An averaging principle for neutral stochastic fractional order differential equations with variable delays driven by Lévy noise *

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Abstract

In this paper, we establish an averaging principle for neutral stochastic fractional differential equations with non-Lipschitz coefficients and with variable delays, driven by Lévy noise. Our result shows that the solutions of the equations concerned can be approximated by the solutions of averaged neutral stochastic fractional differential equations in the sense of convergence in mean square. As an application, we present an example with numerical simulations to explore the established averaging principle.

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1. Introduction

Stochastic differential equations driven by Brownian motion play an important role in many branches of science and industry, such as biology, physics, economics, engineering and financial markets, cf. e.g., [16], [15]. While, it is worthwhile pointing out that Brownian motion is a stochastic process with continuous paths and it can not be used to describe certain discontinuous systems whose structures are subject to stochastic abrupt changes, which may result from abrupt phenomena such as stochastic failures, repairs of the components, changes in the interconnections and sudden environment changes. For such discontinuous systems, stochastic differential equations driven by Lévy noise are recognised to cope with the feature of discontinuity. Many excellent investigations have been done, see e.g., [5], [26], [11], [13] and references therein. On the other hand, fractional calculus becomes more and more interested in link with stochastic calculus, creating hot topics such as fractional Brownian motion, stochastic fractional differential equations, fractional Laplacian as Markov generators, and so on, just mention a few. Stochastic differential equations combining with fractional calculus provide suitable models for many systems and evolutionary processes, such as viscoelastic system (see [6], [17]). Since then, it becomes an active research topic in the study of stochastic dynamic systems (see [1], [2], [3], [12], [24]).

The averaging principle, initiated by Khasminskii in [10], is a very efficient and important tool in study of stochastic fractional differential equations for modelling problems arising in many practical research ares. It in fact provides a powerful tool for simplifying dynamical systems, and obtains approximate solutions to differential equations. The averaging principle enables us to study complex equations with

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related averaging equations, which paves a convenient and easy way to study many important properties (see, e.g., [20], [21], [22], [7], [2], [23], [18]).

Inspired by the aforementioned works, in this paper, we are concerned with an averaging principle for neutral stochastic fractional differential equation driven by Lévy process with variable delay, that is, on a given complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, for $0 < \alpha < 1$, we consider the following

$$
\begin{aligned}
&d(X(t) - D(X(t-\delta(t))))\\
&= u(X(t-), X(t-\delta(t)), t)dt + b(X(t-), X(t-\delta(t)), t)dB(t)\\
&\quad + \sigma(X(t-), X(t-\delta(t)), t)(dt)^\alpha + \int_{|y|<c} h(X(t-), X(t-\delta(t)), y, t)\tilde{N}(dt, dy), t \in [0, T]
\end{aligned}
\tag{1.1}
$$

with initial condition $\xi(0) \in \mathbb{R}^n$ and initial value $X(0) = \xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$ being an $\mathcal{F}_0$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random variable such that $\mathbb{E}\|\xi\|^2 < \infty$, where $\tau > 0$ and $C([-\tau, 0]; \mathbb{R}^n)$ is the totality of continuous $\mathbb{R}^n$-valued function φ defined on $[-\tau, 0]$ with norm $\|\varphi\| := \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$, $\delta : [0, T] \to [0, \tau]$ and the mappings $D : \mathbb{R}^n \to \mathbb{R}^n$, $u, \sigma : \mathbb{R}^n \times \mathbb{R}^n \times [0, T] \longrightarrow \mathbb{R}^n$, $b : \mathbb{R}^n \times \mathbb{R}^n \times [0, T] \longrightarrow \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \times [0, T] \to \mathbb{R}^n$ are continuous functions, $B(t) = (B_1(t), B_2(t), \cdots, B_m(t))^T$ is an m-dimensional $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion, $\widetilde{N}(dt, dy) := N(dt, dy) - \nu(dy)dt$ is the compensated martingale measure associated with the $\{\mathcal{F}_t\}$-adapted Poisson random measure $N : \mathbb{R}^+ \times (\mathbb{R}^n \setminus \{0\})$ with density measure $dt\nu$ determined by a given Lévy measure ν on $(\mathbb{R}^n \setminus \{0\}, \mathcal{B}(\mathbb{R} \setminus \{0\}))$, i.e., satisfying $\int_{\mathbb{R}^n\setminus\{0\}} (|y|^2 \wedge 1)\nu(dy) < \infty$, and the constant c is the maximum allowable jump size,. We suppose that $N(dt, dy)$ and $\widetilde{N}(dt, dy)$ are independent of the Brownian motion $B(t)$.

For simplicity throughout this paper, let C stand for positive constants and their value may be different in different appearances.

The paper is organised as follows. In Section 2, we present some preliminaries and our assumptions for this paper. In Section 3, we will prove an approximation theorem as an averaging principle for the solutions of the considered stochastic fractional differential equations with Lévy noise. We end up our study by providing an example with numerical simulations to explicate our obtained theory in Section 4.

2. Preliminaries

In this section, we briefly recall the definitions of Riemann-Liouville fractional integrals and derivatives.

Definition 2.1. (Riemann-Liouville fractional integrals [19]) For any $\alpha \in (0, 1)$, and a function $f \in L^1([a, b]; \mathbb{R}^n)$, the left sided and right sided Riemann-Liouville fractional integrals of order α are defined for almost all $a < t < b$ respectively by

$$(I^\alpha_{a+} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \qquad t > a,$$

and

$$(I^\alpha_{b-} f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-s)^{\alpha-1} f(s) ds, \qquad t < b,$$

where $\Gamma(\alpha) = \int_0^\infty s^{\alpha-1} e^{-s} ds$ stands for the Gamma function and $L^1[a, b]$ is the space of all Lebesgue integrable functions on a finite interval $[a, b]$ of the $\mathbb{R}$.

