RECOLLEMENTS FOR DERIVED CATEGORIES OF ENRICHED FUNCTORS AND TRIANGULATED CATEGORIES OF MOTIVES

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ABSTRACT. We investigate certain categorical aspects of Voevodsky's triangulated categories of motives. For this, various recollements for Grothendieck categories of enriched functors and their derived categories are established. In order to extend these recollements further with respect to Serre's localization, the concept of the (strict) Voevodsky property for Serre localizing subcategories is introduced. This concept is inspired by the celebrated Voevodsky theorem on homotopy invariant presheaves with transfers. As an application, it is shown that Voevodsky's triangulated categories of motives fit into recollements of derived categories of associated Grothendieck categories of Nisnevich sheaves with specific transfers.

1. INTRODUCTION

Triangulated categories of motives $\mathbf{DM}^{eff}_{\mathscr{C}}(k)$ over a (perfect) field k constructed by Voevodsky [25] are of fundamental importance in motivic homotopy theory (here $\mathscr{C}$ is a reasonable category of correspondences on smooth algebraic varieties Sm/k). By definition [25], it is the full triangulated subcategory of the derived category $\mathbf{D}(Shv(\mathscr{C}))$ of complexes of Nisnevich $\mathscr{C}$-sheaves whose cohomology sheaves are homotopy invariant. By a theorem of Voevodsky [25], the inclusion $\mathbf{DM}^{eff}_{\mathscr{C}}(k) \to \mathbf{D}(Shv(\mathscr{C}))$ has the left adjoint given by the Sulin complex functor C_*. The category $\mathbf{DM}^{eff}_{\mathscr{C}}(k)$ is also equivalent to the quotient category of $\mathbf{D}(Shv(\mathscr{C}))$ with respect to the localizing subcategory $\mathscr{T}_{\mathbb{A}^1}$ generated by complexes of the form $\mathscr{C}(-,X\times\mathbb{A}^1)_{nis} \to \mathscr{C}(-,X)_{nis}$ (see [22, 25] for details).

Inspired by Voevodsky's constructions [25], we investigate certain categorical aspects of $\mathbf{DM}^{eff}_{\mathscr{C}}(k)$. For this purpose, we work in the framework of Grothendieck categories of enriched functors in the sense of [1] and their derived categories [9]. We shall convert some fundamental Voevodsky's theorems into the language of enriched category theory and arrive at some categorical concepts and results which are of independent interest. Afterwards, we shall translate these concepts and results back into the motivic language.

First, recall from [1] that the category of enriched functors $[\mathscr{C},\mathscr{V}]$, where $\mathscr{V}$ is a closed symmetric monoidal Grothendieck category and $\mathscr{C}$ is a small $\mathscr{V}$-category, is a Grothendieck $\mathscr{V}$-category with a set of generators $\{\mathscr{V}(c,-)\oslash g_i \mid c\in \mathrm{Ob}\mathscr{C}, i\in I\}$, where $\{g_i\}_I$ is a set of generators of $\mathscr{V}$. As in [1], we shall refer to the category $[\mathscr{C},\mathscr{V}]$ as the *Grothendieck category of enriched functors*. Our first result is as follows (see Theorem 3.4).

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Theorem 1. *Suppose $\mathscr{A}$ is a full $\mathscr{V}$-subcategory of a small $\mathscr{V}$-category $\mathscr{C}$. Define a localizing subcategory $\mathscr{S}_{\mathscr{A}} := \{Y \in [\mathscr{C},\mathscr{V}] \mid Y(a) = 0 \text{ for all } a \in \mathscr{A}\} \subset [\mathscr{C},\mathscr{V}]$. There is a recollement*

$$\begin{tikzcd} \mathscr{S}_{\mathscr{A}} \arrow[r, "i"] & {[\mathscr{C},\mathscr{V}]} \arrow[l, bend right, "i_L"'] \arrow[l, bend left, "i_R"] \arrow[r, "r"] & {[\mathscr{A},\mathscr{V}]} \arrow[l, bend right, "r_L"'] \arrow[l, bend left, "r_R"] \end{tikzcd}$$

with functors i, r being the canonical inclusion and restriction functors respectively. The functors r_L, r_R are the enriched left and right Kan extensions respectively, i_R is the torsion functor associated with the localizing subcategory $\mathscr{S}_{\mathscr{A}}$.

Using this theorem, we can extend [1, Theorem 5.3] into a recollement (see Theorem 3.5).

Theorem 2. *There is a recollement*

$$\begin{tikzcd} \mathscr{S}_{\mathscr{A}} \arrow[r, "i"] & {[\mathscr{C},\mathscr{V}]} \arrow[l, bend right, "i_L"'] \arrow[l, bend left, "i_R"] \arrow[r, "\ell"] & {[\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}}} \arrow[l, bend right, "\ell_L"'] \arrow[l, bend left, "\ell_R"] \end{tikzcd}$$

with functors i_L, i, i_R being from the preceding theorem. The functor ℓ is the localization functor, ℓ_R is the inclusion and $\ell_L := r_L \circ (\ell \circ r_L)^{-1}$.

Next, we shall pass to derived categories. In what follows in the introduction, we shall always assume that the derived category $\mathbf{D}(\mathscr{V})$ is compactly generated with some reasonable assumptions on compact generators (more precisely, we assume Theorem 4.1 to be satisfied). We prove that there is a recollement as follows (see Theorem 4.11).

Theorem 3. *There exists a recollement of triangulated categories*

$$\begin{tikzcd} \mathscr{E}_{\mathscr{A}} \arrow[r, "\iota"] & \mathbf{D}[\mathscr{C},\mathscr{V}] \arrow[l, bend right, "\iota_L"'] \arrow[l, bend left, "\iota_R"] \arrow[r, "\rho"] & \mathbf{D}[\mathscr{A},\mathscr{V}] \arrow[l, bend right, "\rho_L"'] \arrow[l, bend left, "\rho_R"] \end{tikzcd}$$

where $\mathscr{A} \subset \mathscr{C}$, $\mathscr{E}_{\mathscr{A}} := \{Y \in \mathscr{D}[\mathscr{C},\mathscr{V}] \mid H_n(Y(a)) = 0 \text{ for all } a \in \mathscr{A} \text{ and } n \in \mathbb{Z}\}$, ι is the inclusion, ρ is the restriction.

Subsequently, we consider a Serre localizing subcategory $\mathscr{Q} \subset [\mathscr{C},\mathscr{V}]$. It is not readily apparent when recollements as above are compatible with $\mathscr{Q}$-localization. Precisely, after constructing a recollement of $\mathbf{D}[\mathscr{C},\mathscr{V}]$ in the preceding theorem, it is important for applications to extend it further to a recollement of $\mathbf{D}([\mathscr{C},\mathscr{V}]/\mathscr{Q})$. Therefore to make the desired extension of the recollement to $\mathbf{D}([\mathscr{C},\mathscr{V}]/\mathscr{Q})$ possible, we need to find the right conditions on the localizing subcategories $\mathscr{S}_{\mathscr{A}}$ and $\mathscr{Q}$ of $[\mathscr{C},\mathscr{V}]$. These conditions originate in the fundamental Voevodsky theorem [24], which says that the Nisnevich sheaf F_{nis} associated with a homotopy invariant presheaf with transfers F is a homotopy invariant sheaf with transfers and that it is strictly homotopy invariant whenever the base field is perfect. We translate this theorem into the language of Serre and Bousfield localization theory in Grothendieck categories and their derived categories. Namely, we formulate the desired conditions in Definitions 5.1 and 5.7 that $\mathscr{S}_{\mathscr{A}}$ and $\mathscr{Q}$ must satisfy, that allow us to extend Theorem 3. We shall call these conditions the “Voevodsky properties” owing to the influence of Voevodsky’s fundamental theorem [24] on homotopy invariant presheaves with transfers. We construct the following recollement (see Theorem 5.10).

Theorem 4. *Suppose $\mathscr{A} \subset \mathscr{C}$, and $\mathscr{Q} \subset [\mathscr{C}, \mathscr{V}]$ is a localizing subcategory such that:*

- *$\mathbf{D}\left([\mathscr{C}, \mathscr{V}]/\mathscr{Q}\right)$ is compactly generated and the functor induced by the exact $\mathscr{Q}$-localization functor respects compact objects;*
- *the localizing subcategory $\mathscr{S}_{\mathscr{A}}$ satisfies the strict Voevodsky property with respect to $\mathscr{Q}$ (see Definition 5.7).*

Then there exists a recollement of triangulated categories

$$\begin{array}{ccccc} & \xleftarrow{\ \iota_L^{\mathscr{Q}}\ } & & \xleftarrow{\ \lambda_L^{\mathscr{Q}}\ } & \\ \mathscr{E}_{\mathscr{A}}^{\mathscr{Q}} & \xrightarrow{\ \iota^{\mathscr{Q}}\ } & \mathbf{D}\big([\mathscr{C}, \mathscr{V}]/\mathscr{Q}\big) & \xrightarrow{\ \lambda^{\mathscr{Q}}\ } & \mathbf{D}\big([\mathscr{C}, \mathscr{V}]/\mathscr{J}_{\mathscr{A}}\big), \\ & \xleftarrow[\ \iota_R^{\mathscr{Q}}\ ]{} & & \xleftarrow[\ \lambda_R^{\mathscr{Q}}\ ]{} & \end{array}$$

where $\mathscr{E}_{\mathscr{A}}^{\mathscr{Q}}$ is the full subcategory containing chain complexes with homology belonging to the $\mathscr{Q}$-localization of $\mathscr{S}_{\mathscr{A}}$, and $\mathscr{J}_{\mathscr{A}}$ is the smallest localizing category containing $\mathscr{Q}$ and $\mathscr{S}_{\mathscr{A}}$ in $[\mathscr{C}, \mathscr{V}]$.

It is worth mentioning that in order to prove the previous theorem, we develop a new technique combining Serre localization in Grothendieck categories and Bousfield localization in compactly generated triangulated categories. This technique is of independent interest.

The next step is to convert the previous theorem back into the motivic language, and we are now in a position to discuss its main application. In practice the derived category for the Grothendieck category $Shv(\mathscr{C})$ of Nisnevich sheaves with reasonable transfers $\mathscr{C}$ on smooth algebraic varieties Sm/k plays the role of the middle category. In this language the left category will be nothing but Voevodsky’s [25] triangulated category of motives $\mathbf{DM}_{\mathscr{C}}^{eff}(k)$. Also, the Voevodsky theorem [25, 3.2.6] computes the functor $\iota_L^{\mathscr{Q}}$ as the Suslin complex C_*. We refer the reader to Section 6 for details on how Theorem 4 encodes the above information about $\mathbf{DM}_{\mathscr{C}}^{eff}(k)$.

Therefore Theorem 4 is a kind of categorical framework for the triangulated category of motives $\mathbf{DM}_{\mathscr{C}}^{eff}(k)$. Precisely, the following theorem is true (see Theorem 6.6).

Theorem 5. *Suppose $\mathscr{C}$ is a strict V-category of correspondences in the sense of [8]. There exists a recollement of triangulated categories*

$$\begin{array}{ccccc} & \xleftarrow{\ C_*\ } & & \xleftarrow{\ \lambda_L^{\mathscr{Q}}\ } & \\ \mathbf{DM}_{\mathscr{C}}^{eff}(k) & \xrightarrow{\ \iota^{\mathscr{Q}}\ } & \mathbf{D}\big(Shv(\mathscr{C})\big) & \xrightarrow{\ \lambda^{\mathscr{Q}}\ } & \mathbf{D}\big(Shv(\mathscr{C})/\mathscr{S}_{\mathbb{A}^1}^{\mathscr{Q}}\big), \\ & \xleftarrow[\ \iota_R^{\mathscr{Q}}\ ]{} & & \xleftarrow[\ \lambda_R^{\mathscr{Q}}\ ]{} & \end{array}$$

*where $\mathscr{S}_{\mathbb{A}^1}^{\mathscr{Q}}$ is the localizing Serre subcategory consisting of the homotopy invariant $\mathscr{C}$-sheaves, C_* is the Suslin complex functor, the functor $\lambda^{\mathscr{Q}}$ is induced by the $\mathscr{S}_{\mathbb{A}^1}^{\mathscr{Q}}$-localization functor $Shv(\mathscr{C}) \to Shv(\mathscr{C})/\mathscr{S}_{\mathbb{A}^1}^{\mathscr{Q}}$, $\iota^{\mathscr{Q}}$ is inclusion, and $\lambda_R^{\mathscr{Q}}$ is induced by the **K**-injective resolution functor.*

As an important consequence of the theorem (see Corollary 6.7), the localizing subcategory $\mathscr{T}_{\mathbb{A}^1}$ of $\mathbf{D}(Shv(\mathscr{C}))$ is computed – up to an equivalence of triangulated categories – as the derived category $\mathbf{D}(Shv(\mathscr{C})/\mathscr{S}^{\mathscr{Q}}_{\mathbb{A}^1})$ of the Grothendieck category $Shv(\mathscr{C})/\mathscr{S}^{\mathscr{Q}}_{\mathbb{A}^1}$.

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2. Enriched Category Theory

In this section we collect basic facts about enriched categories we shall need later. We refer the reader to [2] for details. Throughout this paper the quadruple $(\mathscr{V},\otimes,\underline{\mathrm{Hom}},e)$ is a closed symmetric monoidal category with monoidal product $\otimes$, internal Hom-object $\underline{\mathrm{Hom}}$ and monoidal unit e. We sometimes write $[a,b]$ to denote $\underline{\mathrm{Hom}}(a,b)$, where $a,b\in\mathrm{Ob}\,\mathscr{V}$. We have structure isomorphisms

$$a_{abc}:(a\otimes b)\otimes c\to a\otimes(b\otimes c),\quad l_a:e\otimes a\to a,\quad r_a:a\otimes e\to a$$

in $\mathscr{V}$ with $a,b,c\in\mathrm{Ob}\,\mathscr{V}$.

Definition 2.1. A *$\mathscr{V}$-category $\mathscr{C}$*, or *a category enriched over $\mathscr{V}$*, consists of the following data:

(1) a class $\mathrm{Ob}(\mathscr{C})$ of objects;
(2) for every pair $a,b\in\mathrm{Ob}(\mathscr{C})$ of objects, an object $\mathscr{V}_{\mathscr{C}}(a,b)$ of $\mathscr{V}$;
(3) for every triple $a,b,c\in\mathrm{Ob}(\mathscr{C})$ of objects, a composition morphism in $\mathscr{V}$,
$$c_{abc}:\mathscr{V}_{\mathscr{C}}(a,b)\otimes\mathscr{V}_{\mathscr{C}}(b,c)\to\mathscr{V}_{\mathscr{C}}(a,c);$$
(4) for every object $a\in\mathscr{C}$, a unit morphism $u_a:e\to\mathscr{V}_{\mathscr{C}}(a,a)$ in $\mathscr{V}$.

These data must satisfy the natural associativity and unit axioms.

When $\mathrm{Ob}\,\mathscr{C}$ is a set, the $\mathscr{V}$-category $\mathscr{C}$ is called a *small $\mathscr{V}$-category*.

Definition 2.2. Given $\mathscr{V}$-categories $\mathscr{A},\mathscr{B}$, a *$\mathscr{V}$-functor* or an *enriched functor* $F:\mathscr{A}\to\mathscr{B}$ consists in giving:

(1) for every object $a\in\mathscr{A}$, an object $F(a)\in\mathscr{B}$;
(2) for every pair $a,b\in\mathscr{A}$ of objects, a morphism in $\mathscr{V}$,
$$F_{ab}:\mathscr{V}_{\mathscr{A}}(a,b)\to\mathscr{V}_{\mathscr{B}}(F(a),F(b))$$
in such a way that the following axioms hold:
 - ⋄ for all objects $a,a',a''\in\mathscr{A}$, diagram (1) below commutes (composition axiom);
 - ⋄ for every object $a\in\mathscr{A}$, diagram (2) below commutes (unit axiom).

