GROUNDSTATES AND INFINITELY MANY HIGH ENERGY SOLUTIONS TO A CLASS OF NONLINEAR SCHRÖDINGER-POISSON SYSTEMS

TOMAS DUTKO, CARLO MERCURI, AND TERESA MEGAN TYLER

ABSTRACT. We study a nonlinear Schrödinger-Poisson system which reduces to the nonlinear and nonlocal PDE

$$-\Delta u + u + \lambda^2 \left(\frac{1}{\omega |x|^{N-2}} \star \rho u^2 \right) \rho(x) u = |u|^{q-1} u \quad x \in \mathbb{R}^N,$$

where $\omega = (N-2)|\mathbb{S}^{N-1}|$, $\lambda > 0$, $q \in (1, 2^* - 1)$, $\rho : \mathbb{R}^N \to \mathbb{R}$ is nonnegative, locally bounded, and possibly non-radial, $N = 3, 4, 5$ and $2^* = 2N/(N-2)$ is the critical Sobolev exponent. In our setting ρ is allowed as particular scenarios, to either 1) vanish on a region and be finite at infinity, or 2) be large at infinity. We find least energy solutions in both cases, studying the vanishing case by means of a priori integral bounds on the Palais-Smale sequences and highlighting the role of certain positive universal constants for these bounds to hold. Within the Ljusternik-Schnirelman theory we show the existence of infinitely many distinct pairs of high energy solutions, having a min-max characterisation given by means of the Krasnoselskii genus. Our results cover a range of cases where major loss of compactness phenomena may occur, due to the possible unboundedness of the Palais-Smale sequences, and to the action of the group of translations.
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1. INTRODUCTION

This paper is devoted to the nonlinear and nonlocal equation

$$(\mathcal{E}) \qquad -\Delta u + u + \lambda^2 \left(\frac{1}{\omega |x|^{N-2}} \star \rho u^2 \right) \rho(x) u = |u|^{q-1} u \quad x \in \mathbb{R}^N,$$

where $\omega = (N-2)|\mathbb{S}^{N-1}|$, $\lambda > 0$, $q \in (1, 2^* - 1)$, $\rho : \mathbb{R}^N \to \mathbb{R}$ is nonnegative, locally bounded, and possibly non-radial, $N = 3, 4, 5$ and $2^* = 2N/(N-2)$ is the critical Sobolev exponent.

We are mainly concerned with the existence and multiplicity of solutions, together with their variational characterisation. This brings us to addressing issues related to a suitable functional setting and its relevant properties, such as those related to separability and compactness. In particular, the variational formulation of $(\mathcal{E})$ requires in general a functional setting different from the standard Sobolev space $H^1(\mathbb{R}^N)$. This is the case if the right hand side of the classical Hardy-Littlewood-Sobolev inequality

$$\text{(HLS)} \qquad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{|x-y|^{N-2}} \, dx \, dy \lesssim ||\rho u^2||^2_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)},$$

is not finite for some $u \in H^1(\mathbb{R}^N)$. In what follows, we consider separately two assumptions on ρ:

(ρ_1) $\rho^{-1}(0)$ has non-empty interior and there exists $\overline{M} > 0$ such that

$$|x \in \mathbb{R}^N : \rho(x) \leq \overline{M}| < \infty;$$

(ρ_2) for every $M > 0$,

$$|x \in \mathbb{R}^N : \rho(x) \leq M| < \infty.$$

These are reminiscent of analogous assumptions considered in the 'local' context of the nonlinear Schrödinger equation by Bartsch and Wang in [6]. In particular, we will refer to (ρ_1) as to the *vanishing case*, and to (ρ_2) as to the *coercive case*, as the latter assumption is verified if ρ is locally bounded such that $\rho(x) \to \infty$ as $|x| \to \infty$, yielding compactness properties in the functional setting which are stronger than in the other case. It is clear that (ρ_1) is compatible with ρ exploding, as well as with ρ having a finite limit at infinity. The latter is a situation which yields loss of compactness phenomena to occur, in part due, in the present subcritical regime, to the action of the group of translations in $\mathbb{R}^N$. In this vanishing case we prove uniform a priori bounds on suitable sequences of approximated critical points, which allow us to construct nontrivial weak limits having a definite variational nature.

To state and prove our results we define $E(\mathbb{R}^N) \subseteq H^1(\mathbb{R}^N)$ as

$$E(\mathbb{R}^N) := \left\{ u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N) \,:\, \|u\|_{E(\mathbb{R}^N)} < +\infty \right\},$$

with norm

$$\|u\|_{E(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,\mathrm{d}x + \lambda \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{|x-y|^{N-2}}\,\mathrm{d}x\,\mathrm{d}y \right)^{1/2} \right)^{1/2}.$$

Variants of the space $E(\mathbb{R}^N)$ have been studied since the work of P.L. Lions [34], see e.g. [46], and [8],[17], [39]. Solutions to $(\mathcal{E})$ are the critical points of the $C^1(E(\mathbb{R}^N);\mathbb{R})$ energy functional

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) + \frac{\lambda^2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{\omega|x-y|^{N-2}}\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{q+1}\int_{\mathbb{R}^N} |u|^{q+1}. \tag{1.1}$$

One could regard $(\mathcal{E})$ as formally equivalent to a nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \lambda^2 \rho(x)\phi u = |u|^{q-1}u, & x \in \mathbb{R}^N, \\ -\Delta\phi = \rho(x)u^2, & x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

In fact, it is well-known from classical potential theory that if $u^2\rho \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ is such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{|x-y|^{N-2}}\,\mathrm{d}x\,\mathrm{d}y < +\infty, \tag{1.3}$$

then

$$\phi_u(x) = \int_{\mathbb{R}^N} \frac{\rho(y)u^2(y)}{\omega|x-y|^{N-2}}\,\mathrm{d}y \tag{1.4}$$

is the unique weak solution in $D^{1,2}(\mathbb{R}^N)$ of the Poisson equation

$$-\Delta\phi = \rho(x)u^2 \tag{1.5}$$

and it holds that

$$\int_{\mathbb{R}^N} |\nabla\phi_u|^2 = \int_{\mathbb{R}^N} \rho\phi_u u^2\,\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{\omega|x-y|^{N-2}}\,\mathrm{d}x\,\mathrm{d}y. \tag{1.6}$$

Here we set

$$D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\},$$

equipped with norm

$$\|u\|_{D^{1,2}(\mathbb{R}^N)} = \|\nabla u\|_{L^2(\mathbb{R}^N)}.$$

By elliptic regularity, the local boundedness of ρ implies that any pair $(u,\phi) \in E(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ solution to (1.2) is such that u and ϕ are both of class $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. In particular, if $u \geq 0$ is nontrivial, it holds that $u,\phi > 0$. Note that $\inf I_\lambda = -\infty$, however it is an easy exercise to see that I_λ is bounded below on the set of its nontrivial critical points by a positive constant. It therefore makes sense to define a solution $u \in E(\mathbb{R}^N)$ to $(\mathcal{E})$ as a *groundstate* if it is nontrivial, and if it holds that $I_\lambda(u) \leq I_\lambda(v)$ for every nontrivial critical point $v \in E(\mathbb{R}^N)$ of I_λ.

Since the classical work of Ambrosetti-Rabinowitz [4], considerable advances have been made in the understanding of several classes of nonlinear elliptic PDE's in the absence of either the so-called Palais-Smale or the Ambrosetti-Rabinowitz conditions, yet achieving in the spirit of [4] existence and multiplicity results; see e.g. [2, 3, 50, 53]. In addition to those of Strauss [48] and Berestycki-Lions [13], which have been a breakthrough in the study of autonomous scalar field equations on the whole of $\mathbb{R}^N$, a great deal of work, certainly inspired by that of Floer and Weinstein [26], has been devoted to the study of nonlinear Schrödinger equations with nonradial potentials and involving various classes of nonlinearities:

$$-\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N. \tag{1.7}$$

The classical works of Rabinowitz [44] and Benci-Cerami [10] have provided a penetrating analysis on equations like (1.7), and inspired the work on various remarkable variants of it, under different hypotheses on V and f which may allow loss of compactness phenomena to occur. Authors have contributed to understand these phenomena in a min-max setting, in analogy to what had been discovered and highlighted in the context of minimisation problems by P.L. Lions in [35] and related papers. An interesting case has been considered by Bartsch and Wang [6] who proved existence and multiplicity of solutions to (1.7) for $V(x) = 1 + \lambda^2 \rho(x)$, and with ρ satisfying either (ρ_1) or (ρ_2). Years later Jeanjean and Tanaka in [31] and related papers, have looked into cases where $f(x,u)$ may violate the Ambrosetti-Rabinowitz condition. Remarkably, they have been able to overcome the possible unboundedness of the Palais-Smale sequences, with an approach which is reminiscent of the 'monotonicity trick' introduced for a different problem by Struwe [49].

Even though our equation $(\mathcal{E})$ is formally a nonlocal variant of the above nonlinear Schrödinger equation, there are some specific variational features that we wish to highlight, which are not shared with (1.7). Firstly, although our nonlinearity $f(x,u) = |u|^{q-1}u$ does satisfy the Ambrosetti-Rabinowitz condition, to the best of our knowledge it is still not known whether the boundedness of the Palais-Smale sequences holds for $q \in (2,3)$. We stress that for this reason and in this range of exponents, it is not known whether the Palais-Smale condition holds, even with $\rho \equiv 1$ and working with the subspace of radial functions in $H^1(\mathbb{R}^N)$. In the range $q \in (2,3]$, the relation between the mountain-pass level and the infimum over the Nehari manifold for the functional I_λ associated to $(\mathcal{E})$ seems non-straightforward; we recall that these levels coincide when dealing with (1.7) for a fairly broad class of nonlinearities f, see e.g. [53, p. 73]. In the case of pure power nonlinearities and $q \in (1,2]$, and unlike for the action functional associated with (1.7), the variational properties of I_λ are particularly sensitive to λ, yielding existence, multiplicity (of a local minimiser and at the same time of a mountain-pass solution) and nonexistence results, see e.g. [45, 46] and [38]. Finally, a natural functional setting associated to $(\mathcal{E})$ may not be necessarily a Hilbert space. In fact note that assumption (ρ_1) is compatible with a situation where $\rho(x) \to \rho_\infty > 0$ as $|x| \to \infty$, in which the space $E(\mathbb{R}^N) \simeq H^1(\mathbb{R}^N)$, as well as with the case $\rho(x) \to \infty$ as $|x| \to \infty$, in which $E(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)$; we tackle the case of vanishing ρ with a unified approach for these particular sub-cases.

Variants of $(\mathcal{E})$ appear in the study of the quantum many–body problem, see [9], [20], [36]. The convolution term represents a repulsive interaction between particles, whereas the local nonlinearity $|u|^{q-1}u$ is a generalisation of the $u^{5/3}$ term introduced by Slater [47] as local approximation of the exchange term in Hartree–Fock type models, see e.g. [15], [37]. In the last few decades, nonlocal equations like $(\mathcal{E})$ have received increasing attention on questions related to existence, non-existence, variational setting and singular limit in the presence of a parameter. We draw the reader's attention to [1], [11], [20] and references therein, for a broader mathematical picture on questions related to Schrödinger-Poisson type systems. Relevant contributions to the existence of positive solutions, mostly for $q > 3 = N$, such as [23, 21], are based on the classification of positive solutions given by Kwong [32] to

$$-\Delta u + u = u^q, \qquad x \in \mathbb{R}^3,$$

regarded as a 'limiting' PDE when $\rho(x) \to 0$, as $|x| \to \infty$. Recently in [51, 40], in the case $\rho(x) \to 1$, as $|x| \to \infty$, the relation between (1.2) and

$$\begin{cases} -\Delta u + u + \lambda^2 \phi u = |u|^{q-1}u, & \mathbb{R}^3 \\ -\Delta \phi = u^2 & \mathbb{R}^3 \end{cases} \tag{1.8}$$

as a limiting problem, has been studied, though a full understanding of the set of positive solutions to (1.8) has not yet been achieved.

Considerably fewer results have been obtained in relation to the multiplicity of solutions. It is worth mentioning [5] whose (radial) approach is suitable in the presence of constant potentials. More precisely Ambrosetti-Ruiz [5] have studied the problem (1.8) with $\lambda > 0$ and $1 < q < 5$. When $q \in (1,2) \cap (3,5)$ their approach relies on the symmetric version of the Mountain-Pass Theorem [4], whereas for $q \in (2,3]$ and in the spirit of [31, 49], they develop a min-max approach to the multiplicity which in fact improves upon [4] and is based on the existence of bounded Palais-Smale sequences at specific levels associated with the perturbed functional

$$I_{\mu,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) + \frac{\lambda^2}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{\omega|x-y|}\,\mathrm{d}x\,\mathrm{d}y - \frac{\mu}{q+1}\int_{\mathbb{R}^3}|u|^{q+1}\mathrm{d}x,$$

for a dense set of values $\mu \in \left[\frac{1}{2}, 1\right)$.

1.1. **Main Results.** In the vanishing case (ρ_1) our main result is the following.

Theorem 1.1 (**Groundstates for $q \geq 3$ under under (ρ_1)**). *Let $N = 3$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ be nonnegative, satisfying (ρ_1), and $q \in [3, 2^* - 1)$. There exists a positive constant $\lambda_* = \lambda_*(q, \overline{M})$ such that for every $\lambda \geq \lambda_*$, $(\mathcal{E})$ admits a positive groundstate solution $u \in E(\mathbb{R}^3)$. For $q > 3$, u is a mountain-pass solution.*

We point out that by construction $\lambda_* = \max\{\lambda_0, \lambda_1\}$, where λ_0 and λ_1 are universal constants defined in Proposition 5 and in Proposition 6, which ensure that, for every $\lambda \geq \lambda_*$, certain Palais-Smale sequences possess weak limits with a precise variational characterisation.
This result extends to a nonlocal equation that of Bartsch-Wang [6], as we are able to show in the spirit of their work that for λ large, there are no Palais-Smale sequences at the mountain-pass level which are weakly convergent to zero, in a context where the embedding of $E(\mathbb{R}^3)$ into $L^{q+1}(\mathbb{R}^3)$ is in general non-compact. This is the case if for instance $\rho(x) \to \rho_\infty > \overline{M}$, as $|x| \to \infty$. In this case $E(\mathbb{R}^3) \simeq H^1(\mathbb{R}^3)$, with equivalent norms by (HLS), and the non-compactness of the embedding is a well-known fact. Under (ρ_1), a condition 'at infinity' for certain Palais-Smale sequences to be relatively compact is given in Proposition 7 and Proposition 8.
It is worth observing that the arguments of Proposition 6 can be adapted to the original result of Bartsch and Wang [6, Section 5] on the nonlinear Schrödinger equation to prove in their setting, for the whole range of exponents and for λ large enough, the existence of a mountain-pass solution and hence, using the Nehari characterisation of the mountain-pass level [53, p. 73], the existence of a groundstate solution. We prove Proposition 6 highlighting how the 'interaction' between λ and $\overline{M}$ appearing in (ρ_1) yields the desired estimates. To this aim we carry out a Brezis-Lieb type splitting argument in the spirit of [18], combining it with a simple weighted L^3 estimate given in

Lemma 2.3, together with the relation between the mountain-pass level and the infimum on the Nehari manifold, which may be sensitive to whether $q = 3$ or $q > 3$.
To prove Theorem 1.1 we follow a Nehari constraint approach, paying attention to the more delicate case $q = 3$. For this exponent, it is not clear whether the mountain-pass level is critical. From a variational perspective, this is another point that makes our work different from [6]; see also Theorem 1.4 below. We stress here that (ρ_1) may not be enough for the right continuity of the mountain-pass levels $c_\lambda(q)$ to hold as $q \to 3^+$.

In the coercive case (ρ_2) we show that $E(\mathbb{R}^N)$ is compactly embedded in $L^p(\mathbb{R}^N)$ for any $2 < p < 2^*$ and $\lambda > 0$. This is used to prove the following.

Theorem 1.2 (**Groundstates for $q \geq 3$ under (ρ_2)**). *Let $N = 3$, $\rho \in L^\infty_{loc}(\mathbb{R}^3)$ be nonnegative, satisfying (ρ_2), and $q \in [3, 2^*-1)$. Then, for any fixed $\lambda > 0$, $(\mathcal{E})$ has both a positive mountain-pass solution and a positive groundstate solution in $E(\mathbb{R}^3)$, whose energy levels coincide for $q > 3$.*

Note that in this case the compact embedding result provided by Lemma 4.1 allows us to have a 'variationally' stronger result for $q = 3$, to be compared with Theorem 1.1. Namely, we can show that the mountain-pass level is critical, using that the Palais-Smale condition is satisfied under (ρ_2) and for $3 \leq q < 5$. A positive mountain-pass solution, which may not be a groundstate for $q = 3$, is constructed as a strong limit of a Palais-Smale sequence living nearby the positive cone in $E(\mathbb{R}^3)$.

When dealing with the range $q \in (2, 3)$, we overcome the possible unboundedness of the Palais-Smale sequences, combining tools developed in this paper, with the approach of Jeanjean and Tanaka [31]. Roughly speaking, the proof is based on constructing a sequence $(u_n)_{n\in\mathbb{N}}$ of critical points to suitable approximated functionals

$$I_n(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2) + \frac{\lambda^2}{4}\int_{\mathbb{R}^N}\rho(x)\phi_u u^2 - \frac{\mu_n}{q+1}\int_{\mathbb{R}^N}|u|^{q+1},$$

which accumulates around the desired solution when letting $\mu_n \to 1^-$ as a result of satisfying a Pohozaev-type condition stated in Lemma 2.4 (which guarantees its boundedness), and by the compactness property provided by Lemma 4.1. More precisely, we have the following.

Theorem 1.3 (**Groundstates for $q < 3$ under (ρ_2)**). *Let $N = 3, 4, 5$, $q \in (2, 3)$ if $N = 3$ and $q \in (2, 2^* - 1)$ if $N = 4, 5$. Let $\lambda > 0$, and assume $\rho \in L^\infty_{loc}(\mathbb{R}^N) \cap W^{1,1}_{loc}(\mathbb{R}^N)$ is nonnegative and satisfies (ρ_2). Moreover suppose that $k\rho(x) \leq (x, \nabla\rho)$ for some $k > \frac{-2(q-2)}{(q-1)}$. Then, $(\mathcal{E})$ has a mountain-pass solution $u \in E(\mathbb{R}^N)$. Moreover, there exists a groundstate solution.*

Remark 1.1. The same proof when working instead with the functional

$$I_+(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + u^2\right) + \frac{\lambda^2}{4}\int_{\mathbb{R}^N}\rho(x)\phi_u u^2 - \frac{1}{q+1}\int_{\mathbb{R}^N}u_+^{q+1},$$

allows to show that mountain-pass and groundstate critical points exist for this functional, and are positive by construction.

Under (ρ_2) and for $q \leq 3$, the relation between mountain-pass solutions and groundstates found in Theorem 1.2 for $q = 3$ and Theorem 1.3 for $q < 3$ seems not obvious; in particular, it is not clear whether they actually coincide. We are able to get more insight about the variational nature of these solutions in the case ρ is homogeneous of a suitable order $\bar{k} > 0$, as shown in the following theorem. It is worth pointing out that this homogeneity condition is not compatible with ρ vanishing on a region.

Theorem 1.4 (**Homogeneous case for $q \leq 3$: mountain-pass solutions vs. groundstates**). *Let $N = 3, 4, 5$, $q \in (2, 3]$ if $N = 3$ and $q \in (2, 2^* - 1)$ if $N = 4, 5$. Suppose $\lambda > 0$ and $\rho \in L^\infty_{loc}(\mathbb{R}^N) \cap W^{1,1}_{loc}(\mathbb{R}^N)$ is nonnegative, satisfies (ρ_2), and is homogeneous of degree $\bar{k}$, namely $\rho(tx) = t^{\bar{k}}\rho(x)$ for all $t > 0$, for some*

$$\bar{k} > \left(\max\left\{\frac{N}{4}, \frac{1}{q-1}\right\} \cdot (3-q) - 1\right)_+.$$

Then, the mountain-pass solutions that we find in Theorem 1.2 ($q = 3$) and Theorem 1.3 ($q < 3$) are groundstates.

We prove the above theorem analysing some relevant scaling properties of I_λ in Proposition 4, which allows us to characterise the mountain-pass level in terms of the infimum over a certain manifold, defined as a suitable combination of the Nehari and Pohozaev identities. We believe that this manifold is a natural constraint. We point the reader to Remark 2.2, in which we give an explanation of the lower bound assumption on $\bar{k}$.