Definition 2.2. (Riemann-Liouville fractional derivatives [19]) For any $\alpha \in (0,1)$ and any well-defined absolutely continuous function f on an interval $[a,b]$, the left sided and right sided Riemann-Liouville fractional derivatives defined respectively by

$$(D^{\alpha}_{a+}f)(t) = \frac{1}{\Gamma(1-\alpha)}[\frac{f(a)}{(t-a)^{\alpha}} + \int_a^t (t-s)^{-\alpha} f'(s)ds],$$

and

$$(D^{\alpha}_{b-}f)(t) = \frac{1}{\Gamma(1-\alpha)}[\frac{f(b)}{(b-t)^{\alpha}} - \int_t^b (s-t)^{-\alpha} f'(s)ds].$$

Lemma 2.3. *([9]) Let $\sigma(t)$ be a continuous function, then its integration with respect to $(dt)^{\alpha}$, $0 < \alpha \leq 1$, is defined by*

$$\int_0^t \sigma(s)(ds)^{\alpha} = \alpha \int_0^t (t-s)^{\alpha-1}\sigma(s)ds. \tag{2.1}$$

Lemma 2.4. *([14]) Let $p \geq 2, r > 0$ and $a, b \in \mathbb{R}$, then*

$$|a+b|^p \leq \left[1 + r^{\frac{1}{p-1}}\right]^{p-1} \left(|a|^p + \frac{|b|^p}{r}\right).$$

Definition 2.5. An $\mathbb{R}^n$-valued stochastic process $\{X(t)\}_{-\tau \leq t \leq T}$ is called a solution to equation (1.1) if it has the following properties:

(i) $\{X(t)\}$ is right continuous with a left limit and $\{\mathcal{F}_t\}$-adapted and $\mathbb{E}(\int_{-\tau}^T |X(t)|^2 dt) < \infty$;

(ii) $X(0) = \xi$ and for all $t \in [0,T]$, the equation

$$\begin{aligned} X(t) =& \xi(0) + D(X(t-\delta(t))) - D(\xi(-\delta(t))) + \int_0^t u(X(s-), X(s-\delta(s)), s)ds \\ &+ \int_0^t b(X(s-), X(s-\delta(s)), s)dB(s) + \alpha \int_0^t \sigma(X(s-), X(s-\delta(s)), s)(t-s)^{\alpha-1}ds \\ &+ \int_0^t \int_{|y|<c} h(X(s-), X(s-\delta(s)), y, s)\widetilde{N}(ds, dy), \end{aligned}$$

holds with probability 1.

(iii) for any other solution $\overline{X}(t)$, we have

$$\mathbb{P}\{X(t) = \overline{X}(t), \quad for \quad any \quad -\tau \leq t \leq T\} = 1.$$

In what follows, we impose some conditions on the coefficients of the equation (1.1) to get the existence and uniqueness of solutions.

Assumption 2.6. *For any fixed $t \geq 0$ and x_1, x_2, y_1, $y_2 \in \mathbb{R}^n$, the following inequalities hold*

$$\begin{aligned} &|u(x_1,y_1,t) - u(x_2,y_2,t)|^2 + |b(x_1,y_1,t) - b(x_2,y_2,t)|^2 + |\sigma(x_1,y_1,t) - \sigma(x_2,y_2,t)|^2 \\ &+ \int_{|y|<c} |h(x_1,y_1,t,y) - h(x_2,y_2,t,y)|^2 \nu(dy) \leq \phi(t)\kappa(|x_1-x_2|^2 + |y_1-y_2|^2), \end{aligned} \tag{2.2}$$

$$|u(x_1,x_2,t)|^2 + |b(x_1,x_2,t)|^2 + |\sigma(x_1,x_2,t)|^2 + \int_{|y|<c} |h(x_1,x_2,y,t)|^2 \nu(dy) \leq \lambda(t)\kappa((x_1)^2 + (x_2)^2), \tag{2.3}$$

where ϕ, λ are bounded functions, $\kappa(z) : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, concave, nondecreasing function. Moreover, $\kappa(0) = 0, \kappa(z) > 0$ for $z > 0$, $\int_{0^+} \frac{dz}{\kappa(z)} = \infty$.

Assumption 2.7. *For all $x_1, x_2 \in \mathbb{R}^n$, there exists a constant $L_3 \in (0, 1)$ such that*

$$|D(x_1) - D(x_2)| \leq L_3 |x_1 - x_2|, \quad \textit{and} \quad D(0) = 0.$$

Under the Assumptions 2.6-2.7, one can show that there exists a unique solution $X(t)$ to equation (1.1) and $\mathbb{E}(\sup_{0 \leq s \leq t} |X(t)|^2)$ is bounded by positive constant C. The proof is pretty similar to the proof of [3] and we omitted it here.

Remark 2.8. Let $\delta \in (0, 1)$ be sufficiently small, we can define the concrete examples for $\kappa(z)$.
$\kappa_1(z) = Kz, z \geq 0$.
$\kappa_2(z) = \begin{cases} z \log(z^{-1}), & 0 \leq z \leq \delta; \\ \delta \log(\delta^{-1}) + \kappa'_2(\delta-)(z - \delta), & z > \delta. \end{cases}$
$\kappa_3(z) = \begin{cases} z \log(z^{-1}) \log\log(z^{-1}), & 0 \leq z \leq \delta; \\ \delta \log(\delta^{-1}) \log\log(\delta^{-1}) + \kappa'_3(\delta-)(z - \delta), & z > \delta. \end{cases}$
where κ' denotes the derivative of the function κ. They are all concave nondecreasing functions satisfying $\int_{0^+} \frac{dz}{\kappa_i(z)} = \infty$. Furthermore, we observe that the Lipschitz condition is a special case of our proposed condition.

3. AVERAGING PRINCIPLE

In this section, we will prove that the solution of the considered equations (1.1) can be approximated by solutions of averaged neutral stochastic fractional differential equations in the sense of convergence in mean square under non-Lipschitz coefficients.