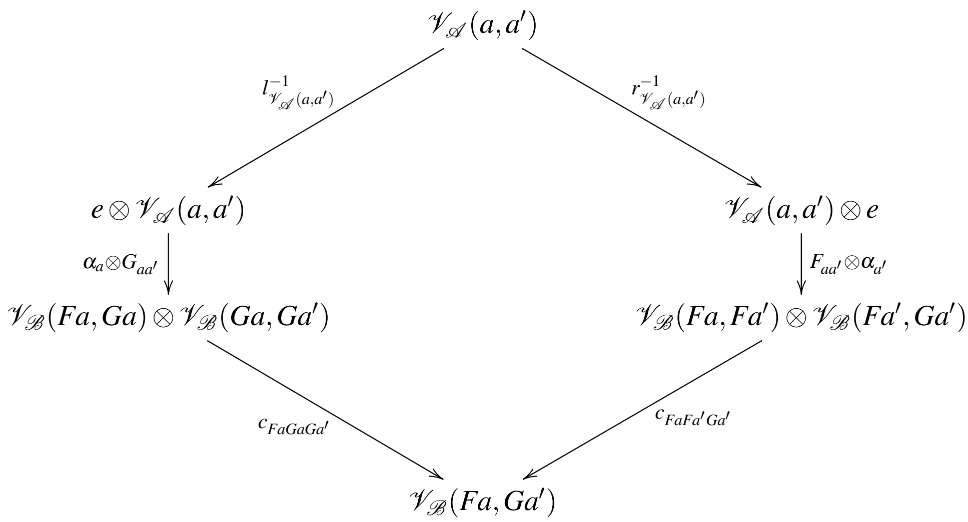
$$\begin{array}{ccc}\mathscr{V}_{\mathscr{A}}(a,a')\otimes\mathscr{V}_{\mathscr{A}}(a',a'') & \xrightarrow{c_{aa'a''}} & \mathscr{V}_{\mathscr{A}}(a,a'') \\ \downarrow{\scriptstyle F_{aa'}\otimes F_{a'a''}} & & \downarrow{\scriptstyle F_{aa''}} \\ \mathscr{V}_{\mathscr{B}}(Fa,Fa')\otimes\mathscr{V}_{\mathscr{B}}(Fa',Fa'') & \xrightarrow[c_{Fa,Fa',Fa''}]{} & \mathscr{V}_{\mathscr{B}}(Fa,Fa'')\end{array}\qquad(1)$$

$$\begin{array}{ccc} e & \xrightarrow{u_a} & \mathscr{V}_{\mathscr{A}}(a,a) \\ & {\scriptstyle u_{Fa}}\searrow & \downarrow{\scriptstyle F_{aa}} \\ & & \mathscr{V}_{\mathscr{B}}(Fa,Fa)\end{array}\qquad(2)$$

Definition 2.3. Let $\mathscr{A},\mathscr{B}$ be two $\mathscr{V}$-categories and $F,G:\mathscr{A}\to\mathscr{B}$ two $\mathscr{V}$-functors. A *$\mathscr{V}$-natural transformation* $\alpha:F\Rightarrow G$ consists in giving, for every object $a\in\mathscr{A}$, a morphism

$$\alpha_a:e\to\mathscr{V}_{\mathscr{B}}(F(a),G(a))$$

in $\mathscr{V}$ such that diagram below commutes, for all objects $a, a' \in \mathscr{A}$.



Any $\mathscr{V}$-category $\mathscr{C}$ defines an ordinary category $\mathscr{U}\mathscr{C}$, also called the *underlying category*. Its class of objects is $\mathrm{Ob}\,\mathscr{C}$, the morphism sets are $\mathrm{Hom}_{\mathscr{U}\mathscr{C}}(a,b) := \mathrm{Hom}_{\mathscr{V}}(e, \mathscr{V}_{\mathscr{C}}(a,b))$ (see [2, p. 316]).

Let $\mathscr{C}, \mathscr{D}$ be two $\mathscr{V}$-categories. The *monoidal product* $\mathscr{C} \otimes \mathscr{D}$ is the $\mathscr{V}$-category, where

$$\mathrm{Ob}(\mathscr{C} \otimes \mathscr{D}) := \mathrm{Ob}\,\mathscr{C} \times \mathrm{Ob}\,\mathscr{D}$$

and

$$\mathscr{V}_{\mathscr{C}\otimes\mathscr{D}}((a,x),(b,y)) := \mathscr{V}_{\mathscr{C}}(a,b) \otimes \mathscr{V}_{\mathscr{D}}(x,y), \quad a,b \in \mathscr{C}, x,y \in \mathscr{D}.$$

Definition 2.4. A $\mathscr{V}$-category $\mathscr{C}$ is a *right $\mathscr{V}$-module* if there is a $\mathscr{V}$-functor $\mathrm{act} : \mathscr{C} \otimes \mathscr{V} \to \mathscr{C}$, denoted $(c,A) \mapsto c \oslash A$ and a $\mathscr{V}$-natural unit isomorphism $r_c : \mathrm{act}(c,e) \to c$ subject to the following conditions:

(1) there are coherent natural associativity isomorphisms $c \oslash (A \otimes B) \to (c \oslash A) \otimes B$;
(2) the isomorphisms $c \oslash (e \otimes A) \rightrightarrows c \oslash A$ coincide.

A right $\mathscr{V}$-module is *closed* if there is a $\mathscr{V}$-functor

$$\mathrm{coact} : \mathscr{V}^{\mathrm{op}} \otimes \mathscr{C} \to \mathscr{C}$$

such that for all $A \in \mathrm{Ob}\,\mathscr{V}$, and $c \in \mathrm{Ob}\,\mathscr{C}$, the $\mathscr{V}$-functor $\mathrm{act}(-,A) : \mathscr{C} \to \mathscr{C}$ is left $\mathscr{V}$-adjoint to $\mathrm{coact}(A,-)$ and $\mathrm{act}(c,-) : \mathscr{V} \to \mathscr{C}$ is left $\mathscr{V}$-adjoint to $\mathscr{V}_{\mathscr{C}}(c,-)$.

If $\mathscr{C}$ is a small $\mathscr{V}$-category, $\mathscr{V}$-functors from $\mathscr{C}$ to $\mathscr{V}$ and their $\mathscr{V}$-natural transformations form the category $[\mathscr{C}, \mathscr{V}]$ of $\mathscr{V}$-functors from $\mathscr{C}$ to $\mathscr{V}$. If $\mathscr{V}$ is complete, then $[\mathscr{C}, \mathscr{V}]$ is also a $\mathscr{V}$-category whose morphism $\mathscr{V}$-object $\mathscr{V}_{[\mathscr{C},\mathscr{V}]}(X,Y)$ is the end

$$\int_{\mathrm{Ob}\,\mathscr{C}} \mathscr{V}(X(c), Y(c)).$$

Lemma 2.5. *Let $\mathscr{V}$ be a complete closed symmetric monoidal category, and $\mathscr{C}$ be a small $\mathscr{V}$-category. Then $[\mathscr{C}, \mathscr{V}]$ is a closed $\mathscr{V}$-module.*

Proof. See [5, 2.4]. □

Given $c \in \mathrm{Ob}\mathscr{C}$, $X \mapsto X(c)$ defines the $\mathscr{V}$-functor $\mathrm{Ev}_c : [\mathscr{C},\mathscr{V}] \to \mathscr{V}$ called *evaluation at* c. The assignment $c \mapsto \mathscr{V}_{\mathscr{C}}(c,-)$ from $\mathscr{C}$ to $[\mathscr{C},\mathscr{V}]$ is again a $\mathscr{V}$-functor $\mathscr{C}^{\mathrm{op}} \to [\mathscr{C},\mathscr{V}]$, called the *$\mathscr{V}$-Yoneda embedding*. $\mathscr{V}_{\mathscr{C}}(c,-)$ is a representable functor, represented by c.

Lemma 2.6 (The Enriched Yoneda Lemma)**.** *Let $\mathscr{V}$ be a complete closed symmetric monoidal category and $\mathscr{C}$ a small $\mathscr{V}$-category. For every $\mathscr{V}$-functor $X : \mathscr{C} \to \mathscr{V}$ and every $c \in \mathrm{Ob}\mathscr{C}$, there is a $\mathscr{V}$-natural isomorphism $X(c) \cong \mathscr{V}_{\mathscr{F}}(\mathscr{V}_{\mathscr{C}}(c,-),X)$.*

Lemma 2.7. *If $\mathscr{V}$ is a bicomplete closed symmetric monoidal category and $\mathscr{C}$ is a small $\mathscr{V}$-category, then $[\mathscr{C},\mathscr{V}]$ is bicomplete. (Co)limits are formed pointwise.*

Proof. See [2, 6.6.17]. $\square$

Corollary 2.8. *Assume $\mathscr{V}$ is bicomplete, and let $\mathscr{C}$ be a small $\mathscr{V}$-category. Then any $\mathscr{V}$-functor $X : \mathscr{C} \to \mathscr{V}$ is $\mathscr{V}$-naturally isomorphic to the coend*

$$X \cong \int^{\mathrm{Ob}\mathscr{C}} \mathscr{V}_{\mathscr{C}}(c,-) \oslash X(c).$$

Proof. See [2, 6.6.18]. $\square$

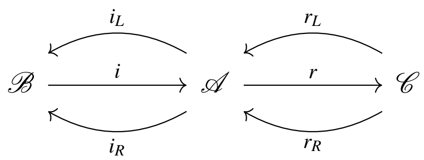
3. Recollements for Grothendieck categories of enriched functors

An *adjoint triple* is a collection of functors $F,H : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ such that F is left adjoint to G and G is left adjoint to H. We denote this information $F \dashv G \dashv H$.

The following lemma is proven in [13, 7.4.1].

Lemma 3.1. *Given an adjoint triple of functors $F \dashv G \dashv H$, F is fully faithful if and only if H is fully faithful.*

Definition 3.2. A *recollement of an Abelian category $\mathscr{A}$ by Abelian categories $\mathscr{B}$ and $\mathscr{C}$* is a diagram of additive functors



such that $i_L \dashv i \dashv i_R$, $r_L \dashv r \dashv r_R$ are adjoint triples, the functors i, r_L and r_R are fully faithful and $\mathrm{Im}\, i = \mathrm{Ker}\, r$.

We refer the reader to [19, 2.8] for the following proposition.

Proposition 3.3. *Given a recollement of Abelian categories as above, for all $A \in \mathscr{A}$ there are $B, B' \in B$ such that the following sequences*

$$0 \to iB \to r_L \circ rA \to A \to i \circ i_L A \to 0,$$

$$0 \to i \circ i_R A \to A \to r_R \circ rA \to iB' \to 0$$

induced by the unit and counit morphisms of the adjunctions are exact in $\mathscr{A}$. Furthermore, $\mathscr{B}$ is a Serre subcategory of $\mathscr{A}$ and r is naturally equivalent to the quotient functor $\mathscr{A} \to \mathscr{A}/\mathscr{B}$. In particular, $\mathscr{C} \cong \mathscr{A}/\mathscr{B}$.

In what follows we fix a closed symmetric monoidal Grothendieck category $\mathscr{V}$ with a set of generators $\{g_i\}_{i \in I}$, and we also choose $\mathscr{C}$ to be a small $\mathscr{V}$-category. By [1, 4.2] the category of enriched functors $[\mathscr{C}, \mathscr{V}]$ is Grothendieck with the set of generators $\{\mathscr{V}_{\mathscr{C}}(c, -) \oslash g_i \mid c \in \mathscr{C}, i \in I\}$.

Consider a full $\mathscr{V}$-subcategory $\mathscr{A}$ of the $\mathscr{V}$-category $\mathscr{C}$. This means that $\mathscr{A}$ is some subcollection of objects of $\mathscr{C}$ and $\mathscr{V}$-morphism objects $\mathscr{V}_{\mathscr{A}}(a,b) = \mathscr{V}_{\mathscr{C}}(a,b)$ for all objects $a, b \in \mathscr{A}$. We shall consider the full subcategory $\mathscr{S}_{\mathscr{A}} := \{F \in [\mathscr{C}, \mathscr{V}] \mid F(a) = 0 \text{ for all } a \in \mathscr{A}\}$. Obviously, the full subcategory $\mathscr{S}_{\mathscr{A}}$ is localizing.

The following result fits $\mathscr{S}_{\mathscr{A}}$ and the Grothendieck categories of enriched functors $[\mathscr{C}, \mathscr{V}]$, $[\mathscr{A}, \mathscr{V}]$ in a recollement.

Theorem 3.4. *Suppose $\mathscr{A}$ is a full $\mathscr{V}$-subcategory of a small $\mathscr{V}$-category $\mathscr{C}$. Then there is a recollement*

$$\xymatrix{\mathscr{S}_{\mathscr{A}} \ar[r]^{i} & [\mathscr{C},\mathscr{V}] \ar@/_2pc/[l]_{i_L} \ar@/^2pc/[l]^{i_R} \ar[r]^{r} & [\mathscr{A},\mathscr{V}] \ar@/_2pc/[l]_{r_L} \ar@/^2pc/[l]^{r_R}}$$

with functors i, r being the canonical inclusion and restriction functors respectively. The functors r_L, r_R are the enriched left and right Kan extensions respectively, i_R is the torsion functor associated with the localizing subcategory $\mathscr{S}_{\mathscr{A}}$.

Proof. It is clear that i is fully faithful and $\operatorname{Im}(i) = \operatorname{Ker}(r)$, as r is the same as precomposition with the inclusion $I : \mathscr{A} \hookrightarrow \mathscr{C}$. We shall construct each functor in turn, beginning with $r_R : [\mathscr{A}, \mathscr{V}] \to [\mathscr{C}, \mathscr{V}]$. Define the functor r_R as the enriched right Kan extension

$$H \mapsto \int_{b \in \mathscr{A}} [\mathscr{V}_{\mathscr{C}}(-, Ib), H(b)].$$

This is well defined as $\mathscr{C}$ is small and $\mathscr{V}$ is bicomplete. Further, we see that $r_R H(Ia) = H(a)$ for all $a \in \mathscr{A}$, by the fully faithfulness of I and the Yoneda lemma. We establish that this is indeed right adjoint, or more precisely, there exists the following natural isomorphism

$$\operatorname{Hom}_{[\mathscr{A},\mathscr{V}]}(G \circ I, H) \cong \operatorname{Hom}_{[\mathscr{C},\mathscr{V}]}(G, r_R(H)).$$

We shall prove it in the enriched case, from which the desired isomorphism will follow. Consider the $\mathscr{V}$-morphism object

$$\mathscr{V}_{[\mathscr{C},\mathscr{V}]}(G, r_R(H)) := \int_{c \in \mathscr{C}} [G(c), r_R H(c)] \cong \int_{c \in \mathscr{C}} \Big[G(c), \int_{b \in \mathscr{A}} [\mathscr{V}_{\mathscr{C}}(c, Ib), H(b)]\Big].$$

As the internal Hom of $\mathscr{V}$ preserves $\mathscr{V}$-ends and the fact that we may swap limits, we see that this is naturally isomorphic to

$$\int_{b \in \mathscr{A}} \int_{c \in \mathscr{C}} \Big[G(c), [\mathscr{V}_{\mathscr{C}}(c, Ib), H(b)]\Big].$$

Next, we use that $\mathscr{V}$ is closed, and the limit preserving properties again to show that the latter object is isomorphic to

$$\int_{b \in \mathscr{A}} \int_{c \in \mathscr{C}} \Big[\mathscr{V}_{\mathscr{C}}(c, Ib) \oslash G(c), H(b)\Big] \cong \int_{b \in \mathscr{A}} \left[\int^{c \in \mathscr{C}} \mathscr{V}_{\mathscr{C}}(c, Ib) \oslash G(c), H(b)\right].$$

This object is naturally isomorphic to

$$\int_{b \in \mathscr{A}} [G(Ib), H(b)] = \mathscr{V}_{[\mathscr{A},\mathscr{V}]}(G \circ I, H).$$

Hence we can deduce the desired adjunction.

We define the enriched left Kan extension $r_L : [\mathscr{A},\mathscr{V}] \to [\mathscr{C},\mathscr{V}]$ to be the functor that acts on objects as

$$F \mapsto \int^{a\in\mathscr{A}} \mathscr{V}_{\mathscr{C}}(Ia,-) \oslash F(a).$$

It is left adjoint to r by [1, 5.2]. Having demonstrated that $r_L \dashv r \dashv r_R$, it is enough to know that either r_L or r_R is fully faithful to determine the other is also fully faithful (see Lemma 3.1). We prove the full and faithfulness of r_R as an enriched functor with the following natural isomorphisms

$$\begin{aligned}\mathscr{V}_{[\mathscr{C},\mathscr{V}]}(r_R X, r_R Y) &\cong \mathscr{V}_{[\mathscr{A},\mathscr{V}]}(r_R(X)\circ I, Y)\\ &\cong \mathscr{V}_{[\mathscr{A},\mathscr{V}]}\left(\int_{b\in\mathscr{A}} [\mathscr{V}_{\mathscr{C}}(-,Ib), X(b)]\circ I, Y\right)\\ &\cong \mathscr{V}_{[\mathscr{A},\mathscr{V}]}\left(\int_{b\in\mathscr{A}} [\mathscr{V}_{\mathscr{C}}(I(-),Ib), X(b)], Y\right).\end{aligned}$$

Since I is a fully faithful enriched functor, we have a natural isomorphism to

$$\begin{aligned}&\cong \mathscr{V}_{[\mathscr{A},\mathscr{V}]}\left(\int_{b\in\mathscr{A}} [\mathscr{V}_{\mathscr{A}}(-,b), X(b)], Y\right)\\ &= \int_{a\in\mathscr{A}} \left[\int_{b\in\mathscr{A}} [\mathscr{V}_{\mathscr{A}}(a,b), X(b)], Y(a)\right]\\ &\cong \int_{a\in\mathscr{A}} \left[\mathscr{V}_{[\mathscr{A},\mathscr{V}]}(\mathscr{V}_{\mathscr{A}}(a,-), X), Y(a)\right]\\ &\cong \int_{a\in\mathscr{A}} [X(a), Y(a)] =: \mathscr{V}_{[\mathscr{A},\mathscr{V}]}(X,Y).\end{aligned}$$

Thus we conclude that both r_L, r_R are fully faithful.

Since $\mathscr{S}_{\mathscr{A}}$ is a localizing subcategory of the Grothendieck category $[\mathscr{C},\mathscr{V}]$, its inclusion always has a right adjoint. This right adjoint is simply the $\mathscr{S}_{\mathscr{A}}$-torsion functor. However, it is known that the adjoint triple $r_L \dashv r \dashv r_R$ can be used to define the adjoint functors i_L, i_R explicitly. For completeness, we demonstrate this.

