Existence and qualitative properties of solutions to nonlinear Schrödinger-Poisson systems

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Abstract

Our main equation of study is the nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \rho(x)\phi u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = \rho(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

with $p \in (2,5)$ and $\rho : \mathbb{R}^3 \to \mathbb{R}$ a nonnegative measurable function. In the spirit of the classical work of P. H. Rabinowitz [55] on nonlinear Schrödinger equations, we first prove existence of positive mountain-pass solutions and least energy solutions to this system under different assumptions on ρ at infinity. Our results cover the range $p \in (2,3)$ where the lack of compactness phenomena may be due to the combined effect of the invariance by translations of a 'limiting problem' at infinity and of the possible unboundedness of the Palais-Smale sequences. In the case of a coercive ρ, namely $\rho(x) \to +\infty$ as $|x| \to +\infty$, we then prove the existence of infinitely many distinct pairs of solutions. For $p \in (3,5)$ we exploit the symmetry of the problem by the action of $\mathbb{Z}_2$ as well as some well-known properties of the Krasnoselskii-genus, whereas for $p \in (2,3]$ we use an appropriate abstract min-max scheme, which requires some additional assumptions on ρ.

After establishing these existence and multiplicity results, we are then interested in the qualitative properties of solutions the singularly perturbed problem

$$\begin{cases} -\varepsilon^2\Delta u + \lambda u + \rho(x)\phi u = |u|^{p-1}u, & x \in \mathbb{R}^3 \\ -\Delta \phi = \rho(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

with $\rho : \mathbb{R}^3 \to \mathbb{R}$ a nonnegative measurable function, $\lambda \in \mathbb{R}$, and $\lambda > 0$, taking advantage of a shrinking parameter $\varepsilon \ll 1$. In particular, we seek to understand the concentration phenomena purely driven by ρ. To this end, we first find necessary conditions for concentration at points to occur for solutions in various functional settings which are suitable for both variational and perturbation methods. We then discuss a variational/penalisation method, which has been exploited in the case of nonlinear Schrödinger equations, and discuss its applications to the present nonlinear Schrödinger-Poisson context, in the attempt of showing that the necessary conditions are, in fact, sufficient conditions on ρ for point concentration of solutions. Finally, we present some preliminary results in this direction that elicit interesting standalone qualitative properties of the solutions.

Declaration and Statements

DECLARATION

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

Signed: [redacted] (candidate)

Date: 15/04/2020

STATEMENT 1

This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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STATEMENT 2

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Contents

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Symbols and Notation

The following symbols and notation will be used throughout the thesis. Any other notation that is used, will be clearly defined in the thesis.

- Δ is the classical Laplacian operator.
- $|x|$ is the Euclidean norm of $x \in \mathbb{R}^N$.
- $L^p(\Omega)$, with $\Omega \subseteq \mathbb{R}^3$ and $p \geq 1$, is the usual Lebesgue space. $L^p(\mathbb{R}^3) = L^p$.
- The Hölder space $C^{k,\alpha}(\Omega)$, with $\Omega \subseteq \mathbb{R}^3$ and $\alpha \in (0,1]$, is the set of functions on Ω that are k-fold differentiable and whose k-fold derivatives are Hölder continuous of order α.
- H^1, $W^{m,p}$ are classical Sobolev spaces.
- X^* and $H^{-1}(\mathbb{R}^3) = H^{-1}$ denotes the dual space of X and $H^1(\mathbb{R}^3)$, respectively.
- $\mathscr{D}(\mathbb{R}^3)$ is the space of test functions.
- $\mathscr{D}'(\mathbb{R}^3)$ is the dual space of $\mathscr{D}(\mathbb{R}^3)$.
- $D^{1,2}(\mathbb{R}^3) = D^{1,2}$ is the space defined as

$$D^{1,2}(\mathbb{R}^3) := \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\},$$

  and equipped with norm

$$||u||_{D^{1,2}(\mathbb{R}^3)} := ||\nabla u||_{L^2(\mathbb{R}^3)}.$$

- $E(\mathbb{R}^3) = E$ is the space defined as

$$E(\mathbb{R}^3) := \left\{u \in D^{1,2}(\mathbb{R}^3) \,:\, ||u||_E < +\infty\right\},$$

  where

$$||u||_E^2 := \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\,\mathrm{d}x + \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}.$$

- We set

$$\phi_u(x) := \int_{\mathbb{R}^3} \frac{\rho(y)u^2(y)}{4\pi|x-y|}\,\mathrm{d}y,$$

and

$$\bar{\phi}_u(x) := \int_{\mathbb{R}^3} \frac{\rho_\infty u^2(y)}{4\pi|x-y|}\,\mathrm{d}y.$$

- ω is the usual normalization factor for the Green function of the Laplacian in $\mathbb{R}^N$; in this thesis, $\omega = 4\pi$ since we work in $\mathbb{R}^3$

- For any $\eta > 0$ and any $z \in \mathbb{R}^3$, $B_\eta(z)$ is the ball of radius η centered at z. For any $\eta > 0$, B_η is the ball of radius η centered at 0.

- $S_{p+1} := \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{||u||^2_{H^1(\mathbb{R}^3)}}{||u||^2_{L^{p+1}(\mathbb{R}^3)}}$ is the best Sobolev constant for the embedding of $H^1(\mathbb{R}^3)$ into $L^{p+1}(\mathbb{R}^3)$.

- Let $A \subset \mathbb{R}^3$. Then, we define

$$\chi_A(x) := \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

- C, C_1, C', etc., denote generic positive constants.

- Asymptotic Notation: For real valued functions $f(t), g(t) \geq 0$, we write:

  - $f(t)g(t)$ if there exists $C > 0$ independent of t such that $f(t) \leq Cg(t)$.
  - $f(t) = o(g(t))$ as $t \to +\infty$ if and only if $g(t) \neq 0$ and $\lim_{t\to+\infty} \frac{f(t)}{g(t)} = 0$.
  - $f(t) = O(g(t))$ as $t \to +\infty$ if and only if there exists $C_1 > 0$ such that $f(t) \leq C_1 g(t)$ for t large.

1 Introduction

In this thesis, we study existence of positive solutions to the following nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \rho(x)\phi u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = \rho(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

with $p \in (2,5)$ and $\rho : \mathbb{R}^3 \to \mathbb{R}$ a nonnegative measurable function which represents a non-constant 'charge' corrector to the density u^2. In the context of the so-called Density Functional Theory, variants of system (1.1) appear as mean field approximations of quantum many–body systems, see [9], [23], [45]. The positive Coulombic potential, ϕ, represents a repulsive interaction between particles, whereas the local nonlinearity $|u|^{p-1}u$ generalises the $u^{5/3}$ term introduced by Slater [58] as local approximation of the exchange potential in Hartree–Fock type models, see e.g. [16], [46].

Throughout this thesis, we work in $\mathbb{R}^N$ with $N = 3$. We note that the same analysis can be performed in every dimension $N \neq 2$, provided N is such that the functional space that we work in embeds compactly into $L_{loc}^{\frac{4N}{N+2}}(\mathbb{R}^N)$, yielding a constraint on the dimension. Namely, we would need the technical assumption that $\frac{4N}{N+2}$ is subcritical with respect to the classical Sobolev exponent $\frac{2N}{N-2}$, which holds if and only if $N = 3, 4, 5$. Because of the different structure of the Green function of the Laplacian, the case $N = 2$ requires different techniques and is mostly open.

Within a min-max setting and in the spirit of Rabinowitz [55], we study existence and qualitative properties of the solutions to (1.1), highlighting those phenomena which are driven by ρ. The system (1.1) 'interpolates' the classical equation

$$-\Delta u + u = u^p, \qquad x \in \mathbb{R}^3, \tag{1.2}$$

whose positive solutions have been classified by Kwong [41], with

$$\begin{cases} -\Delta u + u + \phi u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

studied by several authors in relation to existence, nonexistence, multiplicity and behaviour of the solutions in the semi-classical limit; see e.g. [2], [13], [23], and references therein. In the case $\rho(x) \to 0$ as $|x| \to +\infty$, (1.2) has been exploited as limiting equation to tackle existence/compactness questions related to certain classes of systems similar to (1.1), see e.g. [25] and [26]. In the present paper we consider instances where the convergence of approximating solutions to (1.1) is not characterised by means of (1.2), namely the cases where, as $|x| \to +\infty$, it holds either that $\rho \to +\infty$ ('coercive case'), or that $\rho \to \rho_\infty > 0$ ('non-coercive case'). The latter corresponds to the case where nontrivial solutions of (1.3) (up to coefficients) cause lack of compactness phenomena to occur. The main difficulty in this context is that, despite the extensive literature, a full understanding of the set of positive solutions to (1.3) has not yet been achieved (symmetry, non-degeneracy, etc.).

The autonomous system (1.3), as well as (1.1), presents various mathematical features which are not shared with nonlinear Schrödinger type equations, mostly related to lack of compactness phenomena. In a pioneering work [57], radial functions and constrained minimisation techniques have been used, over a certain natural constraint manifold defined combining the Pohozaev and Nehari identities, yielding existence results of positive radial solutions to (1.3) for all $p \in (2,5)$. Again in a radial setting, a variant of system (1.3) has been studied more recently in [40]. When $p \leq 2$ the change in geometry of the associated energy functional causes differing phenomena to occur. In [57] existence, nonexistence and multiplicity results have been shown to be sensitive to a multiplicative factor for the Poisson term. Nonexistence results for (1.3) have also been obtained in $\mathbb{R}^3$ in the range $p \geq 5$ (see e.g. [28]). In the presence of potentials, however, existence may occur when $p = 5$, as it has been recently shown in [24]. Ambrosetti and Ruiz [7] improved upon these early results by using the so-called 'monotonicity trick' introduced by Struwe [59] and formulated in the context of the nonlinear Schrödinger equations by Jeanjean [38] and Jeanjean-Tanaka [39], in order to show the existence of multiple bound state solutions to (1.3).

Related problems involving a non-constant charge density ρ, and in the presence of potentials, have been studied. The vast majority of works involve the range $p > 3$ since, when $p \leq 3$, one has to face two major obstacles in applying the minimax methods: constructing bounded Palais-Smale sequences and proving that the Palais-Smale condition holds, see e.g [57], and [7], [47]. Cerami and Molle [25] and Cerami and Vaira [26] studied the system

$$\begin{cases} -\Delta u + V(x)u + \lambda \rho(x)\phi u = K(x)|u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = \rho(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $\lambda > 0$ and $V(x)$, $\rho(x)$ and $K(x)$ are nonnegative functions in $\mathbb{R}^3$ such that

$$\lim_{|x|\to+\infty} \rho(x) = 0, \lim_{|x|\to+\infty} V(x) = V_\infty > 0, \lim_{|x|\to+\infty} K(x) = K_\infty > 0, \tag{1.5}$$

and, under suitable assumptions on the potentials, proved the existence of positive ground state and bound state solutions for $p \in (3,5)$. In [19] existence of positive solutions to (1.4) in the range $p \in [3,5)$ has been proved under suitable assumptions on the potentials that guarantee some compact embeddings of weighted Sobolev spaces into weighted L^{p+1} spaces. Vaira [62] also studied system (1.4), in the case that

$$\lim_{|x|\to+\infty} \rho(x) = \rho_\infty > 0, \; V(x) \equiv 1, \lim_{|x|\to+\infty} K(x) = K_\infty > 0, \tag{1.6}$$

and, assuming $\lambda > 0$ and $K(x) \not\equiv 1$, proved the existence of positive ground state solutions for $p \in (3,5)$. In a recent and interesting paper, Sun, Wu and Feng (see Theorem 1.4 of [61]) have shown the existence of a solution to (1.4) for $p \in (1,3]$, assuming (1.6) and $K(x) \equiv 1$, provided λ is sufficiently small and $\int_{\mathbb{R}^3} \rho(x)\phi_{\rho,w_\lambda} w_\lambda^2 < \int_{\mathbb{R}^3} \rho_\infty \phi_{\rho_\infty,w_\lambda} w_\lambda^2$, where $(w_\lambda, \phi_{\rho_\infty,w_\lambda})$ is a positive solution to

$$\begin{cases} -\Delta u + u + \lambda \rho_\infty \phi u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = \rho_\infty u^2, & x \in \mathbb{R}^3. \end{cases}$$

Their results are obtained using the fact that all nontrivial solutions to (1.4) lie in a certain manifold M_λ^- (see Lemma 6.1 in [61]) to show that the energy functional J_λ is bounded from below on the set of nontrivial solutions to (1.4). We believe that this is necessary to prove Corollary 4.3 in [61], and, ultimately, to prove Theorem 1.4 in [61], and thus the existence result is only viable in the reduced range $2.18 \approx \frac{-2+\sqrt{73}}{3} < p \leq 3$ and provided the additional assumption $\frac{3p^2+4p-23}{2(5-p)}\rho(x) + \frac{p-1}{2}(\nabla\rho(x), x) \geq 0$ also holds. In this range of p and under these assumptions, as observed in [61], solutions are ground states.

1.1 Main original results

In light of the above results, in this thesis we aim to study existence and qualitative properties of solutions to (1.1), in the various functional settings corresponding to different hypotheses on the behaviour of ρ at infinity. The main original results of this thesis are contained in Theorem 4.2, Corollary 4.3, Proposition 4.4, Theorem 4.3, Theorem 4.4, Corollary 4.6, Theorem 5.1, Theorem 5.3, Theorem 6.1, Theorem 6.2, Proposition 7.3, and Theorem 7.3.

1.1.1 Existence

We first study the case of coercive ρ, namely $\rho(x) \to +\infty$ as $|x| \to +\infty$, and work in the natural setting for this problem, $E(\mathbb{R}^3)$. When $p \in (2,3)$, we make use of the aforementioned 'monotonicity trick' exploiting the structure of our functional, in order to construct bounded Palais-Smale sequences for small perturbations of (1.1). We are able to prove that these sequences converge using a compact embedding established in Lemma 4.5. We finally show that these results extend to the original problem and obtain the existence of a positive mountain pass solution in Theorem 4.2. After establishing these results, we prove the existence of positive least energy solutions for all $p \in (2,5)$ in Corollary 4.3. When $p \in (3,5)$ the existence follows relatively straightforwardly using the Nehari characterisation of the mountain pass level, and for $p \in (2,3]$, we make use of a minimising sequence in order to obtain the result.

We then focus on the case of non-coercive ρ, namely when $\rho(x) \to \rho_\infty > 0$ as $|x| \to +\infty$. For this problem, $E(\mathbb{R}^3)$ coincides with the larger space $H^1(\mathbb{R}^3)$. Our method to look for solutions of (1.1) in this case relies on an *a posteriori* compactness analysis of bounded Palais-Smale sequences (in the spirit of the classical book of M. Willem [64]), in which we find that any possible lack of compactness is related to the invariance by translations of the subcritical 'problem at infinity' associated to (1.1). This *a posteriori* compactness analysis is provided by Proposition 4.4. There are several compactness results of similar flavour since the pioneering works of P.L. Lions [44] and Benci-Cerami [12], which include more recent contributions in the context of Schrödinger-Poisson systems, see e.g. [26], [62], [24]. We point out that these recent results are mostly in the range $p > 3$, for Palais-Smale sequences constrained on Nehari manifolds, and for functionals without positive parts, unlike our result. In the case $p \in (2,3)$, we use Proposition 4.4 together with a Pohozaev type inequality and Nehari's identity to show that a sequence of approximated critical points, constructed by means of the 'monotonicity trick', is relatively compact. This enables us to obtain the existence of a positive mountain pass solution in Theorem 4.3. The non-coercive case turns out to be more 'regular' with respect to compactness issues when $p \geq 3$. In fact, we can show that the Palais-Smale condition holds at the mountain pass level and as a consequence we obtain Theorem 4.4. We follow up the previous two theorems with Corollary 4.6, which gives the existence of positive least energy solutions in the non-coercive case.

1.1.2 Multiplicity

After proving the existence of mountain pass and least energy solutions (which may or may not coincide), we then study the existence of multiple distinct solutions to (1.1). In the case of a coercive ρ and for $p > 3$, the existence of infinitely many pairs of distinct solutions to (1.1) (Theorem 5.1) follows relatively straightforwardly using the results of [6], which rely on the structure of the associated energy functional as well as some well-known properties of the Krasnoselskii-genus. As usual, however, when $p \leq 3$, we face the additional difficulties of constructing bounded Palais-Smale sequences, as well as proving that the Palais-Smale condition holds. To overcome this, we need a more robust approach than given in [6]. Namely, in the case of a coercive ρ and $p \leq 3$, inspired by [7], we use an appropriate abstract min-max scheme and the aforementioned 'monotonicity trick', in order to obtain the existence of infinitely many pairs of distinct solutions to (1.1) (Theorem 5.3).

1.1.3 Necessary conditions for point concentration

After establishing these existence and multiplicity results, it is natural to ask if the non-locality of the Schrödinger-Poisson system allows us to find localised solutions. Moreover, we are interested in removing any compactness condition on ρ. For these reasons we focus on the equation

$$\begin{cases} -\varepsilon^2 \Delta u + \lambda u + \rho(x)\phi u = |u|^{p-1}u, & x \in \mathbb{R}^3 \\ -\Delta \phi = \rho(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.7}$$

with $\rho : \mathbb{R}^3 \to \mathbb{R}$ a nonnegative measurable function, $\lambda \in \mathbb{R}$, and $\lambda > 0$, taking advantage of a shrinking parameter $\varepsilon \sim \hbar \ll 1$ which behaves like the Planck constant in the so-called 'semiclassical limit'. In this direction, Ianni and Vaira [37] notably showed that concentration of semiclassical solutions to

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u + \rho(x)\phi u = |u|^{p-1}u, & 1 < p < 5, \quad x \in \mathbb{R}^3 \\ -\Delta \phi = \rho(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

occurs at stationary points of the external potential V using a Lyapunov-Schmidt approach (in the spirit of the Ambrosetti-Malchiodi monograph [5] on perturbation methods), whereas in [18] concentration results have been obtained using a variational/penalisation approach in the spirit of del Pino and Felmer [30]. In particular, in [18] the question of studying concentration phenomena which are purely driven by ρ has been raised. None of the aforementioned contributions have dealt with necessary conditions for concentration at points in the case $V \equiv$ constant and in the presence of a variable charge density function ρ. We manage to fill this gap, by obtaining a necessary condition, related to ρ, for the concentration at points for solutions to (6.1) both in $E(\mathbb{R}^3)$ (Theorem 6.1) and in $H^1(\mathbb{R}^3)$ (Theorem 6.2), which are the suitable settings for the study of concentration phenomena with variational and perturbative techniques, respectively. These results are obtained in the spirit of [63] using classical blow-up analysis, uniform decay estimates, and Pohozaev type identities.

1.1.4 Sufficient conditions for point concentration

As a natural next step, by adapting the penalisation method of del Pino and Felmer [30], we are interested in showing that the necessary conditions for the point concentration of solutions to (1.7) that we found, are, in fact, also sufficient conditions on ρ. At this stage, we have only partial results in this direction, however at least two of the results we have obtained thus far elicit interesting standalone qualitative properties of the solutions. Namely, in Proposition 7.3, we show that if we rescale the solutions u_ε to (6.1) as $u_\varepsilon(x_\varepsilon + \varepsilon \cdot)$ around a well-chosen family of points x_ε, then the rescaled solutions have a strong limit in $C^{1,\alpha}_{loc}(\mathbb{R}^3)$. Then, in Theorem 7.3, we show that under the additional assumption $\rho \in L^\infty(\mathbb{R}^3)$, we can find a uniform L^∞ bound on the rescaled solutions.

1.2 Organisation of Thesis

Chapter 2 consists of a background discussion of existence results for the nonlinear Schrödinger equation. In Section 2.1, we detail two constrained minimisation arguments that are used to obtain existence results for this equation when the nonlinearity is $|u|^{p-1}u$. In Section 2.2, we present the well-known Mountain Pass Theorem which allows us to work with unconstrained energy functionals, even in instances when the functional is unbounded from below. In Section 2.3, we revisit the nonlinear Schrödinger equation, but this time with a more general nonlinearity. We list two existence results due to Rabinowitz [55], based on variants of the Mountain Pass Theorem, which hold when the nonlinearity satisfies the Ambrosetti-Rabinowitz condition. We then shift the focus to the case when the nonlinearity does not satisfy this condition. We discuss the technique that Jeanjean [38] and Jeanjean and Tanaka [39] developed, inspired by Struwe's 'monotonicity trick' [59], to overcome the difficulty of constructing bounded Palais-Smale sequences and obtain an existence result in this case. Finally, in Section 2.4 we relate this discussion back to the nonlinear Schrödinger-Poisson system in order to better understand the unique features associated with it in comparison to the nonlinear Schrödinger equation.

Chapter 3 includes the preliminary results on the nonlinear Schrödinger-Poisson system that will be used throughout this thesis. In Section 3.1, we define the space $D^{1,2}(\mathbb{R}^3)$ and the explicit representation of ϕ_u that allows us to reduce the Schrödinger-Poisson system to one equation. Then, in Section 3.2 we define the energy functional associated with the nonlinear Schrödinger-Poisson system and discuss the functional settings that are used throughout the thesis, as well as some associated properties. In Section 3.3, we prove a result which gives the regularity and positivity of solutions to the Schrödinger-Poisson system. Lastly, a Pohozaev type inequality is established in Section 3.4 that gives a necessary condition satisfied by solutions to the Schrödinger-Poisson system under suitable assumptions on ρ.

Chapter 4 consists of the results that give the existence of mountain pass and least energy solutions to the Schrödinger-Poisson system. The results in this chapter are from [50]. In Section 4.1, we outline the min-max setting and define the levels c_μ and c_μ^∞, c, and c^∞, which are relevant for both the coercive and non-coercive cases. We then find lower bounds on the functions I_μ and I_μ^∞, when restricted to the set of nontrivial solutions which are fundamental in relation to compactness properties of Palais-Smale sequences. In Section 4.2, we study the case of a coercive ρ and establish that this coercivity is a sufficient condition for the compactness of the embedding $E(\mathbb{R}^3) \hookrightarrow L^{p+1}(\mathbb{R}^3)$. This enables us, using the min-max setting of Section 4.1, to prove existence of positive mountain pass solutions in the coercive case for $p \in (2,3)$ (Theorem 4.1). We then use a minimisation argument to prove the existence of positive least energy solutions (Corollary 4.3). In Section 4.3, we focus on a non-coercive ρ, and we first establish a representation result for bounded Palais-Smale sequence for I_μ in Proposition 4.4. Using the min-max setting of Section 4.1 and the lower bounds found in this section, we prove existence of positive mountain pass solutions for $p \in (2,3)$ (Theorem 4.3). We then show that for $p \geq 3$ the Palais-Smale condition holds for I at the level c, following which the proof of Theorem 4.4 easily follows. We finally prove the existence of positive least energy solutions in the non-coercive case for $p \in (2,5)$ (Corollary 4.6).

Chapter 5 includes the multiplicity results that we have obtained in the case of a coercive ρ. The results in this chapter are from [31]. In Section 5.1, we provide some background on the Krasnoselskii-genus and define the min-max levels b_m that are relevant when $p > 3$, in order to obtain the existence of infinitely many pairs of distinct solutions in this case (Theorem 5.1). Then, in Section 5.2 we discuss the abstract min-max setting that is used when $p \leq 3$. We include a technical lemma that enables us to use this min-max setting, along with the 'monotonicity trick', to prove the existence of infinitely many pairs of distinct solutions for low p (Theorem 5.3).

Chapter 6 focuses on a singularly perturbed Schrödinger-Poisson system and the concentration behaviour of its solutions in the semiclassical limit. The results in this chapter are from [50]. In Section 6.1 and 6.2, we obtain necessary conditions for the concentration at points in $E(\mathbb{R}^3)$ (Theorem 6.1) and $H^1(\mathbb{R}^3)$ (Theorem 6.2), respectively.

Chapter 7 then focuses on the aim of showing that the necessary conditions obtained in the previous chapter for solutions to the singularly perturbed Schrödinger-Poisson system are also sufficient conditions on ρ for point concentration (Conjecture 7.1). The results in this chapter are from [51]. In Section 7.1, we first look at sufficient conditions for concentration of solutions to the nonlinear Schrödinger equation, as a model problem. We discuss the penalisation method of del Pino and Felmer [30] and highlight the main elements of their proof of such sufficient conditions. In Section 7.2, we introduce the penalisation scheme for the nonlinear Schrödinger-Poisson system and highlight the difficulties that we face and must overcome in trying to adapt the method discussed in the previous section. In Section 7.3, we then list and prove the initial results that we have obtained for the nonlinear Schrödinger-Poisson system. These results include those established in the spirit of [30], as well as some further qualitative properties of the rescaled solutions that we believe will be used going forward to obtain some vital estimates on the energy levels.

2 Background

In this chapter, we begin by providing a background discussion of existence results for the nonlinear Schrödinger equation under different assumptions on the potential and the nonlinearity, which ultimately enables us to understand the unique features associated with the nonlinear Schrödinger-Poisson system. The results included in this section are based on lecture material presented by Carlo Mercuri.

2.1 Nonlinear Schrödinger equation with nonlinearity $|u|^{p-1}u$

We first focus on the nonlinear Schrödinger equation

$$-\Delta u + V(x)u = |u|^{p-1}u, \quad x \in \mathbb{R}^3, \tag{2.1}$$

with $V \in C(\mathbb{R}^3, \mathbb{R})$ and $V(x) \geq c > 0$. Defining the space

$$H_V^1 := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)|u|^2 < +\infty \right\},$$

with norm

$$||u||_{H_V^1}^2 := \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 \right),$$

we search for weak solutions of (2.1) as critical points of the functional $J : H_V^1 \to \mathbb{R}$ defined by

$$J(u) := \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 \right) - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1}.$$

2.1.1 Coercive potential V

We consider a coercive potential V, namely $V(x) \to +\infty$ as $|x| \to +\infty$. In this case, the following result holds.

Lemma 2.1. *Assume $V(x) \to +\infty$ as $|x| \to +\infty$. Then, H_V^1 is compactly embedded in $L^{p+1}(\mathbb{R}^3)$ for all $p \in [1,5)$.*

This compactness result enable us to prove the existence of a solution to (2.1) in the case of a coercive V. Namely, we have the following lemma.

Lemma 2.2. *Assume $V \in C(\mathbb{R}^3, \mathbb{R})$, $V(x) \geq c$ for some $c > 0$, and $V(x) \to +\infty$ as $|x| \to +\infty$. Then, (2.1) has a positive weak solution $u \in H^1_V$ for all $p \in [1,5)$.*

Proof. Set

$$\alpha = \inf_{\substack{u \in H^1_V \\ ||u||^{p+1}_{L^{p+1}}=1}} I(u), \quad \text{where } I(u) := \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right).$$

Take a minimising sequence $(u_n)_{n\in\mathbb{N}} \subset H^1_V$ such that $||u_n||_{L^{p+1}} = 1$ and $I(u_n) \to \alpha$. It follows that $||u_n||_{H^1_V} \leq C_1$ and so, by the Banach-Alaoglu Theorem, up to a subsequence, $u_n \rightharpoonup u$ in H^1_V and, by Lemma 2.1, $u_n \to u$ in $L^{p+1}(\mathbb{R}^3)$ for $p \in [1,5)$. Hence, $||u||_{L^{p+1}} = 1$ and, by the weak lower semicontinuity of the norm,

$$I(u) \leq \liminf_{n\to+\infty} I(u_n) = \alpha.$$

By the definition of α it therefore holds that $I(u) = \alpha$. So, by the Lagrange Multiplier Rule, there exists $\lambda \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^3} \nabla u \nabla \phi + \int_{\mathbb{R}^3} V(x)u\phi = \lambda(p+1)\int_{\mathbb{R}^3} |u|^{p-1}u\phi, \quad \forall \phi \in H^1_V.$$

Setting $u_\lambda = [\lambda(p+1)]^{\frac{1}{p-1}}u$, it holds that

$$\int_{\mathbb{R}^3} \nabla u_\lambda \nabla \phi + \int_{\mathbb{R}^3} V(x)u_\lambda\phi = \int_{\mathbb{R}^3} |u_\lambda|^{p-1}u_\lambda\phi, \quad \forall \phi \in H^1_V.$$

Namely $J'(u_\lambda) = 0$, and so $u_\lambda \in H^1_V$ is a weak solution to (2.1). Since u_λ is real, then $|u_\lambda| \in H^1_V$ and

$$\int_{\mathbb{R}^3} |\nabla|u_\lambda||^2 = \int_{\mathbb{R}^3} |\nabla u_\lambda|^2,$$

and so $J(u_\lambda) = J(|u_\lambda|)$. Therefore, $u_\lambda \geq 0$. Further, by standard regularity arguments we can show $u \in C^1(\mathbb{R}^3)$ and we can then use the maximum principle to show $u_\lambda > 0$. $\square$

2.1.2 Non-coercive potential V

We now consider (2.1) with a non-coercive potential V, namely $V(x) \to V_\infty > 0$ as $|x| \to +\infty$. We note that the proof of Lemma 2.2 relied on the compact embedding of H^1_V into L^{p+1} in order to obtain the strong convergence of the minimising sequence in L^{p+1}. However, in the case of a non-coercive potential V, we no longer have this compact embedding and therefore we require some additional compactness analysis in order to prove the existence of a weak solution. Namely, we will require the following well-known compactness result due to P. L. Lions.

Lemma 2.3 (Concentration-Compactness Lemma [44]). *Let $q \in [2,6)$ and $r > 0$. Suppose $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ is bounded and*

$$\sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |u_n|^q \to 0, \quad \textit{as } \ n \to +\infty.$$

Then, $u_n \to 0$ in $L^p(\mathbb{R}^3)$ for all $p \in (2,6)$.

With this in place, we can now prove an existence result in the case of a non-coercive potential V.

Lemma 2.4. *Assume $V \in C(\mathbb{R}^3, \mathbb{R})$, $V(x) \geq c$, and $V(x) \leq \liminf_{|y|\to+\infty} V(y) < C$ for all $x \in \mathbb{R}^3$ and some c, $C > 0$. Then, (2.1) has a positive weak solution $u \in H^1$ for all $p \in (1,5)$.*

Proof. We first note that since $c < V(x) < C$ for all $x \in \mathbb{R}^3$ by assumption, it follows that the H^1 and H^1_V norm are equivalent and so we work in H^1. We set

$$\alpha = \inf_{\substack{u\in H^1 \\ ||u||^{p+1}_{L^{p+1}}=1}} I(u), \quad \text{where } \ I(u) := \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right), \tag{2.2}$$

and notice that α can be written as

$$\alpha = \inf_{0\not\equiv u\in H^1} I\left(\frac{u}{||u||_{L^{p+1}}}\right) = \inf_{0\not\equiv u\in H^1} \frac{I(u)}{||u||^2_{L^{p+1}}}.$$

So, it holds that

$$I(u) \geq \alpha ||u||^2_{L^{p+1}}, \quad \forall u \in H^1. \tag{2.3}$$

We now take a minimising sequence $(u_n)_{n\in\mathbb{N}} \subset H^1$ such that $||u_n||_{L^{p+1}} = 1$ and $I(u_n) \to \alpha$. It follows that $||u_n||_{H^1} \leq C_1$ and so, by the Banach-Alaoglu Theorem, up to a subsequence, $u_n \rightharpoonup u$ in H^1. Due to this weak convergence, it holds that

$$\begin{aligned} I(u_n) &= \frac{1}{2}\left[||u_n - u||^2_{H^1} + ||u||^2_{H^1} - 2(u_n - u, u)\right] \\ &= I(u_n - u) + I(u) + o(1), \quad \text{as } n \to +\infty. \end{aligned}$$

Namely, we have

$$I(u_n - u) + I(u) \to \alpha, \quad \text{as } n \to +\infty. \tag{2.4}$$

We now break the proof into three claims.

Claim 1. *The following alternative holds:* $||u||^{p+1}_{L^{p+1}} = 1$ *or* $u \equiv 0$.

We first note that since $||u_n||_{H^1} \leq C_1$, then $u_n \to u$ almost everywhere in $\mathbb{R}^3$, and so, by Fatou's Lemma,

$$||u||^{p+1}_{L^{p+1}} \leq \liminf_{n\to+\infty} ||u_n||^{p+1}_{L^{p+1}} = 1.$$

That is,

$$\beta := ||u||^{p+1}_{L^{p+1}} \in [0,1].$$

By the Brezis-Lieb Lemma, as $n \to +\infty$, it holds that

$$\beta + ||u_n - u||^{p+1}_{L^{p+1}} = ||u||^{p+1}_{L^{p+1}} + ||u_n - u||^{p+1}_{L^{p+1}} \to ||u_n||^{p+1}_{L^{p+1}} = 1.$$

Combining this with (2.3) and (2.4), we find that as $n \to +\infty$,

$$\alpha \geq \alpha(1-\beta)^{\frac{2}{p+1}} + \alpha\beta^{\frac{2}{p+1}}.$$

That is, as $n \to +\infty$,

$$1 \geq (1-\beta)^{\frac{2}{p+1}} + \beta^{\frac{2}{p+1}}, \quad \beta \in [0,1].$$

From Figure 1, we can see that $f : [0,1] \to \mathbb{R}$ defined by $f(\beta) := (1-\beta)^{\frac{2}{p+1}} + \beta^{\frac{2}{p+1}}$ is such that $f(\beta) \leq 1$ exactly when $\beta = \{0,1\}$.

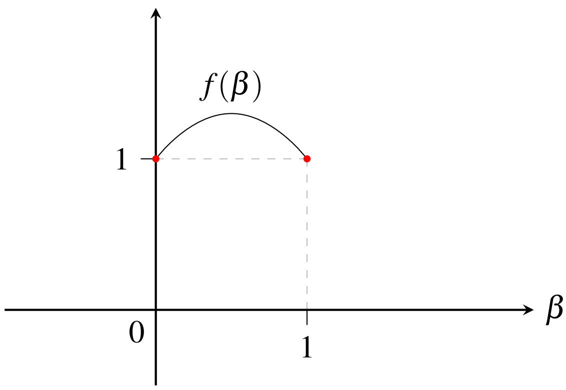

Figure 1: Sketch of $f(\beta) := (1-\beta)^{\frac{2}{p+1}} + \beta^{\frac{2}{p+1}}$.

This proves the claim.

Claim 2. *If $||u||_{L^{p+1}}^{p+1} = 1$, we are done. If $u \equiv 0$, then there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}^3$ such that $v_n := u_n(\cdot - y_n)$ does not have any subsequences which weakly converge to zero.*

If $||u||_{L^{p+1}}^{p+1} = 1$, then we can argue exactly as in Lemma 2.2 using the weak lower semi-continuity of the norm and the Lagrange multiplier rule in order to obtain a positive solution $u \in H^1$.

If $u \equiv 0$, we first recall that $||u_n||_{L^{p+1}} = 1$ and so by Lemma 2.3, it holds that

$$\sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^2 \nrightarrow 0.$$

Namely, there exists $C_2 > 0$ and a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}^3$ such that

$$\int_{B_1(y_n)} |u_n|^2 \geq C_2 > 0.$$

Setting $v_n := u_n(\cdot - y_n)$, we find that

$$\int_{B_1(0)} v_n^2 \geq C_2 > 0, \quad \forall n \in \mathbb{N},$$

which implies that v_n does not have any subsequences which weakly converge to zero.

