Stochastic differential equations driven by fractional Brownian motion with locally Lipschitz drift and their implicit Euler approximation

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Abstract

In this paper, a class of one-dimensional stochastic differential equations driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ is studied. The drift term of the equation is locally Lipschitz and unbounded in the neighborhood of the origin. The existence, uniqueness and positivity of the solutions are proved. We estimate moments including the negative power moments. We also develop the implicit Euler scheme, proved that the scheme is positivity preserving and strong convergent, and obtain rate of convergence. Furthermore, we show that our results can be applied to stochastic interest rate models such as mean-reverting stochastic volatility model and strongly nonlinear Aït-Sahalia type model by using Lamperti transformation.

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1 Introduction

In this paper, we shall consider a one-dimensional stochastic differential equation (in short SDE) driven by fractional Brownian motion:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma \mathrm{d}B_t^H, \ X_0 \geq 0, \tag{1.1}$$

where B_t^H is a fractional Brownian motion (fBM for short) with Hurst parameter $H \in (1/2, 1)$ and the drift term $b(t, x)$ is only local Lipschitz in $x \in (0, \infty)$ and unbounded in the neighborhood of 0.

There are many stochastic interest models which are characterized by stochastic differential equations driven by Brownian motion. For example: the well known Cox-Ingersoll-Ross (C-I-R for short) model [9] and Ait-Sahalia model [1]. However, in order to capture the real world more precisely, due to the memory effects of fractional Brownian motion, it would be reasonable to replace Brownian motion by fBM if there are inert investors in this market, see for instance [24]. Additionally, motivated by studying the fractional C-I-R model, a singular fractional SDEs has been discussed in [18] under some conditions. Recently, a general fractional C-I-R with Hurst parameter $H \in (0, 1)$ was introduced in [21]. However, some models cannot be covered by the conditions introduced recently in [18], see e.g. the equation (3) in [21] or Example 4.1 and Example 4.2 below. Hence one aim of the present paper is to give more general conditions to cover more stochastic interest rate models by using the Lamperti transformation, even if their coefficients have super-linear growth, see e.g. Example 4.2 for the Ait-Sahalia-type interest rate model for details.

Numerical approximations of SDEs arising from finance are of great interest. For instance, the strong approximation of C-I-R model based on the Euler-type method was shown in [11] and optimal convergence rate was obtained; the strong convergence of Euler-Maruyama (EM) type approximations for Aït-Sahalia type model was given [25]; in [19], the EM approximations for a general mean-reverting stochastic volatility model under regime-switching was presented. For the EM scheme of SDEs with non-Lipschitz coefficients, one can see [11, 25, 5, 13] and references therein. However, the numerical issues for SDEs driven by fBM have not been well studied, comparing with SDEs driven by Brownian motion. Recently, the authors in [15] obtained optimal strong convergence rate of backward Euler scheme for C-I-R model driven by fBM. For more details on numerical scheme for fractional SDEs, we refer reader to [15, 16, 17, 22] and references therein. In the present paper, after a general discussion on (1.1), we investigate the numerical approximation of the solution to this equation when X_0 is positive. The strong convergence of the numerical scheme is obtained. Based on the Lamperti transformation used as in [11, 15, 18], our results can cover more interesting models in mathematical finance, such as mean-reverting stochastic volatility model (Example 4.1):

$$\mathrm{d}Z_t = (a_1 - a_2 Z_t)\mathrm{d}t + \sigma Z_t^{\gamma} \mathrm{d}B_t^H, \ Z_0 > 0, \tag{1.2}$$

where $\gamma \in [1/2, 1)$; and Aït-Sahalia type model (Example 4.2):

$$\mathrm{d}Z_t = \left(a_{-1}Z_t^{-1} - a_0 + a_1 Z_t - a_2 Z_t^r\right)\mathrm{d}t + \sigma Z_t^\rho \mathrm{d}B_t^H, Z_0 > 0, \tag{1.3}$$

where $\rho > 1$, $r + 1 > 2\rho$ and $r \geq 2 \wedge \rho + 1$. The stochastic integral in these two models is in the sense of pathwise Riemann-Stieltjes integral developed by Zähle in [27]. Replacing fBM by Brownian motion in equations (1.2) and (1.3), the first model was studied in [19] under regime-switching, and the convergence rate is obtained, the second model was studied in [25], where the convergence rate is not clear. Following the study in [11, 15], the positivity preserving implicit Euler-type method is adopted in our paper. Here, not only is the strong convergence shown, but also the convergence rate is obtained. For concrete examples presented above, the convergence order of the mean-reverting stochastic volatility model is the Hurst parameter H up to a logarithmic term, which is an extension of [11]; while the convergence order for the Aït-Sahalia type model is $(2H-1)\left(\frac{1}{\rho-1} \wedge 1\right)$ up to a logarithmic term.

This paper is structured as follows. In Section 2, we shall recall some basic facts on fractional Brownian motion. Section 3 is devoted to general discussions on (1.1), including existence and uniqueness of solutions to the equation; (negative-power) moments and modular of continuity estimates. In Section 4, we shall present our results on the numerical approximations of (1.1) and their applications on concrete examples.

2 Preliminaries

We shall recall some basic facts about fractional Brownian motion. For more details, we refer readers to [6, 23, 26].

Let $B^H = \{B_t^H, t \in [0, T]\}$ be a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, i.e. B^H is a Gaussian process which is centered with the covariance function

$$\mathbb{E}\left(B_t^H B_s^H\right) = R_H(t, s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

For each $t \in [0, T]$, let $\mathcal{F}_t$ be the σ-algebra generated by the random variables $\{B_s^H : s \in [0, t]\}$ and the sets of probability zero. Furthermore, one can show that $\mathbb{E}|B_t^H - B_s^H|^p = C(p)|t-s|^{pH}$ for all $p \geq 1$. As a consequence of the Kolmogorov continuity criterion, B^H has $(H-\epsilon)$-order Hölder continuous paths for all $\epsilon > 0$. Indeed, the studies on the sample path property of fractional Brownian motion, see for instance [26], show that

$$|B_t^H - B_s^H| \leq A|t-s|^H \sqrt{\log\left(1 + (t-s)^{-1}\right)}$$

where A is a random variable depending on H only and there is some $c > 0$ such that $\mathbb{E}e^{cA^2} < \infty$.

Denote by $\mathscr{E}$ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathscr{E}$ with respect to the scalar product

$$\langle I_{[0,t]}, I_{[0,s]}\rangle_{\mathcal{H}} := \alpha_H \int_0^T \int_0^T \mathbb{1}_{[0,t]}(u)\mathbb{1}_{[0,s]}(v)|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v = R_H(t,s),$$

where $\alpha_H = H(2H-1)$. By the bounded linear transformation theorem, the mapping $I_{[0,t]} \mapsto B_t^H$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space associated with B^H. Denote this isometry by $\phi \mapsto B^H(\phi)$.

On the other hand, the covariance kernel $R_H(t,s)$ can be written as

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,r)K_H(s,r)\mathrm{d}r,$$

where K_H is a square integrable kernel given by

$$K_H(t,s) = \frac{s^{1/2-H}}{\Gamma(H-1/2)}\int_s^t r^{H-1/2}(r-s)^{H-3/2}\mathrm{d}r\mathbb{1}_{[0,t]}(s)$$

in which $\Gamma(\cdot)$ is the Gamma function. Using this kernel, we can define a map from $L^2([0,T])$ to the reproducing kernel space $\mathscr{H}$ defined as follows: let

$$\mathscr{H} = \overline{\mathrm{span}\{R_H(t,\cdot) \mid t\in[0,T]\}}^{\langle\cdot,\cdot\rangle_R},$$

where $\langle R_H(t,\cdot), R_H(s,\cdot)\rangle_R = R_H(t,s),\ s,t\in[0,T]$, and for any $\phi\in L^2([0,T])$, define

$$(K_H\phi)(t) = \int_0^t K_H(t,s)\phi(s)\mathrm{d}s,\ t\in[0,T].$$

It has been proved in [4, 10] that K_H is an isomorphism from $L^2([0,T])$ to $\mathscr{H}$.

Now, define the linear operator $K_H^*: \mathscr{E} \to L^2([0,T])$ by

$$(K_H^*\phi)(s) = K_H(T,s)\phi(s) + \int_s^T (\phi(r)-\phi(s))\frac{\partial K_H}{\partial r}(r,s)\mathrm{d}r.$$

By integration by parts, one can see that

$$(K_H^*\phi)(s) = \int_s^T \phi(r)\frac{\partial K_H}{\partial r}(r,s)\mathrm{d}r.$$

It is clear that $\left(K_H^*\mathbb{1}_{[0,t]}\right)(s) = K_H(t,s)\mathbb{1}_{[0,t]}(s)$. K_H^* is the dual operator of K_H in the following sense: for any $\psi\in\mathscr{E}$ and $h\in L^2([0,T])$,

$$\int_0^T (K_H^*\phi)(r)h(t)\mathrm{d}r = \int_0^T \phi(r)(K_Hh)(\mathrm{d}r). \tag{2.1}$$

Due to [3], for all $\phi,\psi\in\mathscr{E}$, it holds that $\langle K_H^*\phi, K_H^*\psi\rangle_{L^2([0,T])} = \langle\phi,\psi\rangle_{\mathcal{H}}$ and then K_H^* can be extended to an isometry between $\mathcal{H}$ and $L^2([0,T])$. Hence, according to [3] again, the process $\{W_t = B^H((K_H^*)^{-1}\mathrm{I}_{[0,t]}), t \in [0,T]\}$ is a Wiener process, and B^H has the following integral representation

$$B_t^H = \int_0^T (K_H^* \mathbb{1}_{[0,t]})(s)\mathrm{d}W_s = \int_0^t K_H(t,s)\mathrm{d}W_s.$$

With linear operators K_H and K_H^* in hand, there exists an isometry from $\mathcal{H}$ to $\mathscr{H}$ defined by the operator $K_H K_H^*$. Then $\mathscr{H}$ can be charactered by $\mathcal{H}$ with the isometry $K_H K_H^*$. It follows from the integral representation of fBM that $\mathscr{H}$ is the fractional version of the Cameron-Martin space. This was shown rigorously in [10]. The Malliavin derivative of the functional of fBM is defined as an $\mathcal{H}$-valued random variable. For more details on the Malliavin calculus for fBM, one can consult [23].