First, given that $r_L \dashv r$, denote the counit of this adjunction by $\varepsilon : r_L \circ r \Rightarrow \mathrm{id}$, and the unit $\mu : \mathrm{id} \Rightarrow r\circ r_L$. Given the triangle identities, namely that $r(\varepsilon_X)\circ \mu_{r(X)} = \mathrm{id}_X$, we have that $r(\varepsilon_X)$ is an epimorphism, and with r exact, it follows that $r(\mathrm{Coker}(\varepsilon_X)) \cong \mathrm{Coker}(r(\varepsilon_X)) \cong 0$. Thus $\mathrm{Coker}(\varepsilon_X)$ belongs to $\mathrm{Ker}(r) = \mathscr{S}_{\mathscr{A}}$ for all $X \in [\mathscr{C},\mathscr{V}]$. Hence we may define a functor

$$i_L : [\mathscr{C},\mathscr{V}] \to \mathscr{S}_{\mathscr{A}}, \quad X \mapsto \mathrm{Coker}(\varepsilon_X).$$

In order to show that this is indeed adjoint, we consider the canonical map $X \to \mathrm{Coker}(\varepsilon_X)$. Take any map $f : X \to Y$ for some object $Y \in \mathscr{S}_{\mathscr{A}}$. Since $Y \in \mathscr{S}_{\mathscr{A}}$ we have that $\varepsilon_Y : 0 \to Y$, then by naturality we see that $f \circ \varepsilon_X = 0$. Thus there exists a unique map $\mathrm{Coker}(\varepsilon_X) \to Y$ that factors f. Therefore we conclude that i_L is left adjoint to i. Conversely, we define the right adjoint $i_R : Y \mapsto \mathrm{Ker}(\eta_Y)$ where η is the unit of the adjunction $r \dashv r_R$, the proof is entirely dual. The construction of the desired recollement is completed. □

The preceding theorem and Proposition 3.3 imply $[\mathscr{A},\mathscr{V}] \cong [\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}}$. An equivalence of these categories was also proven in [1, 5.3] by using the following explicit functors. Denote by $\ell : [\mathscr{C},\mathscr{V}] \to [\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}}$ the $\mathscr{S}_{\mathscr{A}}$-localization functor. Then the composite functor

$$\varkappa := \ell \circ r_L : [\mathscr{A},\mathscr{V}] \to [\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}} \tag{3}$$

is an equivalence of Grothendieck categories by [1, 5.3].

Theorem 3.5. *Suppose $\mathscr{A}$ is a full $\mathscr{V}$-subcategory of a small $\mathscr{V}$-category $\mathscr{C}$. Then there is a recollement*

$$\begin{array}{ccccc} & \xleftarrow{\;i_L\;} & & \xleftarrow{\;\ell_L\;} & \\ \mathscr{S}_{\mathscr{A}} & \xrightarrow{\;i\;} & [\mathscr{C},\mathscr{V}] & \xrightarrow{\;\ell\;} & [\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}} \\ & \xleftarrow[\;i_R\;]{} & & \xleftarrow[\;\ell_R\;]{} & \end{array}$$

with functors i_L, i, i_R being from Theorem 3.4. The functor ℓ_R is the inclusion and $\ell_L := r_L \circ \varkappa^{-1}$.

Proof. One has $\mathrm{Im}(i) = \mathrm{Ker}(\ell)$ and ℓ_R is a right adjoint of ℓ by the general localization theory of Grothendieck categories. To see that ℓ_L is left adjoint to ℓ, let $X \in [\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}}$ and $Y \in [\mathscr{C},\mathscr{V}]$. The proof of [1, 5.3] shows that the adjunction morphism $r_L r(Y) \to Y$ induces an isomorphism $\ell r_L r(Y) \cong \ell(Y)$. Then

$$\mathrm{Hom}_{[\mathscr{C},\mathscr{V}]}(\ell_L(X),Y) = \mathrm{Hom}_{[\mathscr{C},\mathscr{V}]}(r_L \circ \varkappa^{-1}(X),Y) \cong \mathrm{Hom}_{[\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}}}(X,\varkappa r(Y)).$$

But $\varkappa r(Y) = \ell r_L r(Y) \cong \ell(Y)$. It follows that

$$\mathrm{Hom}_{[\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}}}(X,\varkappa r(Y)) \cong \mathrm{Hom}_{[\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}}}(X,\ell(Y)),$$

and hence ℓ_L is left adjoint to ℓ. Since ℓ_R is fully faithful then so is ℓ_L by Lemma 3.1. □

Example 3.6. Theorem 3.4 recovers an important example of a recollement [21, Chapter IV]. First, consider a ring R and assume there exists some idempotent $e = e^2 \in R$. Then there exists a recollement

$$\begin{array}{ccccc} & \longleftarrow & & \longleftarrow & \\ R/ReR\text{-Mod} & \longrightarrow & R\text{-Mod} & \longrightarrow & eRe\text{-Mod}. \\ & \longleftarrow & & \longleftarrow & \end{array}$$

To derive this well known example from Theorem 3.4, we shall need to define a further preadditive category $\mathscr{R}$ with two objects $\mathscr{E}$, $\mathscr{E}^*$. The morphism groups are given by

$$\mathscr{R}(\mathscr{E},\mathscr{E}) := eRe \qquad \mathscr{R}(\mathscr{E}^*,\mathscr{E}) := eR(1-e)$$

$$\mathscr{R}(\mathscr{E},\mathscr{E}^*) := (1-e)Re \qquad \mathscr{R}(\mathscr{E}^*,\mathscr{E}^*) := (1-e)R(1-e).$$

We define composition by multiplication in R. Set $\mathscr{S}_{\mathscr{E}} := \{F \in [\mathscr{R},\mathbf{Ab}] \mid F(\mathscr{E}) = 0\} \subset [\mathscr{R},\mathbf{Ab}]$. Notice that $[\mathscr{R},\mathbf{Ab}]$ is the ordinary category of additive functors from $\mathscr{R}$ to Abelian groups. By Theorem 3.4 one has the following recollement

$$\begin{array}{ccccc} & \xleftarrow{\;i_L\;} & & \xleftarrow{\;r_L\;} & \\ \mathscr{S}_{\mathscr{E}} & \xrightarrow{\;i\;} & [\mathscr{R},\mathbf{Ab}] & \xrightarrow{\;r\;} & [\mathscr{E},\mathbf{Ab}]. \\ & \xleftarrow[\;i_R\;]{} & & \xleftarrow[\;r_R\;]{} & \end{array}$$

We wish to relate this recollement to the module categories above. We begin with the observation that $[\mathscr{E},\mathbf{Ab}]$ is isomorphic to the category of left eRe-modules, as $\mathrm{End}(\mathscr{E}) = eRe$, and write $M(\mathscr{E}) = M \in \text{Mod-}eRe$. Further, $[\mathscr{R},\mathbf{Ab}]$ is equivalent to the category of left R-modules. This follows from the fact that $[\mathscr{R},\mathbf{Ab}]$ is equivalent to the category of preadditive functors from a single object with morphism group the generalised matrix ring $\begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}$. The proof of this fact follows from a theorem of Mitchell [14, Theorem 7.1]. It says in particular that for a preadditive category $\mathscr{C}$ with finitely many objects, a preadditive functor $F:\mathscr{C}\to\mathbf{Ab}$ is equivalent to the module $\bigoplus_{X\in\mathscr{C}} F(X)$. We also note that this matrix ring is the Peirce decomposition of R. Hence the matrix ring is isomorphic to R and $[\mathscr{R},\mathbf{Ab}]$ is equivalent to the category of left R-modules. Next, $\mathscr{S}_{\mathscr{E}}$ is a full subcategory of $[\mathscr{R},\mathbf{Ab}]$, so given any $S\in\mathscr{S}_{\mathscr{E}}$ we have a canonical isomorphism

$$S \cong \int_{Y\in\mathscr{R}} \mathrm{Hom}_{\mathbf{Ab}}\left(\mathscr{R}(-,Y),S(Y)\right).$$

This is a restatement of the enriched Yoneda lemma. If we require that $S(\mathscr{E}) = 0$ then this is the same as

$$0 = S(\mathscr{E}) \cong \int_{Y\in\mathscr{R}} \mathrm{Hom}_{\mathbf{Ab}}\left(\mathscr{R}(\mathscr{E},Y),S(Y)\right) \cong \mathrm{Hom}_{[\mathscr{R},\mathbf{Ab}]}(\mathscr{R}(\mathscr{E},-),S).$$

As an R-module, $\mathscr{R}(\mathscr{E},-)$ is equivalent to $\mathscr{R}(\mathscr{E},\mathscr{E})\oplus\mathscr{R}(\mathscr{E},\mathscr{E}^*) = eRe\oplus(1-e)Re = Re$. Thus we derive that $S\in\mathscr{S}_{\mathscr{E}}$ if and only if

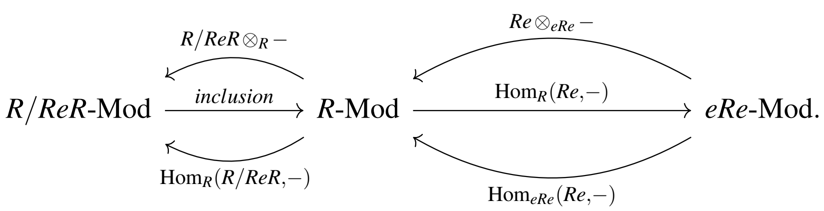
$$\mathrm{Hom}_R(Re,S) \cong \mathrm{Hom}_{[\mathscr{R},\mathbf{Ab}]}(\mathscr{R}(\mathscr{E},-),S) \cong 0.$$

$\mathscr{S}_{\mathscr{E}}$ is equivalent to the subcategory of R-modules S with $\mathrm{Hom}_R(Re,S) = eS = 0$. This subcategory can be identified with the category of R/ReR-modules [21, p. 42].

We calculate r_L, r, r_R as follows. We use the enriched Yoneda lemma again, writing I for the inclusion $\mathscr{E}\hookrightarrow\mathscr{R}$ and derive that

$$r : X \mapsto X\circ I \cong \int_{Y\in\mathscr{R}} \mathrm{Hom}_{\mathbf{Ab}}\left(\mathscr{R}(I(-),Y),X(Y)\right).$$

As an eRe-module it is isomorphic to $\mathrm{Hom}_R(Re,X)$, and so $r = \mathrm{Hom}_R(Re,-)$ in terms of modules. Thus we are able to identify our gluing with the well known gluing (see, e.g., [21, 4] and [18, 2.7])



Our next goal is to extend recollements of Theorems 3.4 and 3.5 to triangulated recollements of associated derived categories.

4. Recollements for the derived categories of enriched functors

We refer the reader to [20] for the notions and basic properties of **K**-projective and **K**-injective resolutions. In what follows we always assume that the following theorem is satisfied, which was proven by the authors in [9, 6.2].

Theorem 4.1. *Let $(\mathscr{V}, \otimes, e)$ be a closed symmetric monoidal Grothendieck category such that the derived category of $\mathscr{V}$ is a compactly generated triangulated category with compact generators $\{P_j\}_{j\in J}$. Further, suppose we have a small $\mathscr{V}$-category $\mathscr{C}$ and that any one of the following conditions is satisfied:*

1. *each P_j is **K**-projective;*
2. *for every **K**-injective $Y \in \mathbf{Ch}[\mathscr{C}, \mathscr{V}]$ and every $c \in \mathscr{C}$, the complex $Y(c) \in \mathbf{Ch}(\mathscr{V})$ is **K**-injective;*
3. *$\mathbf{Ch}(\mathscr{V})$ has a model structure, with quasi-isomorphisms being weak equivalences, such that for every injective fibrant complex $Y \in \mathbf{Ch}[\mathscr{C}, \mathscr{V}]$ the complex $Y(c)$ is fibrant in $\mathbf{Ch}(\mathscr{V})$.*

Then $\mathbf{D}[\mathscr{C}, \mathscr{V}]$ is a compactly generated triangulated category with compact generators $\{\mathscr{V}_{\mathscr{C}}(c, -) \oslash Q_j \mid c \in \mathscr{C}, j \in J\}$ where, if we assume either (1) or (2), $Q_j = P_j$ or if we assume (3) then $Q_j = P_j^c$ a cofibrant replacement of P_j.

Lemma 4.2. *Under the assumptions of Theorem 4.1 let $\mathscr{A}$ be a full $\mathscr{V}$-subcategory of $\mathscr{C}$. Then $\mathbf{Ch}[\mathscr{A}, \mathscr{V}]$ also satisfies either condition (1), (2) or (3). In particular, $\mathbf{D}[\mathscr{A}, \mathscr{V}]$ is a compactly generated triangulated category with compact generators $\{\mathscr{V}_{\mathscr{A}}(a, -) \oslash Q_j \mid a \in \mathscr{A}, j \in J\}$ where, if we assume either (1) or (2), $Q_j = P_j$ or if we assume (3) then $Q_j = P_j^c$ a cofibrant replacement of P_j.*

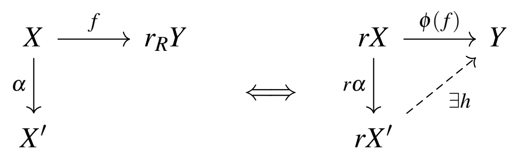
Proof. Assume (1) holds for $\mathbf{Ch}[\mathscr{C}, \mathscr{V}]$. As this is a condition solely reliant on $\mathbf{D}(\mathscr{V})$ it holds automatically for $\mathbf{Ch}[\mathscr{A}, \mathscr{V}]$ as well.

Assume (2) holds for $\mathbf{Ch}[\mathscr{C}, \mathscr{V}]$. Given any **K**-injective $Y \in \mathbf{Ch}[\mathscr{A}, \mathscr{V}]$, consider $r_R Y$ in $\mathbf{Ch}[\mathscr{C}, \mathscr{V}]$. Given an acyclic complex $X \in \mathbf{Ch}[\mathscr{C}, \mathscr{V}]$, rX is acyclic in $\mathbf{Ch}[\mathscr{A}, \mathscr{V}]$. We see that

$$\mathbf{K}[\mathscr{C}, \mathscr{V}](X, r_R Y) \cong \mathbf{K}[\mathscr{A}, \mathscr{V}](rX, Y) \cong 0.$$

Thus $r_R Y$ is **K**-injective in $\mathbf{Ch}[\mathscr{C}, \mathscr{V}]$. By assumption for all $c \in \mathscr{C}$ we have $r_R Y(c)$ is **K**-injective in $\mathbf{Ch}(\mathscr{V})$. Since $r_R Y(a) = Y(a)$ on all $a \in \mathscr{A}$ at every degree (see the proof of Theorem 3.4), it follows that $Y(a)$ is **K**-injective. We deduce that (2) holds for $\mathbf{Ch}[\mathscr{A}, \mathscr{V}]$.

Assume (3) holds for $\mathbf{Ch}[\mathscr{C}, \mathscr{V}]$. Consider an injective fibrant $Y \in \mathbf{Ch}[\mathscr{A}, \mathscr{V}]$. Given $X, X' \in \mathbf{Ch}[\mathscr{C}, \mathscr{V}]$ and an injective quasi-isomorphism $\alpha : X \to X'$, let $f : X \to r_R Y$ be any map. Then $r_R Y$ is injective fibrant if and only if there exists a lift $X' \to r_R Y$ that factors f in $\mathbf{Ch}[\mathscr{C}, \mathscr{V}]$. Consider the following diagrams



where $\phi(f)$ is adjoint to f. We see that the existence of a map $X' \to r_R Y$ extending f follows from the existence of h on the right. Since r is exact, it takes injective quasi-isomorphisms to injective quasi-isomorphisms. Further, Y is injective fibrant in $\mathbf{Ch}[\mathscr{A}, \mathscr{V}]$ by assumption. Therefore a lift $h : r(X') \to Y$ exists. Hence we have a lift $X' \to r_R Y$ and deduce $r_R Y$ is injective fibrant in $\mathbf{Ch}[\mathscr{C}, \mathscr{V}]$. By assumption, it follows that $r_R Y(c)$ is fibrant in $\mathbf{Ch}(\mathscr{V})$ for all $c \in \mathscr{C}$. In particular, for all $a \in \mathscr{A}$, we see that $Y(a) = r_R Y(a)$ is fibrant. Consequently (3) holds for $\mathbf{Ch}[\mathscr{A}, \mathscr{V}]$. □

Definition 4.3. A *recollement of a triangulated category $\mathscr{D}$ by triangulated categories $\mathscr{B}$ and $\mathscr{C}$* is a diagram of triangulated functors

$$\begin{array}{ccccc} & \xleftarrow{\iota_L} & & \xleftarrow{\rho_L} & \\ \mathscr{B} & \xrightarrow{\iota} & \mathscr{A} & \xrightarrow{\rho} & \mathscr{C} \\ & \xleftarrow[\iota_R]{} & & \xleftarrow[\rho_R]{} & \end{array}$$

such that $\iota_L \dashv \iota \dashv \iota_R$, $\rho_L \dashv \rho \dashv \rho_R$ are adjoint triples, the functors ι, ρ_L and ρ_R are fully faithful and $\operatorname{Im} \iota = \operatorname{Ker} \rho$.

Proposition 4.4. *Given a recollement of triangulated categories as above, for all $A \in \mathscr{A}$ we have triangles*

$$\rho_L \rho A \to A \to \iota \iota_L A \to (\rho_L \rho A)[1], \quad \textit{and} \quad \iota \iota_R A \to A \to \rho_R \rho A \to (\iota \iota_R A)[1]$$

induced by the unit and counit morphisms of the adjunctions in $\mathscr{A}$.

