on X realized in a symmetric Fock space $\mathcal{F}(\mathcal{G})$ over a separable Hilbert space $\mathcal{G}$. We start with a $\mathcal{G}^2$-valued function $(L_1(x), L_2(x))$ that is defined for σ-a.a. $x \in X$ and satisfies the assumptions (15), (16) below. We then define operators $\Phi(x)$ in the Fock space $\mathcal{F}(\mathcal{G})$ by formula (17). Theorem 2.1 states that the operators $\Phi(x)$ form a Fock-space realization of a Gaussian field $G(x)$ that is defined for σ-a.a. $x \in X$. (Note, however, that the set of those $x \in X$ for which $G(x)$ is defined can be smaller than the set of those $x \in X$ for which the function $(L_1(x), L_2(x))$ was defined.) The covariance and pseudo-covariance of the Gaussian field $G(x)$ are given by formulas (19) and (20), respectively.

As a consequence of our considerations, in Example 2.4, we derive a Fock-space realization of a proper Gaussian field. The operators $\Phi(x)$ in this case resemble the classical Fock-space realization of a real-valued Gaussian field. The main difference is that, in the case of a real-valued Gaussian field, the creation and annihilation operators use same real vectors, whereas in the case of a proper Gaussian field, the creation and annihilation operators use orthogonal copies of same complex vectors.

In Section 3, we briefly recall the definition of a quasi-free state on the CCR algebra and a quasi-free representation of the CCR.

Next, in Section 4, we recall in Theorem 4.1 a result from [26] which gives sufficient conditions for a family of commuting Hermitian operators, $(\rho(\Delta))_{\Delta\in\mathcal{B}_0(X)}$, in a separable complex Hilbert space, to be essentially self-adjoint and have a point process μ in X as their joint spectral measure. The key condition of Theorem 4.1 is that the family of operators, $(\rho(\Delta))_{\Delta\in\mathcal{B}_0(X)}$, should possess certain correlation measures $\theta^{(n)}$, whose definition is given in Section 4. These measures $\theta^{(n)}$ are then also the correlation measures of the point process μ. We also present formal considerations about the form of the correlation measures $\theta^{(n)}$ when $\rho(\Delta)$ is a particle density given by (11).

In Section 5, we apply Theorem 4.1 to show that a Poisson point process is the joint spectral measure of the operators $(\rho(\Delta))_{\Delta\in\mathcal{B}_0(X)}$, where $\rho(\Delta)$ is the particle density of the trivial quasi-free representation of the CCR in which the creation and annihilation operators are given by (12).

The main results of the paper are in Section 6. Using the $\mathcal{G}^2$-valued function $(L_1(x), L_2(x))$ from Section 2, we construct a quasi-free representation of the CCR in the symmetric Fock space $\mathcal{F}(\mathcal{H}\oplus\mathcal{G})$. We prove that the corresponding particle density is well defined as a family of commuting Hermitian operators, $(\rho(\Delta))_{\Delta\in\mathcal{B}_0(X)}$ (Corollary 6.5). Theorem 6.6 states that these operators satisfy the assumptions of Theorem 4.1 and their joint spectral measure μ is the Cox process Π_R, where $R(x) = |G(x)|^2$ and $G(x)$ is the Gaussian field as in in Theorem 2.1. In particular, μ is a hafnian point process.

2 Fock-space realization of complex Gaussian fields

Let $\mathcal{G}$ be a separable Hilbert space with an antilinear involution $\mathcal{J}$ satisfying $(\mathcal{J}f, \mathcal{J}g)_{\mathcal{G}} = (g, f)_{\mathcal{G}}$ for all $f, g \in \mathcal{G}$. Let $\mathcal{G}^{\odot n}$ denote the nth symmetric tensor power of $\mathcal{G}$. For $n \in \mathbb{N}$, let $\mathcal{F}_n(\mathcal{G}) := \mathcal{G}^{\odot n}n!$, i.e., $\mathcal{F}_n(\mathcal{G})$ coincides with $\mathcal{G}^{\odot n}$ as a set and the inner product in $\mathcal{F}_n(\mathcal{G})$ is equal to $n!$ times the inner product in $\mathcal{G}^{\odot n}$. Let also $\mathcal{F}_0(\mathcal{G}) := \mathbb{C}$. Then $\mathcal{F}(\mathcal{G}) = \bigoplus_{n=0}^{\infty}\mathcal{F}_n(\mathcal{G})$ is called the *symmetric Fock space over* $\mathcal{G}$. The vector $\Omega = (1, 0, 0, \dots) \in \mathcal{F}(\mathcal{G})$ is called the *vacuum*.

Let $\mathcal{F}_{\mathrm{fin}}(\mathcal{G})$ denote the (dense) subspace of $\mathcal{F}(\mathcal{G})$ consisting of all finite vectors

$f = (f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, \dots)$ $(n \in \mathbb{N})$. We equip $\mathcal{F}_{\mathrm{fin}}(\mathcal{G})$ with the topology of the topological direct sum of the $\mathcal{F}_n(\mathcal{G})$ spaces.

For topological vector spaces V and W, we denote by $\mathcal{L}(V, W)$ the space of all linear continuous operators $A : V \to W$. We also denote $\mathcal{L}(V) := \mathcal{L}(V, V)$.

Let $g \in \mathcal{G}$. We define a *creation operator* $a^+(g) \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{G}))$ by $a^+(g)\Omega := g$, and for $f^{(n)} \in \mathcal{F}_n(\mathcal{G})$ $(n \in \mathbb{N})$, $a^+(g)f^{(n)} := g \odot f^{(n)} \in \mathcal{F}_{n+1}(\mathcal{G})$. Next, we define an *annihilation operator* $a^-(g) \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{G}))$ that satisfies $a^-(g)\Omega := 0$ and for any $f_1, \dots, f_n \in \mathcal{G}$,

$$a^-(g) f_1 \odot \cdots \odot f_n = \sum_{i=1}^{n} (f_i, \mathcal{J} g)_{\mathcal{G}}\, f_1 \odot \cdots \odot f_{i-1} \odot f_{i+1} \odot \cdots \odot f_n.$$

We have $a^+(g)^* \restriction_{\mathcal{F}_{\mathrm{fin}}(\mathcal{G})} = a^-(\mathcal{J} g)$ and the operators $a^+(g)$, $a^-(g)$ satisfy the CCR:

$$[a^+(f), a^+(g)] = [a^-(f), a^-(g)] = 0, \quad [a^-(f), a^+(g)] = (g, \mathcal{J} f)_{\mathcal{G}} \tag{14}$$

for all $f, g \in \mathcal{G}$.

Let $D \in \mathcal{B}(X)$ be such that $\sigma(X \setminus D) = 0$. Let $D \ni x \mapsto (L_1(x), L_2(x)) \in \mathcal{G}^2$ be a measurable mapping. We assume that

$$(L_1(x), \mathcal{J} L_2(y))_{\mathcal{G}} = (L_1(y), \mathcal{J} L_2(x))_{\mathcal{G}}, \tag{15}$$

$$(L_1(x), L_1(y))_{\mathcal{G}} = (L_2(x), L_2(y))_{\mathcal{G}} \quad \text{for all } x, y \in D. \tag{16}$$

Define

$$\Phi(x) := a^+(L_1(x)) + a^-(L_2(x)). \tag{17}$$

Let $\Psi(x) := \Phi(x)^* \restriction_{\mathcal{F}_{\mathrm{fin}}(\mathcal{G})}$. Then

$$\Psi(x) = a^+(\mathcal{J} L_2(x)) + a^-(\mathcal{J} L_1(x)). \tag{18}$$

It follows from (14) that conditions (15), (16) are necessary and sufficient in order that $[\Phi(x), \Phi(y)] = [\Psi(x), \Psi(y)] = [\Phi(x), \Psi(y)] = 0$ for all $x, y \in D$.

Below, for each $\Lambda \subset D$, we denote by $\mathbb{F}_\Lambda$ the subspace of the Fock space $\mathcal{F}(\mathcal{G})$ that is the closed linear span of the set

$$\big\{\Psi(x_1)^{k_1} \cdots \Psi(x_m)^{k_m} \Phi(y_1)^{l_1} \cdots \Phi(y_n)^{l_n} \Omega \mid x_1, \dots, x_m, y_1, \dots, y_n \in \Lambda,\\ k_1, \dots, k_m, l_1, \dots, l_n \in \mathbb{N}_0,\ m, n \in \mathbb{N}\big\}.$$

Here $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$.