In this paper, the stochastic integral of fractional Brownian motion is defined by the techniques of fractional calculus developed by Zähle in [27]. We cite the following results on the Riemann-Stieltjes integral and chain rule as a proposition for future use.

Proposition 2.1. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $F \in C^1(\mathbb{R})$.*

(1) Suppose $f \in C^\lambda(a,b)$ and $g \in C^\mu(a,b)$, where $C^\lambda(a,b)$ and $C^\mu(a,b)$ are Hölder continuous functions with order λ and μ respectively. If $\lambda + \mu > 1$, then the Riemann-Stieltjes integral $\int_a^b f\mathrm{d}g$ exists.

(2) Suppose $f \in C^\lambda(a,b)$ such that $F' \circ f \in C^\mu(a,b)$ with $\lambda + \mu > 1$. Then

$$F(f(t)) - F(f(s)) = \int_s^t F' \circ f(r)\mathrm{d}f(r), \ \ s,t \in (a,b).$$

Finally, we shall recall a result on the relationship of stochastic integral and the Skorohod integral w.r.t. fractional Brownian motion. Let

$$|\mathcal{H}| = \left\{ \psi \in \mathcal{H} \;\middle|\; \|\psi\|_{|\mathcal{H}|}^2 = \alpha_H \int_0^T \int_0^T |\psi(s)||\psi(t)||t-s|^{2H-2}\mathrm{d}s\mathrm{d}t < \infty \right\}$$

and $|\mathcal{H}| \otimes |\mathcal{H}|$ be the set of all measurable function such that

$$\|\psi\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2 := \alpha_H^2 \int_{[0,T]^4} |\psi(u,s)||\psi(v,t)||u-v|^{2H-2}|t-s|^{2H-2}\mathrm{d}u\mathrm{d}v\mathrm{d}t\mathrm{d}s < \infty.$$

For $p > 1$, we denote by $\mathbb{D}_{|\mathcal{H}|}^{1,p}$ all the random variables u such that $u \in |\mathcal{H}|$ a.s., its Malliavin derivative $Du \in |\mathcal{H}| \otimes |\mathcal{H}|$ a.s., and

$$\mathbb{E}\|u\|_{|\mathcal{H}|}^p + \mathbb{E}\|Du\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^p < \infty.$$

Then we have the following proposition on the link between stochastic integral w.r.t. fBM and the Skorohod integral, see [23, Proposition 5.2.3 and Page 293] and [2].

Proposition 2.2. *Let $H > \frac{1}{2}$, and let $\{u_t\}_{t\in[0,T]}$ be a stochastic process in $\mathbb{D}^{1,2}_{|\mathcal{H}|}$ such that a.s.*

$$\int_0^T \int_0^T |D_s u_t||t-s|^{2H-2}\mathrm{d}t\mathrm{d}s < \infty.$$

Then

$$\int_0^T u_t \mathrm{d}B_t^H = \delta(u) + \alpha_H \int_0^T \int_0^T D_s u_t |t-s|^{2H-2}\mathrm{d}t\mathrm{d}s.$$

For $p > \frac{1}{H}$,

$$\mathbb{E}\left(\sup_{t\in[0,T]} |\delta(u\mathbb{1}_{[0,t]})|^p\right) \le C\left(\mathbb{E}\int_0^T |u_s|^p \mathrm{d}s + \mathbb{E}\int_0^T \left(\int_0^T |D_r u_s|^{\frac{1}{H}}\mathrm{d}r\right)^{pH} \mathrm{d}s\right).$$

Remark 2.1. *The Proposition 5.2.3 in [23] concerns Stratonovich integral w.r.t. the fBM. For a process u_t satisfying the assumption of Proposition 2.2, the same as that of Proposition 5.2.3 in [23], the Stratonovich integral of u w.r.t. the fBM coincides with the forward integrals, see [23, Remark 2 in Page 292]. By the definition of the forward integrals w.r.t. fBM and the link between fractional and stochastic calculus, see [6, Subsection 5.2], the forward integral w.r.t. fBM with $H \in (\frac{1}{2}, 1)$ is an extension of fractional integral introduced in [27]. In particular, if u_t is a λ-Hölder continuous paths with $\lambda + H > 1$, then the integration $\int_0^T u_t \mathrm{d}B_t^H$ coincides with the pathwise Riemann-Stieltjes integral.*

The second claim of this proposition is the maximal inequality for the divergence integral, see (i) in Page 293 in [23] or [2, Theorem 4]. In the following discussion, we shall make use of the notation

$$\int_0^t u_s \delta B_s^H = \delta(u\mathbb{1}_{[0,t]}).$$

3 A study of SDEs driven by fractional Brownian motion

In this section, we shall consider (1.1) following [18]. Fix $T > 0$. We prove the existence and uniqueness of this equation on $[0,T]$ under the following assumptions.

(A1) The drift term $b : [0,T] \times (0,\infty) \to \mathbb{R}$ is continuous and has continuous derivative w.r.t. the second variable. There exists $K \ge 0$ such that

$$b'_x(t,x) \le K,\ x \in (0,\infty), t \in [0,T],$$

where b'_x is the partial derivative of b w.r.t. the second variable.

(A2) There exist $x_1 > 0$, $\alpha > \frac{1}{H} - 1$ and $h_1 > 0$ such that

$$b(t,x) \ge h_1 x^{-\alpha},\ t \in [0,T], x \le x_1.$$

(A3) There are $x_2 > 0$ and $h_2 > 0$ such that

$$b(t,x) \leq h_2(x+1),\ x \geq x_2, t \geq 0.$$

Remark 3.1. The Assumption **(A2)** is similar to (ii) in [18]. The assumptions **(A1)** and **(A3)** are weaker than (i) and (iii) in [18]. For example, let $b(t,x) = a_1 x^{-\gamma} - a_2 x$ with $\gamma \in (1, +\infty)$, $a_1 > 0$ and $a_2 < 0$. It is clear that b satisfies **(A1)** and **(A3)**, while breaks (i) and (iii) in [18].

Under **(A1)-(A3)**, we prove in Theorem 3.1 below that (1.1) has a positive solution on $(0, T]$ a.s. [21] studies positiveness of solutions to fractional C-I-R models with $H \in (0,1)$ under the framework of pathwise Stratonovich integral. Our result covers [21, Theorem 2], where the authors deal with fractional C-I-R model with $H > \frac{1}{2}$. However, we do not discuss positiveness of solutions to (1.1) with $H < \frac{1}{2}$.

The existence and uniqueness of solutions to (1.1) follows from the existence and uniqueness of the equation below:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}w_t,\ X_0 \geq 0, \tag{3.1}$$

where $w \in C^{\beta}([0,T], \mathbb{R})$ for all $T > 0$ with $\beta \in (\frac{1}{2}, H)$ such that $\alpha > \frac{1}{\beta} - 1$. We say f is a β-Hölder continuous function on $[s,t]$ if

$$\|f\|_{s,t,\beta} := \sup_{s \leq s' < t' \leq t} \frac{|f(s') - f(t')|}{(t' - s')^{\beta}} < \infty.$$

Sometimes, we use $\|\cdot\|_{\beta}$ for simplicity's sake. For a continuous function f on $[s,t]$, we define

$$\|f\|_{s,t,\infty} = \sup_{s \leq r \leq t} |f_r|.$$

Our existence and uniqueness theorem for (3.1) reads as follows.

Theorem 3.1. *Assume that* **(A1)**-**(A3)** *hold.*

(1) For all $X_0 > 0$, it holds that the equation (3.1) *has a unique solution X_t and $X \in C^{\beta}([0,T], (0,\infty))$*

(2) For $X_0 = 0$, if there exists $t_0 > 0$ such that $b(t,\cdot)$ is non-increasing on $(0, x_1)$ for all $0 \leq t \leq t_0$, then (3.1) *has a unique solution X_t and $X_t \in (0,\infty)$ for all $t \in [0,T]$.*

Proof. We first prove the uniqueness. Let $X_t^{[1]}$ and $X_t^{[2]}$ be two solutions of equation (3.1) with the same initial values, then

$$X_t^{[1]} - X_t^{[2]} = X_s^{[1]} - X_s^{[2]} + \int_s^t \left(b(r, X_r^{[1]}) - b(r, X_r^{[2]})\right) \mathrm{d}r,\ s \leq t \leq T.$$

Combining this with **(A1)**, we have

$$\mathrm{d}\left(X_t^{[1]} - X_t^{[2]}\right)^2 = 2\left(b(t, X_t^{[1]}) - b(t, X_t^{[2]})\right)\left(X_t^{[1]} - X_t^{[2]}\right)\mathrm{d}t$$

$$\leq 2K\left(X_t^{[1]} - X_t^{[2]}\right)^2 \mathrm{d}t,$$

which yields

$$\left(X_t^{[1]} - X_t^{[2]}\right)^2 \leq \exp{(2Kt)}\,(X_0^{[1]} - X_0^{[2]})^2.$$

Thus, it $X_t^{[1]} - X_t^{[2]} = 0$ for all $t \in [0, T]$.

We assume that $X_0 > 0$. Since $b : [0, T] \times (0, \infty) \to \mathbb{R}$ is continuous and has continuous derivative w.r.t. the second variable, it is clear that (3.1) has a continuous local solution. Next, we shall prove that $X_t \in (0, \infty)$ for all $t \in [0, T]$. Let

$$\tau_0 = \inf\{t \in [0, T] \mid X_t = 0\}, \qquad \tau_n = \inf\{t \in [0, T] \mid X_t \geq n\},\ n \in \mathbb{N},$$

with the convention that $\inf \emptyset := T+$, where $T+$ is an artificial added element larger than T, but smaller than any $a > T$. We shall prove $\tau_0 = T+$ and $\lim\limits_{n\to\infty} \tau_n = T+$.