In the spirit of Ambrosetti-Rabinowitz [4] and under (ρ_2) we show that (1.2) possesses infinitely many high energy solutions. In our context it seems appropriate to distinguish the cases $q \in (3, 5)$ and $q \in (2, 3]$ when working within the Ljusternik-Schnirelman theory. Since for $q \in (3, 5)$ Lemma 4.1 implies that the Palais-Smale condition is satisfied, we can use the $\mathbb{Z}_2$-equivariant Mountain-Pass theorem, adapting to $E(\mathbb{R}^N)$ arguments similar to those developed for a different functional setting by Szulkin; see [52]. To this aim, in Lemma 2.1 we prove that for $N \geq 3$ $E(\mathbb{R}^N)$ is a separable Banach space, by constructing a suitable linear isometry of $E(\mathbb{R}^N)$ onto the Cartesian product of $H^1(\mathbb{R}^N)$ with some of the mixed norm Lebesgue spaces studied by Benedek and Panzone [12], namely $L^4(\mathbb{R}^N; L^2(\mathbb{R}^N))$. As a consequence of this identification, we can show that $E(\mathbb{R}^N)$ admits a Markushevic basis, that is a set of elements $\{(e_m, e^*_m)\}_{m\in\mathbb{N}} \subset E(\mathbb{R}^N) \times E^*(\mathbb{R}^N)$ such that the duality product $< e_n, e^*_m >= \delta_{nm}$ for all $n, m \in \mathbb{N}$, the e_m's are linearly dense in $E(\mathbb{R}^N)$, and the weak*-closure of span$\{e^*_m\}_{m\in\mathbb{N}}$ is $E^*(\mathbb{R}^N)$. We use this, combined with Lemma 4.1 to obtain lower bounds on the energy which allow us to show the divergence of a sequence of min-max critical levels defined by means of the classical notion of Krasnoselskii genus; see Lemma 5.1 below. This yields the following

Theorem 1.5 (**Infinitely many high energy solutions for** $q > 3$). *Let $N = 3$, $q \in (3, 2^* - 1)$ and $\lambda > 0$. Suppose $\rho \in L^\infty_{loc}(\mathbb{R}^3)$ is nonnegative and satisfies (ρ_2). Then, there exist infinitely many distinct pairs of critical points $\pm u_m \in E(\mathbb{R}^N)$, $m \in \mathbb{N}$, for I_λ such that $I_\lambda(u_m) \to +\infty$ as $m \to +\infty$.*

When $q < 3$, the above construction is not directly applicable because of the possible unboundedness of the Palais-Smale sequences. Here we use a deformation lemma due to Ambrosetti and Ruiz [5], in the flavor of the work of Jeanjean and Tanaka, which is suitable for Ljusternik-Schnirelman type results. Assuming that $\rho(x)$ is homogeneous of some order $\bar{k} > 0$, allows us to define as in [5], certain classes of admissible subsets of $E(\mathbb{R}^N)$ and hence of min-max levels; see Lemma 2.6 and Lemma 5.6 below. This together with the aforementioned Pohozaev type inequality (which in the present homogeneous case becomes an identity by Euler's classical theorem) and Lemma 4.1, allows us to show that these min-max levels are critical, and that they are arbitrarily large, by Lemma 5.1 again. We therefore have the following.

Theorem 1.6 (**Infinitely many high energy solutions for** $q \leq 3$). *Let $N = 3, 4, 5$. Assume $q \in (2, 3]$ if $N = 3$ and $q \in (2, 2^* - 1)$ if $N = 4, 5$. Suppose $\lambda > 0$ and $\rho \in L^\infty_{loc}(\mathbb{R}^N) \cap W^{1,1}_{loc}(\mathbb{R}^N)$ is nonnegative, satisfies (ρ_2), and is homogeneous of degree $\bar{k}$, namely, $\rho(tx) = t^{\bar{k}}\rho(x)$ for all $t > 0$,* *for some*

$$\bar{k} > \left(\max\left\{ \frac{N}{4}, \frac{1}{q-1} \right\} \cdot (3-q) - 1 \right)_+ .$$

Then, there exist infinitely many distinct pairs of critical points, $\pm u_m \in E(\mathbb{R}^N)$, $m \in \mathbb{N}$, for I_λ such that $I_\lambda(u_m) \to +\infty$ as $m \to +\infty$.

Remark 1.2. For $q = 3$, the homogeneity assumption on ρ is not used to prove the boundedness of the Palais-Smale sequences which holds for this exponent, but rather it is used in the construction of the min-max levels.

Remark 1.3. For $N = 3, 4$, we can cover all $\bar{k} > 0$. For $N = 5$, the threshold for $\bar{k}$ is sensitive to the range of q. Namely, if $q \in (\frac{11}{5}, 2^* - 1)$, we can cover all $\bar{k} > 0$, however if $q \in (2, \frac{11}{5})$, this is not the case.

1.2. **Outline.** The paper is organised as follows. In Section 2 we deal with general facts about the functional setting, we prove the separability of $E(\mathbb{R}^N)$ and other properties that will be used throughout, comprising positivity and regularity. We prove a Pohozaev type necessary condition that will be extensively used in the existence proofs, and that in this section is applied to a nonexistence result for $q = 2^* - 1$. Here we also discuss the min-max setting and related properties, which hold for a generic locally bounded ρ.

In Section 3 we work under the vanishing assumption (ρ_1). Here we develop a set of uniform integral estimates which hold for all the values of λ above certain lower thresholds. We conclude the section with the proof of the existence of groundstates, Theorem 1.1, and provide with Proposition 7 and Proposition 8, some new compactness results on sequences of approximated critical points of I_λ.

Section 4 is devoted to the coercive case (ρ_2). For any fixed arbitrary $\lambda > 0$ we prove the compactness Lemma 4.1, and the existence results Theorem 1.2, Theorem 1.3, and Theorem 1.4.

Section 5 is entirely devoted to the multiplicity of high energy solutions. In particular, in Section 5.1 we prove Lemma 5.1 that is key to show later in the proofs the existence of a blowing up sequence of infinitely many distinct critical levels of high energy. In Section 5.2 we recall the notion of the Krasnoselskii genus and its properties, and deal with the proof of Theorem 1.5. Finally, in Section 5.3 we prove Theorem 1.6.

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2. Preliminaries

We introduce the functional setting for our problem and provide a few preliminary lemmas that hold for all nonnegative $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ and will be used under all assumptions on ρ.

2.1. **Functional setting.** For what follows, we will need some properties of the functional setting which are contained in the next lemma.

Lemma 2.1 (**Properties of $E(\mathbb{R}^N)$**)**.** *Assume $N \geq 3$, and $\rho \geq 0$ is a measurable function. The space $E(\mathbb{R}^N)$ is a separable Banach space that admits a Markushevic basis, that is a fundamental and total biorthogonal system, $\{(e_m, e_m^*)\}_{m\in\mathbb{N}} \subset E(\mathbb{R}^N) \times E^*(\mathbb{R}^N)$. Namely, $< e_n, e_m^* >= \delta_{nm}$ for all $n, m \in \mathbb{N}$, the e_m's are linearly dense in $E(\mathbb{R}^N)$, and the weak*-closure of span$\{e_m^*\}_{m\in\mathbb{N}}$ is $E^*(\mathbb{R}^N)$.*

Proof. Following [39], we note that we can equip $E(\mathbb{R}^N)$ with the equivalent norm

$$\|u\|_1 = \left(\|u\|^2_{H^1(\mathbb{R}^N)} + \lambda \left(\int_{\mathbb{R}^N} \left| I_1 \star (\sqrt{\rho}|u|)^2 \right|^2 \right)^{1/2} \right)^{1/2}. \tag{2.1}$$

Here, we have set $\alpha = 1$ in $I_\alpha : \mathbb{R}^N \to \mathbb{R}$, the Riesz potential of order $\alpha \in (0, N)$, defined for $x \in \mathbb{R}^N \setminus \{0\}$ as

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}, \quad A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha},$$

and the choice of normalisation constant A_α ensures that the kernel I_α enjoys the semigroup property

$$I_{\alpha+\beta} = I_\alpha \star I_\beta \text{ for each } \alpha, \beta \in (0, N) \text{ such that } \alpha + \beta < N.$$

We first notice that the operator $T : E(\mathbb{R}^N) \to H^1(\mathbb{R}^N) \times L^4(\mathbb{R}^N; L^2(\mathbb{R}^N))$ defined by

$$(Tu)(x_0, x_1, x_2) = [u(x_0), (\lambda I_1(x_2 - x_1)\rho(x_1))^{\frac{1}{2}} u(x_1)],$$

is a linear isometry from $E(\mathbb{R}^N)$ into the product space $H^1(\mathbb{R}^N) \times L^4(\mathbb{R}^N; L^2(\mathbb{R}^N))$, endowed with the norm

$$\|[u, v]\|_\times = \left(\|u\|^2_{H^1(\mathbb{R}^N)} + \|v\|^2_{L^4(\mathbb{R}^N; L^2(\mathbb{R}^N))} \right)^{1/2}.$$

Here $L^4(\mathbb{R}^N; L^2(\mathbb{R}^N))$ is the mixed norm Lebesgue space of functions $v : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ such that

$$||v||_{L^4(\mathbb{R}^N; L^2(\mathbb{R}^N))} = \left(\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |v(x_1, x_2)|^2 \, \mathrm{d}x_1 \right)^2 \mathrm{d}x_2 \right)^{1/4} < +\infty,$$

see [12]. Since $L^4(\mathbb{R}^N; L^2(\mathbb{R}^N))$ is a separable (see e.g. [43, p. 107]) Banach space (see e.g. [12]), it follows that the linear subspace $T(E(\mathbb{R}^N)) \subseteq H^1(\mathbb{R}^N) \times L^4(\mathbb{R}^N; L^2(\mathbb{R}^N))$, and hence $E(\mathbb{R}^N)$, also satisfies each of these properties. Since every separable Banach space admits a Markushevic basis (see e.g. [29]), the proof is complete. □

Reasoning as in [46] and [39] it is easy to see that $C_c^\infty(\mathbb{R}^N)$ is dense in $E(\mathbb{R}^N)$ and that the unit ball in $E(\mathbb{R}^N)$ is weakly compact; in fact this space is uniformly convex and hence is reflexive. The following variant to the classical Brezis-Lieb lemma will be useful to study the convergence of bounded sequences in $E(\mathbb{R}^N)$; see e.g. [7], [39].

Lemma 2.2 (**Nonlocal Brezis-Lieb lemma).** *Assume $N \geq 3$ and $\rho(x) \in L^\infty_{loc}(\mathbb{R}^N)$ is nonnegative. Let $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ be a bounded sequence such that $u_n \to u$ almost everywhere in $\mathbb{R}^N$. Then it holds that*

$$\lim_{n\to\infty} \left[\|\nabla\phi_{u_n}\|^2_{L^2(\mathbb{R}^N)} - \|\nabla\phi_{(u_n-u)}\|^2_{L^2(\mathbb{R}^N)} \right] = \|\nabla\phi_u\|^2_{L^2(\mathbb{R}^N)}.$$

The next simple estimate is based on an observation of P.-L. Lions, given in [36] for $\rho \equiv 1$; see also [46], and [8], [39].

Lemma 2.3 (**Coulomb-Sobolev inequality).** *Assume $N \geq 3$, $\rho(x) \in L^\infty_{loc}(\mathbb{R}^N)$ is nonnegative. Then the following inequality holds for all $u \in E(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} \rho(x)|u|^3 \leq \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |\nabla\phi_u|^2 \right)^{\frac{1}{2}}. \tag{2.2}$$

Proof. Testing the Poisson equation (1.5) with $|u|$, the statement follows immediately by Cauchy-Schwarz inequality. □

2.2. **Regularity and positivity.** Using standard elliptic regularity theory and the maximum principle, we now provide a result giving the regularity and positivity of the solutions to the Schrödinger-Poisson system.

Proposition 1. *[**Regularity and positivity**] Let $N \in [3,6]$ and $q \in [1, 2^*-1]$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ be nonnegative and $\rho(x) \not\equiv 0$ and $(u, \phi_u) \in E(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ be a nontrivial weak solution to*

$$\begin{cases} -\Delta u + bu + c\rho(x)\phi u = d|u|^{q-1}u, & x \in \mathbb{R}^N, \\ -\Delta \phi = \rho(x)u^2, & x \in \mathbb{R}^N, \end{cases} \tag{2.3}$$

with $b, c, d \in \mathbb{R}_+$. Then, u, $\phi_u \in W^{2,s}_{loc}(\mathbb{R}^N)$, for every $s \geq 1$, and so u, $\phi_u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$; moreover $\phi_u > 0$. If, in addition, $u \geq 0$, then $u > 0$ everywhere.

Proof. Under the hypotheses of the proposition, both u and ϕ_u have weak second derivatives in $L^s_{\text{loc}}(\mathbb{R}^N)$ for all $s < +\infty$. To show this, note that from the first equation in (2.3), we have that $-\Delta u = g(x,u)$, where

$$\begin{aligned} |g(x,u)| &= |(-bu - c\rho(x)\phi_u u + d|u|^{q-1}u| \\ &\leq C(1 + |\rho\phi_u| + |u|^{q-1})(1 + |u|) \\ &=: h(x)(1+|u|). \end{aligned}$$

Using our assumptions on ρ, ϕ_u, u, and that $q \leq 2^*-1$, we can show that $h \in L^{N/2}_{\text{loc}}(\mathbb{R}^N)$, which implies that $u \in L^s_{\text{loc}}(\mathbb{R}^N)$ for all $s < +\infty$ (see e.g. [50, p.270]). Note that here the restriction on the dimension implies that $\phi_u \in L^{N/2}_{\text{loc}}(\mathbb{R}^N)$. Since $u^2\rho \in L^s_{\text{loc}}(\mathbb{R}^N)$ for all $s < +\infty$, then by the second equation in (2.3) and the Calderón-Zygmund estimates, we have that $\phi_u \in W^{2,s}_{\text{loc}}(\mathbb{R}^N)$ (see e.g. [28]). This then enables us to show that $g \in L^s_{\text{loc}}(\mathbb{R}^N)$ for all $s < +\infty$, which implies, by Calderón-Zygmund estimates, that $u \in W^{2,s}_{\text{loc}}(\mathbb{R}^N)$ (see e.g. [28]). The $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$ regularity of both u, ϕ_u is a consequence of Morrey's embedding theorem. Finally, the strict positivity is a consequence of the strong maximum principle with $L^\infty_{\text{loc}}(\mathbb{R}^N)$ coefficients [42], and this concludes the proof. □

2.3. **Nonexistence.** The following lemma, proved in the Appendix, will be extensively used.

Lemma 2.4. *[**Pohozaev-type condition**] Assume $N \in [3,6]$, $q \in [1, 2^*-1]$, $\rho \in L^\infty_{loc}(\mathbb{R}^N) \cap W^{1,1}_{loc}(\mathbb{R}^N)$ is nonnegative, and $k\rho(x) \leq (x, \nabla\rho)$ for some $k \in \mathbb{R}$. Let $(u, \phi_u) \in E(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ be a weak solution to (2.3). Then, it holds that*

$$\begin{aligned} &\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x + \frac{Nb}{2}\int_{\mathbb{R}^N} u^2\,\mathrm{d}x \\ &\quad + \frac{(N+2+2k)c}{4}\int_{\mathbb{R}^N} \rho\phi_u u^2\,\mathrm{d}x - \frac{Nd}{q+1}\int_{\mathbb{R}^N} |u|^{q+1}\,\mathrm{d}x \leq 0. \end{aligned} \tag{2.4}$$

In particular the above is an identity, provided $k\rho(x) = (x, \nabla\rho)$ (by Euler's theorem, this is the case if ρ is homogeneous of order k, see e.g. [27, p. 296]*).*

Although we will use the above necessary condition mainly for existence purposes, this also allows us to find a family of nonexistence results in a certain range of the parameters N, q, λ, k.

Proposition 2 (**Nonexistence: the critical case $q = 2^*-1$**)**.** *Assume $N \in [3,6]$, $q = 2^*-1$, $\rho \in L^\infty_{loc}(\mathbb{R}^N) \cap W^{1,1}_{loc}(\mathbb{R}^N)$ nonnegative, $k\rho(x) \leq (x, \nabla\rho)$ for some $k \geq \frac{N-6}{2}$, and $\lambda > 0$. Let $(u, \phi_u) \in E(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ be a weak solution to (1.2). Then, $(u, \phi_u) = (0,0)$.*

Proof. Combining the Nehari identity $I'_\lambda(u)(u) = 0$ with Lemma 2.4 yields

$$\Big(\frac{N-2}{2} - \frac{N}{q+1}\Big) \int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x + \Big(\frac{N}{2} - \frac{N-2}{2}\Big) \int_{\mathbb{R}^N} u^2 \,\mathrm{d}x + \Big(\frac{2k+6-N}{4}\Big)\lambda^2 \int_{\mathbb{R}^N} \rho\phi_u u^2 \,\mathrm{d}x \leq 0.$$

Hence,

$$\int_{\mathbb{R}^N} u^2 \,\mathrm{d}x \leq 0,$$

and this concludes the proof. □

Remark 2.1. Similar nonexistence results have been obtained in the case of constant potentials and for $N = 3$, in [25]. We point out that the in the above proposition $\lambda > 0$ is arbitrary and the condition on ρ is compatible with (ρ_1), as well as with (ρ_2). It is interesting to note that for $N = 6$ we have $q = 2^* - 1 = 2$, namely nonexistence occurs in a 'low-q' regime, under both conditions (ρ_1) and (ρ_2). The proof shows also that for supercritical exponents $q + 1 > 2^*$ and higher dimensions, under further regularity assumptions required for Lemma 2.4 to hold, nonexistence also occurs.

Proposition 3 (**Nonexistence: the case** $q \in (1, 2]$)**.** *Assume $N \geq 3$, $q \in (1, 2]$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ and $\rho(x) \geq 1$ almost everywhere and $\lambda \geq \frac{1}{2}$. Let $u \in E(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$ satisfy*

$$-\Delta u + u + \lambda^2 \left(\frac{1}{\omega|x|^{N-2}} \star \rho u^2\right) \rho(x) u = |u|^{q-1}u, \qquad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{2.5}$$

Then, $u \equiv 0$.

We note that this proposition is stated to cover also the dimensions $N > 2\left(\frac{q+1}{q-1}\right)$, namely the supercritical cases $3 \geq q + 1 > 2^*$ where $E(\mathbb{R}^N)$ does not embed in $L^{q+1}(\mathbb{R}^N)$.

Proof. By density we can test (2.5) by u and so we obtain

$$\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 + \lambda^2 \rho(x)\phi_u u^2 - |u|^{q+1} = 0. \tag{2.6}$$

Following [45, Theorem 4.1], by Lemma 2.3 and Young's inequality we have

$$\int_{\mathbb{R}^N} \rho(x)|u|^3 \leq \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{4}\int_{\mathbb{R}^N} \rho(x)\phi_u u^2. \tag{2.7}$$

Combining (2.6) and (2.7), we have for all $\lambda \geq \frac{1}{2}$

$$0 \geq \int_{\mathbb{R}^N} u^2 + \rho(x)|u|^3 - |u|^{q+1} \geq \int_{\mathbb{R}^N} f(u),$$

where $f(u) = u^2 + |u|^3 - |u|^{q+1}$ is positive except at zero. Hence $u \equiv 0$, and this concludes the proof. □

2.4. **Min-max setting.** The present section is devoted to the min-max properties of I_λ, which will be used in our existence results.

Lemma 2.5 (**Mountain-Pass Geometry for** I_λ)**.** *Assume $N = 3, 4, 5$, $\rho(x) \in L^\infty_{loc}(\mathbb{R}^N)$ is nonnegative and $q \in (2, 2^* - 1]$. Then, it holds that*

(i) $I_\lambda(0) = 0$ and there exist constants $r, a > 0$ such that $I_\lambda(u) \geq a$ if $\|u\|_{E(\mathbb{R}^N)} = r$;
(ii) there exist $v \in E(\mathbb{R}^N)$ with $\|v\|_{E(\mathbb{R}^N)} > r$ such that $I_\lambda(v) \leq 0$.