Theorem 2.1. *There exists a measurable subset $\Lambda \subset D$ with $\sigma(X \setminus \Lambda) = 0$ and a mean-zero complex-valued Gaussian field $\{G(x)\}_{x \in \Lambda}$ on a probability space $(\Xi, \mathfrak{A}, P)$ such that:*

(i) The Gaussian field $\{G(x)\}_{x\in\Lambda}$ has the covariance

$$\mathcal{K}_1(x,y) = (L_1(x), L_1(y))_{\mathcal{G}}, \quad x,y\in\Lambda, \tag{19}$$

and the pseudo-covariance

$$\mathcal{K}_2(x,y) = (L_1(x), \mathcal{J}L_2(y))_{\mathcal{G}}, \quad x,y\in\Lambda. \tag{20}$$

(ii) There exists a unique unitary operator $\mathcal{I}:\mathbb{F}_\Lambda\to L^2(\Xi,P)$ that satisfies

$$\begin{aligned}&\mathcal{I}\Psi(x_1)^{k_1}\cdots\Psi(x_m)^{k_m}\Phi(y_1)^{l_1}\cdots\Phi(y_n)^{l_n}\Omega\\ &\quad=\overline{G(x_1)}^{k_1}\cdots\overline{G(x_m)}^{k_m}G(y_1)^{l_1}\cdots G(y_n)^{l_n}\end{aligned} \tag{21}$$

for all $x_1,\dots,x_m,y_1,\dots,y_n\in\Lambda$, $k_1,\dots,k_m,l_1,\dots,l_n\in\mathbb{N}_0$, $m,n\in\mathbb{N}$.

Proof. We define $L_1(x)=L_2(x)=0$ for all $x\in X\setminus D$. Then

$$X\ni x\mapsto (L_1(x),L_2(x))\in\mathcal{G}^2 \tag{22}$$

is measurable and satisfies (15), (16) for all $x,y\in X$. By Lusin's theorem (see e.g. [8, 26.7 Theorem]), there exists a sequence of mutually disjoint compact sets $(\Lambda_n)_{n=1}^\infty$ such that $\sigma\big(X\setminus\bigcup_{n=1}^\infty\Lambda_n\big)=0$, and the restriction of the mapping (22) to each Λ_n is continuous. Denote $\Lambda:=\bigcup_{n=1}^\infty\Lambda_n$ and choose a countable subset $\Lambda'\subset\Lambda$ such that, for each $n\in\mathbb{N}$, the set $\Lambda'\cap\Lambda_n$ is dense in Λ_n. As easily seen by approximation, $\mathbb{F}_{\Lambda'}=\mathbb{F}_\Lambda$.

Let us consider the real and imaginary parts of the operators $\Phi(x)$:

$$\begin{aligned}\Re(\Phi(x)) &:= \frac{1}{2}(\Phi(x)+\Psi(x)) = \frac{1}{2}\left(a^+\big(L_1(x)+\mathcal{J}L_2(x)\big)+a^-\big(\mathcal{J}L_1(x)+L_2(x)\big)\right),\\ \Im(\Phi(x)) &:= \frac{1}{2i}(\Phi(x)-\Psi(x)) = \frac{1}{2i}\left(a^+\big(L_1(x)-\mathcal{J}L_2(x)\big)-a^-\big(\mathcal{J}L_1(x)-L_2(x)\big)\right).\end{aligned} \tag{23}$$

These operators are Hermitian and commuting.

It is a standard fact that, for each $g\in\mathcal{G}$,

$$\|a^+(g)\|_{\mathcal{L}(\mathcal{F}_k(\mathcal{G}),\mathcal{F}_{k+1}(\mathcal{G}))} = \|a^-(g)\|_{\mathcal{L}(\mathcal{F}_{k+1}(\mathcal{G}),\mathcal{F}_k(\mathcal{G}))} = \sqrt{k+1}\,\|g\|_{\mathcal{G}}. \tag{24}$$

From here it easily follows that each $f\in\mathcal{F}_{\mathrm{fin}}(\mathcal{G})$ is an analytic vector for each $\Re(\Phi(x))$ and $\Im(\Phi(x))$ ($x\in X$), and the projection-valued measures of the closures of all these operators commute, see [10, Chapter 5, Theorem 1.15].

We now apply the projection spectral theorem [10, Chapter 3, Theorems 2.6 and 3.9 and Section 3.1] to the closures of the operators $\Re(\Phi(x))$ and $\Im(\Phi(x))$ with $x\in\Lambda'$. This implies the existence of a probability space $(\Xi,\mathfrak{A},P)$, real-valued random variables $G_1(x)$ and $G_2(x)$ ($x\in\Lambda'$) and a unique unitary operator $\mathcal{I}:\mathbb{F}_\Lambda\to L^2(\Xi,P)$ that satisfies

$$\begin{aligned}&\mathcal{I}\,\Re(\Phi(x_1))^{k_1}\cdots\Re(\Phi(x_m))^{k_m}\Im(\Phi(y_1))^{l_1}\cdots\Im(\Phi(y_n))^{l_n}\Omega\\ &\quad= G_1(x_1)^{k_1}\cdots G_1(x_m)^{k_m}G_2(y_1)^{l_1}\cdots G_2(y_n)^{l_n}\end{aligned} \tag{25}$$

for all $x_1,\dots,x_m,y_1,\dots,y_n\in\Lambda'$, $k_1,\dots,k_m,l_1,\dots,l_n\in\mathbb{N}_0$, $m,n\in\mathbb{N}$.

Remark 2.2. In fact, $\Xi = \{\omega : \Lambda' \to \mathbb{R}^2\}$, $\mathfrak{A}$ is the cylinder σ-algebra on Ξ (equivalently the countable product of the Borel σ-algebras $\mathcal{B}(\mathbb{R})$), and $P(\cdot) = (E(\cdot)\Omega, \Omega)_{\mathcal{F}(\mathcal{G})}$. Here, E is the projection-valued measure on $(\Xi, \mathfrak{A})$ that is constructed as the countable product of the projection-valued measures of the closures of the operators $\Re(\Phi(x))$ and $\Im(\Phi(x))$ with $x \in \Lambda'$. Furthermore, for each $x \in \Lambda'$, $(G_i(x))(\omega) = \omega_i(x)$ for $\omega = (\omega_1, \omega_2) \in \Xi$.

Next, let $n \in \mathbb{N}$ and $x \in \Lambda_n \setminus \Lambda'$. Then we can find a sequence $(x_k)_{k=1}^\infty$ in $\Lambda' \cap \Lambda_n$ such that $x_k \to x$, hence, by continuity, $(L_1(x_k), L_2(x_k)) \to (L_1(x), L_2(x))$ in $\mathcal{G}^2$. It follows from (25) that $(G_i(x_k))_{k=1}^\infty$ is a Cauchy sequence in $L^2(\Xi, P)$ $(i = 1, 2)$, so we define $G_i(x) := \lim_{k\to\infty} G_i(x_k)$. Then we easily see by approximation that (25) remains true for all $x_1, \dots, x_m, y_1, \dots, y_n \in \Lambda$.

Let Z be an arbitrary finite linear combination (with real coefficients) of random variables from $\{G_1(x),\, G_2(x) \mid x \in \Lambda\}$. Then it follows from (23) that the moments of Z are given by

$$\mathbb{E}(Z^k) = \left((a^+(g) + a^-(\mathcal{J}g))^k\, \Omega, \Omega\right)_{\mathcal{F}(\mathcal{G})}$$

for some $g \in \mathcal{G}$. But this implies that the random variable Z has a Gaussian distribution, see e.g. [10, Chapter 3, Subsection 3.8]. Hence, $\{G_1(x),\, G_2(x) \mid x \in \Lambda\}$ is a Gaussian field.

Finally, for each $x \in \Lambda$, we define $G(x) := G_1(x) + iG_2(x)$. Then $\{G(x)\}_{x\in\Lambda}$ is a complex-valued Gaussian field. Formula (25) implies (21). This, in turn, gives us the covariance and the pseudo-covariance of the Gaussian field $\{G(x)\}_{x\in\Lambda}$. □

Let $\mathcal{H} = L^2(X, \sigma)$ and define an antilinear involution $J : \mathcal{H} \to \mathcal{H}$ by $(Jh)(x) := \overline{h(x)}$. Let us consider a measurable mapping $x \mapsto L(x) \in \mathcal{H}$ defined σ-a.e. on X, and let $K(x, y) := (L(x), L(y))_{\mathcal{H}}$. We will now consider two examples of complex-valued Gaussian fields with covariance $K(x, y)$.