If $\tau_0 \leq T$, then there is $\hat{\tau}_0 \in (0, \tau_0)$ such that $X_t \leq x_1$ for all $t \in (\hat{\tau}_0, \tau_0]$. Since $b(t, x) > 0$ for $x \in (0, x_1), t \geq 0$ and

$$0 = X_{\tau_0} = X_t + \int_t^{\tau_0} b(s, X_s)\mathrm{d}s + w_{\tau_0} - w_t, \tag{3.2}$$

it follows that

$$X_t \leq |w_{\tau_0} - w_t| \leq \|w\|_\beta (\tau_0 - t)^\beta,\ t \in (\hat{\tau}_0, \tau_0). \tag{3.3}$$

On the other hand, by **(A2)**, (3.2) and (3.3), we have

$$\begin{aligned}\|w\|_\beta(\tau_0 - t)^\beta \geq |w_{\tau_0} - w_t| \geq \int_t^{\tau_0} b(s, X_s)\mathrm{d}s& \\ \geq h_1 \int_t^{\tau_0} X_s^{-\alpha}\mathrm{d}s \geq \frac{h_1}{\|w\|_\beta^\alpha} \int_t^{\tau_0} \frac{1}{(\tau_0 - s)^{\alpha\beta}}\mathrm{d}s,\ t > \hat{\tau}_0.&\end{aligned} \tag{3.4}$$

If $\alpha\beta \geq 1$, then $\int_t^{\tau_0} \frac{1}{(\tau_0 - s)^{\alpha\beta}}\mathrm{d}s = \infty$ which leads to a contradiction. If $\alpha\beta < 1$, then

$$\|w\|_\beta(\tau_0 - t)^\beta \geq \frac{(\tau_0 - t)^{1-\alpha\beta} h_1}{(1 - \alpha\beta)\|w\|_\beta^\alpha},\ t > \hat{\tau}_0. \tag{3.5}$$

This, together with $\alpha > \frac{1}{\beta} - 1$, implies that

$$0 = \lim_{t \to \tau_0^-} (\tau_0 - t)^{\alpha\beta + \beta - 1} \geq \frac{h_1}{\|w\|_\beta^{\alpha+1}} > 0,$$

which is a contradiction. Hence $\tau_0 = T+$.

If $\tau_\infty := \lim_{n\to\infty} \tau_n \le T$, then either there exists $\hat{\tau}_1$ such that $X_{\hat{\tau}_1} = x_2 + X_0$ and $X_t \ge x_2 + X_0$ for all $t \in (\hat{\tau}_1, \tau_\infty)$, or for all $n \in \mathbb{N}$ with $n > x_2 + X_0$ and $\epsilon > 0$ there exits an interval $(\hat{\tau}_1, \hat{\tau}_2) \subset (\tau_\infty - \epsilon, \tau_\infty)$ such that $X_{\hat{\tau}_1} = x_2 + X_0$ and

$$x_2 + X_0 \le \inf_{t\in(\hat{\tau}_1,\hat{\tau}_2)} X_t \le n \le \sup_{t\in(\hat{\tau}_1,\hat{\tau}_2)} X_t.$$

In both cases,

$$\begin{aligned} X_t &= X_{\hat{\tau}_1} + \int_{\hat{\tau}_1}^t b(s, X_s)\mathrm{d}s + w_t - w_{\hat{\tau}_1} \\ &\le x_2 + X_0 + \int_{\hat{\tau}_1}^t h_2(X_s + 1)\mathrm{d}s + w_t - w_{\hat{\tau}_1} \\ &\le x_2 + X_0 + \|w\|_\beta \tau_\infty^\beta + h_2\tau_\infty + h_2 \int_{\hat{\tau}_1}^t X_s \mathrm{d}s. \end{aligned}$$

where we use **(A3)** in the second inequality. It follows from Grönwall's inequality that for all $t \in (\hat{\tau}_1, \hat{\tau}_2)$ or $t \in (\hat{\tau}_1, \tau_\infty)$

$$\begin{aligned} X_t &\le \left(x_2 + X_0 + \|w\|_\beta \tau_\infty^\beta + h_2\tau_\infty\right) \exp\left\{(t - \hat{\tau}_1)h_2\right\} \\ &\le \left(x_2 + X_0 + \|w\|_\beta T^\beta + h_2 T\right) e^{Th_2}. \end{aligned}$$

Taking supremum of the left hand side in the above inequality: for all $t \in (\hat{\tau}_1, \tau_\infty)$ in the first case or for all $t \in (\hat{\tau}_1, \hat{\tau}_2)$, $\epsilon \in (0, 1)$, and $n \ge 1$ for the second case, the left hand side is infinite but the right hand side is a finite constant. This is a contradiction. Hence, $\tau_\infty = T+$.

Finally, we deal with the case $X_0 = 0$. For $n \in \mathbb{N}$, let $X_t^{[n]}$ be the solution of (3.1) with $X_0^{[n]} = 1/n$. For $n, m \in \mathbb{N}$, $n < m$, let $\tau = \inf\{t \in [0, T] \mid X_t^{[n]} = X_t^{[m]}\}$. By the uniqueness, $X_t^{[n]} = X_t^{[m]}$ for all $t \ge \tau$, or $\tau = T+$. It is clear that $X_t^{[n]} > X_t^{[m]}$ if $t < \tau$. Thus, the sequence $\{X_t^{[n]}\}_{n\in\mathbb{N}}$ is non-increasing and nonnegative. Let $n_0 \in \mathbb{N}$ be such that $\frac{1}{n_0} < x_1$, and let $\tau^{n_0} = \inf\{t \in [0, T] \mid X_t^{[n_0]} \ge x_1\}$. Set $X_t = \lim_{n\to\infty} X_t^{[n]}$. Then

$$X_{t\wedge\tau^{n_0}} \le X_{t\wedge\tau^{n_0}}^{[n]} \le X_{t\wedge\tau^{n_0}}^{[n_0]} \le x_1, \ n \ge n_0.$$

Set $b(t, 0) = +\infty$ for $t \in [0, t_0]$. Then

$$b(t, X_t) = \lim_{n\to+\infty} b(t, X_t^{[n]}), \ t \in (0, t_0 \wedge \tau^{n_0}]$$

Since for any $t \in [0, t_0]$, $b(t, x)$ is non-increasing for $x \in (0, x_1)$, the following equality follows from the monotone convergence theorem

$$\lim_{n\to\infty} \int_0^{t\wedge t_0\wedge\tau^{n_0}} b(s, X_s^{[n]})\mathrm{d}s = \int_0^{t\wedge t_0\wedge\tau^{n_0}} b(s, X_s)\mathrm{d}s.$$

Taking into account that $X_t^{[n]}$ satisfies (3.1), we have

$$X_{t\wedge t_0\wedge\tau^{n_0}} = \int_0^{t\wedge t_0\wedge\tau^{n_0}} b(s,X_s)\mathrm{d}s + w_{t\wedge t_0\wedge\tau^{n_0}} - w_0.$$

Moreover, this inequality yields that

$$\int_0^{t\wedge t_0\wedge\tau^{n_0}} b(s,X_s)\mathrm{d}s < \infty.$$

Thus, $b(s,X_s) < \infty$ a.e. $s \in [0, t\wedge t_0\wedge\tau^{n_0}]$. By **(A2)**, $X_s > 0$ a.e. $s \in [0, t\wedge t_0\wedge\tau^{n_0}]$. Starting from any $X_s > 0$ with $s \in (0, t\wedge t_0\wedge\tau^{n_0})$, there exists unique solution to (3.1) which is positive. Thus, $X_s > 0$ for all $s \in (0, t\wedge t_0\wedge\tau^{n_0}]$. According to the proof above, $\{X_t\}_{t\in[0,t\wedge t_0\wedge\tau^{n_0}]}$ can be extended to a solution for all $t > 0$ and $X_t > 0$ for all $t \in (0,T]$.

□

Remark 3.2. It is clear that $X \in C^\beta([s,T])$ for all $0 \le s < T$ only if $X_0 > 0$ or $s > 0$. In [18, Remark 2.2], the authors stated that the solution of (2.1) in [18] is in $C^\beta([0,T])$ even if the initial value is 0. However, we should point out that the positiveness of the initial value is necessary. In fact, the solution X_t with $X_0 = 0$ is not β-Hölder continuous on the interval which contains 0. Otherwise, there is $C > 0$ and $t_1 > 0$ such that $X_t \le Ct^\beta$ for $t \in [0,t_1]$. Letting $\tau_{x_1} = \inf\{t \ge 0 \mid X_t \ge x_1\}$, just as (3.4), it follows from (3.1) and **(A2)** that for $t \le \tau_{x_1}$ we have

$$X_t = X_0 + \int_0^t b(s,X_s)\mathrm{d}s + w_t - w_0 \ge h_1 \int_0^t \frac{1}{C^\alpha s^{\beta\alpha}}\mathrm{d}s - \|w\|_\beta t^\beta. \tag{3.6}$$

As in proof of $\tau_0 = T+$, (3.6) leads to a contradiction if $\alpha\beta \ge 1$. For $\alpha\beta < 1$, it is similar to (3.5) that (3.6) and $X_t \le Ct^\beta$ lead to the following contradiction

$$C + \|w\|_\beta \ge \overline{\lim_{t\to 0^+}} \frac{X_t + \|w\|_\beta t^\beta}{t^\beta} \ge \frac{h_1}{C} \overline{\lim_{t\to 0^+}} t^{1-\alpha\beta-\beta} = \infty.$$

According to this theorem, the stochastic equation (1.1) has a unique pathwise solution. Next, we shall study the Malliavin differentiablity of X_t.

Lemma 3.2. *Assume* **(A1)**, **(A2)** *and* **(A3)** *hold. Let X_t be the solution of* (1.1). *Then for all $t > 0$, $X_t \in \mathbb{D}^{1,2}_{|\mathcal{H}|}$ with*

$$D_s X_t = \sigma \exp\left\{\int_s^t b'_x(r,X_r)\,\mathrm{d}r\right\} \mathbb{1}_{[0,t]}(s),$$

and the law of X_t has density w.r.t. the Lebesgue measure on $\mathbb{R}$.

The proof just follows the line of [18, Theorem 3.3.], and the outline of the proof is presented here for the convenience of readers.