Proof. Statement (i) follows reasoning as in Lemma 2.8. To show (ii), pick $u \in C^1(\mathbb{R}^N)$, supported in the unit ball, B_1. Setting $v_t(x) := t^2u(tx)$ we find that

$$\begin{aligned} I_\lambda(v_t) = \frac{t^{6-N}}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + \frac{t^{4-N}}{2}\int_{\mathbb{R}^N} u^2 + \frac{t^{6-N}}{4}\lambda^2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(y)\rho(\frac{y}{t})u^2(x)\rho(\frac{x}{t})}{\omega|x-y|^{N-2}}\,\mathrm{d}y\,\mathrm{d}x \\ - \frac{t^{(2q+2-N)}}{q+1}\int_{\mathbb{R}^N}|u|^{q+1}. \end{aligned} \tag{2.8}$$

Since for every $t \geq 1$ and for almost every $x \in B_1$ we have $\rho(x/t) \leq ||\rho||_{L^\infty(B_1)}$, the fact that $2q+2 > 6$ in (2.8) yields $I_\lambda(v_t) \to -\infty$ as $t \to +\infty$, and this is enough to conclude the proof. ☐

To prove our results for $q < 3$, we will need to work with a perturbed functional, $I_{\mu,\lambda} : E(\mathbb{R}^N) \to \mathbb{R}^N$, defined by

$$I_{\mu,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2) + \frac{\lambda^2}{4}\int_{\mathbb{R}^N}\rho\phi_u u^2 - \frac{\mu}{q+1}\int_{\mathbb{R}^N}|u|^{q+1}, \quad \mu \in \left[\frac{1}{2}, 1\right]. \tag{2.9}$$

As in Lemma 2.5, $I_{\mu,\lambda}$ has the mountain-pass geometry in $E(\mathbb{R}^N)$ for all $\mu \in \left[\frac{1}{2}, 1\right]$. This, as well as the monotonicity of $I_{\mu,\lambda}$ with respect to μ, imply that we can define the min-max level associated with $I_{\mu,\lambda}$ as

$$c_{\mu,\lambda} = \inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]} I_{\mu,\lambda}(\gamma(t)), \quad \mu \in \left[\frac{1}{2}, 1\right] \tag{2.10}$$

where

$$\Gamma_\lambda = \{\gamma \in C([0,1], E(\mathbb{R}^N)) : \gamma(0) = 0,\, I_{\frac{1}{2},\lambda}(\gamma(1)) < 0\}. \tag{2.11}$$

Since the mapping $[1/2, 1] \ni \mu \mapsto c_{\mu,\lambda}$ is non-increasing and left-continuous in μ (see [5, Lemma 2.2]) and the non-perturbed functional I_λ has the mountain-pass geometry by Lemma 2.5, we are now in position to define the min-max level associated with I_λ for all $q \in (2, 2^* - 1)$.

Definition 1 (**Definition of mountain-pass level for I_λ**)**.** We set

$$c_\lambda = \begin{cases} c_{1,\lambda}, & q \in (2,3), \\ \inf\limits_{\gamma\in\bar{\Gamma}_\lambda}\max\limits_{t\in[0,1]} I_\lambda(\gamma(t)), & q \in [3, 2^*-1), \end{cases} \tag{2.12}$$

where $c_{1,\lambda}$ is given by (2.10) and $\bar{\Gamma}_\lambda$ is the family of paths defined as

$$\bar{\Gamma}_\lambda = \left\{\gamma \in C([0,1]; E(\mathbb{R}^N)) : \gamma(0) = 0, I_\lambda(\gamma(1)) < 0\right\}. \tag{2.13}$$

The remainder of this subsection is devoted to further characterisations of the min-max level c_λ for $q \leq 3$. We first require the following technical lemma.

Lemma 2.6. *Suppose $N \geq 3$, $q > 2$ and $\nu > \max\left\{\frac{N}{2}, \frac{2}{q-1}\right\}$. Let $\bar{k} \in \left(\frac{\nu(3-q)-2}{2}, \frac{4\nu-N-2}{2}\right)$. Define $f : \mathbb{R}_0^+ \to \mathbb{R}$ as*

$$f(t) = at^{2\nu+2-N} + bt^{2\nu-N} + ct^{4\nu-N-2-2\bar{k}} - dt^{\nu(q+1)-N}, \quad t \geq 0,$$

where $a, b, c, d \in \mathbb{R}$ are such that $a, b, d > 0$, $c \geq 0$. Then, f has a unique critical point corresponding to its maximum.

Remark 2.2. We point out that our range of parameters ensures that $f(t) \to -\infty$ as $t \to +\infty$ and it holds that

$$\left(\frac{\nu(3-q)-2}{2}, \frac{4\nu-N-2}{2}\right) \bigcap \left(\frac{(\nu+1)(3-q)-2}{2}, \frac{4(\nu+1)-N-2}{2}\right) \neq \emptyset.$$

In Theorem 1.4 and Theorem 1.6, we use Lemma 2.6, assuming

$$\bar{k} > \max\left\{\frac{N}{4}, \frac{1}{q-1}\right\}(3-q) - 1$$

for $\bar{k}$ to belong to one of these intervals.

Proof of Lemma 2.6. Note that by our assumptions, we can write

$$f(t) = \sum_{i=1}^{k} a_i t^{p_i} - t^p,$$

where $a_i \geq 0$, $0 \leq p_i < p$ and both a_i, $p_i \neq 0$ for some i. Setting $s = t^p$, we find

$$f(s) = \sum_{i=1}^{k} a_i s^{\frac{p_i}{p}} - s.$$

It follows that $f(s)$ is strictly concave and has a unique critical point, which is a maximum. Since our assumptions ensure that $f(t) \to -\infty$ as $t \to +\infty$, we can conclude. □

To state our next result, for any $\nu \in \mathbb{R}$, we set

$$\bar{\mathcal{M}}_{\lambda,\nu} = \left\{u \in E(\mathbb{R}^N) \setminus \{0\} : J_{\lambda,\nu}(u) = 0\right\}, \tag{2.14}$$

where $J_{\lambda,\nu} : E(\mathbb{R}^N) \to \mathbb{R}^N$ is defined as

$$\begin{aligned} J_{\lambda,\nu}(u) = \frac{2\nu + 2 - N}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{2\nu - N}{2} \int_{\mathbb{R}^N} u^2 \\ + \frac{4\nu - N - 2 - 2\bar{k}}{4} \cdot \lambda^2 \int_{\mathbb{R}^N} \rho \phi_u u^2 - \frac{\nu(q+1) - N}{q+1} \int_{\mathbb{R}^N} |u|^{q+1}. \end{aligned} \tag{2.15}$$

Notice that, if ρ is homogeneous of order $\bar{k}$, $J_{\lambda,\nu}(u)$ is the derivative of the polynomial $f(t) = I_\lambda(t^\nu u(t\cdot))$ at $t = 1$.

Proposition 4 (Mountain-pass characterisation of groundstates). *Let $N = 3, 4, 5$, $q \in (2, 3]$ if $N = 3$ and $q \in (2, 2^* - 1)$ if $N = 4, 5$. Suppose $\rho \in L^\infty_{loc}(\mathbb{R}^N) \cap W^{1,1}_{loc}(\mathbb{R}^N)$ is nonnegative and is homogeneous of degree $\bar{k}$, namely $\rho(tx) = t^{\bar{k}}\rho(x)$ for all $t > 0$, for some*

$$\bar{k} > \max\left\{\frac{N}{4}, \frac{1}{q-1}\right\}(3-q) - 1.$$

Then, there exists $\nu > \max\{\frac{N}{2}, \frac{2}{q-1}\}$ such that

$$c_\lambda = \inf_{u \in \bar{\mathcal{M}}_{\lambda,\nu}} I_\lambda(u) = \inf_{u \in E(\mathbb{R}^N)\setminus\{0\}} \max_{t \geq 0} I_\lambda(t^\nu u(t\cdot)),$$

where c_λ and $\bar{\mathcal{M}}_{\lambda,\nu}$ are defined in (2.12) and (2.14), respectively.

Proof. We first note that under the assumptions on the parameters, it holds that

$$\frac{4\nu - N - 2}{2} > \frac{(\nu+1)(3-q) - 2}{2}.$$

It follows from this and the lower bound assumption on $\bar{k}$ that we can always find at least one interval

$$\left(\frac{\nu(3-q) - 2}{2}, \frac{4\nu - N - 2}{2}\right), \quad \text{with } \nu > \max\left\{\frac{N}{2}, \frac{2}{q-1}\right\},$$

that contains $\bar{k}$. We fix ν corresponding to such an interval. We break the remainder of the proof into a series of claims.

Claim 1. $\inf_{u\in E(\mathbb{R}^N)\setminus\{0\}} \max_{t\geq 0} I_\lambda(t^\nu u(t\cdot)) \leq \inf_{u\in \bar{\mathcal{M}}_{\lambda,\nu}} I_\lambda(u)$

To see this, let $u \in E(\mathbb{R}^N)\setminus\{0\}$ be fixed and consider the function

$$\begin{aligned} g(t) &= I_\lambda(t^\nu u(t\cdot)) \\ &= at^{2\nu+2-N} + bt^{2\nu-N} + ct^{4\nu-N-2-2\bar{k}} - dt^{\nu(q+1)-N}, \quad t\geq 0, \end{aligned} \tag{2.16}$$

where

$$a = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2,\ b=\frac{1}{2}\int_{\mathbb{R}^N}u^2,\ c=\frac{\lambda^2}{4}\int_{\mathbb{R}^N}\rho\phi_u u^2,\ d=\frac{1}{q+1}\int_{\mathbb{R}^N}|u|^{q+1}.$$

By Lemma 2.6, it holds that g has a unique critical point, $t=\tau_u$, corresponding to its maximum. Moreover, we can see that

$$\begin{aligned} g'(t) &= \frac{\mathrm{d}I_\lambda(t^\nu u(t\cdot))}{\mathrm{d}t} \\ &= \frac{2\nu+2-N}{2}\cdot t^{2\nu+1-N}\int_{\mathbb{R}^N}|\nabla u|^2 + \frac{2\nu-N}{2}\cdot t^{2\nu-N-1}\int_{\mathbb{R}^N}u^2 \\ &\quad + \frac{4\nu-N-2-2\bar{k}}{4}\cdot t^{4\nu-N-3-2\bar{k}}\cdot\lambda^2\int_{\mathbb{R}^N}\rho\phi_u u^2 - \frac{\nu(q+1)-N}{q+1}\cdot t^{\nu(q+1)-N-1}\int_{\mathbb{R}^N}|u|^{q+1}, \end{aligned}$$

and so

$$g'(t)=0 \iff t^\nu u(t\cdot)\in\bar{\mathcal{M}}_{\lambda,\nu}.$$

Taken together, we have shown that for any $u\in E(\mathbb{R}^N)\setminus\{0\}$, there exists a unique $t=\tau_u$ such that $\tau_u^\nu u(\tau_u\cdot)\in\bar{\mathcal{M}}_{\lambda,\nu}$ and the maximum of $I_\lambda(t^\nu u(t\cdot))$ for $t\geq 0$ is achieved at τ_u. Thus, it holds that

$$\inf_{u\in E(\mathbb{R}^N)\setminus\{0\}}\max_{t\geq 0} I_\lambda(t^\nu u(t\cdot)) \leq \max_{t\geq 0} I_\lambda(t^\nu u(t\cdot)) = I_\lambda(\tau_u^\nu u(\tau_u\cdot)), \quad \forall u\in E(\mathbb{R}^N)\setminus\{0\},$$

from which we can deduce that the claim holds.

Claim 2. $c_\lambda \leq \inf_{u\in E(\mathbb{R}^N)\setminus\{0\}}\max_{t\geq 0} I_\lambda(t^\nu u(t\cdot))$.

By the assumptions on our parameters, we can deduce that $\nu(q+1)-N > 2\nu+2-N$ and $\nu(q+1)-N > 4\nu-N-2-2\bar{k}$. It follows that $I_\lambda(t^\nu u(t\cdot))<0$ for every $u\in E(\mathbb{R}^N)\setminus\{0\}$ and t large. Similarly, $I_{\frac{1}{2},\lambda}(t^\nu u(t\cdot))<0$ for every $u\in E(\mathbb{R}^N)\setminus\{0\}$ and t large. Therefore, we obtain

$$c_\lambda \leq \max_{t\geq 0} I_\lambda(t^\nu u(t\cdot)), \quad \forall u\in E(\mathbb{R}^N)\setminus\{0\},$$

and the claim follows.

Claim 3. $\inf_{u\in\bar{\mathcal{M}}_{\lambda,\nu}} I_\lambda(u)\leq c_\lambda$.

We define

$$A_{\lambda,\nu} = \{u\in E(\mathbb{R}^N)\setminus\{0\} : J_{\lambda,\nu}(u)>0\}\cup\{0\},$$

and first note that $A_{\lambda,\nu}$ contains a small ball around the origin. Indeed, arguing as in the proof of Lemma 2.8, we can show that for every $u\in E(\mathbb{R}^N)\setminus\{0\}$ and any $\beta>0$, we have

$$\begin{aligned} J_{\lambda,\nu}(u) \geq \frac{2\nu-N}{2}||u||^2_{H^1(\mathbb{R}^N)} - \left(\frac{4\nu-N-2-2\bar{k}}{\omega}\right)\left(\frac{\beta-1}{4}\right)||u||^4_{H^1(\mathbb{R}^N)} \\ + \left(\frac{4\nu-N-2-2\bar{k}}{\omega}\right)\left(\frac{\beta-1}{4\beta}\right)||u||^4_{E(\mathbb{R}^N)} - \frac{S_{q+1}^{-(q+1)}(\nu(q+1)-N)}{q+1}||u||^{q+1}_{H^1(\mathbb{R}^N)}. \end{aligned}$$

We now pick $\delta = \left(\frac{(2\nu-N)(q+1)S_{q+1}^{q+1}}{4(\nu(q+1)-N)}\right)^{1/(q-1)}$ and note that since $\nu > \frac{N}{2}$, it follows that $\delta > 0$. We assume $||u||_{E(\mathbb{R}^N)} < \delta$ and choosing $\beta > 1$ sufficiently near 1 we obtain

$$
\begin{aligned}
J_{\lambda,\nu}(u) &\geq \left[\frac{2\nu - N}{4} - \left(\frac{4\nu - N - 2 - 2\bar{k}}{\omega}\right)\left(\frac{\beta-1}{4}\right)\delta^2\right]||u||^2_{H^1(\mathbb{R}^N)} + \left(\frac{4\nu - N - 2 - 2\bar{k}}{\omega}\right)\left(\frac{\beta-1}{4\beta}\right)||u||^4_{E(\mathbb{R}^N)} \\
&\geq \left(\frac{4\nu - N - 2 - 2\bar{k}}{\omega}\right)\left(\frac{\beta-1}{4\beta}\right)||u||^4_{E(\mathbb{R}^N)},
\end{aligned}
$$

which is strictly positive by our choice of ν. This is enough to prove that $A_{\lambda,\nu}$ contains a small ball around the origin. Now, notice that if $u \in A_{\lambda,\nu}$, then $g'(1) > 0$, where g is defined in (2.16). Since $g(0) = 0$ and we showed in Claim 1 that τ_u is the unique critical point of g corresponding to its maximum, it follows that $1 < \tau_u$. Using the facts that $I_\lambda(0) = 0$ and $g'(t) = \frac{\mathrm{d}I_\lambda(t^\nu u(t\cdot))}{\mathrm{d}t} \geq 0$ for all $t \in [0, \tau_u]$, we obtain that $I_\lambda(t^\nu u(t\cdot)) \geq 0$ for all $t \in [0, \tau_u]$ and, in particular, at $t = 1$. Thus, we have shown $I_\lambda(u) \geq 0$, which also implies that $I_{\frac{1}{2},\lambda}(u) \geq 0$, for every $u \in A_{\lambda,\nu}$. Therefore, every $\gamma \in \Gamma_\lambda$ and every $\gamma \in \bar{\Gamma}_\lambda$, where Γ_λ and $\bar{\Gamma}_\lambda$ are given by (2.11) and (2.13) respectively, has to cross $\bar{\mathcal{M}}_{\lambda,\nu}$, and so the claim holds.

Conclusion. Putting the claims together, it is clear that the statement holds. □

2.5. **Palais-Smale sequences.** We recall that a sequence $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ is said to be a Palais-Smale sequence for I_λ at some level $c \in \mathbb{R}$ if

$$I(u_n) \to c, \quad I'(u_n) \to 0, \quad \text{as } n \to \infty.$$

If any such a sequence is relatively compact in the $E(\mathbb{R}^N)$ topology, then we say that the functional I_λ satisfies the Palais-Smale condition at level c.

Lemma 2.7 (**Boundedness of Palais-Smale sequences**). *Assume $N = 3, 4$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ is nonnegative, $q \in [3, 2^* - 1]$, and $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ is a Palais-Smale sequence for I_λ at any level $c > 0$. Then, for any fixed $\lambda > 0$, $(u_n)_{n\in\mathbb{N}}$ is bounded in $E(\mathbb{R}^N)$.*

We stress that our assumption on N yields $3 \leq 2^* - 1$.

Proof. For convenience, set

$$a_n = ||u_n||_{H^1(\mathbb{R}^N)}, \qquad b_n = \lambda\left(\int_{\mathbb{R}^N} \phi_{u_n} u_n^2 \rho(x)\right)^{\frac{1}{2}}, \qquad c_q = \min\left\{\left(\frac{q-1}{2}\right), \left(\frac{q-3}{4}\right)\right\}$$

and note that, as $n \to +\infty$,

$$C_1 + o(1)||u_n||_{E(\mathbb{R}^N)} \geq (q+1)I_\lambda(u_n) - I'_\lambda(u_n)(u_n) = \left(\frac{q-1}{2}\right)a_n^2 + \left(\frac{q-3}{4}\right)b_n^2 \tag{2.17}$$

for some $C_1 > 0$. Assuming $||u_n||_{E(\mathbb{R}^N)} \to +\infty$, we show a contradiction in each of the cases:

(i) a_n, $b_n \to +\infty$,
(ii) a_n bounded and $b_n \to +\infty$,
(iii) $a_n \to +\infty$ and b_n bounded.