For arbitrarily fixed $T > 0$, we investigate the following standard integral formulation of equation (1.1)

$$\begin{aligned} X_\epsilon(t) \; = \; & \xi(0) + D(X_\epsilon(t - \delta(t))) - D(\xi(-\delta(t))) + \epsilon \int_0^t u(X_\epsilon(s-), X_\epsilon(s - \delta(s)), s) ds \\ & + \sqrt{\epsilon} \int_0^t b(X_\epsilon(s-), X_\epsilon(s - \delta(s)), s) dB(s) + \epsilon\alpha \int_0^t \sigma(X_\epsilon(s-), X_\epsilon(s - \delta(s)), s)(t - s)^{\alpha - 1} ds \\ & + \sqrt{\epsilon} \int_0^t \int_{|y| < c} h(X_\epsilon(s-), X_\epsilon(s - \delta(s)), y, s) \widetilde{N}(ds, dy) \end{aligned} \tag{3.1}$$

where $\epsilon \in (0, \epsilon_0]$ is a positive small parameter with $\epsilon_0 \in (0, \frac{1}{2})$ that is a fixed number and initial valued $X_\epsilon(t) = \xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\} \in C([-\tau, 0]; \mathbb{R}^n)$, the coefficients have the same conditions as in equation (1.1). Hence the equation (3.1) also has a unique solution $X_\epsilon(t)$, we will examine whether the solution process $X_\epsilon(t)$ can be approximated by the solution process $Z_\epsilon(t)$ of the averaged equation

$$\begin{aligned} Z_\epsilon(t) \; = \; & \xi(0) + D(Z_\epsilon(t - \delta(t))) - D(\xi(-\delta(t))) + \epsilon \int_0^t \overline{u}(Z_\epsilon(s-), Z_\epsilon(s - \delta(s))) ds \\ & + \sqrt{\epsilon} \int_0^t \overline{b}(Z_\epsilon(s-), Z_\epsilon(s - \delta(s))) dB(s) + \epsilon\alpha \int_0^t \overline{\sigma}(Z_\epsilon(s-), Z_\epsilon(s - \delta(s)))(t - s)^{\alpha - 1} ds \\ & + \sqrt{\epsilon} \int_0^t \int_{|y| < c} \overline{h}(Z_\epsilon(s-), Z_\epsilon(s - \delta(s)), y) \widetilde{N}(ds, dy). \end{aligned} \tag{3.2}$$

We assume that the following inequalities hold.

Assumption 3.1. *(Averaging condition) For any $T_1 \in [0,T]$, then we have*

(i) $\left|\frac{1}{T_1}\int_0^{T_1} u(x_1,x_2,s)-\overline{u}(x_1,x_2)ds\right|^2 \le \varphi_1(T_1)\overline{\kappa}(|x_1|^2+|x_2|^2)$,

(ii) $\frac{1}{T_1}\int_0^{T_1} |b(x_1,x_2,s)-\overline{b}(x_1,x_2)|^2 ds \le \varphi_2(T_1)\overline{\kappa}(|x_1|^2+|x_2|^2)$,

(iii) $\frac{1}{T_1}\int_0^{T_1} |\sigma(x_1,x_2,s)-\overline{\sigma}(x_1,x_2)|^2 ds \le \varphi_3(T_1)\overline{\kappa}(|x_1|^2+|x_2|^2)$,

(iv) $\frac{1}{T_1}\int_0^{T_1} |h(x_1,x_2,y,s)-\overline{h}(x_1,x_2,y)|^2\nu(dy) ds \le \varphi_4(T_1)\overline{\kappa}(|x_1|^2+|x_2|^2)$,

where $\varphi_i(T_1)$ are bounded positive functions with $\lim_{T_1\to\infty}\varphi_i(T_1)=0$, $i=1,2,3,4$ and $\overline{\kappa}(\cdot)$ is continuous nondecreasing concave function. Moreover, $\overline{\kappa}(0)=0, \overline{\kappa}(z)>0$ for $z>0$, $\int_{0^+}\frac{dz}{\overline{\kappa}(z)}=\infty$.

Remark 3.2. This condition

$$\lim_{T\to\infty}\varphi_i(T)=0, i=1,2,3,4,$$

is to ensure the existence and uniqueness of the solution to the averaged equation (3.2). For example, for every $x_1,x_2,y_1,y_2\in\mathbb{R}^n$, we have

$$\begin{aligned}|\bar{b}(x_1,y_1)-\bar{b}(x_2,y_2)|^2 \le\ & 3\frac{1}{T}\int_0^T |\bar{b}(x_1,y_1)-b(x_1,y_1,s)|^2 ds + 3\frac{1}{T}\int_0^T |\bar{b}(x_2,y_2)-b(x_2,y_2,s)|^2 ds\\ &+3\frac{1}{T}\int_0^T |b(x_1,y_1,s)-b(x_2,y_2,s)|^2 ds\\ \le\ & 3\varphi_2(T)\bar{\kappa}(|x_1|^2+|y_1|^2)+3\varphi_2(T)\bar{\kappa}(|x_2|^2+|y_2|^2)\\ &+3C\kappa(|x_1-x_2|^2+|y_1-y_2|^2).\end{aligned}$$

By taking T tending to infinity, we see $\bar{b}$ satisfies (2.2). Similarity, one can prove that $\bar{b}$ satisfies (2.3). Using the same method, we have $\overline{u},\overline{\sigma},\overline{h}$ satisfy the Assumption 2.6. Hence, there is a unique solution $Z_\epsilon(t)$ to the averaged equation (3.2).