Claim 3. *If $u \equiv 0$, then the sequence $(v_n)_{n\in\mathbb{N}}$ constructed in the previous claim is a minimising sequence for (2.2).*

We first note that

$$\int_{\mathbb{R}^3} |\nabla v_n|^2 = \int_{\mathbb{R}^3} |\nabla u_n|^2, \tag{2.5}$$

$$\int_{\mathbb{R}^3} |v_n|^2 = \int_{\mathbb{R}^3} |u_n|^2, \tag{2.6}$$

$$\int_{\mathbb{R}^3} |v_n|^{p+1} = \int_{\mathbb{R}^3} |u_n|^{p+1} = 1. \tag{2.7}$$

Moreover, $||v_n||_{L^2}^2 = ||u_n||_{L^2}^2 < C_1$, and so by Bolzano-Weierstrass Theorem, there exists some $k \in \mathbb{R}$ such that

$$||v_n||_{L^2}^2 = \|u_n||_{L^2}^2 \to k, \quad \text{as } n \to +\infty. \tag{2.8}$$

Setting

$$\gamma := \liminf_{|y|\to+\infty} V(y) \geq V(x),$$

we find that

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^3} V(x) v_n^2 \leq \limsup_{n\to\infty} \gamma \int_{\mathbb{R}^3} v_n^2 = \gamma k. \tag{2.9}$$

Now, since $u_n \rightharpoonup u \equiv 0$ then, by Rellich Theorem and the assumption that $V(x) \leq C$, it follows that for any $R>0$,

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^3} V(x)u_n^2 = \liminf_{n\to\infty}\int_{|x|>R} V(x)u_n^2. \tag{2.10}$$

We note that by the definition of γ, it holds that for every $\varepsilon>0$, there exists $R_\varepsilon>0$ such that $V(x)>\gamma-\varepsilon$ for all $|x|>R_\varepsilon$, and therefore we can write

$$\int_{|x|>R_\varepsilon} V(x)u_n^2 \geq (\gamma-\varepsilon)\int_{|x|>R_\varepsilon} u_n^2. \tag{2.11}$$

Combining (2.8), (2.10) and (2.11) and once again using Rellich Theorem, it follows that

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^3} V(x)u_n^2 \geq (\gamma-\varepsilon)k. \tag{2.12}$$

Hence, from (2.9) and (2.12), we can see that

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^3} V(x)v_n^2 \leq \liminf_{n\to+\infty}\int_{\mathbb{R}^3} V(x)u_n^2,$$

which, along with (2.5), (2.6), and (2.7), implies that v_n is a minimising sequence for (2.2).

Conclusion. In summary, if we let $(u_n)_{n\in\mathbb{N}}$ be a minimising sequence for (2.2), then by Claim 1, $u_n \rightharpoonup u$ in H^1 with $||u||_{L^{p+1}}^{p+1}=1$ or $u\equiv 0$. If $||u||_{L^{p+1}}^{p+1}=1$ we are done by Claim 2. If $u\equiv 0$, then we construct $(v_n)_{n\in\mathbb{N}}$ as in Claim 2 and have that v_n is a minimising sequence for (2.2) by Claim 3. Moreover, up to a subsequence, $v_n \rightharpoonup v$ in H^1, by Banach-Alaoglu. Then, either $||v||_{L^{p+1}}^{p+1}=1$ or $v\equiv 0$ by Claim 1, but by Claim 2 the second alternative cannot hold and so we know $||v||_{L^{p+1}}^{p+1}=1$. Thus, we can argue as in Lemma 2.2 using the weak lower semi-continuity of the norm and the Lagrange multiplier rule in order to obtain a positive solution $v\in H^1$. □

2.2 Mountain Pass Theorem

The existence results that we discussed in the previous section relied on a constrained minimisation argument. In this section we will discuss the well-known Mountain Pass Theorem due to Ambrosetti-Rabinowitz [6], which gives us a set of conditions for the energy functional to satisfy in order to guarantee the existence of a critical point at a certain critical level. The Mountain Pass Theorem allows us to work with unconstrained energy functionals, even in instances when the functional is unbounded from below. In order to state the theorem we will need some preliminary definitions.

Definition 1 (Mountain Pass Geometry)**.** *Let X be a Banach space. A functional $J \in C^1(X,\mathbb{R})$ has the Mountain Pass Geometry if the following conditions hold:*

(i) J(0)=0

(ii) There exists r, $a > 0$ such that $J(u) \geq a$ if $||u||_X = r$

(iii) There exists $v \in X$ with $||v||_X > r$ such that $J(v) \leq 0$.

Definition 2 (Palais-Smale sequence at level c)**.** *Let X be a Banach space, $J \in C^1(X,\mathbb{R})$ and $c \in \mathbb{R}$. A sequence $(u_k)_{k\in\mathbb{N}} \subset X$ is a Palais-Smale sequence at level c for J, denoted $(PS)_c$ sequence, if the following hold:*

(i) $J(u_k) \to c$

(ii) $J'(u_k) \to 0$ in X^*.

Remark 2.1. *It has been shown that if a functional J has the Mountain Pass Geometry, then there exists a $(PS)_c$ sequence at the level*

$$c := \inf_{g\in\Gamma} \max_{0\leq t\leq 1} J(g(t)),$$

where

$$\Gamma = \{g \in C([0,1],X) : g(0) = 0, J(g(1)) \leq 0\}.$$

The next definition gives a compactness condition that must hold in order to find solutions using the Mountain Pass Theorem.

Definition 3 (Palais-Smale condition)**.** *Let X be a Banach space, $J \in C^1(X,\mathbb{R})$ and $c \in \mathbb{R}$. The $(PS)_c$ condition holds if every $(PS)_c$ sequence has a strongly convergent subsequence.*

With these definitions in place, we can now state the Mountain Pass Theorem.

Theorem 2.1 (Mountain Pass Theorem [6])**.** *Let X be a Banach space. Assume $J \in C^1(X, \mathbb{R})$ has the Mountain Pass Geometry. Define*

$$c := \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} J(g(t)),$$

where

$$\Gamma = \{g \in C([0,1], X) : g(0) = 0, J(g(1)) \leq 0\}.$$

If J satisfies the $(PS)_c$ condition at the level c defined above, then c is a critical value of J.

For the sake of brevity, the proof of the Mountain Pass Theorem will be omitted. For a complete proof see e.g. p. 478–480 in [32]. The proof relies heavily on a so called Deformation Theorem. The idea of the proof is to show that if c is not a critical level, then for some sufficiently small $\delta > 0$ we can nicely deform the set

$$A_{c+\delta} = \{u \in X : J(u) \leq c + \delta\}$$

into the set

$$A_{c-\delta} = \{u \in X : J(u) \leq c - \delta\}.$$

In order to do so, one solves an appropriate ordinary differential equation and follows the resulting flow "downhill".

Remark 2.2. *We note the the Mountain Pass Theorem gives rise to nontrivial solutions. Namely, by the definition of a critical value, there exists $u \in X$ such that*

$$J'(u) = 0,$$

which implies u is a solution, and

$$J(u) = c = \inf_{g \in \Gamma} \max_{0 \le t \le 1} J(g(t)) > 0,$$

where the positivity is guaranteed by the Mountain Pass Geometry and implies that u is nontrivial.

2.3 Nonlinear Schrödinger equation with a more general nonlinearity

In this section, the nonlinear Schrödinger equation will be considered with a more general nonlinearity $f(x,u)$. Namely, we consider

$$-\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^3. \tag{2.13}$$

The ideas and results from this section will be used for our purposes later in the thesis.

2.3.1 Nonlinearity satisfying the Ambrosetti-Rabinowitz condition

In a crucial paper, Rabinowitz [55] listed assumptions on the nonlinearity f so that variational methods based on variants of the Mountain Pass Theorem could be used in order to obtain existence results for (2.13) in the case of both a coercive and non-coercive potential V. In order to state the results, we first list certain hypotheses on V and f that will need to be used, namely:

(V_1) $V \in C^1(\mathbb{R}^3, \mathbb{R})$ and there is a $c > 0$ such that $V(x) \ge c$ for all $x \in \mathbb{R}^3$,
(V_2) $V(x) \to +\infty$ as $|x| \to +\infty$,
(V_3) $\liminf_{|x| \to +\infty} V(x) = V_\infty$,
(V_4) $V_\infty \ge V(x)$ for all $x \in \mathbb{R}^3$ with $V_\infty \not\equiv V(x)$,
(f_1) $f \in C^2(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$,
(f_2) $f(x,0) = 0 = f_z(x,0)$,
(f_3) there are constants a_1, $a_2 > 0$ and $s \in (1,5)$ such that for all $x \in \mathbb{R}^3$ and $z \in \mathbb{R}$,

$$|f_z(x,z)| \le a_1 + a_2 |z|^{s-1},$$

(f_4) there is a constant $\mu > 2$ such that

$$0 < \mu F(x,z) := \mu \int_0^z f(x,t)\,\mathrm{d}t \le z f(x,z)$$

for all $x \in \mathbb{R}^3$ and $z \in \mathbb{R} \setminus \{0\}$.

(f_5) $t^{-1} z f(z,tz)$ is an increasing function of $t > 0$ for all $x \in \mathbb{R}^3$ and $z \in \mathbb{R} \setminus \{0\}$.

We now state the existence result for a coercive potential V.

Theorem 2.2 ([55]). *If (V_1)-(V_2) and (f_1)-(f_4) are satisfied, then* (2.13) *possesses a nontrivial classical solution $u \in H^1$.*

Proof. See Theorem 1.7 in [55]. □

We follow up this theorem with the existence result for a non-coercive potential V.

Theorem 2.3 ([55]). *Suppose (V_1), (V_3) and (V_4) are satisfied. Assume further that f is independent of x and (f_1)-(f_5) hold. Define $\mathscr{J} : H_V^1 \to \mathbb{R}$ by*

$$\mathscr{J}(u) := \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x) u^2 \right) - \int_{\mathbb{R}^3} F(x,u)\,\mathrm{d}x.$$

Then,

$$c = \inf_{g \in \Gamma} \max_{0 \le t \le 1} \mathscr{J}(g(t)),$$

with

$$\Gamma = \{ g \in C([0,1],X) : g(0) = 0,\ \mathscr{J}(g(1)) \le 0 \},$$

is a critical value of $\mathscr{J}$.

Proof. See Theorem 4.27 in [55]. □

Remark 2.3. *Observe that the nonlinearity* $f(z) = |z|^{p-1}z$ *satisfies* (f_1)*-*(f_5) *for all* $p \in (1,5)$*, and so the results of Lemma 2.2 and 2.4 can be seen as specific cases of Theorem 2.2 and 2.3, respectively.*

As anticipated earlier, the proofs of these results are based on variants of the Mountain Pass Theorem. Namely, the assumptions on the nonlinearity f are first used to show that the Mountain Pass Geometry holds. As per Remark 2.1, this guarantees the existence of a $(\mathrm{PS})_c$ sequence. Assumption (f_4), known as the global Ambrosetti-Rabinowitz condtion, is then used in a vital way in order to show the boundedness of the $(\mathrm{PS})_c$ sequence, which, up to a subsequence, implies the existence of a weak limit by the Banach-Alaoglu Theorem. This weak limit can then be shown to be a weak solution to (2.13). The main difficulty in this problem is that it is unknown how to prove that this weak limit is in fact a strong limit, namely that the $(\mathrm{PS})_c$ condition holds, because of the lack of compactness due to the nonlinearity. As a consequence, it is not immediately known if the weak solution is nontrivial. To overcome this difficulty, Rabinowitz [55] argued by contradiction in the case of a coercive potential V and by using comparison arguments in the case of a non-coercive potential V in order to obtain the above results.

2.3.2 Nonlinearity not satisfying the Ambrosetti-Rabinowitz condition

The natural next step was to relax the global assumption on the nonlinearity f, namely to relax the global Ambrosetti-Rabinowitz condition (f_4). However, in doing so, a new difficulty needed to be faced on top of the lack of compactness due to the nonlinearity; namely, constructing bounded Palais-Smale sequences. Jeanjean [38] and Jeanjean and Tanaka [39] developed a technique to overcome this issue. Namely, Jeanjean [38] and Jeanjean and Tanaka [39] formulated and proved the following theorem, which generalises Struwe's 'monotonicity trick' [59] to the context of nonlinear Schrödinger equations. We include the full statement of the result, as it will be used for our purposes later in the thesis.

Theorem 2.4 ([38], [39])**.** *Let* X *be a Banach space equipped with a norm* $||\cdot||_X$ *and let* $J \subset \mathbb{R}^+$ *be an interval. We consider a family* $(\mathscr{J}_\mu)_{\mu\in J}$ *of* C^1*-functionals on* X *of the form*

$$\mathscr{J}_\mu(u) = A(u) - \mu B(u), \quad \forall \mu \in J,$$

where $B(u) \geq 0$ for all $u \in X$, and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $||u||_X \to +\infty$. We assume there are two points v_1, v_2 in X such that

$$c_\mu = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathscr{I}_\mu(\gamma(t)) > \max\{\mathscr{I}_\mu(v_1), \mathscr{I}_\mu(v_2)\}, \quad \forall \mu \in J,$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\mu \in J$, there is a sequence $\{v_n\} \subset X$ such that

(i) $\{v_n\}$ *is bounded,*

(ii) $\mathscr{I}_\mu(v_n) \to c_\mu$,

(iii) $\mathscr{I}'_\mu(v_n) \to 0$ *in* X^*.

The proof of this result will be omitted for the sake of brevity, but it is important to note that it depends in a crucial way on the monotonicity of the functional $\mathscr{I}_\mu$ with respect to μ and the almost everywhere differentiability of the mapping $\mu \mapsto c_\mu$. This theorem then enabled Jeanjean and Tanaka [39] to obtain the following existence result for nonlinearities which do not satisfy the global Ambrosetti-Rabinowitz condition.

Theorem 2.5 ([39])**.** *Assume $f \in C(\mathbb{R}^+, \mathbb{R})$ is such that:*

(i) $f(0) = 0$ and $f'(0)$ defined as $\lim_{s \to 0^+} f(s)s^{-1}$ exists,
(ii) there is $p < 5$ such that $\lim_{s \to +\infty} f(s)s^{-p} = 0$,
(iii) $\lim_{s \to +\infty} f(s)s^{-1} = +\infty$.

Suppose further that the potential $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies the following conditions:

(i) $f'(0) < \inf \sigma(-\Delta + V(x))$, where $\sigma(-\Delta + V(x))$ denotes the spectrum of the self-adjoint operator $-\Delta + V(x) : H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, i.e.,

$$\inf \sigma(-\Delta + V(x)) = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x}{\int_{\mathbb{R}^3} |u|^2}$$

(ii) $V(x) \to V_\infty \in \mathbb{R}$ *as* $|x| \to +\infty$,

(iii) $V(x) \le V_\infty$ *a.e.,*

(iv) there exists a function $\varphi \in L^2(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$ *such that*

$$|x||\nabla V(x)| \le \varphi^2(x), \quad \forall x \in \mathbb{R}^3.$$

Then, (2.13) *possesses a positive solution* $u \in H^1$.

Although we will not include a formal proof of this result, we will try to give the general ideas. The proof relies on investigating the "perturbed problem,"

$$-\Delta u + V(x)u = \mu f(x,u), \quad x \in \mathbb{R}^3,\ \mu \in \left[\frac{1}{2}, 1\right] \tag{2.14}$$

with the associated family of functionals $\mathscr{I}_\mu : H^1 \to \mathbb{R}$ given by

$$\mathscr{I}_\mu(u) := \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) - \mu \int_{\mathbb{R}^3} F(x,u)\,\mathrm{d}x, \quad \mu \in \left[\frac{1}{2}, 1\right],$$

where

$$F(x,z) := \int_0^z f(x,t)\,\mathrm{d}t.$$

Under the assumptions of Theorem 2.5, it can be proved that $\mathscr{I}_\mu$ has the Mountain Pass Geometry for each $\mu \in \left[\frac{1}{2}, 1\right]$. Using this and the monotonicity of $\mathscr{I}_\mu$ with respect to μ, the min-max levels $c_\mu > 0$ can be defined as in Theorem 2.4, namely in such a way that the class Γ does not depend on μ. Applying Theorem 2.4 with $J = [\frac{1}{2}, 1]$, it follows that there exists a bounded Palais-Smale sequence for $\mathscr{I}_\mu$ at level c_μ for almost every $\mu \in \left[\frac{1}{2}, 1\right]$. Through compactness analysis, it can then be shown that each of these levels c_μ correspond to a critical level of $\mathscr{I}_\mu$, thus giving a sequence of critical points of the perturbed functionals. The assumed decay assumption on $\nabla V(x)$ enables the derivation of a Pohozaev type identity, that is a necessary condition satisfied by critical points u of $\mathscr{I}_\mu$ obtained by using $(x, \nabla u)$ as a test function in (2.14). The Pohozaev identity can then be used in order to show the boundedness of a selected sequence of critical points of the perturbed functionals. Finally, it is possible to prove that this sequence of critical points actually converges to a solution of the original problem (2.13). In fact, this sequence of critical points is, indeed, a bounded Palais-Smale sequence for $\mathscr{J}$.

2.4 Nonlinear Schrödinger equation versus nonlinear Schrödinger-Poisson system

In this thesis, we will be interested in the nonlinear Schrödinger-Poisson system,

$$\begin{cases} -\Delta u + u + \rho(x)\phi u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = \rho(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

with $p \in (2,5)$ and $\rho : \mathbb{R}^3 \to \mathbb{R}$ a nonnegative measurable function. Within a min-max setting and in the spirit of Rabinowitz [55], we study existence and qualitative properties of the solutions to this system highlighting those phenomena which are driven by ρ, in both the case of a coercive and non-coercive ρ. This system presents various mathematical features, sensitive to the range of p, which are not shared with the nonlinear Schrödinger equation. Indeed, the vast majority of previous works involve the range $p > 3$ since, when $p \leq 3$, one has to face two major obstacles in applying the minimax methods: constructing bounded Palais-Smale sequences and proving that the Palais-Smale condition holds, see e.g [57], and [7], [47].

We are immediately able to highlight a feature which is not shared with the nonlinear Schrödinger equation; even though the nonlinearity in the Schrödinger-Poisson system satisfies the global Ambrosetti-Rabinowitz condition, the boundedness of the Palais-Smale sequences is not automatic. Indeed, it is the structure of the functional associated with the nonlinear Schrödinger-Poisson system when $p < 3$ that causes the possible unboundedness of the Palais-Smale sequences. It remains an open question as to whether the Palais-Smale sequences are bounded for $p < 3$ and it is, in fact, suspected to be untrue. For $p \leq 3$, one must also overcome the obstacle of showing that the Palais-Smale condition holds. In order to prove this condition, some compactness analysis is needed. In the case $\rho(x) \to \rho_\infty > 0$ as $|x| \to +\infty$ it is the invariance by translations of the limiting problem at infinity that causes the lack of compactness phenomena to occur. This type of lack of compactness has been tackled in pioneering works of [44] and [12] in the context of minimisation problems and Schrödinger type equations, respectively. More recent contributions in the context of Schrödinger-Poisson systems are mostly in the range $p > 3$ (see e.g. [26], [62], [24]).

The results contained in this thesis cover this difficult range $p \leq 3$. We generalise previous results that we have discussed in this chapter to the context of the nonlinear Schrödinger-Poisson system, in order to overcome the important technical differences associated with this system in comparison to the nonlinear Schrödinger equation. Namely, we make use of the aforementioned 'monotonicity trick' [59], [38], [39] exploiting the structure of our functional to construct bounded Palais-Smale sequences for small perturbations of the nonlinear Schrödinger-Poisson system. We then prove that these sequences converge using compact embeddings and an a posteriori compactness analysis of bounded Palais-Smale sequences (in the spirit of the classical book of M. Willem [64]) in the case of a coercive and non-coercive potential ρ, respectively.

3 Preliminaries

In this section, we discuss some preliminary results on the nonlinear Schrödinger-Poisson system that will be used throughout this thesis. The results included in this section are a critical elaboration on lecture material presented by Carlo Mercuri.

3.1 The space $D^{1,2}(\mathbb{R}^3)$ and the explicit representation of ϕ_u

We set $D^{1,2}(\mathbb{R}^3) = D^{1,2}$ as the space defined as

$$D^{1,2}(\mathbb{R}^3) := \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\},$$

and equipped with norm

$$||u||_{D^{1,2}(\mathbb{R}^3)} := ||\nabla u||_{L^2(\mathbb{R}^3)}.$$

With this definition in place, we can state the next result, which will ultimately allow us to reduce the nonlinear Schrödinger-Poisson system to one equation. This result is classical from potential theory, but we include the statement and proof for the reader's convenience.

Theorem 3.1. *Assume $\rho : \mathbb{R}^3 \to \mathbb{R}$ is a nonnegative measurable function. If $u^2\rho \in L^1_{loc}(\mathbb{R}^3)$ is such that*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y < +\infty, \tag{3.1}$$

then,

$$\phi_u(x) := \int_{\mathbb{R}^3} \frac{\rho(y)u^2(y)}{4\pi|x-y|}\,\mathrm{d}y \in D^{1,2}(\mathbb{R}^3)$$

is the unique weak solution in $D^{1,2}(\mathbb{R}^3)$ of the Poisson equation

$$-\Delta\phi = \rho(x)u^2$$

and it holds that

$$\int_{\mathbb{R}^3} |\nabla \phi_u|^2 = \int_{\mathbb{R}^3} \rho \phi_u u^2 \,\mathrm{d}x = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{4\pi|x-y|} \,\mathrm{d}x\,\mathrm{d}y. \tag{3.2}$$

Proof. We will break the proof into a series of claims.

Claim 1. *Define* $f_n(x) := \min(\rho^{\frac{1}{2}}(x)u(x), n) \cdot \chi_{B_n(0)}(x)$. *Then,*

$$\phi_n(x) := \int_{\mathbb{R}^3} \frac{f_n^2(y)}{4\pi|x-y|} \,\mathrm{d}y$$

solves

$$-\Delta \phi_n = f_n^2$$

in the sense of distributions. Namely, it holds that

$$-\int_{\mathbb{R}^3} \phi_n \Delta \psi = \int_{\mathbb{R}^3} f_n^2 \psi,$$

for all $\psi \in C_c^\infty(\mathbb{R}^3)$. *Moreover,* $\phi_n \in C^1(\mathbb{R}^3)$ *and* ϕ_n *is uniformly bounded in* $D^{1,2}(\mathbb{R}^3)$.

We first note that by definition f_n has the following properties:

(i) f_n has compact support,

(ii) f_n is nondecreasing,

(iii) $f_n \nearrow \rho^{\frac{1}{2}} u$ almost everywhere,

(iv) $f_n \in L^\infty(\mathbb{R}^3)$.

Thus, by Theorem 6.21 in [42], it follows that

$$\phi_n(x) := \int_{\mathbb{R}^3} \frac{f_n^2(y)}{4\pi|x-y|} \,\mathrm{d}y$$

solves

$$-\Delta\phi_n = f_n^2,$$

in the sense of distributions. It further follows from this theorem that ϕ_n has a distributional derivative that is given, for almost every x, by

$$\partial_i \phi_n(x) = \int_{\mathbb{R}^3} \frac{\partial G_y(x)}{\partial x_i} f_n^2(y)\,\mathrm{d}y, \tag{3.3}$$

where

$$G_y(x) = \frac{1}{4\pi|x-y|},$$

$$\frac{\partial G_y(x)}{\partial x_i} = \frac{-1}{4\pi}|x-y|^{-3}(x_i - y_i),$$

and

$$-\Delta G_y = \delta_y \quad \text{in } \mathscr{D}'(\mathbb{R}^3),$$

meaning that

$$\int_{\mathbb{R}^3} G_y(x)\Delta\psi(x)\,\mathrm{d}x = -\psi(y) \quad \forall \psi \in C_c^\infty(\mathbb{R}^3). \tag{3.4}$$

Now, by (iv) and Theorem 10.2 in [42], it follows that $\phi_n \in C^1(\mathbb{R}^3)$. Finally, using (3.3) and (iii), it holds that

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla\phi_n|^2 &= \sum_{i=1}^{3} \int_{\mathbb{R}^3} (\partial_i \phi_n)^2 \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f_n^2(x) f_n^2(y)}{4\pi|x-y|} \\ &\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{4\pi|x-y|},\end{aligned}$$

which is bounded by assumption, and so the claim is proved.

Claim 2. *Since ϕ_n is uniformly bounded in $D^{1,2}(\mathbb{R}^3)$ by the previous claim, then up to a subsequence $\phi_n \rightharpoonup \phi$ in $D^{1,2}(\mathbb{R}^3)$. We claim that*

$$-\Delta\phi = \rho(x)u^2$$

in the sense of distributions. Namely, it holds that

$$-\int_{\mathbb{R}^3} \phi\Delta\psi = \int_{\mathbb{R}^3} \rho u^2 \psi,$$

for all $\psi \in C_c^\infty(\mathbb{R}^3)$.

We first note that by Theorem 8.6 in [42] we have that $\phi_n \to \phi$ in $L^p_{loc}(\mathbb{R}^3)$ for all $p < 6$ and $\phi_n \to \phi$ a.e. on $\mathbb{R}^3$. On the other hand, by the Monotone Convergence Theorem, as $n \to +\infty$, it holds that

$$\phi_n(x) = \int_{\mathbb{R}^3} \frac{f_n^2(y)}{4\pi|x-y|} \to \int_{\mathbb{R}^3} \frac{\rho(y)u^2(y)}{4\pi|x-y|} =: \phi_u(x).$$

Therefore, we have that $\phi_n \to \phi_u$ a.e. and, so by the uniqueness of the limit, $\phi_u \equiv \phi \in D^{1,2}(\mathbb{R}^3)$. By Claim 1, we have that

$$-\int_{\mathbb{R}^3} \phi_n \Delta\psi = \int_{\mathbb{R}^3} f_n^2 \psi, \tag{3.5}$$

for all $\psi \in C_c^\infty(\mathbb{R}^3)$. Now, since $\phi_n \in C^1(\mathbb{R}^3)$, $\phi_n \rightharpoonup \phi$ in $D^{1,2}(\mathbb{R}^3)$, and ψ has compact support, it holds that

$$-\int_{\mathbb{R}^3} \phi_n \Delta\psi = \int_{\mathbb{R}^3} \nabla\phi_n \nabla\psi \to \int_{\mathbb{R}^3} \nabla\phi\nabla\psi, \quad \text{as } n \to +\infty. \tag{3.6}$$

Further, by the Monotone Convergence Theorem, as $n \to +\infty$,

$$\int_{\mathbb{R}^3} f_n^2 \psi \to \int_{\mathbb{R}^3} \rho u^2 \psi. \tag{3.7}$$

Therefore, putting together (3.5), (3.6), and (3.7) we have shown that for all $\psi \in C_c^\infty(\mathbb{R}^3)$,

$$-\int_{\mathbb{R}^3} \phi \Delta \psi = \int_{\mathbb{R}^3} \rho u^2 \psi.$$

Claim 3. *For each $n \in \mathbb{N}$, the following hold:*

(a) $\int_{\mathbb{R}^3} \nabla \phi_n \nabla \psi_k = \int_{\mathbb{R}^3} f_n^2 \psi_k, \quad \forall \psi_k \in C_c^\infty(\mathbb{R}^3),$

(b) $\int_{\mathbb{R}^3} \nabla \phi_n \nabla \psi = \int_{\mathbb{R}^3} f_n^2 \psi, \quad \forall \psi \in D^{1,2}(\mathbb{R}^3).$

To prove (a), we take $\psi_k \in C_c^\infty(\mathbb{R}^3)$. Then, using (3.3) and Fubini's Theorem, it holds that

$$\begin{aligned}
\int_{\mathbb{R}^3} \nabla \phi_n \nabla \psi_k &= \int_{\mathbb{R}^3} \sum_{i=1}^{3} \left(\int_{\mathbb{R}^3} \frac{\partial G_y}{\partial x_i}(x) f_n^2(y)\,\mathrm{d}y \right) \partial_i \psi_k \,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \sum_{i=1}^{3} \left(\int_{\mathbb{R}^3} \frac{\partial G_y}{\partial x_i}(x) \partial_i \psi_k \,\mathrm{d}x \right) f_n^2(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^3} \psi_k f_n^2,
\end{aligned}$$

where we have used integration by parts and (3.4) to obtain the final equality.

To prove (b), we let $\psi \in D^{1,2}(\mathbb{R}^3)$ be arbitrary. We pick $\psi_k \in C_c^\infty(\mathbb{R}^3)$ such that $\psi_k \to \psi$ in $D^{1,2}(\mathbb{R}^3)$. This is possible because $D^{1,2}(\mathbb{R}^3)$ can be defined as the closure of $C_c^\infty(\mathbb{R}^3)$ with respect to $||\nabla \cdot ||_{L^2}$. Then, by (a), it holds that

$$\int_{\mathbb{R}^3} \nabla \phi_n \nabla \psi_k = \int_{\mathbb{R}^3} \psi_k f_n^2. \tag{3.8}$$

We note that by the Cauchy Schwarz Inequality,

$$\begin{aligned}
\left| \int_{\mathbb{R}^3} \nabla \phi_n \nabla (\psi_k - \psi) \right| &\leq ||\nabla \phi_n||_{L^2} ||\nabla(\psi_k - \psi)||_{L^2} \\
&= ||\phi_n||_{D^{1,2}} ||\psi_k - \psi||_{D^{1,2}} \to 0, \quad \text{as } k \to +\infty,
\end{aligned} \tag{3.9}$$

since ϕ_n is bounded in $D^{1,2}(\mathbb{R}^3)$ by Claim 1. Moreover, by Hölder's and Sobolev's

Inequality,

$$\begin{aligned}\left|\int_{\mathbb{R}^3}(\psi_k-\psi)f_n^2\right| &\le ||f_n||_{L^\infty}^2\int_{\operatorname{supp} f_n}|\psi_k-\psi| \\ &\le ||f_n||_{L^\infty}^2|\operatorname{supp} f_n|^{\frac{5}{6}}||\psi_k-\psi||_{L^6} \\ &\le C||f_n||_{L^\infty}^2|\operatorname{supp} f_n|^{\frac{5}{6}}||\nabla(\psi_k-\psi)||_{L^2} \\ &= C||f_n||_{L^\infty}^2|\operatorname{supp} f_n|^{\frac{5}{6}}||\psi_k-\psi||_{D^{1,2}}\to 0, \quad \text{as } k\to+\infty, \end{aligned} \tag{3.10}$$

since f_n is compactly supported and bounded in $L^\infty(\mathbb{R}^3)$ by Claim 1. Putting together (3.8), (3.9), and (3.10), we have shown

$$\int_{\mathbb{R}^3}\nabla\phi_n\nabla\psi=\int_{\mathbb{R}^3}f_n^2\psi.$$

Claim 4. *The following hold:*

(a) $\int_{\mathbb{R}^3}\nabla\phi\nabla\psi_k=\int_{\mathbb{R}^3}\rho u^2\psi_k, \quad \forall\psi_k\in C_c^\infty(\mathbb{R}^3),$

(b) $\int_{\mathbb{R}^3}\nabla\phi\nabla\psi=\int_{\mathbb{R}^3}\rho u^2\psi, \quad \forall\psi\in D^{1,2}(\mathbb{R}^3).$

Since $-\Delta\phi=\rho u^2$ in the sense of distributions by Claim 2, the proof of (a) follows similarly to that of Claim 3. To prove (b), we let $\psi\in D^{1,2}(\mathbb{R}^3)$ be arbitrary. We pick $\psi_k\in C_c^\infty(\mathbb{R}^3)$ such that $\psi_k\to\psi$ in $D^{1,2}(\mathbb{R}^3)$. Then, by (a), it holds that

$$\int_{\mathbb{R}^3}\nabla\phi\nabla\psi_k=\int_{\mathbb{R}^3}\rho u^2\psi_k. \tag{3.11}$$

As in Claim 3, by Cauchy Schwarz Inequality, we have that

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\nabla\phi\nabla(\psi_k-\psi)\right| &\le ||\nabla\phi||_{L^2}||\nabla(\psi_k-\psi)||_{L^2} \\ &= ||\phi||_{D^{1,2}}||\psi_k-\psi||_{D^{1,2}}\to 0, \quad \text{as } k\to+\infty, \end{aligned} \tag{3.12}$$

since ϕ is bounded in $D^{1,2}(\mathbb{R}^3)$ by Claim 2. We now notice that since ϕ_n is bounded in $D^{1,2}(\mathbb{R}^3)$ by Claim 1 and $|\psi_k|\in D^{1,2}(\mathbb{R}^3)$, then using Claim 3 (b) and the Cauchy Schwarz Inequality, it holds that

$$\begin{aligned}\int_{\mathbb{R}^3} f_n^2|\psi_k| &= \int_{\mathbb{R}^3} \nabla\phi_n \nabla|\psi_k| \\ &\le ||\nabla\phi_n||_{L^2}||\nabla\psi_k||_{L^2} \\ &\le C||\psi_k||_{D^{1,2}}. \end{aligned} \tag{3.13}$$

Moreover, for each fixed k, by the Monotone Convergence Theorem,

$$\int_{\mathbb{R}^3} f_n^2|\psi_k| \to \int_{\mathbb{R}^3} \rho u^2|\psi_k|, \quad \text{as } n \to +\infty. \tag{3.14}$$

Putting (3.13) and (3.14) together, we have that

$$\int_{\mathbb{R}^3} \rho u^2|\psi_k| \le C||\psi_k||_{D^{1,2}}.$$

Using this and Fatou's Lemma, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^3} \rho u^2\psi\right| &\le \int_{\mathbb{R}^3} \rho u^2|\psi| \\ &\le \liminf_{k\to+\infty}\int_{\mathbb{R}^3} \rho u^2|\psi_k| \\ &\le C\liminf_{k\to+\infty}||\psi_k||_{D^{1,2}} \\ &= C||\psi||_{D^{1,2}}.\end{aligned}$$

Therefore, we have shown that $\psi \mapsto \int \rho u^2\psi$ is a linear and continuous functional in $D^{1,2}(\mathbb{R}^3)$. Hence, since $\psi_k \to \psi$ in $D^{1,2}(\mathbb{R}^3)$, it follows that

$$\int_{\mathbb{R}^3} \rho u^2\psi_k \to \int_{\mathbb{R}^3} \rho u^2\psi,$$

and so putting this, (3.12) and (3.11) together, we have proved the claim.

Claim 5. *It holds that*

$$\phi(x) \equiv \phi_u(x) := \int_{\mathbb{R}^3} \frac{\rho(y)u^2(y)}{4\pi|x-y|}\,\mathrm{d}y$$

is the unique weak solution in $D^{1,2}(\mathbb{R}^3)$ *of the Poisson equation*

$$-\Delta\phi = \rho(x)u^2.$$

It follows from Claim 2 that $\phi \equiv \phi_u$, where, by Claim 4 (b), ϕ is a weak solution in $D^{1,2}(\mathbb{R}^3)$ of the Poisson equation $-\Delta\phi = \rho(x)u^2$. It remains to prove the uniqueness of ϕ. Assume there exists ϕ_1, $\phi_2 \in D^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta\phi_i = \rho(x)u^2, \quad \text{for } i = 1,2.$$

It follows that

$$\Delta(\phi_1 - \phi_2) = 0.$$

Testing this equation with $(\phi_1 - \phi_2)$, we obtain

$$\int_{\mathbb{R}^3} |\nabla(\phi_1 - \phi_2)|^2 = ||\nabla(\phi_1 - \phi_2)||_{L^2}^2 = 0.$$

Using this and the Sobolev Inequality, we can see that

$$C||\phi_1 - \phi_2||_{L^6}^2 \leq ||\nabla(\phi_1 - \phi_2)||_{L^2}^2 = 0,$$

which implies $\phi_1 \equiv \phi_2$, thus proving uniqueness.