First consider $q > 3$. If $b_n \to +\infty$, for large n we have $b_n^2 \geq b_n$ and by (2.17) we get

$$C_1 + o(1)||u_n||_{E(\mathbb{R}^N)} \geq c_q||u_n||^2_{E(\mathbb{R}^N)},\ n \to +\infty,$$

a contradiction in case (i) and (ii). If $a_n \to +\infty$ and b_n is bounded, then $||u_n||_{E(\mathbb{R}^N)} \sim a_n$, hence

$$C_1 + o(1)a_n \geq c_q a_n^2,\ n \to +\infty,$$

a contradiction in case (iii). This makes the proof complete for $q > 3$.
Consider now $q = 3$. By Sobolev inequality we have

$$C_2 \geq I_\lambda(u_n) \geq \frac{1}{2}a_n^2 + \frac{1}{4}b_n^2 - C_3 a_n^4,$$

for some C_2, $C_3 > 0$, which yields a contradiction in case (ii). On the other hand if $a_n \to +\infty$, from the same estimate we have

$$b_n \lesssim a_n^2,\ n \to +\infty. \tag{2.18}$$

Note that (2.17) yields

$$C_1 + o(1)||u_n||_{E(\mathbb{R}^N)} \geq a_n^2,\ \ n \to +\infty. \tag{2.19}$$

Dividing by $||u_n||_{E(\mathbb{R}^N)} = \left(a_n^2 + b_n\right)^{\frac{1}{2}}$, we get $\frac{a_n^4}{a_n^2+b_n} = o(1)$, $n \to +\infty$, hence

$$b_n \gtrsim a_n^4,\ \ n \to +\infty,$$

a contradiction in case (iii). This and (2.18), give

$$a_n^4 \lesssim a_n^2,\ \ n \to +\infty,$$

a contradiction in case (i). This completes the proof. □

Lemma 2.8 (**Lower bound uniform in λ for PS sequences at level c_λ**)**.** *Assume $N = 3, 4, 5$, $\lambda > 0$, $q \in (2, 2^*-1]$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ is nonnegative. There exists a universal constant $\alpha = \alpha(q) > 0$ independent of λ such that for any Palais-Smale sequence $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ for I_λ at level c_λ, it holds that*

$$\liminf_{n\to\infty} \|u_n\|^{q+1}_{L^{q+1}(\mathbb{R}^N)} \geq \alpha.$$

Proof. For every $u \in E(\mathbb{R}^N)$, denoting S_{q+1} the best constant such that $S_{q+1}\|u\|_{L^{q+1}(\mathbb{R}^N)} \leq \|u\|_{H^1(\mathbb{R}^N)}$, we have

$$I_\lambda(u) \geq \frac{1}{2}||u||^2_{H^1(\mathbb{R}^N)} + \frac{\lambda^2}{4}\int_{\mathbb{R}^3} \rho\phi_u u^2 - \frac{S_{q+1}^{-(q+1)}}{q+1}||u||^{q+1}_{H^1(\mathbb{R}^N)}.$$

Since $\omega\lambda^2 \int_{\mathbb{R}^N} \rho\phi_u u^2 = \left(||u||^2_{E(\mathbb{R}^N)} - ||u||^2_{H^1(\mathbb{R}^N)}\right)^2$, estimating the term $||u||^2_{E(\mathbb{R}^N)}||u||^2_{H^1(\mathbb{R}^N)}$ with Young's inequality, we have for any $\beta > 0$

$$I_\lambda(u) \geq \frac{1}{2}||u||^2_{H^1(\mathbb{R}^N)} - \frac{1}{\omega}\left(\frac{\beta-1}{4}\right)||u||^4_{H^1(\mathbb{R}^N)} + \frac{1}{\omega}\left(\frac{\beta-1}{4\beta}\right)||u||^4_{E(\mathbb{R}^N)} - \frac{S_{q+1}^{-(q+1)}}{q+1}||u||^{q+1}_{H^1(\mathbb{R}^N)}.$$

We now pick $\delta = \left(\frac{(q+1)S_{q+1}^{q+1}}{4}\right)^{1/(q-1)}$ and assume $||u||_{E(\mathbb{R}^N)} < \delta$, which also implies that $||u||_{H^1(\mathbb{R}^N)} < \delta$. Then, choosing $\beta > 1$ sufficiently near 1 we obtain

$$\begin{aligned} I_\lambda(u) &\geq \left[\frac{1}{4} - \frac{1}{\omega}\left(\frac{\beta-1}{4}\right)\delta^2\right]||u||^2_{H^1(\mathbb{R}^N)} + \frac{1}{\omega}\left(\frac{\beta-1}{4\beta}\right)||u||^4_{E(\mathbb{R}^N)} \\ &\geq \frac{1}{\omega}\left(\frac{\beta-1}{4\beta}\right)||u||^4_{E(\mathbb{R}^N)}. \end{aligned}$$

We note here that both δ and β depend on q but not on λ. Thus, we have shown that if $||u||_{E(\mathbb{R}^N)} = \delta/2$, then $I_\lambda(u) \geq \underline{c}$, for some $\underline{c} > 0$ independent of λ. So, since every path connecting the origin to where the functional I_λ is negative crosses the sphere of radius $\delta/2$, it follows that

$$c_\lambda \geq \underline{c} \text{ for every } \lambda \geq 0.$$

For convenience, set

$$a_n = ||u_n||_{H^1(\mathbb{R}^N)}, \qquad b_n^2 = \lambda \left(\int_{\mathbb{R}^N} \phi_{u_n} u_n^2 \rho(x) \right)^{\frac{1}{2}},$$

where $(u_n)_{n\in\mathbb{N}}$ is an arbitrary Palais-Smale sequence at the level c_λ. It holds that

$$\begin{aligned} c_\lambda + o(1) - \|I'_\lambda(u_n)\|_{E'(\mathbb{R}^N)} \|u_n\|_{E(\mathbb{R}^N)} &\leq I_\lambda(u_n) - I'_\lambda(u_n) u_n \\ &= \left(\frac{1}{2} - 1\right) a_n^2 + \left(\frac{1}{4} - 1\right) b_n^4 + \left(1 - \frac{1}{q+1}\right) \|u_n\|_{q+1}^{q+1}. \end{aligned}$$

By concavity note that $\|u_n\|_{E(\mathbb{R}^N)} \leq a_n + b_n$, hence the above yields

$$\underline{c} + o(1) \underbrace{- \|I'_\lambda(u_n)\|_{E'(\mathbb{R}^N)}(a_n + b_n) + \frac{1}{2}\left(a_n^2 + b_n^4\right)}_{c_n} \leq \|u_n\|_{q+1}^{q+1},$$

and it is easy to see that $\liminf c_n \geq 0$. The conclusion follows then with $\alpha := \underline{c}$.

□

3. The case of ρ vanishing on a region

Throughout this section we will make the assumption that

(ρ_1) $\rho^{-1}(0)$ has non-empty interior and there exists $\overline{M} > 0$ such that

$$|x \in \mathbb{R}^N : \rho(x) \leq \overline{M}| < \infty.$$

In what follows it is convenient to set

$$A(R) = \{x \in \mathbb{R}^N : |x| > R,\ \rho(x) \geq \overline{M}\},$$

$$B(R) = \{x \in \mathbb{R}^N : |x| > R,\ \rho(x) < \overline{M}\},$$

for any $R > 0$.

Lemma 3.1 (Key vanishing property). *Suppose ρ is a measurable function and that for some $\overline{M} \in \mathbb{R}$ it holds that*

$$\overline{B} := |x \in \mathbb{R}^N : \rho(x) < \overline{M}| < \infty.$$

Then

$$\lim_{R\to\infty} |B(R)| = 0.$$

Proof. The conclusion follows by the dominated convergence theorem as $B(R) \subseteq \overline{B}$ yields

$$|B(R)| = \int_{\overline{B}} \chi_{B(R)}(x)\, \mathrm{d}x \leq |\overline{B}|.$$

□

Lemma 3.2 (Uniform bounds in λ for PS sequences at level c_λ). *Assume $N = 3, 4$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ is nonnegative, satisfying (ρ_1), $q \in [3, 2^*-1]$, $\lambda > 0$. There exists a universal constant $\overline{C} = \overline{C}(q, N) > 0$ independent of λ, such that for any Palais-Smale sequence $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ for I_λ at level c_λ it holds that $\|u_n\|_{E(\mathbb{R}^N)} < \overline{C}$.*

Proof. Let $v \in C_c^\infty(\mathbb{R}^N) \setminus \{0\}$ have support in $\rho^{-1}(0)$. Pick $t_v > 0$ such that $I_0(t_v v) < 0$ and set $v_t = tt_v v$. Then, by definition of c_λ,

$$c_\lambda \leq \max_{t\in[0,1]} I_\lambda(v_t) = \max_{t\geq 0} I_0(tv) =: \bar{c}\ ^{1}. \tag{3.1}$$

Note that since (u_n) is bounded by Lemma 2.7, it holds that

$$\begin{aligned} c_\lambda &= \lim_{n\to\infty} (I_\lambda(u_n) - \frac{1}{q+1} I'_\lambda(u_n)\cdot u_n) \\ &= \lim_{n\to\infty} \Big(\Big(\frac{1}{2} - \frac{1}{q+1}\Big) \|u_n\|^2_{H^1(\mathbb{R}^N)} + \lambda^2 \Big(\frac{1}{4} - \frac{1}{q+1}\Big) \int_{\mathbb{R}^N} \phi_{u_n}\rho(x) u_n^2 \Big). \end{aligned}$$

The conclusion follows immediately in the case $q > 3$. For $q = 3$ the above yields a uniform bound independent on λ for the $H^1(\mathbb{R}^N)$ norm and hence for the $L^{q+1}(\mathbb{R}^N)$ norm as well by Sobolev's inequality. Since

$$\lambda^2 \limsup_{n\to\infty} \int_{\mathbb{R}^N} \phi_{u_n} u_n^2 \rho(x) \leq 4 \left(c_\lambda + \limsup_{n\to\infty} \left(\|u_n\|^2_{H^1(\mathbb{R}^N)} + \|u_n\|^{q+1}_{L^{q+1}(\mathbb{R}^N)} \right) \right),$$

this concludes the proof. □

Lemma 3.3 (Control on the tails of uniformly bounded sequences). *Assume $N = 3, 4, 5$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ is nonnegative, satisfying (ρ_1), and $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ is bounded uniformly with respect to λ. Then, for every $\beta > 0$ there exists $\lambda_\beta > 0$ and $R_\beta > 0$ such that for $\lambda > \lambda_\beta$ and $R > R_\beta$,*

$$||u_n||^3_{L^3(\mathbb{R}^N\setminus B_R)} < \beta.$$

Proof. By Lemma 2.3 we have

$$\lambda \int_{\mathbb{R}^N} \rho(x)|u_n|^3 \leq C\|u_n\|^3_{E(\mathbb{R}^N)} \leq C', \tag{3.2}$$

for some positive constant C' independent of λ. Hence

$$\int_{A(R)} |u_n|^3 \leq \frac{C'}{\lambda \overline{M}}.$$

Also observe that by Hölder's inequality and Lemma 3.1 we have

$$\begin{aligned} \int_{B(R)} |u_n|^3 &\leq \Big(\int_{\mathbb{R}^N} |u_n|^{2^*} \Big)^{\frac{3}{2^*}} \Big(\int_{B(R)} 1 \Big)^{\frac{2^*-3}{2^*}} \\ &\leq C''\|u_n\|^3_{E(\mathbb{R}^N)} \cdot |B(R)|^{\frac{2^*-3}{2^*}} \\ &\leq C'''|B(R)|^{\frac{2^*-3}{2^*}} \to 0. \end{aligned}$$

as $R \to \infty$, again for some uniform constant $C''' > 0$. Note that our assumption on N yields $3 < 2^*$. This is enough to conclude the proof. □

Proposition 5 (Nonzero weak limits of PS sequences at level c_λ for λ large). *Let $N = 3$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ be nonnegative, satisfying (ρ_1), and $q \in [3, 5)$. There exist universal positive constants $\lambda_0 = \lambda_0(q, \overline{M})$ and $\alpha_0 = \alpha_0(q)$, such that if for some $\lambda \geq \lambda_0$, $u \in E(\mathbb{R}^3)$ is the weak limit of a Palais-Smale sequence for I_λ at level c_λ, then it holds that*

$$\int_{\mathbb{R}^3} |u|^3 \,\mathrm{d}x > \alpha_0.$$

[1]In fact this bound holds in dimensions $N = 3, 4, 5$ and every $q \in (2, 2^* - 1]$.

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^3)$ be an arbitrary Palais-Smale sequence at level c_λ. Note that we can pick $\alpha(q) > 0$ independent of λ and of the sequence such that

$$\liminf_{n\to\infty} \|u_n\|^3_{L^3(\mathbb{R}^3)} \geq \alpha(q).$$

Indeed by interpolation

$$\int_{\mathbb{R}^3} |u_n|^{q+1} \leq \Big(\int_{\mathbb{R}^3} |u_n|^3\Big)^{\frac{5-q}{3}} \Big(\int_{\mathbb{R}^3} |u_n|^6\Big)^{\frac{q-2}{3}}$$

and the claim follows by Sobolev inequality and the uniform bound given by Lemma 3.2 and by Lemma 2.8. In particular, recall that by Lemma 3.2, there exists a universal constant $\overline{C} = \overline{C}(q, N) > 0$ independent of λ and of the sequence, such that $\|u_n\|_{E(\mathbb{R}^N)} < \overline{C}$. By Lemma 3.3, it follows than that we can pick $\lambda_0(q, \overline{M})$ and $R_\alpha > 0$ such that such that for every $\lambda \geq \lambda_0$ and every $R > R_\alpha$ we have

$$\limsup_{n\to\infty} \|u_n\|^3_{L^3(\mathbb{R}^3\setminus B_R)} < \frac{\alpha}{2}.$$

By the classical Rellich theorem, passing if necessary to a subsequence, we can assume that $u_n \to u$ in $L^3_{\text{loc}}(\mathbb{R}^3)$. Therefore, for every $R > R_\alpha$, we have

$$\|u\|^3_{L^3(B_R)} = \lim_{n\to\infty} \|u_n\|^3_{L^3(B_R)} \geq \liminf_{n\to\infty} \|u_n\|^3_{L^3(\mathbb{R}^3)} - \limsup_{n\to\infty} \|u_n\|^3_{L^3(\mathbb{R}^3\setminus B_R)} > \frac{\alpha}{2}.$$

The conclusion follows with $\alpha_0 = \alpha/2$. □

Proposition 6 **(Energy estimates for λ large).** *Let $N = 3$, $\rho \in L^\infty_{loc}(\mathbb{R}^3)$ be nonnegative, satisfying (ρ_1), and $q \in [3, 5)$. Let λ_0 be defined as in Proposition 5. There exists a universal constant $\lambda_1 = \lambda_1(q, \overline{M}) > 0$ such that, if $\lambda \geq \max(\lambda_0, \lambda_1)$ and u is the nontrivial weak limit in $E(\mathbb{R}^3)$ of some Palais-Smale sequence $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^3)$ for I_λ at level c_λ, then it holds that*

- $I_\lambda(u) = c_\lambda$, *for* $q \in (3, 5)$,
- $\inf_{v\in\mathcal{N}_\lambda} I_\lambda(v) \leq I_\lambda(u) \leq c_\lambda$, *for* $q = 3$.

In particular, for all $\lambda \geq \max(\lambda_0, \lambda_1)$, the mountain-pass level c_λ is critical for $q \in (3, 5)$, as well as the level $I_\lambda(u)$ for $q = 3$.

Proof. By Proposition 5, for every $q \in [3, 2^* - 1)$ and $\lambda \geq \lambda_0$, passing if necessary to a subsequence, we can assume that $u_n \rightharpoonup u \in E(\mathbb{R}^3) \setminus \{0\}$ weakly in $E(\mathbb{R}^3)$ and almost everywhere, for some Palais-Smale sequence $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^3)$ for I_λ at level c_λ. By a standard argument u is a critical point of I_λ. For sake of clarity we break the proof into two steps.

Step 1: We first show that there exists a universal constant $C = C(q) > 0$ such that for every $\lambda \geq \lambda_0$, $R > 0$ and $n \in \mathbb{N}$, it holds that

$$\begin{aligned} I_\lambda(u_n - u) \geq &\left(\frac{1}{4} - S_\lambda S^{-1}\left(\int_{A(R)} |u_n - u|^6\right)^{\frac{2}{3}}\right) \int_{\mathbb{R}^3} |\nabla(u_n - u)|^2 \\ &- C|B(R)|^{\frac{5-q}{6}} - \frac{1}{q+1} \int_{|x|<R} |u_n - u|^{q+1}, \end{aligned} \tag{3.3}$$

where

$$S_\lambda := (q-2)[3(q+1)]^{\frac{-3}{q-2}} \left(\frac{2(5-q)}{\lambda\overline{M}}\right)^{\frac{5-q}{q-2}},$$

$S = 3(\pi/2)^{4/3}$ is the Sobolev constant, and $\overline{M}$ is defined as in (ρ_1). Reasoning as in Lemma 2.3 and by Lemma 3.3 we obtain,

$$\begin{aligned} I_\lambda(u_n - u) &\geq \frac{1}{4}\int_{\mathbb{R}^3} |\nabla(u_n-u)|^2 + \frac{1}{4}\int_{\mathbb{R}^3} |\nabla(u_n-u)|^2 \\ &\qquad + \frac{\lambda^2}{4}\int_{\mathbb{R}^3} \phi_{(u_n-u)}(u_n-u)^2\rho(x) - \frac{1}{q+1}\int_{\mathbb{R}^3}|u_n-u|^{q+1} \\ &\geq \frac{1}{4}\int_{\mathbb{R}^3} |\nabla(u_n-u)|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^3}\rho(x)|u_n-u|^3 - \frac{1}{q+1}\int_{\mathbb{R}^3}|u_n-u|^{q+1} \\ &\geq \frac{1}{4}\int_{\mathbb{R}^3} |\nabla(u_n-u)|^2 + \frac{\lambda\overline{M}}{2}\int_{A(R)}|u_n-u|^3 - \frac{1}{q+1}\int_{\mathbb{R}^3}|u_n-u|^{q+1}. \end{aligned} \tag{3.4}$$

Note that

$$\int_{\mathbb{R}^3}|u_n-u|^{q+1} = \int_{|x|<R} \dots + \int_{A(R)} \dots + \int_{B(R)} \dots$$

Using that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $E(\mathbb{R}^3)$ and arguing as in Lemma 3.3 and by Sobolev inequality, we have

$$\int_{B(R)}|u_n-u|^{q+1} \leq C_1\|u_n-u\|^{q+1}_{L^6(\mathbb{R}^3)}|B(R)|^{\frac{5-q}{6}} \leq C_2|B(R)|^{\frac{5-q}{6}}. \tag{3.5}$$

By the interpolation and Young's inequalities we obtain for all $\delta > 0$ that

$$\begin{aligned} \frac{1}{q+1}\int_{A(R)}|u_n-u|^{q+1} &\leq \frac{1}{q+1}\left(\int_{A(R)}|u_n-u|^3\right)^{\frac{5-q}{3}}\left(\int_{A(R)}|u_n-u|^6\right)^{\frac{q-2}{3}} \\ &\leq \left(\frac{5-q}{3}\right)\left(\frac{\delta}{q+1}\right)^{\frac{3}{5-q}}\int_{A(R)}|u_n-u|^3 + \left(\frac{q-2}{3}\right)\delta^{\frac{-3}{q-2}}\int_{A(R)}|u_n-u|^6. \end{aligned}$$

In particular, we can set

$$\delta = \left(\frac{\lambda\overline{M}}{2}\cdot\frac{3}{5-q}\right)^{\frac{5-q}{3}}(q+1).$$

Hence

$$\begin{aligned} \frac{1}{q+1}\int_{A(R)}|u_n-u|^{q+1} &\leq \frac{\lambda\overline{M}}{2}\int_{A(R)}|u_n-u|^3 + S_\lambda\int_{A(R)}|u_n-u|^6 \\ &\leq \frac{\lambda\overline{M}}{2}\int_{A(R)}|u_n-u|^3 + S_\lambda S^{-1}\left(\int_{A(R)}|u_n-u|^6\right)^{\frac{2}{3}}\int_{\mathbb{R}^3}|\nabla(u_n-u)|^2, \end{aligned} \tag{3.6}$$

where we have used Sobolev's inequality written as

$$S\left(\int_{A(R)}|u_n-u|^6\right)^{\frac{1}{3}} \leq \int_{\mathbb{R}^3}|\nabla(u_n-u)|^2.$$

Putting together (3.4), (3.5) and (3.6), the claim (3.3) follows.
Step 2: Conclusion. By the classical Brezis-Lieb lemma and Lemma 2.2 we have

$$c_\lambda = \lim_{n\to\infty} I_\lambda(u_n) = I_\lambda(u) + \lim_{n\to\infty} I_\lambda(u_n-u). \tag{3.7}$$

Note that there exists a positive constant $\lambda_1 = \lambda_1(q,\overline{M})$ such that for every $\lambda \geq \lambda_1$ it holds that

$$\frac{1}{4} - S_\lambda S^{-3}\overline{C}^4 \geq 0, \tag{3.8}$$

where $\overline{C}$ is defined via Lemma 3.2 by the property $\|u_n\|_{E(\mathbb{R}^3)} < \overline{C}$. Note that, again by the Brezis-Lieb lemma, we have

$$\int_{A(R)} |u_n - u|^6 = \int_{A(R)} |u_n|^6 - \int_{A(R)} |u|^6 + o_n(R),$$

with $\lim_{n\to\infty} o_n(R) = 0$ for any fixed $R > 0$; since by Sobolev's inequality it holds that

$$\int_{A(R)} |u_n|^6 \leq S^{-3} \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^3 \leq S^{-3}\overline{C}^6,$$

we obtain the estimate

$$\limsup_{R\to\infty} \limsup_{n\to\infty} \int_{A(R)} |u_n - u|^6 \leq S^{-3}\overline{C}^6. \tag{3.9}$$

We conclude, by (3.3), (3.8), (3.9) and the classical Rellich theorem that

$$\begin{aligned}\lim_{n\to\infty} I_\lambda(u_n - u) &\geq \liminf_{R\to\infty} \liminf_{n\to\infty} \left(\frac{1}{4} - S_\lambda S^{-1} \left(\int_{A(R)} |u_n - u|^6 \right)^{\frac{2}{3}} \right) \int_{\mathbb{R}^3} |\nabla(u_n - u)|^2 \\ &\geq \left[\frac{1}{4} - S_\lambda S^{-3}\overline{C}^4 \right] \liminf_{n\to\infty} \int_{\mathbb{R}^3} |\nabla(u_n - u)|^2 \geq 0,\end{aligned}$$

and hence by (3.7) that $I_\lambda(u) \leq c_\lambda$. On the other hand, since $u \in \mathcal{N}_\lambda$, it holds that

$$\inf_{v\in\mathcal{N}_\lambda} I_\lambda(v) \leq I_\lambda(u) \leq c_\lambda,$$

and this completes the proof for $q = 3$. For $q \in (3, 2^* - 1)$, since

$$c_\lambda = \inf_{v\in\mathcal{N}_\lambda} I_\lambda(v),$$

it follows that $I_\lambda(u) = c_\lambda$, and this concludes the proof. □

Remark 3.1 (On the Palais-Smale condition). When $q > 3$, the fact that $\lim I_\lambda(u_n - u) = 0$ for λ large suggests that the Palais-Smale condition at the mountain-pass level c_λ can be recovered in some cases. To illustrate this, note that the assumption (ρ_1) is compatible with having, say $\rho(x) \to 2\overline{M}$, as $|x| \to \infty$, namely a situation where lack of compactness phenomena may occur for the system (1.2) as a consequence of the invariance by translations of (1.8), which plays the role of a 'problem at infinity', see e.g. [40]. We stress here that ρ may approach its limit from below as well as from above. To see that in this case the Palais-Smale condition is satisfied for λ large, denote by $I_\lambda^{\rho\equiv 2\overline{M}}$ the functional associated to $(\mathcal{E})$ with $\rho \equiv 2\overline{M}$, and observe that in this situation $E(\mathbb{R}^3) \simeq H^1(\mathbb{R}^3)$, with equivalent norms by (HLS). Reasoning as in [40, Proposition 1.6], there exist $l \in \mathbb{N}\cup\{0\}$, functions $(v_1, \ldots, v_l) \subset H^1(\mathbb{R}^3)$, and sequences of points $(y_n^j)_{n\in\mathbb{N}} \subset \mathbb{R}^3$, $1 \leq j \leq l$, such that, passing if necessary to a subsequence,

- v_j are possibly nontrivial critical points of $I_\lambda^{\rho\equiv 2\overline{M}}$ for $1 \leq j \leq l$,
- $|y_n^j| \to +\infty$, $|y_n^j - y_n^{j'}| \to +\infty$ as $n \to +\infty$ if $j \neq j'$,
- $||u_n - u - \sum_{j=1}^l v_j(\cdot - y_n^j)||_{H^1(\mathbb{R}^3)} \to 0$ as $n \to +\infty$,
- $c_\lambda = I_\lambda(u) + \sum_{j=1}^l I_\lambda^{\rho\equiv 2\overline{M}}(v_j)$.