Proof. This follows from [18, 7.3(ii)]. □

Definition 4.5. Define the following full subcategories of $\mathbf{D}[\mathscr{C}, \mathscr{V}]$. Given a full $\mathscr{V}$-subcategory $\mathscr{A} \subset \mathscr{C}$, denote by $\mathscr{T}_{\mathscr{A}}$ the smallest localizing subcategory of $\mathbf{D}[\mathscr{C}, \mathscr{V}]$ generated by the subcollection of compact generators

$$\mathscr{T}_{\mathscr{A}} := \langle \mathscr{V}_{\mathscr{C}}(a, -) \oslash Q_j \mid a \in \mathscr{A}, Q_j \in \mathbf{D}(\mathscr{V}) \rangle,$$

where Q_j are those compact generators of $\mathbf{D}(\mathscr{V})$ as formulated in Theorem 4.1. Define a further full subcategory of $\mathbf{D}[\mathscr{C}, \mathscr{V}]$ as

$$\mathscr{E}_{\mathscr{A}} := \{Y \in \mathscr{D}[\mathscr{C}, \mathscr{V}] \mid H_n(Y(a)) = 0 \text{ for all } a \in \mathscr{A} \text{ and } n \in \mathbb{Z}\}.$$

Lemma 4.6. $\mathscr{T}_{\mathscr{A}}^{\perp} = \mathscr{E}_{\mathscr{A}}$.

Proof. By the proof of [9, 6.2] there is a natural isomorphism

$$\operatorname{Hom}_{\mathbf{D}[\mathscr{C}, \mathscr{V}]}(\mathscr{V}_{\mathscr{C}}(a, -) \oslash Q_j, Y) \cong \operatorname{Hom}_{\mathbf{D}(\mathscr{V})}(Q_j, Y(a)). \tag{4}$$

Since Q_j-s are compact generators of $\mathbf{D}(\mathscr{V})$, it follows that $\operatorname{Hom}_{\mathbf{D}[\mathscr{C}, \mathscr{V}]}(\mathscr{V}_{\mathscr{C}}(a, -) \oslash Q_j, Y) = 0$ for all Q_j-s if and only if $Y(a)$ is acyclic. Thus if we assume that Y belongs to $\mathscr{E}_{\mathscr{A}}$, then $Y(a) = 0$ for all $a \in \mathscr{A}$, and hence $\operatorname{Hom}_{\mathbf{D}[\mathscr{C}, \mathscr{V}]}(\mathscr{V}_{\mathscr{C}}(a, -) \oslash Q_j, Y) = 0$ for all Q_j-s. Therefore $\mathscr{E}_{\mathscr{A}} \subset \mathscr{T}_{\mathscr{A}}^{\perp}$, because by a theorem of Neeman [17, 2.1] $\mathscr{T}_{\mathscr{A}}$ is the smallest triangulated full subcategory containing $\mathscr{V}_{\mathscr{C}}(a, -) \oslash Q_j$-s. Likewise, assuming $X \in \mathscr{T}_{\mathscr{A}}^{\perp}$ then $\operatorname{Hom}_{\mathbf{D}[\mathscr{C}, \mathscr{V}]}(\mathscr{V}_{\mathscr{C}}(a, -) \oslash Q_j, Y) = 0$ for all $a \in \mathscr{A}$ and Q_j-s. Since Q_j-s are compact generators of $\mathbf{D}(\mathscr{V})$, it follows from the isomorphism (4) that $Y(a)$ is acyclic for all $a \in \mathscr{A}$. Therefore $\mathscr{T}_{\mathscr{A}}^{\perp} \subset \mathscr{E}_{\mathscr{A}}$. We conclude that $\mathscr{T}_{\mathscr{A}}^{\perp} = \mathscr{E}_{\mathscr{A}}$. □

Lemma 4.7. *There is a recollement of triangulated categories*

$$\begin{array}{ccccc} & \xleftarrow{\iota_L} & & \xleftarrow{\tau_L} & \\ \mathscr{T}_{\mathscr{A}}^{\perp} & \xrightarrow{\iota} & \mathbf{D}[\mathscr{C}, \mathscr{V}] & \xrightarrow{\tau} & \mathscr{T}_{\mathscr{A}}, \\ & \xleftarrow[\iota_R]{} & & \xleftarrow[\tau_R]{} & \end{array}$$

in which ι, τ_L are inclusions.

Proof. This follows from [12, 5.6.1] if we observe that the subcategory $\mathscr{T}_{\mathscr{A}}$ is compactly generated by a subcollection of compact generators. □

Lemma 4.8. *Let $\mathscr{S}$ and $\mathscr{T}$ be compactly generated triangulated categories. Suppose there exists a set of compact generators Σ in $\mathscr{S}$ and a triangulated functor $F:\mathscr{S}\to\mathscr{T}$ that preserves direct sums such that*

1. *the collection $\{F(X)|X\in\Sigma\}$ is a set of compact generators in $\mathscr{T}$,*
2. *for any X,Y in Σ, the induced map*

$$F_{X,Y[n]}:\mathrm{Hom}_{\mathscr{S}}\big(X,Y[n]\big)\to\mathrm{Hom}_{\mathscr{T}}\big(FX,FY[n]\big)$$

is an isomorphism for all $n\in\mathbb{Z}$.

Then F is an equivalence of triangulated categories.

Proof. To show that F is an equivalence, we need to demonstrate that it is fully faithful and essentially surjective. Without loss of generality, we may assume that Σ is closed under direct summands and shifts. Denote the collection of compact objects belonging to $\mathscr{S}$ by $\mathscr{S}^c$. Then $\mathscr{S}^c$ is recovered as $\bigcup_{n\geq 0}\Sigma_n$ where $\Sigma_0=\Sigma$ and for $n>0$, we set Σ_n to consist of the direct summands of objects in $\{Z\in\mathscr{S}^c|$ there exists a triangle $X\to Y\to Z\to X[1]$ with $X,Y\in\Sigma_{n-1}\}$. We also use here Neeman's Theorem [17, 2.1]. We have that F is fully faithful on Σ_0 by assumption.

Using a long exact sequence and the five lemma we also have that (2) holds for $X\in\Sigma_0,Y\in\Sigma_1$, similarly any $X\in\Sigma_1,Y\in\Sigma_0$. Suppose $Z,Z'\in\Sigma_1$, then there is a triangle $X\to Y\to Z\to X[1]$ with $X,Y\in\Sigma_0$. Consider a commutative diagram

$$\begin{array}{ccccccccc}
(Z',X) & \longrightarrow & (Z',Y) & \longrightarrow & (Z',Z) & \longrightarrow & (Z',X[1]) & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
(FZ',FX) & \longrightarrow & (FZ',FY) & \longrightarrow & (FZ',FZ) & \longrightarrow & (FZ',FX[1]) & \longrightarrow & \dots
\end{array}$$

where the vertical arrows are those induced by the functor F. Applying the five lemma, we see that the dashed arrow is an isomorphism and can conclude that (2) holds for Σ_1. Proceeding by induction, we see that (2) holds for Σ_n, for all $n\in\mathbb{Z}$, and further holds for $\mathscr{S}^c$. We also see that F takes objects in $\mathscr{S}^c$ to compact objects in $\mathscr{T}$.

Let $X\in\mathscr{S}^c$ and let $\mathscr{S}'$ be the full subcategory in $\mathscr{S}$ of those objects $Z\in\mathscr{S}$ for which the homomorphism

$$F_{X,Z[n]}:\mathrm{Hom}_{\mathscr{S}}\big(X,Z[n]\big)\to\mathrm{Hom}_{\mathscr{T}}\big(FX,FZ[n]\big)$$

is an isomorphism for all $n\in\mathbb{Z}$. We have shown that $\mathscr{S}^c\subset\mathscr{S}'$. Using the five lemma, one can show as above that $\mathscr{S}'$ is triangulated. We claim that $\mathscr{S}'$ is closed under direct sums. Indeed, let $\{Z_i\}_I$ be a family of objects in $\mathscr{S}'$. Consider the following commutative diagram

$$\begin{array}{ccc}
(X,\oplus_I Z_i) & \xrightarrow{\cong} & \oplus_I(X,Z_i) \\
\downarrow & & \downarrow\cong \\
(FX,\oplus_I FZ_i) & \xrightarrow{\cong} & \oplus_I(FX,FZ_i).
\end{array}$$

Since F preserves direct sums by assumption, the dashed arrow is an isomorphism and $\mathscr{S}'$ is closed under direct sums. By Neeman's Theorem [17, 2.1] this implies that $\mathscr{S}'=\mathscr{S}$.

Now let $\mathscr{S}''$ be the full subcategory in $\mathscr{S}$ of those objects $Y\in\mathscr{S}$ for which the map

$$F_{Y,Z}:\mathrm{Hom}_{\mathscr{S}}(Y,Z)\to\mathrm{Hom}_{\mathscr{T}}(FY,FZ)$$

is an isomorphism for all $Z \in \mathscr{S}$. We have shown that $\mathscr{S}^c \subset \mathscr{S}''$. $\mathscr{S}''$ is plainly closed under direct sums. Using the five lemma, one can show as above that $\mathscr{S}'$ is triangulated. By Neeman's Theorem [17, 2.1] $\mathscr{S}'' = \mathscr{S}$, and hence F is fully faithful.

It remains to show that F is essentially surjective. Let $\mathscr{T}'$ be the essential image of F. Then $\mathscr{T}^c \subset \mathscr{T}'$, because $F(\mathscr{S}^c)$ is a full subcategory of compact generators and one can use the same induction arguments as above to show that any compact objects in $\mathscr{T}$ is isomorphic to the image of a compact object in $\mathscr{S}$. Clearly, $\mathscr{T}'$ is triangulated and closed under direct sums. By Neeman's Theorem [17, 2.1] $\mathscr{T}' = \mathscr{T}$, and hence F is an equivalence. □

Recall that $\mathbf{D}[\mathscr{C},\mathscr{V}]$ is compactly generated by objects $\{\mathscr{V}_{\mathscr{C}}(c,-) \oslash Q_j \mid c \in \mathscr{C}\}$ where $\{Q_j\}_J$ is a family of compact generators of $\mathbf{D}(\mathscr{V})$ (see Theorem 4.1). Given a collection of objects $\mathscr{A} \subset \mathscr{C}$, the exact restriction functor $r : [\mathscr{C},\mathscr{V}] \to [\mathscr{A},\mathscr{V}]$ of Theorem 3.4 induces a triangulated functor $\rho : \mathbf{D}[\mathscr{C},\mathscr{V}] \to \mathbf{D}[\mathscr{A},\mathscr{V}]$ that applies r degreewise to any complex. By Lemma 4.2 $\{\mathscr{V}_{\mathscr{A}}(a,-) \oslash Q_j \mid a \in \mathscr{A}, j \in J\}$ is a family of compact generators of $\mathbf{D}[\mathscr{A},\mathscr{V}]$.

Lemma 4.9. *For all $a \in \mathscr{A}$, $X \in \mathbf{D}[\mathscr{C},\mathscr{V}]$ and $Q \in \{Q_j\}_J$, we have a natural isomorphism $\rho : \mathrm{Hom}_{\mathbf{D}[\mathscr{C},\mathscr{V}]}(\mathscr{V}_{\mathscr{C}}(a,-) \oslash Q, X) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{D}[\mathscr{A},\mathscr{V}]}(\mathscr{V}_{\mathscr{A}}(a,-) \oslash Q, \rho X)$ induced by the triangulated functor ρ.*

Proof. The proof of this lemma reduces to verifying commutativity of the following diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{D}[\mathscr{C},\mathscr{V}]}(\mathscr{V}_{\mathscr{C}}(a,-) \oslash Q_j, X) & \xrightarrow{\rho} & \mathrm{Hom}_{\mathbf{D}[\mathscr{A},\mathscr{V}]}(\mathscr{V}_{\mathscr{A}}(a,-) \oslash Q_j, \rho X) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_{\mathbf{D}(\mathscr{V})}(Q_j, X(a)) & \xrightarrow{=} & \mathrm{Hom}_{\mathbf{D}(\mathscr{V})}(Q_j, \rho X(a)), \end{array}$$

where the vertical arrows are the isomorphisms (4).

Consider a map

$$\alpha \circ q^{-1} : \quad \mathscr{V}_{\mathscr{C}}(a,-) \oslash Q_j \xrightarrow{\alpha} Y \xleftarrow{q} X$$

belonging to $\mathrm{Hom}_{\mathbf{D}[\mathscr{C},\mathscr{V}]}(\mathscr{V}_{\mathscr{C}}(a,-) \oslash Q_j, X)$, where α is a chain map and q is a quasi-isomorphism. If we apply ρ, then we have a diagram

$$\rho(\alpha \circ q^{-1}) : \quad \mathscr{V}_{\mathscr{A}}(a,-) \oslash Q_j \xrightarrow{r(\alpha)} \rho Y \xleftarrow{r(q)} \rho X$$

We consider the image of this morphism across the adjunction. We see that $\rho(\alpha \circ q^{-1})$ is taken to a map

$$Q_j \xrightarrow{\overline{r(\alpha)}} \rho Y(a) \xleftarrow{r(q)_a} \rho X(a)$$

in $\mathbf{D}(\mathscr{V})$, where $\overline{r(\alpha)}$ is the image of $r(\alpha)$ under the isomorphism (4). On the other hand, consider the image of $\alpha \circ q^{-1}$ under the isomorphism (4). It is is mapped to

$$Q_j \xrightarrow{\overline{\alpha}} Y(a) \xleftarrow{q_a} X(a) \quad = \quad Q_j \xrightarrow{\overline{\alpha}} \rho Y(a) \xleftarrow{r(q)_a} \rho X(a)$$

Clearly, $r(q)_a = q_a$, and we draw the following diagram

$$\begin{array}{ccc}
Q_j & \xrightarrow{=} & Q_j \\
\downarrow{\scriptstyle u_{\mathscr{C}}} & & \downarrow{\scriptstyle u_{\mathscr{A}}} \\
\underline{\mathrm{Hom}}_{\mathbf{Ch}(\mathscr{V})}\big(\mathscr{V}_{\mathscr{C}}(a,-), \mathscr{V}_{\mathscr{C}}(a,-) \oslash Q_j\big) & \xrightarrow{r} & \underline{\mathrm{Hom}}_{\mathbf{Ch}(\mathscr{V})}\big(\mathscr{V}_{\mathscr{A}}(a,-), \mathscr{V}_{\mathscr{A}}(a,-) \oslash Q_j\big) \\
\downarrow{\scriptstyle \underline{\mathrm{Hom}}_{\mathbf{Ch}(\mathscr{V})}(\mathscr{V}_{\mathscr{C}}(a,-),\alpha)} & & \downarrow{\scriptstyle \underline{\mathrm{Hom}}_{\mathbf{Ch}(\mathscr{V})}(\mathscr{V}_{\mathscr{A}}(a,-),r\alpha)} \\
\underline{\mathrm{Hom}}_{\mathbf{Ch}(\mathscr{V})}\big(\mathscr{V}_{\mathscr{C}}(a,-), Y\big) & \xrightarrow{r} & \underline{\mathrm{Hom}}_{\mathbf{Ch}(\mathscr{V})}\big(\mathscr{V}_{\mathscr{A}}(a,-), \rho Y\big) \\
\downarrow & & \downarrow \\
Y(a) & \xrightarrow{=} & \rho Y(a)
\end{array}$$

where the left most path is $\overline{\alpha}$ and the right most path is $\overline{r(\alpha)}$. We identify the canonical morphism $u_{\mathscr{C}}$ associated with the identity on $\mathscr{V}_{\mathscr{C}}(a,-) \oslash Q_j$, and similarly we identify $u_{\mathscr{A}}$ with the canonical morphism associated with the identity on and $\mathscr{V}_{\mathscr{A}}(a,-) \oslash Q_j$. The first square of the diagram commutes as the unit morphisms $e \to \mathscr{V}_{\mathscr{C}}(a,a)$ and $e \to \mathscr{V}_{\mathscr{A}}(a,a)$ are equal. The second square is obviously commutative. The third square commutes by the Enriched Yoneda Lemma. We conclude that $\overline{r(\alpha)}$ equals $\overline{\alpha}$. Hence the lemma. □

Lemma 4.10. *We define the following triangulated functor $F : \mathscr{T}_{\mathscr{A}} \hookrightarrow \mathbf{D}[\mathscr{C}, \mathscr{V}] \to \mathbf{D}[\mathscr{A}, \mathscr{V}]$ which consists of the inclusion τ_L of Lemma 4.7 followed by $\rho : \mathbf{D}[\mathscr{C}, \mathscr{V}] \to \mathbf{D}[\mathscr{A}, \mathscr{V}]$. Then $F : \mathscr{T}_{\mathscr{A}} \to \mathbf{D}[\mathscr{A}, \mathscr{V}]$ is an equivalence of categories.*

Proof. We use standard arguments regarding compactly generated triangulated categories. By Lemma 4.2 $\mathbf{D}[\mathscr{A}, \mathscr{V}]$ is compactly generated.