Claim 6. *The following are equivalent:*

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2 = \int_{\mathbb{R}^3} \rho\phi_u u^2 \,\mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{4\pi|x-y|} \,\mathrm{d}x\,\mathrm{d}y.$$

Testing the equation $-\Delta\phi_u = \rho(x)u^2$ with ϕ_u, we obtain

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2 = \int_{\mathbb{R}^3} \rho\phi_u u^2.$$

Therefore, plugging in

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{\rho(y)u^2(y)}{4\pi|x-y|}\,\mathrm{d}y,$$

it follows that

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2 = \int_{\mathbb{R}^3} \rho\phi_u u^2\,\mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{4\pi|x-y|}\,\mathrm{d}x\,\mathrm{d}y.$$

□

3.2 The associated energy functional and functional settings

Using the explicit representation of ϕ_u given in Theorem 3.1 reduces the nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \rho(x)\phi u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = \rho(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

to the problem

$$-\Delta u + u + \rho(x)\phi_u u = |u|^{p-1}u. \tag{3.15}$$

Positive solutions of this problem are critical points of the functional

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) + \frac{1}{4}\int_{\mathbb{R}^3}\rho\phi_u u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} u_+^{p+1}, \tag{3.16}$$

which is natural to define in $E(\mathbb{R}^3) \subseteq H^1(\mathbb{R}^3)$

$$E(\mathbb{R}^3) := \left\{u \in D^{1,2}(\mathbb{R}^3) : ||u||_E < +\infty\right\},$$

where

$$||u||_E := \left(\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)\,\mathrm{d}x + \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

Variants of this space have been studied since the work of P.L. Lions [43], see e.g.

[56], and [11], [19], [48]. We recall that by the classical Hardy-Littlewood-Sobolev inequality, it holds that

$$\left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{4\pi|x-y|} \mathrm{d}x\,\mathrm{d}y \right| \leq C||\rho u^2||^2_{L^{\frac{6}{5}}(\mathbb{R}^3)}, \tag{3.17}$$

for some $C > 0$. Thus, if $u \in H^1(\mathbb{R}^3)$, we see that, depending on the assumptions on ρ, we may not be able to control the Coulomb integral using the natural bound provided by Hardy-Littlewood-Sobolev inequality. This may be the case if e. g. $\rho(x) \to +\infty$ as $|x| \to +\infty$. For these reasons, in the present thesis we analyse those instances where $E(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$ do not coincide.

3.2.1 The space $E(\mathbb{R}^3)$

Let us assume that ρ is continuous and nonnegative. It is easy to see that $E(\mathbb{R}^3)$ is a uniformly convex Banach space. As a consequence it is reflexive and, in particular, the unit ball is weakly compact. Reasoning as in Proposition 2.4 in [56] and Proposition 2.10 in [48], we have the following important result.

Lemma 3.1. *Assume ρ is continuous and nonnegative. A sequence $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^3)$ is weakly convergent to u in $E(\mathbb{R}^3)$ if and only if it is bounded in $E(\mathbb{R}^3)$ and converges strongly to u in $L^1_{loc}(\mathbb{R}^3)$. In particular, $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$.*

Proof. We first note that if $u_n \rightharpoonup u$ in $E(\mathbb{R}^3)$, then $||u_n||_{E(\mathbb{R}^3)} < C$ by weak lower semicontinuity of the norm and so $u_n \to \tilde{u}$ in $L^1_{loc}(\mathbb{R}^3)$ by Rellich Theorem. On the other hand, if we assume $u_n \to \tilde{u}$ in $L^1_{loc}(\mathbb{R}^3)$ and $||u_n||_{E(\mathbb{R}^3)} < C$, then $u_n \rightharpoonup u$ in $E(\mathbb{R}^3)$ since $E(\mathbb{R}^3)$ is reflexive. Thus, to prove the first part of the lemma, it suffices to show that $u = \tilde{u}$ a.e. We first note that if we take $\varphi \in C_c^\infty(\mathbb{R}^3)$ and $v \in E(\mathbb{R}^3)$, then

$$\left| \int_{\mathbb{R}^3} \varphi v \right| \leq C \int_{supp\,\varphi} v \leq C||v||_{E(\mathbb{R}^3)},$$

for some $C > 0$. Thus, $v \mapsto \int \varphi v$ is a linear and continuous functional on $E(\mathbb{R}^3)$. So, since $u_n \rightharpoonup u$ in $E(\mathbb{R}^3)$, it follows that

$$\int_{\mathbb{R}^3} \varphi u_n \to \int_{\mathbb{R}^3} \varphi u.$$

Moreover, since $u_n \to \tilde{u}$ in $L^1_{loc}(\mathbb{R}^3)$, it holds that

$$\int_{\mathbb{R}^3} \varphi(u_n - \tilde{u}) \leq C \int_{supp\,\varphi} (u_n - \tilde{u}) \to 0.$$

Therefore, we have shown

$$\int_{\mathbb{R}^3} \varphi u = \int_{\mathbb{R}^3} \varphi \tilde{u},$$

and so since $\varphi \in C_c^\infty(\mathbb{R}^3)$ was arbitrary, we conclude that $u = \tilde{u}$ a.e.

It remains to show $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$. Let $\psi \in D^{1,2}(\mathbb{R}^3)$ be arbitrary. Pick $\psi_k \in C_c^\infty(\mathbb{R}^3)$ such that $\psi_k \to \psi$ in $D^{1,2}(\mathbb{R}^3)$. Then, since ϕ_{u_n} solves $-\Delta \phi_{u_n} = \rho(x) u_n^2$, it holds that

$$\int_{\mathbb{R}^3} \nabla \phi_{u_n} \nabla \psi_k = \int_{\mathbb{R}^3} \rho(x) u_n^2 \psi_k. \tag{3.18}$$

Now, since ϕ_{u_n} is bounded in $D^{1,2}(\mathbb{R}^3)$ by assumption, then, up to a subsequence, $\phi_{u_n} \rightharpoonup \phi$ in $D^{1,2}(\mathbb{R}^3)$, for some $\phi \in D^{1,2}(\mathbb{R}^3)$. Thus, it automatically follows from the weak convergence that

$$\int_{\mathbb{R}^3} \nabla \phi_{u_n} \nabla \psi_k \to \int_{\mathbb{R}^3} \nabla \phi \nabla \psi_k, \quad \text{as } n \to +\infty. \tag{3.19}$$

Moreover, since $u_n^2 \to u^2$ in $L^1_{loc}(\mathbb{R}^3)$ by Rellich Theorem and since ρ is continuous by assumption, it holds that

$$\left| \int_{\mathbb{R}^3} \rho(u_n^2 - u^2) \psi_k \right| \leq C \int_{\text{supp } \psi_k} |u_n^2 - u^2| \to 0, \quad \text{as } n \to +\infty. \tag{3.20}$$

Therefore, combining (3.18), (3.19) and (3.20) we have now shown that

$$\int_{\mathbb{R}^3} \nabla \phi \nabla \psi_k = \int_{\mathbb{R}^3} \rho(x) u^2 \psi_k. \tag{3.21}$$

Now, arguing exactly as in the proof of Claim 4 of Theorem 3.1, we obtain

$$\left| \int_{\mathbb{R}^3} \nabla \phi \nabla (\psi_k - \psi) \right| \to 0, \quad \text{as } k \to +\infty, \tag{3.22}$$

since $\phi \in D^{1,2}(\mathbb{R}^3)$. Furthermore, using the facts that

$$\int_{\mathbb{R}^3} \rho u_n^2 \psi_k \to \int_{\mathbb{R}^3} \rho u^2 \psi_k, \quad \text{as } n \to +\infty,$$

by (3.20) and ϕ_n is bounded in $D^{1,2}(\mathbb{R}^3)$ by assumption, we can argue similarly to the proof of Claim 4 of Theorem 3.1 to show that $\psi \mapsto \int \rho u^2 \psi$ is a linear and continuous functional on $D^{1,2}(\mathbb{R}^3)$. Hence, since $\psi_k \to \psi$ in $D^{1,2}(\mathbb{R}^3)$, it follows that

$$\int_{\mathbb{R}^3} \rho u^2 \psi_k \to \int_{\mathbb{R}^3} \rho u^2 \psi, \quad \text{as } k \to +\infty,$$

and so putting this, (3.22) and (3.21) together, we obtain

$$\int_{\mathbb{R}^3} \nabla \phi \nabla \psi = \int_{\mathbb{R}^3} \rho(x) u^2 \psi.$$

Namely, we have shown that $\phi \in D^{1,2}(\mathbb{R}^3)$ solves $-\Delta\phi = \rho(x)u^2$, and so by the uniqueness of the solution in $D^{1,2}(\mathbb{R}^3)$ (Theorem 3.1) we have $\phi = \phi_u$. □

The following nonlocal Brezis-Lieb lemma, which is stated without proof, will also be very useful to study the compactness of Palais-Smale sequences.

Lemma 3.2 ([10], [48]). *[**Nonlocal Brezis-Lieb lemma**] Assume ρ is continuous and nonnegative. Let $(u_n)_{n\in\mathbb{N}} \subset E(\mathbb{R}^3)$ be a bounded sequence such that $u_n \to u$ almost everywhere in $\mathbb{R}^3$. Then it holds that*

$$\lim_{n\to\infty} \left[\|\nabla \phi_{u_n}\|^2_{L^2(\mathbb{R}^3)} - \|\nabla \phi_{u_n - u}\|^2_{L^2(\mathbb{R}^3)} \right] = \|\nabla \phi_u\|^2_{L^2(\mathbb{R}^3)}.$$

3.3 Regularity and positivity

Using standard regularity theory and the maximum principle, we now provide a result giving the regularity and positivity of the solutions to the Schrödinger-Poisson system.

Proposition 3.1. *[**Regularity and positivity**] Let $p \in [1,5]$, $\rho \in C(\mathbb{R}^3) \cap L^\infty_{loc}(\mathbb{R}^3) \setminus \{0\}$ be nonnegative and $(u, \phi_u) \in E(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ be a weak solution of the problem*

$$\begin{cases} -\Delta u + bu + c\rho(x)\phi u = d|u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = \rho(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{3.23}$$

with $b, c, d \in \mathbb{R}_+$. Assume that u is nonnegative. Then, u, $\phi_u \in W^{2,q}_{loc}(\mathbb{R}^3)$, for every $q \geq 1$, and so u, $\phi_u \in C^{1,\alpha}_{loc}(\mathbb{R}^3)$. If, in addition, $u \not\equiv 0$, then u, $\phi_u > 0$ everywhere.

Proof. Under the hypotheses of the proposition, both u and ϕ_u have weak second derivatives in L^q_{loc} for all $q < \infty$. In fact, note that from the first equation in (3.23), we have that $-\Delta u = g(x,u)$, where

$$\begin{aligned} |g(x,u)| &= |(-bu - c\rho(x)\phi u + d|u|^{p-1}u| \\ &\leq C(1 + |\rho\phi_u| + |u|^{p-1})(1 + |u|) \\ &:= h(x)(1 + |u|). \end{aligned}$$

Using our assumptions on ρ, ϕ_u, and u, we can show that $h \in L^{3/2}_{\text{loc}}(\mathbb{R}^3)$, which implies that $u \in L^q_{\text{loc}}(\mathbb{R}^3)$ for all $q < +\infty$ (see e.g. p. 270 in [60]). Note that since $u^2\rho \in L^q_{\text{loc}}(\mathbb{R}^3)$ for all $q < +\infty$, then by the second equation in (3.23) and the Calderón-Zygmund estimates, we have that $\phi_u \in W^{2,q}_{\text{loc}}(\mathbb{R}^3)$ (see e.g. [35]). This then enables us to show that $g \in L^q_{\text{loc}}(\mathbb{R}^3)$ for all $q < +\infty$, which implies, by Calderón-Zygmund estimates, that $u \in W^{2,q}_{\text{loc}}(\mathbb{R}^3)$ (see e.g. [35]). The $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^3)$ regularity of both u, ϕ_u is a consequence of Morrey's embedding theorem. Finally, the strict positivity is a consequence of the strong maximum principle, and this concludes the proof. □

Remark 3.1. *If, in addition, $\rho \in C^{0,\alpha}_{loc}(\mathbb{R}^3)$, then, by Schauder's estimates on both equations, it holds that $u, \phi_u \in C^{2,\alpha}_{loc}(\mathbb{R}^3)$.*

3.4 Pohozaev inequality

To complete the preliminaries on the nonlinear Schrödinger-Poisson system, we establish a useful Pohozaev type inequality, that gives a necessary condition satisfied by solutions under suitable assumptions on ρ. Since we do not find a precise reference, we give a proof for the reader's convenience. It is interesting to note that Pohozaev type identities are often used to prove nonexistence results by showing that the necessary condition provided by the identity is violated, see the pioneering works of Pohozaev [53] in the context of bounded domains and of Berestycki and Lions [15] in $\mathbb{R}^N$. However, unlike these arguments, we will use the Pohozaev inequality that we establish in the next lemma to produce existence results, in the spirit of [57].

Lemma 3.3. *[**Pohozaev inequality**] Assume $p \in [1,5]$, $\rho \in L^\infty_{loc}(\mathbb{R}^3) \cap W^{1,1}_{loc}(\mathbb{R}^3)$ is nonnegative, and $k\rho(x) \leq (x, \nabla\rho)$ for some $k > \frac{-2(p-2)}{(p-1)}$. Let $(u, \phi_u) \in E(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ be a weak solution of the problem (3.23). Then, it holds that*

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3b}{2}\int_{\mathbb{R}^3} u^2 + \frac{(5+2k)c}{4}\int_{\mathbb{R}^3} \rho\phi_u u^2 - \frac{3d}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} \leq 0. \tag{3.24}$$

Proof. With the regularity remarks of Section 3.3 in place, we now multiply the first equation in (3.23) by $(x, \nabla u)$ and integrate on $B_R(0)$ for some $R > 0$. We will compute each integral separately. We first note that by Lemma 3.1 in [28] it holds that

$$\begin{aligned}\int_{B_R} -\Delta u(x, \nabla u)\,\mathrm{d}x = &-\frac{1}{2}\int_{B_R} |\nabla u|^2\,\mathrm{d}x \\ &-\frac{1}{R}\int_{\partial B_R} |(x,\nabla u)|^2\,\mathrm{d}\sigma + \frac{R}{2}\int_{\partial B_R} |\nabla u|^2\,\mathrm{d}\sigma.\end{aligned} \tag{3.25}$$

Fixing $i = 1,2,3$, integrating by parts and using the divergence theorem, we then see that,

$$\begin{aligned}\int_{B_R} bu(x_i\partial_i u)\,\mathrm{d}x &= b\left[-\frac{1}{2}\int_{B_R} u^2\,\mathrm{d}x + \frac{1}{2}\int_{B_R} \partial_i(u^2 x_i)\,\mathrm{d}x\right] \\ &= b\left[-\frac{1}{2}\int_{B_R} u^2\,\mathrm{d}x + \frac{1}{2}\int_{\partial B_R} u^2\frac{x_i^2}{|x|}\,\mathrm{d}\sigma\right].\end{aligned}$$

So, summing over i, we get

$$\int_{B_R} bu(x,\nabla u)\,\mathrm{d}x = b\left[-\frac{3}{2}\int_{B_R} u^2\,\mathrm{d}x + \frac{R}{2}\int_{\partial B_R} u^2\,\mathrm{d}\sigma\right]. \tag{3.26}$$

Again, fixing $i = 1,2,3$, integrating by parts and using the divergence theorem, we find that,

$$\begin{aligned}\int_{B_R} c\rho\phi_u u x_i(\partial_i u)\,\mathrm{d}x &= c\bigg[-\frac{1}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x - \frac{1}{2}\int_{B_R}\phi_u u^2 x_i(\partial_i\rho)\,\mathrm{d}x \\ &\qquad -\frac{1}{2}\int_{B_R}\rho u^2 x_i(\partial_i\phi_u)\,\mathrm{d}x + \frac{1}{2}\int_{B_R}\partial_i(\rho\phi_u u^2 x_i)\,\mathrm{d}x\bigg] \\ &= c\bigg[-\frac{1}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x - \frac{1}{2}\int_{B_R}\phi_u u^2 x_i(\partial_i\rho)\,\mathrm{d}x \\ &\qquad -\frac{1}{2}\int_{B_R}\rho u^2 x_i(\partial_i\phi_u)\,\mathrm{d}x + \frac{1}{2}\int_{\partial B_R}\rho\phi_u u^2\frac{x_i^2}{|x|}\,\mathrm{d}\sigma\bigg].\end{aligned}$$

Thus, summing over i, we get

$$\begin{aligned}\int_{B_R} c\rho\phi_u u(x,\nabla u)\,\mathrm{d}x = c\bigg[&-\frac{3}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x - \frac{1}{2}\int_{B_R}\phi_u u^2(x,\nabla\rho)\,\mathrm{d}x \\ &-\frac{1}{2}\int_{B_R}\rho u^2(x,\nabla\phi_u)\,\mathrm{d}x + \frac{R}{2}\int_{\partial B_R}\rho\phi_u u^2\,\mathrm{d}\sigma\bigg]. \qquad (3.27)\end{aligned}$$

Finally, once more fixing $i = 1,2,3$, integrating by parts and using the divergence theorem, we find that,

$$\int_{B_R} d|u|^{p-1}u(x_i\partial_i u)\,\mathrm{d}x = d\left[\frac{-1}{p+1}\int_{B_R}|u|^{p+1}\,\mathrm{d}x + \frac{1}{p+1}\int_{\partial B_R}|u|^{p+1}\frac{x_i^2}{|x|}\,\mathrm{d}\sigma\right],$$

and so, summing over i, we see that

$$\begin{aligned}\int_{B_R} d|u|^{p-1}u(x,\nabla u)\,\mathrm{d}x = d\bigg[&\frac{-3}{p+1}\int_{B_R}|u|^{p+1}\,\mathrm{d}x \\ &+\frac{R}{p+1}\int_{\partial B_R}|u|^{p+1}\,\mathrm{d}\sigma\bigg]. \qquad (3.28)\end{aligned}$$

Putting (3.25), (3.26), (3.27) and (3.28) together, we see that

$$
\begin{aligned}
-\frac{1}{2}\int_{B_R}|\nabla u|^2\,\mathrm{d}x-\frac{1}{R}\int_{\partial B_R}|(x,\nabla u)|^2\,\mathrm{d}\sigma+\frac{R}{2}\int_{\partial B_R}|\nabla u|^2\,\mathrm{d}\sigma \\
+b\left[-\frac{3}{2}\int_{B_R}u^2\,\mathrm{d}x+\frac{R}{2}\int_{\partial B_R}u^2\,\mathrm{d}\sigma\right] \\
+c\Bigg[-\frac{3}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x-\frac{1}{2}\int_{B_R}\phi_u u^2(x,\nabla\rho)\,\mathrm{d}x \\
-\frac{1}{2}\int_{B_R}\rho u^2(x,\nabla\phi_u)\,\mathrm{d}x+\frac{R}{2}\int_{\partial B_R}\rho\phi_u u^2\,\mathrm{d}\sigma\Bigg] \\
-d\left[\frac{-3}{p+1}\int_{B_R}|u|^{p+1}\,\mathrm{d}x+\frac{R}{p+1}\int_{\partial B_R}|u|^{p+1}\,\mathrm{d}\sigma\right]=0.
\end{aligned}
\tag{3.29}
$$

We now multiply the second equation in (3.23) by $(x,\nabla\phi_u)$ and integrate on $B_R(0)$ for some $R>0$. Using Lemma 3.1 in [28] we see that

$$
\begin{aligned}
\int_{B_R}\rho u^2(x,\nabla\phi_u)\,\mathrm{d}x &= \int_{B_R}-\Delta\phi_u(x,\nabla\phi_u)\,\mathrm{d}x \\
&= -\frac{1}{2}\int_{B_R}|\nabla\phi_u|^2\,\mathrm{d}x-\frac{1}{R}\int_{\partial B_R}|(x,\nabla\phi_u)|^2\,\mathrm{d}\sigma \\
&\quad +\frac{R}{2}\int_{\partial B_R}|\nabla\phi_u|^2\,\mathrm{d}\sigma.
\end{aligned}
$$

Substituting this into (3.29) and rearranging, we get

$$
\begin{aligned}
&\frac{1}{2}\int_{B_R}|\nabla u|^2\,\mathrm{d}x+\frac{3b}{2}\int_{B_R}u^2\,\mathrm{d}x+\frac{(3+k)c}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x \\
&\qquad -\frac{c}{4}\int_{B_R}|\nabla\phi_u|^2\,\mathrm{d}x-\frac{3d}{p+1}\int_{B_R}|u|^{p+1}\,\mathrm{d}x \\
&\le \frac{1}{2}\int_{B_R}|\nabla u|^2\,\mathrm{d}x+\frac{3b}{2}\int_{B_R}u^2\,\mathrm{d}x+\frac{3c}{2}\int_{B_R}\rho\phi_u u^2\,\mathrm{d}x \\
&\qquad +\frac{c}{2}\int_{B_R}\phi_u u^2(x,\nabla\rho)\,\mathrm{d}x-\frac{c}{4}\int_{B_R}|\nabla\phi_u|^2\,\mathrm{d}x-\frac{3d}{p+1}\int_{B_R}|u|^{p+1}\,\mathrm{d}x \\
&= -\frac{1}{R}\int_{\partial B_R}|(x,\nabla u)|^2\,\mathrm{d}\sigma+\frac{R}{2}\int_{\partial B_R}|\nabla u|^2\,\mathrm{d}\sigma+\frac{bR}{2}\int_{\partial B_R}u^2\,\mathrm{d}\sigma \\
&\qquad +\frac{cR}{2}\int_{\partial B_R}\rho\phi_u u^2\,\mathrm{d}\sigma+\frac{c}{2R}\int_{\partial B_R}|(x,\nabla\phi_u)|^2\,\mathrm{d}\sigma \\
&\qquad -\frac{cR}{4}\int_{\partial B_R}|\nabla\phi_u|^2\,\mathrm{d}\sigma-\frac{dR}{p+1}\int_{\partial B_R}|u|^{p+1}\,\mathrm{d}\sigma,
\end{aligned}
\tag{3.30}
$$

where we have used to assumption $k\rho(x)\le(x,\nabla\rho)$ for some $k>\frac{-2(p-2)}{(p-1)}$ to obtain the first inequality. We now call the right hand side of (3.30) I_R, namely

$$I_R := -\frac{1}{R}\int_{\partial B_R} |(x,\nabla u)|^2 \,\mathrm{d}\sigma + \frac{R}{2}\int_{\partial B_R} |\nabla u|^2 \,\mathrm{d}\sigma + \frac{bR}{2}\int_{\partial B_R} u^2 \,\mathrm{d}\sigma + \frac{cR}{2}\int_{\partial B_R} \rho\phi_u u^2 \,\mathrm{d}\sigma + \frac{c}{2R}\int_{\partial B_R} |(x,\nabla\phi_u)|^2 \,\mathrm{d}\sigma - \frac{cR}{4}\int_{\partial B_R} |\nabla\phi_u|^2 \,\mathrm{d}\sigma - \frac{dR}{p+1}\int_{\partial B_R} |u|^{p+1} \,\mathrm{d}\sigma.$$

We note that $|(x,\nabla u)| \leq R|\nabla u|$ and $|(x,\nabla\phi_u)| \leq R|\nabla\phi_u|$ on ∂B_R, so it holds that

$$|I_R| \leq \frac{3R}{2}\int_{\partial B_R} |\nabla u|^2 \,\mathrm{d}\sigma + \frac{bR}{2}\int_{\partial B_R} u^2 \,\mathrm{d}\sigma + \frac{cR}{2}\int_{\partial B_R} \rho\phi_u u^2 \,\mathrm{d}\sigma + \frac{3cR}{4}\int_{\partial B_R} |\nabla\phi_u|^2 \,\mathrm{d}\sigma + \frac{dR}{p+1}\int_{\partial B_R} |u|^{p+1} \,\mathrm{d}\sigma.$$

Now, since $|\nabla u|^2, u^2 \in L^1(\mathbb{R}^3)$ because $u \in E(\mathbb{R}^3) \subset H^1(\mathbb{R}^3)$, $\rho\phi_u u^2, |\nabla\phi_u|^2 \in L^1(\mathbb{R}^3)$ because $\int_{\mathbb{R}^3} \rho\phi_u u^2 \,\mathrm{d}x = \int_{\mathbb{R}^3} |\nabla\phi_u|^2 \,\mathrm{d}x$ and $\phi_u \in D^{1,2}(\mathbb{R}^3)$, and $|u|^{p+1} \in L^1(\mathbb{R}^3)$ because $E(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ for all $q \in [2,6]$, then it holds that $I_{R_n} \to 0$ as $n \to +\infty$ for a suitable sequence $R_n \to +\infty$ (see e.g. [28]). Moreover, since (3.30) holds for any $R > 0$, it follows that

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x + \frac{3b}{2}\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x + \frac{(3+k)c}{2}\int_{\mathbb{R}^3} \rho\phi_u u^2 \,\mathrm{d}x - \frac{c}{4}\int_{\mathbb{R}^3} |\nabla\phi_u|^2 \,\mathrm{d}x - \frac{3d}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} \,\mathrm{d}x \leq 0,$$

and so, we obtain

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x + \frac{3b}{2}\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x + \frac{(5+2k)c}{4}\int_{\mathbb{R}^3} \rho\phi_u u^2 \,\mathrm{d}x - \frac{3d}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} \,\mathrm{d}x \leq 0,$$

using the fact that $\int_{\mathbb{R}^3} |\nabla\phi_u|^2 \,\mathrm{d}x = \int_{\mathbb{R}^3} \rho\phi_u u^2 \,\mathrm{d}x$. □

4 Existence

In this chapter, we prove the existence of both mountain pass solutions and least energy solutions to the nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \rho(x)\phi u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = \rho(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{4.1}$$

with $p \in (2,5)$ and $\rho : \mathbb{R}^3 \to \mathbb{R}$ a nonnegative measurable function, in the case of a coercive and non-coercive ρ. The results in this chapter are from [50].

4.1 The min-max setting: definition of c_μ, c_μ^∞, c, and c^∞

In what is to come, we will first examine the existence of solutions of (4.1) in the case of a coercive potential ρ (see Section 4.2). The appropriate setting for this problem will be the space $E(\mathbb{R}^3) \subset H^1(\mathbb{R}^3)$. We begin by recalling that solving (4.1) reduces to solving

$$-\Delta u + u + \rho(x)\phi_u u = |u|^{p-1}u, \tag{4.2}$$

with $\phi_u(x) := \int_{\mathbb{R}^3} \frac{u^2(y)\rho(y)}{4\pi|x-y|}dy \in D^{1,2}(\mathbb{R}^3)$. Positive solutions of this equation are critical points of the functional $I : E(\mathbb{R}^3) \to \mathbb{R}$, defined as

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) + \frac{1}{4}\int_{\mathbb{R}^3}\rho\phi_u u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} u_+^{p+1}. \tag{4.3}$$

It will also be useful to introduce a perturbation of (4.2), namely

$$-\Delta u + u + \rho(x)\phi_u u = \mu|u|^{p-1}u, \qquad \mu \in \left[\frac{1}{2}, 1\right]. \tag{4.4}$$

Similarly, the positive solutions of this perturbed problem are critical points of the corresponding functional $I_\mu : E(\mathbb{R}^3) \to \mathbb{R}$, defined as

$$I_\mu(u) := \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) + \frac{1}{4}\int_{\mathbb{R}^3}\rho\phi_u u^2 - \frac{\mu}{p+1}\int_{\mathbb{R}^3} u_+^{p+1}, \qquad \mu \in \left[\frac{1}{2}, 1\right]. \tag{4.5}$$

We will now show that I_μ has the mountain pass geometry in E for each $\mu \in [\frac{1}{2}, 1]$.

Lemma 4.1. *[**Mountain-Pass Geometry for** I_μ] Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative and $p \in (2,5]$. Then, for each $\mu \in [\frac{1}{2},1]$, it holds:*

(i) $I_\mu(0) = 0$ *and there exists constants* $r, a > 0$ *such that* $I_\mu(u) \geq a$ *if* $||u||_E = r$.

(ii) *There exists* $v \in E$ *with* $||v||_E > r$, *such that* $I_\mu(v) \leq 0$.

Proof. We follow Lemma 14 in [19]. To prove (i) note that since $H^1(\mathbb{R}^3) \hookrightarrow L^{p+1}(\mathbb{R}^3)$ then for some constant $C > 0$, it holds that

$$I_\mu(u) \geq \frac{1}{2}||u||^2_{H^1} + \frac{1}{4}\int_{\mathbb{R}^3} \rho \phi_u u^2 - C\mu ||u||^{p+1}_{H^1}.$$

Now, from the definition of the norm in E we can see that

$$4\pi \int_{\mathbb{R}^3} \rho \phi_u u^2 = \left(||u||^2_E - ||u||^2_{H^1}\right)^2.$$

Therefore, we have that

$$\begin{aligned} I_\mu(u) &\geq \frac{1}{2}||u||^2_{H^1} + \frac{1}{16\pi}\left(||u||^2_E - ||u||^2_{H^1}\right)^2 - C\mu ||u||^{p+1}_{H^1} \\ &= \frac{1}{2}||u||^2_{H^1} + \frac{1}{4\pi}\left(\frac{1}{4}||u||^4_E - \frac{1}{2}||u||^2_E ||u||^2_{H^1} + \frac{1}{4}||u||^4_{H^1}\right) - C\mu ||u||^{p+1}_{H^1}. \end{aligned}$$

For some $\alpha \neq 0$, using the elementary inequality

$$\frac{1}{2}||u||^2_E ||u||^2_{H^1} \leq \frac{\alpha^2}{4}||u||^4_{H^1} + \frac{1}{4\alpha^2}||u||^4_E$$

we have

$$\begin{aligned} I_\mu(u) \geq \frac{1}{2}||u||^2_{H^1} &+ \frac{1}{4\pi}\left(\frac{1}{4}||u||^4_E - \frac{\alpha^2}{4}||u||^4_{H^1} - \frac{1}{4\alpha^2}||u||^4_E + \frac{1}{4}||u||^4_{H^1}\right) \\ &- C\mu ||u||^{p+1}_{H^1} \end{aligned}$$

$$= \frac{1}{2}||u||_{H^1}^2 - \frac{1}{4\pi}\left(\frac{\alpha^2-1}{4}\right)||u||_{H^1}^4 + \frac{1}{4\pi}\left(\frac{\alpha^2-1}{4\alpha^2}\right)||u||_E^4 - C\mu||u||_{H^1}^{p+1}. \tag{4.6}$$

We now assume $||u||_E < \delta$ for some $\delta > 0$, which also implies that $||u||_{H^1}^2 < \delta^2$, and we take $\alpha > 1$. Then, from (4.6), we see that

$$\begin{aligned} I_\mu(u) &\geq \left[\frac{1}{2} - \frac{1}{4\pi}\left(\frac{\alpha^2-1}{4}\right)\delta^2 - C\mu\delta^{p-1}\right]||u||_{H^1}^2 + \frac{1}{4\pi}\left(\frac{\alpha^2-1}{4\alpha^2}\right)||u||_E^4 \\ &\geq \frac{1}{4\pi}\left(\frac{\alpha^2-1}{4\alpha^2}\right)||u||_E^4, \quad \text{for } \delta \text{ sufficiently small.} \end{aligned}$$

Hence, we have shown that the origin is a strict local minimum for I_μ in E if $p \in [2,5]$.

To show (ii), pick $u \in C^1(\mathbb{R}^3)$, supported in the unit ball, B_1. Setting $v_t(x) := t^2u(tx)$ we find that

$$I_\mu(v_t) = \frac{t^3}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{t}{2}\int_{\mathbb{R}^3}u^2 + \frac{t^3}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(y)\rho(\frac{y}{t})u^2(x)\rho(\frac{x}{t})}{4\pi|x-y|}\,\mathrm{d}y\,\mathrm{d}x - \frac{\mu t^{2p-1}}{p+1}\int_{\mathbb{R}^3}u_+^{p+1}. \tag{4.7}$$

Since the Poisson term is uniformly bounded, namely for $t > 1$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(y)\rho(\frac{y}{t})u^2(x)\rho(\frac{x}{t})}{4\pi|x-y|}\,\mathrm{d}y\,\mathrm{d}x \leq ||\rho||_{L^\infty(B_1)}^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(y)u^2(x)}{4\pi|x-y|}\,\mathrm{d}y\,\mathrm{d}x < +\infty,$$

the fact that $2p-1 > 3$ in (4.7) yields $I_\mu(v_t) \to -\infty$ as $t \to +\infty$, and this is enough to prove (ii). This concludes the proof. □

The previous lemma, as well as the monotonicity of I_μ with respect to μ, imply that there exists $\bar{v} \in E \setminus \{0\}$ such that

$$I_\mu(\bar{v}) \leq I_{\frac{1}{2}}(\bar{v}) \leq 0, \qquad \forall \mu \in \left[\frac{1}{2}, 1\right].$$

Thus, we can define, in the spirit of Ambrosetti-Rabinowitz [6], the min-max level associated with I_μ as

$$c_\mu := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\mu(\gamma(t)), \tag{4.8}$$

where Γ is the family of paths

$$\Gamma := \{\gamma \in C([0,1],E) : \gamma(0) = 0,\, \gamma(1) = \bar{v}\}.$$

It is worth emphasising that to apply the monotonicity trick [38] and [39] it is essential that the above class Γ does not depend on μ.

Lemma 4.2. *Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative and $p \in (2,5)$. Then:*

(i) The mapping $\left[\frac{1}{2},1\right] \ni \mu \mapsto c_\mu$ is non-increasing and left-continuous.

(ii) For almost every $\mu \in [\frac{1}{2},1]$, there exists a bounded Palais-Smale sequence for I_μ at the level c_μ. That is, there exists a bounded sequence $(u_n)_{n\in\mathbb{N}} \subset E$ such that $I_\mu(u_n) \to c_\mu$ and $I'_\mu(u_n) \to 0$.

Proof. The proof of (i) follows from Lemma 2.2 in [7]. To prove (ii), we notice that by Lemma 4.1, it holds that

$$c_\mu = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\mu(\gamma(t)) > 0 \geq \max\{I_\mu(0), I_\mu(\bar{v})\}, \qquad \forall \mu \in \left[\frac{1}{2},1\right].$$

Thus, the result follows by Theorem 2.4. □

With this result in place, we can define the set

$$\mathscr{M} := \left\{\mu \in \left[\frac{1}{2},1\right] : \exists \text{ bounded Palais-Smale sequence for } I_\mu \text{ at the level } c_\mu\right\}. \tag{4.9}$$

We can now state the following corollary which will be used to obtain the existence of solutions to the non-perturbed problem.

Corollary 4.1. *The set $\mathscr{M}$ defined in* (4.9) *is dense in* $\left[\frac{1}{2},1\right]$.