It is standard to see that $I_\lambda^{\rho\equiv 2\overline{M}}$ is uniformly bounded below on the set of its nontrivial critical points by a positive constant, independent on λ. It then follows that for all $\lambda \geq \max(\lambda_0, \lambda_1)$, Proposition 6 and the above yield $c_\lambda = I_\lambda(u)$ and at the same time $l = 0$; as a consequence the Palais-Smale condition is satisfied at the level c_λ. These considerations yield the following

Proposition 7 (**Palais-Smale condition under** (ρ_1)). *Let $N = 3 < q$ and $\rho \geq 0$ be locally bounded such that (ρ_1) is satisfied and such that $\rho(x) \to \rho_\infty > \overline{M}$ as $|x| \to \infty$. Let λ_0 and λ_1 be as in Proposition 6. Then, for all $\lambda \geq \max(\lambda_0, \lambda_1)$, I_λ satisfies the Palais-Smale condition at the mountain-pass level c_λ.*

It is not obvious how to prove the above proposition in the case $q = 3$; nevertheless the same considerations on strong convergence apply instead to approximated critical points of I_λ constrained on the Nehari manifold, see the proof Theorem 1.1 and Proposition 8 below.

3.1. **Proof of Theorem 1.1.** Now that we have the necessary preliminaries we present the proof of Theorem 1.1.

Proof of Theorem 1.1. We recall that

$$\mathcal{N}_\lambda \coloneqq \left\{u \in E(\mathbb{R}^3) \setminus \{0\} : G_\lambda(u) = 0\right\},$$

where

$$G_\lambda(u) = I'_\lambda(u)(u) = ||u||^2_{H^1(\mathbb{R}^3)} + \lambda^2 \int_{\mathbb{R}^3} \rho\phi_u u^2 - ||u||^{q+1}_{L^{q+1}(\mathbb{R}^3)}.$$

We note that for all $q \in [3, 2^* - 1)$, it is standard to see that $\mathcal{N}_\lambda$ is nonempty. Moreover, we claim that the conditions

(i) $\exists r > 0 : B_r \cap \mathcal{N}_\lambda = \emptyset$,
(ii) $G'_\lambda(u)(u) \neq 0, \quad \forall u \in \mathcal{N}_\lambda$,

are satisfied, and so, by standard arguments, it follows that the Nehari manifold $\mathcal{N}_\lambda$ is a natural constraint (see e.g. [3]). Indeed, for (i), we notice that if $u \in \mathcal{N}_\lambda$, then

$$0 = ||u||^2_{H^1(\mathbb{R}^3)} + \lambda^2 \int_{\mathbb{R}^3} \rho\phi_u u^2 - ||u||^{q+1}_{L^{q+1}(\mathbb{R}^3)} \geq ||u||^2_{H^1(\mathbb{R}^3)} - S_{q+1}^{-(q+1)}||u||^{q+1}_{H^1(\mathbb{R}^3)},$$

from which it follows that

$$||u||_{E(\mathbb{R}^3)} \geq ||u||_{H^1(\mathbb{R}^3)} \geq S_{q+1}^{(q+1)/(q-1)}, \quad \forall u \in \mathcal{N}_\lambda. \tag{3.10}$$

Setting $r = S_{q+1}^{(q+1)/(q-1)} - \delta$ for some small $\delta > 0$ yields (i). For (ii), we notice that if $u \in \mathcal{N}_\lambda$, then by the definition of the Nehari manifold, the assumption $q \geq 3$ and (3.10), it holds that

$$\begin{aligned} G'_\lambda(u)(u) &= 2||u||^2_{H^1(\mathbb{R}^3)} + 4\lambda^2 \int_{\mathbb{R}^3} \rho\phi_u u^2 - (q+1)||u||^{q+1}_{L^{q+1}(\mathbb{R}^3)} \\ &= (1-q)||u||^2_{H^1(\mathbb{R}^3)} + (3-q)\lambda^2 \int_{\mathbb{R}^3} \rho\phi_u u^2 \\ &\leq (1-q)S_{q+1}^{2(q+1)/(q-1)} \\ &< 0. \end{aligned} \tag{3.11}$$

Thus, the claim holds and so the Nehari manifold is a natural constraint. Now, if $q \in (3, 2^* - 1)$, setting $\lambda_* = \max\{\lambda_0, \lambda_1\}$, the conclusion follows immediately from Proposition 6 and the following characterisation of the mountain-pass level,

$$c_\lambda = \inf_{v \in \mathcal{N}_\lambda} I_\lambda(v).$$

On the other hand, assume $q = 3$ and $\lambda \geq \max\{\lambda_0, \lambda_1\}$. We note that

$$c_\lambda^* \coloneqq \inf_{v \in \mathcal{N}_\lambda} I_\lambda(v)$$

is well-defined since $\mathcal{N}_\lambda$ is nonempty, and so, we take $(\tilde{w}_n)_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda$ to be a minimising sequence for I_λ on $\mathcal{N}_\lambda$, namely, $I_\lambda(\tilde{w}_n) \to c_\lambda^*$. By the Ekeland variational principle (see e.g. [24]), there exists another minimising sequence $(w_n)_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda$ and $\xi_n \in \mathbb{R}$ such that

$$I_\lambda(w_n) \to c_\lambda^*, \tag{3.12}$$

$$I'_\lambda(w_n)(w_n) = 0, \tag{3.13}$$

and

$$I'_\lambda(w_n) - \xi_n G'_\lambda(w_n) \to 0, \qquad \text{in } (E(\mathbb{R}^3))'. \tag{3.14}$$

Now, by Proposition 6, (3.1), (3.12) and (3.13), it holds that

$$\lim_{n\to+\infty}\left(I_\lambda(w_n) - \frac{1}{q+1}I'_\lambda(w_n)(w_n)\right) = c_\lambda^* \leq c_\lambda \leq \bar{c},$$

for some $\bar{c}$ independent of λ. We can therefore argue as in Lemma 3.2 to show that

$$||w_n||_{E(\mathbb{R}^3)} < \bar{C}, \tag{3.15}$$

where $\bar{C} > 0$ is the same constant independent of λ given by Lemma 3.2. Moreover, since $(w_n)_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda$, it follows using (3.10) that

$$||w_n||^4_{L^4(\mathbb{R}^3)} = ||w_n||^2_{H^1(\mathbb{R}^3)} + \lambda^2 \int_{\mathbb{R}^3} \rho\phi_{w_n} w_n^2 \geq ||w_n||^2_{H^1(\mathbb{R}^3)} \geq S_4^4 > 0.$$

Thus, by interpolation it holds

$$S_4^4 \leq \int_{\mathbb{R}^3} |w_n|^4 \leq \left(\int_{\mathbb{R}^3} |w_n|^3\right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^3} |w_n|^6\right)^{\frac{1}{3}},$$

and so, by the Sobolev inequality and (3.15), it follows that we can pick $\alpha > 0$ independent of λ such that

$$\liminf_{n\to\infty} \|w_n\|^3_{L^3(\mathbb{R}^3)} \geq \alpha.$$

Moreover, by Lemma 3.3, we can set $\lambda_* = \max\{\lambda_0, \lambda_1\}$ and $R_\alpha > 0$ such that such that for every $\lambda \geq \lambda_*$ and every $R > R_\alpha$ we have

$$\limsup_{n\to\infty} \|w_n\|^3_{L^3(\mathbb{R}^3\setminus B_R)} < \frac{\alpha}{2}.$$

Now, since $(w_n)_{n\in\mathbb{N}}$ is bounded, passing if necessary to a subsequence, we can assume that $w_n \rightharpoonup w$ in $E(\mathbb{R}^3)$ and $w_n \to w$ in $L^3_{\text{loc}}(\mathbb{R}^3)$. It follows that for every $\lambda \geq \lambda_*$ and $R > R_\alpha$,

$$\|w\|^3_{L^3(B_R)} \geq \liminf_{n\to\infty} \|w_n\|^3_{L^3(\mathbb{R}^3)} - \limsup_{n\to\infty} \|w_n\|^3_{L^3(\mathbb{R}^3\setminus B_R)} > \frac{\alpha}{2},$$

and so $w \not\equiv 0$. We now notice that by (3.13), (3.14), and (3.15), it holds, up to a constant independent of λ, that

$$\begin{aligned} o(1) &= ||I'_\lambda(w_n) - \xi_n G'_\lambda(w_n)||_{(E(\mathbb{R}^3))'} \\ &\gtrsim |I'_\lambda(w_n)(w_n) - \xi_n G'_\lambda(w_n)(w_n)| \\ &= |\xi_n G'_\lambda(w_n)(w_n)|, \end{aligned}$$

for some $\xi_n \in \mathbb{R}$. Since $(w_n) \subset \mathcal{N}_\lambda$, by (3.11), we have that $G'_\lambda(w_n)(w_n) < -2S_4^4 < 0$, and so the above yields $\xi_n \to 0$. Moreover, using (3.15) and the inequality

$$|D(f,g)|^2 \leq D(f,f)D(g,g),$$

where

$$D(f,g) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f(x)g(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y,$$

for f, g measurable and nonnegative functions (see [33, p.250]), it follows that $G'_\lambda(w_n)$ is bounded. Taken together, we have that $\xi_n G'_\lambda(w_n) \to 0$, and using this and (3.14), we obtain $I'_\lambda(w_n) \to 0$. Hence, $(w_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence for I_λ at level c^*_λ, and so, since we have also shown that $w_n \rightharpoonup w \not\equiv 0$ in $E(\mathbb{R}^3)$, a standard argument yields that w is a nontrivial critical point of I_λ. Namely, $w \in \mathcal{N}_\lambda$, and thus

$$c^*_\lambda \leq I_\lambda(w). \tag{3.16}$$

On the other hand, arguing as in Proposition 6, replacing u_n, u, and c_λ with w_n, w, and c^*_λ, respectively, for every $\lambda \geq \lambda_*$, we obtain

$$I_\lambda(w) \leq c^*_\lambda. \tag{3.17}$$

For the reader convenience we recall that λ_1 is chosen in Proposition 6 so that for every $\lambda \geq \lambda_1$, it holds that $\frac{1}{4} - S_\lambda S^{-3}\bar{C}^4 \geq 0$, where $\bar{C}$ is defined via Lemma 3.2 by the property $||u_n||_{E(\mathbb{R}^3)} < \bar{C}$. Going through the same argument with $(w_n)_{n\in\mathbb{N}}$, since $(w_n)_{n\in\mathbb{N}}$ is bounded by precisely the same uniform constant, namely $||w_n||_{E(\mathbb{R}^3)} < \bar{C}$, we conclude that (3.17) holds for every $\lambda \geq \lambda_*$, as $\lambda_* \geq \lambda_1$ by construction. Putting (3.16) and (3.17) together yields

$$I_\lambda(w) = \inf_{v\in\mathcal{N}_\lambda} I_\lambda(v).$$

Since $I_\lambda(w) = I_\lambda(|w|)$ and $w \in \mathcal{N}_\lambda$ if and only if $|w| \in \mathcal{N}_\lambda$, we can assume $w \geq 0$, and it follows that $w > 0$ by Proposition 1. This completes the proof. □

As a byproduct of the above proof, we have the following

Proposition 8 (Constrained Palais-Smale condition under (ρ_1)). *Let $N = 3 = q$ and $\rho \geq 0$ be locally bounded such that (ρ_1) is satisfied and such that $\rho(x) \to \rho_\infty > \overline{M}$ as $|x| \to \infty$. Let λ_0 and λ_1 be as in Proposition 6. Then, for all $\lambda \geq \max(\lambda_0, \lambda_1)$, the restriction $I_\lambda|_{\mathcal{N}_\lambda}$ satisfies the Palais-Smale condition at the level*

$$c^*_\lambda = \inf_{v\in\mathcal{N}_\lambda} I_\lambda(v).$$

That is, every sequence $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^3) \simeq H^1(\mathbb{R}^3)$ such that

$$I(u_n) \to c^*_\lambda, \qquad \nabla I_\lambda(u_n)|_{\mathcal{N}_\lambda} \to 0 \text{ in } H^{-1}(\mathbb{R}^3)$$

is relatively compact.

Proof. The proof follows reasoning exactly as in Remark 3.1. We leave out the details. □

4. The case of coercive ρ

In the present section $\lambda > 0$ is an arbitrary fixed value, and on ρ we make the assumption that

(ρ_2) For every $M > 0$,

$$|x \in \mathbb{R}^N : \rho(x) \leq M| < \infty.$$

Lemma 4.1 (Compactness property). *Let $N = 3, 4, 5$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ be nonnegative, satisfying (ρ_2), and $q \in (1, 2^* - 1)$. Then, $E(\mathbb{R}^N)$ is compactly embedded into $L^{q+1}(\mathbb{R}^N)$.*

Proof. By Lemma 2.3, multiplying by λ we obtain

$$\lambda \int_{\mathbb{R}^N} \rho(x)|u|^3 \leq \left(\frac{1}{\omega}\right)^{\frac{1}{2}} \|u\|^3_{E(\mathbb{R}^N)}. \tag{4.1}$$

Set

$$A(R) = \{x \in \mathbb{R}^N : |x| > R,\ \rho(x) \geq M\},$$

$$B(R) = \{x \in \mathbb{R}^N : |x| > R,\ \rho(x) < M\}.$$

Without loss of generality, assume that $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ is such that $u_n \rightharpoonup 0$. For convenience, write

$$\int_{\mathbb{R}^N \setminus B_R} |u_n|^3 = \int_{A(R)} |u_n|^3 + \int_{B(R)} |u_n|^3$$

where B_R is a ball of radius R centred at the origin. Fix $\delta > 0$ and pick M, r, C, such that $M > \frac{2}{\lambda\delta}\left(\frac{1}{\omega}\right)^{\frac{1}{2}} \sup_n \|u_n\|^3_{E(\mathbb{R}^N)}$, $r = \frac{2^*}{3} > 1$ and

$$C \geq \sup_{u \in E(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^3_{L^{2^*}(\mathbb{R}^N)}}{\|u\|^3_{E(\mathbb{R}^N)}}.$$

Let $\frac{1}{r} + \frac{1}{r'} = 1$. By Lemma 3.1, for every $M > 0$, and every $R > 0$ large enough, it holds that

$$|B(R)| \leq \Big[\frac{\delta}{2C \sup_n \|u_n\|^3_{E(\mathbb{R}^N)}}\Big]^{r'}. \tag{4.2}$$

Since $N = 3, 4, 5$, we can pick $r = \frac{2^*}{3} > 1$ such that by Hölder inequality it holds that

$$\begin{aligned}\int_{B(R)} |u_n|^3 &\leq \Big(\int_{B(R)} |u_n|^{2^*}\Big)^{\frac{1}{r}} \Big(\int_{B(R)} 1\Big)^{\frac{1}{r'}} \\ &\leq \|u_n\|^3_{L^{2^*}(\mathbb{R}^N)} \cdot |B(R)|^{\frac{1}{r'}} \\ &\leq C\|u_n\|^3_{E(\mathbb{R}^N)} \cdot |B(R)|^{\frac{1}{r'}} \leq \frac{\delta}{2},\end{aligned}$$

Moreover, by our choice of M and (4.1), we see that

$$\int_{A(R)} |u_n|^3 \leq \frac{1}{\lambda M}\left(\frac{1}{\omega}\right)^{\frac{1}{2}} ||u_n||^3_{E(\mathbb{R}^N)} \leq \frac{\delta}{2}.$$

By the classical Rellich theorem, and since δ was arbitrary, this is enough to prove our lemma for $q = 2$. By interpolation the case $q \neq 2$ follows immediately, and this concludes the proof. □

Using the above lemma, and for $q \geq 3$, it is easy to see that the Palais-Smale condition holds for I_λ at any level.