Proof. Let $\epsilon \in (0,1)$, $h \in \mathcal{H}$ with $h_0 = 0$ and

$$X_t^\epsilon = X_0 + \int_0^t b(r, X_r^\epsilon)\mathrm{d}r + \sigma B_t^H + \sigma\epsilon K_H K_H^* h(t).$$

Then by **(A1)**,

$$\begin{aligned} X_t^\epsilon - X_t &= \int_0^t (b(r, X_r^\epsilon) - b(r, X_r))\,\mathrm{d}r + \sigma\epsilon K_H K_H^* h(t) \\ &= \int_0^t b'_x\left(r, X_r^\xi\right)(X_r^\epsilon - X_r)\mathrm{d}r + \sigma\epsilon K_H K_H^* h(t),\ t > 0, \end{aligned}$$

where $X_r^\xi = X_r + \xi_s^\epsilon(X_r^\epsilon - X_r)$ and $\xi_s^\epsilon \in (0,1)$ depends on s and ϵ. This equality, along with (2.1) (see also [6, Lemma 2.1.9]), implies

$$\begin{aligned} X_t^\epsilon - X_t &= \sigma\epsilon \int_0^t \exp\left\{\int_s^t b'_x\left(r, X_r^\xi\right)\mathrm{d}r\right\}(K_H K_H^* h)(\mathrm{d}s) \\ &= \sigma\epsilon \int_0^T K_H^*\left(\exp\left\{\int_\cdot^t b'_x\left(r, X_r^\xi\right)\mathrm{d}r\right\}\mathbb{1}_{[0,t]}(\cdot)\right)(s) K_H^* h(s)\mathrm{d}s. \end{aligned}$$

Since the continuity of $b'_x(t,\cdot)$, **(A1)** and $K_H^* h \in L^2([0,T])$, it follows from the dominated convergence theorem that the limit

$$\begin{aligned} \lim_{\epsilon\to 0^+} \frac{X_t^\epsilon - X_t}{\epsilon} &= \sigma \int_0^T K_H^*\left(\exp\left\{\int_\cdot^t b'_x\left(r, X_r\right)\mathrm{d}r\right\}\mathbb{1}_{[0,t]}(\cdot)\right)(s) K_H^* h(s)\mathrm{d}s \\ &= \sigma\left\langle \exp\left\{\int_\cdot^t b'_x\left(r, X_r\right)\mathrm{d}r\right\}\mathbb{1}_{[0,t]}(\cdot), h\right\rangle_{\mathcal{H}} \end{aligned}$$

holds almost sure and in $L^2(\Omega)$. Consequently,

$$D_\cdot X_t = \sigma \exp\left\{\int_\cdot^t b'_x\left(r, X_r\right)\mathrm{d}r\right\}\mathbb{1}_{[0,t]}(\cdot).$$

It is clear that $\|DX_t\|_{\mathcal{H}} > 0$, and $\mathbb{E}\|DX_t\|_{\mathcal{H}}^2 < \infty$ follows from **(A1)**. Then the existence of density w.r.t. the Lebesgue measure follows from the classical result of Malliavin calculus, see e.g. [23, Theorem 2.1.2 or Theorem 2.1.3]. □

Next, we shall study the moment estimates of solutions to (1.1). To this end, we introduce the following assumption.

(A2') The condition **(A2)** holds with $\alpha \geq 1$. There exist $\theta > 0$ and $h_3 > 0$ such that

$$b(t,x) \leq h_3(1 + x + x^{-\theta}),\ t \in [0,T], x > 0. \tag{3.7}$$

It should be noted that $\theta \geq \alpha$ by **(A2)** and (3.7), and **(A2')** implies **(A3)**. This assumption is used for positive moment estimate. To give the negative moment estimate, we introduce the following

(A3') there exists a $q > 0$ and $h_4 > 0$ such that

$$(b(t,x))^- \leq h_4(1 + x^q), \;\; s \in [0,T], x > 0 \tag{3.8}$$

where $(b(t,x))^-$ denotes the negative part of $b(t,x)$.

We first consider the negative moments for the solution to (1.1).

Lemma 3.3. *Assume* **(A1)**, **(A2')** *and* **(A3')**. *Let X_t be a solution to* (1.1) *with $X_0 > 0$.*

(1) Suppose $\alpha = 1$. Then for $p \geq 1$ and

$$h_1 \geq ((p+1) \vee q) H T^{2H-1} e^{KT}, \tag{3.9}$$

one has

$$\sup_{s\in[0,T]} \mathbb{E} X_s^{-p} < \infty. \tag{3.10}$$

If (3.9) *holds with p replaced by $2(p+2)$, then*

$$\mathbb{E} \sup_{s\in[0,T]} X_s^{-p} < \infty.$$

(2) Suppose $\alpha > 1$. Then for all $p > 0$,

$$\mathbb{E} \sup_{s\in[0,T]} X_s^{-p} < \infty.$$

Proof. We divide the proof into two steps

Step (i). We first prove (3.10). In fact, due to the Hölder inequality, we only need to prove the claim for large p. Thus we assume that $p+1 \geq q$. Since X_t is β-Hölder continuous for $\beta < H$, applying Proposition 2.1, Proposition 2.2 and Lemma 3.2, we obtain that

$$\begin{aligned}
(X_t + \epsilon)^{-p} &= (X_0 + \epsilon)^{-p} - p \int_0^t \frac{b(s, X_s)}{(\epsilon + X_s)^{p+1}} \mathrm{d}s - \sigma p \int_0^t (\epsilon + X_s)^{-(p+1)} \mathrm{d}B_s^H \\
&\leq (X_0 + \epsilon)^{-p} - p \int_0^t \frac{b(s, X_s)}{(\epsilon + X_s)^{p+1}} \mathrm{d}s - \sigma p \int_0^t (\epsilon + X_s)^{-(p+1)} \delta B_s^H \\
&\quad + \sigma p(p+1)\alpha_H \int_0^t \int_0^s \frac{D_r X_s |s-r|^{2H-2}}{(\epsilon + X_s)^{p+2}} \mathrm{d}r \mathrm{d}s \\
&\leq (X_0 + \epsilon)^{-p} - p \int_0^t \frac{b(s, X_s) X_s - \sigma^2 (p+1) H s^{2H-1} e^{Ks}}{(\epsilon + X_s)^{p+2}} \mathrm{d}s \\
&\quad - \sigma p \int_0^t (\epsilon + X_s)^{-(p+1)} \delta B_s^H.
\end{aligned}$$

Let

$$\tilde{x}_1 = x_1 \wedge 1 \wedge \left(\frac{e^{-KT} h_1}{\sigma^2 (p+1) H T^{2H-1}} \right)^{\frac{1}{\alpha - 1}} \mathbb{1}_{[\alpha>1]} + (x_1 \wedge 1) \mathbb{1}_{[\alpha=1]}.$$

Then

$$-\frac{b(s,x)}{(\epsilon+x)^{p+2}} \leq -\mathbb{1}_{[x\leq\tilde{x}_1]}\frac{h_1}{x^\alpha(\epsilon+x)^{p+2}} + \mathbb{1}_{[x\geq\tilde{x}_1]}\frac{h_4(1+x^q)}{(\epsilon+x)^{p+2}}$$
$$\leq h_4\left(\frac{1}{\tilde{x}_1^{p+2}} + \frac{1}{\tilde{x}_1^{p+2-q}}\right)$$

and

$$\begin{aligned}
&-\frac{b(s,x)x - \sigma^2(p+1)Hs^{2H-1}e^{\int_0^s K_u^+ \mathrm{d}u}}{(\epsilon+x)^{p+2}} \\
&\leq -\frac{h_1 x^{-(\alpha-1)} - \sigma^2(p+1)Hs^{2H-1}e^{Ks}}{(\epsilon+x)^{p+2}}\mathbb{1}_{[x\leq\tilde{x}_1]} \\
&\quad + \frac{h_4(1+x^q)x + (p+1)Hs^{2H-1}e^{Ks}}{(\epsilon+x)^{p+2}}\mathbb{1}_{[x\geq\tilde{x}_1]} \\
&\leq -\frac{h_1\tilde{x}_1^{-\alpha+1} - \sigma^2(p+1)Hs^{2H-1}e^{Ks}}{x^{p+2}}\mathbb{1}_{[x\leq\tilde{x}_1]} \\
&\quad + (p+1)Hs^{2H-1}e^{Ks}\tilde{x}_1^{-p-2} + h_4\left(\tilde{x}_1^{-(p+1)} + \tilde{x}_1^{-p-1+q}\right).
\end{aligned}$$

Since (3.9) and the definition of $\tilde{x}_1$, there exists $C > 0$ depending on $\tilde{x}_1, p, q, \sigma$ such that

$$(X_t+\epsilon)^{-p} \leq (X_0+\epsilon)^{-p} + C\int_0^t (h_4 + s^{2H-1})\mathrm{d}s - \sigma p\int_0^t (\epsilon+X_s)^{-(p+1)}\delta B_s^H.$$

Taking expectation and letting $\epsilon \to 0$, (3.10) is proved.

Step (ii) We shall complete the rest of the proof. If $\alpha > 1$ or (3.9) holds with p replaced by $2(p+2)$, then

$$\sup_{[0,T]}\mathbb{E}X_t^{-2(p+2)} < \infty.$$

Then $X^{-(p+1)} \in \mathbb{D}^{1,2}_{|\mathcal{H}|}$, (see Page 2 for the definition of $\mathbb{D}^{1,2}_{|\mathcal{H}|}$), due to the following inequality

$$\begin{aligned}
&\int_{[0,T]^2}\int_{[0,T]^2} \mathbb{E}(D_sX_t)X_t^{-(p+2)}(D_vX_u)X_u^{-(p+2)}|u-v|^{2H-2}|t-s|^{2H-2}\mathrm{d}u\mathrm{d}v\mathrm{d}s\mathrm{d}t \\
&\quad \leq \sigma^2 e^{2KT}\sup_{[0,T]}\mathbb{E}X_t^{-2(p+2)}\int_{[0,T]^2}\int_{[0,T]^2}|u-v|^{2H-2}|t-s|^{2H-2}\mathrm{d}u\mathrm{d}v\mathrm{d}s\mathrm{d}t \\
&\quad < \infty.
\end{aligned}$$

By Proposition 2.1, Proposition 2.2 and Lemma 3.2 again, there is some $C > 0$ depending on $\tilde{x}_1, p, q, \sigma$ such that

$$X_t^{-p} \leq X_0^{-p} + C\int_0^t (h_4 + s^{2H-1})\mathrm{d}s - p\int_0^t X_s^{-(p+1)}\delta B_s^H.$$

It follows from the maximal inequality of the Skorohod integral (see e.g. [23, Page 293] or Proposition 2.2) that

$$\begin{aligned}
&\left(\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^r X_s^{-(p+1)}\delta B_s^H\right|^2\right)^{\frac{1}{2}}\\
&\le C\left(\int_0^t \mathbb{E}X_s^{-2(p+1)}\mathrm{d}s+\mathbb{E}\int_0^t\left(\int_0^s (p+1)^{\frac{1}{H}}X_s^{-\frac{p+2}{H}}|D_rX_s|^{\frac{1}{H}}\mathrm{d}r\right)^{2H}\mathrm{d}s\right)^{\frac{1}{2}}\\
&\le C_{p,H}(1+t^H)e^{Kt}\left(\int_0^t \mathbb{E}X_s^{-2(p+2)}\left(1+X_s^2\right)\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}\sup_{s\in[0,T]}X_s^{-p}\le X_0^{-p}+C\int_0^T(h_4+s^{2H-1})\mathrm{d}s&\\
+C_{p,H}(1+T^H)e^{KT}\left(\int_0^T \mathbb{E}X_s^{-2(p+2)}\left(1+X_s^2\right)\mathrm{d}s\right)^{\frac{1}{2}}&,
\end{aligned}$$

which implies the required conclusion. □

If (3.7) holds with $b(t,x)$ replaced by $|b(t,x)|$, then we can obtain moment estimates of $|X|_{0,T,\infty}$ by applying [12, Theorem 3.1] to $X_t^{1+\theta}$. However, if (3.8) holds, that is we allow that $|b(t,x)|$ has super-linear growth near infinity, then the following lemma can not be covered by [12]. For $g\in C([0,T],\mathbb{R}^d)$, we denote by $\mathbb{M}_{g,T}(\cdot)$ the modulus of continuity of g on $[0,T]$, i.e.

$$\mathbb{M}_{g,T}(h)=\sup_{0\le s,t\le T,|s-t|\le h}|g_t-g_s|.$$

We now give the estimates for the positive moments of the solution to (1.1).