Proof. Recall that $\mathscr{M}$ is dense in $\left[\frac{1}{2},1\right]$ if and only if for all $x \in \left[\frac{1}{2},1\right]$ and for all $\delta > 0$ there exists $y \in \mathscr{M}$ such that $y \in (x-\delta, x+\delta)$. Let $x \in \left[\frac{1}{2},1\right]$ and $\delta > 0$ be arbitrary. Then,

$$\lambda\left(\left[\frac{1}{2},1\right] \cap (x-\delta, x+\delta)\right) \in [\delta, 2\delta],$$

where here λ denotes the Lebesgue measure. We note that $\left[\frac{1}{2},1\right] \setminus \mathscr{M}$ has zero Lebesgue measure by Lemma 4.2 (ii). Therefore, it holds that

$$\begin{aligned}
\lambda\left(\mathscr{M} \cap (x-\delta, x+\delta)\right) &= \lambda\left(\left[\frac{1}{2},1\right] \cap (x-\delta, x+\delta)\right) \\
&\quad - \lambda\left(\left(\left[\frac{1}{2},1\right] \setminus \mathscr{M}\right) \cap (x-\delta, x+\delta)\right) \\
&= \lambda\left(\left[\frac{1}{2},1\right] \cap (x-\delta, x+\delta)\right) \\
&\in [\delta, 2\delta].
\end{aligned}$$

So, since $\mathscr{M} \cap (x-\delta, x+\delta)$ has positive measure, it is necessarily nonempty. In particular, we can choose $y \in \mathscr{M} \cap (x-\delta, x+\delta)$. This completes the proof. □

Now, we note that since I has the mountain pass geometry by Lemma 4.1, using (i) of Lemma 4.2, we can define the min-max level associated with I as

$$c := \begin{cases} c_1, & p \in (2,3), \\ \inf_{\gamma \in \bar{\Gamma}} \max_{t \in [0,1]} I(\gamma(t)), & p \in [3,5), \end{cases} \tag{4.10}$$

where c_1 is defined in (4.8) and $\bar{\Gamma}$ is the family of paths

$$\bar{\Gamma} := \{\gamma \in C([0,1], E(\mathbb{R}^3)) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

This finalises the preliminary min-max scheme for the case of a coercive ρ.

In Section 4.3, we will then focus on the case of non-coercive ρ, namely $\rho(x) \to \rho_\infty$ as $|x| \to +\infty$, and the appropriate setting for this problem will be the space $H^1(\mathbb{R}^3)$. It will once again be useful to introduce a perturbation of (4.2), namely, (4.4), and to recall that the positive solutions of this perturbed problem are critical points of the corresponding functional, $I_\mu : H^1(\mathbb{R}^3) \to \mathbb{R}$, defined in (4.5). We note that Lemma 4.1 and Lemma 4.2 hold with $E(\mathbb{R}^3) = H^1(\mathbb{R}^3)$, and thus $\mathscr{M}$ can be defined as in (4.9). We now introduce the problem at infinity related to (4.2) and (4.4) in this case, namely

$$-\Delta u + u + \rho_\infty \bar{\phi}_u u = |u|^{p-1}u, \tag{4.11}$$

and

$$-\Delta u + u + \rho_\infty \bar{\phi}_u u = \mu |u|^{p-1}u, \qquad \mu \in \left[\frac{1}{2}, 1\right], \tag{4.12}$$

respectively, where $\bar{\phi}_u(x) := \int_{\mathbb{R}^3} \frac{\rho_\infty u^2(y)}{4\pi|x-y|} dy \in D^{1,2}(\mathbb{R}^3)$. Positive solutions of (4.11) are critical points of $I^\infty : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$I^\infty(u) := \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) + \frac{1}{4}\int_{\mathbb{R}^3} \rho_\infty \bar{\phi}_u u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} u_+^{p+1}. \tag{4.13}$$

Similarly, positive solutions of (4.12) are critical points of the corresponding functional, $I_\mu^\infty : H^1(\mathbb{R}^3) \to \mathbb{R}$, defined as

$$I_\mu^\infty(u) := \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) + \frac{1}{4}\int_{\mathbb{R}^3} \rho_\infty \bar{\phi}_u u^2 - \frac{\mu}{p+1}\int_{\mathbb{R}^3} u_+^{p+1}, \quad \mu \in \left[\frac{1}{2}, 1\right]. \tag{4.14}$$

It can be shown that I_μ^∞ satisfies the geometric conditions of the mountain-pass theorem, using similar arguments as those used in the proof of Lemma 4.1. We therefore define the min-max level associated with I_μ^∞ as

$$c_\mu^\infty := \inf_{\gamma \in \Gamma^\infty} \max_{t \in [0,1]} I_\mu^\infty(\gamma(t)), \tag{4.15}$$

where

$$\Gamma^\infty := \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, I^\infty_{\frac{1}{2}}(\gamma(1)) < 0\}.$$

Moreover, we define the min-max level associated with I^∞ as

$$c^\infty := \begin{cases} c_1^\infty, & p \in (2,3), \\ \inf_{\gamma \in \bar{\Gamma}^\infty} \max_{t \in [0,1]} I^\infty(\gamma(t)), & p \in [3,5), \end{cases} \tag{4.16}$$

where c_1^∞ is given by (4.15) and $\bar{\Gamma}^\infty$ is the family of paths

$$\bar{\Gamma}^\infty := \{\gamma \in C([0,1], E(\mathbb{R}^3)) : \gamma(0) = 0, I^\infty(\gamma(1)) < 0\}.$$

4.1.1 Lower bounds for I and I^∞

In the next two lemmas, we establish lower bounds on I_μ and I_μ^∞, when restricted to nonnegative and nontrivial solutions of (4.4) and (4.12), respectively. These bounds will be used on numerous occasions.

Lemma 4.3. *Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative and $\mu \in [\frac{1}{2}, 1]$. Define $\mathscr{A} := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : u$ is a nonnegative solution to (4.4)$\}$. Then, if $p \in [3,5)$, it holds that*

$$\inf_{u \in \mathscr{A}} I_\mu(u) \geq \frac{p-1}{2(p+1)} \left(S_{p+1}\right)^{\frac{p+1}{p-1}} > 0.$$

If $p \in (2,3)$, suppose, in addition, $\rho \in W^{1,1}_{loc}(\mathbb{R}^3)$ and $k\rho(x) \leq (x, \nabla\rho)$ for some $k > \frac{-2(p-2)}{(p-1)}$. Then, it holds that

$$\inf_{u \in \mathscr{A}} I_\mu(u) \geq C(k,p),$$

with

$$C(k,p) := \left(\frac{2(p-2) + k(p-1)}{(3+2k)(p+1)}\right) \left(S_{p+1}\right)^{\frac{p+1}{p-1}} > 0.$$

Proof. Let $\bar{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ be an arbitrary nonnegative solution of (4.4) such that $I_\mu(\bar{u}) = \bar{c}$, for some $\bar{c} \in \mathbb{R}$. Using the Sobolev embedding theorem and the fact that $I'_\mu(\bar{u})(\bar{u}) = 0$, we see that

$$S_{p+1}||\bar{u}||^2_{L^{p+1}} \leq ||\bar{u}||^2_{H^1} \leq ||\bar{u}||^2_{H^1} + \int_{\mathbb{R}^3} \rho \phi_{\bar{u}} \bar{u}^2 = \mu ||\bar{u}||^{p+1}_{L^{p+1}}.$$

Since $\mu \leq 1$ it follows that

$$\left(S_{p+1}\right)^{\frac{p+1}{p-1}} \leq ||\bar{u}||^2_{H^1}. \tag{4.17}$$

If $p \in [3,5)$, using the definition of $\bar{c}$ and Nehari's condition, we can see that

$$\left(\frac{1}{2} - \frac{1}{p+1}\right)||\bar{u}||^2_{H^1} \leq \bar{c},$$

and so the bound on $\bar{c}$ immediately follows from (4.17). If $p \in (2,3)$, we first note that since $I_\mu(\bar{u}) = \bar{c}$, $I'_\mu(\bar{u})(\bar{u}) = 0$, and $\bar{u} = (\bar{u})_+$, then $\bar{u}$ satisfies

$$\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla \bar{u}|^2 + \bar{u}^2) + \frac{1}{4}\int_{\mathbb{R}^3} \rho \phi_{\bar{u}} \bar{u}^2 - \frac{\mu}{p+1}\int_{\mathbb{R}^3} \bar{u}^{p+1} = \bar{c}, \tag{4.18}$$

and

$$\int_{\mathbb{R}^3}(|\nabla \bar{u}|^2 + \bar{u}^2) + \int_{\mathbb{R}^3} \rho \phi_{\bar{u}} \bar{u}^2 - \mu\int_{\mathbb{R}^3} \bar{u}^{p+1} = 0. \tag{4.19}$$

Moreover, since $\bar{u}$ solves (4.4) then, as a consequence of Lemma 3.3, it holds that

$$\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla \bar{u}|^2 + \bar{u}^2) + \left(\frac{5+2k}{4}\right)\int_{\mathbb{R}^3} \rho \phi_{\bar{u}} \bar{u}^2 - \frac{3\mu}{p+1}\int_{\mathbb{R}^3} \bar{u}^{p+1} \leq 0. \tag{4.20}$$

For ease, we now set $\alpha = ||\bar{u}||^2_{H^1}$, $\gamma = \int_{\mathbb{R}^3} \rho \phi_{\bar{u}} \bar{u}^2$, and $\delta = \mu \int_{\mathbb{R}^3} \bar{u}^{p+1}$, and note that α, γ, $\delta \geq 0$. From (4.18), (4.19), and (4.20), we can see that α, γ, and δ satisfy

$$\begin{cases} \frac{1}{2}\alpha + \frac{1}{4}\gamma - \frac{1}{p+1}\delta = \bar{c}, \\ \alpha + \gamma - \delta = 0, \\ \frac{1}{2}\alpha + \left(\frac{5+2k}{4}\right)\gamma - \frac{3}{p+1}\delta \leq 0, \end{cases}$$

and so, it holds that

$$\delta \leq \frac{\bar{c}(3+2k)(p+1)}{2(p-2)+k(p-1)},$$

and

$$\alpha = \delta - \gamma.$$

Since γ is nonnegative, we find

$$\alpha \leq \alpha + \gamma = \delta \leq \frac{\bar{c}(3+2k)(p+1)}{2(p-2)+k(p-1)}.$$

This and (4.17) implies the statement, since $k > \frac{-2(p-2)}{(p-1)} > \frac{-3}{2}$ for $p \in (2,3)$. This concludes the proof. □

Lemma 4.4. *If $p \in (2,5)$, $\mu \in [\frac{1}{2}, 1]$ and $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ is a nonnegative solution of (4.12), then, it holds that*

$$I_\mu^\infty(u) \geq c_\mu^\infty > 0.$$

Moreover, if $p \in (2,5)$ and $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ is a nonnegative solution of (4.11), then

$$I^\infty(u) \geq c^\infty > 0.$$

In both cases, $u > 0$.

Proof. The lower bounds follow easily by similar arguments to those used in the proof of Proposition 3.4 in [36]. Since u is nonnegative and nontrivial, then it is strictly positive by the strong maximum principle, and this concludes the proof. □

4.2 The case of coercive $\rho(x)$

In this section we will examine the existence of solutions of (4.1) in the case of a coercive potential ρ, namely $\rho(x) \to +\infty$ as $|x| \to +\infty$.

4.2.1 Preliminary results

In the following lemma, we establish that this coercivity is indeed a sufficient condition for the compactness of the embedding $E \hookrightarrow L^{p+1}(\mathbb{R}^3)$.

Lemma 4.5. *Assume $\rho(x) \to +\infty$ as $|x| \to +\infty$. Then, E is compactly embedded in $L^{p+1}(\mathbb{R}^3)$ for all $p \in (1,5)$.*

Proof. We first recall that for any $u \in E$, it holds that

$$-\Delta\phi_u = \rho u^2,$$

where $\phi_u(x) := \int_{\mathbb{R}^3} \frac{\rho(y)u^2(y)}{4\pi|x-y|} dy \in D^{1,2}(\mathbb{R}^3)$. Testing this equation with u_+ and u_- and using the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^3} \rho|u|^3 &= \int_{\mathbb{R}^3} \nabla|u|\nabla\phi_u \\
&\leq \left(\int_{\mathbb{R}^3} |\nabla|u||^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\nabla\phi_u|^2\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{4\pi|x-y|}\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{4\pi}\right)^{\frac{1}{2}} ||u||_E^3.
\end{aligned}$$

Thus, if $\rho > 0$, this implies the continuous embedding $E \hookrightarrow L^3_\rho(\mathbb{R}^3)$, where $L^3_\rho(\mathbb{R}^3) := \{u : \rho^{\frac{1}{3}}u \in L^3(\mathbb{R}^3)\}$, equipped with norm $||u||_{L^3_\rho} := ||\rho^{\frac{1}{3}}u||_{L^3}$.

Without loss of generality, assume $u_n \rightharpoonup 0$ in E. Since $\rho(x) \to +\infty$ as $|x| \to +\infty$, then for any $\varepsilon > 0$, there exists an $R > 0$ such that

$$\int_{\mathbb{R}^3\setminus B_R} |u_n|^3 = \int_{\mathbb{R}^3\setminus B_R} \frac{\rho}{\rho}|u_n|^3 < \varepsilon \int_{\mathbb{R}^3\setminus B_R} \rho|u_n|^3 < \varepsilon C, \tag{4.21}$$

for some $C > 0$. This and the classical Rellich theorem implies that, passing if necessary to a subsequence,

$$\int_{\mathbb{R}^3} |u_n|^3 \to 0. \tag{4.22}$$

Therefore, we have proved the lemma for $p = 2$. Now, if $p \in (1,2)$, then, by interpolation, for some $\alpha \in (0,1)$, it holds that

$$||u_n||_{L^{p+1}(\mathbb{R}^3)} \leq ||u_n||^{\alpha}_{L^2(\mathbb{R}^3)} ||u_n||^{1-\alpha}_{L^3(\mathbb{R}^3)} \to 0,$$

as the $L^2(\mathbb{R}^3)$ norm is bounded. The case $p \in (2,5)$ is similar using Sobolev's inequality, and this concludes the proof. □

As a consequence of the previous lemma, we have the following warm-up theorem regarding existence in the coercive case for $p \geq 3$.

Theorem 4.1. *[**Coercive case: existence of mountain pass solution for** $p \geq 3$] Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative and $\rho(x) \to +\infty$ as $|x| \to +\infty$. Then, for any $p \in [3,5)$, there exists a solution, $(u, \phi_u) \in E(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, of (4.1), whose components are positive functions. In particular, u is a mountain pass critical point of I at level c, where c is the min-max level defined in (4.10).*

Proof. Since $\rho(x) \to +\infty$ as $|x| \to +\infty$, then E is compactly embedded in $L^{p+1}(\mathbb{R}^3)$ by Lemma 4.5, and therefore the existence of a Mountain Pass solution u to (4.1) is provided by Theorem 1 of [19]. Both u, ϕ_u are positive by the strong maximum principle, and this concludes the proof. □

It is also worth finding conditions such that the term ρu^2 goes to zero at infinity, since the whole right hand side of the Poisson equation is classically interpreted as a 'charge density'. This is provided by the following.

Proposition 4.2. *[**Decay of** u **and** ρu^2] Let $\rho : \mathbb{R}^3 \to \mathbb{R}$ be continuous and nonnegative, $p \in [1,5]$, and $(u, \phi_u) \in E(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ be solution to (4.1). Assume that u is nonnegative. Then, for every $\gamma \in (0,1)$, there exists $C > 0$ such that*

$$u(x) \leq Ce^{-\gamma(1+|x|)} \qquad (L^2\text{-decay}).$$

If, in addition, ρ is such that

(i) $\liminf_{|x|\to\infty} \rho(x)|x|^{1-2\alpha} > A$

(ii) $\limsup_{|x|\to\infty} \rho(x)e^{-\beta(1+|x|)^{\alpha}} \leq B$

for some $\alpha, \beta, A, B > 0$, with $\beta < 2\sqrt{A}$, then, for some constant $C > 0$, it holds that

(a) $u(x) \leq Ce^{-\sqrt{A}(1+|x|)^{\alpha}}$

and therefore

(b) $\rho(x)u^2(x) = O(e^{(\beta-2\sqrt{A})(1+|x|)^{\alpha}}), \qquad$ *as* $|x| \to +\infty$.

Proof. The conclusion easily follows by Theorem 6 in [19] (see also [21]). More precisely, setting $W(x) := 1 + \frac{\rho(x)}{|x|}$, the L^2-decay follows as $W(x) \geq 1$ and therefore

$$\liminf_{|x|\to+\infty} W(x) > \gamma^2$$

is automatically satisfied for every $\gamma \in (0,1)$. Moreover, note that by (i) it follows that

$$\liminf_{|x|\to+\infty} W(x)|x|^{2-2\alpha} \geq \liminf_{|x|\to+\infty} \rho(x)|x|^{1-2\alpha} > A$$

which yields (a) again by Theorem 6 in [19]. This concludes the proof. □

4.2.2 Theorem 4.2

We now prove the following theorem regarding existence of mountain pass solutions in the coercive case for $p < 3$.

Theorem 4.2. *[**Coercive case: existence of mountain pass solution for** $p \in (2,3)$] Suppose $\rho \in C(\mathbb{R}^3) \cap W^{1,1}_{loc}(\mathbb{R}^3)$ is nonnegative and $\rho(x) \to +\infty$ as $|x| \to +\infty$. Suppose further that $k\rho(x) \leq (x, \nabla\rho)$ for some $k > \frac{-2(p-2)}{(p-1)}$. Then, for any $p \in (2,3)$, there exists a solution, $(u, \phi_u) \in E(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, of (4.1), whose components are positive* *functions. In particular, u is a mountain pass critical point of I at level c, where c is the min-max level defined in* (4.10)*.*

Proof. We first note that by Corollary 4.1, the set $\mathscr{M}$, defined in (4.9), is dense in $\left[\frac{1}{2},1\right]$. We now break the proof into a series of claims.

Claim 1. *The values c_μ are critical levels of I_μ for all $\mu \in \mathscr{M}$. Namely, there exists $u \in E$ such that $I_\mu(u) = c_\mu$ and $I'_\mu(u) = 0$.*

By definition, for each $\mu \in \mathscr{M}$, there exists a bounded sequence $(u_n)_{n\in\mathbb{N}} \subset E$ such that $I_\mu(u_n) \to c_\mu$ and $I'_\mu(u_n) \to 0$. Since $(u_n)_{n\in\mathbb{N}}$ is bounded, there exists $u \in E$ such that, up to a subsequence, $u_n \rightharpoonup u$ in E. Using this and the fact that E is compactly embedded in $L^{p+1}(\mathbb{R}^3)$ by Lemma 4.5, arguing as in Lemma 16 in [19], with $V(x) = 1$ and $K(x) = \mu$, we see that for all $\delta > 0$, there exists a ball $B \subset \mathbb{R}^3$ such that

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^3\setminus B} \rho\phi_{u_n} u_n^2 < \delta, \tag{4.23}$$

and

$$\limsup_{n\to+\infty} \left| \int_{\mathbb{R}^3\setminus B} \rho\phi_{u_n} u_n u \right| < \delta. \tag{4.24}$$

We then note that since $(u_n)_{n\in\mathbb{N}}$ is bounded in E, we also have that, up to a subsequence, $u_n \rightharpoonup u$ in H^1. Now, using this and the fact that $(u_n)_{n\in\mathbb{N}}$ is a bounded Palais Smale sequence for I_μ, as well as (4.23), (4.24), and Lemma 4.5, we can reason as in Lemma 18 in [19], with $V(x) = 1$ and $K(x) = \mu$, to see that

$$\int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2) \to \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2). \tag{4.25}$$

Thus, using (4.23) and the boundedness of $(u_n)_{n\in\mathbb{N}}$, we can argue as in the proof of Theorem 1 in [19], to see that

$$\int_{\mathbb{R}^3} \rho\phi_{u_n} u_n^2 \to \int_{\mathbb{R}^3} \rho\phi_u u^2, \tag{4.26}$$

which, when combined with (4.25) and Lemma 4.5, implies that

$$I_\mu(u_n) \to I_\mu(u).$$

Therefore, we have shown

$$I_\mu(u) = c_\mu.$$

Moreover, by standard arguments, using the weak convergence $u_n \rightharpoonup u$ in E, we can show

$$I'_\mu(u) = 0.$$

We finally note that, by putting (4.25) and (4.26) together, we have that $||u_n||_E^2 \to ||u||_E^2$, and so by Lemma 3.2, it follows that $u_n \to u$ in E. This concludes the proof of Claim 1.

Claim 2. *Let $(\mu_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathscr{M}$ such that $\mu_n \to 1$ and assume $(u_n)_{n\in\mathbb{N}} \subset E$ is such that $I_{\mu_n}(u_n) = c_{\mu_n}$ and $I'_{\mu_n}(u_n) = 0$ for each n. Then, there exists $u \in E$ such that, up to a subsequence, $u_n \to u$ in E, $I(u) = c$, and $I'(u) = 0$.*

We first note that testing the equation $I'_{\mu_n}(u_n) = 0$ with $(u_n)_-$, one sees that $u_n \geq 0$ for each n. Therefore, it holds that u_n satisfies

$$-\Delta u_n + u_n + \rho(x)\phi_{u_n}u_n = \mu_n u_n^p, \tag{4.27}$$

$$\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_n|^2 + u_n^2) + \frac{1}{4}\int_{\mathbb{R}^3}\rho\phi_{u_n}u_n^2 - \frac{\mu_n}{p+1}\int_{\mathbb{R}^3}u_n^{p+1} = c_{\mu_n}, \tag{4.28}$$

and

$$\int_{\mathbb{R}^3}(|\nabla u_n|^2 + u_n^2) + \int_{\mathbb{R}^3}\rho\phi_{u_n}u_n^2 - \mu_n\int_{\mathbb{R}^3}u_n^{p+1} = 0. \tag{4.29}$$

Moreover, since u_n solves (4.27) then, as a consequence of Lemma 3.3, we see that

$$\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_n|^2+u_n^2)+\left(\frac{5+2k}{4}\right)\int_{\mathbb{R}^3}\rho\phi_{u_n}u_n^2-\frac{3\mu_n}{p+1}\int_{\mathbb{R}^3}u_n^{p+1}\leq 0. \tag{4.30}$$

Setting $\alpha_n=\int_{\mathbb{R}^3}(|\nabla u_n|^2+u_n^2)$, $\gamma_n=\int_{\mathbb{R}^3}\rho\phi_{u_n}u_n^2$, and $\delta_n=\mu_n\int_{\mathbb{R}^3}u_n^{p+1}$, we can see from (4.28), (4.29), and (4.30), that α_n, γ_n, $\delta_n\geq 0$ satisfy

$$\begin{cases} \frac{1}{2}\alpha_n & + & \frac{1}{4}\gamma_n & - & \frac{1}{p+1}\delta_n & = & c_{\mu_n}, \\ \alpha_n & + & \gamma_n & - & \delta_n & = & 0, \\ \frac{1}{2}\alpha_n & + & \left(\frac{5+2k}{4}\right)\gamma_n & - & \frac{3}{p+1}\delta_n & \leq & 0. \end{cases} \tag{4.31}$$

Solving the system, we find that

$$\delta_n\leq\frac{c_{\mu_n}(3+2k)(p+1)}{2(p-2)+k(p-1)},$$

$$\gamma_n\leq\frac{-2c_{\mu_n}(p-5)}{2(p-2)+k(p-1)},$$

and

$$\alpha_n=\delta_n-\gamma_n.$$

Since c_{μ_n} is bounded, $k>\frac{-2(p-2)}{(p-1)}>\frac{-3}{2}$, and δ_n, γ_n, and α_n are all nonnegative, we can deduce that δ_n, γ_n, and α_n are all bounded. Hence, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in E and so there exists $u\in E$ such that, up to a subsequence, $u_n\rightharpoonup u$ in E.

We now follow a similar procedure to that of Claim 1. Using the facts that $I'_{\mu_n}(u_n)=0$, u_n is bounded in E, E is compactly embedded in $L^{p+1}(\mathbb{R}^3)$ by Lemma 4.5, and $\mu_n\to 1$, by an easy argument similar to the proof of Lemma 16 in [19], with $V(x)=1$ and $K(x)=\mu_n$, we have that for all $\delta>0$, there exists a ball $B\subset\mathbb{R}^3$ such that

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^3\setminus B}\rho\phi_{u_n}u_n^2<\delta, \tag{4.32}$$

and

$$\limsup_{n\to+\infty}\left|\int_{\mathbb{R}^3\setminus B}\rho\phi_{u_n}u_nu\right|<\delta. \tag{4.33}$$

Now, using the facts that $I'_{\mu_n}(u_n)=0$ and $\mu_n\to 1$, as well as (4.32), (4.33), and Lemma 4.5, we can adapt the proof of Lemma 18 in [19], with $V(x)=1$ and $K(x)=\mu_n$, to see that

$$\int_{\mathbb{R}^3}(|\nabla u_n|^2+u_n^2)\to\int_{\mathbb{R}^3}(|\nabla u|^2+u^2). \tag{4.34}$$

Finally, using (4.32), (4.34), the boundedness of u_n, Lemma 4.5, and the fact that $\mu_n\to 1$, we can easily adapt the proof of Theorem 1 in [19], to see that

$$\int_{\mathbb{R}^3}\rho\phi_{u_n}u_n^2\to\int_{\mathbb{R}^3}\rho\phi_u u^2, \tag{4.35}$$

$$c_{\mu_n}=I_{\mu_n}(u_n)\to I(u), \tag{4.36}$$

and

$$0=I'_{\mu_n}(u_n)\to I'(u).$$

As in Claim 1, we see that (4.34) and (4.35) imply that $||u_n||_E^2\to||u||_E^2$, and so by Lemma 3.2, it follows that $u_n\to u$ in E. We now recall that, for $p\in(2,3)$, it holds that $c_{\mu_n}\to c$ as $\mu_n\nearrow 1$ by definition (4.10). Thus, from (4.36) it follows that $I(u)=c$.

Conclusion. Let $(\mu_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathscr{M}$ such that $\mu_n\to 1$. By Claim 1, we can choose $(u_n)_{n\in\mathbb{N}}\subset E$ such that $I_{\mu_n}(u_n)=c_{\mu_n}$ and $I'_{\mu_n}(u_n)=0$ for each n. By Claim 2, it follows that that there exists $u\in E$ such that, up to a subsequence, $u_n\to u$ in E, $I(u)=c$, and $I'(u)=0$. Namely, we have shown $(u,\phi_u)\in E(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)$ is a solution of (4.1). By the strong maximum principle ϕ_u is strictly positive. Testing the equation $I'(u)=0$ with u_- one sees that $u\geq 0$ and, in fact, strictly positive as a consequence of the strong maximum principle. This concludes the proof. □

4.2.3 Corollary 4.3

We now prove the existence of least energy solutions for all $p \in (2,5)$. It is important to note that for $p \in (3,5)$ the solutions provided by the following corollary coincide with those provided by Theorem 4.1. For $p \in (2,3]$, we make use of a minimising sequence in order to obtain the result, however we do not know whether the least energy solutions provided by what follows are the same as those provided by Theorem 4.1 ($p = 3$) and Theorem 4.2.

Corollary 4.3. *[**Coercive case: existence of a least energy solution for** $p \in (2,5)$] Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative and $\rho(x) \to +\infty$ as $|x| \to +\infty$. If $p \in (2,3)$, suppose, in addition, that $\rho \in W^{1,1}_{loc}(\mathbb{R}^3)$ and $k\rho(x) \leq (x, \nabla\rho)$ for some $k > \frac{-2(p-2)}{(p-1)}$. Then, for all $p \in (2,5)$, there exists a solution, $(u, \phi_u) \in E(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, of (4.1), whose components are positive functions, such that u is a least energy critical point of I.*

Proof. When $p > 3$ it is standard to see that the Mountain Pass level c has the following characterisation

$$c = \inf_{u \in \mathcal{N}} I(u), \qquad \mathcal{N} = \{u \in E \setminus \{0\} \mid I'(u)u = 0\}, \tag{4.37}$$

see e.g. Theorem 5 in [19]. It follows that the mountain pass solution u found in Theorem 4.1 is a least energy solution of I in this case. If $p \in (2,3]$, define

$$c^* := \inf_{u \in \mathscr{A}} I(u),$$

where

$$\mathscr{A} := \{u \in E(\mathbb{R}^3) \setminus \{0\} : u \text{ is a nonnegative solution to (4.2)}\}.$$

When $p = 3$, we notice that the mountain pass critical point, u, that we found in Theorem 4.1 is such that $u \in \mathscr{A}$. Similarly, when $p \in (2,3)$, the mountain pass critical point that we found in Theorem 4.2 is in $\mathscr{A}$. Therefore, in both cases, $\mathscr{A}$ is nonempty and c^* is well-defined. Now, let $(w_n)_{n\in\mathbb{N}} \subset \mathscr{A}$ be a minimising sequence for I on $\mathscr{A}$, namely $I(w_n) \to c^*$ as $n \to +\infty$ and $I'(w_n) = 0$. If $p = 3$, it holds that

$$c + 1 + o(1)||w_n||_{E(\mathbb{R}^3)} \geq (p+1)I(w_n) - I'(w_n)w_n \geq \|w_n\|^2_{H^1(\mathbb{R}^3)},$$

and it follows from standard arguments that $(w_n)_{n\in\mathbb{N}}$ is bounded (see e.g. Lemma 7.14). If $p\in(2,3)$, setting $\alpha_n=\int_{\mathbb{R}^3}(|\nabla w_n|^2+w_n^2)$, $\gamma_n=\int_{\mathbb{R}^3}\rho\phi_{w_n}w_n^2$, and $\delta_n=\int_{\mathbb{R}^3}w_n^{p+1}$, and arguing as in Theorem 4.2 Claim 2, we see that α_n, γ_n, and δ_n satisfy the system (4.31) with $d_n:=I(w_n)$ in the place of c_{μ_n}. Thus, solving this system and arguing as in Theorem 4.2 Claim 2, we can obtain that α_n, γ_n, and δ_n are all bounded since $(d_n)_{n\in\mathbb{N}}$ is uniformly bounded. It follows that $(w_n)_{n\in\mathbb{N}}$ is also bounded in this case. Therefore, for all $p\in(2,3]$, there exists $w_0\in E$ such that, up to a subsequence, $w_n\rightharpoonup w_0$ in E. Arguing as in the proof of Theorem 4.2 Claim 1, we can show $w_n\to w_0$ in E, $I(w_0)=c^*$, and $I'(w_0)=0$. We note that by Lemma 4.3, it holds that $c^*\geq C>0$ for some uniform constant $C>0$, and so w_0 is nontrivial. Finally, reasoning as in the conclusion of Theorem 4.2, we see that both w_0,ϕ_{w_0} are positive, and this concludes the proof. □

Remark 4.1. *If we define*

$$\mathscr{I}(u):=\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)+\frac{1}{4}\int_{\mathbb{R}^3}\rho\phi_u u^2-\frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}, \tag{4.38}$$

then, under the same assumptions on ρ as in Corollary 4.3, we can prove the existence of a least energy critical point for $\mathscr{I}$ for all $p\in(2,5)$ by following similar techniques to those used in the proof of Corollary 4.3. Since for $p>3$ the mountain pass level is equal to the infimum on the Nehari manifold, in this range it is possible to select a positive groundstate critical point for $\mathscr{I}$. It is not clear whether this is also the case for $p\in(2,3]$.

4.3 The case of non-coercive $\rho(x)$

We now turn our attention to the problem of finding solutions when ρ is non-coercive, namely when $\rho(x)\to\rho_\infty>0$ as $|x|\to+\infty$. In this setting, $E(\mathbb{R}^3)$ coincides with the larger space $H^1(\mathbb{R}^3)$, and so we look for solutions $(u,\phi_u)\in H^1(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)$ of (4.1).

4.3.1 Bounded PS sequences: proof of Proposition 4.4

Before moving forward, we will need some useful preliminary lemmas.

Lemma 4.6 ([52]). *Let $p \geq 0$ and $(u_n)_{n\in\mathbb{N}} \subset L^{p+1}(\mathbb{R}^3)$ be such that $u_n \to u$ almost everywhere on $\mathbb{R}^3$, $\sup_n ||u_n||_{L^{p+1}} < +\infty$, and $(u_n)_- \to 0$ in $L^{p+1}(\mathbb{R}^3)$. Then, $u \in L^{p+1}(\mathbb{R}^3)$, $u \geq 0$,*

$$(u_n - u)_- \to 0 \qquad \textit{in } L^{p+1}(\mathbb{R}^3),$$

and

$$||(u_n - u)_+||_{L^{p+1}}^{p+1} = ||(u_n)_+||_{L^{p+1}}^{p+1} - ||u_+||_{L^{p+1}}^{p+1} + o(1).$$

Lemma 4.7. *Let $p > 0$ and set*

$$F(u) = \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}, \quad F_+(u) = \frac{1}{p+1}\int_{\mathbb{R}^3} u_+^{p+1}.$$

Assume $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ is such that $u_n \to u$ a.e. on $\mathbb{R}^3$ and $\sup_n ||u_n||_{H^1} < +\infty$. Then, it holds that

$$F'(u_n) - F'(u_n - u) - F'(u) = o(1), \qquad \textit{in } H^{-1}(\mathbb{R}^3).$$

If, in addition, $(u_n)_- \to 0$ in $L^{p+1}(\mathbb{R}^3)$, then

$$F_+'(u_n) - F_+'(u_n - u) - F_+'(u) = o(1), \qquad \textit{in } H^{-1}(\mathbb{R}^3).$$

Proof. The result follows as a consequence of Lemma 3.2 in [52], Lemma 4.6, and Hölder's inequality. □

The final preliminary result that we need is a splitting lemma for the nonlocal part of the derivative of the energy functional along bounded sequences. The proof follows by convexity estimates and Fatou's lemma, adapting similar arguments of Section 3 in [49] and Lemma 4.2 in [29] to a nonlocal context.

Lemma 4.8. *[**Nonlocal splitting lemma**] Assume $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ is bounded and $u_n \to v_0$ almost everywhere. Suppose further $\rho \in C(\mathbb{R}^3)$ is nonnegative and*

$\rho(x) \to \rho_\infty \geq 0$ *as* $|x| \to +\infty$. *Then, the following hold:*

(i) $\rho\phi_{(u_n-v_0)}(u_n - v_0) - \rho_\infty\bar{\phi}_{(u_n-v_0)}(u_n - v_0) = o(1)$ *in* $H^{-1}(\mathbb{R}^3)$

(ii) $\rho\phi_{u_n}u_n - \rho\phi_{(u_n-v_0)}(u_n - v_0) - \rho\phi_{v_0}v_0 = o(1)$ *in* $H^{-1}(\mathbb{R}^3)$.