Lemma 4.2 (**Palais-Smale condition**). *Let $N = 3$, $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ be nonnegative, satisfying (ρ_2), and $q \in [3, 2^* - 1)$. Then, I_λ satisfies the Palais-Smale condition at every level $c \in \mathbb{R}$.*

Proof. Since by Lemma 4.1 the embedding of $E(\mathbb{R}^3)$ into $L^{q+1}(\mathbb{R}^3)$ is compact, using Lemma 2.2, the conclusion follows arguing as in [17, p. 1077]. □

4.1. **Proof of Theorem 1.2 and Theorem 1.3.**

Proof of Theorem 1.2. Using Lemma 2.5 and Lemma 4.2, the Mountain-Pass Theorem yields the existence a mountain-pass type solution for all $q \in [3, 2^* - 1)$. Namely, there exists $u \in E(\mathbb{R}^N)$ such that $I_\lambda(u) = c_\lambda$ and $I'_\lambda(u) = 0$, where c_λ is given in (2.12). For $q > 3$, the mountain-pass level c_λ has the characterisation

$$c_\lambda = \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u), \quad \mathcal{N}_\lambda := \{u \in E(\mathbb{R}^N) \setminus \{0\} \mid I'_\lambda(u)(u) = 0\},$$

and it follows that u is a groundstate solution of I_λ. Since $I_\lambda(u) = I_\lambda(|u|)$, we can assume $u \geq 0$, and so $u > 0$ by the strong maximum principle, Proposition 1. For $q = 3$, we can show the existence of a positive mountain-pass solution applying the general min-max principle [53, p.41], and observing that, in our context, we can restrict to admissible curves γ's which map into the positive cone $P \coloneqq \{u \in E(\mathbb{R}^3) : u \geq 0\}$. In fact, arguing as in [41, p.481], since I_λ satisfies the mountain-pass geometry by Lemma 2.5, it is possible to select a Palais-Smale sequence $(u_n)_{n\in\mathbb{N}}$ at the level c_λ such that

$$\operatorname{dist}(u_n, P) \to 0,$$

from which it follows that $(u_n)_- \to 0$ in $L^6(\mathbb{R}^3)$, see also [16, Lemma 2.2]. Then, by construction and up to a subsequence, there exists a weak limit $u \geq 0$, and hence, by Lemma 4.2 a nontrivial nonnegative solution, the positivity of which holds by Proposition 1.
The existence of a positive groundstate can be shown with a mild modification to the proof of Theorem 1.1, using here that all the relevant convergence statements hold for any fixed $\lambda > 0$ as a consequence of assumption (ρ_2) and Lemma 4.1. This is enough to conclude. □

Proof of Theorem 1.3. We can argue as in [40, Theorem 1.3], based on [31] and on the compactness of the embedding of $E(\mathbb{R}^N)$ into $L^{q+1}(\mathbb{R}^N)$. The latter is provided in our context by Lemma 4.1. By these, there exists an increasing sequence $\mu_n \to 1$ and $(u_n)_{n\in\mathbb{N}} \in E(\mathbb{R}^N)$ such that $I_{\mu_n,\lambda}(u_n) = c_{\mu_n,\lambda}$ and $I'_{\mu_n,\lambda}(u_n) = 0$, where $I_{\mu_n,\lambda}$ and $c_{\mu_n,\lambda}$ are defined as in (2.9) and (2.10). By Lemma 2.4, we see that

$$\frac{N-2}{2}\int_{\mathbb{R}^N}(|\nabla u_n|^2 + u_n^2) + \left(\frac{N+2+2k}{4}\right)\int_{\mathbb{R}^N}\rho(x)\phi_{u_n}u_n^2 - \frac{N\mu_n}{q+1}\int_{\mathbb{R}^N}|u_n|^{q+1} \leq 0. \tag{4.3}$$

Setting $\alpha_n = \int_{\mathbb{R}^N}(|\nabla u_n|^2 + u_n^2)$, $\gamma_n = \lambda^2\int_{\mathbb{R}^N}\rho(x)\phi_{u_n}u_n^2$, $\delta_n = \mu_n\int_{\mathbb{R}^N}|u_n|^{q+1}$, we can put together the equalities $I_{\mu_n,\lambda}(u_n) = c_{\mu_n,\lambda}$ and $I'_{\mu_n,\lambda}(u_n)(u_n) = 0$ with (4.3) obtaining the system

$$\begin{cases} \alpha_n + \gamma_n - \delta_n = 0, \\ \frac{1}{2}\alpha_n + \frac{1}{4}\gamma_n - \frac{1}{q+1}\delta_n = c_{\mu_n,\lambda}, \\ \frac{N-2}{2}\alpha_n + \left(\frac{N+2+2k}{4}\right)\gamma_n - \frac{N}{q+1}\delta_n \leq 0, \end{cases} \tag{4.4}$$

which yields

$$\delta_n \leq \frac{c_{\mu_n,\lambda}(6-N+2k)(q+1)}{2(q-2)+k(q-1)}, \quad \gamma_n \leq \frac{2c_{\mu_n,\lambda}\big(2(q+1)-N(q-1)\big)}{2(q-2)+k(q-1)}, \quad \text{and } \alpha_n = \delta_n - \gamma_n.$$

We note that $k > \frac{-2(q-2)}{(q-1)} > \frac{N-6}{2}$ since $q < 2^* - 1$, and so since $\alpha_n, \gamma_n, \delta_n$ are all nonnegative, it follows that $\alpha_n, \gamma_n, \delta_n$ are all bounded. Hence the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded and there exists $u \in E(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \rightharpoonup u$ in $E(\mathbb{R}^N)$. Using Lemma 4.1 and arguing as in [17, Theorem 1] we obtain that $\|u_n\|^2_{E(\mathbb{R}^N)} \to \|u\|^2_{E(\mathbb{R}^N)}$ and

$$c_{\mu_n,\lambda} = I_{\mu_n,\lambda}(u_n) \to I_\lambda(u). \tag{4.5}$$

It follows that $u_n \to u$ in $E(\mathbb{R}^N)$, which combined with the left-continuity property of the levels [5, Lemma 2.2], namely $c_{\mu_n,\lambda} \to c_{1,\lambda} = c_\lambda$ as $\mu_n \nearrow 1$, yields $I_\lambda(u) = c_\lambda$. Since u is a critical point by the weak convergence, it follows that u is mountain-pass solution. Finally, the existence of a groundstate solution is based on minimising over the set of nontrivial critical points of I_λ, and carrying out an identical argument to the above to show the strong convergence of such a minimising sequence, again using Lemma 4.1. This concludes the proof. □

4.2. **Proof of Theorem 1.4.** Under an additional hypotheses on ρ, we now prove that the energy level of the groundstate solutions coincide with the mountain-pass level.

Proof of Theorem 1.4. By Proposition 4, it holds that

$$c_\lambda = \inf_{u\in\bar{\mathcal{M}}_{\lambda,\nu}} I_\lambda(u),$$

where $\bar{\mathcal{M}}_{\lambda,\nu}$ is defined in (2.14). Since $J_{\lambda,\nu}(u) = 0$ is equivalent to the Pohozaev equation given by Lemma 2.4 minus the equation $\nu I'_\lambda(u)(u) = 0$, it is clear that $\bar{\mathcal{M}}_{\lambda,\nu}$ contains all of the critical points of I_λ, and thus the mountain-pass solutions that we find in Theorem 1.2 ($q = 3$) and Theorem 1.3 ($q < 3$) are groundstates. This completes the proof. □

5. Multiplicity results: coercive ρ

In the current section, we discuss the existence of high energy solutions in the case ρ satisfies (ρ_2). Throughout what follows, we denote the unit ball in $E(\mathbb{R}^N)$ by B_1. Moreover, since λ does not play any role and can be fixed arbitrarily under assumption (ρ_2), we set $\lambda \equiv 1$ for the sake of simplicity and define

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2) + \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)\rho(x)u^2(y)\rho(y)}{|x-y|^{N-2}}\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{q+1}\int_{\mathbb{R}^N}|u|^{q+1}.$$

5.1. Preliminaries.

We will now discuss some preliminaries that will be used in proving both Theorem 1.5 and 1.6. Following [4] we set

$$\hat{A}_0 = \{u \in E(\mathbb{R}^N) : I(u) \geq 0\}, \tag{5.1}$$

and

$$\Gamma^* = \{h \in C(E(\mathbb{R}^N), E(\mathbb{R}^N)) : h(0) = 0, h \text{ is an odd homeomorphism of } E(\mathbb{R}^N) \text{ onto } E(\mathbb{R}^N), h(B_1) \subset \hat{A}_0\}. \tag{5.2}$$

In the next lemma, we establish a result that allows us to obtain high energy solutions in our Banach space setting. Before stating the lemma, we note that since the biorthogonal system given by Lemma 2.1 is fundamental, then, for any $m \in \mathbb{N}$, it holds that

$$E(\mathbb{R}^N) = \mathrm{span}\{e_1, \dots, e_m\} \oplus \overline{\mathrm{span}}\{e_{m+1}, \dots\}.$$

Thus, throughout what follows we set

$$E_m = \mathrm{span}\{e_1, \dots, e_m\},$$
$$E_m^\perp = \overline{\mathrm{span}}\{e_{m+1}, \dots\},$$

and note that, for any $m \in \mathbb{N}$, E_m and $E_m^\perp$ define algebraically and topologically complementary subspaces of $E(\mathbb{R}^N)$.

Lemma 5.1 (**Divergence of min-max levels d_m**)**.** *Let $N \geq 3$ and $q > 1$. Suppose $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ is nonnegative, satisfying (ρ_2). Define*

$$d_m := \sup_{h \in \Gamma^*} \inf_{u \in \partial B_1 \cap E_{m-1}^\perp} I(h(u)), \tag{5.3}$$

where Γ^* is given by (5.2). Then, $d_m \to +\infty$ as $m \to +\infty$.

Proof. First we set

$$T = \left\{u \in E(\mathbb{R}^N) \setminus \{0\} : ||u||^2_{H^1(\mathbb{R}^N)} = ||u||^{q+1}_{L^{q+1}(\mathbb{R}^N)}\right\}$$

and

$$\tilde{d}_m = \inf_{u \in T \cap E_m^\perp} ||u||_{E(\mathbb{R}^N)},$$

and claim that $\tilde{d}_m \to +\infty$ as $m \to +\infty$. To see this, assume to the contrary that there exists $u_m \in T \cap E_m^\perp$ and some $d > 0$ such that $||u_m||_{E(\mathbb{R}^N)} \leq d$ for all $m \in \mathbb{N}$. Since $< e_n^*, u_m >= 0$ for all $m \geq n$ and the e_n^*'s are total by Lemma 2.1, then it follows that $u_m \rightharpoonup 0$ in $E(\mathbb{R}^N)$ (see e.g. [52]). Since $E(\mathbb{R}^N)$ is compactly embedded into $L^{q+1}(\mathbb{R}^N)$ by Lemma 4.1, it follows that $u_m \to 0$ in $L^{q+1}(\mathbb{R}^N)$. However, since $u_m \in T$, it follows from the Sobolev inequality that

$$||u_m||^{q+1}_{H^1(\mathbb{R}^N)} \geq S^{q+1}_{q+1}||u_m||^{q+1}_{L^{q+1}(\mathbb{R}^N)} = S^{q+1}_{q+1}||u_m||^2_{H^1(\mathbb{R}^N)},$$

from which we deduce

$$||u_m||^{q+1}_{L^{q+1}(\mathbb{R}^N)} \geq S_{q+1}^{2(q+1)/(q-1)} > 0.$$

This shows that u_m is bounded away from 0 in $L^{q+1}(\mathbb{R}^N)$, a contradiction, and so we have proved that

$$\tilde{d}_m \to +\infty \text{ as } m \to +\infty. \tag{5.4}$$

Now notice that since E_m and $E_m^{\perp}$ are complementary subspaces, it holds that there exists a $\bar{C} \geq 1$ such that each $u \in B_1$ can be uniquely written as

$$u = v + w, \text{ with } v \in E_m, w \in E_m^{\perp}, \tag{5.5}$$

$$||v||_{E(\mathbb{R}^N)} \leq \bar{C}||u||_{E(\mathbb{R}^N)} \leq \bar{C}, \tag{5.6}$$

$$||w||_{E(\mathbb{R}^N)} \leq \bar{C}||u||_{E(\mathbb{R}^N)} \leq \bar{C}, \tag{5.7}$$

as a consequence of the open mapping theorem, see [19, p.37]. Define $h_m : E_m^{\perp} \to E_m^{\perp}$ by

$$h_m(u) = (\bar{C}K)^{-1}\tilde{d}_m u,$$

where

$$K > \max\left\{1, \left(\frac{4}{q+1}\right)^{\frac{1}{q-1}}\right\},$$

and note that h_m is an odd homeomorphism of $E_m^{\perp}$ onto $E_m^{\perp}$. Now, for any $u \in E(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $\beta(u) > 0$ such that $\beta(u)u \in T$, namely

$$\beta(u) = \left(\frac{||u||^2_{H^1(\mathbb{R}^N)}}{||u||^{q+1}_{L^{q+1}(\mathbb{R}^N)}}\right)^{\frac{1}{q-1}}. \tag{5.8}$$

If we define

$$I_0(u) = \frac{1}{2}||u||^2_{H^1(\mathbb{R}^N)} - \frac{1}{q+1}||u||^{q+1}_{L^{q+1}(\mathbb{R}^N)},$$

then for each $u \in E(\mathbb{R}^N) \setminus \{0\}$, it holds that

$$I_0(tu) = \frac{t^2}{2}||u||^2_{H^1(\mathbb{R}^N)} - \frac{t^{q+1}}{q+1}||u||^{q+1}_{L^{q+1}(\mathbb{R}^N)}$$

is a monotone increasing function for $t \in [0, \beta(u)]$ with a maximum at $t = \beta(u)$. Note that for each $u \in (E_m^{\perp} \cap B_{\bar{C}}) \setminus \{0\}$, by the definition of $\tilde{d}_m$ and $\beta(u)$, we have

$$\bar{C}^{-1}\tilde{d}_m \leq \bar{C}^{-1}||\beta(u)u||_{E(\mathbb{R}^N)} \leq \beta(u), \tag{5.9}$$

and so since $K \geq 1$, it holds that

$$(\bar{C}K)^{-1}\tilde{d}_m \leq \bar{C}^{-1}\tilde{d}_m \leq \beta(u), \quad \text{for all } u \in (E_m^{\perp} \cap B_{\bar{C}}) \setminus \{0\}.$$

Putting everything together, it follows that

$$I_0(h_m(u)) = I_0((\bar{C}K)^{-1}\tilde{d}_m u) > 0 \quad \text{for all } u \in (E_m^{\perp} \cap B_{\bar{C}}) \setminus \{0\}.$$

Moreover,

$$h_m(0) = 0.$$

Therefore,

$$h_m(E_m^{\perp} \cap B_{\bar{C}}) \subset \left\{u \in E(\mathbb{R}^N) : I_0(u) \geq 0\right\}. \tag{5.10}$$

Now, for each $m \in \mathbb{N}$ and some $\delta > 0$, define $\tilde{h}_m : E_m \times E_m^{\perp} \to E_m \times E_m^{\perp}$ by

$$\tilde{h}_m([v, w]) = [\delta v, (\bar{C}K)^{-1}\tilde{d}_m w].$$

Notice that $\tilde{h}_m$ is an odd homeomorphism of $E_m \times E_m^{\perp}$ onto $E_m \times E_m^{\perp}$. Moreover, by (5.5), the function $g_m : E_m \times E_m^{\perp} \to E(\mathbb{R}^N)$ defined by

$$g_m([v,w]) = v + w,$$

is an odd homeomorphism. Hence, defining $H_m : E(\mathbb{R}^N) \to E(\mathbb{R}^N)$ as

$$H_m = g_m \circ \tilde{h}_m \circ g_m^{-1},$$

we see that H_m is an odd homeomorphism of $E(\mathbb{R}^N)$ onto $E(\mathbb{R}^N)$. By (5.5)-(5.7), it holds that

$$B_1 \subseteq g_m(\{E_m \cap B_{\bar{C}}\} \times \{E_m^{\perp} \cap B_{\bar{C}}\}),$$

and so

$$\begin{aligned} H_m(B_1) &\subseteq H_m(g_m(\{E_m \cap B_{\bar{C}}\} \times \{E_m^{\perp} \cap B_{\bar{C}}\})) \\ &= g_m(\tilde{h}_m(\{E_m \cap B_{\bar{C}}\} \times \{E_m^{\perp} \cap B_{\bar{C}}\})) \\ &= g_m(\{\delta(E_m \cap B_{\bar{C}})\} \times \{\bar{C}^{-1}K^{-1}\tilde{d}_m(E_m^{\perp} \cap B_{\bar{C}})\}) \\ &= \Big\{u \in E(\mathbb{R}^N) : u = v + w, v \in \delta(E_m \cap B_{\bar{C}}), w \in \bar{C}^{-1}K^{-1}\tilde{d}_m(E_m^{\perp} \cap B_{\bar{C}})\Big\} \\ &=: Z_{m,\delta}. \end{aligned} \tag{5.11}$$

Now, fix $m \in \mathbb{N}$. We claim that

$$Z_{m,\delta} \subset \big\{u \in E(\mathbb{R}^N) : I_0(u) > 0\big\} \cup \{0\}$$

for some $\delta = \delta(m) > 0$. To see this, assume, by contradiction, that there exists $\delta_j \to 0$ and $u_j \notin \big\{u \in E(\mathbb{R}^N) : I_0(u) > 0\big\} \cup \{0\}$ such that $u_j \in Z_{m,\delta_j}$. Then, by definition of Z_{m,δ_j}, it holds that

$$||u_j||_{E(\mathbb{R}^N)} \leq ||v_j||_{E(\mathbb{R}^N)} + ||w_j||_{E(\mathbb{R}^N)} \leq \delta_j \bar{C} + K^{-1}\tilde{d}_m,$$

which implies u_j is bounded. Thus, up to a subsequence $u_j \rightharpoonup \bar{u}$ in $E(\mathbb{R}^N)$ and so it follows that $u_j \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^N)$. Moreover, since $E(\mathbb{R}^N)$ is compactly embedded into $L^{q+1}(\mathbb{R}^N)$ by Lemma 4.1, it follows that $u_j \to \bar{u}$ in $L^{q+1}(\mathbb{R}^N)$, with $||\bar{u}||^{q+1}_{L^{q+1}(\mathbb{R}^N)} > 0$ by previous arguments. Thus, by the weak lower semicontinuity of the $H^1(\mathbb{R}^N)$ norm and the strong convergence in $L^{q+1}(\mathbb{R}^N)$, we deduce that

$$\frac{1}{2}||\bar{u}||^2_{H^1(\mathbb{R}^N)} \leq \frac{1}{q+1}||\bar{u}||^{q+1}_{L^{q+1}(\mathbb{R}^N)},$$

which implies $\bar{u} \notin \big\{u \in E(\mathbb{R}^N) : I_0(u) > 0\big\} \cup \{0\}$. On the other hand, since $\delta_j \to 0$, then $v_j \to 0$. It follows from this and (5.10) that $\bar{u} \in \bar{C}^{-1}K^{-1}\tilde{d}_m(E_m^{\perp} \cap B_{\bar{C}}) \subset \big\{u \in E(\mathbb{R}^N) : I_0(u) > 0\big\} \cup \{0\}$. Hence, we have reached a contradiction and so the claim holds. Thus, using this and (5.11), for each $m \in \mathbb{N}$, we pick $\delta = \delta(m) > 0$ so that

$$H_m(B_1) \subset \big\{u \in E(\mathbb{R}^N) : I_0(u) > 0\big\} \cup \{0\} \subset \big\{u \in E(\mathbb{R}^N) : I(u) \geq 0\big\} = \hat{A}_0,$$

namely $H_m \in \Gamma^*$, where $\hat{A}_0$ and Γ^* are given by (5.1) and (5.2), respectively. We can therefore see that

$$d_{m+1} = \sup_{h \in \Gamma^*} \inf_{u \in \partial B_1 \cap E_m^{\perp}} I(h(u)) \geq \inf_{u \in \partial B_1 \cap E_m^{\perp}} I(H_m(u)). \tag{5.12}$$

Now take $u \in \partial B_1 \cap E_m^{\perp}$. Then, using (5.8), (5.9) and the fact that $\int_{\mathbb{R}^N} \rho\phi_u u^2 = \omega^{-1}(1 - ||u||^2_{H^1(\mathbb{R}^N)})^2$, it holds that

$$I(H_m(u)) = \frac{1}{2}(\bar{C}^{-1}K^{-1}\tilde{d}_m)^2||u||^2_{H^1(\mathbb{R}^N)} + \frac{1}{4}(\bar{C}^{-1}K^{-1}\tilde{d}_m)^4 \int_{\mathbb{R}^N} \rho\phi_u u^2$$

$$
\begin{aligned}
&\quad - \frac{1}{q+1}(\bar{C}K)^{-q-1}\tilde{d}_m^{q+1}||u||^{q+1}_{L^{q+1}(\mathbb{R}^N)} \\
&= \frac{1}{2}(\bar{C}^{-1}K^{-1}\tilde{d}_m)^2||u||^2_{H^1(\mathbb{R}^N)} + \frac{1}{4}(\bar{C}^{-1}K^{-1}\tilde{d}_m)^4\int_{\mathbb{R}^N}\rho\phi_u u^2 \\
&\quad - \frac{(\bar{C}K)^{-q-1}\tilde{d}_m^2}{q+1}\left(\frac{\tilde{d}_m}{\beta(u)}\right)^{q-1}||u||^2_{H^1(\mathbb{R}^N)} \\
&\geq \frac{1}{2}(\bar{C}^{-1}K^{-1}\tilde{d}_m)^2\left(1-\frac{2K^{1-q}}{q+1}\right)||u||^2_{H^1(\mathbb{R}^N)} \\
&\quad + \frac{1}{4\omega}(\bar{C}^{-1}K^{-1}\tilde{d}_m)^4\left(1-||u||^2_{H^1(\mathbb{R}^N)}\right)^2 \\
&\geq \min\left\{K_1\tilde{d}_m^2, K_2\tilde{d}_m^4\right\}\left(||u||^4_{H^1(\mathbb{R}^N)} - ||u||^2_{H^1(\mathbb{R}^N)} + 1\right) \\
&\geq \frac{3}{4}\min\left\{K_1\tilde{d}_m^2, K_2\tilde{d}_m^4\right\},
\end{aligned}
$$

where $K_1 \geq \frac{1}{4\bar{C}^2K^2}$ by our choice of K and $K_2 = \frac{1}{4\omega\bar{C}^4K^4}$. Finally, using this, (5.12), and (5.4), we obtain

$$
\begin{aligned}
d_{m+1} &\geq \inf_{u\in\partial B_1\cap E_m^\perp} I(H_m(u)) \\
&\geq \frac{3}{4}\min\left\{K_1\tilde{d}_m^2, K_2\tilde{d}_m^4\right\} \to +\infty, \qquad \text{as } m\to+\infty.
\end{aligned}
$$

This completes the proof. □

5.2. **Proof of Theorem 1.5.** In order to prove Theorem 1.5, we will need some background material including the notion of the Krasnoselskii-genus and its properties. Throughout what follows we let G be a compact topological group. Following [24], we begin with a number of definitions that we will need before introducing the notion of the Krasnoselskii-genus.