Lemma 3.4. *Assume* **(A1)**, **(A2')** *and* **(A3')**. *Let* $\{X_t\}_{t\in[0,T]}$ *be a solution of* (1.1) *with* $X_0>0$.

(1) If $\alpha>1$, *then for any* $p>0$, *we have*

$$\mathbb{E}\|X\|_{0,T,\infty}^p<\infty, \tag{3.11}$$

and

$$\left(\mathbb{E}\mathbb{M}_{X,T}^p(h)+\mathbb{E}\mathbb{M}_{X^{-1},T}^p(h)\right)^{\frac{1}{p}}\le C_{p,T}\left(h+h^H\sqrt{\log(1+1/h)}\right). \tag{3.12}$$

(2) If $\alpha=1$, *then for* $p>0$, *there exists* $T>0$ *such that* (3.11) *and* (3.12) *hold.*

Proof. Suppose $\alpha > 1$. We first prove that

$$\mathbb{E}\left(X_t^p + \int_0^t X_s^p \mathrm{d}s\right) < \infty,\ t \geq 0, p > 0. \tag{3.13}$$

By (2) of Proposition 2.1, Lemma 3.2 and Proposition 2.2, for any $n > 0$

$$\begin{aligned}
&\frac{nX_t^p}{n+X_t^p} - \frac{nX_0^p}{n+X_0^p} \\
&\quad = \int_0^t \frac{pn^2 X_s^{p-1}}{(n+X_s^p)^2} b(s,X_s)\mathrm{d}s + \int_0^t \frac{\sigma pn^2 X_s^{p-1}}{(n+X_s^p)^2}\mathrm{d}B_s^H \\
&\quad \leq \int_0^t \frac{pn^2 h_3 X_s^{p-1}(1+X_s+X_s^{-\theta})}{(n+X_s^p)^2}\mathrm{d}s + \int_0^t \frac{\sigma pn^2 X_s^{p-1}}{(n+X_s^p)^2}\delta B_s^H \\
&\qquad + \int_0^t\int_0^s \frac{\alpha_H \sigma pn^2 X_s^{p-2}(n(p-1)^+ - (p+1)X_s^p)}{(n+X_s^p)^3} D_r X_s |r-s|^{2H-2}\mathrm{d}r\mathrm{d}s \\
&\quad \leq \int_0^t \left(\frac{2pnh_3 X_s^p}{n+X_s^p} + ph_3(1+X_s^{-\theta})\right)\mathrm{d}s + \int_0^t \frac{\sigma pn^2 X_s^{p-1}}{(n+X_s^p)^2}\delta B_s^H \\
&\qquad + C_H e^{Kt}\int_0^t \frac{\sigma^2 p(p-1)^+ n X_s^{p-2}}{n+X_s^p} s^{2H-1}\mathrm{d}s \\
&\quad \leq C_{t,p,K,H,\sigma}\int_0^t \left(\frac{nX_s^p}{n+X_s^p}(h_3 + s^{2H-1})\right)\mathrm{d}s + \int_0^t \frac{\sigma pn^2 X_s^{p-1}}{(n+X_s^p)^2}\delta B_s^H \\
&\qquad + C_{t,p,K,H,\sigma}\int_0^t \left(h_3(1+X_s^{-\theta}) + (p-1)^+ s^{2H-1} X_s^{-2}\right)\mathrm{d}s,
\end{aligned}$$

where $C_{t,p,K,H,t}$ is locally bounded in t. Then it follows from the Grönwall lemma and Lemma 3.3 that

$$\mathbb{E}\frac{nX_t^p}{n+X_t^p} \leq X_0^p + e^{C_{t,p,K,H,\sigma}t}\int_0^t \mathbb{E}\left(h_3(1+X_s^{-\theta}) + \frac{(p-1)^+ s^{2H-1}}{X_s^2}\right)\mathrm{d}s < \infty,$$

which implies (3.13) by letting $n \to \infty$ and using Fatou's lemma.

Next, we shall prove that

$$\mathbb{E}\sup_{s\in[0,t]} X_s^p < \infty,\ t \geq 0, p > 0. \tag{3.14}$$

Indeed, by chain rule, (3.13) and Lemma 3.2, we have $X_t^{p-1} \in \mathbb{D}_{|\mathcal{H}|}^{1,2}$ and

$$\begin{aligned}
X_t^p &= X_0^p + \int_0^t X_s^{p-1} b(s,X_s)\mathrm{d}s + \sigma\int_0^t X_s^{p-1}\mathrm{d}B_s^H \\
&\leq X_0^p + \int_0^t h_3(X_s^{p-1} + X_s^p + X_s^{p-\theta-1})\mathrm{d}s + \sigma\int_0^t X_s^{p-1}\delta B_s^H \\
&\quad + \sigma(p-1)\alpha_H\int_0^t\int_0^s X_s^{p-s} D_r X_s |r-s|^{2H-2}\mathrm{d}r\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\le X_0^p + C\int_0^t h_3(1+X_s^p)\mathrm{d}s + |\sigma|\left|\int_0^t X_s^{p-1}\delta B_s^H\right| \\
&\quad + C_{H,\sigma,p}e^{Kt}\int_0^t X_s^{p-2}|s|^{2H-1}\mathrm{d}s. \qquad (3.15)
\end{aligned}
$$

The maximal inequality of Skorohod integral yields that the following inequality holds

$$
\begin{aligned}
&\left(\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s X_r^{p-1}\delta B_r^H\right|^2\right)^{\frac{1}{2}} \\
&\quad\le C\left(\int_0^t \mathbb{E}X_r^{2(p-1)}\mathrm{d}r + \mathbb{E}\int_0^t\left(\int_0^r (p-1)^{\frac{1}{H}}X_r^{\frac{p-2}{H}}|D_uX_r|^{\frac{1}{H}}\mathrm{d}u\right)^{2H}\mathrm{d}r\right)^{\frac{1}{2}} \\
&\quad\le C_{H,p}(1+t^H)e^{Kt}\left(\int_0^t \mathbb{E}X_r^{2(p-2)}\left(1+X_s^2\right)\mathrm{d}r\right)^{\frac{1}{2}}. \qquad (3.16)
\end{aligned}
$$

Combining (3.15) and (3.16) with (3.13), we get (3.14).

Next, we shall give the estimates of modulus of continuity. By **(A2')** and **(A3')**, we have

$$
|b(s,x)| \le (h_4\vee h_3)(1+x^q+x^{-\theta}) \equiv \tilde{h}(1+x^q+x^{-\theta}).
$$

Then for any $t>s\ge 0$,

$$
\begin{aligned}
|X_t-X_s| &\le \int_s^t |b(r,X_r)|\mathrm{d}r + |\sigma(B_t^H-B_s^H)| \\
&\le \int_s^t \tilde{h}\left(1+X_r^q+X_r^{-\theta}\right)\mathrm{d}r + |\sigma|\mathbb{M}_{B^H,T}((t-s)) \\
&\le \tilde{h}\left(1+\|X\|_{s,t,\infty}^q+\|X^{-1}\|_{s,t,\infty}^\theta\right)(t-s) \\
&\quad + |\sigma|\mathbb{M}_{B^H,T}((t-s)), \qquad (3.17)
\end{aligned}
$$

which implies for any $p>0$,

$$
\mathbb{E}\mathbb{M}_{X,T}(h)^p \le C_{T,p}\left(1+\mathbb{E}\|X\|_{0,T,\infty}^{pq}+\mathbb{E}\|X^{-1}\|_{0,T,\infty}^{\theta p}\right)h^p + C_p|\sigma|^p\mathbb{E}(\mathbb{M}_{B^H,T}(h))^p.
$$

It follows from $\alpha>1$, Lemma 3.3 and the modulus of continuity of B^H (see e.g. [26, Theorem 4.2] or [20, Theorem 6.3.3]) that

$$
\mathbb{E}\mathbb{M}_{X,T}(h)^p \le C_{p,T}\left\{h^p + h^{pH}\left(\log\left(1+\frac{1}{h}\right)\right)^{\frac{p}{2}}\right\}.
$$

By the Hölder inequality and the following inequality

$$
\sup_{|t-s|\le h,s,t\le T}|X_t^{-1}-X_s^{-1}| \le \sup_{0\le s,t\le T}\left(\frac{1}{X_tX_s}\right)\sup_{|t-s|\le h,s,t\le T}|X_t-X_s|
$$

$$\leq \left(\sup_{0\leq t\leq T} X_t^{-2}\right) \mathbb{M}_{X,T}(h),$$

we get the moment estimate of the modulus of continuity of X^{-1}.

For $\alpha = 1$, one can repeat the argument for $\alpha > 1$, and note that negative power moments in (1) of Lemma 3.3 hold for small T depending on p. □

4 Numerical Approximation

In this section, we shall consider the numerical approximation of the following equation

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma \mathrm{d}B_t^H,\ X_0 > 0. \tag{4.1}$$

The drift term $b(\cdot)$ satisfies **(A1)**, **(A2')** and **(A3')**, and all these conditions are independent of time. To ensure the positivity of the numerical scheme, we shall use the backward Euler method as in [15]. Moments estimates obtained in the previous section will be used here.