Example 2.3. Let $\mathcal{G} = \mathcal{H}$, $\mathcal{J} = J$, and let $L_1(x) = L_2(x) = L(x)$. Obviously, conditions (15), (16) are satisfied. Then,

$$\Phi(x) = a^+(L(x)) + a^-(L(x)), \quad \Psi(x) = a^+(JL(x)) + a^-(JL(x)).$$

By Theorem 2.1, the corresponding Gaussian field $G(x)$ has the covariance $\mathcal{K}_1(x, y) = K(x, y)$ and the pseudo-covariance $\mathcal{K}_2(x, y) = (L(x), JL(y))_{\mathcal{H}}$. If $L(x)$ is a real-valued function for σ-a.a. $x \in X$, $\mathcal{K}_1(x, y) = \mathcal{K}_2(x, y) = K(x, y)$, while the function $K(x, y)$ is symmetric. Hence, as discussed in Introduction, $R(x) := |G(x)|^2$ is a 2-permanental process, defined for σ-a.a. $x \in X$. If $L(x)$ is not real-valued on a set of positive σ measure, then the moments of $R(x)$ are given by (8), (9) with $\mathcal{K}_1(x, y)$, $\mathcal{K}_2(x, y)$ as above.

Example 2.4. Let $\mathcal{G} = \mathcal{H} \oplus \mathcal{H}$, $\mathcal{J} = J \oplus J$, and let $L_1(x) = (L(x), 0)$, $L_2(x) = (0, L(x))$. As easily seen, conditions (15), (16) are satisfied. We define, for each $h \in \mathcal{H}$, $a_1^+(h) := a^+(h, 0)$, $a_2^+(h) := a^+(0, h)$ and similarly $a_1^-(h)$, $a_2^-(h)$. Then

$$\Phi(x) = a_1^+(L(x)) + a_2^-(L(x)), \quad \Psi(x) = a_2^+(JL(x)) + a_1^-(JL(x)).$$

For the corresponding Gaussian field $G(x)$, $\mathcal{K}_1(x,y)=K(x,y)$, while $\mathcal{K}_2(x,y)=0$, i.e., $G(x)$ is a proper Gaussian field. Hence, as discussed in Introduction, $R(x):=|G(x)|^2$ is a permanental process, defined for σ-a.a. $x\in X$.

Remark 2.5. Let $G_1(x)$ and $G_2(x)$ be two independent copies of the Gaussian field from Example 2.3. Then, the Gaussian field $G(x)$ from Example 2.4 can be constructed as $G(x)=\frac{1}{\sqrt{2}}\big(G_1(x)+iG_2(x)\big)$.

The following example generalizes the constructions in Examples 2.3 and 2.4.

Example 2.6. Let $\mathcal{G}=\mathcal{H}\oplus\mathcal{H}$ be as in Example 2.4 and consider a measurable mapping $x\mapsto(\alpha(x),\beta(x))\in\mathcal{G}^2$ defined σ-a.e. on X. Let

$$L_1(x):=\left(\frac{\alpha(x)+\beta(x)}{2},\frac{\alpha(x)-\beta(x)}{2}\right),\quad L_2(x):=\left(\frac{\alpha(x)-\beta(x)}{2},\frac{\alpha(x)+\beta(x)}{2}\right).$$

As easily seen, conditions (15) and (16) are satisfied. For the corresponding Gaussian field $G(x)$,

$$\begin{aligned}\mathcal{K}_1(x,y)&=\frac{1}{2}\big((\alpha(x),\alpha(y))_{\mathcal{H}}+(\beta(x),\beta(y))_{\mathcal{H}}\big),\\ \mathcal{K}_2(x,y)&=\frac{1}{2}\big((\alpha(x),J\alpha(y))_{\mathcal{H}}-(\beta(x),J\beta(y))_{\mathcal{H}}\big).\end{aligned}$$

In the special case where $L(x)=\alpha(x)=\beta(x)$, this is just the construction from Example 2.4. When choosing $\alpha(x)=\sqrt{2}\,L(x)$ and $\beta(x)=0$, the corresponding Gaussian field $G(x)$ has the same finite-dimensional distributions as the Gaussian field from Example 2.3.

Let the conditions of Theorem 2.1 be satisfied and $R(x)=|G(x)|^2$. To construct the Cox process Π_R with correlation functions given by (7), we further assume that, for each $\Delta\in\mathcal{B}_0(X)$, $\int_\Delta\mathbb{E}(R(x))\sigma(dx)<\infty$. By (16) and (19), this is equivalent to the condition

$$\int_\Delta\|L_1(x)\|^2_{\mathcal{G}}\,\sigma(dx)=\int_\Delta\|L_2(x)\|^2_{\mathcal{G}}\,\sigma(dx)<\infty. \tag{26}$$

Example 2.3 (continued). Since $\mathcal{G}=\mathcal{H}$, we define $L(x,y):=(L(x))(y)$. By (26),

$$\int_{\Delta\times X}|L(x,y)|^2\sigma^{\otimes 2}(dx\,dy)<\infty.$$

Hence, for each $\Delta\in\mathcal{B}_0(X)$, we can define a Hilbert–Schmidt operator L^Δ in $\mathcal{H}$ with integral kernel $\chi_\Delta(x)L(x,y)$. Here χ_Δ denotes the indicator function of the set Δ. Define $K^\Delta:=L^\Delta(L^\Delta)^*$. This operator is nonnegative ($K^\Delta\geq 0$), trace-class, and has integral kernel $K^\Delta(x,y)=(L(x),L(y))_{\mathcal{H}}$ for $x,y\in\Delta$. (Note that $K^\Delta(x,y)$ vanishes outside Δ^2). Thus, for $x,y\in\Delta$, the covariance $\mathcal{K}_1(x,y)$ of the Gaussian $G(x)$ is equal to $K^\Delta(x,y)$.

Next, for a bounded linear operator $A \in \mathcal{H}$, we define the *transposed* of A by $A^T := JA^*J$. If A is an integral operator with integral kernel $A(x,y)$, then A^T is the integral operator with integral kernel $A^T(x,y) = A(y,x)$. Hence, for all $x, y \in \Delta$, the pseudo-covariance $\mathcal{K}_2(x,y)$ of the Gaussian $G(x)$ is equal to the integral kernel $Q^\Delta(x,y)$ of the operator $Q^\Delta := L^\Delta(L^\Delta)^T$.

In the case where $L(x,y)$ is an integral kernel of a bounded linear operator L in $\mathcal{H}$, we can define $K := LL^*$ and $Q := LL^T$, and $\mathcal{K}_1(x,y) = K(x,y)$, $\mathcal{K}_2(x,y) = Q(x,y)$, where $K(x,y)$ and $Q(x,y)$ are the integral kernels of the operators K and Q, respectively.

Example 2.4 (continued). We proceed similarly to Example 2.3. However, in this case, the moments of the Gaussian field $G(x)$ are determined by the covariance $\mathcal{K}_1(x,y)$ only. Hence, if $L(x,y)$ is an integral kernel of a bounded linear operator L in $\mathcal{H}$, the moments are determined by (the integral kernel of) the operator $K := LL^*$. Hence, without loss of generality, we may assume that $L = \sqrt{K}$, equivalently the operator L is self-adjoint.

Example 2.6 (continued). Since $\mathcal{G} = \mathcal{H}$, we define $\alpha(x,y) := (\alpha(x))(y)$ and similarly $\beta(x,y)$. In this case, condition (26) means that, for each $\Delta \in \mathcal{B}_0(X)$,

$$\int_{\Delta \times X} (|\alpha(x,y)|^2 + |\beta(x,y)|^2)\sigma^{\otimes 2}(dx\,dy) < \infty,$$

and we can proceed similarly to Example 2.3. Assume additionally that $\alpha(x,y)$ and $\beta(x,y)$ are integral kernels of operators $A, B \in \mathcal{L}(\mathcal{H})$, respectively. Then the covariance $\mathcal{K}_1(x,y)$ of the Gaussian field $G(x)$ is the integral kernel of the operator $\frac{1}{2}(AA^* + BB^*)$, while the pseudo-covariance $\mathcal{K}_2(x,y)$ is the integral kernel of the operator $\frac{1}{2}(AA^T - BB^T)$.