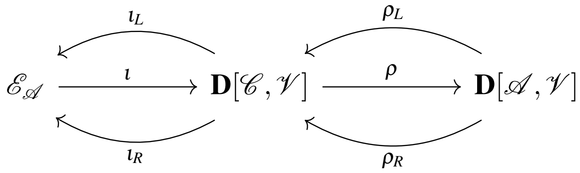
Following Lemma 4.8 it is enough to show that F sends compact generators to compact generators and induces an isomorphism between their Hom-sets. By definition, $\{\mathscr{V}_{\mathscr{C}}(a,-) \oslash Q_j \mid a \in \mathscr{A}, j \in J\}$ is a family of compact generators of $\mathscr{T}_{\mathscr{A}}$. The functor F maps $\mathscr{V}_{\mathscr{C}}(a,-) \oslash Q_j$ to $\mathscr{V}_{\mathscr{A}}(a,-) \oslash Q_j$, a compact generator of $\mathbf{D}[\mathscr{A}, \mathscr{V}]$. Next, as τ_L is inclusion, we need only verify whether the map

$$\mathrm{Hom}_{\mathbf{D}[\mathscr{C},\mathscr{V}]}(\mathscr{V}_{\mathscr{C}}(a,-) \oslash Q_j, \mathscr{V}_{\mathscr{C}}(b,-) \oslash Q_k) \xrightarrow{\rho} \mathrm{Hom}_{\mathbf{D}[\mathscr{A},\mathscr{V}]}(\mathscr{V}_{\mathscr{A}}(a,-) \oslash Q_j, \mathscr{V}_{\mathscr{A}}(b,-) \oslash Q_k)$$

is an isomorphism. This follows from Lemma 4.9. We conclude that F is an equivalence by Lemma 4.8. □

The following result is an extension of the recollement of Theorem 3.4 to derived categories.

Theorem 4.11. *Under the assumptions of Theorem 4.1 there exists a recollement of triangulated categories*



where $\mathscr{A} \subset \mathscr{C}$, $\mathscr{E}_{\mathscr{A}} := \{Y \in \mathscr{D}[\mathscr{C},\mathscr{V}] \mid H_n(Y(a)) = 0$ *for all* $a \in \mathscr{A}$ *and* $n \in \mathbb{Z}\}$, ι *is the inclusion,* ρ *is the restriction. The functor* ρ_R *takes* Y *to* $r_R(kY)$, *where* kY *is a* $\mathbf{K}$*-injective resolution of* Y *in* $\mathbf{D}[\mathscr{A},\mathscr{V}]$ *and* r_R *is the functor of Theorem 3.4 which applies to* kY *degreewise. The functor* $\rho_L := \tau_L \circ F^{-1}$, *where* $\tau_L : \mathscr{T}_{\mathscr{A}} \to \mathbf{D}[\mathscr{C},\mathscr{V}]$ *is the inclusion and* $F : \mathscr{T}_{\mathscr{A}} \to \mathbf{D}[\mathscr{A},\mathscr{V}]$ *is the equivalence of Lemma 4.10.*

Proof. Using Lemma 4.6, we have the triangulated functors ι_L, ι_R as defined in Lemma 4.7. It is clear that $\mathscr{E}_{\mathscr{A}}$ is the kernel of ρ. The functor $\rho_R : \mathbf{D}[\mathscr{A},\mathscr{V}] \to \mathbf{D}[\mathscr{C},\mathscr{V}]$ is right adjoint to ρ as

$$\mathbf{D}[\mathscr{C},\mathscr{V}](X, r_R kY) \cong \mathbf{K}[\mathscr{C},\mathscr{V}](X, r_R kY)$$

by the fact that r_R preserves $\mathbf{K}$-injectivity (see proof of Lemma 4.2) and

$$\mathbf{K}[\mathscr{C},\mathscr{V}](X, r_R kY) \cong \mathbf{K}[\mathscr{A},\mathscr{V}](rX, kY) \cong \mathbf{D}[\mathscr{A},\mathscr{V}](\rho X, kY) \cong \mathbf{D}[\mathscr{A},\mathscr{V}](\rho X, Y).$$

It is well known that the adjoint to a triangulated functor is also triangulated, thus ρ_R is a triangulated functor. Further, fully faithfulness follows as

$$\mathbf{D}[\mathscr{C},\mathscr{V}](r_R kX, r_R kY) \cong \mathbf{D}[\mathscr{A},\mathscr{V}](r r_R kX, Y)$$

by adjunction and

$$\mathbf{D}[\mathscr{A},\mathscr{V}](r r_R kX, Y) \cong \mathbf{D}[\mathscr{A},\mathscr{V}](kX, Y) \cong \mathbf{D}[\mathscr{A},\mathscr{V}](X, Y)$$

by the fact that $r r_R \Rightarrow \mathrm{id}_{[\mathscr{A},\mathscr{V}]}$ is an isomorphism of functors.

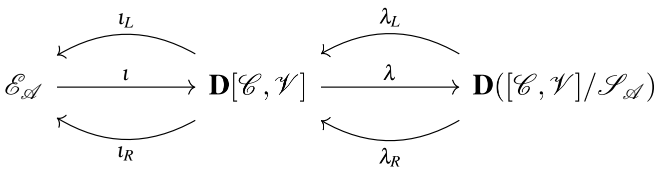
Now we claim that ρ_L is left adjoint to ρ. By construction, $F = \rho \circ \tau_L$. By Lemma 4.7 we have

$$\mathbf{D}[\mathscr{C},\mathscr{V}](\tau_L \circ (\rho \circ \tau_L)^{-1} X, Y) \cong \mathbf{D}[\mathscr{A},\mathscr{V}](X, (\rho \circ \tau_L) \circ \tau Y),$$

natural in $X \in \mathbf{D}[\mathscr{A},\mathscr{V}]$ and $Y \in \mathbf{D}[\mathscr{C},\mathscr{V}]$. By Proposition 4.4 there is a distinguished triangle $\iota \iota_L Y[-1] \to \tau_L \tau Y \to Y \to \iota \iota_L Y$. One has $\iota_L Y \in \mathscr{E}_{\mathscr{A}}$. Applying ρ to the triangle, one gets a triangle $0 \to \rho \tau_L \tau Y \to \rho Y \to 0$. Thus $\rho \circ \tau_L \circ \tau Y \cong \rho Y$. We have the desired adjunction. Since ρ_L is the composition of fully faithful functors, it is itself fully faithful. This completes the construction of the recollement. □

The following result is an extension of the recollement of Theorem 3.5 to derived categories.

Theorem 4.12. *Under the assumptions of Theorem 4.1 there exists a recollement of triangulated categories*



where $\mathscr{A} \subset \mathscr{C}$, ι_L, ι, ι_R *are those of Theorem 4.11. The functor* λ *applies the* $\mathscr{S}_{\mathscr{A}}$*-localization exact* *functor to any complex of* $\mathbf{D}[\mathscr{C},\mathscr{V}]$. *The functor* $\lambda_L := \rho_L \circ \varkappa^{-1}$, *where* ρ_L *is from Theorem 4.11 and* $\varkappa = \ell \circ r_L : [\mathscr{A},\mathscr{V}] \xrightarrow{\cong} [\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}}$ *is the equivalence (3). The functor* λ_R *takes* Y *to* $\ell_R(kY)$, *where* kY *is a* $\mathbf{K}$*-injective resolution of* Y *in* $\mathbf{D}([\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}})$ *and* ℓ_R *is the inclusion functor* $[\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}} \to [\mathscr{C},\mathscr{V}]$ *of Theorem 3.5 which applies to* kY *degreewise.*

Proof. We have that $\mathscr{E}_{\mathscr{A}}$ is the kernel of λ and λ_L is plainly fully faithful. Further consider the natural isomorphisms

$$\begin{aligned}\mathbf{D}[\mathscr{C},\mathscr{V}](\lambda_L X, Y) &= \mathbf{D}[\mathscr{C},\mathscr{V}](\rho_L \circ (\ell \circ r_L)^{-1} X, Y)\\ &\cong \mathbf{D}([\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}})(X, (\ell \circ r_L) \circ \rho Y)\\ &\cong \mathbf{D}([\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}})(X, (\ell \circ r_L \circ r) Y).\end{aligned}$$

By Proposition 3.3 and Theorem 3.4 there is an exact sequence in $\mathbf{Ch}[\mathscr{C},\mathscr{V}]$

$$0 \to i(A) \to r_L \circ rY \to Y \to i \circ i_L Y \to 0$$

for some $A \in \mathbf{Ch}\,\mathscr{S}_{\mathscr{A}} \subset \mathscr{E}_{\mathscr{A}}$. Applying the $\mathscr{S}_{\mathscr{A}}$-localization functor λ to this exact sequence, we get that the morphism of functors $\lambda \circ r_L \circ r \to \lambda$ is pointwise an isomorphism in $\mathbf{Ch}([\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}})$. Hence we can conclude that

$$\mathbf{D}([\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}})(X, \ell \circ r_L \circ rY) \cong \mathbf{D}([\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}})(X, \lambda Y)$$

and $\lambda_L \dashv \lambda$. We also have $\lambda \dashv \lambda_R$ by observing that ℓ_R preserves $\mathbf{K}$-injective resolutions and applying the same arguments when proving that $\rho \dashv \rho_R$ in Theorem 4.11. Now λ_R is fully faithful by Lemma 3.1. □

5. Recollements for triangulated categories and Serre localization

After constructing a recollement of $\mathbf{D}[\mathscr{C},\mathscr{V}]$ in Theorem 4.12, it is important to extend it further to a recollement of $\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)$, where $\mathscr{Q}$ is a Serre localizing subcategory of $[\mathscr{C},\mathscr{V}]$. We are motivated to investigate the categorical aspects of Voevodsky's triangulated categories of motives $\mathbf{DM}^{eff}_{\mathscr{A}}(k)$, where $\mathscr{A}$ is a reasonable category of correspondences on smooth algebraic varieties (see Section 6 below). However, such an extension of the recollement of Theorem 4.12 to $\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)$ is hardly possible for any localizing subcategory $\mathscr{Q}$ of $[\mathscr{C},\mathscr{V}]$, because certain adjoint functors do not seem to be constructible for general $\mathscr{Q}$.

Therefore to make the desired extension of the recollement to $\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)$ possible, we need to find the right conditions on the localizing subcategories $\mathscr{S}_{\mathscr{A}}$ and $\mathscr{Q}$ of $[\mathscr{C},\mathscr{V}]$. These conditions originate in the fundamental Voevodsky theorem [24], which says that the Nisnevich sheaf F_{nis} associated with a homotopy invariant presheaf with transfers F is a homotopy invariant sheaf with transfers and that it is strictly homotopy invariant whenever the base field is perfect. Translating this theorem into the language of Serre and Bousfield localization theory in Grothendieck categories and their derived categories, we make the following

Definition 5.1. Consider a Grothendieck category $\mathscr{D}$, and any two Serre localizing subcategories $\mathscr{Q},\mathscr{S} \subset \mathscr{D}$. We say that $\mathscr{S}$ *satisfies the Voevodsky property with respect to* $\mathscr{Q}$ or just the *V-property* ("V" for Voevodsky) if the full subcategory in $\mathscr{D}$ of $\mathscr{Q}$-local objects $\mathscr{S}^{\mathscr{Q}} = \{S_{\mathscr{Q}} \mid S \in \mathscr{S}\}$ is a subcategory of $\mathscr{S}$. In other words, the $\mathscr{Q}$-localization $S_{\mathscr{Q}}$ of any object $S \in \mathscr{S}$ is in $\mathscr{S}$.

Example 5.2. Let $\mathscr{D}$ be the Grothendieck category PwT/k of Nisnevich presheaves with transfers over a field k, $\mathscr{S}$ be the Serre localizing subcategory of homotopy invariant presheaves with transfers, and $\mathscr{Q}$ be the localizing subcategory of Nisnevich locally trivial presheaves with transfers. Then the Voevodsky theorem [24] implies that $\mathscr{S}$ satisfies the Voevodsky property with respect to

$\mathscr{Q}$, because the $\mathscr{Q}$-localization functor is nothing but the Nisnevich sheafification as follows from the following lemma.

Lemma 5.3. *The quotient category $(PwT/k)/\mathscr{Q}$ of $\mathscr{Q}$-local presheaves with transfers equals the category SwT/k of Nisnevich sheaves with transfers.*

Proof. Given $F \in PwT/k$ its Nisnevich sheaf F_{nis} is in PwT/k and the morphism $F \to F_{nis}$ is a morphism in PwT/k by [24] (see [22, 1.2] as well). Obviously, F_{nis} is $\mathscr{Q}$-torsionfree. Let

$$F_{nis} \hookrightarrow G \twoheadrightarrow X$$

be a short exact sequence in PwT/k with $X \in \mathscr{Q}$. Since $X_{nis} = 0$, it follows that the composite map $F_{nis} \hookrightarrow G \to G_{nis}$ is an isomorphism in PwT/k, hence the short exact sequence splits. We see that $\mathrm{Ext}^1_{PwT/k}(X, F_{nis}) = 0$, and so F_{nis} is $\mathscr{Q}$-local. Thus every Nisnevich sheaf with transfers is $\mathscr{Q}$-local.

Conversely, suppose $F \in PwT$ is $\mathscr{Q}$-local. Consider a long exact sequence in the category $\mathbb{Z}Pre(Sm/k)$ of ordinary presheaves of Abelian groups

$$A \hookrightarrow F \to F_{nis} \twoheadrightarrow X,$$

where A, X are Nisnevich locally trivial. Since $F \to F_{nis}$ is a map of PwT/k, it follows that $A, X \in PwT$ (limits/colimits in PwT/k are computed in $\mathbb{Z}Pre(Sm/k)$). Therefore $A, X \in \mathscr{Q}$. But F is $\mathscr{Q}$-local, and so $A = 0$. We get a short exact sequence $F \hookrightarrow F_{nis} \twoheadrightarrow X$ in PwT/k. Since $F \in PwT$ is $\mathscr{Q}$-local and $X \in \mathscr{Q}$, this short exact sequence splits. We have $F_{nis} = F \oplus X$. But F_{nis} is $\mathscr{Q}$-torsionfree, so $X = 0$ and $F = F_{nis}$. Thus every $\mathscr{Q}$-local presheaf with transfers is a Nisnevich sheaf with transfers. □

Since we work with two localizing subcategories, we need the notion of their join.

Definition 5.4. Consider a Grothendieck category $\mathscr{D}$, and any two Serre localizing subcategories $\mathscr{Q}, \mathscr{S} \subset \mathscr{D}$. We define the *join* $\mathscr{J} := \sqrt{(\mathscr{Q} \cup \mathscr{S})}$ of $\mathscr{Q}$ and $\mathscr{S}$ as the smallest localizing subcategory containing them.

Below we shall need the following lemma proven in [7, Lemma 5].

Lemma 5.5. *Given two localizing subcategories $\mathscr{Q}, \mathscr{S}$ of a Grothendieck category $\mathscr{D}$, the full subcategory $\mathscr{S}^{\mathscr{Q}}$ of objects of the form $X_{\mathscr{Q}}$ with $X \in \mathscr{S}$ is closed under direct sums, subobjects, and quotient objects in $\mathscr{D}/\mathscr{Q}$. Moreover, let $\mathscr{J}$ be the unique localizing subcategory of $\mathscr{D}$ containing $\mathscr{Q}$ such that the quotient Grothendieck category $\mathscr{D}/\mathscr{J} = (\mathscr{D}/\mathscr{Q})/\sqrt{\mathscr{S}^{\mathscr{Q}}}$, where $\sqrt{\mathscr{S}^{\mathscr{Q}}}$ is the smallest localizing category in $\mathscr{D}/\mathscr{Q}$ containing $\mathscr{S}^{\mathscr{Q}}$. Then $\mathscr{J} = \sqrt{(\mathscr{Q} \cup \mathscr{S})}$, the join of $\mathscr{Q}, \mathscr{S}$ in $\mathscr{D}$.*

Lemma 5.6. *Let $\mathscr{S}, \mathscr{Q}$ be two localizing subcategories of a Grothendieck category $\mathscr{D}$. If $\mathscr{S}$ satisfies the Voevodsky property with respect to $\mathscr{Q}$, then the subcategory $\mathscr{S}^{\mathscr{Q}}$ of Lemma 5.5 is localizing in the quotient Grothendieck category $\mathscr{D}/\mathscr{Q}$.*

Proof. By definition, $\mathscr{S}^{\mathscr{Q}} \subset \sqrt{\mathscr{S}^{\mathscr{Q}}}$. Further, $\mathscr{S}^{\mathscr{Q}}$ is closed under direct sums, subobjects and quotients in $\mathscr{D}/\mathscr{Q}$ by Lemma 5.5. Consider an exact sequence $X \hookrightarrow Y \overset{g}{\twoheadrightarrow} Z$ with $X, Z \in \mathscr{S}^{\mathscr{Q}}$. By assumption, $\mathscr{S}^{\mathscr{Q}} \subset \mathscr{S}$ and so $X, Z \in \mathscr{S}$. Moreover, the image of g in $\mathscr{D}$ also belongs to $\mathscr{S}$ as $\mathrm{Im}(g) \subset Z$. Thus there is a short exact sequence $X \hookrightarrow Y \overset{g}{\twoheadrightarrow} \mathrm{Im}(g)$ in $\mathscr{D}$ with $X, \mathrm{Im}(g) \in \mathscr{S}$. This implies that $Y \in \mathscr{S}$, hence $Y = Y_{\mathscr{Q}} \in \mathscr{S}^{\mathscr{Q}}$ and we have that $\mathscr{S}^{\mathscr{Q}}$ is closed under extensions. We conclude that $\mathscr{S}^{\mathscr{Q}}$ is localizing in $\mathscr{D}/\mathscr{Q}$. □

Definition 5.7. Under the assumptions of the preceding lemma denote by $\mathscr{E} := \{Y \in \mathbf{D}(\mathscr{D}) \mid H^*(Y) \in \mathscr{S}\}$ and $\mathscr{E}^{\mathscr{Q}} := \{Z \in \mathbf{D}(\mathscr{D}/\mathscr{Q}) \mid H^*_{\mathscr{D}/\mathscr{Q}}(Z) \in \mathscr{S}^{\mathscr{Q}}\}$, where $H^*_{\mathscr{D}/\mathscr{Q}}$ refers to cohomology computed in the quotient category $\mathscr{D}/\mathscr{Q}$ (note that $\mathscr{S}^{\mathscr{Q}}$ is localizing in $\mathscr{D}/\mathscr{Q}$ by the preceding lemma). We have a canonical triangulated functor between derived categories

$$k : \mathbf{D}(\mathscr{D}/\mathscr{Q}) \to \mathbf{D}(\mathscr{D})$$

taking a complex $Z \in \mathbf{D}(\mathscr{D}/\mathscr{Q})$ to its $\mathbf{K}$-injective resolution kY in $\mathbf{D}(\mathscr{D}/\mathscr{Q})$ and regarding kY as a complex of $\mathbf{D}(\mathscr{D})$. Notice that k preserves $\mathbf{K}$-injective objects.