Let $T > 0$, $N \in \mathbb{N}$ such that $h := \frac{T}{N} < (h_3 \vee K)^{-1}$, $t_n = nh$, and let $\Delta B_{n+1}^H = B_{t_{n+1}}^H - B_{t_n}^H$. We introduce the backward Euler scheme. Define $Y_0 = X_0 > 0$, and consider

$$Y_{n+1} = Y_n + b(Y_{n+1})h + \sigma \Delta B_{n+1}^H,\ n \in \mathbb{N} \cup \{0\}. \tag{4.2}$$

The equation (4.2) has a unique positive solution Y_{n+1}, $n \geq 0$. To prove this, we consider the following function

$$U(x) = b(x)h - x,\ x > 0.$$

By **(A2)** and $h > 0$, $\lim\limits_{x\to 0^+} U(x) = +\infty$. By **(A2')** and $h_3 h < 1$,

$$\overline{\lim_{x\to+\infty}} U(x) \leq \lim_{x\to+\infty} ((h_3 h - 1)x) = -\infty.$$

Moreover, by **(A1)** and $hK < 1$, we have

$$U'(x) = b'(x)h - 1 \leq Kh - 1 < 0.$$

Then for any $c \in \mathbb{R}$, there exists unique x such that $U(x) = c$. Hence, there exists Y_{n+1} such that

$$U(Y_{n+1}) = b(Y_{n+1})h - Y_{n+1} = -Y_n - \sigma \Delta B_{n+1}^H,$$

which is equivalent to (4.2). Let

$$Y_t^h = \frac{t_{n+1} - t}{t_{n+1} - t_n} Y_n + \frac{t - t_n}{t_{n+1} - t_n} Y_{n+1},\ t_n \leq t \leq t_{n+1}.$$

For a random variable ξ, we denote $\|\xi\|_p = (\mathbb{E}|\xi|^p)^{\frac{1}{p}}$. In addition to **(A1)**, **(A2')** and **(A3')**, we shall impose the following assumptions.

(H1) The drift term $b \in C^2(\mathbb{R})$, and there are nonnegative constants p_1, p_2 and $C > 0$ such that

$$|b'(x)| + |b''(x)| \le C(1 + x^{p_1} + x^{-p_2}), \ x > 0. \tag{4.3}$$

Our result on numerical approximation of (4.1) reads as follows.

Theorem 4.1. *Assume* **(A1)***,* **(A2')***,* **(A3')** *and* **(H1)** *hold. Let* $h_0 = (h_3 \vee K)^{-1}$ *and* $h < h_0$.
(1) If $\alpha > 1$*, then*

$$\mathbb{E} \sup_{0 \le n \le N-1} |X_{t_{n+1}} - Y_{n+1}|^p \le C_{T,X_0,\theta,H,p,B} h^{pH}, \tag{4.4}$$

$$\mathbb{E} \sup_{t \in [0,T]} \left|X_t - Y_t^h\right|^p \le C_{T,X_0,\theta,H,p,B} h^{pH} \left(\log\left(1 + \frac{1}{h}\right)\right)^{p/2}. \tag{4.5}$$

(2) If $\alpha = 1$*, then for* $p > 0$*, there is* $T > 0$ *such that* (4.4) *and* (4.5) *hold.*

Proof. We only prove the claim for $\alpha > 1$. For $\alpha = 1$, the negative power moments estimates hold for T depending on the given $p > 0$ (see Lemma 3.3). Then for T small enough, the arguments for $\alpha > 1$ work well in the small interval, and the claim then can be obtained.

(1) We first prove (4.4). It follows from the definition of Y_{n+1} and the mean value theorem that that

$$\begin{aligned} X_{t_{n+1}} - Y_{n+1} &= X_{t_n} - Y_n + \int_{t_n}^{t_{n+1}} b(X_s)\mathrm{d}s - b(Y_{n+1})h \\ &= X_{t_n} - Y_n + \big(b(X_{t_{n+1}}) - b(Y_{n+1})\big)\, h \\ &\quad - \int_{t_n}^{t_{n+1}} \big(b(X_{t_{n+1}}) - b(X_s)\big)\, \mathrm{d}s \\ &= X_{t_n} - Y_n + b'(Y_{n+1} + \xi_{n+1}(X_{t_{n+1}} - Y_{n+1}))h(X_{t_{n+1}} - Y_{n+1}) \\ &\quad - \int_{t_n}^{t_{n+1}} \left(\int_s^{t_{n+1}} b'(X_r)b(X_r)\mathrm{d}r + \sigma \int_s^{t_{n+1}} b'(X_r)\mathrm{d}B_r^H\right) \mathrm{d}s, \end{aligned} \tag{4.6}$$

where $\xi_{n+1} \in (0,1)$. By **(A1)** and letting

$$\Delta_{n+1} = b'(Y_{n+1} + \xi_{n+1}(X_{t_{n+1}} - Y_{n+1})),$$

we have

$$1 - \Delta_{n+1}h \ge 1 - Kh > 0$$

holds for $h < h_0$. On the other hand, it follows from the Fubini theorem that

$$\int_{t_n}^{t_{n+1}} \left(\int_s^{t_{n+1}} b'(X_r)b(X_r)\mathrm{d}r + \sigma \int_s^{t_{n+1}} b'(X_r)\mathrm{d}B_r^H\right) \mathrm{d}s$$

$$= \int_{t_n}^{t_{n+1}} (r - t_n)\, b'(X_r) b(X_r) \mathrm{d}r + \sigma \int_{t_n}^{t_{n+1}} (r - t_n) b'(X_r) \mathrm{d}B_r^H.$$

Substituting this into (4.6), letting $\Upsilon_{n+1} = X_{t_{n+1}} - Y_{n+1}$ and

$$Q_{n+1} = - \int_{t_n}^{t_{n+1}} (r - t_n)\, b'(X_r) b(X_r) \mathrm{d}r - \sigma \int_{t_n}^{t_{n+1}} (r - t_n) b'(X_r) \mathrm{d}B_r^H,$$

we get that

$$\Upsilon_{n+1} = (1 - \Delta_{n+1} h)^{-1} \Upsilon_n + (1 - \Delta_{n+1} h)^{-1} Q_{n+1}.$$

Consequently,

$$\Upsilon_{n+1} = \sum_{i=1}^{n+1} Q_i \prod_{k=i}^{n+1} (1 - \Delta_k h)^{-1} =: \sum_{i=1}^{n+1} Q_i \rho_i. \tag{4.7}$$

Next, we shall estimate the right hand side of the above equality. Since

$$\prod_{k=i}^{n+1} (1 - \Delta_k h)^{-1} \le (1 - Kh)^{-n+i-2} \le e^{(n+1) \log \frac{1}{1-Kh}} \le e^{\frac{(n+1)Kh}{1-Kh}} = e^{\frac{KT}{1-Kh}},$$

it follows from (4.7) that

$$\mathbb{E} \sup_{1 \le n \le N} |\Upsilon_n|^p \le C_{T,K} \mathbb{E} \left(\sum_{i=1}^{N} |Q_i| \right)^p. \tag{4.8}$$

By the definition of Q_i, there are two integrals to be estimated. For the ordinary integral, it follows from **(A2')**, **(A3')** and **(H1)** that

$$\begin{aligned}
\left\| \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (t_{i-1} - r) b'(X_r) b(X_r) \mathrm{d}r \right\|_p &\le C_{K,T} h \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \| b'(X_r) b(X_r) \|_p \mathrm{d}r \\
&\le C_T h \int_0^T \| 1 + X_r^{-(\theta + p_1)} + X_r^{q \vee (1+p_2)} \|_p \mathrm{d}r \\
&\le C_{T,\theta,X_0,p_1,p_2,q,K} h.
\end{aligned} \tag{4.9}$$

For the stochastic integration, by Proposition 2.2 or [23, Theorem 5.2.3]

$$\begin{aligned}
&\sum_{i=1}^{N} \left| \int_{t_{i-1}}^{t_i} (r - t_{i-1}) b'(X_r) \mathrm{d}B_r^H \right| \\
&\quad \le \sum_{i=1}^{N} \left| \int_0^T (t - t_{i-1}) b'(X_t) \mathbb{1}_{(t_{i-1}, t_i]} \delta B_t^H \right| \\
&\qquad + \sum_{i=1}^{N} \left| \int_0^T \int_0^T (r - t_{i-1}) b''(X_r) D_s X_r |s - r|^{2H-2} \mathbb{1}_{(t_{i-1}, t_i]}(r) \mathbb{1}_{(0,r]}(s) \mathrm{d}s \mathrm{d}r \right|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \left| \int_0^T (r - t_{i-1}) b'(X_r) \mathbb{1}_{(t_{i-1}, t_i]}(r) \delta B_r^H \right| \\
&\quad + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_0^r (r - t_{i-1}) \left| b''(X_r) \right| |D_s X_r| |s - r|^{2H-2} \mathrm{d}s \mathrm{d}r \\
&=: I_1 + I_2.
\end{aligned} \tag{4.10}
$$

For I_2, it follows from (4.3) that

$$
\begin{aligned}
\|I_2\|_p &\leq \left\| \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_0^r (r - t_{i-1}) |b''(X_r)| |D_s X_r| |s - r|^{2H-2} \mathrm{d}s \mathrm{d}r \right\|_p \\
&\leq Ch \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_0^r \|X_r^{-p_1} + X_r^{p_2}\|_p e^{\int_s^r K \mathrm{d}u} |s - r|^{2H-2} \mathrm{d}s \mathrm{d}r \\
&\leq C_{T,K,H} h \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|X_r^{-p_1} + X_r^{p_2}\|_p r^{2H-1} \mathrm{d}s \mathrm{d}r \\
&\leq C_{T,\theta,H,X_0,K} h.
\end{aligned} \tag{4.11}
$$

For I_1, it follows from Minkowski's inequality that

$$
\begin{aligned}
\|I_1\|_p &= \left\| \sum_{i=1}^{N} \left| \int_0^T (r - t_{i-1}) b'(X_r) \mathbb{1}_{(t_{i-1}, t_i]} \delta B_r^H \right| \right\|_p \\
&\leq \sum_{i=1}^{N} \left\| \int_0^T (r - t_{i-1}) b'(X_r) \mathbb{1}_{(t_{i-1}, t_i]} \delta B_r^H \right\|_p .
\end{aligned}
$$