Definition 2 (**Isometric representation**)**.** The set $\{T(g) : g \in G\}$ is an isometric representation of G on E if $T(g) : E \to E$ is an isometry for each $g \in G$ and the following hold:

(i) $T(g_1 + g_2) = T(g_1)\circ T(g_2)$ for all $g_1, g_2 \in G$
(ii) $T(0) = I$, where $I : E \to E$ is the identity map on E
(iii) $(g, u) \mapsto T(g)(u)$ is continuous.

Definition 3 (**Invariant subset**)**.** A subset $A \subset E$ is invariant if $T(g)A = A$ for all $g \in G$.

Definition 4 (**Equivariant mapping**)**.** A mapping R between two invariant subsets A_1 and A_2, namely $R : A_1 \to A_2$, is said to be equivariant if $R\circ T(g) = T(g)\circ R$ for all $g \in G$.

Definition 5 (**The class $\mathcal{A}$**)**.** We denote the class of all closed and invariant subsets of E by $\mathcal{A}$. Namely,

$$\mathcal{A} := \{A \subset E : A \text{ closed}, T(g)A = A\ \forall g \in G\}.$$

Definition 6 (**G-index with respect to $\mathcal{A}$**)**.** A G-index on E with respect to $\mathcal{A}$ is a mapping $\text{ind} : \mathcal{A} \to \mathbb{N}\cup\{+\infty\}$ such that the following hold:

(i) $\text{ind}(A) = 0$ if and only if $A = \emptyset$.
(ii) If $R : A_1 \to A_2$ is continuous and equivariant, then $\text{ind}(A_1) \leq \text{ind}(A_2)$.
(iii) $\text{ind}(A_1 \cup A_2) \leq \text{ind}(A_1) + \text{ind}(A_2)$.
(iv) If $A \in \mathcal{A}$ is compact, then there exists a neighbourhood N of A such that $N \in \mathcal{A}$ and $\text{ind}(N) = \text{ind}(A)$.

With these definitions in place, we are ready to introduce the concept of the Krasnoselskii-genus.

Lemma 5.2 (**The Krasnoselskii-genus**). *Let $G = \mathbb{Z}_2 = \{0, 1\}$ and define $T(0) = I$, $T(1) = -I$, where $I : E \to E$ is the identity map on E. Given any closed and symmetric with respect to the origin subset $A \in \mathcal{A}$, define $\gamma(A) = k \in \mathbb{N}$ if k is the smallest integer such that there exists some odd mapping $\varphi \in C(A, \mathbb{R}^k \setminus \{0\})$. Moreover, define $\gamma(A) = +\infty$ if no such mapping exists and $\gamma(\emptyset) = 0$. Then, the mapping $\gamma : \mathcal{A} \to \mathbb{N} \cup \{+\infty\}$ is a $\mathbb{Z}_2$-index on E, called the Krasnoselskii-genus.*

Proof. See the proof of Proposition 2.1 in [24]. □

The next lemma gives a property of the Krasnoselskii-genus relevant for us to obtain our multiplicity result.

Lemma 5.3 (**Multiplicity from the Krasnoselskii-genus**). *Assume $A \in \mathcal{A}$ is such that $0 \notin A$ and $\gamma(A) \geq 2$. Then, A has infinitely many points.*

Proof. See the proof of Proposition 2.2 in [24]. □

For the proof of Theorem 1.5, we recall a classical result of Ambrosetti and Rabinowitz, [4].

Theorem 5.1 ([4]; **Min-max setting high q**). *Let $I \in C^1(E(\mathbb{R}^N), \mathbb{R}^N)$ satisfy the following:*

- *(i) $I(0) = 0$ and there exists constants $R, a > 0$ such that $I(u) \geq a$ if $||u||_{E(\mathbb{R}^N)} = R$*
- *(ii) If $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ is such that $0 < I(u_n)$, $I(u_n)$ bounded above, and $I'(u_n) \to 0$, then $(u_n)_{n\in\mathbb{N}}$ possesses a convergent subsequence*
- *(iii) $I(u) = I(-u)$ for all $u \in E(\mathbb{R}^N)$*
- *(iv) For a nested sequence $E_1 \subset E_2 \subset \cdots$ of finite dimensional subspaces of $E(\mathbb{R}^N)$ of increasing dimension, it holds that $E_i \cap \hat{A}_0$ is bounded for each $i = 1, 2, \ldots$, where $\hat{A}_0$ is given by (5.1)*

Define

$$b_m = \inf_{K\in\Gamma_m} \max_{u\in K} I(u),$$

with

$$\Gamma_m = \{K \subset E(\mathbb{R}^N) : K \text{ is compact and symmetric with respect to the origin and for all } h \in \Gamma^*, \text{ it holds that } \gamma(K \cap h(\partial B_1)) \geq m\},$$

where Γ^* is given by (5.2). Then, for each $m \in \mathbb{N}$, it holds that $0 < a \leq b_m \leq b_{m+1}$ and b_m is a critical value of I. Moreover, if $b_{m+1} = \cdots = b_{m+r} = b$, then $\gamma(K_b) \geq r$, where

$$K_b := \{u \in E(\mathbb{R}^N) : I(u) = b,\, I'(u) = 0\},$$

is the set of critical points at any level $b > 0$.

Proof. See [4, Theorem 2.8]. □

We are now in position to prove Theorem 1.5.

Proof of Theorem 1.5. We aim to apply Theorem 5.1 and therefore must verify that I satisfies assumptions (i)-(iv) of this theorem. By Lemma 2.5, I satisfies the Mountain-Pass Geometry and thus (i) holds. By Lemma 4.2, (ii) holds. Clearly, (iii) holds due to the structure of the functional I. We now must show that (iv) holds. We first notice by straightforward calculations that for any $u \in \partial B_1$ and any for $t > 0$, it holds that

$$\begin{aligned} I(tu) &= \frac{t^2}{2}||u||^2_{H^1(\mathbb{R}^N)} + \frac{t^4}{4}\int_{\mathbb{R}^N} \rho\phi_u u^2 - \frac{t^{q+1}}{q+1}\int_{\mathbb{R}^N} |u|^{q+1} \\ &= \frac{t^2}{2}\left(||u||^2_{H^1(\mathbb{R}^N)} + \frac{t^2}{2}\int_{\mathbb{R}^N} \rho\phi_u u^2 - \frac{2t^{q-1}}{q+1}\int_{\mathbb{R}^N} |u|^{q+1}\right). \end{aligned}$$

We now set

$$\alpha := ||u||^2_{H^1(\mathbb{R}^N)} > 0, \quad \beta := \frac{1}{2}\int_{\mathbb{R}^N} \rho\phi_u u^2 \geq 0, \quad \gamma := \frac{2}{q+1}\int_{\mathbb{R}^N} |u|^{q+1} > 0,$$

and look for positive solutions of

$$\frac{t^2}{2}(\alpha + \beta t^2 - \gamma t^{q-1}) = 0.$$

Since $q > 3$, it holds that $\alpha + \beta t^2 - \gamma t^{q-1} = 0$ has a unique solution $t = t_u > 0$. That is, we have shown that for each $u \in \partial B_1$, there exists a unique $t = t_u > 0$ such that I satisfies

$$\begin{aligned} &I(t_u u) = 0 \\ &I(tu) > 0, \ \forall t < t_u \\ &I(tu) < 0, \ \forall t > t_u. \end{aligned}$$

Now, for any $m \in \mathbb{N}$, we choose E_m a m-dimensional subspace of $E(\mathbb{R}^N)$ in such a way that $E_m \subset E_{m'}$ for $m < m'$. Moreover, for any $m \in \mathbb{N}$, we set

$$W_m := \{w \in E(\mathbb{R}^N) : v = tu, \ t \geq 0, \ u \in \partial B_1 \cap E_m\}.$$

Then, the function $h : E_m \to W_m$ given by

$$h(z) = t\frac{z}{||z||}, \quad \text{with } t = ||z||$$

defines a homeomorphism from E_m onto W_m, and so $W_1 \subset W_2 \subset \cdots$ is a nested sequence of finite dimensional subspaces of $E(\mathbb{R}^N)$ of increasing dimension. We also notice that

$$T_m := \sup_{u \in \partial B_1 \cap E_m} t_u < +\infty$$

since $\partial B_1 \cap E_m$ is compact. So, for all $t > T_m$ and $u \in \partial B_1 \cap E_m$, it holds that $I(tu) < 0$, and thus $W_m \cap \hat{A}_0$ is bounded, where $\hat{A}_0$ is given by (5.1). Since this holds for arbitrary $m \in \mathbb{N}$, we have shown that (iv) holds. Hence, we have shown that Theorem 5.1 applies to the functional I. If b_m are distinct for $m = 1, \ldots, j$ with $j \in \mathbb{N}$, we obtain j distinct pairs of critical points corresponding to critical levels $0 < b_1 < b_2 < \cdots < b_j$. If $b_{m+1} = \cdots = b_{m+r} = b$, then $\gamma(K_b) \geq r \geq 2$. Moreover, $0 \notin K_b$ since $b > 0 = I(0)$. Further, K_b is invariant since I is an invariant functional and K_b is closed since I satisfies the Palais-Smale condition, and so $K_b \in \mathcal{A}$. Therefore, by Lemma 5.3, K_b possesses infinitely many points. Finally, we note that by [4, Theorem 2.13], for each $m \in \mathbb{N}$, it holds that

$$d_m \leq b_m,$$

where d_m is defined in (5.3). It therefore follows from Lemma 5.1 that

$$b_m \to +\infty, \text{ as } m \to +\infty.$$

This concludes the proof. □

5.3. **Proof of Theorem 1.6.** Before proving Theorem 1.6, we must establish some preliminary results that we will need to use. The first lemma that we recall will give us an abstract definition of the min-max levels and some properties.

Lemma 5.4 ([5]; **Abstract min-max setting for low** q**).** *Consider a Banach space E, and a functional $\Phi_\mu : E \to \mathbb{R}$ of the form $\Phi_\mu(u) = \alpha(u) - \mu\beta(u)$, with $\mu > 0$. Suppose that α, $\beta \in C^1$ are even functions, $\lim_{||u|| \to +\infty} \alpha(u) = +\infty$, $\beta(u) \geq 0$, and β, β' map bounded sets onto bounded sets. Suppose further that there exists $K \subset E$ and a class $\mathcal{F}$ of compact sets in E such that:*

($\mathcal{F}$.1) $K \subset A$ for all $A \in \mathcal{F}$ and $\sup_{u\in K} \Phi_\mu(u) < c_\mu$, where c_μ is defined as:

$$c_\mu \coloneqq \inf_{A\in\mathcal{F}} \max_{u\in A} \Phi_\mu(u). \tag{5.13}$$

($\mathcal{F}$.2) If $\eta \in C([0,1]\times E, E)$ is an odd homotopy such that

- *$\eta(0,\cdot) = I$, where $I : E \to E$ is the identity map on E*
- *$\eta(t,\cdot)$ is a homeomorphism*
- *$\eta(t,x) = x$ for all $x \in K$,*

then $\eta(1,A) \in \mathcal{F}$ for all $A \in \mathcal{F}$.

Then, it holds that the mapping $\mu \mapsto c_\mu$ is non-increasing and left-continuous, and therefore is almost everywhere differentiable.

Proof. See [5, Lemma 2.2]. □

Under the hypotheses of the previous lemma, we can now define the set of values of $\mu \in \left[\frac{1}{2}, 1\right]$ such that c_μ, given by (5.13), is differentiable. Namely, we define

$$\mathcal{J} \coloneqq \left\{\mu \in \left[\frac{1}{2}, 1\right] \ : \text{the mapping } \mu \mapsto c_\mu \text{ is differentiable}\right\}.$$

Corollary 1 (On density of perturbation values μ). *The set $\mathcal{J}$ is dense in $\left[\frac{1}{2}, 1\right]$.*

Proof. Fix $x \in \left[\frac{1}{2}, 1\right]$ and $\delta > 0$, and denote by $|\cdot|$ the Lebesgue measure. Since $\left[\frac{1}{2}, 1\right] \setminus \mathcal{J}$ has zero Lebesgue measure by Lemma 5.4, we have

$$|\mathcal{J} \cap (x-\delta, x+\delta)| = \left|\left[\frac{1}{2}, 1\right] \cap (x-\delta, x+\delta)\right| > 0.$$

It follows that $\mathcal{J} \cap (x-\delta, x+\delta)$ is nonempty and so we can choose $y \in \mathcal{J} \cap (x-\delta, x+\delta)$. Since x and δ are arbitrary, this completes the proof. □

With the definition of $\mathcal{J}$ in place, we can also recall another vital result from [5], which will be used to obtain the boundedness of our Palais-Smale sequences.

Lemma 5.5 ([5]; **Boundedness of Palais-Smale sequences at level c_μ).** *For any $\mu \in \mathcal{J}$, there exists a bounded Palais-Smale sequence for Φ_μ at the level c_μ defined by (5.13). That is, there exists a bounded sequence $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^N)$ such that $\Phi_\mu(u_n) \to c_\mu$ and $\Phi_\mu'(u_n) \to 0$.*

Proof. See [5, Proposition 2.3]. □

Moving toward a less abstract setting, for any $\mu \in \left[\frac{1}{2}, 1\right]$, we define the perturbed functional $I_\mu : E(\mathbb{R}^N) \to \mathbb{R}^N$ as

$$I_\mu(u) \coloneqq \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) + \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{|x-y|^{N-2}}\,\mathrm{d}x\,\mathrm{d}y - \frac{\mu}{q+1}\int_{\mathbb{R}^N} |u|^{q+1}. \tag{5.14}$$

The next result that we will need in order to prove Theorem 1.6, follows as a result of Lemma 2.6.

Lemma 5.6 (On the sign of the energy level of I_μ along certain curves). *Assume $N = 3,4,5$ and $q \in (2, 2^*-1]$. Suppose further that ρ is homogeneous of degree $\bar{k}$, namely, $\rho(tx) = t^{\bar{k}}\rho(x)$ for all $t > 0$, for some*

$$\bar{k} > \max\left\{\frac{N}{4}, \frac{1}{q-1}\right\} \cdot (3-q) - 1.$$

Then, there exists $\nu > \max\left\{\frac{N}{2}, \frac{2}{q-1}\right\}$ such that for each fixed $\mu \in \left[\frac{1}{2}, 1\right]$ and each $u \in E(\mathbb{R}^N)\setminus\{0\}$, there exists a unique $t = t_u > 0$ with the property that

$$I_\mu(t_u^\nu u(t_u \cdot)) = 0,$$

$$I_\mu(t^\nu u(t\cdot)) > 0,\ \forall t < t_u,$$
$$I_\mu(t^\nu u(t\cdot)) < 0,\ \forall t > t_u,$$

where I_μ is defined in (5.14).

Proof. We first note that under the assumptions on the parameters, we can show that

$$\frac{4\nu - N - 2}{2} > \frac{(\nu+1)(3-q) - 2}{2}.$$

It follows from this and the lower bound assumption on $\bar{k}$ that we can always find at least one interval

$$\left(\frac{\nu(3-q)-2}{2}, \frac{4\nu - N - 2}{2}\right), \quad \text{with } \nu > \max\left\{\frac{N}{2}, \frac{2}{q-1}\right\},$$

that contains $\bar{k}$. We pick ν corresponding to such an interval and fix $\mu \in \left[\frac{1}{2}, 1\right]$. Then, for any $u \in E(\mathbb{R}^N) \setminus \{0\}$ and for any $t > 0$, using the assumption that ρ is homogeneous of degree $\bar{k}$, we find that

$$\begin{aligned} I_\mu(t^\nu u(t\cdot)) &= \frac{t^{2\nu+2-N}}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{t^{2\nu-N}}{2}\int_{\mathbb{R}^N} u^2 + \frac{t^{4\nu-N-2}}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(y)\rho(\frac{y}{t})u^2(x)\rho(\frac{x}{t})}{\omega|x-y|^{N-2}} \\ &\quad - \frac{\mu t^{\nu(q+1)-N}}{q+1}\int_{\mathbb{R}^N} |u|^{q+1} \\ &= \frac{t^{2\nu+2-N}}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{t^{2\nu-N}}{2}\int_{\mathbb{R}^N} u^2 + \frac{t^{4\nu-N-2-2\bar{k}}}{4}\int_{\mathbb{R}^N} \rho\phi_u u^2 - \frac{\mu t^{\nu(q+1)-N}}{q+1}\int_{\mathbb{R}^N} |u|^{q+1}. \end{aligned}$$

We therefore set

$$a = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2,\ b = \frac{1}{2}\int_{\mathbb{R}^N} u^2,\ c = \frac{1}{4}\int_{\mathbb{R}^N} \rho\phi_u u^2,\ d = \frac{\mu}{q+1}\int_{\mathbb{R}^N} |u|^{q+1},$$

and consider the polynomial

$$f(t) = at^{2\nu+2-N} + bt^{2\nu-N} + ct^{4\nu-N-2-2\bar{k}} - dt^{\nu(q+1)-N}, \quad t \geq 0.$$

Since $u \in E(\mathbb{R}^N) \setminus \{0\}$, we can deduce that $a, b, d > 0$ and $c \geq 0$, and so, by Lemma 2.6, it holds that f has a unique critical point corresponding to its maximum. Thus, since $I_\mu(t^\nu u(t\cdot)) = f(t)$ and, by assumptions, $\nu(q+1) - N > 2\nu + 2 - N$ and $\nu(q+1) - N > 4\nu - N - 2 - 2\bar{k}$, it follows that there exists a unique $t = t_u > 0$ such that the conclusion holds. □

With the previous results established, we are finally in position to prove Theorem 1.6.