Consider a Grothendieck category $\mathscr{D}$ and any two Serre localizing subcategories $\mathscr{Q}, \mathscr{S} \subset \mathscr{D}$. We say that $\mathscr{S}$ *satisfies the strict Voevodsky property with respect to* $\mathscr{Q}$ or just the *strict V-property* if $\mathscr{S}$ satisfies the Voevodsky property with respect to $\mathscr{Q}$ and the restriction of $k : \mathbf{D}(\mathscr{D}/\mathscr{Q}) \to \mathbf{D}(\mathscr{D})$ to $\mathscr{E}^{\mathscr{Q}}$ lands in $\mathscr{E}$. In other words, the $\mathscr{D}$-homology objects of the $\mathbf{K}$-injective resolutions of the complexes from $\mathscr{E}^{\mathscr{Q}}$ belong to $\mathscr{S}$.

Example 5.8. Under the notation of Example 5.2 $\mathscr{D}/\mathscr{Q}$ is the derived category $\mathbf{D}(SwT/k)$ of the Grothendieck category SwT/k of Nisnevich sheaves with transfers by Lemma 5.3. If the base field k is perfect, the Voevodsky theorem [24] and [15, 6.2.7] imply that $\mathscr{S}$ satisfies the strict Voevodsky property with respect to $\mathscr{Q}$.

Proposition 5.9. *Let $\mathscr{J}$ be the join of two localizing subcategories $\mathscr{Q}, \mathscr{S}$ of a Grothendieck category $\mathscr{D}$. If we consider quotient categories as full subcategories of local objects then the following relation is true: $\mathscr{D}/\mathscr{J} = (\mathscr{D}/\mathscr{S}) \cap (\mathscr{D}/\mathscr{Q})$. In other words, $\mathscr{D}/\mathscr{J}$ is the full subcategory of those objects that are both $\mathscr{Q}$- and $\mathscr{S}$-local.*

Proof. Since $\mathscr{D}/\mathscr{J} = (\mathscr{D}/\mathscr{S})/\sqrt{\mathscr{Q}^{\mathscr{S}}}$ by Lemma 5.5, this implies that every object in $\mathscr{D}/\mathscr{J}$ is $\mathscr{S}$-local. In the same manner, every object in $\mathscr{D}/\mathscr{J}$ is $\mathscr{Q}$-local and hence $\mathscr{D}/\mathscr{J} \subset (\mathscr{D}/\mathscr{S}) \cap (\mathscr{D}/\mathscr{Q})$. Next, suppose $X \in (\mathscr{D}/\mathscr{S}) \cap (\mathscr{D}/\mathscr{Q})$. It is enough to show that X is $\sqrt{\mathscr{Q}_{\mathscr{S}}}$-local in $\mathscr{D}/\mathscr{S}$. By Lemma 5.5 $\mathscr{Q}_{\mathscr{S}}$ is closed under subobjects, quotients and direct sums. By [7, Corollary 3] X is $\sqrt{\mathscr{Q}_{\mathscr{S}}}$-local in $\mathscr{D}/\mathscr{S}$ if and only if $\operatorname{Hom}_{\mathscr{D}/\mathscr{S}}(W_{\mathscr{S}}, X) = 0 = \operatorname{Ext}^1_{\mathscr{D}/\mathscr{S}}(W_{\mathscr{S}}, X)$ for all $W_{\mathscr{S}} \in \mathscr{Q}_{\mathscr{S}}$, where $W_{\mathscr{S}} \in \mathscr{Q}_{\mathscr{S}}$ is the $\mathscr{S}$-localization of some $W \in \mathscr{Q}$. We have $\operatorname{Hom}_{\mathscr{D}/\mathscr{S}}(W_{\mathscr{S}}, X) = \operatorname{Hom}_{\mathscr{D}}(W, X) = 0$, as X is $\mathscr{Q}$-local. Now, for $\operatorname{Ext}^1_{\mathscr{D}/\mathscr{S}}(W_{\mathscr{S}}, X) = 0$, we show that every short exact sequence splits.

Suppose we have a short exact sequence $X \hookrightarrow M \twoheadrightarrow W_{\mathscr{S}}$ in $\mathscr{D}/\mathscr{S}$. If we consider the canonical morphism λ associated with $\mathscr{S}$-localization, then there is the following diagram in $\mathscr{D}$

$$\begin{array}{ccccc} X & \overset{i}{\hookrightarrow} & N & \overset{\alpha}{\longrightarrow} & W \\ \downarrow{=} & & \downarrow{\lambda_N} & & \downarrow{\lambda_W} \\ X & \overset{i'}{\hookrightarrow} & M & \longrightarrow & W_{\mathscr{S}}. \end{array}$$

As $\mathscr{Q}$ is localizing, if we take the image of α in W then $\operatorname{Im}\alpha \in \mathscr{Q}$. Since X is $\mathscr{Q}$-local then $\operatorname{Ext}^1_{\mathscr{D}}(\operatorname{Im}\alpha, X) = 0$. Thus the short exact sequence $X \hookrightarrow N \twoheadrightarrow \operatorname{Im}\alpha$ splits in $\mathscr{D}$. Therefore there exists some map $\beta : N \to X$ with $\beta \circ i = 1_X$, and since X is $\mathscr{S}$-local and $M = N_{\mathscr{S}}$, we have that β factors uniquely through λ_N. Denote this factor $\beta' : M \to X$ with $\beta' \circ \lambda_N = \beta$. We see that β' splits the bottom short exact sequence in $\mathscr{D}/\mathscr{S}$ as $1_X = \beta i = (\beta' \lambda_N) i = \beta' i'$. Hence X is $\sqrt{\mathscr{Q}_{\mathscr{S}}}$-local. It follows that X is $\mathscr{J}$-local and $(\mathscr{D}/\mathscr{S}) \cap (\mathscr{D}/\mathscr{Q}) \subset \mathscr{D}/\mathscr{J}$. We conclude that $\mathscr{D}/\mathscr{J} = (\mathscr{D}/\mathscr{S}) \cap (\mathscr{D}/\mathscr{Q})$. □

We are now in a position to formulate the main result of this section.

Theorem 5.10. *Under the assumptions of Theorem 4.1 suppose $\mathscr{A} \subset \mathscr{C}$ and $\mathscr{Q} \subset [\mathscr{C}, \mathscr{V}]$ is a localizing subcategory such that:*

- $\mathbf{D}\left([\mathscr{C},\mathscr{V}]/\mathscr{Q}\right)$ *is compactly generated and the functor* $\mathbf{D}[\mathscr{C},\mathscr{V}] \to \mathbf{D}\left([\mathscr{C},\mathscr{V}]/\mathscr{Q}\right)$ *induced by the exact* $\mathscr{Q}$*-localization functor* $(\cdot)_{\mathscr{Q}} : [\mathscr{C},\mathscr{V}] \to [\mathscr{C},\mathscr{V}]/\mathscr{Q}$ *respects compact objects;*
- *the localizing subcategory* $\mathscr{S}_{\mathscr{A}} = \left\{Y \in [\mathscr{C},\mathscr{V}] \mid Y(a) = 0 \text{ for all } a \in \mathscr{A}\right\}$ *satisfies the strict Voevodsky property with respect to* $\mathscr{Q}$.

Then there exists a recollement of triangulated categories

$$\mathscr{E}^{\mathscr{Q}}_{\mathscr{A}} \xrightarrow{\iota^{\mathscr{Q}}} \mathbf{D}\left([\mathscr{C},\mathscr{V}]/\mathscr{Q}\right) \xrightarrow{\lambda^{\mathscr{Q}}} \mathbf{D}\left([\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}\right), \qquad \iota^{\mathscr{Q}}_L,\ \iota^{\mathscr{Q}}_R : \mathbf{D}\left([\mathscr{C},\mathscr{V}]/\mathscr{Q}\right) \to \mathscr{E}^{\mathscr{Q}}_{\mathscr{A}}, \quad \lambda^{\mathscr{Q}}_L,\ \lambda^{\mathscr{Q}}_R : \mathbf{D}\left([\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}\right) \to \mathbf{D}\left([\mathscr{C},\mathscr{V}]/\mathscr{Q}\right)$$

where $\mathscr{E}^{\mathscr{Q}}_{\mathscr{A}}$ *is as in Definition 5.7,* $\mathscr{J}_{\mathscr{A}}$ *is the join of* $\mathscr{Q}$ *and* $\mathscr{S}_{\mathscr{A}}$ *in* $[\mathscr{C},\mathscr{V}]$*, the functor* $\lambda^{\mathscr{Q}}$ *is induced by the* $\mathscr{S}^{\mathscr{Q}}_{\mathscr{A}}$*-localization functor* $[\mathscr{C},\mathscr{V}]/\mathscr{Q} \to [\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}$ *associated with the localizing subcategory* $\mathscr{S}^{\mathscr{Q}}_{\mathscr{A}} = \{X_{\mathscr{Q}} \mid X \in \mathscr{S}_{\mathscr{A}}\}$ *of Lemmas 5.5-5.6,* $\iota^{\mathscr{Q}}$ *is inclusion, and* $\lambda^{\mathscr{Q}}_R$ *is induced by the* **K***-injective resolution functor.*

Remark 5.11. In our major application (see Section 6) the middle category will be the derived category of Nisnevich sheaves with reasonable transfers $\mathscr{B}$ on smooth algebraic varieties Sm/k, the left category is Voevodsky's [25] triangulated category of motives $\mathbf{DM}^{eff}_{\mathscr{B}}(k)$. The celebrated Voevodsky theorem [25, 3.2.6] computes the functor $\iota^{\mathscr{Q}}_L$ for this example as the Suslin complex C_*. Theorem 5.10 is a kind of the categorical framework for the triangulated category of motives $\mathbf{DM}^{eff}_{\mathscr{B}}(k)$.

We postpone the proof of Theorem 5.10. First we establish a number of facts. Note that $[\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}} \cong \left([\mathscr{C},\mathscr{V}]/\mathscr{Q}\right)/\mathscr{S}^{\mathscr{Q}}_{\mathscr{A}}$ by Lemma 5.5. By Proposition 5.9 $[\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}$ consists of those objects of $[\mathscr{C},\mathscr{V}]$ which are both $\mathscr{Q}$- and $\mathscr{S}_{\mathscr{A}}$-local. Recall from Theorem 4.1 that $\mathbf{D}[\mathscr{C},\mathscr{V}]$ is compactly generated with compact generators $\left\{\mathscr{V}_{\mathscr{C}}(c,-) \oslash Q_j\right\}$.

Lemma 5.12. *Under the assumptions of Theorem 5.10 denote by* $\mathscr{T}^{\mathscr{Q}}_{\mathscr{A}}$ *the full subcategory of* $\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)$ *generated by the compact objects* $\left\{\left(\mathscr{V}_{\mathscr{C}}(a,-) \oslash Q_j\right)_{\mathscr{Q}} \mid a \in \mathscr{A}\right\}$*. Then there is a recollement of triangulated categories*

$$\mathscr{E}^{\mathscr{Q}}_{\mathscr{A}} \xrightarrow{\iota^{\mathscr{Q}}} \mathbf{D}([\mathscr{C},\mathscr{V}]/\mathscr{Q}) \xleftarrow{\tau^{\mathscr{Q}}_L} \mathscr{T}^{\mathscr{Q}}_{\mathscr{A}}, \qquad \iota^{\mathscr{Q}}_L,\ \iota^{\mathscr{Q}}_R : \mathbf{D}([\mathscr{C},\mathscr{V}]/\mathscr{Q}) \to \mathscr{E}^{\mathscr{Q}}_{\mathscr{A}}, \quad \tau^{\mathscr{Q}} : \mathbf{D}([\mathscr{C},\mathscr{V}]/\mathscr{Q}) \to \mathscr{T}^{\mathscr{Q}}_{\mathscr{A}}, \quad \tau^{\mathscr{Q}}_R : \mathscr{T}^{\mathscr{Q}}_{\mathscr{A}} \to \mathbf{D}([\mathscr{C},\mathscr{V}]/\mathscr{Q})$$

where $\iota^{\mathscr{Q}}, \tau^{\mathscr{Q}}_L$ *are inclusions.*

Proof. Let us show that $\mathscr{E}^{\mathscr{Q}}_{\mathscr{A}} = (\mathscr{T}^{\mathscr{Q}}_{\mathscr{A}})^{\perp}$. Suppose $X \in \mathscr{E}^{\mathscr{Q}}_{\mathscr{A}}$, then

$$\begin{aligned}\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)\Big(\big(\mathscr{V}_{\mathscr{C}}(a,-) \oslash Q_j\big)_{\mathscr{Q}}, X\Big) &= \mathbf{K}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)\Big(\big(\mathscr{V}_{\mathscr{C}}(a,-) \oslash Q_j\big)_{\mathscr{Q}}, kX\Big) \\ &= \mathbf{K}[\mathscr{C},\mathscr{V}]\Big(\mathscr{V}_{\mathscr{C}}(a,-) \oslash Q_j, kX\Big),\end{aligned}$$

where kX is the **K**-injective resolution of X in $\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)$. Since $\mathscr{S}_{\mathscr{A}}$ satisfies the strict V-property by assumption, it follows from Lemma 4.6 that $kX \in \mathscr{E}_{\mathscr{A}} = \mathscr{T}^{\perp}_{\mathscr{A}}$ and then

$\mathbf{K}[\mathscr{C},\mathscr{V}]\big(\mathscr{V}_{\mathscr{C}}(a,-)\oslash Q_j,kX\big)=0$. Therefore $X\in(\mathscr{T}_{\mathscr{A}}^{\mathscr{Q}})^{\perp}$, because by a theorem of Neeman [17, 2.1] $\mathscr{T}_{\mathscr{A}}^{\mathscr{Q}}$ is the smallest triangulated full subcategory closed under direct sums containing $\big(\mathscr{V}_{\mathscr{C}}(a,-)\oslash Q_j\big)_{\mathscr{Q}}$-s. Next, given $Y\in(\mathscr{T}_{\mathscr{A}}^{\mathscr{Q}})^{\perp}$ then

$$\begin{aligned}
0&=\mathbf{D}\left([\mathscr{C},\mathscr{V}]/\mathscr{Q}\right)\Big(\big(\mathscr{V}_{\mathscr{C}}(a,-)\oslash Q_j\big)_{\mathscr{Q}},kY\Big)\\
&=\mathbf{K}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)\Big(\big(\mathscr{V}_{\mathscr{C}}(a,-)\oslash Q_j\big)_{\mathscr{Q}},kY\Big)\\
&=\mathbf{K}[\mathscr{C},\mathscr{V}]\Big(\mathscr{V}_{\mathscr{C}}(a,-)\oslash Q_j,kY\Big)\\
&=\mathbf{D}[\mathscr{C},\mathscr{V}]\Big(\mathscr{V}_{\mathscr{C}}(a,-)\oslash Q_j,kY\Big)\\
&\cong\mathbf{D}(\mathscr{V})\Big(Q_j,kY(a)\Big).
\end{aligned}$$

We use here the isomorphism (4) and the fact that kY is $\mathbf{K}$-injective in $\mathbf{D}[\mathscr{C},\mathscr{V}]$. This implies that $kY(a)$ is acyclic, because Q_j-s are compact generators of $\mathbf{D}(\mathscr{V})$. Thus kY has cohomology belonging to $\mathscr{S}_{\mathscr{A}}$ and therefore its $\mathscr{Q}$-localized cohomology belongs to $\mathscr{S}_{\mathscr{A}}^{\mathscr{Q}}$. Since $H^*_{[\mathscr{C},\mathscr{V}]/\mathscr{Q}}(Y)=H^*_{[\mathscr{C},\mathscr{V}]/\mathscr{Q}}(kY)\in\mathscr{S}_{\mathscr{A}}^{\mathscr{Q}}$ we have that $Y\in\mathscr{E}_{\mathscr{A}}^{\mathscr{Q}}$. We conclude that $\mathscr{E}_{\mathscr{A}}^{\mathscr{Q}}=\big(\mathscr{T}_{\mathscr{A}}^{\mathscr{Q}}\big)^{\perp}$. Since $\mathscr{T}_{\mathscr{A}}^{\mathscr{Q}}$ is compactly generated, our statement now follows from [12, 5.6.1]. □

Lemma 5.13. *Under the assumptions of Theorem 5.10 let $X\in\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)$ and $Y=\lambda^{\mathscr{Q}}(X)$. Then the natural morphism $f_a:kX(a)\to kY(a)$ induced by $X\to Y$ is a quasi-isomorphism in $\mathbf{Ch}(\mathscr{V})$ for all $a\in\mathscr{A}$, where kX,kY are $\mathbf{K}$-injective resolutions of X,Y in $\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)$. Furthermore, if $k'Y$ is a $\mathbf{K}$-injective resolution of Y in $\mathbf{D}([\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}})$, then the induced morphisms $g_a:kY(a)\to k'Y(a)$ and $g_af_a:kX(a)\to k'Y(a)$ are quasi-isomorphisms in $\mathbf{Ch}(\mathscr{V})$ for all $a\in\mathscr{A}$.*

Proof. Consider a triangle $Z\to X\xrightarrow{\phi}Y\to Z[1]$ in $\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)$. We have then $Z\in\mathscr{E}_{\mathscr{A}}^{\mathscr{Q}}$, as $\operatorname{Ker}\lambda^{\mathscr{Q}}=\mathscr{E}_{\mathscr{A}}^{\mathscr{Q}}$ and $\lambda^{\mathscr{Q}}(\phi)$ is an isomorphism in $\mathbf{Ch}\big([\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}\big)$. Apply the triangulated functor of Definition 5.7 $k:\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)\to\mathbf{D}[\mathscr{C},\mathscr{V}]$ and consider the triangle $kZ\to kX\xrightarrow{f}kY\to kZ[1]$ in $\mathbf{D}[\mathscr{C},\mathscr{V}]$, where $f=k(\phi)$. We have $kZ\in\mathscr{E}_{\mathscr{A}}$ in $\mathbf{D}[\mathscr{C},\mathscr{V}]$ by an assumption of Theorem 5.10. Hence $kZ(a)\cong 0$ and $f_a:kX(a)\xrightarrow{\cong}kY(a)$ in $\mathbf{D}(\mathscr{V})$. We use here the triangulated evaluation functor $\mathbf{D}[\mathscr{C},\mathscr{V}]\to\mathbf{D}(\mathscr{V})$ induced by the exact evaluation functor $B\in[\mathscr{C},\mathscr{V}]\mapsto B(a)\in\mathscr{V}$.