Theorem 3.3. *Assume that the original and averaged equation* (3.1) *and* (3.2) *satisfy the Assumptions 2.6-2.7 and 3.1. Then for any small number $\delta_1>0$, there exist $L>0$, $\beta\in(0,1)$ and $\epsilon_1\in(0,\epsilon_0]$ such that for all $\epsilon\in(0,\epsilon_1]$,*

$$\mathbb{E}(\sup_{t\in[0,L\epsilon^{-\beta}(1-L_3)^2]}|x_\epsilon(t)-Z_\epsilon(t)|^2)\le\delta_1.$$

Proof. By equations (3.1) and (3.2), we obtain

$$X_\epsilon(t)-Z_\epsilon(t)=D(X_\epsilon(t-\delta(t)))-D(Z_\epsilon(t-\delta(t)))+\Lambda(t), \tag{3.3}$$

with

$$\begin{aligned}\Lambda(t)=&\epsilon\int_0^t[u(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-\overline{u}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))]ds\\ &+\sqrt{\epsilon}\int_0^t[b(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-\overline{b}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))]dB(s)\\ &+\alpha\epsilon\int_0^t(t-s)^{\alpha-1}[\sigma(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-\overline{\sigma}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))]ds\\ &+\sqrt{\epsilon}\int_0^t\int_{|y|<c}[h(X_\epsilon(s-),X_\epsilon(s-\delta(s)),y,s)-\overline{h}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),y)]\widetilde{N}(ds,dy).\end{aligned} \tag{3.4}$$

By Lemma 2.4, we have

$$|X_\epsilon(t)-Z_\epsilon(t)|^2\le(1+r)\left(|\Lambda(t)|^2+\frac{|D(X_\epsilon(t-\delta(t)))-D(Z_\epsilon(t-\delta(t)))|^2}{r}\right).$$

Letting $r=\frac{L_3}{1-L_3}$ and using Assumption 2.7, one can obtain

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le t\le v}|X_\epsilon(t)-Z_\epsilon(t)|^2\right)\\&\le L_3\mathbb{E}\left(\sup_{0\le t\le v}|X_\epsilon(t)-Z_\epsilon(t)|^2\right)+\frac{1}{1-L_3}\mathbb{E}\left(\sup_{0\le t\le v}|\Lambda(t)|^2\right).\end{aligned}$$

That is

$$(1-L_3)\mathbb{E}\left(\sup_{0\le t\le v}|X_\epsilon(t)-Z_\epsilon(t)|^2\right)\le\frac{1}{1-L_3}\mathbb{E}\left(\sup_{0\le t\le v}|\Lambda(t)|^2\right).$$

Therefore

$$\mathbb{E}\left(\sup_{0\le t\le v}|X_\epsilon(t)-Z_\epsilon(t)|^2\right)\le\frac{1}{(1-L_3)^2}\mathbb{E}\left(\sup_{0\le t\le v}|\Lambda(t)|^2\right).$$

Using the simple inequality

$$|x_1+x_2+\cdots+x_m|^2\le m(|x_1|^2+|x_2|^2+\cdots+|x_m|^2),\tag{3.5}$$

we have

$$\begin{aligned}&\sup_{0\le t\le v}|\Lambda(t)|^2\\&\le 4\epsilon^2\sup_{0\le t\le v}(\int_0^t[u(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-\overline{u}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))]ds)^2\\&\quad+4\epsilon\sup_{0\le t\le v}(\int_0^t[b(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-\overline{b}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))]dB(s))^2\\&\quad+4\epsilon^2\alpha^2\sup_{0\le t\le v}(\int_0^t(t-s)^{\alpha-1}[\sigma(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-\overline{\sigma}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))]ds)^2\\&\quad+4\epsilon\sup_{0\le t\le v}(\int_0^t\int_{|y|<c}[h(X_\epsilon(s-),X_\epsilon(s-\delta(s)),y,s)-\overline{h}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),y)]\widetilde{N}(ds,dy))^2.\end{aligned}$$

So,

$$\begin{aligned}&\mathbb{E}(\sup_{0\le t\le v}|X_\epsilon(t)-Z_\epsilon(t)|^2)\\\le\;&\frac{4\epsilon^2}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}(\int_0^t[u(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-\overline{u}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))]ds)^2\\&+\frac{4\epsilon}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}(\int_0^t[b(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-\overline{b}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))]dB(s))^2\\&+\frac{4\epsilon^2\alpha^2}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}(\int_0^t(t-s)^{\alpha-1}[\sigma(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-\overline{\sigma}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))]ds)^2\end{aligned}$$

$$+\frac{4\epsilon}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}(\int_0^t\int_{|y|<c}[h(X_\epsilon(s-),X_\epsilon(s-\delta(s)),y,s)-\overline{h}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),y)]\widetilde{N}(ds,dy))^2$$

$$=:\ \sum_{i=1}^{4}I_i, \tag{3.6}$$

where $v\in[0,T]$. Now we present some useful estimates for I_i, $i=1,2,3,4$. First, we use inequality (3.5) to obtain

$$\begin{aligned}
I_1 &= \frac{4\epsilon^2}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}(\int_0^t[u(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-\overline{u}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))]ds)^2\\
&\le \frac{8\epsilon^2}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}|\int_0^t[u(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-u(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),s)]ds|^2\\
&\quad+\frac{8\epsilon^2}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}|\int_0^t[u(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),s)-\overline{u}(Z_\epsilon(s),Z_\epsilon(s-\delta(s)))]ds|^2\\
&=: I_{11}+I_{12}.
\end{aligned}$$

For the term I_{11}, by Hölder inequality and Assumption 2.6, we get

$$\begin{aligned}
I_{11} &\le \frac{8\epsilon^2}{(1-L_3)^2}\mathbb{E}\left(\sup_{0\le t\le v}t\int_0^t|u(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-u(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),s)|^2ds\right)\\
&\le \frac{8Cv\epsilon^2}{(1-L_3)^2}\mathbb{E}\int_0^v\kappa(|X_\epsilon(s-)-Z_\epsilon(s-)|^2+|X_\epsilon(s-\delta(s))-Z_\epsilon(s-\delta(s))|^2)ds\\
&\le \frac{8Cv\epsilon^2}{(1-L_3)^2}\int_0^v\kappa(2\mathbb{E}(\sup_{0\le s_1\le s}|X_\epsilon(s_1)-Z_\epsilon(s_1)|^2)ds\\
&\le \frac{8Cv\epsilon^2}{(1-L_3)^2}\int_0^v\kappa(2\mathbb{E}(\sup_{0\le s_1\le v}|X_\epsilon(s_1)-Z_\epsilon(s_1)|^2)dv.
\end{aligned}$$