3 Quasi-free states on the CCR algebra

In this section, we assume that $\mathcal{H}$ is a separable complex Hilbert space with an anti-linear involution J in $\mathcal{H}$ satisfying $(Jf, Jh)_{\mathcal{H}} = (h, f)_{\mathcal{H}}$ for all $f, h \in \mathcal{H}$. Let $\mathcal{V}$ be a dense subspace of $\mathcal{H}$ that is invariant for J. Let $\mathfrak{F}$ be a separable Hilbert space and $\mathfrak{D}$ a dense subspace of $\mathfrak{F}$. For each $h \in \mathcal{V}$, let $A^+(h)$, $A^-(h) : \mathfrak{D} \to \mathfrak{D}$ be linear operators satisfying the following assumptions:

(i) $A^+(h)$ and $A^-(h)$ depend linearly on $h \in \mathcal{V}$;

(ii) the domain of the adjoint operator of $A^+(h)$ in $\mathfrak{F}$ contains $\mathfrak{D}$ and $A^+(h)^* \restriction \mathfrak{D} = A^-(Jf)$;

(iii) the operators $A^+(h)$, $A^-(h)$ satisfy the CCR:

$$[A^+(f), A^+(h)] = [A^-(f), A^-(h)] = 0, \quad [A^-(f), A^+(h)] = (h, Jf)_{\mathcal{H}} \tag{27}$$

for all $f, h \in \mathcal{V}$.

Let $\mathbb{A}$ denote the complex unital $*$-algebra generated by the operators $A^+(h)$, $A^-(h)$ ($h \in \mathcal{V}$). Let $\tau : \mathbb{A} \to \mathbb{C}$ be a state on $\mathbb{A}$, i.e., τ is linear, $\tau(\mathbf{1}) = 1$ and $\tau(a^*a) \geq 0$ for all $a \in \mathbb{A}$. For each $h \in \mathcal{V}$, we define a Hermitian operator

$$B(h) := A^+(h) + A^-(Jh). \tag{28}$$

These operators satisfy the commutation relation

$$[B(f), B(h)] = 2\Im(h, f)_{\mathcal{H}}, \quad h, f \in \mathcal{H}. \tag{29}$$

Note that

$$A^+(h) = \frac{1}{2}(B(h) - iB(ih)), \quad A^-(h) = \frac{1}{2}(B(Jh) + iB(Jh)).$$

Therefore, we can think of the algebra $\mathbb{A}$ as generated by the operators $B(h)$ ($h \in \mathcal{V}$), subject to the commutation relation (29). Hence, the state τ is completely determined by the functionals $T^{(n)} : \mathcal{V}^n \to \mathbb{C}$ ($n \geq 1$), where $T^{(1)}(h) := \tau(B(h))$ and

$$T^{(n)}(h_1, \dots, h_n) := \tau\big((B(h_1) - T^{(1)}(h_1)) \cdots (B(h_n) - T(h_n))\big), \quad n \geq 2.$$

The state τ is called *quasi-free* if, for each $n \in \mathbb{N}$, $T^{(2n+1)} = 0$ and

$$T^{(2n)}(h_1, \dots, h_{2n}) = \sum T^{(2)}(h_{i_1}, h_{j_1}) \cdots T^{(2)}(h_{i_n}, h_{j_n})$$

where the summation is over all partitions $\{i_1, j_1\}, \dots, \{i_n, j_n\}$ of $\{1, \dots, 2n\}$ with $i_k < j_k$ ($k = 1, \dots, n$), see e.g. [14, Section 5.2].

Remark 3.1. Let $\phi : \mathcal{V} \to \mathbb{C}$ be a linear functional. For each $h \in \mathcal{V}$, we define operators $\mathbf{A}^+(h) := A^+(h) + \phi'(h)$ and $\mathbf{A}^-(h) := A^-(h) + \phi(h)$, where $\phi'(h) := \overline{\phi(Jh)}$. The operators $\mathbf{A}^+(h)$, $\mathbf{A}^-(h)$ also satisfy the conditions (i)–(iii) discussed above. Obviously, the algebra generated by the operators $\mathbf{A}^+(h)$, $\mathbf{A}^-(h)$ coincides with $\mathbb{A}$. If $\tau : \mathbb{A} \to \mathbb{C}$ is a quasi-free state with respect to the operators $A^+(h)$, $A^-(h)$, then τ is also quasi-free with respect to the operators $\mathbf{A}^+(h)$, $\mathbf{A}^-(h)$.

Let us now present an explicit construction of a representation of the CCR algebra $\mathbb{A}$ and a quasi-free state τ on it. This construction resembles the *Bogoliubov transformation*, see e.g. [14, Subsection 5.2.2.2] or [12, Section 4][2].

Let $\mathcal{E}$ be a separable Hilbert space with an antilinear involution $\mathcal{J}$ satisfying $(\mathcal{J}f, \mathcal{J}g)_{\mathcal{E}} = (g, f)_{\mathcal{E}}$ for all $f, g \in \mathcal{E}$. Let $K_i \in \mathcal{L}(\mathcal{H}, \mathcal{E})$ ($i = 1, 2$). Denote

$$K_i' := \mathcal{J} K_i J \in \mathcal{L}(\mathcal{H}, \mathcal{E}) \tag{30}$$

and assume that

$$(K_2')^* K_1 - (K_1')^* K_2 = 0,$$

[2] In [12], a Bogoliubov transformation is called a *linear canonical transformation*.

$$K_2^* K_2 - K_1^* K_1 = 1. \tag{31}$$

For each $h \in \mathcal{H}$, we define, in the symmetric Fock space $\mathcal{F}(\mathcal{E})$, the operators

$$A^+(h) := a^+(K_2 h) + a^-(K_1 h), \quad A^-(h) := a^-(K_2' h) + a^+(K_1' h) \tag{32}$$

with domain $\mathcal{F}_{\mathrm{fin}}(\mathcal{E})$. Here $a^+(\cdot)$ and $a^-(\cdot)$ are the creation and annihilation operators in $\mathcal{F}(\mathcal{E})$, respectively. It follows from (14), (31), and (32) that $A^+(h)$ and $A^-(h)$ satisfy the conditions (i)–(iii) with $\mathcal{V} = \mathcal{H}$, $\mathfrak{F} = \mathcal{F}(\mathcal{E})$, and $\mathfrak{D} = \mathcal{F}_{\mathrm{fin}}(\mathcal{E})$.

Let $\mathbb{A}$ denote the corresponding CCR algebra and let $\tau : \mathbb{A} \to \mathbb{C}$ be the *vacuum state on* $\mathbb{A}$, i.e., $\tau(a) := (a\Omega, \Omega)_{\mathcal{F}(\mathcal{E})}$. For each $h \in \mathcal{H}$,

$$B(h) = A^+(h) + A^-(Jh) = a^+((K_2 + \mathcal{J} K_1)h) + a^-((K_1 + \mathcal{J} K_2)h). \tag{33}$$

In particular, $\tau(B(h)) = 0$. Hence, it easily follows from (33) that τ is a quasi-free state with

$$T^{(2)}(f, h) = \big((K_1 + \mathcal{J} K_2)f, (K_1 + \mathcal{J} K_2)h\big)_{\mathcal{E}}. \tag{34}$$

Remark 3.2. Note that, in the classical Bogoliubov transformation, one chooses $\mathcal{E} = \mathcal{H}$.

Remark 3.3. Choosing $\mathcal{E} = \mathcal{H}$, $K_1 = 1$ and $K_2 = 0$, we get $A^+(h) = a^+(h)$, $A^-(h) = a^-(h)$. In this case, the vacuum state is quasi-free, with $T^{(1)} = 0$ and $T^{(2)}(f, h) = (f, Jh)_{\mathcal{H}}$.

4 Particle density and correlation functions

Let $\mathfrak{F}$ be a separable Hilbert space and let $\mathfrak{D}$ be a dense subspace of $\mathfrak{F}$. For each $\Delta \in \mathcal{B}_0(X)$, let $\rho(\Delta) : \mathfrak{D} \to \mathfrak{D}$ be a linear Hermitian operator in $\mathfrak{F}$. We further assume that the operators $\rho(\Delta)$ commute, i.e., $[\rho(\Delta_1), \rho(\Delta_2)] = 0$ and for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$. Let $\mathcal{A}$ denote the complex unital (commutative) $*$-algebra generated by $\rho(\Delta)$ ($\Delta \in \mathcal{B}_0(X)$). Let Ω be a fixed vector in $\mathfrak{F}$ with $\|\Omega\|_{\mathfrak{F}} = 1$, and let a state $\tau : \mathcal{A} \to \mathbb{C}$ be defined by $\tau(a) := (a\Omega, \Omega)_{\mathfrak{F}}$ for $a \in \mathcal{A}$.