Proof. For the proof of (i), we set

$$\phi_u^*(x) := \int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|}\,\mathrm{d}y.$$

Take any $h \in H^1$, and note that

$$\begin{aligned} \left|\int_{\mathbb{R}^3} \big(\rho\phi_{(u_n-v_0)}(u_n - v_0) - \rho_\infty\bar{\phi}_{(u_n-v_0)}(u_n - v_0)\big)h\right| \\ &\leq \left|\int_{\mathbb{R}^3} (\rho - \rho_\infty)\phi_{(u_n-v_0)}(u_n - v_0)h\right| \\ &\quad + \left|\int_{\mathbb{R}^3} \rho_\infty(\phi_{(u_n-v_0)} - \bar{\phi}_{(u_n-v_0)})(u_n - v_0)h\right| \\ &=: I_1 + I_2. \end{aligned} \tag{4.39}$$

Now, by assumption, for every $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $|\rho - \rho_\infty| < \varepsilon$ for all $|x| > R_\varepsilon$. So, using Hölder's and Sobolev's inequalities, we can see that

$$\begin{aligned} I_1 &\leq \left|\int_{B_{R_\varepsilon}} (\rho - \rho_\infty)\phi_{(u_n-v_0)}(u_n - v_0)h\right| \\ &\quad + \left|\int_{|x|>R_\varepsilon} (\rho - \rho_\infty)\phi_{(u_n-v_0)}(u_n - v_0)h\right| \\ &\leq ||\rho||_{L^\infty}||\phi_{(u_n-v_0)}||_{L^6}||u_n - v_0||_{L^2(B_{R_\varepsilon})}||h||_{L^3} \\ &\quad + \varepsilon||\phi_{(u_n-v_0)}||_{L^6}||u_n - v_0||_{L^2}||h||_{L^3} \\ &\lesssim (||\rho||_{L^\infty}||\nabla\phi_{(u_n-v_0)}||_{L^2}||u_n - v_0||_{L^2(B_{R_\varepsilon})} \\ &\quad + \varepsilon||\nabla\phi_{(u_n-v_0)}||_{L^2}||u_n - v_0||_{L^2})||h||_{H^1}. \end{aligned} \tag{4.40}$$

Moreover, by using Hölder's and Sobolev's inequalities once again, we have

$$\begin{aligned} I_2 &\leq \rho_\infty ||\phi_{(u_n-v_0)} - \bar{\phi}_{(u_n-v_0)}||_{L^6} ||u_n - v_0||_{L^2} ||h||_{L^3} \\ &\lesssim \rho_\infty ||\phi_{(u_n-v_0)} - \bar{\phi}_{(u_n-v_0)}||_{L^6} ||u_n - v_0||_{L^2} ||h||_{H^1}, \end{aligned} \tag{4.41}$$

and, by Minkowski's, Sobolev's, and Hardy-Littlewood-Sobolev inequalities, for every $\varepsilon > 0$, it holds

$$\begin{aligned} ||\phi_{(u_n-v_0)} &- \bar{\phi}_{(u_n-v_0)}||_{L^6} \\ &= \left(\int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} \frac{(\rho(y)-\rho_\infty)(u_n-v_0)^2(y)}{4\pi|x-y|} \mathrm{d}y \right|^6 \mathrm{d}x \right)^{\frac{1}{6}} \\ &\leq \left(\int_{\mathbb{R}^3} \left(\int_{B_{R_\varepsilon}} \frac{|\rho(y)-\rho_\infty|(u_n-v_0)^2(y)}{4\pi|x-y|} \mathrm{d}y \right)^6 \mathrm{d}x \right)^{\frac{1}{6}} \\ &\qquad + \left(\int_{\mathbb{R}^3} \left(\int_{|x|>R_\varepsilon} \frac{|\rho(y)-\rho_\infty|(u_n-v_0)^2(y)}{4\pi|x-y|} \mathrm{d}y \right)^6 \mathrm{d}x \right)^{\frac{1}{6}} \\ &\leq ||\rho||_{L^\infty} \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \frac{(u_n-v_0)^2(y)\chi^2_{B_{R_\varepsilon}}(y)}{4\pi|x-y|} \mathrm{d}y \right)^6 \mathrm{d}x \right)^{\frac{1}{6}} \\ &\qquad + \varepsilon \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \frac{(u_n-v_0)^2(y)}{4\pi|x-y|} \mathrm{d}y \right)^6 \mathrm{d}x \right)^{\frac{1}{6}} \\ &= ||\rho||_{L^\infty} ||\phi^*_{(u_n-v_0)\chi_{B_{R_\varepsilon}}}||_{L^6} + \varepsilon ||\phi^*_{(u_n-v_0)}||_{L^6} \\ &\lesssim ||\rho||_{L^\infty} ||\nabla\phi^*_{(u_n-v_0)\chi_{B_{R_\varepsilon}}}||_{L^2} + \varepsilon ||\nabla\phi^*_{(u_n-v_0)}||_{L^2} \\ &\lesssim ||\rho||_{L^\infty} ||(u_n-v_0)\chi_{B_{R_\varepsilon}}||^2_{L^{\frac{12}{5}}} + \varepsilon ||\nabla\phi^*_{(u_n-v_0)}||_{L^2}. \end{aligned} \tag{4.42}$$

So, putting together (4.39), (4.40), (4.41), and (4.42), we obtain, for every $\varepsilon > 0$,

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} \left(\rho\phi_{(u_n-v_0)}(u_n-v_0) - \rho_\infty\bar{\phi}_{(u_n-v_0)}(u_n-v_0)\right)h \right| \\ &\qquad \leq C(||\rho||_{L^\infty} ||\nabla\phi_{(u_n-v_0)}||_{L^2} ||u_n-v_0||_{L^2(B_{R_\varepsilon})} + \varepsilon ||\nabla\phi_{(u_n-v_0)}||_{L^2} ||u_n-v_0||_{L^2} \end{aligned}$$

$$
\begin{aligned}
&+\rho_\infty||\rho||_{L^\infty}||(u_n-v_0)\chi_{B_{R_\varepsilon}}||^2_{L^{\frac{12}{5}}}||u_n-v_0||_{L^2}\\
&\quad+\rho_\infty\varepsilon||\nabla\phi^*_{(u_n-v_0)}||_{L^2}||u_n-v_0||_{L^2})||h||_{H^1},
\end{aligned}
$$

for some $C>0$. Since $\rho\in L^\infty$, $\phi_{(u_n-v_0)},\phi^*_{(u_n-v_0)}$ are uniformly bounded in $D^{1,2}$, u_n-v_0 is uniformly bounded in L^2, and $u_n-v_0\to 0$ in L^2_{loc} and $L^{\frac{12}{5}}_{\text{loc}}$, then we have proven (i).

To prove (ii), we first take any $h\in H^1$, and note that by Hölder's and Sobolev's inequalities, it holds that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}(\rho\phi_{u_n}u_n-\rho\phi_{(u_n-v_0)}(u_n-v_0)-\rho\phi_{v_0}v_0)h\right|\\
&\qquad\le||\rho||_{L^\infty}||\phi_{u_n}u_n-\phi_{(u_n-v_0)}(u_n-v_0)-\phi_{v_0}v_0||_{L^{\frac{3}{2}}}||h||_{L^3}\\
&\qquad\le C||\rho||_{L^\infty}||\phi_{u_n}u_n-\phi_{(u_n-v_0)}(u_n-v_0)-\phi_{v_0}v_0||_{L^{\frac{3}{2}}}||h||_{H^1},
\end{aligned}
\tag{4.43}
$$

for some $C>0$. Now, by convexity, iterating the inequality

$$
|a+b|^{\frac{3}{2}}\le\sqrt{2}\left(|a|^{\frac{3}{2}}+|b|^{\frac{3}{2}}\right),
$$

we can obtain

$$
\begin{aligned}
F_n&:=\left|\phi_{u_n}u_n-\phi_{(u_n-v_0)}(u_n-v_0)-\phi_{v_0}v_0\right|^{\frac{3}{2}}\\
&\le 2\left(\left|\left(\phi_{u_n}-\phi_{(u_n-v_0)}\right)u_n\right|^{\frac{3}{2}}+\left|\phi_{(u_n-v_0)}v_0\right|^{\frac{3}{2}}+|\phi_{v_0}v_0|^{\frac{3}{2}}\right).
\end{aligned}
\tag{4.44}
$$

Then, using the Cauchy-Schwarz inequality, we notice that

$$
\begin{aligned}
\left|\phi_{u_n}-\phi_{(u_n-v_0)}\right|&\le\int_{\mathbb{R}^3}\frac{\rho|2u_n-v_0||v_0|}{4\pi|x-y|}\,\mathrm{d}y\\
&\le\left(\int_{\mathbb{R}^3}\frac{\rho|2u_n-v_0|^2}{4\pi|x-y|}\,\mathrm{d}y\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}\frac{\rho|v_0|^2}{4\pi|x-y|}\,\mathrm{d}y\right)^{\frac{1}{2}}\\
&=\phi^{\frac{1}{2}}_{(2u_n-v_0)}\phi^{\frac{1}{2}}_{v_0},
\end{aligned}
$$

and so, using this and applying Young's inequality twice, we see that, for every $\varepsilon > 0$,

$$\begin{aligned}\left|\left(\phi_{u_n} - \phi_{(u_n - v_0)}\right) u_n\right|^{\frac{3}{2}} &\leq \phi_{(2u_n - v_0)}^{\frac{3}{4}} \phi_{v_0}^{\frac{3}{4}} |u_n|^{\frac{3}{2}} \\ &\leq \varepsilon^{\frac{8}{7}} \phi_{(2u_n - v_0)}^{\frac{6}{7}} |u_n|^{\frac{12}{7}} + \varepsilon^{-8} \phi_{v_0}^6 \\ &\leq \varepsilon^{\frac{8}{7}} \left(\phi_{(2u_n - v_0)}^6 + |u_n|^2\right) + \varepsilon^{-8} \phi_{v_0}^6. \end{aligned} \tag{4.45}$$

Moreover, again using Young's inequality, it holds, for every $\varepsilon > 0$,

$$\left|\phi_{(u_n - v_0)} v_0\right|^{\frac{3}{2}} \leq \varepsilon^4 \phi_{(u_n - v_0)}^6 + \varepsilon^{-\frac{4}{3}} v_0^2. \tag{4.46}$$

Combining (4.44), (4.45), and (4.46), we see that, for all $\varepsilon > 0$,

$$\begin{aligned} F_n &\leq 2\left(\varepsilon^{\frac{8}{7}} \left(\phi_{(2u_n - v_0)}^6 + |u_n|^2\right) + \varepsilon^{-8} \phi_{v_0}^6 + \varepsilon^4 \phi_{(u_n - v_0)}^6 + \varepsilon^{-\frac{4}{3}} v_0^2 + |\phi_{v_0} v_0|^{\frac{3}{2}}\right) \\ &=: G_n, \end{aligned}$$

and so $G_n - F_n \geq 0$. We recall that by assumption $u_n \to v_0$ almost everywhere, and so it follows that $\phi_{(u_n - v_0)} \to 0$, $\phi_{u_n} \to \phi_{v_0}$, and $\phi_{(2u_n - v_0)} \to \phi_{v_0}$ almost everywhere. Thus, applying Fatou's Lemma to $G_n - F_n$, we obtain

$$\begin{aligned} 2 \int_{\mathbb{R}^3} &\left(\varepsilon^{\frac{8}{7}} (\phi_{v_0}^6 + |v_0|^2) + \varepsilon^{-8} \phi_{v_0}^6 + \varepsilon^{-\frac{4}{3}} v_0^2 + |\phi_{v_0} v_0|^{\frac{3}{2}}\right) \\ &\leq 2\Bigg(\varepsilon^{\frac{8}{7}} \sup_{n \geq 1} \int_{\mathbb{R}^3} \left(\phi_{(2u_n - v_0)}^6 + |u_n|^2\right) + \varepsilon^{-8} \int_{\mathbb{R}^3} \phi_{v_0}^6 + \varepsilon^4 \sup_{n \geq 1} \int_{\mathbb{R}^3} \phi_{(u_n - v_0)}^6 \\ &\qquad + \varepsilon^{-\frac{4}{3}} \int_{\mathbb{R}^3} v_0^2 + \int_{\mathbb{R}^3} |\phi_{v_0} v_0|^{\frac{3}{2}}\Bigg) - \limsup_{n \to +\infty} \int_{\mathbb{R}^3} F_n. \end{aligned}$$

Therefore, after cancelations and using Sobolev's inequality, we see that

$$\begin{aligned}
\limsup_{n\to+\infty}\int_{\mathbb{R}^3} F_n \leq 2\bigg(&\varepsilon^{\frac{8}{7}}\sup_{n\geq 1}\int_{\mathbb{R}^3}\left(\phi^6_{(2u_n-v_0)}+|u_n|^2\right)\\
&+\varepsilon^4\sup_{n\geq 1}\int_{\mathbb{R}^3}\phi^6_{(u_n-v_0)}-\varepsilon^{\frac{8}{7}}\int_{\mathbb{R}^3}\left(\phi^6_{v_0}+|v_0|^2\right)\bigg)\\
=2\bigg(&\varepsilon^{\frac{8}{7}}\sup_{n\geq 1}\left(||\phi_{(2u_n-v_0)}||^6_{L^6}+||u_n||^2_{L^2}\right)\\
&+\varepsilon^4\sup_{n\geq 1}||\phi_{(u_n-v_0)}||^6_{L^6}-\varepsilon^{\frac{8}{7}}\left(||\phi_{v_0}||^6_{L^6}+||v_0||^2_{L^2}\right)\bigg)\\
\leq C\bigg(&\varepsilon^{\frac{8}{7}}\sup_{n\geq 1}\left(||\nabla\phi_{(2u_n-v_0)}||^6_{L^2}+||u_n||^2_{L^2}\right)\\
&+\varepsilon^4\sup_{n\geq 1}||\nabla\phi_{(u_n-v_0)}||^6_{L^2}-\varepsilon^{\frac{8}{7}}\left(||\phi_{v_0}||^6_{L^6}+||v_0||^2_{L^2}\right)\bigg),
\end{aligned}$$

for some $C>0$ and for all $\varepsilon>0$. We note that u_n, v_0 are uniformly bounded in L^2 and $\phi_{(u_n-v_0)}, \phi_{(2u_n-v_0)}$ are uniformly bounded in $D^{1,2}$ since $u_n-v_0, 2u_n-v_0$ are uniformly bounded in H^1. Moreover, since $v_0\in H^1$, it follows that $||\phi_{v_0}||^6_{L^6}$ is bounded by Sobolev's inequality. Hence, since $\varepsilon>0$ is arbitrary, it holds that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} F_n = 0,$$

which combined with (4.43) yields (ii), and this concludes the proof. □

With these preliminaries in place, we now prove a useful 'splitting' proposition for bounded Palais-Smale sequences for I_μ, that highlights the connection to the problem at infinity. There are several compactness results of similar flavour since the pioneering works of P.L. Lions [44] and Benci-Cerami [12], which include more recent contributions in the context of Schrödinger-Poisson systems, see e.g. [26], [62], [24]. We point out that these recent results are mostly in the range $p>3$, for Palais-Smale sequences constrained on Nehari manifolds, and for functionals without positive parts, unlike our result.

Proposition 4.4. *[***Global compactness for bounded PS sequences***] Suppose $\rho\in C(\mathbb{R}^3)$ is nonnegative and $\rho(x)\to\rho_\infty\geq 0$ as $|x|\to+\infty$. Let $p\in(2,5)$ and $\mu\in\left[\frac{1}{2},1\right]$ and assume $(u_n)_{n\in\mathbb{N}}\subset H^1(\mathbb{R}^3)$ is a bounded Palais-Smale sequence for I_μ. Then, there exist $l\in\mathbb{N}$, a finite sequence $(v_0,\ldots,v_l)\subset H^1(\mathbb{R}^3)$, and l sequences of points*

$(y_n^j)_{n\in\mathbb{N}} \subset \mathbb{R}^3$, $1 \le j \le l$, satisfying, up to a subsequence of $(u_n)_{n\in\mathbb{N}}$,

(i) v_0 *is a nonnegative solution of* (4.4),

(ii) v_j *are nonnegative, and possibly nontrivial, solutions of* (4.12) *for* $1 \le j \le l$,

(iii) $|y_n^j| \to +\infty$, $|y_n^j - y_n^{j'}| \to +\infty$ *as* $n \to +\infty$ *if* $j \neq j'$,

(iv) $||u_n - v_0 - \sum_{j=1}^{l} v_j(\cdot - y_n^j)||_{H^1(\mathbb{R}^3)} \to 0$ *as* $n \to +\infty$,

(v) $||u_n||^2_{H^1(\mathbb{R}^3)} \to \sum_{j=0}^{l} ||v_j||^2_{H^1(\mathbb{R}^3)}$ *as* $n \to +\infty$,

(vi) $I_\mu(u_n) = I_\mu(v_0) + \sum_{j=1}^{l} I_\mu^\infty(v_j) + o(1)$.

Remark 4.2. *In the case* $\rho_\infty = 0$*, the limiting equation* (4.12) *reduces to coincide with the classical nonlinear Schrödinger equation* $-\Delta u + u = u^p$ *, whose positive solutions have been classified by Kwong [41].*

Proof of Proposition 4.4. Since $(u_n)_{n\in\mathbb{N}}$ is bounded in H^1, we may assume $u_n \rightharpoonup v_0$ in H^1 and $u_n \to v_0$ a.e. in $\mathbb{R}^3$. We set $u_n^1 := u_n - v_0$, and we first note that

$$||u_n^1||^2_{H^1} = ||u_n - v_0||^2_{H^1} = ||u_n||^2_{H^1} - ||v_0||^2_{H^1} + o(1). \tag{4.47}$$

We now prove three claims involving the sequence $(u_n^1)_{n\in\mathbb{N}}$.

Claim 1. $I_\mu^\infty(u_n^1) = I_\mu(u_n) - I_\mu(v_0) + o(1)$.

Testing $I'_\mu(u_n)$ with $(u_n)_-$ we have

$$I'_\mu(u_n)(u_n)_- = \int_{\mathbb{R}^3} (\nabla u_n \nabla((u_n)_-) + u_n(u_n)_-) + \int_{\mathbb{R}^3} \rho\phi_{u_n} u_n (u_n)_- \\ - \mu \int_{\mathbb{R}^3} (u_n)_+^p (u_n)_-$$

$$= ||(u_n)_-||^2_{H^1} + \int_{\mathbb{R}^3} \rho\phi_{u_n}(u_n)^2_-.$$

Since $(u_n)_{n\in\mathbb{N}}$ is bounded, $I'_\mu(u_n)(u_n)_- = o(1)$, which implies

$$(u_n)_- \to 0 \text{ in } H^1,$$

and by Sobolev's embedding

$$(u_n)_- \to 0 \text{ in } L^{p+1}\ \forall p \in [1,5].$$

Now, using this and the boundedness of $(u_n)_{n\in\mathbb{N}}$ in L^{p+1}, it holds, by Lemma 4.6, that

$$||(u_n^1)_+||^{p+1}_{L^{p+1}} = ||(u_n)_+||^{p+1}_{L^{p+1}} - ||(v_0)_+||^{p+1}_{L^{p+1}} + o(1).$$

Therefore, using this and (4.47), we can see that

$$\begin{aligned} I^\infty_\mu(u_n^1) = \frac{1}{2}(||u_n||^2_{H^1} - ||v_0||^2_{H^1}) + \frac{1}{4}\int_{\mathbb{R}^3} \rho_\infty\bar{\phi}_{(u_n-v_0)}(u_n-v_0)^2 \\ - \frac{\mu}{p+1}\left(||(u_n)_+||^{p+1}_{L^{p+1}} - ||(v_0)_+||^{p+1}_{L^{p+1}}\right) + o(1). \end{aligned} \tag{4.48}$$

We now notice that since, by symmetry,

$$\int_{\mathbb{R}^3} \rho_\infty(u_n-v_0)^2\phi_{(u_n-v_0)} = \int_{\mathbb{R}^3} \rho(u_n-v_0)^2\bar{\phi}_{(u_n-v_0)},$$

then it holds that

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} \rho\phi_{(u_n-v_0)}(u_n-v_0)^2 - \int_{\mathbb{R}^3} \rho_\infty\bar{\phi}_{(u_n-v_0)}(u_n-v_0)^2\right| \\ &\quad\le \int_{\mathbb{R}^3} \phi_{(u_n-v_0)}(u_n-v_0)^2|\rho(x)-\rho_\infty| + \int_{\mathbb{R}^3} \bar{\phi}_{(u_n-v_0)}(u_n-v_0)^2|\rho(x)-\rho_\infty| \\ &\quad=: I_1 + I_2. \end{aligned}$$

We note that for all $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that $|\rho(x) - \rho_\infty| < \varepsilon$ for all $|x| > R_\varepsilon$. Thus, we can see,

$$
\begin{aligned}
I_1 \leq & \int_{B_{R_\varepsilon}} \phi_{(u_n - v_0)}(u_n - v_0)^2 |\rho(x) - \rho_\infty| \\
& \qquad + \int_{|x| > R_\varepsilon} \phi_{(u_n - v_0)}(u_n - v_0)^2 |\rho(x) - \rho_\infty| \\
\leq & C\Big(||\rho||_{L^\infty} || \, ||\nabla \phi_{(u_n - v_0)}||_{L^2} ||u_n - v_0||^2_{L^{\frac{12}{5}}(B_{R_\varepsilon})} \\
& \qquad + \varepsilon ||\nabla \phi_{(u_n - v_0)}||_{L^2} ||u_n - v_0||^2_{L^{\frac{12}{5}}} \Big),
\end{aligned}
$$

where $C > 0$ is a constant. Since $\rho \in L^\infty$, $\phi_{(u_n - v_0)}$ is uniformly bounded in $D^{1,2}$ and $u_n - v_0 \to 0$ in $L^{12/5}_{\text{loc}}$, the above shows that $I_1 \to 0$ as $n \to +\infty$. Similarly, we can see that

$$
\begin{aligned}
I_2 \leq C' \Big(& ||\rho||_{L^\infty} || \, ||\nabla \bar{\phi}_{(u_n - v_0)}||_{L^2} ||u_n - v_0||^2_{L^{\frac{12}{5}}(B_{R_\varepsilon})} \\
& \qquad + \varepsilon ||\nabla \bar{\phi}_{(u_n - v_0)}||_{L^2} ||u_n - v_0||^2_{L^{\frac{12}{5}}} \Big),
\end{aligned}
$$

and so $I_2 \to 0$ as $n \to +\infty$. Therefore, we have shown that

$$
\int_{\mathbb{R}^3} \rho_\infty \bar{\phi}_{(u_n - v_0)}(u_n - v_0)^2 = \int_{\mathbb{R}^3} \rho \phi_{(u_n - v_0)}(u_n - v_0)^2 + o(1),
$$

and thus, by the nonlocal Brezis-Lieb Lemma 3.2, it holds that

$$
\int_{\mathbb{R}^3} \rho_\infty \bar{\phi}_{(u_n - v_0)}(u_n - v_0)^2 = \int_{\mathbb{R}^3} \rho \phi_{u_n} u_n^2 - \int_{\mathbb{R}^3} \rho \phi_{v_0} v_0^2 + o(1).
$$

Putting this together with (4.48), we see that $I^\infty_\mu(u^1_n) = I_\mu(u_n) - I_\mu(v_0) + o(1)$, and the claim is proved.

Claim 2. *$I'_\mu(v_0) = 0$ and $v_0 \geq 0$.*

We notice that for all $\psi \in C^\infty_c(\mathbb{R}^3)$, it holds that

$$I'_\mu(u_n)(\psi) = \int_{\mathbb{R}^3} (\nabla u_n \nabla \psi + u_n \psi) + \int_{\mathbb{R}^3} \rho \phi_{u_n} u_n \psi - \mu \int_{\mathbb{R}^3} (u_n)_+^p \psi.$$

Using the fact that $u_n \rightharpoonup v_0$ in H^1 and a local compactness argument, we can show that $I'_\mu(u_n)(\psi) = I'_\mu(v_0)(\psi) + o(1)$. So, since $I'_\mu(u_n) \to 0$ by the definition of a Palais-Smale sequence, it holds that $I'_\mu(v_0) = 0$ by density. We note that by testing this equation with $(v_0)_-$, we obtain that $v_0 \geq 0$.

Claim 3. $(I^\infty_\mu)'(u_n^1) \to 0$.

We first note that by Lemma 4.8, it holds that

$$\begin{aligned} &\rho \phi_{u_n} u_n - \rho_\infty \bar{\phi}_{(u_n - v_0)}(u_n - v_0) - \rho \phi_{v_0} v_0 \\ &\qquad = \rho \phi_{u_n} u_n - \rho \phi_{(u_n - v_0)}(u_n - v_0) - \rho \phi_{v_0} v_0 + o(1) \\ &\qquad = o(1) \qquad \text{in } H^{-1}(\mathbb{R}^3). \end{aligned} \tag{4.49}$$

Moreover, since we have showed in Claim 1 that $(u_n)_- \to 0$ in L^{p+1}, then, by Lemma 4.7, it follows that

$$(u_n)_+^p - (u_n - v_0)_+^p - (v_0)^p = o(1), \qquad \text{in } H^{-1}(\mathbb{R}^3). \tag{4.50}$$

Therefore, using (4.49) and (4.50), we can conclude that

$$(I^\infty_\mu)'(u_n^1) = I'_\mu(u_n) - I'_\mu(v_0) + o(1),$$

and so

$$(I^\infty_\mu)'(u_n^1) = o(1)$$

since $I'_\mu(u_n) \to 0$ by the definition of Palais-Smale sequence and $I'_\mu(v_0) = 0$ by Claim 2. This completes the proof of the claim.

Partial conclusions. With these results in place, we now define

$$\delta := \limsup_{n\to+\infty}\left(\sup_{y\in\mathbb{R}^3}\int_{B_1(y)}|u_n^1|^{p+1}\right).$$

We can see that $\delta \geq 0$. If $\delta = 0$, the P. L. Lions Lemma [44] implies $u_n^1 \to 0$ in L^{p+1}. Since it holds that

$$(I_\mu^\infty)'(u_n^1)(u_n^1) = ||u_n^1||_{H^1}^2 + \int_{\mathbb{R}^3}\rho_\infty\bar{\phi}_{u_n^1}(u_n^1)^2 - \mu\int_{\mathbb{R}^3}(u_n^1)_+^{p+1},$$

and $(I_\mu^\infty)'(u_n^1) \to 0$ by Claim 3, then, if $u_n^1 \to 0$ in L^{p+1}, it follows that $u_n^1 \to 0$ in H^1. In this case, we are done since we have $u_n \to v_0$ in H^1. Therefore, we assume $\delta > 0$. This implies that there exists $(y_n^1)_{n\in\mathbb{N}} \subset \mathbb{R}^3$ such that

$$\int_{B_1(y_n^1)}|u_n^1|^{p+1} > \frac{\delta}{2}.$$

We now define $v_n^1 := u_n^1(\cdot + y_n^1)$. We may assume $v_n^1 \rightharpoonup v_1$ in H^1 and $v_n^1 \to v_1$ a.e. in $\mathbb{R}^3$. Then, since

$$\int_{B_1(0)}|v_n^1|^{p+1} > \frac{\delta}{2},$$

it follows from Rellich Theorem that $v_1 \not\equiv 0$. Since $u_n^1 \rightharpoonup 0$ in H^1, then $(y_n^1)_{n\in\mathbb{N}}$ must be unbounded and so we assume, up to a subsequence, $|y_n^1| \to +\infty$. We set $u_n^2 := u_n^1 - v_1(\cdot - y_n^1)$, and, using (4.47), we note that

$$||u_n^2||_{H^1}^2 = ||u_n^1||_{H^1}^2 - ||v_1||_{H^1}^2 + o(1) = ||u_n||_{H^1}^2 - ||v_0||_{H^1}^2 - ||v_1||_{H^1}^2 + o(1). \quad (4.51)$$

We now prove three claims regarding the sequence $(u_n^2)_{n\in\mathbb{N}}$.

Claim 4. $I_\mu^\infty(u_n^2) = I_\mu(u_n) - I_\mu(v_0) - I_\mu^\infty(v_1) + o(1)$.

Arguing similarly as in Claim 1, namely testing $(I_\mu^\infty)'(u_n^1)$ with $(u_n^1)_-$, we can show that $(u_n^1)_- \to 0$ in L^{p+1}, and so $(u_n^1(\cdot + y_n^1))_- \to 0$ in L^{p+1}. Thus, once again using Lemma 4.6, we can see that

$$
\begin{aligned}
||(u_n^1)_+||_{L^{p+1}}^{p+1} &= ||(u_n^1(\cdot + y_n^1) - v_1)_+||_{L^{p+1}}^{p+1} + ||(v_1)_+||_{L^{p+1}}^{p+1} + o(1) \\
&= ||(u_n^1 - v_1(\cdot - y_n^1))_+||_{L^{p+1}}^{p+1} + ||(v_1)_+||_{L^{p+1}}^{p+1} + o(1) \\
&= ||(u_n^2)_+||_{L^{p+1}}^{p+1} + ||(v_1)_+||_{L^{p+1}}^{p+1} + o(1),
\end{aligned}
$$

and so

$$
||(u_n^2)_+||_{L^{p+1}}^{p+1} = ||(u_n^1)_+||_{L^{p+1}}^{p+1} - ||(v_1)_+||_{L^{p+1}}^{p+1} + o(1).
$$

Therefore, using this and (4.51), we have that

$$
\begin{aligned}
I_\mu^\infty(u_n^2) = ||u_n^1||_{H^1}^2 - ||v_1||_{H^1}^2 + \frac{1}{4}\int_{\mathbb{R}^3} \rho_\infty \bar{\phi}_{(u_n^1 - v_1(\cdot - y_n^1))}(u_n^1 - v_1(x - y_n^1))^2 \\
- \frac{\mu}{p+1}\left(||(u_n^1)_+||_{L^{p+1}}^{p+1} - ||(v_1)_+||_{L^{p+1}}^{p+1} \right) + o(1).
\end{aligned}
$$

We can show, by changing variables and using Lemma 3.2, that

$$
\int_{\mathbb{R}^3} \rho_\infty \bar{\phi}_{(u_n^1 - v_1(\cdot - y_n^1))}(u_n^1 - v_1(x - y_n^1))^2 = \int_{\mathbb{R}^3} \rho_\infty \bar{\phi}_{u_n^1}(u_n^1)^2 - \int_{\mathbb{R}^3} \rho_\infty \bar{\phi}_{v_1} v_1^2 + o(1).
$$

Thus, by combining the last two equations and using Claim 1, we see that $I_\mu^\infty(u_n^2) = I_\mu^\infty(u_n^1) - I_\mu^\infty(v_1) + o(1) = I_\mu(u_n) - I_\mu(v_0) - I_\mu^\infty(v_1) + o(1)$, and so the claim is proved.

Claim 5. $(I_\mu^\infty)'(v_1) = 0$ *and* $v_1 \geq 0$.

Let $h \in H^1(\mathbb{R}^3)$ and set $h_n := h(\cdot - y_n^1)$. By a change of variables, we can see that

$$
(I_\mu^\infty)'(u_n^1(x + y_n^1))(h) = (I_\mu^\infty)'(u_n^1)(h_n),
$$

and so, since $(I_\mu^\infty)'(u_n^1) \to 0$ by Claim 3, we have that

$$(I_\mu^\infty)'(u_n^1(x+y_n^1)) \to 0. \tag{4.52}$$

We now note, for any $\psi \in C_c^\infty(\mathbb{R}^3)$, it holds that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^3} \rho_\infty \bar{\phi}_{u_n^1}(x+y_n^1) u_n^1(x+y_n^1)\psi - \int_{\mathbb{R}^3} \rho_\infty \bar{\phi}_{v_1} v_1 \psi \right| \\
&\quad \leq \left| \int_{\mathbb{R}^3} \rho_\infty \bar{\phi}_{u_n^1}(x+y_n^1)(u_n^1(x+y_n^1) - v_1)\psi \right| \\
&\qquad\qquad + \left| \int_{\mathbb{R}^3} \rho_\infty (\bar{\phi}_{u_n^1}(x+y_n^1) - \bar{\phi}_{v_1}) v_1 \psi \right| \\
&\quad \leq \rho_\infty ||\bar{\phi}_{u_n^1}(\cdot+y_n^1)||_{L^6} ||u_n^1(\cdot+y_n^1) - v_1||_{L^2(\operatorname{supp}\psi)} ||\psi||_{L^3} \\
&\qquad\qquad + \rho_\infty ||\bar{\phi}_{u_n^1}(\cdot+y_n^1) - \bar{\phi}_{v_1}||_{L^2(\operatorname{supp}\psi)} ||v_1||_{L^6} ||\psi||_{L^3},
\end{aligned}$$

and so since $u_n^1(\cdot+y_n^1) - v_1 \to 0$ in L^2_{loc} and $\bar{\phi}_{u_n^1}(\cdot+y_n^1) - \bar{\phi}_{v_1} \to 0$ in L^2_{loc}, and all of the other terms in the final equation are bounded, then we have shown that

$$\int_{\mathbb{R}^3} \rho_\infty \bar{\phi}_{u_n^1}(x+y_n^1) u_n^1(x+y_n^1)\psi \to \int_{\mathbb{R}^3} \rho_\infty \bar{\phi}_{v_1} v_1 \psi.$$

Using this and the fact that $u_n^1(\cdot+y_n^1) \rightharpoonup v_1$ in H^1, it follows by standard arguments that $(I_\mu^\infty)'(u_n^1(x+y_n^1))(\psi) = (I_\mu^\infty)'(v_1)(\psi) + o(1)$. This implies that $(I_\mu^\infty)'(v_1) = 0$, by (4.52) and density. Testing this equation with $(v_1)_-$, shows that $v_1 \geq 0$.

Claim 6. $(I_\mu^\infty)'(u_n^2) \to 0$.

We take any $h \in H^1(\mathbb{R}^3)$ and set $h_n := h(\cdot+y_n^1)$. We note that, by a change of variables, it holds that

$$(I_\mu^\infty)'(u_n^2)(h) = (I_\mu^\infty)'(u_n^1(\cdot+y_n^1) - v_1)(h_n). \tag{4.53}$$

Now, arguing as we did in the proof of (ii) of Lemma 4.8, we can show that

$$\begin{aligned}\rho_\infty\bar{\phi}_{u_n^1}(\cdot+y_n^1)u_n^1(\cdot+y_n^1)\\ -\rho_\infty\bar{\phi}_{(u_n^1(\cdot+y_n^1)-v_1)}(u_n^1(\cdot+y_n^1)-v_1)-\rho_\infty\bar{\phi}_{v_1}v_1=o(1),\end{aligned}\tag{4.54}$$

in $H^{-1}(\mathbb{R}^3)$. Moreover, since we showed $(u_n^1(\cdot+y_n^1))_-\to 0$ in L^{p+1} in Claim 4, we can once again use Lemma 4.7 to conclude that

$$(u_n^1(\cdot+y_n^1))_+^p-(u_n^1(\cdot+y_n^1)-v_1)_+^p-(v_1)_+^p=o(1),\qquad \text{in } H^{-1}(\mathbb{R}^3).$$

It follows from this and (4.54) that

$$(I_\mu^\infty)'(u_n^1(\cdot+y_n^1)-v_1)=(I_\mu^\infty)'(u_n^1(\cdot+y_n^1))-(I_\mu^\infty)'(v_1)+o(1),\quad \text{in } H^{-1}(\mathbb{R}^3).\tag{4.55}$$

Since, by Claim 5 and a change of variables, it holds that

$$(I_\mu^\infty)'(u_n^1(\cdot+y_n^1))(h_n)-(I_\mu^\infty)'(v_1)(h_n)=(I_\mu^\infty)'(u_n^1)(h),$$

then combining this, (4.53) and (4.55), we see that

$$(I_\mu^\infty)'(u_n^2)=(I_\mu^\infty)'(u_n^1)+o(1),\qquad \text{in } H^{-1}(\mathbb{R}^3).$$

It therefore follows that $(I_\mu^\infty)'(u_n^2)\to 0$ since $(I_\mu^\infty)'(u_n^1)\to 0$ by Claim 3, and we are done.