For the term I_{12}, by Assumption 3.1 (i), we get

$$\begin{aligned}
I_{12} &= \frac{8\epsilon^2}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}t^2\left|\frac{1}{t}\int_0^t[u(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),s)-\overline{u}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))]ds\right|^2\\
&\le \frac{8v^2\epsilon^2}{(1-L_3)^2}\sup_{0\le t\le v}\varphi_1(t)\overline{\kappa}\{\sup_{0\le s\le t}[\mathbb{E}(\sup_{0\le s_1\le s}|Z_\epsilon(s_1)|^2)+\mathbb{E}|Z_\epsilon(s-\delta(s))|^2]\}\\
&\le \frac{8v^2\epsilon^2}{(1-L_3)^2}\sup_{0\le t\le v}\{\varphi_1(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\},
\end{aligned}$$

since

$$\mathbb{E}(\sup_{0\le s\le t}|X(s-\delta(s))|^2)\le\mathbb{E}(\sup_{-\tau\le s\le t}|X(s)|^2)\le\mathbb{E}\|\xi\|^2+\mathbb{E}(\sup_{0\le s\le t}|X(s)|^2).$$

So we get

$$\begin{aligned}
\mathbb{E}I_1 &\le \frac{8Cv\epsilon^2}{(1-L_3)^2}\int_0^v\kappa(2\mathbb{E}(\sup_{0\le s_1\le v}|X_\epsilon(s_1)-Z_\epsilon(s_1)|^2)dv\\
&\quad+\frac{8v^2\epsilon^2}{(1-L_3)^2}\sup_{0\le t\le v}\{\varphi_1(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\}.
\end{aligned}\tag{3.7}$$

For the term I_2, by (3.5), we get

$$\begin{aligned}
I_2 &= \frac{4\epsilon}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}\bigg(\int_0^t [b(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-\overline{b}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))]dB(s)\bigg)^2\\
&\le \frac{8\epsilon}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}\bigg(\int_0^t [b(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-b(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)]dB(s)\bigg)^2\\
&\quad+\frac{8\epsilon}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}\bigg(\int_0^t [b(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-\overline{b}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))]dB(s)\bigg)^2\\
&=: I_{21}+I_{22}.
\end{aligned}$$

For the term I_{21}, by Doob's martingale inequalitiy, Itô isometry and Assumption 2.6, we get

$$\begin{aligned}
I_{21} &\le \frac{32\epsilon}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}(\int_0^t \phi(s)\kappa(|X_\epsilon(s-)-Z_\epsilon(s-)|^2+|X_\epsilon(s-\delta(s))-Z_\epsilon(s-\delta(s))|^2)ds)\\
&\le \frac{32C\epsilon}{(1-L_3)^2}\int_0^v \kappa(2\mathbb{E}(\sup_{0\le s_1\le s}|X_\epsilon(s_1)-Z_\epsilon(s_1)|^2)ds\\
&\le \frac{32C\epsilon}{(1-L_3)^2}\int_0^v \kappa(2\mathbb{E}(\sup_{0\le s_1\le v}|X_\epsilon(s_1)-Z_\epsilon(s_1)|^2)dv.
\end{aligned}$$

For the term I_{22}, by Assumption 3.1 (ii), we get

$$\begin{aligned}
\mathbb{E}I_{22} &= \frac{32\epsilon}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v} t(\frac{1}{t}\int_0^t |b(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),s)-\overline{b}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))|^2ds)\\
&\le \frac{32v\epsilon}{(1-L_3)^2}\sup_{0\le t\le v}\varphi_2(t)\overline{\kappa}\{\sup_{0\le s\le t}[\mathbb{E}(\sup_{0\le s_1\le s}|Z_\epsilon(s_1)|^2)+\mathbb{E}|Z_\epsilon(s-\delta(s))|^2]\}\\
&\le \frac{32v\epsilon}{(1-L_3)^2}\sup_{0\le t\le v}\{\varphi_2(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\}.
\end{aligned}$$

So,

$$\begin{aligned}
I_2 \le{}& \frac{32C\epsilon}{(1-L_3)^2}\int_0^v \kappa(2\mathbb{E}(\sup_{0\le s_1\le v}|X_\epsilon(s_1)-Z_\epsilon(s_1)|^2)dv\\
&+\frac{32v\epsilon}{(1-L_3)^2}\sup_{0\le t\le v}\{\varphi_2(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\}.
\end{aligned}\tag{3.8}$$

For the term I_3, by (3.5), we get

$$\begin{aligned}
I_3 &= \frac{4\epsilon^2\alpha^2}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}(\int_0^t (t-s)^{\alpha-1}[\sigma(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-\overline{\sigma}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))]ds)^2\\
&\le \frac{8\epsilon^2\alpha^2}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}(\int_0^t (t-s)^{\alpha-1}[\sigma(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-\sigma(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),s)]ds)^2\\
&\quad+\frac{8\epsilon^2\alpha^2}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}(\int_0^t (t-s)^{\alpha-1}[\sigma(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),s)-\overline{\sigma}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))]ds)^2\\
&=: I_{31}+I_{32}.
\end{aligned}$$

For the term I_{31}, by Hölder's inequality, Itô-isometry and Assumption 2.6, we get

$$\begin{aligned}
I_{31} &= \frac{8\epsilon^2\alpha^2}{(1-L_3)^2}\frac{v^{2\alpha-1}}{2\alpha-1}\mathbb{E}\sup_{0\le t\le v}\int_0^t|\sigma(X_\epsilon(s-),X_\epsilon(s-\delta(s)),s)-\sigma(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),s)|^2ds\\
&\le \frac{8C\epsilon^2\alpha^2v^{2\alpha-1}}{(1-L_3)^2(2\alpha-1)}\mathbb{E}\sup_{0\le t\le v}\int_0^t\kappa(|X_\epsilon(s-)-Z_\epsilon(s-)|^2+|X_\epsilon(s-\delta(s))-Z_\epsilon(s-\delta(s))|^2)ds\\
&\le \frac{8C\epsilon^2\alpha^2v^{2\alpha-1}}{(1-L_3)^2(2\alpha-1)}\int_0^v\kappa(2\mathbb{E}(\sup_{0\le s_1\le s}|X_\epsilon(s_1)-Z_\epsilon(s_1)|^2)ds\\
&\le \frac{8C\epsilon^2\alpha^2v^{2\alpha-1}}{(1-L_3)^2(2\alpha-1)}\int_0^v\kappa(2\mathbb{E}(\sup_{0\le s_1\le v}|X_\epsilon(s_1)-Z_\epsilon(s_1)|^2)dv.
\end{aligned}$$