We define *Wick polynomials in* $\mathcal{A}$ by the following recurrence formula:

$$\begin{aligned} {:}\rho(\Delta){:} &= \rho(\Delta), \\ {:}\rho(\Delta_1) \cdots \rho(\Delta_{n+1}){:} &= \rho(\Delta_{n+1})\, {:}\rho(\Delta_1) \cdots \rho(\Delta_n){:} \\ &\quad - \sum_{i=1}^{n} {:}\rho(\Delta_1) \cdots \rho(\Delta_{i-1}) \rho(\Delta_i \cap \Delta_{n+1}) \rho(\Delta_{i+1}) \cdots \rho(\Delta_n){:}\,, \end{aligned} \tag{35}$$

where $\Delta, \Delta_1, \dots, \Delta_{n+1} \in \mathcal{B}_0(X)$ and $n \in \mathbb{N}$. It is easy to see that, for each permutation $\pi \in \mathfrak{S}_n$,

$${:}\rho(\Delta_1) \cdots \rho(\Delta_n){:} = {:}\rho(\Delta_{\pi(1)}) \cdots \rho(\Delta_{\pi(n)}){:}\,.$$

We assume that, for each $n\in\mathbb{N}$, there exists a symmetric measure $\theta^{(n)}$ on X^n that is concentrated on $X^{(n)}$ (i.e., $\theta^{(n)}(X^n\setminus X^{(n)})=0$) and satisfies[3]

$$\theta^{(n)}\big(\Delta_1\times\dots\times\Delta_n\big)=\frac{1}{n!}\,\tau\big({:}\rho(\Delta_1)\cdots\rho(\Delta_n){:}\big),\quad \Delta_1,\dots,\Delta_n\in\mathcal{B}_0(X). \tag{36}$$

Furthermore, we assume that, for each $\Delta\in\mathcal{B}_0(X)$, there exists a constant $C_\Delta>0$ such that

$$\theta^{(n)}(\Delta^n)\le C_\Delta^n,\quad n\in\mathbb{N}, \tag{37}$$

and for any sequence $\{\Delta_l\}_{l\in\mathbb{N}}\subset\mathcal{B}_0(X)$ such that $\Delta_l\downarrow\varnothing$ (i.e., $\Delta_1\supset\Delta_2\supset\Delta_3\supset\cdots$ and $\bigcap_{l=1}^\infty\Delta_l=\varnothing$), we have $C_{\Delta_l}\to 0$ as $l\to\infty$.

Theorem 4.1 ([26]). (i) *Under the above assumptions, there exists a unique point process μ in X whose correlation measures are $(\theta^{(n)})_{n=1}^\infty$.*

(ii) *Let $\mathfrak{D}':=\{a\Omega\mid a\in\mathcal{A}\}$ and let $\mathfrak{F}'$ denote the closure of $\mathfrak{D}'$ in $\mathfrak{F}$. Then each operator $(\rho(\Delta),\mathfrak{D}')$ is essentially self-adjoint in $\mathfrak{F}'$, and the operator-valued measures of the closures of the operators $(\rho(\Delta),\mathfrak{D}')$ commute. Furthermore, there exists a unique unitary operator $U:\mathfrak{F}'\to L^2(\Gamma(X),\mu)$ satisfying $U\Omega=1$ and*

$$U(\rho(\Delta_1)\cdots\rho(\Delta_n)\Omega)=\gamma(\Delta_1)\cdots\gamma(\Delta_n) \tag{38}$$

for any $\Delta_1,\dots,\Delta_n\in\mathcal{B}_0(X)$ ($n\in\mathbb{N}$). In particular,

$$\tau\big(\rho(\Delta_1)\cdots\rho(\Delta_n)\big)=\int_{\Gamma(X)}\gamma(\Delta_1)\cdots\gamma(\Delta_n)\,\mu(d\gamma). \tag{39}$$

We finish this section with a formal observation. Let again $\mathcal{H}=L^2(X,\sigma)$ and the antilinear involution J in $\mathcal{H}$ be given by $(Jf)(x):=\overline{f(x)}$. Let $A^+(h)$ and $A^-(h)$ ($h\in\mathcal{V}$) be operators satisfying the CCR, and let the corresponding operators $A^+(x)$, $A^-(x)$ ($x\in X$) be derfined by (10). For each $\Delta\in\mathcal{B}_0(X)$, let $\rho(\Delta)$ be given by (11). The CCR (27) and formulas (11), (35) imply that, for any $\Delta_1,\dots,\Delta_n\in\mathcal{B}_0(X)$,

$${:}\rho(\Delta_1)\cdots\rho(\Delta_n){:}=\int_{\Delta_1\times\dots\times\Delta_n}A^+(x_n)\cdots A^+(x_1)A^-(x_1)\cdots A^-(x_n)\sigma^{\otimes n}(dx_1\cdots dx_n). \tag{40}$$

Thus, the Wick polynomials correspond to the Wick normal ordering, in which all the creation operators are to the left of all the annihilation operators. Hence, by (36) and (40), we formally obtain

$$\theta^{(n)}\big(\Delta_1\times\dots\times\Delta_n\big)=\frac{1}{n!}\int_{\Delta_1\times\dots\times\Delta_n}\tau\big(A^+(x_n)\cdots A^+(x_1)A^-(x_1)\cdots A^-(x_n)\big)\sigma^{\otimes n}(dx_1\cdots dx_n).$$

[3] In view of formulas (35), (36), it is natural to call $\theta^{(n)}$ the *n-th correlation measure of the family of operators* $(\rho(\Delta))_{\Delta\in\mathcal{B}_0(X)}$.

Therefore, by (3), the point process μ from Theorem 4.1 has the correlation functions

$$k^{(n)}(x_1,\dots,x_n)=\tau\big(A^+(x_n)\cdots A^+(x_1)A^-(x_1)\cdots A^-(x_n)\big).$$

Below we will see that, in the case of a Cox process Π_R, where $R(x)=|G(x)|^2$ and $G(x)$ is a complex-valued Gaussian field from Section 2, the above formal calculations can be given a rigorous meaning. We will start, however, with the simpler case of a Poisson point process.

5 Application of Theorem 4.1 to Poisson point processes

Recall Remark 3.1. Let $\mathcal V$ denote the (dense) subspace of $\mathcal H=L^2(X,\sigma)$ consisting of all measurable bounded (versions of) functions $h:X:\to\mathbb C$ with compact support. Let us fix a function $\lambda\in L^2_{\mathrm{loc}}(X,\sigma)$ and define a functional $\phi:\mathcal V\to\mathbb C$ by

$$\phi(h):=\int_X h(x)\lambda(x)\sigma(dx). \tag{41}$$

Note that

$$\phi'(h):=\int_X h(x)\overline{\lambda(x)}\,\sigma(dx). \tag{42}$$

For each $h\in\mathcal V$, we define operators $A^+(h),A^-(h)\in\mathcal L(\mathcal F_{\mathrm{fin}}(\mathcal H))$ by

$$A^+(h):=a^+(h)+\phi'(h),\quad A^-(h):=a^-(h)+\phi(h). \tag{43}$$

Here $a^+(h)$ and $a^-(h)$ are the creation and annihilation operators in $\mathcal F(\mathcal H)$. By Remarks 3.1 and 3.3, the vacuum state τ on the CCR algebra generated by $A^+(h)$, $A^-(h)$ ($h\in\mathcal V$) is quasi-free.

Let $a^+(x)$ and $a^-(x)$ be the operator-valued distributions corresponding to $a^+(h)$ and $a^-(h)$, respectively. Then $A^+(x)=a^+(x)+\overline{\lambda(x)}$ and $A^-(x)=a^-(x)+\lambda(x)$. Hence, the corresponding particle density takes the form

$$\rho(x)=\lambda(x)a^+(x)+\overline{\lambda(x)}\,a^-(x)+a^+(x)a^-(x)+|\lambda(x)|^2.$$

Our next aim is to rigorously define, for each $\Delta\in\mathcal B_0(X)$, an operator $\rho(\Delta)=\int_\Delta A^+(x)A^-(x)\sigma(dx)\in\mathcal L(\mathcal F_{\mathrm{fin}}(\mathcal H))$.

We clearly have, for each $\Delta\in\mathcal B_0(X)$,

$$\int_\Delta\lambda(x)a^+(x)\,\sigma(dx)=\int_X\chi_\Delta(x)\lambda(x)a^+(x)\,\sigma(dx)=a^+(\chi_\Delta\lambda)$$

and similarly

$$\int_\Delta\overline{\lambda(x)}\,a^-(x)\,\sigma(dx)=a^-(\chi_\Delta\overline{\lambda}).$$

(Note that $\chi_\Delta \lambda \in \mathcal{H}$.) Next, for $h \in \mathcal{V}$ and $f^{(n)} \in \mathcal{F}_n(\mathcal{H})$,

$$\big(a^-(h)f^{(n)}\big)(x_1, \dots, x_{n-1}) = n \int_X h(x) f^{(n)}(x, x_1, \dots, x_{n-1})\sigma(dx).$$

Hence

$$(a^-(x)f^{(n)})(x_1, \dots, x_{n-1}) = n f^{(n)}(x, x_1, \dots, x_{n-1}), \tag{44}$$

which implies

$$\bigg(\int_\Delta a^+(x)a^-(x)\sigma(dx) f^{(n)}\bigg)(x_1, \dots, x_n) = \big(\chi_\Delta(x_1) + \dots + \chi_\Delta(x_n)\big) f^{(n)}(x_1, \dots, x_n). \tag{45}$$

Hence, $a^0(\chi_\Delta) := \int_\Delta a^+(x)a^-(x)\sigma(dx) \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{H}))$. The $a^0(\chi_\Delta)$ is called a *neutral operator*.