Now consider a triangle $W\to Y\xrightarrow{\gamma}k'Y\to W[1]$ in $\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)$. Then $\lambda^{\mathscr{Q}}(\gamma)$ is a quasi-isomorphism in $\mathbf{Ch}\big([\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}\big)$, and hence $W\in\mathscr{E}_{\mathscr{A}}^{\mathscr{Q}}$. Note that $k'Y$ is $\mathbf{K}$-injective in $\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)$. As above the induced morphism $g:kY\to k'Y$ is such that $g_a:kY(a)\to k'Y(a)$ is a quasi-isomorphisms in $\mathbf{Ch}(\mathscr{V})$ for all $a\in\mathscr{A}$, and hence so is $g_af_a:kX(a)\to k'Y(a)$. □

Corollary 5.14. *The functor $\lambda^{\mathscr{Q}}$ induces an isomorphism*

$$\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)\Big(\big(\mathscr{V}_{\mathscr{C}}(a,-)\oslash P\big)_{\mathscr{Q}},X\Big)\xrightarrow{\cong}\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}\big)\Big(\big(\mathscr{V}_{\mathscr{C}}(a,-)\oslash P\big)_{\mathscr{J}_{\mathscr{A}}},Y\Big)$$

for all $a\in\mathscr{A}$, $P\in\{Q_j\}_J$, $X\in\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)$ and $Y=\lambda^{\mathscr{Q}}(X)$.

Proof. This follows from Lemma 5.13 isomorphism (4) and the commutativity of the following diagram, in which all vertical arrows are isomorphisms:

$$\begin{array}{ccc}
\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)\Big(\big(\mathscr{V}_{\mathscr{C}}(a,-)\oslash P\big)_{\mathscr{Q}},X\Big) & \xrightarrow{\lambda_*^{\mathscr{Q}}} & \mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}\big)\Big(\big(\mathscr{V}_{\mathscr{C}}(a,-)\oslash P\big)_{\mathscr{J}_{\mathscr{A}}},Y\Big) \\
\downarrow\cong & & \downarrow\cong \\
\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)\Big(\big(\mathscr{V}_{\mathscr{C}}(a,-)\oslash P\big)_{\mathscr{Q}},kX\Big) & & \mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}\big)\Big(\big(\mathscr{V}_{\mathscr{C}}(a,-)\oslash P\big)_{\mathscr{J}_{\mathscr{A}}},k'Y\Big) \\
\downarrow\cong & & \downarrow\cong \\
\mathbf{D}[\mathscr{C},\mathscr{V}]\big(\mathscr{V}_{\mathscr{C}}(a,-)\oslash P,kX\big) & & \mathbf{D}[\mathscr{C},\mathscr{V}]\big(\mathscr{V}_{\mathscr{C}}(a,-)\oslash P,k'Y\big) \\
\downarrow\cong & & \downarrow\cong \\
\mathbf{D}(\mathscr{V})\big(P,kX(a)\big) & \xrightarrow{\cong} & \mathbf{D}(\mathscr{V})\big(P,k'Y(a)\big)
\end{array}$$

where kX is a **K**-injective resolution of X in $\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)$ and $k'Y$ is a **K**-injective resolution of Y in $\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}\big)$. □

Corollary 5.15. *The triangulated functor $F^{\mathscr{Q}} := \lambda^{\mathscr{Q}} \circ \tau_L^{\mathscr{Q}} : \mathscr{T}_{\mathscr{A}}^{\mathscr{Q}} \to \mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}\big)$ is fully faithful.*

Proof. It follows from Corollary 5.14 that the functor $F^{\mathscr{Q}}$ is fully faithful on pairs $Z,X \in \mathscr{T}_{\mathscr{A}}^{\mathscr{Q}}$ with Z compact. Moreover, $F^{\mathscr{Q}}$ preserves direct sums, because direct sums in the derived category of a Grothendieck category are formed degreewise [23, 19.13.4] and $\lambda^{\mathscr{Q}}$ preserves degreewise direct sums. Let $S \subset \mathscr{T}_{\mathscr{A}}^{\mathscr{Q}}$ be the full triangulated subcategory of those objects Z such that $(Z,X) \to (F^{\mathscr{Q}}Z, F^{\mathscr{Q}}X)$ is an isomorphism for a fixed $X \in \mathscr{T}_{\mathscr{A}}^{\mathscr{Q}}$. Then $S \supset (\mathscr{T}_{\mathscr{A}}^{\mathscr{Q}})^c$ and S is closed under direct sums. Therefore S is localizing. We conclude from a theorem of Neeman [17, 2.1] that $S = \mathscr{T}_{\mathscr{A}}^{\mathscr{Q}}$. □

Lemma 5.16. *The triangulated functor $F^{\mathscr{Q}} : \mathscr{T}_{\mathscr{A}}^{\mathscr{Q}} \to \mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}\big)$ is an equivalence.*

Proof. As $F^{\mathscr{Q}}$ is fully faithful by the preceding corollary, it remains to show that it is essentially surjective. Take any $Y \in \mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}\big)$ and consider it as an object of $\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)$. Then $\lambda^{\mathscr{Q}}(Y) = Y$. If we take $\tau^{\mathscr{Q}}(Y) \in \mathscr{T}_{\mathscr{A}}^{\mathscr{Q}}$ (see Lemma 5.12), then $F^{\mathscr{Q}}(\tau^{\mathscr{Q}}Y) = \lambda^{\mathscr{Q}}\tau_L^{\mathscr{Q}}\tau^{\mathscr{Q}}Y$. Using Lemma 5.12 and Proposition 4.4, there exists a triangle

$$\tau_L^{\mathscr{Q}}\tau^{\mathscr{Q}}Y \to Y \to \iota^{\mathscr{Q}}\iota_L^{\mathscr{Q}}Y \to (\tau_L^{\mathscr{Q}}\tau^{\mathscr{Q}}Y)[1].$$

Applying the triangulated functor $\lambda^{\mathscr{Q}}$ to it, it follows that $\lambda^{\mathscr{Q}}\tau_L^{\mathscr{Q}}\tau^{\mathscr{Q}}Y \cong \lambda^{\mathscr{Q}}Y = Y$. We have $F^{\mathscr{Q}}(\tau^{\mathscr{Q}}Y) \cong Y$ and thus $F^{\mathscr{Q}}$ is essentially surjective. We conclude that $F^{\mathscr{Q}}$ is an equivalence. □

Proof of Theorem 5.10. We define the functors $\iota_L^{\mathscr{Q}}, \iota^{\mathscr{Q}}, \iota_R^{\mathscr{Q}}$ to be those coming from Lemma 5.12, and $\operatorname{Ker}\lambda^{\mathscr{Q}} = \mathscr{E}_{\mathscr{A}}^{\mathscr{Q}}$ is immediate. We set $\lambda_L^{\mathscr{Q}} := \tau_L^{\mathscr{Q}} \circ (F^{\mathscr{Q}})^{-1}$. Then $\lambda_L^{\mathscr{Q}}$ is fully faithful as it is the composition of fully faithful functors. We have natural isomorphisms

$$\begin{aligned}
\mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)(\lambda_L^{\mathscr{Q}}X,Y) &= \mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big)\big(\tau_L^{\mathscr{Q}} \circ (F^{\mathscr{Q}})^{-1}X,Y\big) \\
&\cong \mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}\big)(X, F^{\mathscr{Q}} \circ \tau^{\mathscr{Q}}Y) \\
&\cong \mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}\big)(X, \lambda^{\mathscr{Q}} \circ \tau_L^{\mathscr{Q}} \circ \tau^{\mathscr{Q}}Y) \\
&\cong \mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}\big)(X, \lambda^{\mathscr{Q}}Y).
\end{aligned}$$

We use here the fact that $\lambda^{\mathscr{Q}}\tau_L^{\mathscr{Q}}\tau^{\mathscr{Q}}Y \cong \lambda^{\mathscr{Q}}Y$ (see the proof of Lemma 5.16). Therefore $\lambda_L^{\mathscr{Q}} \dashv \lambda^{\mathscr{Q}}$. The proof that $\lambda^{\mathscr{Q}} \dashv \lambda_R^{\mathscr{Q}}$ is obvious. Lemma 3.1 now implies that $\lambda_R^{\mathscr{Q}}$ is fully faithful. Hence the theorem. □

We can summarise Theorems 4.12-5.10 by drawing the following diagram

$$\begin{array}{ccccc}
\mathscr{E}_{\mathscr{A}} & \begin{array}{c}\xleftarrow{\iota_L}\\ \xrightarrow{\iota}\\ \xleftarrow[\iota_R]{}\end{array} & \mathbf{D}[\mathscr{C},\mathscr{V}] & \begin{array}{c}\xleftarrow{\lambda_L}\\ \xrightarrow{\lambda}\\ \xleftarrow[\lambda_R]{}\end{array} & \mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}}\big)\\
{\scriptstyle(\cdot)_{\mathscr{Q}}}\downarrow\uparrow{\scriptstyle k} & & {\scriptstyle(\cdot)_{\mathscr{Q}}}\downarrow\uparrow{\scriptstyle k} & & {\scriptstyle(\cdot)'_{\mathscr{Q}}}\downarrow\uparrow{\scriptstyle k'}\\
\mathscr{E}_{\mathscr{A}}^{\mathscr{Q}} & \begin{array}{c}\xleftarrow{\iota_L^{\mathscr{Q}}}\\ \xrightarrow{\iota^{\mathscr{Q}}}\\ \xleftarrow[\iota_R^{\mathscr{Q}}]{}\end{array} & \mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{Q}\big) & \begin{array}{c}\xleftarrow{\lambda_L^{\mathscr{Q}}}\\ \xrightarrow{\lambda^{\mathscr{Q}}}\\ \xleftarrow[\lambda_R^{\mathscr{Q}}]{}\end{array} & \mathbf{D}\big([\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}\big)
\end{array}$$

in which the top and bottom horizontal arrows form recollements, the vertical arrows k,k' take $\mathbf{K}$-injective resolutions in the derived categories of the corresponding quotient Grothendieck categories, the functor $(\cdot)_{\mathscr{Q}}$ is induced by $\mathscr{Q}$-localization and $(\cdot)'_{\mathscr{Q}}$ is induced by the exact $\sqrt{\mathscr{Q}_{\mathscr{S}_{\mathscr{A}}}}$-localization functor $[\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{A}} \to [\mathscr{C},\mathscr{V}]/\mathscr{J}_{\mathscr{A}}$ with $\mathscr{Q}_{\mathscr{S}_{\mathscr{A}}} = \{X_{\mathscr{S}_{\mathscr{A}}} \mid X \in \mathscr{Q}\}$.

We have the following obvious equivalences of functors:

$$(\cdot)'_{\mathscr{Q}} \circ \lambda \cong \lambda^{\mathscr{Q}} \circ (\cdot)_{\mathscr{Q}}, \quad (\cdot)_{\mathscr{Q}} \circ \iota \cong \iota^{\mathscr{Q}} \circ (\cdot)_{\mathscr{Q}}, \quad \lambda_R \circ k' \cong k \circ \lambda_R^{\mathscr{Q}}, \quad \iota \circ k \cong k \circ \iota^{\mathscr{Q}}.$$

Since $(\cdot)_{\mathscr{Q}} \dashv k$, $\iota_L^{\mathscr{Q}} \dashv \iota^{\mathscr{Q}}$, $\iota_L \dashv \iota$, it follows that $\iota_L^{\mathscr{Q}} \circ (\cdot)_{\mathscr{Q}} \dashv k \circ \iota^{\mathscr{Q}}$ and $(\cdot)_{\mathscr{Q}} \circ \iota_L \dashv \iota \circ k$. But $\iota \circ k \cong k \circ \iota^{\mathscr{Q}}$, hence there is an equivalence of functors

$$\iota_L^{\mathscr{Q}} \circ (\cdot)_{\mathscr{Q}} \cong (\cdot)_{\mathscr{Q}} \circ \iota_L.$$

We obtain an equivalence of functors $\iota_L^{\mathscr{Q}} \circ (\cdot)_{\mathscr{Q}} \circ k \cong (\cdot)_{\mathscr{Q}} \circ \iota_L \circ k$. Since $(\cdot)_{\mathscr{Q}} \circ k \cong \mathrm{id}$, we see that $\iota_L^{\mathscr{Q}} \cong (\cdot)_{\mathscr{Q}} \circ \iota_L \circ k$.

As we have noticed in Remark 5.11, the functor $\iota_L^{\mathscr{Q}}$ is of particular importance in Voevodsky's triangulated categories of motives. It was computed by Voevodsky [25, 3.2.6] as the Suslin complex C_*. In order to make this precise, we convert recollements of Theorems 4.12-5.10 into the Voevodsky language of triangulated categories of motives in the next section.

6. Recollements for triangulated categories of motives

Throughout this section k is a field and $\mathscr{V} = \mathbf{Ab}$. So every $\mathscr{V}$-category $\mathscr{C}$ is nothing but a preadditive category. We always work here with preadditive categories whose objects are the k-smooth separated schemes of finite type Sm/k. Such preadditive categories are also called categories of correspondences in motivic homotopy theory. We fix a category of correspondences $\mathscr{C}$ in the sense of [8]. Briefly, $\mathscr{C}$ must have a compatible action of Sm/k and satisfy certain properties with respect to Nisnevich topology.

The category of enriched functors $[\mathscr{C}^{\mathrm{op}},\mathscr{V}]$ is the category of additive contravariant functors from $\mathscr{C}$ to Abelian groups $\mathbf{Ab}$. It is also called the *category of presheaves with $\mathscr{C}$-correspondences*. We shall denote it by $Pre(\mathscr{C})$. It is a Grothendieck category such that $\{\mathscr{C}(-,X)\}_{X\in Sm/k}$ is a family of finitely generated projective generators of $Pre(\mathscr{C})$. Denote by $\mathbf{D}\big(Pre(\mathscr{C})\big)$ its derived category of unbounded complexes. We shall also write $\mathbb{Z}_{\mathscr{C}}(X)$ to denote $\mathscr{C}(-,X)$, where $X \in Sm/k$. The derived category of Abelian groups $\mathbf{D}(\mathbf{Ab})$ has compact generators given by the shifted complexes $\mathbb{Z}[n]$ of $\mathbb{Z}$, which are $\mathbf{K}$-projective in $\mathbf{D}(\mathbf{Ab})$ as well. By Theorem 4.1(1) $\mathbf{D}\big(Pre(\mathscr{C})\big)$ has compact generators given by the shifted complexes $\mathbb{Z}_{\mathscr{C}}(X)[n] = \mathbb{Z}_{\mathscr{C}}(X) \otimes \mathbb{Z}[n]$, $X \in Sm/k$, which are **K**-projective in $\mathbf{D}\big(Pre(\mathscr{C})\big)$ also.