By [23, Proposition 1.5.8],

$$
\begin{aligned}
&\mathbb{E} \left| \int_0^T (r - t_{i-1}) b'(X_r) \mathbb{1}_{(t_{i-1}, t_i]} \delta B_r^H \right|^p \\
&\quad \leq C_p \Bigg(\left(\int_{(t_{i-1}, t_i]^2} (r - t_{i-1})(s - t_{i-1}) |\mathbb{E} b'(X_r)| |\mathbb{E} b'(X_s)| |r - s|^{2H-2} \mathrm{d}r \mathrm{d}s \right)^{\frac{p}{2}} \\
&\qquad + \mathbb{E} \Big(\int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} \int_0^r \int_0^s (r - t_{i-1})(s - t_{i-1}) |b''(X_r)| |b''(X_s)| \\
&\qquad\qquad \times |D_u X_r| |D_v X_s| |u - v|^{2H-2} |s - r|^{2H-2} \mathrm{d}u \mathrm{d}v \mathrm{d}s \mathrm{d}r \Big)^{\frac{p}{2}} \\
&\quad \leq C_{T,p_1,p_2,K} h^p \left(\int_{(t_{i-1}, t_i]^2} |r - s|^{2H-2} \mathrm{d}r \mathrm{d}s \right)^{\frac{p}{2}} \\
&\qquad + C_{T,K,H} h^p \Big(\int_{(t_{i-1}, t_i]^2} \int_0^r \int_0^s \left(\mathbb{E} \left(1 + X_r^{-p_1} + X_r^{p_2} \right)^{\frac{p}{2}} \left(1 + X_s^{-p_1} + X_s^{p_2} \right)^{\frac{p}{2}} \right)^{\frac{2}{p}} \\
&\qquad\qquad \times |u - v|^{2H-2} |r - s|^{2H-2} \mathrm{d}u \mathrm{d}v \mathrm{d}r \mathrm{d}s \Big)^{\frac{p}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\le C_{T,X_0,H,K}h^{p+Hp} \\
&\quad + C_{T,X_0,K,H,p_1,p_2,p}h^p\left(\int_{(t_{i-1},t_i]^2}\int_{(0,T]^2}|u-v|^{2H-2}|r-s|^{2H-2}\mathrm{d}u\mathrm{d}v\mathrm{d}r\mathrm{d}s\right)^{\frac{p}{2}} \\
&\le C_{T,X_0,K,H,p_1,p_2,p}h^{p+pH}.
\end{aligned}
$$

Thus

$$
\|I_1\|_p \le C_{T,X_0,p,H,K}\sum_{i=1}^N h^{1+H} \le C_{T,X_0,p,H,K}h^H. \tag{4.12}
$$

Substituting (4.9), (4.10), (4.11) and (4.12) into (4.8), we obtain

$$
\mathbb{E}\sup_{0\le n\le N-1}|\Upsilon_{n+1}|^p \le C_{T,X_0,\theta,H,p,q,K,p_1,p_2}h^{pH}.
$$

(2) For $t\in[t_n,t_{n+1}]$,

$$
\begin{aligned}
\left|X_t - Y_t^h\right| &= \Big| -\frac{t-t_n}{h}X_{t_{n+1}} - \frac{t_{n+1}-t}{h}X_{t_n} + X_t + \frac{t-t_n}{h}(X_{t_{n+1}} - Y_{n+1}) \\
&\qquad + \frac{t_{n+1}-t}{h}(X_{t_n} - Y_n)\Big| \\
&\le \left|X_{t_{n+1}} - X_t\right| + |X_{t_n} - X_t| + |\Upsilon_{n+1}| + |\Upsilon_n| \\
&\le 2\int_{t_n}^{t_{n+1}}|b(X_r)|\mathrm{d}r + |B^H_{t_{n+1}} - B^H_t| + |B^H_t - B^H_{t_n}| \\
&\qquad + |\Upsilon_{n+1}| + |\Upsilon_n| \\
&\le C\int_{t_n}^{t_{n+1}}\left(1 + X_r^{q\vee 1} + X_r^{-\theta}\right)\mathrm{d}r + 2\mathbb{M}_{H,T}(h) + |\Upsilon_{n+1}| + |\Upsilon_n| \\
&\le C\left(\left(1 + \|X\|_{0,T,\infty}^{q\vee 1}\right)h + \left(\int_0^T X_r^{-\frac{\theta}{1-H}}\mathrm{d}r\right)^{1-H} h^H\right) \\
&\qquad + 2\mathbb{M}_{B^H,T}(h) + 2\sup_{1\le n\le N}|\Upsilon_n| \\
&=: I_1 + I_2 + I_3.
\end{aligned}
$$

Since I_1, I_2 and I_3 are independent of n and t, we have

$$
\mathbb{E}\sup_{t\in[0,T]}\left|X_t - Y_t^h\right|^p \le 3^{p-1}\left(\mathbb{E}I_1^p + \mathbb{E}I_2^p + \mathbb{E}I_3^p\right).
$$

It follows from Lemma 3.3 and Lemma 3.4 that

$$
\mathbb{E}I_1^p \le C_{K,q,T,X_0,p,\theta,H}h^{pH}.
$$

The inequality (4.4) yields that

$$
\mathbb{E}I_3^p = 2^p\mathbb{E}\sup_{1\le n\le N}|\Upsilon_n|^p \le C_{K,q,T,X_0,p,\theta,H}h^{pH}.
$$

The modulus of continuity of B^H (see e.g. [26, Theorem 4.2] or [20, Theorem 6.3.3]) implies that there is a constant $C_{T,p} > 0$ such that

$$\mathbb{E} I_2^p = 2^p \mathbb{E}\mathbb{M}^p_{B^H,T}(h) \le C_{T,p} h^{pH} \left(\log(1+1/h)\right)^{p/2},$$

Hence,

$$\mathbb{E} \sup_{t\in[0,T]} \left|X_t - Y_t^h\right|^p \le C_{K,q,T,X_0,p,\theta,H} h^{pH} \left(\log(1+1/h)\right)^{p/2}.$$

The proof is therefore complete. □

Before we provide examples to illustrate Theorem 4.1, we need following corollary for future use.

Corollary 4.2. *Assume the conditions of Theorem 4.1 hold.*
(1) If $\alpha > 1$, then for any $l > 0$,

$$\left(\mathbb{E} \sup_{t\in[0,T]} |X_t^l - (Y_t^h)^l|^p\right)^{\frac{1}{p}} \le C h^{H(l\wedge 1)} \left(\log(1+1/h)\right)^{\frac{l\wedge 1}{2}}; \qquad (4.13)$$

for any $l \in (0, \alpha]$,

$$\left(\mathbb{E} \sup_{t\in[0,T]} |X_t^{-l} - (Y_t^h)^{-l}|^p\right)^{\frac{1}{p}} \le C_{p,T} h^{(2H-1)(l\wedge 1)} \left(\log(1+1/h)\right)^{l\wedge 1}; \qquad (4.14)$$

(2) If $\alpha = 1$, then for $l > 0$ and $p > 0$, there is $T > 0$ such that (4.13) *holds; for $l \in (0, 1]$ and $p > 0$, there is $T > 0$ such that* (4.14) *holds.*

Proof. For $l \in (0, 1]$, it follows from Lemma 4.1, the basic inequality

$$|a^l - b^l| \le |a-b|^l, \qquad a > 0, b > 0,$$

and Jessen's inequality that

$$\left(\mathbb{E} \sup_{t\in[0,T]} |X_t^l - (Y_t^h)^l|^p\right)^{\frac{1}{p}} \le \left(\mathbb{E} \sup_{t\in[0,T]} |X_t - Y_t^h|^{lp}\right)^{\frac{1}{p}} \le \left(\mathbb{E} \sup_{t\in[0,T]} |X_t - Y_t^h|^p\right)^{\frac{l}{p}}$$
$$\le C h^{lH} \left(\log(1+\frac{1}{h})\right)^{\frac{l}{2}}.$$

For $l > 1$,

$$\left(\mathbb{E} \sup_{t\in[0,T]} |X_t^l - (Y_t^h)^l|^p\right)^{\frac{1}{p}}$$
$$\le \left(\mathbb{E} \sup_{t\in[0,T]} \left(X_t^{p(l-1)} \vee (Y_t^h)^{p(l-1)}\right) |X_t - Y_t^h|^p\right)^{\frac{1}{p}}$$

$$\leq \left(\mathbb{E}\sup_{t\in[0,T]}\left(X_t^{2p(l-1)}\vee (Y_t^h)^{2p(l-1)}\right)\right)^{\frac{1}{2p}}\left(\mathbb{E}\sup_{t\in[0,T]}|X_t-Y_t^h|^{2p}\right)^{\frac{1}{2p}}$$
$$\leq C_{p,T}h^H\left(\log(1+\frac{1}{h})\right)^{\frac{1}{2}}.$$

Hence, we have proved our first claim.