Thus, for each $\Delta \in \mathcal{B}_0(X)$, we have rigorously defined

$$\rho(\Delta) = a^+(\chi_\Delta \lambda) + a^-(\chi_\Delta \overline{\lambda}) + a^0(\chi_\Delta) + \int_\Delta |\lambda(x)|^2 \sigma(dx) \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{H})). \tag{46}$$

Obviously, $\rho(\Delta)$ is a Hermitian operator in $\mathcal{F}(\mathcal{H})$. To construct a state on the corresponding $*$-algebra, we use the vacuum Ω in the Fock space $\mathcal{F}(\mathcal{H})$.

Proposition 5.1. *The operators $(\rho(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ defined by (46) and the vacuum state τ satisfy the assumptions of Theorem 4.1. In this case, $\theta^{(n)} = \frac{1}{n!}(|\lambda|^2\sigma)^{\otimes n}$, so that μ is the Poisson point process with intensity $|\lambda(x)|^2$. Furthermore, we have $\mathfrak{F}' = \mathcal{F}(\mathcal{H})$.*

Remark 5.2. For the Poisson point process μ with intensity $|\lambda|^2$, the existence of the unitary isomorphism $U : \mathcal{F}(\mathcal{H}) \to L^2(\Gamma(X), \mu)$ that satisfies (38), (39) is a well-known fact, see e.g. [34]. Our approach to the construction of the isomorphism U may be compared with paper [21].

Proof of Proposition 5.1. For any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, the commutation $[\rho(\Delta_1), \rho(\Delta_2)] = 0$ follows from the CCR and the commutation relations

$$\big[a^0(\chi_{\Delta_1}), a^+(\chi_{\Delta_2}\lambda)\big] = a^+(\chi_{\Delta_1 \cap \Delta_2}\lambda), \quad \big[a^0(\chi_{\Delta_1}), a^-(\chi_{\Delta_2}\overline{\lambda})\big] = -a^-(\chi_{\Delta_1 \cap \Delta_2}\overline{\lambda}).$$

Next, let $C \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{H}))$. Similarly to (44), (45), we see that, for each $\Delta \in \mathcal{B}_0(X)$, $\int_\Delta a^+(x)Ca^-(x)\sigma(dx)$ determines an operator from $\mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{H}))$. In particular, for $f \in \mathcal{H}$ and $n \in \mathbb{N}$,

$$\int_\Delta a^+(x)Ca^-(x)\sigma(dx) f^{\otimes n} = n(\chi_\Delta f) \odot (Cf^{\otimes(n-1)}). \tag{47}$$

Therefore,

$$\int_\Delta A^+(x)CA^-(x)\sigma(dx)$$

$$= \int_\Delta a^+(x)Ca^-(x)\sigma(dx) + a^+(\chi_\Delta\lambda)C + Ca^-(\chi_\Delta\overline{\lambda}) + \int_\Delta |\lambda(x)|^2\sigma(dx)\,C \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{H})).$$

Hence, we may define, for any $\Delta_1, \dots, \Delta_n \in \mathcal{B}_0(X)$,

$$\int_{\Delta_1\times\dots\times\Delta_n} A^+(x_n)\cdots A^+(x_1)A^-(x_1)\cdots A^-(x_n)\,\sigma^{\otimes n}(dx_1\cdots dx_n)$$
$$:= \int_{\Delta_n} A^+(x_n)\Big(\int_{\Delta_{n-1}} A^+(x_{n-1})\Big(\cdots\int_{\Delta_1} A^+(x_1)A^-(x_1)\sigma(dx_1)\Big)$$
$$\cdots A^-(x_{n-1})\sigma(dx_{n-1})\Big)A^-(x_n)\sigma(dx_n) \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{H})).$$

We next state that formula (40) now holds rigorously. Indeed, a direct calculation shows that, for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$ and $C \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{H}))$,

$$\rho(\Delta_1)\int_{\Delta_2} A^+(x)CA^-(x)\,\sigma(dx)$$
$$= \int_{\Delta_2} A^+(x)\rho(\Delta_1)CA^-(x)\,\sigma(dx) + \int_{\Delta_1\cap\Delta_2} A^+(x)CA^-(x)\,\sigma(dx). \tag{48}$$

Now formula (40) follows by induction from (35) and (48).

Applying the vacuum state τ to (40) , we get

$$\tau\big(:\!\rho(\Delta_1)\cdots\rho(\Delta_n)\!:\big) = \int_{\Delta_1\times\dots\times\Delta_n} |\lambda(x_1)|^2\cdots|\lambda(x_n)|^2\sigma^{\otimes n}(dx_1\cdots dx_n). \tag{49}$$

Since the measure σ is non-atomic, $\sigma^{\otimes n}$ is concentrated on $X^{(n)}$. By (36) and (49), estimate (37) holds with $C_\Delta = \int_\Delta |\lambda(x)|^2\sigma(dx)$. Hence, the assumptions of Theorem 4.1 are satisfied. The form of the correlation measures implies that μ is the Poisson point process with intensity $|\lambda(x)|^2$.

Finally, the proof of the equality $\mathfrak{F}' = \mathcal{F}(\mathcal{H})$ is standard and we leave it to the interested reader. □

6 Application of Theorem 4.1 to hafnian point processes

We will use below the notations from Section 2. We assume that conditions (15), (16), and (26) are satisfied. Let also the subspace $\mathcal{V}$ of $\mathcal{H}$ be as in Section 5.

Let $h \in \mathcal{V}$. By the Cauchy inequality,

$$\int_X |h(x)|\,\|L_i(x)\|_{\mathcal{G}}\,\sigma(dx) < \infty, \quad i = 1, 2.$$

Hence, by using e.g. [11, Chapter 10, Theorem 3.1], we can define

$$\int_X hL_i\,d\sigma, \int_X h\,\mathcal{J}L_i\,d\sigma \in \mathcal{G}$$

as Bochner integrals.

Example 6.1. Recall Examples 2.3 and 2.4. As easily seen, for each $h \in \mathcal{V}$,

$$\int_X hL\, d\sigma = (L^\Delta)^T h, \ \int_X h\, JL\, d\sigma = (L^\Delta)^* h \in \mathcal{H}, \tag{50}$$

where $\Delta \in \mathcal{B}_0(X)$ is chosen so that h vanishes outside Δ. In particular, if $L(x,y)$ is the integral kernel of an operator $L \in \mathcal{L}(\mathcal{H})$, then we can replace L^Δ in formula (50) with L. Furthermore, in the latter case, we could set $\mathcal{V} = \mathcal{H}$.

Denote $\mathcal{E} := \mathcal{H} \oplus \mathcal{G}$. We recall the well-known unitary isomorphism between $\mathcal{F}(\mathcal{H}) \otimes \mathcal{F}(\mathcal{G})$ and $\mathcal{F}(\mathcal{E})$. In view of our considerations in Sections 2 and 5, see, in particular, formulas (17), (18), and (41)–(43), we consider in $\mathcal{F}(\mathcal{E})$ the following linear operators with domain $\mathcal{F}_{\mathrm{fin}}(\mathcal{E})$,

$$\begin{aligned} A^+(h) :=& a^+\left(h, \int_X h\, \mathcal{J} L_2\, d\sigma\right) + a^-\left(0, \int_X h\, \mathcal{J} L_1\, d\sigma\right),\\ A^-(h) :=& a^+\left(0, \int_X hL_1\, d\sigma\right) + a^-\left(h, \int_X hL_2\, d\sigma\right), \quad h \in \mathcal{V}. \end{aligned} \tag{51}$$

Proposition 6.2. *The operators $A^+(h)$, $A^-(h)$ defined by* (51) *satisfy the conditions (i)–(iii) from Section 3 with $\mathfrak{F} = \mathcal{F}(\mathcal{E}) = \mathcal{F}(H \oplus \mathcal{G})$ and $\mathfrak{D} = \mathcal{F}_{\mathrm{fin}}(\mathcal{E})$. The vacuum state on the corresponding CCR algebra is quasi-free with $T^{(1)} = 0$ and*

$$\begin{aligned} T^{(2)}(f,h) =& \int_X \overline{f(x)}\, h(x) \sigma(dx)\\ &+ 2\int_{X^2} \Re\left(f(x)h(y)\overline{\mathcal{K}_2(x,y)} + \overline{f(x)}\, h(y)\mathcal{K}_1(x,y)\right) \sigma^{\otimes 2}(dx\, dy). \end{aligned} \tag{52}$$