Definition 6.1. Let $\mathbb{A}^1$ be the affine line and $X \in Sm/k$. The projection morphism $pr_X : X \times \mathbb{A}^1 \to X$ is left inverse to the inclusion $i_0 : X \to X \times \mathbb{A}^1$ identifying X with $X \times 0$ in $X \times \mathbb{A}^1$. It induces a split epimorphism of finitely generated projective objects in $Pre(\mathscr{C})$

$$pr_{X,*} : \mathbb{Z}_{\mathscr{C}}(X \times \mathbb{A}^1) \to \mathbb{Z}_{\mathscr{C}}(X).$$

We define $\mathbb{Z}_{\mathscr{C}}^{\mathbb{A}^1}(X) := \operatorname{Ker}(pr_{X,*})$. By definition, $\mathbb{Z}_{\mathscr{C}}^{\mathbb{A}^1}(X)$ is finitely generated projective in $Pre(\mathscr{C})$.

We also define the following full subcategory of $Pre(\mathscr{C})$:

$$\mathscr{S}_{\mathbb{A}^1} := \Big\{ F \in Pre(\mathscr{C}) \mid \operatorname{Hom}_{Pre(\mathscr{C})}\big(\mathbb{Z}_{\mathscr{C}}^{\mathbb{A}^1}(X), F\big) = 0 \text{ for all } X \in Sm/k \Big\},$$

which is localizing because the $\mathbb{Z}_{\mathscr{C}}^{\mathbb{A}^1}(X)$-s are projective objects in $Pre(\mathscr{C})$. The full subcategory of $Pre(\mathscr{C})$ whose objects are the $\mathbb{Z}_{\mathscr{C}}^{\mathbb{A}^1}(X)$, $X \in Sm/k$, will be denoted by $\mathscr{C}^{\mathbb{A}^1}$. It will play the same role as the $\mathscr{V}$-full subcategory $\mathscr{A}$ from previous sections. The category of contravariant additive functors from $\mathscr{C}^{\mathbb{A}^1}$ to **Ab** will be denoted by $Pre(\mathscr{C}^{\mathbb{A}^1})$. It will play the same role as $[\mathscr{A}, \mathscr{V}]$ from previous sections.

Recall that a presheaf F of Abelian groups is *homotopy invariant* if $pr_X^* : F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism for all $X \in Sm/k$.

Lemma 6.2. *A presheaf with $\mathscr{C}$-correspondences belongs to $\mathscr{S}_{\mathbb{A}^1}$ if and only if it is homotopy invariant.*

Proof. This immediately follows from the definition of $\mathscr{S}_{\mathbb{A}^1}$. □

Remark 6.3. Given a preadditive category $\mathscr{B}$, the category $(\mathscr{B}^{\mathrm{op}}, \mathbf{Ab})$ is naturally equivalent to $(\mathscr{D}^{\mathrm{op}}, \mathbf{Ab})$, where $\mathscr{D}$ is a full subcategory of any family of finitely generated projectives of $(\mathscr{B}^{\mathrm{op}}, \mathbf{Ab})$ containing the representable functors $\mathscr{B}(-, b)$. In other words, we can always add as many finitely generated projectives as is necessary (see, e.g., [6, Section 4]). Recall that the Yoneda Lemma identifies $\mathscr{B}$ with the full subcategory of representable functors of $(\mathscr{B}^{\mathrm{op}}, \mathbf{Ab})$. For example, we can add $\mathbb{Z}_{\mathscr{C}}^{\mathbb{A}^1}(X)$, $X \in Sm/k$, to $\mathscr{C}$ and form a bigger preadditive category $\widetilde{\mathscr{C}}$ without changing $Pre(\mathscr{C})$ in the above sense that there is a natural equivalence $Pre(\widetilde{\mathscr{C}}) \xrightarrow{\sim} Pre(\mathscr{C})$. This equivalence is induced by the restriction from $\widetilde{\mathscr{C}}$ to $\mathscr{C}$. We shall tacitly use this remark below when applying recollements from preceding sections to the corresponding categories associated with $Pre(\mathscr{C})$.

It follows from Theorem 3.5 that there is a recollement of Abelian categories

$$\mathscr{S}_{\mathbb{A}^1} \underset{i_R}{\overset{i_L}{\leftleftarrows}} \xrightarrow{\;i\;} Pre(\mathscr{C}) \underset{\ell_R}{\overset{\ell_L}{\leftleftarrows}} \xrightarrow{\;\ell\;} Pre(\mathscr{C})/\mathscr{S}_{\mathbb{A}^1}$$

with functors i_L, i, i_R being from Theorem 3.4. The functor ℓ_R is the inclusion and $\ell_L := r_L \circ \varkappa^{-1}$, where $\varkappa : Pre\big(\mathscr{C}^{\mathbb{A}^1}\big) \to Pre(\mathscr{C})/\mathscr{S}_{\mathbb{A}^1}$ is a natural equivalence of Grothendieck categories [6, 4.10] and $r_L : Pre\big(\mathscr{C}^{\mathbb{A}^1}\big) \to Pre(\mathscr{C})$ is the left Kan extension associated with the fully faithful embedding $\mathscr{C}^{\mathbb{A}^1} \to \widetilde{\mathscr{C}}$ composed with the natural equivalence $Pre(\widetilde{\mathscr{C}}) \xrightarrow{\sim} Pre(\mathscr{C})$ of Remark 6.3.

In this particular recollement we are able to give an explicit description of the functor i_L. For this, recall that the *strict homotopization* of a presheaf of Abelian groups F, denoted by $[F]$, is defined as $\mathrm{Coker}\big(i_1^* - i_0^* : F(- \times \mathbb{A}^1) \to F\big)$, where $F(- \times \mathbb{A}^1)$ is the presheaf $X \in Sm/k \mapsto F(X \times \mathbb{A}^1)$ and $i_1 : X \to X \times \mathbb{A}^1$ identifies X with $X \times 1 \subset X \times \mathbb{A}^1$. Notice that $[F]$ is the zeroth homology presheaf of the Suslin complex $C_*(F)$ of F $\big($see [22, Section 1] for the definition of $C_*(F)\big)$. In particular, $[F]$ is homotopy invariant, because all homology presheaves of $C_*(F)$ are homotopy invariant $\big($see [22, Section 1] or [16, Section 2.3] for details$\big)$. By Lemma 6.2 $[F] \in \mathscr{S}_{\mathbb{A}^1}$.

Proposition 6.4. *$i_L : Pre(\mathscr{C}) \to \mathscr{S}_{\mathbb{A}^1}$ is computed as the strict homotopization functor $F \mapsto [F]$.*

Proof. Our statement will follow if we show that the strict homotopization functor is left adjoint to the inclusion $i : \mathscr{S}_{\mathbb{A}^1} \to Pre(\mathscr{C})$. Let $F \in Pre(\mathscr{C})$, $H \in \mathscr{S}_{\mathbb{A}^1}$ and let $f : F \to H$ be a morphism in $Pre(\mathscr{C})$. Consider a commutative diagram

$$\begin{array}{ccccc} F(- \times \mathbb{A}^1) & \xrightarrow{i_1^*-i_0^*} & F & \xrightarrow{j} & [F] \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle [f]} \\ H(- \times \mathbb{A}^1) & \xrightarrow{i_1^*-i_0^*} & H & \longrightarrow & [H] \end{array}$$

By Lemma 6.2 H is homotopy invariant, hence the left bottom arrow equals zero and $H = [H]$. We see that $f = [f] \circ j$, and so the strict homotopization functor is left adjoint to i. □

Next denote by $\mathbf{PDM}_{\mathscr{C}}^{eff}(k)$ the full triangulated subcategory of $\mathbf{D}\big(Pre(\mathscr{C})\big)$ consisting of the complexes with homotopy invariant homology presheaves. Theorem 4.12 and Lemma 6.2 imply that there exists a recollement of triangulated categories

$$\mathbf{PDM}_{\mathscr{C}}^{eff}(k) \underset{\iota_R}{\overset{\iota_L}{\leftleftarrows}} \!\!\xrightarrow{\ \iota\ } \mathbf{D}\big(Pre(\mathscr{C})\big) \underset{\lambda_R}{\overset{\lambda_L}{\leftleftarrows}} \!\!\xrightarrow{\ \lambda\ } \mathbf{D}\big(Pre(\mathscr{C})/\mathscr{S}_{\mathbb{A}^1}\big)$$

The functor ι is inclusion, λ applies the $\mathscr{S}_{\mathbb{A}^1}$-localization exact functor to any complex of $\mathbf{D}\big(Pre(\mathscr{C})\big)$. The functor $\lambda_L := \rho_L \circ \varkappa^{-1}$, where ρ_L is from Theorem 4.11 and $\varkappa : Pre\big(\mathscr{C}^{\mathbb{A}^1}\big) \to Pre(\mathscr{C})/\mathscr{S}_{\mathbb{A}^1}$ is a natural equivalence of Grothendieck categories [6, 4.10]. The functor λ_R takes Y to $\ell_R(kY)$, where kY is a $\mathbf{K}$-injective resolution of Y in $\mathbf{D}\big(Pre(\mathscr{C})/\mathscr{S}_{\mathbb{A}^1}\big)$ and ℓ_R is the inclusion functor $Pre(\mathscr{C})/\mathscr{S}_{\mathbb{A}^1} \to Pre(\mathscr{C})$ which applies to kY degreewise.

Remark 6.5. In fact, there is another description of ι_L and λ_L for this particular recollement. Similar to [16, Section 2.3] one can show that ι_L is given by the Suslin complex functor C_*. Here C_* is a kind of homotopy extension for Proposition 6.4 about the strict homotopization functor. Since $\varkappa : Pre\big(\mathscr{C}^{\mathbb{A}^1}\big) \to Pre(\mathscr{C})/\mathscr{S}_{\mathbb{A}^1}$ is an equivalence of Grothendieck categories [6, 4.10] and $\mathbf{D}\big(Pre(\mathscr{C}^{\mathbb{A}^1})\big)$ has enough $\mathbf{K}$-projectives (in the sense that every complex admits a $\mathbf{K}$-projective resolution), then λ_L takes a complex X to the $\mathbf{K}$-projective resolution $\widetilde{X}$ of $\varkappa^{-1}(X)$ in $\mathbf{D}\big(Pre(\mathscr{C}^{\mathbb{A}^1})\big)$ and applies ℓ_L to $\widetilde{X}$ degreewise. We use here the fact that ℓ_L is extended to a left Quillen functor $\mathbf{Ch}\big(Pre(\mathscr{C}^{\mathbb{A}^1})\big) \to \mathbf{Ch}\big(Pre(\mathscr{C})\big)$ with respect to the standard projective model structure on both categories.

In order to apply Theorem 5.10 to the derived category $\mathbf{D}\big(Shv(\mathscr{C})\big)$ of the Grothendieck category of Nisnevich sheaves with $\mathscr{C}$-correspondences, we now suppose that $\mathscr{C}$ is a strict V-category of correspondences in the sense of [8]. For example, the category of finite correspondences Cor in the sense of Suslin–Voevodsky [22], K_0 and $K_0^{\oplus}$ in the sense of Grayson–Walker [11, 26] are strict V-categories of correspondences whenever the base field k is perfect. The category of Milnor–Witt correspondences $\widetilde{Cor}$ in the sense of Calmès–Fasel [3] is a strict V-category of correspondences if k is infinite perfect with $char(k) \neq 2$. We also refer the reader to [4] for further examples. If $\mathscr{C}$ is a strict V-category of correspondences then so is $\mathscr{C}_R := \mathscr{C} \otimes R$ with R a ring of fractions of $\mathbb{Z}$ like, for example, $\mathbb{Z}[\frac{1}{p}]$ or $\mathbb{Q}$.

Let $(\cdot)_{\mathscr{Q}} : Pre(\mathscr{C}) \to Shv(\mathscr{C})$ be the Nisnevich sheafification functor from presheaves to Nisnevich sheaves with $\mathscr{C}$-correspondences and let $\mathscr{Q} = \operatorname{Ker}\big((\cdot)_{\mathscr{Q}}\big)$ be the localizing subcategory in $Pre(\mathscr{C})$ of Nisnevich locally trivial presheaves. Similar to Lemma 5.3 we have $Shv(\mathscr{C}) = Pre(\mathscr{C})/\mathscr{Q}$. Since $\mathscr{C}$ is a strict V-category of correspondences, then $(\cdot)_{\mathscr{Q}}$ takes homotopy invariant presheaves to homotopy invariant sheaves. Using Lemma 6.2 $\mathscr{S}_{\mathbb{A}^1}$ satisfies the strict Voevodsky property with respect to $\mathscr{Q}$ in the sense of Definition 5.7. By Lemma 5.6 the full subcategory of homotopy invariant $\mathscr{C}$-sheaves $\mathscr{S}_{\mathbb{A}^1}^{\mathscr{Q}}$ is localizing in $Shv(\mathscr{C})$. By Lemma 5.5 one has $Shv(\mathscr{C})/\mathscr{S}_{\mathbb{A}^1}^{\mathscr{Q}} = Pre(\mathscr{C})/\mathscr{J}_{\mathbb{A}^1}$, where $\mathscr{J}_{\mathbb{A}^1}$ is the join of $\mathscr{S}_{\mathbb{A}^1}$ and $\mathscr{Q}$. By Proposition 5.9 $Shv(\mathscr{C})/\mathscr{S}_{\mathbb{A}^1}^{\mathscr{Q}}$ consists of the sheaves which are $\mathscr{S}_{\mathbb{A}^1}$-local in $Pre(\mathscr{C})$.

By [10, Section 6] $\mathbf{D}\big(Shv(\mathscr{C})\big)$ is a compactly generated triangulated category with compact generators given by representable $\mathscr{C}$-sheaves $\{\mathscr{C}(-,X)_{nis}\}_{X\in Sm/k}$. Since $\mathscr{C}$-presheaves $\{\mathscr{C}(-,X)\}_{X\in Sm/k}$ regarded as complexes in single degrees are compact generators of $\mathbf{D}\big(Pre(\mathscr{C})\big)$, it follows that the sheafification functor $(\cdot)_{\mathscr{Q}} : \mathbf{D}\big(Pre(\mathscr{C})\big) \to \mathbf{D}\big(Shv(\mathscr{C})\big)$ respects compact objects.

Following Voevodsky [25], denote by $\mathbf{DM}^{eff}_{\mathscr{C}}(k)$ the full triangulated subcategory of $\mathbf{D}\big(Shv(\mathscr{C})\big)$ consisting of homotopy invariant cohomology sheaves. We call it the *triangulated category of $\mathscr{C}$-motives*. In the notation of Definition 5.7, $\mathbf{DM}^{eff}_{\mathscr{C}}(k) = \mathscr{E}^{\mathscr{Q}}$.

If we document the above arguments and use Theorem 5.10, we have shown the following

Theorem 6.6. *Suppose $\mathscr{C}$ is a strict V-category of correspondences in the sense of [8]. There exists a recollement of triangulated categories*

$$\mathbf{DM}^{eff}_{\mathscr{C}}(k) \underset{\iota_R^{\mathscr{Q}}}{\overset{C_*}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\iota^{\mathscr{Q}}}}}}} \mathbf{D}\big(Shv(\mathscr{C})\big) \underset{\lambda_R^{\mathscr{Q}}}{\overset{\lambda_L^{\mathscr{Q}}}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\lambda^{\mathscr{Q}}}}}}} \mathbf{D}\big(Shv(\mathscr{C})/\mathscr{S}_{\mathbb{A}^1}^{\mathscr{Q}}\big),$$

*where C_* is the Suslin complex functor, the functor $\lambda^{\mathscr{Q}}$ is induced by the $\mathscr{S}_{\mathbb{A}^1}^{\mathscr{Q}}$-localization functor $Shv(\mathscr{C}) \to Shv(\mathscr{C})/\mathscr{S}_{\mathbb{A}^1}^{\mathscr{Q}}$ associated with the localizing subcategory $\mathscr{S}_{\mathbb{A}^1}^{\mathscr{Q}}$ of $Shv(\mathscr{C})$, $\iota^{\mathscr{Q}}$ is inclusion, and $\lambda_R^{\mathscr{Q}}$ is induced by the **K**-injective resolution functor.*

Thus the Voevodsky triangulated category of $\mathscr{C}$-motives $\mathbf{DM}^{eff}_{\mathscr{C}}(k)$ fits into a recollement of explicit triangulated categories. The fact that C_* is left adjoint to the inclusion is a celebrated theorem of Voevodsky [25, 3.2.6] (see [22, 1.12] as well).

Let $\mathscr{T}_{\mathbb{A}^1}$ be the localizing subcategory of $\mathbf{D}\big(Shv(\mathscr{C})\big)$ generated by complexes of the form $\mathscr{C}(-,X\times\mathbb{A}^1)_{nis} \to \mathscr{C}(-,X)_{nis}$. Then $\mathbf{D}\big(Shv(\mathscr{C})\big)/\mathscr{T}_{\mathbb{A}^1}$ is triangle equivalent to $\mathbf{DM}^{eff}_{\mathscr{C}}(k)$ (see [22, 25]). We finish the paper with the following explicit description of $\mathscr{T}_{\mathbb{A}^1}$:

Corollary 6.7. *The composite functor*

$$\mathscr{T}_{\mathbb{A}^1} \hookrightarrow \mathbf{D}\big(Shv(\mathscr{C})\big) \xrightarrow{\lambda^{\mathscr{Q}}} \mathbf{D}\big(Shv(\mathscr{C})/\mathscr{S}_{\mathbb{A}^1}^{\mathscr{Q}}\big)$$

is an equivalence of triangulated categories.

Proof. This follows from Theorem 6.6 and Lemma 5.16. ☐

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