Proof of Theorem 1.6. We first note that by Lemma 5.6, we can choose $\nu > \max\left\{\frac{N}{2}, \frac{2}{q-1}\right\}$, so that for each $u \in \partial B_1$, there exists a unique $t = t_u > 0$ such that I_μ with $\mu = \frac{1}{2}$, defined by (5.14), satisfies

$$\begin{aligned} &I_{\frac{1}{2}}(t_u^\nu u(t_u\cdot)) = 0, \\ &I_{\frac{1}{2}}(t^\nu u(t\cdot)) > 0,\ \forall t < t_u, \\ &I_{\frac{1}{2}}(t^\nu u(t\cdot)) < 0,\ \forall t > t_u. \end{aligned} \tag{5.15}$$

Now, for any $m \in \mathbb{N}$, we choose E_m a m-dimensional subspace of $E(\mathbb{R}^N)$ in such a way that $E_m \subset E_{m'}$ for $m < m'$. Moreover, for any $m \in \mathbb{N}$, we set

$$W_m := \{w \in E(\mathbb{R}^N) : w = t^\nu u(t\cdot),\ t \geq 0,\ u \in \partial B_1 \cap E_m\}.$$

Then, the function $h : E_m \to W_m$ given by

$$h(e) = t^\nu u(t\cdot), \quad \text{with } t = ||e||_{E(\mathbb{R}^N)},\ u = \frac{e}{||e||_{E(\mathbb{R}^N)}},$$

defines an odd homeomorphism from E_m onto W_m. We notice that it holds that

$$T_m := \sup_{u \in \partial B_1 \cap E_m} t_u < +\infty, \tag{5.16}$$

since $\partial B_1 \cap E_m$ is compact. So, the set

$$A_m = \{w \in E(\mathbb{R}^N) : w = t^\nu u(t\cdot),\ t \in [0, T_m],\ u \in \partial B_1 \cap E_m\}$$

is compact. We now define

$$H := \{g : E(\mathbb{R}^N) \to E(\mathbb{R}^N) : g \text{ is an odd homeomorphism and } g(w) = w \text{ for all } w \in \partial A_m\},$$

and

$$G_m := \{g(A_m) : g \in H\}.$$

We aim to verify $(\mathcal{F}.1)$ and $(\mathcal{F}.2)$ of Lemma 5.4. We take G_m as the class $\mathcal{F}$ and $K = \partial A_m$ and define the min-max levels

$$c_{m,\mu} := \inf_{A \in G_m} \max_{u \in A} I_\mu(u).$$

Then, since $T_m \geq t_u$ for all $u \in \partial B_1 \cap E_m$ by definition, it follows from (5.15) that

$$I_\mu(w) \leq I_{\frac{1}{2}}(w) \leq 0, \quad \forall w \in \partial A_m,\ \forall \mu \in \left[\frac{1}{2}, 1\right].$$

Moreover, since $G_m \subset G_{m+1}$ for all $m \in \mathbb{N}$, it holds that $c_{m,\mu} \geq c_{m-1,\mu} \geq \cdots \geq c_{1,\mu} > 0$. Taken together, we have shown that

$$\sup_{w \in \partial A_m} I_\mu(w) \leq 0 < c_{m,\mu}, \tag{5.17}$$

and thus $(\mathcal{F}.1)$ is verified. Moreover, for any η given by $(\mathcal{F}.2)$ and any $g \in H$, it holds that $\tilde{g} = \eta(1, g)$ belongs to H, and so $(\mathcal{F}.2)$ is satisfied. Since $(\mathcal{F}.1)$ and $(\mathcal{F}.2)$ are satisfied, Lemma 5.4 applies. Thus, for any $m \in \mathbb{N}$, we denote by $\mathcal{J}_m$ the set of values $\mu \in \left[\frac{1}{2}, 1\right]$ such that the function $\mu \mapsto c_{m,\mu}$ is differentiable. We then let

$$\mathcal{M} := \bigcap_{m \in \mathbb{N}} \mathcal{J}_m.$$

We note that since

$$\left[\frac{1}{2}, 1\right] \setminus \mathcal{M} = \bigcup_{m \in \mathbb{N}} \left(\left[\frac{1}{2}, 1\right] \setminus \mathcal{J}_m\right)$$

and $[\frac{1}{2}, 1] \setminus \mathcal{J}_m$ has zero Lebesgue measure for each m by Lemma 5.4, then it follows that $\left[\frac{1}{2}, 1\right] \setminus \mathcal{M}$ has zero Lebesgue measure. Arguing as in the proof of Corollary 1, we obtain that $\mathcal{M}$ is dense in $\left[\frac{1}{2}, 1\right]$. We can now apply Proposition 5.5 with $\Phi_\mu = I_\mu$. Namely, for each fixed $m \in \mathbb{N}$ and $\mu \in \mathcal{M}$ we obtain that there exists a bounded sequence $(u_n)_{n \in \mathbb{N}} \subset E(\mathbb{R}^N)$ such that $I_\mu(u_n) \to c_{m,\mu}$ and $I'_\mu(u_n) \to 0$. The embedding of $E(\mathbb{R}^N)$ into $L^{q+1}(\mathbb{R}^N)$ is compact by Lemma 4.1 so, arguing as in the proof of Theorem 1.3, we can show that the values $c_{m,\mu}$ are critical levels of I_μ for each $m \in \mathbb{N}$ and $\mu \in \mathcal{M}$. We then take m fixed, $(\mu_n)_{n \in \mathbb{N}}$ an increasing sequence in $\mathcal{M}$ such that $\mu_n \to 1$, and $(u_n)_{n \in \mathbb{N}} \subset E(\mathbb{R}^N)$ such that $I'_{\mu_n}(u_n) = 0$ and $I_{\mu_n}(u_n) = c_{m,\mu_n}$. We note that since ρ is homogeneous of degree $\bar{k}$ by assumption, it follows from [27, p. 296] that $\bar{k}\rho(x) = (x, \nabla\rho)$.

So, setting $\alpha_n = \int_{\mathbb{R}^N}(|\nabla u_n|^2 + u_n^2)$, $\gamma_n = \int_{\mathbb{R}^N} \rho(x)\phi_{u_n} u_n^2$, $\delta_n = \mu_n \int_{\mathbb{R}^N}|u_n|^{q+1}$ and using the Pohozaev-type condition deduced in Lemma 2.4, we obtain the system

$$\begin{cases} \alpha_n & + & \gamma_n & - & \delta_n & = & 0, \\ \frac{1}{2}\alpha_n & + & \frac{1}{4}\gamma_n & - & \frac{1}{q+1}\delta_n & = & c_{m,\mu_n}, \\ \frac{N-2}{2}\alpha_n & + & \left(\frac{N+2+2k}{4}\right)\gamma_n & - & \frac{N}{q+1}\delta_n & \leq & 0. \end{cases} \tag{5.18}$$

Since the assumptions on $\bar{k}$ guarantee that $\bar{k} > \frac{-2(q-2)}{(q-1)} > \frac{N-6}{2}$ for $q \in (2,3]$ if $N=3$ and for $q \in (2, 2^*-1)$ if $N = 4,5$, it follows that we can solve this system and show that $\alpha_n, \gamma_n, \delta_n$ are all bounded as in the proof of Theorem 1.3. Moreover, continuing to argue as in the proof of this theorem and using the compact embedding of $E(\mathbb{R}^N)$ into $L^{q+1}(\mathbb{R}^N)$, we can then prove that for each fixed m there exists $u \in E(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \to u$ in $E(\mathbb{R}^N)$, $I(u) = I_1(u) = c_{m,1}$, and $I'(u) = I_1'(u) = 0$. It therefore remains to show that $I(u) = c_{m,1} \to +\infty$ as $m \to +\infty$. In order to do so, we define

$$\tilde{\Gamma}_m := \left\{ g \in C(E_m \cap B_1, E(\mathbb{R}^N)) : g \text{ is odd, one-to-one, } I(g(y)) \leq 0 \text{ for all } y \in \partial(E_m \cap B_1) \right\},$$

$$\tilde{G}_m := \left\{ A \subset E(\mathbb{R}^N) : A = g(E_m \cap B_1), g \in \tilde{\Gamma}_m \right\},$$

$$\tilde{b}_m := \inf_{A \in \tilde{G}_m} \max_{u \in A} I(u).$$

We then note that by [4, Corollary 2.16], it holds that

$$d_m \leq \tilde{b}_m,$$

where d_m is given by (5.3). It therefore follows from Lemma 5.1 that

$$\tilde{b}_m \to +\infty, \text{ as } m \to +\infty. \tag{5.19}$$

We will now show $G_m \subseteq \tilde{G}_m$. We take $A \in G_m$. Then, by definition, there exists $g \in H$ such that $A = g(A_m)$. We define an odd homeomorphism $\varphi : E_m \cap B_1 \to A_m$ by

$$\varphi(e) = t^\nu u(t\cdot), \quad \text{with } t = T_m ||e||_{E(\mathbb{R}^N)},\ u = \frac{e}{||e||_{E(\mathbb{R}^N)}},$$

where T_m is defined in (5.16), and set $\tilde{g} = g \circ \varphi$. Since we can write $A = \tilde{g}(E_m \cap B_1)$, then by the definition of $\tilde{G}_m$ we need only to show that $\tilde{g} \in \tilde{\Gamma}_m$. Clearly, $\tilde{g} \in C(E_m \cap B_1, E(\mathbb{R}^N))$ is odd and one-to-one. Moreover, for every $y \in \partial(E_m \cap B_1)$, setting $w = \varphi(y) \in \partial A_m$, we have $I(\tilde{g}(y)) = I(g(w))$. Since $g \in H$ and $w \in \partial A_m$, then by definition it holds that $g(w) = w$. Putting everything together, we have

$$I(\tilde{g}(y)) = I(g(w)) = I(w) \leq \sup_{w \in \partial A_m} I(w) \leq 0,$$

where the final inequality follows from (5.17). Hence, we have shown $\tilde{g} \in \tilde{\Gamma}_m$ and so $G_m \subseteq \tilde{G}_m$. Therefore, for each $m \in \mathbb{N}$, it follows that

$$\tilde{b}_m = \inf_{A \in \tilde{G}_m} \max_{u \in A} I(u) \leq \inf_{A \in G_m} \max_{u \in A} I(u) = c_{m,1},$$

and so, by (5.19), we conclude that

$$c_{m,1} \to +\infty, \text{ as } m \to +\infty,$$

as required. □

Appendix A: Proof of the Pohozaev-type condition

Proof of Lemma 2.4. With the regularity remarks of Proposition 1 in place, we now multiply the first equation in (2.3) by $(x,\nabla u)$ and integrate on $B_R(0)$ for some $R>0$. We will compute each integral separately. We first note that

$$\begin{aligned}\int_{B_R} -\Delta u(x,\nabla u)\,\mathrm{d}x = \frac{2-N}{2}\int_{B_R}|\nabla u|^2\,\mathrm{d}x \\ -\frac{1}{R}\int_{\partial B_R}|(x,\nabla u)|^2\,\mathrm{d}\sigma + \frac{R}{2}\int_{\partial B_R}|\nabla u|^2\,\mathrm{d}\sigma.\end{aligned} \tag{5.20}$$

Fixing $i=1,\dots,N$, integrating by parts and using the divergence theorem, we then see that,

$$\begin{aligned}\int_{B_R} bu(x_i\partial_i u)\,\mathrm{d}x &= b\left[-\frac{1}{2}\int_{B_R}u^2\,\mathrm{d}x + \frac{1}{2}\int_{B_R}\partial_i(u^2x_i)\,\mathrm{d}x\right] \\ &= b\left[-\frac{1}{2}\int_{B_R}u^2\,\mathrm{d}x + \frac{1}{2}\int_{\partial B_R}u^2\frac{x_i^2}{|x|}\,\mathrm{d}\sigma\right].\end{aligned}$$

So, summing over i, we get

$$\int_{B_R} bu(x,\nabla u)\,\mathrm{d}x = b\left[-\frac{N}{2}\int_{B_R}u^2\,\mathrm{d}x + \frac{R}{2}\int_{\partial B_R}u^2\,\mathrm{d}\sigma\right]. \tag{5.21}$$

Again, fixing $i=1,\dots,N$, integrating by parts and using the divergence theorem, we find that,

$$\begin{aligned}\int_{B_R} c\rho\phi_u u x_i(\partial_i u)\,\mathrm{d}x &= c\Big[-\frac{1}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x - \frac{1}{2}\int_{B_R}\phi_u u^2 x_i(\partial_i\rho)\,\mathrm{d}x \\ &\qquad -\frac{1}{2}\int_{B_R}\rho u^2 x_i(\partial_i\phi_u)\,\mathrm{d}x + \frac{1}{2}\int_{B_R}\partial_i(\rho\phi_u u^2 x_i)\,\mathrm{d}x\Big] \\ &= c\Big[-\frac{1}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x - \frac{1}{2}\int_{B_R}\phi_u u^2 x_i(\partial_i\rho)\,\mathrm{d}x \\ &\qquad -\frac{1}{2}\int_{B_R}\rho u^2 x_i(\partial_i\phi_u)\,\mathrm{d}x + \frac{1}{2}\int_{\partial B_R}\rho\phi_u u^2\frac{x_i^2}{|x|}\,\mathrm{d}\sigma\Big].\end{aligned}$$

Thus, summing over i, we get

$$\begin{aligned}\int_{B_R} c\rho\phi_u u(x,\nabla u)\,\mathrm{d}x = c\Big[-\frac{N}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x - \frac{1}{2}\int_{B_R}\phi_u u^2(x,\nabla\rho)\,\mathrm{d}x \\ -\frac{1}{2}\int_{B_R}\rho u^2(x,\nabla\phi_u)\,\mathrm{d}x + \frac{R}{2}\int_{\partial B_R}\rho\phi_u u^2\,\mathrm{d}\sigma\Big].\end{aligned} \tag{5.22}$$

Finally, once more fixing $i=1,\dots,N$, integrating by parts and using the divergence theorem, we find that,

$$\int_{B_R} d|u|^{q-1}u(x_i\partial_i u)\,\mathrm{d}x = d\left[\frac{-1}{q+1}\int_{B_R}|u|^{q+1}\,\mathrm{d}x + \frac{1}{q+1}\int_{\partial B_R}|u|^{q+1}\frac{x_i^2}{|x|}\,\mathrm{d}\sigma\right],$$

and so, summing over i, we see that

$$\begin{aligned}\int_{B_R} d|u|^{q-1}u(x,\nabla u)\,\mathrm{d}x = d\Big[\frac{-N}{q+1}\int_{B_R}|u|^{q+1}\,\mathrm{d}x \\ + \frac{R}{q+1}\int_{\partial B_R}|u|^{q+1}\,\mathrm{d}\sigma\Big].\end{aligned} \tag{5.23}$$

Putting (5.20), (5.21), (5.22) and (5.23) together, we see that

$$
\begin{aligned}
&\frac{2-N}{2}\int_{B_R}|\nabla u|^2\,\mathrm{d}x-\frac{1}{R}\int_{\partial B_R}|(x,\nabla u)|^2\,\mathrm{d}\sigma+\frac{R}{2}\int_{\partial B_R}|\nabla u|^2\,\mathrm{d}\sigma\\
&\quad+b\Big[-\frac{N}{2}\int_{B_R}u^2\,\mathrm{d}x+\frac{R}{2}\int_{\partial B_R}u^2\,\mathrm{d}\sigma\Big]\\
&\qquad+c\Big[-\frac{N}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x-\frac{1}{2}\int_{B_R}\phi_u u^2(x,\nabla\rho)\,\mathrm{d}x\\
&\qquad\quad-\frac{1}{2}\int_{B_R}\rho u^2(x,\nabla\phi_u)\,\mathrm{d}x+\frac{R}{2}\int_{\partial B_R}\rho\phi_u u^2\,\mathrm{d}\sigma\Big]\\
&\qquad\quad-d\left[\frac{-N}{q+1}\int_{B_R}|u|^{q+1}\,\mathrm{d}x+\frac{R}{q+1}\int_{\partial B_R}|u|^{q+1}\,\mathrm{d}\sigma\right]=0.
\end{aligned}
\tag{5.24}
$$

We now multiply the second equation in (2.3) by $(x,\nabla\phi_u)$ and integrate on $B_R(0)$ for some $R>0$. By a simple calculation we see that

$$
\begin{aligned}
\int_{B_R}\rho u^2(x,\nabla\phi_u)\,\mathrm{d}x&=\int_{B_R}-\Delta\phi_u(x,\nabla\phi_u)\,\mathrm{d}x\\
&=\frac{2-N}{2}\int_{B_R}|\nabla\phi_u|^2\,\mathrm{d}x-\frac{1}{R}\int_{\partial B_R}|(x,\nabla\phi_u)|^2\,\mathrm{d}\sigma\\
&\quad+\frac{R}{2}\int_{\partial B_R}|\nabla\phi_u|^2\,\mathrm{d}\sigma.
\end{aligned}
$$

Substituting this into (5.24) and rearranging, we get

$$
\begin{aligned}
&\frac{N-2}{2}\int_{B_R}|\nabla u|^2\,\mathrm{d}x+\frac{Nb}{2}\int_{B_R}u^2\,\mathrm{d}x+\frac{(N+k)c}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x\\
&\quad+\frac{c(2-N)}{4}\int_{B_R}|\nabla\phi_u|^2\,\mathrm{d}x-\frac{Nd}{q+1}\int_{B_R}|u|^{q+1}\,\mathrm{d}x\\
&\le\frac{N-2}{2}\int_{B_R}|\nabla u|^2\,\mathrm{d}x+\frac{Nb}{2}\int_{B_R}u^2\,\mathrm{d}x+\frac{Nc}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x\\
&\quad+\frac{c}{2}\int_{B_R}\phi_u u^2(x,\nabla\rho)\,\mathrm{d}x+\frac{c(2-N)}{4}\int_{B_R}|\nabla\phi_u|^2\,\mathrm{d}x-\frac{Nd}{q+1}\int_{B_R}|u|^{q+1}\,\mathrm{d}x\\
&=-\frac{1}{R}\int_{\partial B_R}|(x,\nabla u)|^2\,\mathrm{d}\sigma+\frac{R}{2}\int_{\partial B_R}|\nabla u|^2\,\mathrm{d}\sigma+\frac{bR}{2}\int_{\partial B_R}u^2\,\mathrm{d}\sigma\\
&\quad+\frac{cR}{2}\int_{\partial B_R}\rho\phi_u u^2\,\mathrm{d}\sigma+\frac{c}{2R}\int_{\partial B_R}|(x,\nabla\phi_u)|^2\,\mathrm{d}\sigma\\
&\qquad-\frac{cR}{4}\int_{\partial B_R}|\nabla\phi_u|^2\,\mathrm{d}\sigma-\frac{dR}{q+1}\int_{\partial B_R}|u|^{q+1}\,\mathrm{d}\sigma,
\end{aligned}
\tag{5.25}
$$

where we have used the assumption $k\rho(x) \leq (x, \nabla\rho)$ for some $k \in \mathbb{R}$ to obtain the first inequality. We now call the right hand side of (5.25) I_R, namely

$$\begin{aligned} I_R := -\frac{1}{R}\int_{\partial B_R} |(x,\nabla u)|^2\,\mathrm{d}\sigma + \frac{R}{2}\int_{\partial B_R} |\nabla u|^2\,\mathrm{d}\sigma + \frac{bR}{2}\int_{\partial B_R} u^2\,\mathrm{d}\sigma \\ + \frac{cR}{2}\int_{\partial B_R} \rho\phi_u u^2\,\mathrm{d}\sigma + \frac{c}{2R}\int_{\partial B_R} |(x,\nabla\phi_u)|^2\,\mathrm{d}\sigma \\ - \frac{cR}{4}\int_{\partial B_R} |\nabla\phi_u|^2\,\mathrm{d}\sigma - \frac{dR}{q+1}\int_{\partial B_R} |u|^{q+1}\,\mathrm{d}\sigma. \end{aligned}$$

We note that $|(x,\nabla u)| \leq R|\nabla u|$ and $|(x,\nabla\phi_u)| \leq R|\nabla\phi_u|$ on ∂B_R, so it holds that

$$\begin{aligned} |I_R| \leq \frac{3R}{2}\int_{\partial B_R} |\nabla u|^2\,\mathrm{d}\sigma + \frac{bR}{2}\int_{\partial B_R} u^2\,\mathrm{d}\sigma \\ + \frac{cR}{2}\int_{\partial B_R} \rho\phi_u u^2\,\mathrm{d}\sigma + \frac{3cR}{4}\int_{\partial B_R} |\nabla\phi_u|^2\,\mathrm{d}\sigma + \frac{dR}{q+1}\int_{\partial B_R} |u|^{q+1}\,\mathrm{d}\sigma. \end{aligned}$$

Now, since $|\nabla u|^2$, $u^2 \in L^1(\mathbb{R}^N)$ as $u \in E(\mathbb{R}^N) \subsetneq H^1(\mathbb{R}^N)$, $\rho\phi_u u^2$, $|\nabla\phi_u|^2 \in L^1(\mathbb{R}^N)$ because $\int_{\mathbb{R}^N} \rho\phi_u u^2\,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla\phi_u|^2\,\mathrm{d}x$ and $\phi_u \in D^{1,2}(\mathbb{R}^N)$, and $|u|^{q+1} \in L^1(\mathbb{R}^N)$ because $E(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ for all $s \in [2, 2^*]$, then it holds that $I_{R_n} \to 0$ as $n \to +\infty$ for a suitable sequence $R_n \to +\infty$. Moreover, since (5.25) holds for any $R > 0$, it follows that

$$\begin{aligned} \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x + \frac{Nb}{2}\int_{\mathbb{R}^N} u^2\,\mathrm{d}x + \frac{(N+k)c}{2}\int_{\mathbb{R}^N} \rho\phi_u u^2\,\mathrm{d}x \\ + \frac{c(2-N)}{4}\int_{\mathbb{R}^N} |\nabla\phi_u|^2\,\mathrm{d}x - \frac{Nd}{q+1}\int_{\mathbb{R}^N} |u|^{q+1}\,\mathrm{d}x \leq 0, \end{aligned}$$

and so, we obtain

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x + \frac{Nb}{2}\int_{\mathbb{R}^N} u^2\,\mathrm{d}x + \frac{(N+2+2k)c}{4}\int_{\mathbb{R}^N} \rho\phi_u u^2\,\mathrm{d}x - \frac{Nd}{q+1}\int_{\mathbb{R}^N} |u|^{q+1}\,\mathrm{d}x \leq 0,$$

using the fact that $\int_{\mathbb{R}^N} |\nabla\phi_u|^2\,\mathrm{d}x = \int_{\mathbb{R}^N} \rho\phi_u u^2\,\mathrm{d}x$. This completes the proof. □

Data availability statement

On behalf of all authors, the corresponding author states that there are no data associated to our manuscripts.

Conflicts of interest/Competing interests

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Department of Mathematics, Computational Foundry, Swansea University, Fabian Way, Swansea, U.K. SA1 8EN

E-mail address: `662536@swansea.ac.uk`

Department of Mathematics, Computational Foundry, Swansea University, Fabian Way, Swansea, U.K. SA1 8EN

E-mail address: `c.mercuri@swansea.ac.uk`

Department of Mathematics, Computational Foundry, Swansea University, Fabian Way, Swansea, U.K. SA1 8EN

E-mail address: `megan.tyler268@outlook.com`