Here $\mathcal{K}_1$ and $\mathcal{K}_2$ are defined by (19) *and* (20)*, respectively.*

Proof. The first statement of the proposition is obvious in view of the commutation of the operators $\Phi(x)$, $\Psi(x)$. By (28) and (51),

$$B(h) = a^+\left(h, \int_X (h\, \mathcal{J} L_2 + (Jh)\, L_1) d\sigma\right) + a^-\left(Jh, \int_X (h\, \mathcal{J} L_1 + (Jh)\, L_2) d\sigma\right).$$

Hence, by (15), (16), (19), and (20), the second statement of the proposition also follows. □

Remark 6.3. Assume that, for $i = 1, 2$, the map $\mathcal{V} \ni h \mapsto \int_X hL_i\, d\sigma \in \mathcal{G}$ extends by continuity to a bounded linear operator $\mathbb{L}_i \in \mathcal{L}(\mathcal{H}, \mathcal{G})$. Then, by (30) and (51),

$$A^+(h) = a^+(h, \mathbb{L}_2' h) + a^-(0, \mathbb{L}_1' h), \quad A^-(h) = a^+(0, \mathbb{L}_1 h) + a^-(h, \mathbb{L}_2 h).$$

This quasi-free representation of the CCR is a special case of (31), (32). By (34),

$$T^{(2)}(f,h) = (h,f)_{\mathcal{H}} + \big((\mathcal{J}\mathbb{L}_1 + \mathbb{L}_2) Jf, (\mathcal{J}\mathbb{L}_1 + \mathbb{L}_2) Jh\big)_{\mathcal{G}}.$$

By (51), the corresponding operator-valued distributions $A^+(x)$ and $A^-(x)$ are given by

$$\begin{aligned} A^+(x) &= a_1^+(x) + a_2^+(\mathcal{J}L_2(x)) + a_2^-(\mathcal{J}L_1(x)), \\ A^-(x) &= a_1^-(x) + a_2^+(L_1(x)) + a_2^-(L_2(x)), \end{aligned} \tag{53}$$

compare with (17), (18). The operators $a_i^\pm(\cdot)$ $(i = 1, 2)$ are defined similarly to Example 2.4.

Proposition 6.4. *Let $A^+(x)$, $A^-(x)$ be given by* (53), *and let $C \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{E}))$. For each $\Delta \in \mathcal{B}_0(X)$, $\int_\Delta A^+(x)CA^-(x)\,\sigma(dx)$ determines an operator from $\mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{E}))$, in the sense explained in the proof.*

Proof. It is sufficient to prove the statement when $C \in \mathcal{L}(\mathcal{F}_n(\mathcal{E}), \mathcal{F}_m(\mathcal{E}))$. We also fix $\Delta \in \mathcal{B}_0(X)$. By (24) and (26),

$$\begin{aligned} &\int_\Delta \|a_2^-(\mathcal{J}L_1(x))Ca_2^+(L_1(x))\|_{\mathcal{L}(\mathcal{F}_{n-1}(\mathcal{E}),\mathcal{F}_{m-1}(\mathcal{E}))}\,\sigma(dx) \\ &\qquad \le \|C\|_{\mathcal{L}(\mathcal{F}_n(\mathcal{E}),\mathcal{F}_m(\mathcal{E}))}\sqrt{nm}\int_\Delta \|L_1(x)\|^2_{\mathcal{G}}\,\sigma(dx) < \infty. \end{aligned}$$

Hence, by [11, Chapter 10, Theorem 3.1], the following Bochner integral is well-defined:

$$\int_\Delta a_2^-(\mathcal{J}L_1(x))Ca_2^+(L_1(x))\sigma(dx) \in \mathcal{L}(\mathcal{F}_{n-1}(\mathcal{E}), \mathcal{F}_{m-1}(\mathcal{E})).$$

Note that, by e.g. [11, Chapter 10, Theorem 3.2], for each $f^{(n-1)} \in \mathcal{F}_{n-1}(\mathcal{E})$,

$$\left(\int_\Delta a_2^-(\mathcal{J}L_1(x))Ca_2^+(L_1(x))\sigma(dx)\right)f^{(n-1)} = \int_\Delta a_2^-(\mathcal{J}L_1(x))Ca_2^+(L_1(x))f^{(n-1)}\sigma(dx),$$

where the right hand side is a Bochner integral with values in $\mathcal{F}_{m-1}(\mathcal{E})$. The proof of existence of the other Bochner integrals of the type $\int_\Delta a_2^\pm(\mathcal{J}L_i(x))Ca_2^\pm(L_j(x))\sigma(dx)$ $(i, j \in \{1, 2\})$ is similar.

Next, we define

$$\int_\Delta a_1^+(x)Ca_1^-(x)\sigma(dx) \in \mathcal{L}(\mathcal{F}_{n+1}(\mathcal{E}), \mathcal{F}_{m+1}(\mathcal{E}))$$

by analogy (47). Let $(e_i)_{i\ge1}$ be an orthonormal basis in $\mathcal{H}$ such that $Je_i = e_i$ for all $i \ge 1$. As easily seen,

$$\int_\Delta a_1^+(x)Ca_1^-(x)\sigma(dx) = \sum_{i,j\ge1} (\chi_\Delta e_i, e_j)_{\mathcal{H}}\, a_1^+(e_j)Ca_1^-(e_i), \tag{54}$$

where the series converges strongly in $\mathcal{L}(\mathcal{F}_{n+1}(\mathcal{E}), \mathcal{F}_{m+1}(\mathcal{E}))$.

By (26), we can define a linear operator $L_2^\Delta \in \mathcal{L}(\mathcal{G}, \mathcal{H})$ by

$$(L_2^\Delta g)(x) := \chi_\Delta(x)(L_2(x), \mathcal{J}g)_{\mathcal{G}}\,.$$

By analogy with (47), we define

$$\int_\Delta a_1^+(x)Ca_2^-(L_2(x))\sigma(dx) \in \mathcal{L}(\mathcal{F}_{n+1}(\mathcal{E}), \mathcal{F}_{m+1}(\mathcal{E})) \tag{55}$$

that satisfies, for each $f = (h, g) \in \mathcal{E}$,

$$\int_\Delta a_1^+(x)Ca_2^-(L_2(x))\sigma(dx)f^{\otimes(n+1)} = (n+1)\big(L_2^\Delta g\big) \odot (Cf^{\otimes n}). \tag{56}$$

Let $(u_j)_{j\ge1}$ be an orthonormal basis in $\mathcal{G}$ such that $\mathcal{J}u_j = u_j$ for all $j \ge 1$. Then, similarly to (54), we obtain

$$\int_\Delta a_1^+(x)Ca_2^-(L_2(x))\sigma(dx) = \sum_{i,j\ge1}(L_2^\Delta u_j, e_i)_{\mathcal{H}}\, a_1^+(e_i)Ca_2^-(u_j), \tag{57}$$

where the series converges strongly in $\mathcal{L}(\mathcal{F}_{n+1}(\mathcal{E}), \mathcal{F}_{m+1}(\mathcal{E}))$.

Next, we note that $\mathcal{H} \otimes \mathcal{G} = L^2(X \to \mathcal{G}, \sigma)$. Hence, by (26), $\chi_\Delta(\cdot)L_1(\cdot) \in \mathcal{H} \otimes \mathcal{G}$. Note also that $\mathcal{H} \otimes \mathcal{G}$ is a subspace of $\mathcal{E}^{\otimes 2} = (\mathcal{H} \oplus \mathcal{G})^{\otimes 2}$. For each $m \in \mathbb{N}$, we denote by $P_m : \mathcal{E}^{\otimes m} \to \mathcal{E}^{\odot m}$ the symmetrization operator. We naturally set, for each $f^{(k)} \in \mathcal{F}_k(\mathcal{E})$,

$$\int_\Delta a_1^+(x)a_2^+(L_1(x))\sigma(dx)f^{(k)} = P_{k+2}\big((\chi_\Delta(\cdot)L_1(\cdot)) \otimes f^{(k)}\big).$$

Hence, we define

$$\int_\Delta a_1^+(x)Ca_2^+(L_1(x))\sigma(dx) \in \mathcal{L}(\mathcal{F}_{n-1}(\mathcal{E}), \mathcal{F}_{m+1}(\mathcal{E}))$$

by

$$\int_\Delta a_1^+(x)Ca_2^+(L_1(x))\sigma(dx)f^{(n-1)} := P_{m+1}(1_{\mathcal{E}} \otimes (CP_n))\big((\chi_\Delta(\cdot)L_1(\cdot)) \otimes f^{(n-1)}\big)$$

for $f^{(n-1)} \in \mathcal{F}_{n-1}(\mathcal{E})$. Here $1_{\mathcal{E}}$ is the identity operator in $\mathcal{E}$. Therefore,

$$\int_\Delta a_1^+(x)Ca_2^+(L_1(x))\sigma(dx) = \sum_{i,j\ge1}(\chi_\Delta(\cdot)L_1(\cdot), e_i \otimes u_j)_{\mathcal{H}\otimes\mathcal{G}}\, a_1^+(e_i)Ca_2^+(u_j), \tag{58}$$

where the series converges strongly in $\mathcal{L}(\mathcal{F}_{n-1}(\mathcal{E}), \mathcal{F}_{m+1}(\mathcal{E}))$.