To consider the negative power approximation, we first give an estimate of Y_n^{-1}. By (4.2), **(A2)** and **(A3')**, there is positive constant C which is independent of n, h such that

$$\begin{aligned}C\left(Y_{n+1}^{-\alpha}h-(Y_{n+1}^q+1)h\right)&\leq b(Y_{n+1})h\\ &\leq |Y_{n+1}-X_{t_{n+1}}|+|X_{t_{n+1}}-X_{t_n}|+|Y_n-X_{t_n}|+|\sigma||B_{n+1}^H|.\end{aligned}$$

Then

$$\begin{aligned}C\sup_{1\leq n\leq N}Y_n^{-\alpha}&\leq C\sup_{1\leq n\leq N}(Y_n^q+1)+\frac{2}{h}\sup_{1\leq n\leq N}|Y_n-X_{t_n}|\\ &\quad+\frac{1}{h}\sup_{1\leq n\leq N}|X_{t_{n+1}}-X_{t_n}|+\frac{|\sigma|}{h}\mathbb{M}^p_{B^H,T}(h)\\ &\leq C\sup_{1\leq n\leq N}(Y_n^q+1)+\frac{2}{h}\sup_{1\leq n\leq N}|Y_n-X_{t_n}|\\ &\quad+\frac{1}{h}\mathbb{M}_{X,T}(h)+\frac{|\sigma|}{h}\mathbb{M}^p_{B^H,T}(h).\end{aligned}\tag{4.15}$$

By (4.4), it is clear that for all $p>0$, we have $\mathbb{E}\sup_{1\leq n\leq N}(Y_n^{qp})<\infty$. Due to Theorem 4.1,

$$\mathbb{E}\sup_{1\leq n\leq N}|Y_n-X_{t_n}|^p\leq h^{Hp}\left(\log(1+\frac{1}{h})\right)^{\frac{p}{2}}.$$

It follows from Lemma 3.4 that

$$\mathbb{E}\mathbb{M}^p_{X,T}(h)\leq h^{Hp}\left(\log(1+\frac{1}{h})\right)^{\frac{p}{2}}.$$

Combining these with (4.15), we obtain

$$\mathbb{E}\sup_{1\leq n\leq N}Y_n^{-p\alpha}\leq C\left(1+h^{(H-1)p}\left(\log(1+\frac{1}{h})\right)^{\frac{p}{2}}\right).\tag{4.16}$$

Then for $l\in[1,\alpha]$, it follows from (4.16) that

$$\begin{aligned}\left(\mathbb{E}\sup_{t\in[0,T]}|X_t^{-l}-(Y_t^h)^{-l}|^p\right)^{\frac{1}{p}}&=\left(\mathbb{E}\sup_{t\in[0,T]}\frac{|X_t^l-(Y_t^h)^l|^p}{X_t^{pl}(Y_t^h)^{pl}}\right)^{\frac{1}{p}}\\ \leq\left(\mathbb{E}\sup_{t\in[0,T]}X_t^{-3pl}\right)^{\frac{1}{3p}}&\left(\mathbb{E}\sup_{t\in[0,T]}(Y_t^h)^{-3pl}\right)^{\frac{1}{3p}}\left(\mathbb{E}\sup_{t\in[0,T]}|X_t^l-(Y_t^h)^l|^{3p}\right)^{\frac{1}{3p}}\end{aligned}$$

$$
\begin{aligned}
&= \left(\mathbb{E} \sup_{t\in[0,T]} X_t^{-3pl}\right)^{\frac{1}{3p}} \left(\mathbb{E} \sup_{1\le n\le N} Y_n^{-3pl}\right)^{\frac{1}{3p}} \left(\mathbb{E} \sup_{t\in[0,T]} |X_t^l - (Y_t^h)^l|^{3p}\right)^{\frac{1}{3p}} \\
&\le C_{p,T} \left(1 + h^{H-1}\left(\log(1+\frac{1}{h})\right)^{\frac{1}{2}}\right) h^H \left(\log(1+\frac{1}{h})\right)^{\frac{1}{2}} \\
&\le C_{p,T} h^{2H-1} \log(1+\frac{1}{h}).
\end{aligned}
$$

For $l < 1$,

$$
\begin{aligned}
\left(\mathbb{E} \sup_{t\in[0,T]} |X_t^{-l} - (Y_t^h)^{-l}|^p\right)^{\frac{1}{p}} &\le \left(\mathbb{E} \sup_{t\in[0,T]} |X_t^{-1} - (Y_t^h)^{-1}|^p\right)^{\frac{l}{p}} \\
&\le C_{p,T,l} h^{(2H-1)l} \left(\log(1+\frac{1}{h})\right)^l .
\end{aligned}
$$

Combining these two cases together, we prove our second conclusion. □

Remark 4.1. *If ϕ is a continuous function on $(0, \infty)$ such that*

$$
|\phi(x) - \phi(y)| \le C|x^l - y^l| \qquad \text{or} \qquad |\phi(x) - \phi(y)| \le C|x^{-l} - y^{-l}|
$$

for l as in Corollary 4.2 and some $C > 0$, then we can approximate $\phi(X_t)$ by $\phi(Y_t^h)$.

For $\alpha = 1$, the convergence of the backward Euler scheme for C-I-R model driven by fractional Brownian motion has been obtained in [15]. Theorem 4.1 and Corollary 4.2 in our paper can also be applied to C-I-R model.

Finally, we apply our results to the two examples introduced in the introduction.

Example 4.1. *We consider the numerical simulation of the following equation*

$$
\mathrm{d}Z_t = (a_1 - a_2 Z_t)\mathrm{d}t + \sigma Z_t^\gamma \mathrm{d}B_t^H, Z_0 > 0 \tag{4.17}
$$

with $\gamma \in (\frac{1}{2}, 1)$, $a_1 > 0$, $a_2 \in \mathbb{R}$ and $\sigma \neq 0$. To study this equation, we consider

$$
\mathrm{d}X_t = (1-\gamma)\left(a_1 X_t^{-\frac{\gamma}{1-\gamma}} - a_2 X_t\right)\mathrm{d}t + \sigma(1-\gamma)\mathrm{d}B_t^H, X_0 = Z_0^{1-\gamma}.
$$

Setting $b(x) = (1-\gamma)a_1 x^{-\frac{\gamma}{1-\gamma}} - a_2(1-\gamma)x$, it is clear that **(A1)**, **(A2')** *and* **(A3')** *hold with $K = a_2^-$, $\theta = \alpha = -\frac{\gamma}{1-\gamma}$, and $q = \mathbb{1}_{[a_2>0]}$. Then this equation has a unique solution by applying Theorem 3.1. Moreover, it follows from the chain rule that $Z_t = X_t^{\frac{1}{1-\gamma}}$ and (4.17) has a unique solution. Let Y_t^h be the numerical solution of X_t and $h_0 = ([(1-\gamma)a_1] \vee a_2^-)^{-1}$. It follows from Corollary 4.2 that*

$$
\left(\mathbb{E} \sup_{t\in[0,T]} |Z_t - (Y_t^h)^{\frac{1}{1-\gamma}}|^p\right)^{\frac{1}{p}} \le C h^H \left(\log(1+1/h)\right)^{\frac{1}{2}}, \ h < h_0.
$$

Example 4.2. *In this example, we investigate the nonlinear Aït-Sahalia-type interest rate model:*

$$\mathrm{d}Z_t = \left(a_{-1}Z_t^{-1} - a_0 + a_1 Z_t - a_2 Z_t^r\right)\mathrm{d}t + \sigma Z_t^\rho \mathrm{d}B_t^H, Z_0 > 0, \tag{4.18}$$

with $r+1 > 2\rho$ and $r \geq 2 \wedge \rho+1 > 2$ and $a_i > 0$, $i = -1,0,1,2$. To investigate the numerical solutions of (4.18)*, we consider*

$$\begin{aligned}\mathrm{d}X_t = (\rho-1)&\left(a_2 X_t^{-\frac{r-\rho}{\rho-1}} - a_1 X_t + a_0 X_t^{\frac{\rho}{\rho-1}} - a_{-1}X_t^{\frac{\rho+1}{\rho-1}}\right)\mathrm{d}t \\ &+ (1-\rho)\sigma \mathrm{d}B_t^H, X_0 = Z_0^{1-\rho}.\end{aligned} \tag{4.19}$$

Set

$$\begin{aligned}b(x) &= (\rho-1)\left(a_2 x^{-\frac{r-\rho}{\rho-1}} - a_1 x + a_0 x^{\frac{\rho}{\rho-1}} - a_{-1}x^{\frac{\rho+1}{\rho-1}}\right) \\ &\equiv b_1 x^{-\frac{r-\rho}{\rho-1}} - b_2 x + b_3 x^{\frac{\rho}{\rho-1}} - b_4 x^{\frac{\rho+1}{\rho-1}}.\end{aligned}$$

Since $\frac{r-\rho}{\rho-1} > 1$, it is clear that **(A1)**, **(A2')** *and* **(A3')** *hold with $\theta = \alpha = \frac{r-\rho}{\rho-1}$, $q = \frac{\rho+1}{\rho-1}$ and some constant K. Then this equation has a unique solution, and so does* (4.18)*. Moreover $Z_t = X_t^{-\frac{1}{\rho-1}}$. It is clear by $\frac{r-\rho}{\rho-1} > 1$ and $\frac{\rho+1}{\rho-1} > \frac{\rho}{\rho-1}$ that for $h > 0$*

$$\lim_{x\to 0^+}(b(x)h - x) = +\infty, \qquad \lim_{x\to+\infty}(b(x)h - x) = -\infty.$$

On the other hand,

$$\begin{aligned}b'(x)h - 1 = &-\frac{b_1(r-\rho)h}{\rho-1}x^{-\frac{r+1}{\rho-1}} - (b_2 h + 1) \\ &+ \frac{b_3\rho h}{\rho-1}x^{\frac{1}{\rho-1}} - \frac{b_4(\rho+1)h}{\rho-1}x^{\frac{2}{\rho-1}}.\end{aligned}$$

Then for $0 < h < \frac{4(\rho-1)b_4(\rho+1)}{b_3^2\rho^2}$, we have

$$\frac{(b_3\rho)^2h^2}{(\rho-1)^2} - 4\left(\frac{b_1(r-\rho)h}{\rho-1}x^{\frac{r+1}{\rho-1}} + b_2 h + 1\right)\frac{b_4(\rho+1)h}{\rho-1} < 0,$$

which implies that $b'(x)h - 1 < 0$. Consequently, **(H1)** *holds. Hence, Theorem 4.1 can be applied to* (4.19) *for $h < \frac{4(\rho-1)b_4(\rho+1)}{b_3^2\rho^2}$.*

Let Y_t^h be the numerical approximation of X_t, since $r+1 > 2\rho$ and $r > 2 \wedge \rho+1$, we have $\frac{1}{\rho-1} \leq \frac{r-\rho}{\rho-1}$. Letting $l = \frac{1}{\rho-1}$ in Corollary 4.2, we then have numerical approximation of Z_t such that

$$\left(\mathbb{E}\sup_{t\in[0,T]}|Z_t - (Y_t^h)^{\frac{1}{\rho-1}}|^p\right)^{\frac{1}{p}} \leq C_{p,T}h^{(2H-1)(\frac{1}{\rho-1}\wedge 1)}\left(\log(1+1/h)\right)^{\frac{1}{\rho-1}\wedge 1},$$

which implies that

$$\lim_{h\to 0^+}\mathbb{E}\sup_{t\in[0,T]}|Z_t - (Y_t^h)^{\frac{1}{\rho-1}}|^p = 0.$$

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