Conclusions. With these results in place we can now see that if $u_n^2\to 0$ in H^1, then we are done. Otherwise, $u_n^2\rightharpoonup 0$ in H^1, but not strongly, and so we repeat the argument. By iterating the procedure, we obtain sequences of points $(y_n^j)_{n\in\mathbb{N}}\subset\mathbb{R}^3$ such that $|y_n^j|\to+\infty$, $|y_n^j-y_n^{j'}|\to+\infty$ as $n\to+\infty$ if $j\neq j'$ and a sequence of functions $(u_n^j)_{n\in\mathbb{N}}$ with $u_n^1=u_n-v_0$ and $u_n^j=u_n^{j-1}-v_{j-1}(\cdot-y_n^{j-1})$ for $j\geq 2$ such that

$$
\begin{aligned}
&u_n^j(x+y_n^j) \rightharpoonup v_j(x) \text{ in } H^1, \\
&||u_n||^2_{H^1(\mathbb{R}^3)} = \sum_{j=0}^{l-1} ||v_j||^2_{H^1(\mathbb{R}^3)} + ||u_n^l||^2_{H^1} + o(1), \\
&||u_n - v_0 - \sum_{j=1}^{l} v_j(\cdot - y_n^j)||_{H^1(\mathbb{R}^3)} \to 0 \text{ as } n \to +\infty, \\
&I_\mu(u_n) = I_\mu(v_0) + \sum_{j=1}^{l-1} I_\mu^\infty(v_j) + I_\mu^\infty(u_n^l) + o(1), \\
&(I_\mu^\infty)'(v_j) = 0 \text{ and } v_j \geq 0 \text{ for } j \geq 1,
\end{aligned} \tag{4.56}
$$

We notice from the last equation that it holds that $(I_\mu^\infty)'(v_j)(v_j) = 0$ for each $j \geq 1$. Using this, the Sobolev embedding theorem and the fact that $\mu \leq 1$, we have that

$$
S_{p+1}||v_j||^2_{L^{p+1}} \leq ||v_j||^2_{H^1} \leq ||v_j||^2_{H^1} + \int_{\mathbb{R}^3} \rho_\infty \bar{\phi}_{v_j}(v_j)^2 = \mu||(v_j)_+||^{p+1}_{L^{p+1}} \leq ||v_j||^{p+1}_{L^{p+1}},
$$

and therefore, we can conclude that, for each $j \geq 1$,

$$
||v_j||^2_{H^1} \geq (S_{p+1})^{\frac{p+1}{p-1}}.
$$

Combining this and the fact $(u_n)_{n\in\mathbb{N}}$ is bounded in H^1, we see from (4.56) that the iteration must stop at some finite index $l \in \mathbb{N}$. □

4.3.2 Theorem 4.3

We are finally in position to establish two sufficient conditions that guarantee the existence of a mountain pass solution to (4.1) in the case of non-coercive ρ. When $p \in (2,3)$, we use Proposition 4.4 together with a Pohozaev type inequality and Nehari's identity to show that a sequence of approximated critical points, constructed by means of the 'monotonicity trick', is relatively compact. This enables us to obtain the following result.

Theorem 4.3. ***[Non-coercive case: existence of mountain pass solution for** $p \in (2,3)$**]*** *Suppose $\rho \in C(\mathbb{R}^3) \cap W^{1,1}_{loc}(\mathbb{R}^3)$ is nonnegative, $\rho(x) \to \rho_\infty > 0$ as $|x| \to +\infty$,* *and* $k\rho(x) \leq (x, \nabla\rho)$ *for some* $k > \frac{-2(p-2)}{(p-1)}$*. Suppose further that*

(i) either $c < c^\infty$*,*

(ii) or $\rho(x) \leq \rho_\infty$ *for all* $x \in \mathbb{R}^3$*, with strict inequality,* $\rho(x) < \rho_\infty$*, on some ball* $B \subset \mathbb{R}^3$*,*

where c *and (resp.)* c^∞ *are min-max levels defined in (4.10) and (resp.) (4.16). Then, for any* $p \in (2,3)$*, there exists a solution,* $(u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$*, of (4.1), whose components are positive functions. In particular,* u *is a mountain pass critical point of* I *at level* c*.*

Proof. We first note that by Corollary 4.1 with $E = H^1$, the set $\mathscr{M}$, defined in (4.9), is dense in $\left[\frac{1}{2}, 1\right]$.

Claim 1. *Under assumptions* (i)*, the values* c_μ *are critical levels of* I_μ *for all* $\mu \in (1-\varepsilon, 1] \cap \mathscr{M}$*, with* $\varepsilon > 0$ *sufficiently small. Namely, there exists a nonnegative* $u \in H^1$ *such that* $I_\mu(u) = c_\mu$ *and* $I'_\mu(u) = 0$ *for all* $\mu \in (1-\varepsilon, 1] \cap \mathscr{M}$*. Under assumptions* (ii)*, the same statement holds for all* $\mu \in \mathscr{M}$*.*

We recall that for all $\mu \in \mathscr{M}$, by definition, there exists a bounded sequence $(u_n)_{n\in\mathbb{N}} \subset H^1$ such that $I_\mu(u_n) \to c_\mu$ and $I'_\mu(u_n) \to 0$. We note that by Proposition 4.4 and the definition of $(u_n)_{n\in\mathbb{N}}$, it holds that

$$c_\mu = I_\mu(v_0) + \sum_{j=1}^{l} I^\infty_\mu(v_j), \tag{4.57}$$

where v_0 is a nonnegative solution of (4.4) and v_j are nonnegative solutions of (4.12) for $1 \leq j \leq l$.

Assume that (i) holds. For $\varepsilon > 0$ small enough, it holds that $c_\mu < c^\infty_\mu$ for all $\mu \in (1-\varepsilon, 1] \cap \mathscr{M}$, by continuity. Pick μ on this set. If v_j is nontrivial for some $1 \leq j \leq l$, it would follow that $I^\infty_\mu(v_j) \geq c^\infty_\mu > c_\mu$ by Lemma 4.4. This is in contradiction with (4.57) since $I_\mu(v_0) \geq 0$, by Lemma 4.3, and so, $v_j \equiv 0$ for all $1 \leq j \leq l$. Therefore, $u_n \to v_0$ in H^1 by (iv) of Proposition 4.4, $I_\mu(v_0) = c_\mu$ by (4.57), and $I'_\mu(v_0) = 0$ since v_0 is a nonnegative solution of (4.4). Thus, we have shown c_μ is a critical level of I_μ in this case.

Now, assume that (ii) holds. We note that this implies that $I_\mu(\gamma(t)) \leq I_\mu^\infty(\gamma(t))$ for each fixed $\gamma \in \Gamma^\infty$, $\mu \in [\frac{1}{2}, 1]$ and $t \in [0,1]$. It therefore follows that $I_{\frac{1}{2}}(\gamma(1)) \leq I_{\frac{1}{2}}^\infty(\gamma(1)) < 0$ for all $\gamma \in \Gamma^\infty$, and so $\Gamma^\infty \subseteq \Gamma$. Using this and Lemma 4.4, we can see that for each nontrivial v_j in (4.57), it holds

$$\begin{aligned} I_\mu^\infty(v_j) &\geq c_\mu^\infty \\ &= \inf_{\gamma \in \Gamma^\infty} \max_{t \in [0,1]} I_\mu^\infty(\gamma(t)), \\ &\geq \inf_{\gamma \in \Gamma^\infty} \max_{t \in [0,1]} I_\mu(\gamma(t)) \\ &\geq \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\mu(\gamma(t)) \\ &= c_\mu. \end{aligned} \tag{4.58}$$

We now assume, by contradiction, $v_0 \equiv 0$ in (4.57), which would imply $I_\mu(v_0) = 0$. Using this and (4.58), we see from (4.57) that there exists one nontrivial v_j, call it v_1, such that v_1 is a nonnegative solution of (4.12) and

$$I_\mu^\infty(v_1) = c_\mu^\infty = c_\mu. \tag{4.59}$$

Define $\bar{v}_t(x) = t^2 v_1(tx)$ and $\gamma : \mathbb{R} \to H^1(\mathbb{R}^3)$, $\gamma(t) = \bar{v}_t$. By Lemma 3.3 in [57, p. 663], the function $f(t) = I_\mu^\infty(\bar{v}_t)$ has a unique critical point corresponding to its maximum, and it can be shown that $f'(1) = 0$ by Nehari's and Pohozaev's identities for v_1. We deduce that

$$\max_{t \in \mathbb{R}} I_\mu^\infty(\gamma(t)) = I_\mu^\infty(v_1),$$

and that there exists $M > 0$ such that

$$I_{\frac{1}{2}}^\infty(\gamma(M)) < 0,$$

and

$$\max_{t\in\mathbb{R}} I_\mu^\infty(\gamma(t)) = \max_{t\in[0,M]} I_\mu^\infty(\gamma(t)).$$

We then define $\gamma_0 : [0,1] \to H^1(\mathbb{R}^3)$, $\gamma_0(t) = \gamma(Mt)$, and see from the above work that $\gamma_0 \in \Gamma^\infty$. Therefore, we have that

$$\begin{aligned} I_\mu^\infty(v_1) &= \max_{t\in\mathbb{R}} I_\mu^\infty(\gamma(t)) \\ &= \max_{t\in[0,M]} I_\mu^\infty(\gamma(t)) \\ &= \max_{t\in[0,1]} I_\mu^\infty(\gamma_0(t)). \end{aligned}$$

Since we have $v_1 > 0$ on B where $\rho(x) < \rho_\infty$ by Lemma 4.4, it follows that

$$\begin{aligned} c_\mu^\infty &= I_\mu^\infty(v_1) \\ &= \max_{t\in[0,1]} I_\mu^\infty(\gamma_0(t)) \\ &> \max_{t\in[0,1]} I_\mu(\gamma_0(t)) \\ &\geq \inf_{\gamma\in\Gamma^\infty} \max_{t\in[0,1]} I_\mu(\gamma(t)) \\ &\geq \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\mu(\gamma(t)) \\ &= c_\mu, \end{aligned}$$

which contradicts (4.59). Therefore, we have shown that $v_0 \not\equiv 0$. Now, since v_0 is a nontrivial and nonnegative solution of (4.4), then $I_\mu(v_0) > 0$ by Lemma 4.3. Putting this and (4.58) together in (4.57), it follows that $v_j \equiv 0$ for all $1 \leq j \leq l$. Therefore, $u_n \to v_0$ in H^1 by (iv) of Proposition 4.4, $I_\mu(v_0) = c_\mu$ by (4.57), and $I'_\mu(v_0) = 0$ since v_0 is a nonnegative solution of (4.4). This concludes the proof of Claim 1.

Claim 2. *Let $\mu_n \to 1$ be an increasing sequence in $(1-\varepsilon, 1] \cap \mathscr{M}$ and (resp.) $\mathscr{M}$ under assumptions (i) and (resp.) (ii). Assume $(u_n)_{n\in\mathbb{N}} \subset H^1$ is such that u_n is nonnegative, $I_{\mu_n}(u_n) = c_{\mu_n}$ and $I'_{\mu_n}(u_n) = 0$ for each n. Then, there exists a nonnegative $u \in H^1$ such that, up to a subsequence, $u_n \to u$ in H^1, $I(u) = c$, and $I'(u) = 0$.*

Since u_n is nonnegative, $(u_n)_+ = u_n$, and so we can see that

$$\begin{aligned} I(u_n) &= I_{\mu_n}(u_n) + \frac{\mu_n - 1}{p+1} \int_{\mathbb{R}^3} u_n^{p+1} \\ &= c_{\mu_n} + \frac{\mu_n - 1}{p+1} \int_{\mathbb{R}^3} u_n^{p+1}, \end{aligned} \tag{4.60}$$

and, for all $v \in H^1(\mathbb{R}^3)$,

$$\begin{aligned} I'(u_n)(v) &= I'_{\mu_n}(u_n)(v) + (\mu_n - 1) \int_{\mathbb{R}^3} u_n^p v \\ &\leq |\mu_n - 1| \, ||u_n||_{L^{p+1}}^p ||v||_{L^{p+1}} \\ &\leq S_{p+1}^{-\frac{1}{2}} |\mu_n - 1| \, ||u_n||_{L^{p+1}}^p ||v||_{H^1}. \end{aligned} \tag{4.61}$$

Set $\alpha_n = \int_{\mathbb{R}^3}(|\nabla u_n|^2 + u_n^2)$, $\gamma_n = \int_{\mathbb{R}^3} \rho \phi_{u_n} u_n^2$, and $\delta_n = \mu_n \int_{\mathbb{R}^3} u_n^{p+1}$. As in Theorem 4.2 Claim 2, we see that α_n, γ_n, $\delta_n \geq 0$ satisfy (4.31), and thus we can obtain that α_n, γ_n, and δ_n are all bounded. Therefore, using this, (4.60), (4.61), and the fact that $c_{\mu_n} \to c$ as $\mu_n \nearrow 1$ by definition (4.10), we can deduce that $||u_n||_{H^1}$ is bounded, $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to +\infty$. That is, we have shown that $(u_n)_{n\in\mathbb{N}}$ is a bounded Palais-Smale sequence for $I = I_1$ at the level $c = c_1$, and so, $1 \in \mathscr{M}$. By Claim 1, it follows that there exists a nonnegative $u \in H^1$ such that, up to a subsequence, $u_n \to u$ in H^1, $I(u) = c$, and $I'(u) = 0$.

Conclusion. Let $\mu_n \to 1$ be an increasing sequence in $(1-\varepsilon, 1] \cap \mathscr{M}$ and (resp.) $\mathscr{M}$ under assumptions (i) and (resp.) (ii). By Claim 1, we can choose $(u_n)_{n\in\mathbb{N}} \subset H^1$ such that u_n is nonnegative, $I_{\mu_n}(u_n) = c_{\mu_n}$ and $I'_{\mu_n}(u_n) = 0$ for each n. By Claim 2, it follows that there exists a nonnegative $u \in H^1$ such that, up to a subsequence, $u_n \to u$ in H^1, $I(u) = c$, and $I'(u) = 0$. That is, we have shown $(u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ solves (4.1). Since u and ϕ_u are nonnegative by construction, by regularity and the strong maximum principle, it follows that they are, in fact, strictly positive. This concludes the proof. □

4.3.3 Theorem 4.4

The non-coercive case turns out to be more 'regular' with respect to compactness issues when $p \geq 3$. In fact, we can show that the Palais-Smale condition holds at the mountain pass level c.

Proposition 4.5. *[**Palais-Smale condition for** $p \geq 3$] Let $\rho \in C(\mathbb{R}^3)$ be nonnegative such that $\rho(x) \to \rho_\infty > 0$ as $|x| \to +\infty$, and suppose one of the following conditions holds:*

(i) either $c < c^\infty$,

(ii) or $\rho(x) \leq \rho_\infty$ for all $x \in \mathbb{R}^3$, with strict inequality, $\rho(x) < \rho_\infty$, on some ball $B \subset \mathbb{R}^3$,

where c and (resp.) c^∞ are defined in (4.10) and (resp.) (4.16). Then, for any $p \in [3,5)$, every Palais-Smale sequence $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ for I, at the level c, is relatively compact. In particular, c is a critical level for I.

Proof. Since, for $p \geq 3$, we have

$$c + 1 + o(1)\|u_n\|_{H^1(\mathbb{R}^3)} \geq (p+1)I(u_n) - I'(u_n)u_n \geq \|u_n\|^2_{H^1(\mathbb{R}^3)},$$

it follows that $(u_n)_{n\in\mathbb{N}}$ is bounded. By the definition of u_n and Proposition 4.4 with $\mu = 1$, it holds that

$$c = I(v_0) + \sum_{j=1}^{l} I^\infty(v_j), \tag{4.62}$$

where v_0 is a nonnegative solution of (4.2) and v_j are nonnegative solutions of (4.11) for $1 \leq j \leq l$. Reasoning as in Claim 1 of Theorem 4.3, setting $\mu = 1$ and replacing c_μ, c_μ^∞, Γ, and Γ^∞ with c, c^∞, $\bar{\Gamma}$, and $\bar{\Gamma}^\infty$, respectively, throughout, the statement follows. This concludes the proof. $\square$

As a consequence of the previous result, we have the following theorem, giving the existence of mountain pass solutions for $p \geq 3$.

Theorem 4.4. *[**Non-coercive case: existence of mountain pass solution for** $p \geq 3$] Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative and $\rho(x) \to \rho_\infty > 0$ as $|x| \to +\infty$. Let one of the following conditions hold:*

(i) either $c < c^\infty$,

(ii) or $\rho(x) \leq \rho_\infty$ for all $x \in \mathbb{R}^3$, with strict inequality, $\rho(x) < \rho_\infty$, on some ball $B \subset \mathbb{R}^3$,

where c and (resp.) c^∞ are minimax levels defined in (4.10) and (resp.) (4.16). Then, for any $p \in [3,5)$ there exists a solution, $(u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, of (4.1), whose components are positive functions. In particular, u is a mountain pass critical point of I at level c.

Proof. The regularity and the strong maximum principle imply that the nontrivial and nonnegative critical point, u, of I, found in Proposition 4.5, is strictly positive. For the same reason, $\phi_u > 0$ everywhere. □

4.3.4 Corollary 4.6

We follow up the previous two existence theorems with a result giving the existence of least energy solutions in the non-coercive case. When $p \in (3,5)$ the existence follows relatively straightforwardly using the Nehari characterisation of the mountain pass level, and when $p \in (2,3]$ we use a minimising sequence together with Proposition 4.4 to obtain the result.

Corollary 4.6. *[**Non-coercive case: existence of least energy solution for** $p \in (2,5)$] Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative, $\rho(x) \to \rho_\infty > 0$ as $|x| \to +\infty$, and one of the following conditions holds:*

(i) either $c < c^\infty$,

(ii) or $\rho(x) \leq \rho_\infty$ for all $x \in \mathbb{R}^3$, with strict inequality, $\rho(x) < \rho_\infty$, on some ball $B \subset \mathbb{R}^3$,

where c and (resp.) c^∞ are minimax levels defined in (4.10) and (resp.) (4.16). If $p \in (2,3)$, suppose in addition that $\rho \in W^{1,1}_{loc}(\mathbb{R}^3)$ and $k\rho(x) \leq (x, \nabla\rho)$ for some $k > \frac{-2(p-2)}{(p-1)}$. Then, for all $p \in (2,5)$, there exists a solution, $(u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, of (4.1), whose components are positive functions, such that u is a least energy critical point of I.

Proof. If $p \in (3,5)$, we can use the Nehari characterisation of the mountain pass level (4.37) with $E = H^1$ to see that the mountain pass solution u found in Theorem 4.4 is a least energy solution for I. If $p \in (2,3]$, we set

$$c^* := \inf_{u \in \mathscr{A}} I(u),$$

where

$$\mathscr{A} := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : u \text{ is a nonnegative solution to } (4.2)\},$$

and can show that $\mathscr{A}$ is nonempty and c^* is well-defined using the mountain pass critical points that we found in Theorem 4.4 and Theorem 4.3 when $p = 3$ and $p \in (2,3)$, respectively. It is important to note that when $p = 3$, the critical point, $u \in \mathscr{A}$, that we found in Theorem 4.4 satisfies $I(u) = c$, which implies $c^* \leq c$. Similarly, when $p \in (2,3)$, we can show $c^* \leq c$ using the critical point that we found in Theorem 4.3. Now, for any $p \in (2,3)$, arguing as in the proof of Corollary 4.3, we can show that there exists a bounded sequence $(w_n)_{n\in\mathbb{N}} \subset \mathscr{A}$ such that $I(w_n) \to c^*$ as $n \to +\infty$ and $I'(w_n) = 0$. By applying Proposition 4.4 with $\mu = 1$ to $(w_n)_{n\in\mathbb{N}}$, we can see that

$$c \geq c^* = I(v_0) + \sum_{j=1}^{l} I^\infty(v_j),$$

where v_0 is a nonnegative solution of (4.2) and v_j are nonnegative solutions of (4.11) for $1 \leq j \leq l$. Reasoning as in Claim 1 of Theorem 4.3 with $\mu = 1$ and replacing c_μ, c_μ^∞, Γ, and Γ^∞ with c, c^∞, $\bar{\Gamma}$, and $\bar{\Gamma}^\infty$, respectively, throughout, we can show $I(v_0) = c^*$ and $I'(v_0) = 0$. We note that by Lemma 4.3, it holds that $c^* \geq C > 0$ for some uniform constant $C > 0$, and so it follows that v_0 is a nontrivial least energy critical point of

I. The strict positivity of v_0 and ϕ_{v_0} follows by regularity and the strong maximum principle since they are nonnegative by construction. This concludes the proof. □

Remark 4.3. *By following similar techniques to those used in the proof of Corollary 4.6, we can show that under the same assumptions as this corollary (with obvious modifications to the minimax levels), there exists a least energy solution for $\mathscr{I}$, defined in (4.38), for all $p \in (2,5)$. As in the coercive case, it is not clear if we can select a positive groundstate for $p \in (2,3]$.*

5 Multiplicity Results

In this chapter, we study the existence of multiple solutions to the same nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \rho(x)\phi u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = \rho(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{5.1}$$

with $p \in (2,5)$ and $\rho : \mathbb{R}^3 \to \mathbb{R}$ a nonnegative measurable function. We obtain results in the case of a coercive ρ for $p > 3$ and $p \leq 3$, in the spirit of Ambrosetti and Rabinowitz [6] and Ambrosetti and Ruiz [7], respectively. The results in this chapter are from [31].

5.1 Main multiplicity result: coercive $\rho(x)$ and $p > 3$

In the case of a coercive ρ and for $p > 3$, the existence of infinitely many solutions to the Schrödinger-Poisson system follows relatively straightforwardly using the results of [6]. Namely, the main multiplicity result we obtain is as follows.

Theorem 5.1. *Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative and $\rho(x) \to +\infty$ as $|x| \to +\infty$. Then, for any $p \in (3,5)$, there exists infinitely many distinct pairs of critical points in $E(\mathbb{R}^3)$ for*

$$\mathscr{I}(u) := \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) + \frac{1}{4}\int_{\mathbb{R}^3}\rho\phi_u u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}.$$

Remark 5.1. *We notice that here we deal with critical points of the functional $\mathscr{I}$ rather than the functional I that we defined in (3.16) as we are no longer solely interested in positive solutions.*

In order to prove this theorem, we will need some background material including the notion of the Krasnoselskii-genus and its properties.

5.1.1 Krasnoselskii-genus

Throughout this section we let E be the usual Banach space defined previously and let G be a compact topological group. Following [27], we begin with a number of definitions that we will need before introducing the notion of the Krasnoselskii-genus.

Definition 4. *The set* $\{T(g) : g \in G\}$ *is an isometric representation of* G *on* E *if* $T(g) : E \to E$ *is an isometry for each* $g \in G$ *and the following hold:*

(i) $T(g_1 + g_2) = T(g_1) \circ T(g_2)$ *for all* $g_1, g_2 \in G$

(ii) $T(0) = I$*, where* $I : E \to E$ *is the identity map on* E

(iii) $(g, u) \mapsto T(g)(u)$ *is continuous.*

Definition 5. *A subset* $A \subset E$ *is invariant if* $T(g)A = A$ *for all* $g \in G$*.*

Definition 6. *A mapping* R *between two invariant subsets* A_1 *and* A_2*, namely* $R : A_1 \to A_2$*, is said to be equivariant if* $R \circ T(g) = T(g) \circ R$ *for all* $g \in G$*.*

Definition 7. *We denote the class of all closed and invariant subsets of* E *by* $\mathscr{A}$*. Namely,*

$$\mathscr{A} := \{A \subset E : A \text{ closed}, T(g)A = A \ \forall g \in G\}.$$

Definition 8. *A* G*-index on* E *with respect to* $\mathscr{A}$ *is a mapping* $ind : \mathscr{A} \to \mathbb{N} \cup \{+\infty\}$ *such that the following hold:*

(i) $ind(A) = 0$ *if and only if* $A = \emptyset$*.*

(ii) If $R : A_1 \to A_2$ *is continuous and equivariant, then* $ind(A_1) \le ind(A_2)$*.*

(iii) $ind(A_1 \cup A_2) \le ind(A_1) + ind(A_2)$*.*

(iv) If $A \in \mathscr{A}$ *is compact, then there exists a neighborhood* N *of* A *such that* $N \in \mathscr{A}$ *and* $ind(N) = ind(A)$*.*

With these definitions in place, we are ready to introduce the concept of the Krasnoselskii-genus.

Lemma 5.1. *Let* $G = \mathbb{Z}_2 = \{0, 1\}$ *and define* $T(0) = I$*,* $T(1) = -I$*, where* $I : E \to E$ *is the identity map on* E*. Given any closed and symmetric with respect to the origin subset* $A \in \mathscr{A}$*, define* $\gamma(A) = k \in \mathbb{N}$ *if* k *is the smallest integer such that there exists some odd mapping* $\varphi \in C(A, \mathbb{R}^k \setminus \{0\})$*. Moreover, define* $\gamma(A) = +\infty$ *if no such mapping exists and* $\gamma(\emptyset) = 0$*. Then, the mapping* $\gamma : \mathscr{A} \to \mathbb{N} \cup \{+\infty\}$ *is a* $\mathbb{Z}_2$*-index on* E*, called the Krasnoselskii-genus.*

Proof. See the proof of Proposition 2.1 in [27]. □

The next lemma gives a property of the Krasnoselskii-genus that will be crucial in obtaining our multiplicity result.

Lemma 5.2. *Assume $A \in \mathscr{A}$ is such that $0 \notin A$ and $\gamma(A) \geq 2$. Then, A has infinitely many points.*

Proof. See the proof of Proposition 2.2 in [27]. □

5.1.2 The min-max setting: definition of b_m

We will now define the min-max levels at which we will find critical points of $\mathscr{I}$. In order to do so, we introduce a number of sets, following [6]. Namely, we let

$$\hat{A}_0 = \{u \in E : \mathscr{I}(u) \geq 0\},$$

$$\Gamma_* = \{h \in C(E,E) : h(0) = 0, h \text{ is a homeomorphism of } E \text{ onto } E, h(B_1) \subset \hat{A}_0\},$$

$$\Gamma^* = \{h \in \Gamma_* : h \text{ is odd}\},$$

and

$$\Gamma_m = \{K \subset E : K \text{ is compact and symmetric with respect to the origin and for all } h \in \Gamma^*, \text{ it holds that } \gamma(K \cap h(\partial B_1)) \geq m\}.$$

We are now in position to define the min-max levels as

$$b_m = \inf_{K \in \Gamma_m} \max_{u \in K} \mathscr{I}(u).$$

For what follows, we will also need to define the set of critical points at any level $b > 0$, namely

$$K_b = \{u \in E : \mathscr{I}(u) = b, \mathscr{I}'(u) = 0\}.$$

5.1.3 Proof of Theorem 5.1

Before completing the proof of Theorem 5.1, we recall a vital result from [6].

Theorem 5.2 ([6]). *Let $\mathscr{I} \in C^1(E,\mathbb{R})$ satisfy the following:*

(i) $\mathscr{I}(0) = 0$ *and there exists constants* $R, a > 0$ *such that* $\mathscr{I}(u) \geq a$ *if* $||u||_E = R$

(ii) *If* $(u_n)_{n\in\mathbb{N}} \subset E$ *is such that* $0 < \mathscr{I}(u_n)$, $\mathscr{I}(u_n)$ *bounded above, and* $\mathscr{I}'(u_n) \to 0$, *then* $(u_n)_{n\in\mathbb{N}}$ *possesses a convergent subsequence*

(iii) $\mathscr{I}(u) = \mathscr{I}(-u)$ *for all* $u \in E$

(iv) *For a nested sequence* $E_1 \subset E_2 \subset \cdots$ *of finite dimensional subspaces of* E *of increasing dimension, it holds that* $E_i \cap \hat{A}_0$ *is bounded for each* $i = 1, 2, \ldots$

Then, for each $m \in \mathbb{N}$, *it holds that* $0 < a \leq b_m \leq b_{m+1}$ *and* b_m *is a critical value of* $\mathscr{I}$. *Moreover, if* $b_{m+1} = \cdots = b_{m+r} = b$, *then* $\gamma(K_b) \geq r$.

Proof. See the proof of Theorem 2.8 in [6]. □

We are now in position to prove Theorem 5.1.

Proof of Theorem 5.1. We aim to apply Theorem 5.2 and therefore must verify that $\mathscr{I}$ satisfies assumptions (i)-(iv) of this theorem. Arguing as in Lemma 4.1, we can show that $\mathscr{I}$ satisfies the Mountain Pass Geometry and thus (i) holds. To show (ii), we note that by the assumptions on $(u_n)_{n\in\mathbb{N}}$ and since $p > 3$, it holds that

$$C + 1 + o(1)||u_n||_E \geq (p+1)\mathscr{I}(u_n) - \mathscr{I}'(u_n)u_n \geq ||u_n||^2_{H^1} + \int_{\mathbb{R}^3} \rho \phi_u u^2,$$

for some $C > 0$, and so it follows by standard arguments that $(u_n)_{n\in\mathbb{N}}$ is bounded (see e.g. Lemma 7.14). Then, arguing as in the proof of Claim 1 of Theorem 4.2, we can show that $(u_n)_{n\in\mathbb{N}}$ possesses a convergent subsequence, and thus $\mathscr{I}$ satisfies the Palais-Smale condition. Namely, (ii) holds. Clearly, (iii) holds due to the structure of the functional $\mathscr{I}$. We now must show that (iv) holds. We first notice by straightforward calculations that for any $u \in \partial B_1$, namely for any $u \in E \setminus \{0\}$ such that $||u||_E = 1$, and any for $t > 0$, it holds that

$$
\begin{aligned}
\mathscr{I}(tu) &= \frac{t^2}{2}||u||_{H^1}^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} \rho\phi_u u^2 - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} \\
&= \frac{t^2}{2}\left(||u||_{H^1}^2 + \frac{t^2}{2}\int_{\mathbb{R}^3} \rho\phi_u u^2 - \frac{2t^{p-1}}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\right).
\end{aligned}
$$

We now set

$$
\alpha := ||u||_{H^1}^2, \quad \beta := \frac{1}{2}\int_{\mathbb{R}^3} \rho\phi_u u^2, \quad \gamma := \frac{2}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} > 0,
$$

and look for positive solutions of

$$
\frac{t^2}{2}(\alpha + \beta t^2 - \gamma t^{p-1}) = 0.
$$

Since $p > 3$, it holds that $\alpha + \beta t^2 - \gamma t^{p-1} = 0$ has a unique solution $t = t(u) > 0$. That is, we have shown that for each $u \in \partial B_1$, there exists a unique $t = t(u) > 0$ such that $\mathscr{I}$ satisfies

$$
\begin{aligned}
&\mathscr{I}(t(u)u) = 0 \\
&\mathscr{I}(tu) > 0, \ \forall t < t(u) \\
&\mathscr{I}(tu) < 0, \ \forall t > t(u).
\end{aligned}
$$

We now consider a nested sequence $E_1 \subset E_2 \subset \cdots$ of finite dimensional subspaces of E of increasing dimension. For any $k \in \mathbb{N}$, we set

$$
V_k := \{v \in E : v = tu, \ t \geq 0, \ u \in \partial B_1 \cap E_k\}.
$$

Then, the function $h : E_k \to V_k$ given by

$$
h(z) = t\frac{z}{||z||}, \quad \text{with } t = ||z||
$$

defines a homeomorphism from E_k onto V_k, and so $V_1 \subset V_2 \subset \cdots$ is a nested sequence of finite dimensional subspaces of E of increasing dimension. We also notice that

$$T_k := \sup_{u \in \partial B_1 \cap E_k} t(u) < +\infty$$

since $\partial B_1 \cap E_k$ is compact. So, for all $t > T_k$ and $u \in \partial B_1 \cap E_k$, it holds that $\mathscr{I}(tu) < 0$, and thus $V_k \cap \hat{A}_0$ is bounded. Since this holds for arbitrary $k \in \mathbb{N}$, we have shown that (iv) holds. Hence, we have shown that Theorem 5.2 applies to the functional $\mathscr{I}$. If b_m are distinct for $m = 1, \dots, k$ with $k \in \mathbb{N}$, we obtain k distinct pairs of critical points corresponding to critical levels $0 < b_1 < b_2 < \cdots < b_k$. If $b_{m+1} = \cdots = b_{m+r} = b$, then $\gamma(K_b) \geq r \geq 2$. Moreover, $0 \notin K_b$ since $b > 0 = \mathscr{I}(0)$. Further, K_b is invariant since $\mathscr{I}$ is an invariant functional and K_b is closed since $\mathscr{I}$ satisfies the Palais-Smale condition, and so $K_b \in \mathscr{A}$. Therefore, by Lemma 5.2, K_b possesses infinitely many points. This concludes the proof. □

5.2 Preliminary result: coercive $\rho(x)$ and $p \leq 3$

In this section, we discuss a preliminary multiplicity result in the case of a coercive ρ and $p \leq 3$, as well as some current ongoing work. As usual, when $p \leq 3$, we face the additional difficulty of constructing bounded Palais-Smale sequences. Because of this, we need a more robust approach than given in [6]. Namely, in the case of a coercive ρ and $p \leq 3$, we can borrow from the approach of [7] in order to obtain the following preliminary multiplicity result.

Theorem 5.3. *Suppose $\rho \in C(\mathbb{R}^3) \cap W^{1,1}_{loc}(\mathbb{R}^3)$ is nonnegative and $\rho(x) \to +\infty$ as $|x| \to +\infty$. Suppose further that ρ is homogeneous of degree $\bar{k}$ for some $\bar{k} \in (0, \frac{1}{2}] \cup \{1\} \cup \{\frac{3}{2}\}$, namely $\rho(tx) = t^{\bar{k}} \rho(x)$ for all $t > 0$. Then, for any $p \in (2,3]$, there exists infinitely many distinct pairs of critical points, $\pm u_k \in E(\mathbb{R}^3)$, $k \in \mathbb{N}$, for*

$$\mathscr{I}(u) := \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) + \frac{1}{4} \int_{\mathbb{R}^3} \rho \phi_u u^2 - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1},$$

such that $\mathscr{I}(u_k) \to +\infty$ as $k \to +\infty$.

Remark 5.2. *We note that Theorem 5.3 can actually be proved for $\bar{k} \in (2-p, 0)$ also, however this range of $\bar{k}$ is incompatible with the coercivity assumption, namely $\rho(x) \to +\infty$ as $|x| \to +\infty$.*

Before proving Theorem 5.3, we must recall some vital results from [7] that will enable us to obtain our result.

5.2.1 The abstract min-max setting

The first lemma that we recall will give us an abstract definition of the min-max levels and some properties.

Lemma 5.3 ([7]). *Consider a Banach space E, and a functional $\Phi_\mu : E \to \mathbb{R}$ of the form $\Phi_\mu(u) = \alpha(u) - \mu\beta(u)$, with $\mu > 0$. Suppose that α, $\beta \in C^1$ are even functions, $\lim_{||u||\to+\infty} \alpha(u) = +\infty$, $\beta(u) \geq 0$, and β, β' map bounded sets onto bounded sets. Suppose further that there exists $K \subset E$ and a class $\mathscr{F}$ of compact sets in E such that:*

($\mathscr{F}$.1) $K \subset A$ for all $A \in \mathscr{F}$ and $\sup_{u\in K} \Phi_\mu(u) < c_\mu$, where c_μ is defined as:

$$c_\mu := \inf_{A\in\mathscr{F}} \max_{u\in A} \Phi_\mu(u).$$

($\mathscr{F}$.2) If $\eta \in C([0,1] \times E, E)$ is an odd homotopy such that

- *$\eta(0,\cdot) = I$, where $I : E \to E$ is the identity map on E*
- *$\eta(t,\cdot)$ is a homeomorphism*
- *$\eta(t,x) = x$ for all $x \in K$,*

then $\eta(1,A) \in \mathscr{F}$ for all $A \in \mathscr{F}$.

Then, it holds that the mapping $\mu \mapsto c_\mu$ is non-increasing and left-continuous, and therefore is almost everywhere differentiable.

Proof. See the proof of Lemma 2.2 in [7]. □

With this lemma in place, we can now recall the second vital result from [7]. The following proposition will be used to obtain the boundedness of our Palais-Smale sequences.

Proposition 5.1 ([7]). *Under the hypotheses of the previous lemma, we denote the set of values of μ such that c_μ is differentiable by $\mathscr{J} \subset (0,+\infty)$. Then, for any $\mu \in \mathscr{J}$, there exists a bounded Palais-Smale sequence for Φ_μ at the level c_μ. That is, there* *exists a bounded sequence $(u_n)_{n\in\mathbb{N}} \subset E$ such that $\Phi_\mu(u_n) \to c_\mu$ and $\Phi'_\mu(u_n) \to 0$.*

Proof. See the proof of Proposition 2.3 in [7]. □

The final result we will need is a technical lemma, which we now state and prove.