For the term I_{32}, by Assumption 3.1 (iii), we get

$$\begin{aligned}
I_{32} &\le \frac{8\epsilon^2\alpha^2}{(1-L_3)^2}\frac{v^{2\alpha-1}}{2\alpha-1}\mathbb{E}\sup_{0\le t\le v}\int_0^t|\sigma(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),s)-\overline{\sigma}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))|^2ds\\
&\le \frac{8\epsilon^2\alpha^2v^{2\alpha-1}}{(1-L_3)^2(2\alpha-1)}\mathbb{E}\sup_{0\le t\le v}t(\frac{1}{t}\int_0^t|\sigma(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),s)-\overline{\sigma}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)))|^2ds)\\
&\le \frac{8\epsilon^2\alpha^2v^{2\alpha}}{(1-L_3)^2(2\alpha-1)}\sup_{0\le t\le v}\varphi_3(t)\overline{\kappa}\{\sup_{0\le s\le t}[\mathbb{E}(\sup_{0\le s_1\le s}|Z_\epsilon(s_1)|^2)+\mathbb{E}|Z_\epsilon(s-\delta(s))|^2]\}\\
&\le \frac{8\epsilon^2\alpha^2v^{2\alpha}}{(1-L_3)^2(2\alpha-1)}\sup_{0\le t\le v}\{\varphi_3(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\}.
\end{aligned}$$

So,

$$\begin{aligned}
I_3 \le &\frac{8C\epsilon^2\alpha^2v^{2\alpha-1}}{(1-L_3)^2(2\alpha-1)}\int_0^v\kappa(2\mathbb{E}(\sup_{0\le s_1\le v}|X_\epsilon(s_1)-Z_\epsilon(s_1)|^2)dv\\
&+\frac{8\epsilon^2\alpha^2v^{2\alpha}}{(1-L_3)^2(2\alpha-1)}\sup_{0\le t\le v}\{\varphi_3(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\}.
\end{aligned}\tag{3.9}$$

For the term I_4, by (3.5), we get

$$\begin{aligned}
I_4 &= \frac{4\epsilon}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}(\int_0^t\int_{|y|<c}[h(X_\epsilon(s-),X_\epsilon(s-\delta(s)),y,s)-\overline{h}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),y)]\widetilde{N}(ds,dy))^2\\
&\le \frac{8\epsilon}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}(\int_0^t\int_{|y|<c}[h(X_\epsilon(s-),X_\epsilon(s-\delta(s)),y,s)-h(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),y,s)]\widetilde{N}(ds,dy))^2\\
&+\frac{8\epsilon}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}(\int_0^t\int_{|y|<c}[h(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),y,s)-\overline{h}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),y)]\widetilde{N}(ds,dy))^2\\
&=: I_{41}+I_{42}.
\end{aligned}$$

For the term I_{41}, by Doob's martingale inequalitiy, Itô-isometry and Assumption 2.6, we get

$$\begin{aligned}
I_{41} &\le \frac{32C\epsilon}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v}\int_0^t\kappa(|X_\epsilon(s-)-Z_\epsilon(s-)|^2+|X_\epsilon(s-\delta(s))-Z_\epsilon(s-\delta(s))|^2)ds\\
&\le \frac{32C\epsilon}{(1-L_3)^2}\int_0^v\kappa(2\mathbb{E}(\sup_{0\le s_1\le s}|X_\epsilon(s_1)-Z_\epsilon(s_1)|^2)ds\\
&\le \frac{32C\epsilon}{(1-L_3)^2}\int_0^v\kappa(2\mathbb{E}(\sup_{0\le s_1\le v}|X_\epsilon(s_1)-Z_\epsilon(s_1)|^2)dv.
\end{aligned}$$

For the term I_{42}, by Assumption 3.1 (iv), we get

$$
\begin{aligned}
I_{42} &\le \frac{32\epsilon}{(1-L_3)^2}\mathbb{E}\sup_{0\le t\le v} t(\frac{1}{t}\int_0^t\int_{|y|<c}|h(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),y,s)-\overline{h}(Z_\epsilon(s-),Z_\epsilon(s-\delta(s)),y)|^2\nu(dy)ds) \\
&\le \frac{32v\epsilon}{(1-L_3)^2}\sup_{0\le t\le v}\varphi_4(t)\overline{\kappa}\{\sup_{0\le s\le t}[\mathbb{E}(\sup_{0\le s_1\le s}|Z_\epsilon(s_1)|^2)+\mathbb{E}|Z_\epsilon(s-\delta(s))|^2]\} \\
&\le \frac{32v\epsilon}{(1-L_3)^2}\sup_{0\le t\le v}\{\varphi_4(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\}.
\end{aligned}
$$

So,

$$
\begin{aligned}
I_4 \le \frac{32C\epsilon}{(1-L_3)^2}&\int_0^v\kappa(2\mathbb{E}(\sup_{0\le s_1\le v}|X_\epsilon(s_1)-Z_\epsilon(s_1)|^2)dv \\
&+\frac{32v\epsilon}{(1-L_3)^2}\sup_{0\le t\le v}\{\varphi_4(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\}.
\end{aligned}
\tag{3.10}
$$