Similarly to Remark 6.3, for $i = 1, 2$, we define $\mathbb{L}_i^\Delta \in \mathcal{L}(\mathcal{H}, \mathcal{G})$ by

$$\mathbb{L}_i^\Delta h := \int_\Delta h(x) L_i(x) \sigma(dx), \quad h \in \mathcal{H}$$

(in the sense of Bochner integration).

Similarly to (55), (56), we define

$$\int_\Delta a_2^+(\mathcal{J} L_2(x)) C a_1^-(x) \sigma(dx) \in \mathcal{L}(\mathcal{F}_{n+1}(\mathcal{E}), \mathcal{F}_{m+1}(\mathcal{E}))$$

by

$$\int_\Delta a_2^+(\mathcal{J} L_2(x)) C a_1^-(x) \sigma(dx) f^{\otimes(n+1)} := (n+1)\big((\mathbb{L}_2^\Delta)' h\big) \odot (C f^{\otimes n}), \quad f = (h, g) \in \mathcal{E}.$$

Similarly to (57),

$$\int_\Delta a_2^+(\mathcal{J} L_2(x)) C a_1^-(x) \sigma(dx) = \sum_{i,j \geq 1} \big((\mathbb{L}_2^\Delta)' e_i, u_j\big)_{\mathcal{G}}\, a_2^+(u_j) C a_1^-(e_i), \tag{59}$$

where the series converges strongly in $\mathcal{L}(\mathcal{F}_{n+1}(\mathcal{E}), \mathcal{F}_{m+1}(\mathcal{E}))$.

Finally, we define

$$\int_\Delta a_2^-(\mathcal{J} L_1(x)) C a_1^-(x) \sigma(dx) \in \mathcal{L}(\mathcal{F}_{n+1}(\mathcal{E}), \mathcal{F}_{m-1}(\mathcal{E}))$$

by

$$\int_\Delta a_2^-(\mathcal{J} L_1(x)) C a_1^-(x) \sigma(dx) f^{\otimes(n+1)} := (n+1) a_2^-\big((\mathbb{L}_1^\Delta)' h\big)(C f^{\otimes n}), \quad f = (h, g) \in \mathcal{E}.$$

Hence,

$$\int_\Delta a_2^-(\mathcal{J} L_1(x)) C a_1^-(x) \sigma(dx) = \sum_{i,j \geq 1} \big((\mathbb{L}_1^\Delta)' e_i, u_j\big)_{\mathcal{G}}\, a_2^-(u_j) C a_1^-(e_i), \tag{60}$$

where the series converges strongly in $\mathcal{L}(\mathcal{F}_{n+1}(\mathcal{E}), \mathcal{F}_{m-1}(\mathcal{E}))$. □

Proposition 6.5. *For each $\Delta \in \mathcal{B}_0(X)$, the particle density $\rho(\Delta) = \int_\Delta A^+(x) A^-(x)\, \sigma(dx)$ is a well-defined Hermitian operator in $\mathcal{F}(\mathcal{E})$ with domain $\mathcal{F}_{\mathrm{fin}}(\mathcal{E})$. Furthermore, for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, $[\rho(\Delta_1),\, \rho(\Delta_2)] = 0$.*

Proof. By Proposition 6.4, we have $\rho(\Delta) \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{E}))$. The fact that $\rho(\Delta)$ is a Hermitian operator in $\mathcal{F}(\mathcal{E})$ easily follows from the proof of Proposition 6.4.

To prove the commutation, one uses the corresponding Bochner integrals, formulas (54), (57)–(60). In doing so, one uses the fact that every strongly convergent sequence of bounded linear operators is norm-bounded. Hence, for every strongly convergent sums of bounded linear operators, $A = \sum_{i \geq 1}^\infty A_i$ and $B = \sum_{j \geq 1}^\infty B_j$, one has $AB = \sum_{i,j \geq 1} A_i B_j$, where the latter double series converges strongly. □

Theorem 6.6. *The operators $(\rho(\Delta))_{\Delta\in\mathcal{B}_0(X)}$ defined by Propositions 6.4, 6.5 and the state τ defined by the vacuum vector Ω satisfy the assumptions of Theorem 4.1. The corresponding point process μ is the Cox process Π_R, where $R(x)=|G(x)|^2$, and $G(s)$ is the Gaussian field from Theorem 2.1. The point process Π_R is hafnian with the correlation kernel $\mathbb{K}(x,y)$ given by* (9)*, where $\mathcal{K}_1(x,y)$ and $\mathcal{K}_2(x,y)$ are given by* (19) *and* (20)*, respectively.*

Proof. Direct calculations show that, for any $\Delta_1,\Delta_2\in\mathcal{B}_0(X)$ and $C\in\mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\mathcal{E}))$, formula (48) holds, which implies formula (40). We apply the vacuum state τ to (40). Using formulas (8), (9), (17), (18), (53), Theorem 2.1, and Proposition 6.4, we conclude that the measure $\theta^{(n)}$ is given by

$$
\begin{aligned}
\theta^{(n)}(dx_1\cdots dx_n) &= \frac{1}{n!}\,\tau\big(\Psi(x_n)\cdots\Psi(x_1)\Phi(x_1)\cdots\Phi(x_n)\big)\sigma^{\otimes n}(dx_1\cdots dx_n)\\
&= \frac{1}{n!}\,\mathbb{E}\big(\overline{G(x_n)}\cdots\overline{G(x_1)}G(x_1)\cdots G(x_n)\big)\sigma^{\otimes n}(dx_1\cdots dx_n) \qquad (61)\\
&= \frac{1}{n!}\,\mathbb{E}\big(|G(x_1)|^2\cdots|G(x_n)|^2\big)\sigma^{\otimes n}(dx_1\cdots dx_n) \qquad (62)\\
&= \frac{1}{n!}\,\operatorname{haf}[\mathbb{K}(x_i,x_j)]_{i,j=1,\dots,n}\,\sigma^{\otimes n}(dx_1\cdots dx_n). \qquad (63)
\end{aligned}
$$

In particular, $\theta^{(n)}$ is a positive measure that is concentrated on $X^{(n)}$.

If $\mathcal{Y}=G(x)$ or $\overline{G(x)}$ and $\mathcal{Z}=G(y)$ or $\overline{G(y)}$, then

$$
|\mathbb{E}(\mathcal{Y}\mathcal{Z})|\le\big(\mathbb{E}(|\mathcal{Y}|^2)\mathbb{E}(|\mathcal{Z}|^2)\big)^{1/2}=\big(\mathbb{E}(|G(x)|^2)\,\mathbb{E}(|G(y)|^2)\big)^{1/2}=\|L_1(x)\|_{\mathcal{G}}\,\|L_1(y)\|_{\mathcal{G}}.
$$

The number of all partitions $\{i_1,j_1\},\dots,\{i_n,j_n\}$ of $\{1,\dots,2n\}$ is $\frac{(2n)!}{(n!)2^n}\le 2^n n!$. Hence, by (61) and the formula for the moments of Gaussian random variables,

$$
\theta^{(n)}(\Delta^n)\le\left(2\int_\Delta\|L_1(x)\|_{\mathcal{G}}^2\,\sigma(dx)\right)^n,\quad \Delta\in\mathcal{B}_0(X).
$$

Thus, the operators $(\rho(\Delta))_{\Delta\in\mathcal{B}_0(X)}$ satisfy the assumptions of Theorem 4.1.

The statement of the theorem about the arising point process μ follows immediately from (62) and (63). $\square$

Acknowledgments

The authors are grateful to the referee for valuable comments that improved the manuscript.

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