Lemma 5.4. *Assume a, b, c, d are positive constants, $p > 2$ and $\bar{k} \in (2-p, \frac{1}{2}] \cup \{1\} \cup \{\frac{3}{2}\}$. Define $f : \mathbb{R}_0^+ \to \mathbb{R}$ as*

$$f(t) = at^3 + bt + ct^{3-2\bar{k}} - dt^{2p-1}, \quad t \geq 0.$$

Then, f has a unique critical point corresponding to its maximum.

Proof. We will go through the details for the case $\bar{k} \in (0, \frac{1}{2})$. We begin by computing some derivatives of f. Namely,

$$f'(t) = 3at^2 + b + (3-2\bar{k})ct^{2-2\bar{k}} - (2p-1)dt^{2p-2},$$

$$f''(t) = 6at + (2-2\bar{k})(3-2\bar{k})ct^{1-2\bar{k}} - (2p-1)(2p-2)dt^{2p-3},$$

$$f'''(t) = 6a + (1-2\bar{k})(2-2\bar{k})(3-2\bar{k})ct^{-2\bar{k}} - (2p-1)(2p-2)(2p-3)dt^{2p-4},$$

$$\begin{aligned} f^{(4)}(t) = &(-2\bar{k})(1-2\bar{k})(2-2\bar{k})(3-2\bar{k})ct^{-2\bar{k}-1} \\ &- (2p-1)(2p-2)(2p-3)(2p-4)dt^{2p-5}. \end{aligned}$$

Now, since $\bar{k} \in (0, \frac{1}{2})$ and $p > 2$, it holds that $f'''(t) \to +\infty$ as $t \to 0$ and $f'''(t) \to -\infty$ as $t \to +\infty$. Moreover, it holds that $f^{(4)}(t) = 0$ if and only if

$$t^{2\bar{k}+2p-4} = \frac{(-2\bar{k})(1-2\bar{k})(2-2\bar{k})(3-2\bar{k})c}{(2p-1)(2p-2)(2p-3)(2p-4)d}.$$

Thus, since

$$\frac{(-2\bar{k})(1-2\bar{k})(2-2\bar{k})(3-2\bar{k})c}{(2p-1)(2p-2)(2p-3)(2p-4)d} < 0,$$

it follows that f''' has no positive critical points. Taken together, we have shown f''' is strictly decreasing and so there exists $t_3 > 0$ such that $f'''(t_3) = 0$ and $f'''(t)(t_3 - t) > 0$ for $t \neq t_3$. We now look at f'' and notice that since $f''(0) = 0$ and f'' is increasing for $t < t_3$, it follows that f'' takes positive values at least for $t \in (0, t_3)$. Moreover, $f''(t) \to -\infty$ as $t \to +\infty$ and f'' is decreasing for $t > t_3$, and so there exists $t_2 > t_3$ such that $f''(t_2) = 0$ and $f''(t)(t_2 - t) > 0$ for $t \neq t_2$. Then, since $f'(0) = b > 0$ and $f'(t) \to -\infty$ as $t \to +\infty$, we can repeat these arguments to show that there exists $t_1 > t_2$ such that $f'(t_1) = 0$, namely t_1 is a critical point of f, and $f'(t)(t_1 - t) > 0$ for $t \neq t_1$, namely t_1 is unique. Clearly, t_1 corresponds to a maximum of f since $f(0) = 0$ and $f(t) \to -\infty$ as $t \to +\infty$. The cases $\bar{k} \in (2-p, 0)$, $\bar{k} = 0$, $\bar{k} = \frac{1}{2}$, $\bar{k} = 1$, and $\bar{k} = \frac{3}{2}$ follow by similar analysis on the derivatives of f, with slight modifications.

□

5.2.2 Proof of Theorem 5.3

We are now in position to prove Theorem 5.3.

Proof of Theorem 5.3. Since ρ is coercive, we will work in the usual Banach space $E(\mathbb{R}^3)$. We aim to apply Proposition 5.1 in the case of the perturbed functional $\mathscr{I}_\mu : E(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathscr{I}_\mu(u) := \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) + \frac{1}{4}\int_{\mathbb{R}^3}\rho\phi_u u^2 - \frac{\mu}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}, \quad \mu \in \left[\frac{1}{2}, 1\right].$$

We must verify ($\mathscr{F}$.1) and ($\mathscr{F}$.2). For any $u \in E(\mathbb{R}^3)$ and for any $t > 0$, we set $u_t(x) := u(tx)$. Evaluating $\mathscr{I}_\mu$ at $t^2 u_t$ and using the assumption that ρ is homogeneous of degree $\bar{k}$, we find that

$$\mathscr{I}_\mu(t^2 u_t) = \frac{t^3}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{t}{2}\int_{\mathbb{R}^3}u^2 + \frac{t^3}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(y)\rho(\frac{y}{t})u^2(x)\rho(\frac{x}{t})}{4\pi|x-y|} - \frac{\mu t^{2p-1}}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}$$

$$= \frac{t^3}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{t}{2}\int_{\mathbb{R}^3}u^2 + \frac{t^{3-2\bar{k}}}{4}\int_{\mathbb{R}^3}\rho\phi_u u^2 - \frac{\mu t^{2p-1}}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}.$$

We now set

$$a := \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2,\ b := \frac{1}{2}\int_{\mathbb{R}^3}u^2,\ c := \frac{1}{4}\int_{\mathbb{R}^3}\rho\phi_u u^2,\ d := \frac{\mu}{p+1}\int_{\mathbb{R}^3}|u|^{p+1},$$

and, noting that $a, b, c, d > 0$, consider the polynomial

$$f(t) = at^3 + bt + ct^{3-2\bar{k}} - dt^{2p-1}, \quad t \geq 0.$$

By Lemma 5.4, it follows that for each $u \in \partial B_1$, there exists a unique $t = t(u) > 0$ such that $\mathscr{I}_\mu$ with $\mu = \frac{1}{2}$ satisfies

$$\begin{aligned}
&\mathscr{I}_{\frac{1}{2}}(t(u)^2 u_{t(u)}) = 0\\
&\mathscr{I}_{\frac{1}{2}}(t^2 u_t) > 0,\ \forall t < t(u)\\
&\mathscr{I}_{\frac{1}{2}}(t^2 u_t) < 0,\ \forall t > t(u).
\end{aligned} \tag{5.2}$$

We can now follow Subsection 2.2 in [7]. We include the details for the sake of completeness. We begin by considering a nested sequence $E_1 \subset E_2 \subset \cdots$ of finite dimensional subspaces of E of increasing dimension. For any $k \in \mathbb{N}$, we set

$$V_k := \{v \in E : v = t^2 u_t,\ t \geq 0,\ u \in \partial B_1 \cap E_k\}.$$

Then, the function $h : E_k \to V_k$ given by

$$h(e) = t^2 u_t, \quad \text{with } t = ||e||,\ u = \frac{e}{||e||},$$

defines an odd homeomorphism from E_k onto V_k. We notice that it holds that

$$T_k := \sup_{u \in \partial B_1 \cap E_k} t(u) < +\infty,$$

since $\partial B_1 \cap E_k$ is compact. So, the set

$$A_k := \{v \in E : v = t^2 u_t,\, t \in [0, T_k],\, u \in \partial B_1 \cap E_k\}$$

is compact and symmetric. We now define

$$H := \{g : E \to E : g \text{ is an odd homeomorphism and } g(v) = v \text{ for all } v \in \partial A_k\},$$

and

$$G_k := \{g(A_k) : g \in H\}.$$

We are finally in position to verify ($\mathscr{F}$.1) and ($\mathscr{F}$.2). We take G_k as the class $\mathscr{F}$ and $K = \partial A_k$ and define the min-max levels

$$c_{k,\mu} := \inf_{A \in G_k} \max_{u \in A} \mathscr{I}_\mu(u).$$

Then, since $T_k \geq t(u)$ for all $u \in \partial B_1 \cap E_k$ by definition, it follows from (5.2) that

$$\mathscr{I}_\mu(v) \leq \mathscr{I}_{\frac{1}{2}}(v) \leq 0, \quad \forall v \in \partial A_k,\, \forall \mu \in \left[\frac{1}{2}, 1\right].$$

Moreover, since $G_k \subset G_{k+1}$ for all $k \in \mathbb{N}$, it holds that $c_{k,\mu} \geq c_{k-1,\mu} \geq \cdots \geq c_{1,\mu} > 0$. Taken together, we have shown that

$$\sup_{v \in \partial A_k} \mathscr{I}_\mu(v) \leq 0 < c_{k,\mu},$$

and thus ($\mathscr{F}$.1) is verified. Moreover, for any η given by ($\mathscr{F}$.2) and any $g \in H$, it holds that $\tilde{g} = \eta(1, g)$ belongs to H, and so ($\mathscr{F}$.2) is satisfied.

Since ($\mathscr{F}$.1) and ($\mathscr{F}$.2) are satisfied, Lemma 5.3 applies. Thus, for any $k \in \mathbb{N}$, we denote by $\mathscr{M}_k$ the set of values $\mu \in \left[\frac{1}{2}, 1\right]$ such that the function $\mu \mapsto c_{k,\mu}$ is differentiable. We then let

$$\mathcal{M} := \bigcap_{k\in\mathbb{N}} \mathcal{M}_k.$$

We note that since

$$\left[\frac{1}{2},1\right] \setminus \mathcal{M} = \bigcup_{k\in\mathbb{N}} \left(\left[\frac{1}{2},1\right] \setminus \mathcal{M}_k\right)$$

and $[\frac{1}{2},1] \setminus \mathcal{M}_k$ has zero Lebesgue measure for each k by Lemma 5.3, then it follows that $\left[\frac{1}{2},1\right] \setminus \mathcal{M}$ has zero Lebesgue measure. We can now apply Proposition 5.1 with $\Phi_\mu = \mathscr{I}_\mu$. Namely, for each fixed $k \in \mathbb{N}$ and $\mu \in \mathcal{M}$ we obtain that there exists a bounded sequence $(u_n)_{n\in\mathbb{N}} \subset E$ such that $\mathscr{I}_\mu(u_n) \to c_{k,\mu}$ and $\mathscr{I}_\mu'(u_n) \to 0$. Arguing exactly as in the proof of Claim 1 of Theorem 4.2, we can show that the values $c_{k,\mu}$ are critical levels of $\mathscr{I}_\mu$ for each $k \in \mathbb{N}$ and $\mu \in \mathcal{M}$. We then take k fixed, $(\mu_n)_{n\in\mathbb{N}}$ an increasing sequence in $\mathcal{M}$ such that $\mu_n \to 1$, and $(u_n)_{n\in\mathbb{N}} \subset E$ such that $\mathscr{I}_{\mu_n}(u_n) = c_{k,\mu_n}$ and $\mathscr{I}_{\mu_n}'(u_n) = 0$. We note here that since ρ is homogeneous of degree $\bar{k}$ by assumption, it follows that $\bar{k}\rho(x) = (x, \nabla\rho)$. So, arguing as in Claim 2 of Theorem 4.2, we can show that for each fixed k there exists $u \in E$ such that, up to a subsequence, $u_n \to u$ in E, $\mathscr{I}(u) = \mathscr{I}_1(u) = c_{k,1}$, and $\mathscr{I}'(u) = \mathscr{I}_1'(u) = 0$. Since

$$\mathscr{I}(u) \geq \frac{1}{2}||u||_E^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}, \quad \forall u \in E,$$

and E is compactly embedded in $L^{p+1}(\mathbb{R}^3)$ by Lemma 4.5, we can then argue as in the proof of Lemma 2.7 of [7] to show that $\mathscr{I}(u) = c_{k,1} \to +\infty$ as $k \to +\infty$. This concludes the proof. □

We suspect that the constrained minimisation approach in [61] may help refining the approach we used in the above proof inspired by [7], in order to relax the relation between ρ and $\nabla\rho$ that we imposed in Theorem 5.3. This, as well as obtaining multiplicity results in the case of a non-coercive ρ, is ongoing work.

6 Necessary Conditions for Point Concentration

In this chapter, we focus on the singularly perturbed system

$$\begin{cases} -\varepsilon^2\Delta u + \lambda u + \rho(x)\phi u = |u|^{p-1}u, & x \in \mathbb{R}^3 \\ -\Delta\phi = \rho(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{6.1}$$

with $\rho : \mathbb{R}^3 \to \mathbb{R}$ a nonnegative measurable function, $\lambda \in \mathbb{R}$, and $\lambda > 0$, taking advantage of a shrinking parameter $\varepsilon \sim \hbar \ll 1$ which behaves like the Planck constant in the so-called 'semiclassical limit'. In what is to follow, we are interested in the concentration behaviour of solutions, by which we mean the following.

Definition 9. *Let $(u_\varepsilon, \phi_{u_\varepsilon})$ be a sequence of positive solutions to (6.1). We say that u_ε concentrate at a point x_0 for sufficiently small $\varepsilon > 0$ if for all $\delta > 0$ there exists $\varepsilon_0 > 0$ and $R > 0$ such that $u_\varepsilon(x) \leq \delta$ for $|x - x_0| \geq \varepsilon R$, $\varepsilon < \varepsilon_0$.*

We will prove a necessary condition for the concentration of positive solutions in the semiclassical limit, $\varepsilon \to 0^+$, in both $E(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$ using classical blow-up analysis, uniform decay estimates, and Pohozaev type identities, in the spirit of [63]. The results in this chapter are from [50].

6.1 Theorem 6.1

We first prove a necessary condition for the concentration of positive solutions in $E(\mathbb{R}^3)$.

Theorem 6.1. *[**Necessary conditions in** E] Suppose that $\rho \in C^1(\mathbb{R}^3)$ is nonnegative and $|\nabla\rho(x)| = O(|x|^a e^{b|x|})$ as $|x| \to +\infty$ for some $a > 0$ and some $b \in \mathbb{R}$. Let $p \in [2, 5)$ and let $(u_\varepsilon, \phi_{u_\varepsilon}) \in E(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ be a sequence of positive solutions of (6.1). Assume that u_ε concentrate at a point x_0 for sufficiently small ε. Then, $\nabla\rho(x_0) = 0$.*

Remark 6.1. *Since we deal with concentrating solutions, we use the mean value theorem to control the growth of ρ with the assumption on $\nabla\rho$ in order to apply the dominated convergence theorem in the proof of the theorem (see Claim 5). We note that this assumption is not needed in Theorem 6.2 as we work with a bounded ρ and therefore the application of the dominated convergence theorem is more immediate.*

Remark 6.2. *When $b > 0$ the proof of Theorem 6.1 Claim 5 is sensitive to ε being smaller than the ratio $\frac{\sqrt{\lambda}}{b}$. This ratio arises as the proof consists of balancing the aforementioned growth of ρ and $\nabla\rho$ with the a priori exponential decay of the concentrating solutions in order to apply the dominated convergence theorem.*

Proof of Theorem 6.1. We will break the proof into five claims.

Claim 1. $\sup_{\varepsilon>0} ||u_\varepsilon||_{L^\infty(\mathbb{R}^3)} < +\infty$

We will argue by contradiction. Assume, to the contrary, that there exists a sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ such that $\varepsilon_m \to 0$ as $m \to +\infty$, $u_m := u_{\varepsilon_m}$ solves (6.1) for each m, and it holds

$$||u_m||_{L^\infty(\mathbb{R}^3)} \to +\infty \text{ as } m \to +\infty.$$

Let

$$\alpha_m := \max u_m, \qquad (\alpha_m \to +\infty \text{ as } m \to +\infty),$$
$$\beta_m := \alpha_m^{-(p-1)/2}, \qquad (\beta_m \to 0 \text{ as } m \to +\infty).$$

Define

$$v_m(x) := \frac{1}{\alpha_m} u_m(x_m + \varepsilon_m \beta_m x),$$

where x_m is a global maximum point of u_m. We note that such a point exists because, by regularity theory, u_m are solutions in the classical sense and, moreover, by the concentration assumption, u_m decays to zero uniformly with respect to m. Now, multiplying (6.1) by $\frac{\beta_m^2}{\alpha_m}$, we obtain

$$\begin{aligned}
-\frac{\varepsilon_m^2\beta_m^2}{\alpha_m}\Delta u_m(x_m+\varepsilon_m\beta_m x) + \frac{\beta_m^2}{\alpha_m}\lambda u_m(x_m+\varepsilon_m\beta_m x)& \\
+\beta_m^2\rho(x_m+\varepsilon_m\beta_m x)\phi_{u_m}(x_m+\varepsilon_m\beta_m x)\frac{1}{\alpha_m}u_m(x_m+\varepsilon_m\beta_m x)& \\
=\frac{\beta_m^2}{\alpha_m}u_m^p(x_m+\varepsilon_m\beta_m x).&
\end{aligned}$$

Noting that $\Delta v_m(x) = \varepsilon_m^2\beta_m^2\Delta u_m(x_m+\varepsilon_m\beta_m x)/\alpha_m$ and $\beta_m^2/\alpha_m = 1/\alpha_m^p$, we see that v_m

satisfies

$$-\Delta v_m + \beta_m^2 \lambda v_m + \beta_m^2 \rho(x_m + \varepsilon_m \beta_m x)\phi_{u_m}(x_m + \varepsilon_m \beta_m x) v_m = v_m^p.$$

We further note that

$$\begin{aligned}
\phi_{u_m}(x_m + \varepsilon_m \beta_m x) &= \int_{\mathbb{R}^3} \frac{u_m^2(y)\rho(y)}{4\pi|x_m + \varepsilon_m \beta_m x - y|}\, \mathrm{d}y \\
&= \int_{\mathbb{R}^3} \frac{u_m^2(x_m + \varepsilon_m \beta_m y)\rho(x_m + \varepsilon_m \beta_m y)}{4\pi|x_m + \varepsilon_m \beta_m x - x_m - \varepsilon_m \beta_m y|} \cdot \varepsilon_m^3 \beta_m^3\, \mathrm{d}y \\
&= \varepsilon_m^2 \beta_m^2 \alpha_m^2 \int_{\mathbb{R}^3} \frac{v_m^2(y)\rho(x_m + \varepsilon_m \beta_m y)}{4\pi|x - y|}\, \mathrm{d}y,
\end{aligned}$$

where we have used the change of variables $y \to x_m + \varepsilon_m \beta_m y$ in going from the first to second line. Therefore, v_m satisfies

$$\begin{aligned}
&-\Delta v_m + \beta_m^2 \lambda v_m \\
&\qquad + \beta_m^2 \rho(x_m + \varepsilon_m \beta_m x) \left(\varepsilon_m^2 \beta_m^2 \alpha_m^2 \int_{\mathbb{R}^3} \frac{v_m^2(y)\rho(x_m + \varepsilon_m \beta_m y)}{4\pi|x - y|}\, \mathrm{d}y \right) v_m = v_m^p.
\end{aligned}$$

Namely, since $\beta_m^4 \alpha_m^2 = \alpha_m^{-2(p-1)} \alpha_m^2 = \alpha_m^{4-2p}$ (by the definition of β), we have that v_m satisfies

$$\begin{aligned}
-\Delta v_m = -\varepsilon_m^2 \alpha_m^{4-2p} \rho(x_m + \varepsilon_m \beta_m x) \left(\int_{\mathbb{R}^3} \frac{v_m^2(y)\rho(x_m + \varepsilon_m \beta_m y)}{4\pi|x - y|}\, \mathrm{d}y \right) v_m \\
- \beta_m^2 \lambda v_m + v_m^p.
\end{aligned} \tag{6.2}$$

It is worth noting here that since $\alpha_m \to +\infty$ as $m \to +\infty$, then $\alpha_m^{4-2p} \to 0$ as $m \to +\infty$ for $p > 2$ and $\alpha_m^{4-2p} \to 1$ as $m \to +\infty$ for $p = 2$ [1].

We now fix some compact set K. We notice, by construction, $||v_m||_{L^\infty(\mathbb{R}^3)} = 1$ for all m, and, by assumption, ρ is continuous. We also highlight that due to the con-

[1]This is the only point in which we use the restriction $p \geq 2$.

centration assumption, we have that the sequence of global maximum points x_m is uniformly bounded with respect to m. So, since $v_m^2\rho$ is uniformly bounded in $L^\infty(K)$, then $\int_{\mathbb{R}^3} \frac{v_m^2(y)\rho(x_m+\varepsilon_m\beta_m y)}{4\pi|x-y|}\,dy$ is uniformly bounded in $C^{0,\alpha}(K)$ and consequently, is uniformly bounded in $L^\infty(K)$ (see e.g. p. 260 in [42]; p. 11 in [1]). Thus, the entire right-hand side of (6.2) is uniformly bounded in $L^\infty(K)$ which implies v_m is uniformly bounded in $C^{1,\alpha}(K)$ (see e.g. [35]). It then follows that the right-hand side of (6.2) is uniformly bounded in $C^{0,\alpha}(K)$, and therefore v_m is uniformly bounded in $C^{2,\alpha}(K)$ by Schauder estimates (see e.g. [35]). Namely, for $x, y \in K$, $x \neq y$, and for every m, it holds that

$$|\partial^\beta v_m(x) - \partial^\beta v_m(y)| \leq C_K|x-y|^\alpha, \quad |\beta| \leq 2,$$

for some constant C_K which depends on K but does not depend on m. It follows that uniformly on compact sets and for some $v_0 \in C^2(\mathbb{R}^3)$,

$$\partial^\beta v_m \to \partial^\beta v_0 \text{ as } m \to +\infty, \quad |\beta| \leq 2.$$

Therefore, taking the limit $m \to +\infty$ in (6.2) we get

$$\begin{cases} -\Delta v_0 = v_0^p, & x \in \mathbb{R}^3 \\ v_0(0) = 1, \end{cases}$$

where the second equality has come from the fact that $v_m(0) = u_m(x_m)/\alpha_m = \alpha_m/\alpha_m = 1$ for all m. On the other hand from the equation, by a celebrated result of Gidas-Spruck [34] we infer $v_0 \equiv 0$. So, we have reached a contradiction, and thus

$$\sup_{\varepsilon>0} ||u_\varepsilon||_{L^\infty(\mathbb{R}^3)} < +\infty.$$

Claim 2. *Assume there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that $\varepsilon_k \to 0$ as $k \to +\infty$ and $u_k := u_{\varepsilon_k}$ solves (6.1) for each k. Let $w_k(x) := u_k(x_0+\varepsilon_k x)$, where x_0 is a concentration point for u_k. Then,*

(i) up to a subsequence, $w_k \to$ some w_0 in $C^2_{loc}(\mathbb{R}^3)$,

(ii) $w_0 > 0$.

We begin by proving (i). We first notice that w_k solves

$$\begin{cases} -\Delta w_k + \lambda w_k + \rho(x_0+\varepsilon_k x)\phi_{u_k}(x_0+\varepsilon_k x) w_k = w_k^p, & x \in \mathbb{R}^3 \\ -\Delta \phi_{u_k}(x_0+\varepsilon_k x) = \rho(x_0+\varepsilon_k x) w_k^2, & x \in \mathbb{R}^3. \end{cases} \tag{6.3}$$

We note that

$$\begin{aligned} \phi_{u_k}(x_0+\varepsilon_k x) &= \int_{\mathbb{R}^3} \frac{u_k^2(y)\rho(y)}{4\pi|x_0+\varepsilon_k x - y|}\,\mathrm{d}y \\ &= \int_{\mathbb{R}^3} \frac{u_k^2(x_0+\varepsilon_k y)\rho(x_0+\varepsilon_k y)}{4\pi|x_0+\varepsilon_k x - x_0 - \varepsilon_k y|}\cdot \varepsilon_k^3\,\mathrm{d}y \\ &= \varepsilon_k^2 \int_{\mathbb{R}^3} \frac{w_k^2(y)\rho(x_0+\varepsilon_k y)}{4\pi|x-y|}\,\mathrm{d}y, \end{aligned}$$

where we have used the change of variables $y \to x_0 + \varepsilon_k y$ in going from the first to second line. So, w_k solves

$$-\Delta w_k = -\lambda w_k - \rho(x_0+\varepsilon_k x)\left(\varepsilon_k^2 \int_{\mathbb{R}^3} \frac{w_k^2(y)\rho(x_0+\varepsilon_k y)}{4\pi|x-y|}\,\mathrm{d}y\right) w_k + w_k^p. \tag{6.4}$$

We now once again fix some compact set K. We notice that, by Claim 1, $\sup_{k>0}||w_k||_{L^\infty(\mathbb{R}^3)} < +\infty$, and, by assumption, ρ is continuous. So, since $w_k^2\rho$ is uniformly bounded in $L^\infty(K)$, then $\int_{\mathbb{R}^3} \frac{w_k^2(y)\rho(x_0+\varepsilon_k y)}{4\pi|x-y|}\,\mathrm{d}y$ is uniformly bounded in $C^{0,\alpha}(K)$ and thus, is uniformly bounded in $L^\infty(K)$ (see e.g. p. 260 in [42]; p. 11 in [1]). Therefore, the right-hand side of (6.4) is uniformly bounded in $L^\infty(K)$ which implies w_k is uniformly bounded in $C^{1,\alpha}(K)$ (see e.g. [35]). It follows that the right-hand side of (6.4) is uniformly bounded in $C^{0,\alpha}(K)$, and thus, by Schauder estimates, we have that w_k is uniformly bounded in $C^{2,\alpha}(K)$ (see e.g. [35]). Since this holds for every compact set contained in $\mathbb{R}^3$, arguing the same way as in Claim 1, it follows that uniformly on compact sets and for some $w_0 \in C^2(\mathbb{R}^3)$,

$$\partial^\beta w_k \to \partial^\beta w_0 \text{ as } k \to +\infty, \quad |\beta| \le 2.$$

Therefore, taking the limit $k \to +\infty$ in (6.4), we have

$$-\Delta w_0 + \lambda w_0 = w_0^p, \quad x \in \mathbb{R}^3. \tag{6.5}$$

We now aim to prove (ii). Let x_k be a maximum point of u_k. Since u_k is a solution to (6.1), we have that

$$-\varepsilon_k^2 \Delta u_k(x_k) + \lambda u_k(x_k) + \rho(x_k)\phi_{u_{x_k}}(x_k)u_k(x_k) = u_k^p(x_k).$$

Noting that $\Delta u_k(x_k) \leq 0$ since x_k is a maximum point of u_k, we see that

$$[\lambda + \rho(x_k)\phi_{u_{x_k}}(x_k)]u_k(x_k) \leq u_k^p(x_k),$$

and so

$$u_k(x_k) \geq [\lambda + \rho(x_k)\phi_{u_{x_k}}(x_k)]^{\frac{1}{p-1}} \geq \lambda^{\frac{1}{p-1}} > 0. \tag{6.6}$$

Therefore, the local maximum values of u_k, and hence of w_k, are greater than or equal to $\lambda^{\frac{1}{p-1}}$, and since $w_k \to w_0$ in $C^2_{\text{loc}}(\mathbb{R}^3)$, then $w_0 \not\equiv 0$. In particular, this and (6.5), imply that $w_0 > 0$ by the strong maximum principle.

Claim 3. *For large k, it holds that* $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{w_k^2(y)\rho(x_0+\varepsilon_k y)w_k^2(x)\nabla\rho(x_0+\varepsilon_k x)}{4\pi|x-y|}\,\mathrm{d}y\,\mathrm{d}x = 0.$

We first recall that w_k, as defined in Claim 2, solves (6.3). Multiplying the first equation in (6.3) by ∇w_k and integrating on $B_R(0)$, we get

$$\begin{aligned} 0 = &\int_{B_R} \Delta w_k \nabla w_k \,\mathrm{d}x - \int_{B_R} \lambda \frac{\nabla w_k^2}{2}\,\mathrm{d}x - \frac{1}{2}\int_{B_R} \nabla(\rho(x_0+\varepsilon_k x)\phi_{u_k}(x_0+\varepsilon_k x)w_k^2)\,\mathrm{d}x \\ &+ \frac{\varepsilon_k}{2}\int_{B_R} \rho(x_0+\varepsilon_k x)\nabla\phi_{u_k}(x_0+\varepsilon_k x)w_k^2\,\mathrm{d}x \\ &\quad + \frac{\varepsilon_k}{2}\int_{B_R} \nabla\rho(x_0+\varepsilon_k x)\phi_{u_k}(x_0+\varepsilon_k x)w_k^2\,\mathrm{d}x + \int_{B_R} \frac{\nabla w_k^{p+1}}{p+1}\,\mathrm{d}x. \end{aligned}$$

By using the divergence theorem and rearranging terms, this becomes

$$
\begin{aligned}
&\frac{\varepsilon_k}{2}\int_{B_R}\nabla\rho(x_0+\varepsilon_k x)\phi_{u_k}(x_0+\varepsilon_k x)w_k^2\,\mathrm{d}x \\
&\qquad=\int_{\partial B_R}\left(\lambda\frac{w_k^2}{2}\nu-\frac{w_k^{p+1}}{p+1}\nu+\frac{1}{2}\rho(x_0+\varepsilon_k x)\phi_{u_k}(x_0+\varepsilon_k x)w_k^2\nu\right)\mathrm{d}\sigma \qquad (6.7)\\
&\qquad\quad-\frac{\varepsilon_k}{2}\int_{B_R}\rho(x_0+\varepsilon_k x)\nabla\phi_{u_k}(x_0+\varepsilon_k x)w_k^2\,\mathrm{d}x-\int_{B_R}\Delta w_k\nabla w_k\,\mathrm{d}x,
\end{aligned}
$$

where ν is the exterior normal field on B_R. We now focus on the second integral on the right-hand side of this equality. We begin by noting that if we multiply the second equation in (6.3) by $\nabla\phi_{u_k}(x_0+\varepsilon_k x)$ and integrate on $B_R(0)$, we get

$$
\begin{aligned}
&-\int_{B_R}\rho(x_0+\varepsilon_k x)\nabla\phi_{u_k}(x_0+\varepsilon_k x)w_k^2\,\mathrm{d}x \\
&\qquad\qquad=\int_{B_R}\Delta\phi_{u_k}(x_0+\varepsilon_k x)\nabla\phi_{u_k}(x_0+\varepsilon_k x)\,\mathrm{d}x.
\end{aligned}
\qquad (6.8)
$$

Moreover, using the divergence theorem, we see that

$$
\begin{aligned}
&\frac{\varepsilon_k}{2}\int_{B_R}\Delta\phi_{u_k}(x_0+\varepsilon_k x)\frac{\partial}{\partial x_i}\phi_{u_k}(x_0+\varepsilon_k x)dx \\
&\quad=\frac{1}{2}\int_{B_R}\operatorname{div}\left(\nabla\phi_{u_k}(x_0+\varepsilon_k x)\frac{\partial}{\partial x_i}\phi_{u_k}(x_0+\varepsilon_k x)\right)\mathrm{d}x \\
&\quad\quad-\frac{1}{2}\int_{B_R}\nabla\phi_{u_k}(x_0+\varepsilon_k x)\frac{\partial}{\partial x_i}(\nabla\phi_{u_k}(x_0+\varepsilon_k x))\,\mathrm{d}x \\
&\quad=\frac{1}{2}\int_{\partial B_R}\left(\frac{\partial\phi_{u_k}(x_0+\varepsilon_k x)}{\partial\nu}\frac{\partial}{\partial x_i}\phi_{u_k}(x_0+\varepsilon_k x)-\frac{1}{2}|\nabla\phi_{u_k}(x_0+\varepsilon_k x)|^2\nu_i\right)\mathrm{d}\sigma. \qquad (6.9)
\end{aligned}
$$

Therefore, combining (6.8) and (6.9), we obtain

$$
\begin{aligned}
&-\frac{\varepsilon_k}{2}\int_{B_R}\rho(x_0+\varepsilon_k x)\nabla\phi_{u_k}(x_0+\varepsilon_k x)w_k^2\,\mathrm{d}x \qquad (6.10)\\
&\qquad=\frac{1}{2}\int_{\partial B_R}\left(\nabla\phi_{u_k}(x_0+\varepsilon_k x)\frac{\partial\phi_{u_k}(x_0+\varepsilon_k x)}{\partial\nu}-\frac{1}{2}|\nabla\phi_{u_k}(x_0+\varepsilon_k x)|^2\nu\right)\mathrm{d}\sigma.
\end{aligned}
$$

Turning our attention to the third integral on the right-hand side of (6.7) and by arguing in a similar way as above, we can show that

$$\int_{B_R} \Delta w_k \nabla w_k \,\mathrm{d}x = \int_{\partial B_R} \left(\nabla w_k \frac{\partial w_k}{\partial \nu} - \frac{1}{2}|\nabla w_k|^2 \nu \right) \mathrm{d}\sigma. \tag{6.11}$$

Therefore, using (6.10) and (6.11), we see that (6.7) becomes

$$\begin{aligned} \frac{\varepsilon_k}{2} \int_{B_R} \nabla \rho(x_0+\varepsilon_k x)\phi_{u_k}(x_0+\varepsilon_k x) w_k^2 \,\mathrm{d}x \\ = \int_{\partial B_R} \Bigg(\lambda \frac{w_k^2}{2}\nu - \frac{w_k^{p+1}}{p+1}\nu + \frac{1}{2}\rho(x_0+\varepsilon_k x)\phi_{u_k}(x_0+\varepsilon_k x) w_k^2 \nu \\ + \frac{1}{2}\nabla \phi_{u_k}(x_0+\varepsilon_k x)\frac{\partial \phi_{u_k}(x_0+\varepsilon_k x)}{\partial \nu} - \frac{1}{4}|\nabla \phi_{u_k}(x_0+\varepsilon_k x)|^2 \nu \\ - \nabla w_k \frac{\partial w_k}{\partial \nu} + \frac{1}{2}|\nabla w_k|^2 \nu \Bigg) \mathrm{d}\sigma. \end{aligned} \tag{6.12}$$

Call the integral on the right-hand side of this equation I_R. Then,

$$\begin{aligned} |I_R| \le& \int_{\partial B_R} \Bigg(\lambda \frac{w_k^2}{2} + \frac{w_k^{p+1}}{p+1} + \frac{1}{2}\rho(x_0+\varepsilon_k x)\phi_{u_k}(x_0+\varepsilon_k x) w_k^2 \\ &+ \frac{1}{2}|\nabla \phi_{u_k}(x_0+\varepsilon_k x)|^2 + \frac{1}{4}|\nabla \phi_{u_k}(x_0+\varepsilon_k x)|^2 + |\nabla w_k|^2 + \frac{1}{2}|\nabla w_k|^2 \Bigg) \mathrm{d}\sigma \\ \le& \frac{3}{2}\int_{\partial B_R} \Bigg(\lambda w_k^2 + w_k^{p+1} + \rho(x_0+\varepsilon_k x)\phi_{u_k}(x_0+\varepsilon_k x) w_k^2 \\ &+ |\nabla \phi_{u_k}(x_0+\varepsilon_k x)|^2 + |\nabla w_k|^2 \Bigg) \mathrm{d}\sigma. \end{aligned}$$

So,

$$\begin{aligned} \int_0^{+\infty} |I_R| \le& \int_0^{+\infty} \frac{3}{2}\int_{\partial B_R} \Bigg(\lambda w_k^2 + w_k^{p+1} + \rho(x_0+\varepsilon_k x)\phi_{u_k}(x_0+\varepsilon_k x) w_k^2 \\ &+ |\nabla \phi_{u_k}(x_0+\varepsilon_k x)|^2 + |\nabla w_k|^2 \Bigg) \mathrm{d}\sigma \,\mathrm{d}R \\ =& \frac{3}{2}\int_{\mathbb{R}^3} \Bigg(\lambda w_k^2 + w_k^{p+1} + \rho(x_0+\varepsilon_k x)\phi_{u_k}(x_0+\varepsilon_k x) w_k^2 \end{aligned}$$

$$+ |\nabla \phi_{u_k}(x_0 + \varepsilon_k x)|^2 + |\nabla w_k|^2 \Big) \,\mathrm{d}x$$

$$< +\infty \text{ for each } k,$$

since w_k is a solution to (6.3). Thus, for each fixed k, there exists a sequence $R_m \to +\infty$ as $m \to +\infty$ such that $I_{R_m} \to 0$ as $m \to +\infty$. Letting $R = R_m \to +\infty$ in (6.12) yields

$$\begin{aligned} 0 &= \frac{\varepsilon_k}{2} \int_{\mathbb{R}^3} \nabla \rho(x_0 + \varepsilon_k x) \phi_{u_k}(x_0 + \varepsilon_k x) w_k^2 \,\mathrm{d}x \\ &= \frac{\varepsilon_k}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\varepsilon_k^2 w_k^2(y) \rho(x_0 + \varepsilon_k y) w_k^2(x) \nabla \rho(x_0 + \varepsilon_k x)}{4\pi |x-y|} \,\mathrm{d}y \,\mathrm{d}x. \end{aligned}$$

Since this holds for each fixed k, we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w_k^2(y) \rho(x_0 + \varepsilon_k y) w_k^2(x) \nabla \rho(x_0 + \varepsilon_k x)}{4\pi |x-y|} \,\mathrm{d}y \,\mathrm{d}x = 0. \tag{6.13}$$

Claim 4. *There exists $R_0 > 0$ and $C > 0$ such that, for k sufficiently large, $w_k(x) \leq C|x|^{-1} e^{-\frac{\sqrt{\lambda}}{2}|x|}$ for all $|x| \geq R_0$.*

We first note that, by the concentration assumption, it holds that $w_k \to 0$ as $|x| \to +\infty$. Namely, there exists $R_0 > 0$, $K > 0$ such that

$$w_k \leq \left(\frac{\lambda}{2}\right)^{\frac{1}{p-1}}, \quad \forall |x| \geq R_0, \, \forall k \geq K. \tag{6.14}$$

It follows that

$$w_k^p \leq \frac{\lambda}{2} w_k, \quad \forall |x| \geq R_0, \, \forall k \geq K,$$

and therefore, since w_k solves (6.3), we have, for all $|x| \geq R_0$ and for all $k \geq K$,

$$-\Delta w_k + \lambda w_k \leq -\Delta w_k + (\lambda + \rho(x_0 + \varepsilon_k x) \phi_{u_k}(x_0 + \varepsilon_k x)) w_k = w_k^p \leq \frac{\lambda}{2} w_k. \tag{6.15}$$

Namely, it holds that

$$-\Delta w_k \leq -\frac{\lambda}{2} w_k, \quad \forall\, |x| \geq R_0,\ \forall k \geq K. \tag{6.16}$$

Now, define

$$\omega(x) := C|x|^{-1} e^{-\frac{\sqrt{\lambda}}{2}|x|}, \quad \text{where } C := \left(\frac{\lambda}{2}\right)^{\frac{1}{p-1}} R_0\, e^{\frac{\sqrt{\lambda}}{2} R_0},$$

Then, using this definition and (6.14), we see that

$$w_k(x) \leq \left(\frac{\lambda}{2}\right)^{\frac{1}{p-1}} = \omega(x), \quad \text{for } |x| = R_0,\ \forall k \geq K. \tag{6.17}$$

It can also be checked that,

$$\Delta\omega \leq \frac{\lambda}{4}\omega, \quad \text{for } |x| \neq 0. \tag{6.18}$$

We then define $\bar{\omega}_k(x) := w_k(x) - \omega(x)$. By (6.17) it holds that

$$\bar{\omega}_k(x) \leq 0, \quad \text{for } |x| = R_0,\ \forall k \geq K. \tag{6.19}$$

Moreover, using (6.16) and (6.18), it holds that

$$-\Delta\bar{\omega}_k + \frac{\lambda}{2}\bar{\omega}_k \leq 0, \quad \forall\, |x| \geq R_0,\ \forall k \geq K. \tag{6.20}$$

and

$$\lim_{|x| \to +\infty} \bar{\omega}_k(x) = 0. \tag{6.21}$$

Thus, by the maximum principle on unbounded domains (see e.g. [14]), it follows that,

$$w_k(x) \leq C|x|^{-1} e^{-\frac{\sqrt{\lambda}}{2}|x|}, \quad \forall\, |x| \geq R_0,$$

for k sufficiently large.