Taking (3.7)-(3.10) and Jensen inequality into account, we obtain

$$
\begin{aligned}
&\mathbb{E}(\sup_{0\le t\le v}|X_\epsilon(t)-Z_\epsilon(t)|^2) \\
&\le \frac{8v^2\epsilon^2}{(1-L_3)^2}\sup_{0\le t\le v}\{\varphi_1(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\}+\frac{32v\epsilon}{(1-L_3)^2}\sup_{0\le t\le v}\{\varphi_2(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\} \\
&+\frac{8\epsilon^2\alpha^2v^{2\alpha}}{(1-L_3)^2(2\alpha-1)}\sup_{0\le t\le v}\{\varphi_3(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\}+\frac{32v\epsilon}{(1-L_3)^2}\sup_{0\le t\le v}\{\varphi_4(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\} \\
&+(\frac{8Cv\epsilon^2}{(1-L_3)^2}+\frac{32C\epsilon}{(1-L_3)^2}+\frac{8C\epsilon^2\alpha^2v^{2\alpha-1}}{(1-L_3)^2(2\alpha-1)}+\frac{32C\epsilon}{(1-L_3)^2})\int_0^v\kappa(2\mathbb{E}(\sup_{0\le s_1\le v}|X_\epsilon(s_1)-Z_\epsilon(s_1)|^2)dv.
\end{aligned}
$$

Because $\kappa(z)$ is concave, there exist $a\ge 0$, $b\ge 0$ such that (see Mao [15])

$$
\kappa(z)\le a+bz,\quad z\ge 0. \tag{3.11}
$$

Then,

$$
\begin{aligned}
&\mathbb{E}(\sup_{0\le t\le v}|x_\epsilon(t)-Z_\epsilon(t)|^2) \\
&\quad\le \frac{8v^2\epsilon^2}{(1-L_3)^2}\sup_{0\le t\le v}\{\varphi_1(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\}+\frac{32v\epsilon}{(1-L_3)^2}\sup_{0\le t\le v}\{\varphi_2(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\} \\
&\qquad+\frac{8\epsilon^2\alpha^2v^{2\alpha}}{(1-L_3)^2(2\alpha-1)}\sup_{0\le t\le v}\{\varphi_3(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\}+\frac{32v\epsilon}{(1-L_3)^2}\sup_{0\le t\le v}\{\varphi_4(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\} \\
&\qquad+(\frac{8Cv\epsilon^2}{(1-L_3)^2}+\frac{32C\epsilon}{(1-L_3)^2}+\frac{8C\epsilon^2\alpha^2v^{2\alpha-1}}{(1-L_3)^2(2\alpha-1)}+\frac{32C\epsilon}{(1-L_3)^2})va \\
&\qquad+(\frac{8v\epsilon^2}{(1-L_3)^2}+\frac{32\epsilon}{(1-L_3)^2}+\frac{8\epsilon^2\alpha^2v^{2\alpha-1}}{(1-L_3)^2(2\alpha-1)}+\frac{32\epsilon}{(1-L_3)^2})Cb\int_0^v 2\mathbb{E}(\sup_{0\le t\le v}|x_\epsilon(t)-Z_\epsilon(t)|^2)dv \\
&\quad\le \frac{8v\epsilon}{(1-L_3)^2}[v\epsilon R_1+4R_2+\epsilon R_3R_5+4R_4+(v\epsilon+8+\epsilon R_5)aC]+R_6\int_0^v\mathbb{E}(\sup_{0\le t\le v}|x_\epsilon(t)-Z_\epsilon(t)|^2)dv,
\end{aligned}
$$

where constants

$$
R_1:=\sup_{0\le t\le v}\{\varphi_1(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\},\quad R_2:=\sup_{0\le t\le v}\{\varphi_2(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\},
$$

$$R_3 := \sup_{0\le t\le v}\{\varphi_3(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\},\quad R_4 := \sup_{0\le t\le v}\{\varphi_4(t)\overline{\kappa}(2\mathbb{E}\|\xi\|^2+2C)\},$$

$$R_5 := \frac{\alpha^2 v^{2\alpha-1}}{2\alpha-1},$$

$$R_6 := 2(\frac{8v\epsilon^2}{(1-L_3)^2}+\frac{64\epsilon}{(1-L_3)^2}+\frac{8\epsilon^2R_5}{(1-L_3)^2})Cb,$$

here we have used the concavity of the function $\overline{\kappa}(\cdot)$.

In terms of Gronwall's inequality, we obtain

$$\begin{aligned}&\mathbb{E}(\sup_{0\le t\le v}|X_\epsilon(t)-Z_\epsilon(t)|^2)\\ &\le \frac{8v\epsilon}{(1-L_3)^2}[v\epsilon R_1+4R_2+\epsilon R_3R_5+4R_4+(v\epsilon+8+\epsilon R_5)aC]\exp(vR_6).\end{aligned}$$

Choose $\beta\in(0,1)$ and $L>0$ such that for every $t\in[0,L\epsilon^{-\beta}(1-L_3)^2]\subseteq[0,T]$, we have

$$\mathbb{E}(\sup_{t\in[0,L\epsilon^{-\beta}(1-L_3)^2]}|X_\epsilon(t)-Z_\epsilon(t)|^2)\le R_8L\epsilon^{1-\beta},$$

where

$R_8=8[L\epsilon^{-\beta}(1-L_3)^2\epsilon R_1+4R_2+\epsilon R_3R_5+4R_4+(L\epsilon^{-\beta}(1-L_3)^2\epsilon+8+\epsilon R_5)aC]\exp(L\epsilon^{-\beta}(1-L_3)^2R_6)$.

Consequently, for any number $\delta_1>0$, we can choose $\epsilon_1\in(0,\epsilon_0]$ such that for every $\epsilon\in(0,\epsilon_1]$ and $t\in[0,L\epsilon^{-\beta}(1-L_3)^2]$, the inequality

$$\mathbb{E}(\sup_{t\in[0,L\epsilon^{-\beta}(1-L_3)^2]}|X_\epsilon(t)-Z_\epsilon(t)|^2)\le\delta_1,$$

holds. This completes the proof.

Remark 3.4. Let $\delta(t)=\tau$, τ a positive constant, then equation (1.1) will reduce to the fractional neutral stochastic equation with poisson jumps ([24]). Noting that our results generalize and improve those of [24].

Remark 3.5. Let $\delta(t)=\tau$, and ignore the Lévy noise and neutral term, then equation (1.1) will reduce to the fractional stochastic differential equation with time delays([12]). On the other hand, let $\delta(t)=\tau$, and ignore the fractional and neutral term, then equation (1.1) will reduce to the stochastic differential equation with Lévy noise ([20]). Hence our results also generalize and improve those of [12], [20].

Remark 3.6. Let $\delta(t)=0$, and ignore the Lévy noise and neutral term, then equation (1.1) will reduce to the fractional stochastic differential equation without time delays([25]). Hence our results also generalize and improve those of [25].

Remark 3.7. Theorem 3.3 means the convergence in probability of the original solution $X_\epsilon(t)$ and the averaged solution $Z_\epsilon(t)$.

4. Example

Example 4.1. Consider the following one-dimensional neutral stochastic fractional differential equation driven by Lévy process with variable delay

$$\begin{aligned} d[X_\epsilon(t) - \frac{1}{2}\sin(X_\epsilon(t-1))] =& \epsilon[-X_\epsilon(t) + \frac{1}{2}X_\epsilon(t-1)\cos(t)]dt \\ &+ \sqrt{\epsilon}\lambda dB(t) + \epsilon(dt)^\alpha + \sqrt{\epsilon}\int_{|y|<1} y^2\nu(dy)dt, \end{aligned}$$

with initial value $X_\epsilon(t) = t+1$, $t \in [-1, 0]$, Lévy measure ν satisfies $\nu(dy) = \frac{1}{1+y^2}dy$ and $\lambda \in \mathbb{R}$, here

$$u(x,z,t) = -x + \frac{1}{2}z\cos(t), \quad b(x,z,t) = \lambda, \quad \sigma(x,z,t) = 1, \quad h(x,z,y,t) = y^2.$$

Let

$$\overline{u}(x,z) = \frac{1}{\pi}\int_0^\pi u(x,z,t)dt = -x, \ \overline{b}(x,z) = \lambda, \ \overline{\sigma}(x,z) = 1, \ \overline{h}(x,z,y) = y^2.$$

Hence, we have the corresponding averaged neutral stochastic fractional differential equation driven by Lévy process with variable delay as follows

$$\begin{aligned} d[Z_\epsilon(t) - \frac{1}{2}\sin(Z_\epsilon(t-1))] =& -\epsilon Z_\epsilon(t)dt + \sqrt{\epsilon}\lambda dB(t) \\ &+ \epsilon(dt)^\alpha + \sqrt{\epsilon}\int_{|y|<1} y^2\nu(dy)dt. \end{aligned}$$

When $t \in [0, 1]$, we have

$$\begin{aligned} d[Z_\epsilon(t) - \frac{1}{2}\sin(t)] &= -\epsilon Z_\epsilon(t)dt + \sqrt{\epsilon}\lambda dB(t) \\ &\quad + \epsilon(dt)^\alpha + \sqrt{\epsilon}\int_{|y|<1} y^2\nu(dy)dt \\ &= -\epsilon Z_\epsilon(t)dt + \sqrt{\epsilon}\lambda dB(t) + \epsilon(dt)^\alpha + \frac{(4-\pi)\sqrt{\varepsilon}}{2}dt. \end{aligned}$$

Obviously, $Z_\epsilon(t)$ is nothing but the following well-known mean-reverting Ornstein-Uhlenbeck process

$$\begin{aligned} Z_\epsilon(t) = Z_\epsilon(0)e^{-\epsilon t} &+ \frac{(4-\pi)\sqrt{\epsilon}}{2}\int_0^t e^{-\epsilon(t-s)}ds + \frac{1}{2}\int_0^t e^{-\epsilon(t-s)}\sin(s)ds \\ &+ \sqrt{\epsilon}\lambda\int_0^t e^{-\epsilon(t-s)}dB(s) + \alpha\epsilon\int_0^t (t-s)^{\alpha-1}e^{-\epsilon(t-s)}ds, \end{aligned}$$

Repeating this procedure over the intervals $[1, 2]$, $[2, 3]$, etc, we obtain the solution on the entire interval $[0, T]$. Noting that

$$\left|\frac{1}{T}\int_0^T u(x,z,t) - \overline{u}(x,z)dt\right|^2 = \left|\frac{1}{T}\int_0^T \cos(t)dt\right|^2\left|\frac{1}{4}z\right|^2,$$

and

$$\left|\frac{1}{T}\int_0^T \cos(t)dt\right|^2 = \frac{\sin^2(T)}{T^2} \leq \frac{1}{T^2}.$$

It is easy to see that Assumptions 2.6-2.7 and 3.1 are satisfied, so Theorem 3.3 holds. Let $Er = [|X_\epsilon(t) - Z_\epsilon(t)|^2]^{\frac{1}{2}}$. Figure 1 and 2 demonstrate a good agreement between solutions of the original equation and the averaged equation, hence, the averaging principle for the considered equation is successfully established.

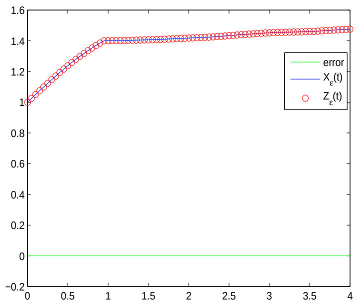

Figure 1: Comparison of the original solution $X_\epsilon(t)$ with the averaged solution $Z_\epsilon(t)$ with $\epsilon = 0.0001, \lambda = 1.5, \alpha = 2/3$

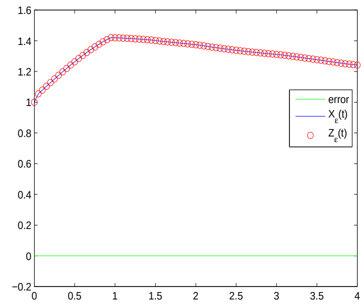

Figure 2: Comparison of the original solution $X_\epsilon(t)$ with the averaged solution $Z_\epsilon(t)$ with $\epsilon = 0.001, \lambda = -2, \alpha = 0.7$

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