Claim 5. $\rho(x_0)\nabla\rho(x_0) = 0$.

We first pick a uniform large constant $C > 0$ such that for all $x \in \mathbb{R}^3$ and large k it holds that

$$w_k(x) \leq \tilde{w}(x) := C(1+|x|)^{-1} e^{-\frac{\sqrt{\lambda}}{2}|x|}. \tag{6.22}$$

We now highlight the fact that due to the concentration assumption, from now on, we can take k large enough and a suitable $\varepsilon_1 > 0$ such that

$$\varepsilon_k < \varepsilon_1 < \min\left\{\varepsilon_0, \frac{\sqrt{\lambda}}{b}\right\}, \quad \text{if } b > 0,$$

and simply

$$\varepsilon_k < \varepsilon_0, \quad \text{if } b \leq 0,$$

where $\varepsilon_0 > 0$ is as in Definition 9. We assume that $b > 0$ as the case $b \leq 0$ is easier and requires only obvious modifications. By the growth assumption on ρ, there exists a uniform constant $C_1 > 0$ such that for all $x \in \mathbb{R}^3$,

$$|\nabla\rho(x_0 + \varepsilon_k x)| \leq C_1(1+|x|)^a e^{b\varepsilon_1|x|} =: g(x).$$

By the mean value theorem we have

$$|\rho(x_0 + \varepsilon_k y)| \leq |\varepsilon_k y||\nabla\rho(x_0 + \theta(\varepsilon_k y))| + |\rho(x_0)|,$$

for some $\theta \in [0,1]$. Combining this with the estimate on $|\nabla\rho(x_0 + \theta(\varepsilon_k y))|$, it follows that for some uniform constant $C_2 > 0$ and for all $y \in \mathbb{R}^3$,

$$|\rho(x_0+\varepsilon_k y)| \le C_2|y|(1+|y|)^a e^{b\varepsilon_1|y|} + |\rho(x_0)| =: f(y).$$

Therefore, putting everything together, we have that, for k sufficiently large,

$$\left|\frac{w_k^2(y)\rho(x_0+\varepsilon_k y)w_k^2(x)\nabla\rho(x_0+\varepsilon_k x)}{(x-y)}\right| \le \frac{\tilde{w}^2(y)f(y)\tilde{w}^2(x)g(x)}{|x-y|}. \tag{6.23}$$

The right hand side is a uniform $L^1(\mathbb{R}^6)$ bound. In fact, using for instance the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\tilde{w}^2(y)f(y)\tilde{w}^2(x)g(x)}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x\right| &\lesssim ||\tilde{w}^2 f||_{L^{6/5}(\mathbb{R}^3)}||\tilde{w}^2 g||_{L^{6/5}(\mathbb{R}^3)} \\ &< +\infty, \end{aligned} \tag{6.24}$$

as the choice of ε_1 implies that $\tilde{w}^2 f, \tilde{w}^2 g \in L^{6/5}(\mathbb{R}^3)$. We now let $k \to +\infty$ in (6.13), and note that by (6.23), (6.24), Claim 2, and the assumption that $\rho \in C^1(\mathbb{R}^3)$, we can use the dominated convergence theorem to obtain

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{w_0^2(y)\rho(x_0)w_0^2(x)\nabla\rho(x_0)}{4\pi|x-y|}\,\mathrm{d}y\,\mathrm{d}x = 0.$$

Then, since $w_0 > 0$ by Claim 2, we have that

$$\rho(x_0)\nabla\rho(x_0) = 0.$$

Since ρ is nonnegative, any zero is a global minimum, and so we have $\nabla\rho(x_0) = 0$. □

6.2 Theorem 6.2

We follow up Theorem 6.1 with a similar result on necessary conditions for concentration of solutions in $H^1(\mathbb{R}^3)$.

Theorem 6.2. *[**Necessary conditions in** H^1] Suppose that $\rho \in C^1(\mathbb{R}^3)$ is nonnegative and that $\rho, \nabla\rho$ are bounded. Let $p \in [2,5)$ and let $(u_\varepsilon, \phi_{u_\varepsilon}) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ be a sequence of positive solutions of (6.1). Assume that u_ε concentrate at a point x_0 for sufficiently small ε. Then, $\nabla\rho(x_0) = 0$.*

Remark 6.3. *It is possible to relax the global boundedness assumption on ρ and/or on $\nabla\rho$ when working in $H^1(\mathbb{R}^3)$, if we make a growth assumption on $\nabla\rho$. Namely, if we work in $H^1(\mathbb{R}^3)$ and have adequate local integrability on ρ to ensure*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y < +\infty,$$

typically identified using the Hardy-Littlewood-Sobolev inequality, the statement of the theorem and the proof is identical to that of Theorem 6.1.

Proof of Theorem 6.2. The proof closely follows that of Theorem 6.1. We assert that the same five claims as were made in the proof of Theorem 6.1 hold, and will only highlight the differences in the proofs of these claims. The proof of Claim 1 and Claim 2 follow similarly as in Theorem 6.1, however since ρ is both continuous and globally bounded in this case, we do not need to fix a specific compact set K in the regularity arguments, but instead it follows directly that v_m is uniformly bounded in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$ and w_k is uniformly bounded in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$. The proof of Claim 3 and Claim 4 follow exactly as in Theorem 6.1. To prove Claim 5, we define the exponentially decaying function $\tilde{w}$ as in (6.22) and since ρ and $\nabla\rho$ are bounded, we have, for k sufficiently large,

$$\left|\frac{w_k^2(y)\rho(x_0+\varepsilon_k y)w_k^2(x)\nabla\rho(x_0+\varepsilon_k x)}{(x-y)}\right| \lesssim \frac{\tilde{w}^2(y)\tilde{w}^2(x)}{|x-y|} \in L^1(\mathbb{R}^6).$$

This is enough to conclude the proof as in Theorem 6.1 using the dominated convergence theorem. □

Remark 6.4. *In the proof of both Theorem 6.1 and Theorem 6.2, one actually finds the condition $\rho(x_0)\nabla\rho(x_0) = 0$. We believe that this may be a necessary condition in the case ρ is allowed to change sign on a small region.*

7 Sufficient Conditions for Point Concentration

In this chapter, we once again focus on the singularly perturbed nonlinear Schrödinger-Poisson problem

$$\begin{cases} -\varepsilon^2\Delta u + u + \rho(x)\phi u = |u|^{p-1}u, & x \in \mathbb{R}^3 \\ -\Delta\phi = \rho(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{7.1}$$

with $\rho : \mathbb{R}^3 \to \mathbb{R}$ a nonnegative measurable function and $\varepsilon \ll 1$. As a natural next step, in the spirit of del Pino and Felmer [30], we are interested in finding sufficient conditions on ρ to guarantee concentration of positive solutions in the semiclassical limit $\varepsilon \to 0^+$. Based on related literature on the nonlinear Schrödinger equation, as well as the necessary conditions that we obtained in the previous chapter, we formulate the following conjecture.

Conjecture 7.1. *Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative and $p \in (2,5)$. Assume that $\Lambda \subset \mathbb{R}^3$ is a bounded open set satisfying*

$$\rho(x_0) := \inf_{x\in\Lambda} \rho(x) < \inf_{x\in\partial\Lambda} \rho(x).$$

Then, there exists a family of solutions $(u_\varepsilon, \phi_{u_\varepsilon})$ of (7.1), whose components are positive functions, such that u_ε concentrate at the point x_0 for sufficiently small ε.

We will discuss the method that we will attempt to adapt in order to prove this conjecture, as well as the preliminary results that we have obtained thus far. The results in this chapter are from [51].

7.1 Model problem: nonlinear Schrödinger equation

Before going into more details about the partial results that we have obtained in the case of the nonlinear Schrödinger-Poisson system (7.1), we will first look at sufficient conditions for concentration of solutions to the nonlinear Schrödinger equation with potentials. The seminal paper on this subject is credited to Floer and Weinstein [33]. In [3] and [30], sufficient conditions were found on a 'nice' potential V, using perturbative and penalisation approaches, respectively. More general and/or different assumptions on the potential V have also been considered, see for example [4], [8], and [20]. We will restrict our discussion to the penalisation method of del Pino and Felmer [30], as our Banach space setting will be most suited to this method. Although the results that we cover in this section are well-established, we include them for the reader's convenience in order to illustrate the method that we aim to adapt in the case of the nonlinear Schrödinger-Poisson system (7.1), as well as to be able to later highlight the technical obstacles in doing so. Thus, in this section we focus on the problem

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = |u|^{p-1}u, & x \in \mathbb{R}^3 \\ u > 0, & x \in \mathbb{R}^3. \end{cases} \tag{7.2}$$

where $\varepsilon > 0$ is a small parameter, $p \in (1,5)$ and $V : \mathbb{R}^3 \to \mathbb{R}$ is a positive potential bounded away from zero, namely $\inf_{x\in\mathbb{R}^3} V(x) > 0$. The main concentration result that we will discuss is as follows.

Theorem 7.1 ([30]). *Assume that V is a positive locally Hölder continuous potential which is bounded away from zero and that $\Lambda \subset \mathbb{R}^3$ is a bounded open set satisfying*

$$V_0 := \inf_{x\in\Lambda} V(x) < \inf_{x\in\partial\Lambda} V(x).$$

Then, there exist $\varepsilon_0 > 0$ and a family of solutions $\{u_\varepsilon \in H^1(\mathbb{R}^3) : 0 < \varepsilon < \varepsilon_0\}$ of (7.2) with the property that each u_ε possesses a single maximum point x_ε such that $V(x_\varepsilon) \to V_0$ as $\varepsilon \to 0$. Moreover,

$$\lim_{\substack{\varepsilon\to 0\\ R\to+\infty}} ||u_\varepsilon||_{L^\infty(\Lambda\setminus B(x_\varepsilon,\varepsilon R))} = 0,$$

and the limiting profile is given by

$$u_\varepsilon(x) = v\left(\frac{x - x_\varepsilon}{\varepsilon}\right) + w_\varepsilon(x),$$

where v is the unique positive radial solution of

$$-\Delta v + V_0 v = |v|^{p-1}v, \quad x \in \mathbb{R}^3,$$

and $w_\varepsilon \to 0$ in $C^2_{loc}(\mathbb{R}^3)$ and in $L^\infty(\mathbb{R}^3)$ as $\varepsilon \to 0$.

While we will not go through a precise proof of this result, we will discuss the crucial steps that are required to do so in the forthcoming sections.

Penalisation Scheme in a Nutshell

The main idea of the approach of del Pino and Felmer [30] is to replace the nonlinearity outside of some compact set by an appropriate penalised nonlinearity (Section 7.1.1). Doing so makes the penalised problem compact, whereas the original problem is, in general, not. Due to this compactness, the existence of mountain pass solutions to the penalised problem can be deduced. The limit problem is then introduced (Section 7.1.2), which will later be used to obtain asymptotic estimates and ultimately to give the limiting profile of solutions to the original problem. The aim is to prove that the solutions of the penalised problem are small outside of the compact set so that they solve the original problem. To do so, estimates of the mountain pass level and asymptotic analyses of solutions to the penalised problem are performed (Section 7.1.3). Barrier functions may be used in order to obtain sharper estimates of decay of solutions to the penalised problem. Finally, the maximum/comparison principle is used to prove that the solutions of the penalised problem solve the original problem, and the asymptotic estimates are used to prove the concentration behaviour of such solutions (Section 7.1.4).

7.1.1 Penalised problem

As alluded to above, the first step is to introduce the penalised nonlinearity $g : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$g(x,s) := \chi_\Lambda(x)s^p + (1-\chi_\Lambda(x)) \min\{\kappa V(x)s, s^p\},$$

where $\kappa \in (0,1)$ and $\Lambda \subset \mathbb{R}^3$ is a bounded open set satisfying

$$V_0 := \inf_{x\in\Lambda} V(x) < \inf_{x\in\partial\Lambda} V(x).$$

This enables the penalised problem

$$-\varepsilon^2 \Delta u + V(x)u = g(x,u), \quad x \in \mathbb{R}^3, \tag{7.3}$$

to be defined. Weak solutions of this problem are critical points of the functional $\mathscr{J}_\varepsilon : H^1_V \to \mathbb{R}^3$ given by

$$\mathscr{J}_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^3} \left(\varepsilon^2|\nabla u|^2 + V(x)|u|^2\right) - \int_{\mathbb{R}^3} G(x,u),$$

where

$$G(x,u) = \int_0^u g(x,\sigma)\,\mathrm{d}\sigma.$$

Here, the Hilbert space H^1_V is defined as

$$H^1_V := \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)|u|^2 < +\infty\right\},$$

endowed with norm

$$||u||_\varepsilon = \left(\int_{\mathbb{R}^3} \left(\varepsilon^2|\nabla u|^2 + V(x)|u|^2\right)\right)^{\frac{1}{2}}.$$

The advantage of introducing the penalisation is that the penalised problem (7.3) is compact, and so it is possible to show that $\mathscr{J}_\varepsilon$ satisfies both the Mountain Pass Geometry and Palais-Smale condition. Therefore, by the Mountain Pass Theorem, the following existence result holds for the penalised problem.

Lemma 7.1. *For every $\varepsilon > 0$, the functional $\mathscr{J}_\varepsilon$ has a critical point $u_\varepsilon \in H^1(\mathbb{R}^3)$ at the min-max level*

$$c_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} \mathscr{J}_\varepsilon(\gamma(t)),$$

where

$$\Gamma_\varepsilon := \{\gamma \in C([0,1], H^1_V) : \gamma(0) = 0,\ \mathscr{J}_\varepsilon(\gamma(1)) < 0\}.$$

In particular, u_ε is a weak positive solution of (7.3).

7.1.2 Limit problem

It is then important to introduce the limit problem associated with (7.2). Namely, for $a > 0$, the limit problem is

$$\begin{cases} -\Delta u + au = |u|^{p-1}u, & x \in \mathbb{R}^3 \\ u > 0, & x \in \mathbb{R}^3. \end{cases} \tag{7.4}$$

Weak solutions are critical points of the functional $\mathscr{I}_a : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathscr{I}_a(u) := \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + au^2\right) - \frac{1}{p+1}\int_{\mathbb{R}^3} u^{p+1}.$$

Any nontrivial critical point $u \in H^1(\mathbb{R}^3)$ of $\mathscr{I}_a$ belongs to the Nehari manifold

$$\mathscr{N}_a := \{u \in H^1(\mathbb{R}^3) : u \not\equiv 0 \text{ and } \mathscr{I}_a'(u)(u) = 0\}.$$

A function $u \in H^1(\mathbb{R}^3)$ is a least-energy solution of (7.4) if

$$\mathscr{I}_a(u) = \inf_{v \in \mathscr{N}_a} \mathscr{I}_a(v).$$

The ground-energy function $\mathscr{E} : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$\mathscr{E}(a) := \inf_{v \in \mathscr{N}_a} \mathscr{I}_a(v),$$

and the concentration function $\mathscr{C} : \mathbb{R}^3 \to (0, +\infty]$ by

$$\mathscr{C}(\xi) = \mathscr{E}(V(\xi)).$$

The following lemma gives the existence of a solution to the limit problem and its relation with the ground-energy function.

Lemma 7.2. *For every $a > 0$, $\mathscr{E}(a)$ is a critical value of $\mathscr{I}_a$. If $u \in \mathscr{N}_a$ and $\mathscr{E}(a) = \mathscr{I}_a(u)$, then $u \in C^1(\mathbb{R}^3)$ and up to translation, u is a radial function with a unique and nondegenerate maximum at the origin.*

7.1.3 Asymptotics of solutions to penalised problem

The next goal is to study the asymptotic behaviour of the solution u_ε of (7.3) as $\varepsilon \to 0$. First, an upper estimate of the critical level c_ε is derived by using a cut-off function supported in Λ, so that the penalisation agrees with u^p. The result is as follows.

Lemma 7.3. *It holds that*

$$\limsup_{\varepsilon \to 0} \varepsilon^{-3} c_\varepsilon \leq \inf_{\Lambda} \mathscr{C}.$$

This lemma, as well as properties of the penalisation, allow the following estimate on the norm of the solutions to be proved.

Corollary 7.1. *There exists $C > 0$ such that, for every ε small enough*

$$||u_\varepsilon||_\varepsilon \leq C\varepsilon^{\frac{3}{2}}.$$

Since this norm is of order $\varepsilon^{\frac{3}{2}}$, it is natural to rescale u_ε as $u_\varepsilon(x_\varepsilon + \varepsilon\cdot)$ around a well-chosen family of points x_ε. For what is to follow, it will be important to observe that such sequences are relatively compact for the uniform C^1-convergence over compact sets, using standard regularity and bootstrap arguments. Namely, the following holds.

Lemma 7.4. *Let $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^3$ be sequences such that $\varepsilon_n \to 0$ and $x_n \to \bar{x}$ as $n \to +\infty$. Set*

$$v_n(x) := u_{\varepsilon_n}(x_n + \varepsilon_n x).$$

Then, there exists $v \in H^1(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$ such that, up to a subsequence, $v_n \to v$ in $C^1_{loc}(\mathbb{R}^3)$. Moreover,

$$\int_{\mathbb{R}^3} |\nabla v|^2 = \lim_{R\to+\infty} \lim_{n\to+\infty} \varepsilon_n^{-3} \int_{B(x_n,\varepsilon_n R)} \varepsilon_n^2 |\nabla u_{\varepsilon_n}|^2,$$

and

$$\int_{\mathbb{R}^3} V(\bar{x}) v^2 = \lim_{R\to+\infty} \lim_{n\to+\infty} \varepsilon_n^{-3} \int_{B(x_n,\varepsilon_n R)} V(x) u_{\varepsilon_n}^2.$$

The next step will be to derive a lower estimate of the energy of the solution u_ε as a function of the number and the location of its local maxima. This will be done in the following three lemmas. First, using the uniform C^1-convergence of the rescaled solutions v_n over compact sets, as well as the Nehari characterisation of the solutions to the limiting problem, the following estimate on small balls is established.

Lemma 7.5. *Let $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^3$ be sequences such that $\varepsilon_n \to 0$, $x_n \to \bar{x}$ as $n \to +\infty$ and*

$$\liminf_{n\to+\infty} u_{\varepsilon_n}(x_n) > 0.$$

Then, up to a subsequence,

$$\liminf_{R\to+\infty}\liminf_{n\to+\infty} \varepsilon_n^{-3}\left(\int_{B_n(R)} \frac{1}{2}\left(\varepsilon_n^2|\nabla u_{\varepsilon_n}|^2 + V(x)u_{\varepsilon_n}^2\right) - G(x,u_{\varepsilon_n})\right) \geq \mathscr{C}(\bar{x}),$$

where $B_n(R) := B(x_n, \varepsilon_n R)$.

Next, using $\eta_{R,\varepsilon_n} u_{\varepsilon_n}$ as a test function in (7.3), where $\eta_{R,\varepsilon_n} \in C^\infty(\mathbb{R}^3)$ is a well chosen cut-off function supported outside small balls, the following estimate outside balls is obtained.

Lemma 7.6. *Let $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $(x_n^i)_{n\in\mathbb{N}} \subset \mathbb{R}^3$, $1 \leq i \leq K$, be sequences such that $\varepsilon_n \to 0$ and $x_n^i \to \bar{x}^i \in \bar{\Lambda}$ as $n \to +\infty$. Then, up to a subsequence,*

$$\liminf_{R\to+\infty}\liminf_{n\to+\infty} \varepsilon_n^{-3}\left(\int_{\mathbb{R}^3\setminus\mathscr{B}_n(R)} \frac{1}{2}\left(\varepsilon_n^2|\nabla u_{\varepsilon_n}|^2 + V(x)u_{\varepsilon_n}^2\right) - G(x,u_{\varepsilon_n})\right) \geq 0,$$

where $\mathscr{B}_n(R) := \cup_{i=1}^K B(x_n^i, \varepsilon_n R)$.

Then, combining Lemma 7.5 and 7.6, the following result is derived, which gives a lower estimate of the energy of the solution u_ε.

Lemma 7.7. *Let $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $(x_n^i)_{n\in\mathbb{N}} \subset \mathbb{R}^3$, $1 \le i \le K$, be sequences such that $\varepsilon_n \to 0$ and $x_n^i \to \bar{x}^i \in \bar{\Lambda}$ as $n \to +\infty$. If for every $1 \le i \le j \le K$, we have*

$$\limsup_{n\to+\infty} \frac{|x_n^i - x_n^j|}{\varepsilon_n} = +\infty,$$

and if for every $1 \le i \le K$,

$$\liminf_{n\to+\infty} u_{\varepsilon_n}(x_n^i) > 0,$$

then

$$\liminf_{n\to+\infty} \varepsilon_n^{-3} \mathscr{J}_{\varepsilon_n}(u_{\varepsilon_n}) \ge \sum_{i=1}^{K} \mathscr{C}(\bar{x}^i).$$

After establishing this estimate from below, it is then shown that there does, in fact, exist a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^3$ satisfying the assumptions of the previous lemma. Namely, using the definition of the penalisation and the positivity of u_ε, the following result is established.

Lemma 7.8. *It holds that*

$$||u_\varepsilon||_{L^\infty(\Lambda)} > (\kappa V_0)^{\frac{1}{p-1}}.$$

7.1.4 Concentration and existence of solutions to original problem

With these results in place, it is possible to prove a first concentration result. The proof of each of the statements in the following lemma rely on contradiction arguments in which the energy of the solution is estimated from below using the assumption $\inf_{x\in\Lambda} V(x) < \inf_{x\in\partial\Lambda} V(x)$ and Lemma 7.7. Ultimately a contradiction is found with the upper estimate of the critical level obtained in Lemma 7.3.

Lemma 7.9. *Let $(x_\varepsilon)_{\varepsilon>0} \subset \Lambda$ be such that*

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0.$$

Then,

(i) $\lim_{\varepsilon\to 0} V(x_\varepsilon) = \inf_{x\in\Lambda} V(x)$,

(ii) $\liminf_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \partial\Lambda) > 0$,

(iii) *for every* $\delta > 0$, *there exists* $\varepsilon_0 > 0$ *and* $R > 0$ *such that, for every* $\varepsilon \in (0, \varepsilon_0)$,

$$||u_\varepsilon||_{L^\infty(\Lambda\setminus B(x_\varepsilon,\varepsilon R))} \leq \delta.$$

Using this lemma, the assumption on the potential V and the maximum principle, it is then shown that $u_\varepsilon^p \leq \kappa V u_\varepsilon$ in $\mathbb{R}^3 \setminus \Lambda$ and so $g(x, u_\varepsilon) = u_\varepsilon^p$ in $\mathbb{R}^3 \setminus \Lambda$. Namely, the solution u_ε to the penalised problem (7.3) is, in fact, a solution to the original problem (7.2). This is summarised in the following result.

Lemma 7.10. *Assume that* V *is a continuous potential bounded away from zero and that* $\Lambda \subset \mathbb{R}^3$ *is a bounded open set satisfying*

$$V_0 := \inf_{x\in\Lambda} V(x) < \inf_{x\in\partial\Lambda} V(x).$$

Then, there exist $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$, *problem* (7.2) *has at least one positive solution* $u_\varepsilon \in H^1(\mathbb{R}^3) \cap C^{2,\alpha}(\mathbb{R}^3)$.

Regularity estimates are then used to improve Lemma 7.4 and show that the rescaled solutions converge to a solution of the limiting equation (7.4), giving the limiting profile of the solutions u_ε.

Lemma 7.11. *Suppose that the assumptions of Lemma 7.10 are satisfied and assume furthermore that* V *is locally Hölder continuous in* Λ. *Let* u_ε *be the positive solution of* (7.2) *found in that lemma and let* $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ *and* $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^3$ *be sequences such that* $\varepsilon_n \to 0$ *and* $x_n \to \bar{x}$ *as* $n \to +\infty$. *Set*

$$v_n(x) := u_{\varepsilon_n}(x_n + \varepsilon_n x).$$

Then, there exists $v \in H^1(\mathbb{R}^3) \cap C^2(\mathbb{R}^3)$ *such that, up to a subsequence,* $v_n \to v$ *in* $C^{2,\alpha}_{loc}(\mathbb{R}^3)$. *In fact,* v *solves* (7.4) *with* $a = V(\bar{x})$.

Next, using the fact that $u_\varepsilon^p \leq \kappa V u_\varepsilon$ in $\mathbb{R}^3 \setminus \Lambda$, it is shown that if x_ε is a local maximum of u_ε, then $x_\varepsilon \in \Lambda$. Using Lemma 7.9 and the non-degeneracy of the maximum of v (Lemma 7.2), an argument by contradiction shows that u_ε has exactly one maximum. Namely, the following result is obtained.

Lemma 7.12. *Suppose that the assumptions of Lemma 7.10 are satisfied and assume furthermore that V is locally Hölder continuous in Λ. Let u_ε be the positive solution of (7.2) found in that lemma. Then, there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, u_ε has exactly one local, and hence global, maximum $x_\varepsilon \in \Lambda$.*

Finally, Theorem 7.1 is the consequence of combining Lemma 7.9, 7.10, 7.11 and 7.12.

7.2 Preliminaries for nonlinear Schrödinger-Poisson system

In this section, we will introduce the penalisation scheme for the nonlinear Schrödinger-Poisson system, and then we will discuss in more detail how we aim to adapt the method of del Pino and Felmer [30] in order to prove Conjecture 7.1. We will see that although we would like to mimic the technique discussed in the previous section, it is not directly applicable in the case of the nonlinear Schrödinger-Poisson system due to a major obstacle; namely, a lack of direct relation with the limiting problem.

7.2.1 Penalised problem

We will first define the penalisation and its properties. Assume there exists a bounded open set $\Lambda \subset \mathbb{R}^3$ satisfying

$$\rho(x_0) := \inf_{x\in\Lambda} \rho(x) < \inf_{x\in\partial\Lambda} \rho(x).$$

Fix $\kappa \in (0, \frac{1}{2})$. Define the penalised nonlinearity $g : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}^+$ as

$$g(x,s) := \chi_\Lambda(x)s^p + (1-\chi_\Lambda(x)) \min\{\kappa s, s^p\}, \tag{7.5}$$

and set

$$G(x,s) := \int_0^s g(x,\sigma)d\sigma.$$

We note that g is a Carathéodory function which satisfies:

(g_1) $g(x,s) = o(s)$ as $s \to 0^+$ uniformly in compact subsets of $\mathbb{R}^3$,

(g_2) there exists $\lambda \in (1,5)$ such that $\lim_{s\to+\infty} \frac{g(x,s)}{s^\lambda} = 0$,

(g_3) (i) $0 < (p+1)G(x,s) \le g(x,s)s$ for all $x \in \Lambda$, $s > 0$,

(ii) $0 < 2G(x,s) \le g(x,s)s \le \kappa s^2$ for all $x \notin \Lambda$, $s > 0$,

(g_4) the function $s \mapsto \frac{g(x,s)}{s}$ in non-decreasing in $\mathbb{R}^3$.

With these definitions in place, we can now work with the penalised problem

$$\begin{cases} -\varepsilon^2 \Delta u + u + \rho(x)\phi u = g(x,u), & x \in \mathbb{R}^3, \\ -\Delta \phi = \rho(x)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{7.6}$$

Using the explicit representation

$$\phi_u(x) := \int_{\mathbb{R}^3} \frac{\rho(y)u^2(y)}{4\pi|x-y|} \, \mathrm{d}y,$$

the system (7.6) reduces to solving the equation

$$-\varepsilon^2 \Delta u + u + \rho(x)\phi_u u = g(x,u). \tag{7.7}$$

Weak solutions of this equation are critical points of the functional $J_\varepsilon : E_\varepsilon(\mathbb{R}^3) \to \mathbb{R}$ given by

$$J_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^3} (\varepsilon^2 |\nabla u|^2 + u^2) + \frac{1}{4}\int_{\mathbb{R}^3} \rho \phi_u u^2 - \int_{\mathbb{R}^3} G(x,u). \tag{7.8}$$

Here, the space $E_\varepsilon(\mathbb{R}^3) = E_\varepsilon$ is defined as

$$E_\varepsilon(\mathbb{R}^3) := \left\{ u \in D^{1,2}(\mathbb{R}^3) : ||u||_{E_\varepsilon} < +\infty \right\},$$

where

$$||u||^2_{E_\varepsilon} := \int_{\mathbb{R}^3} (\varepsilon^2 |\nabla u|^2 + u^2) \, \mathrm{d}x + \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)\rho(x)u^2(y)\rho(y)}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2}.$$

For what follows, we will also need to define

$$||u||^2_{H^1_\varepsilon} := \int_{\mathbb{R}^3} (\varepsilon^2 |\nabla u|^2 + u^2)\, \mathrm{d}x.$$

Assumptions (g_2) and (g_3) imply that J_ε is well-defined and $J_\varepsilon \in C^1(E_\varepsilon, \mathbb{R})$.

7.2.2 Limit problem

We now introduce the associated limit problem, namely,

$$\begin{cases} -\Delta u + u = |u|^{p-1} u, & x \in \mathbb{R}^3 \\ u > 0, & x \in \mathbb{R}^3, \end{cases} \tag{7.9}$$

whose solutions have been classified by Kwong [41].

Remark 7.1. *The limiting equation is identified by using the scaling*

$$u_\varepsilon(\cdot) := u\left(\frac{\cdot - x_0}{\varepsilon}\right)$$

in the Schrödinger-Poisson system (7.1)*. Formally, one obtains the equation*

$$-\Delta u + u + \varepsilon^2 \rho(\varepsilon x + x_0) \left(\int_{\mathbb{R}^3} \frac{u^2(y) \rho(\varepsilon y + x_0)}{|x-y|} \, \mathrm{d}y \right) u = |u|^{p-1} u, \tag{7.10}$$

which becomes (7.9) *as* $\varepsilon \to 0$*.*

Remark 7.2. *Due to the nature of our problem, namely due to the* ε^2 *that appears in front of the Poisson term in* (7.10)*, we lose all information on* ρ *in the limiting equation. This is in stark contrast to the case of the nonlinear Schrödinger equation, in which the limiting problem, ground energy function and concentration function have a direct relation with the limit of the rescaled potential* $V(\varepsilon x + x_0)$*.*

Weak solutions of the limit problem (7.9) are critical points of the functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ given by

$$I(u) := \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1}. \tag{7.11}$$

It is standard to characterise the energy level of ground state solutions of (7.9) by

$$\bar{c} := \inf_{\gamma \in \bar{\Gamma}} \max_{t \in [0,1]} I(\gamma(t)), \tag{7.12}$$

where $\bar{\Gamma}$ is the family of paths

$$\bar{\Gamma} := \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

7.2.3 Perturbed penalised and associated limit problem

For $p \in (2,3)$, it will be useful to define a perturbation of the penalised problem (7.6), namely

$$\begin{cases} -\varepsilon^2 \Delta u + u + \rho(x)\phi u = \mu g(x,u), & x \in \mathbb{R}^3,\ \mu \in \left[\frac{1}{2}, 1\right], \\ -\Delta \phi = \rho(x) u^2, & x \in \mathbb{R}^3, \end{cases} \tag{7.13}$$

which reduces to solving

$$-\varepsilon^2 \Delta u + u + \rho(x)\phi_u u = \mu g(x,u), \quad \mu \in \left[\frac{1}{2}, 1\right], \tag{7.14}$$

with $\phi_u(x) := \int_{\mathbb{R}^3} \frac{\rho(y)u^2(y)}{4\pi|x-y|} \mathrm{d}y$. Weak solutions of this equation are critical points of the functional $J_{\varepsilon,\mu} : E_\varepsilon \to \mathbb{R}$ given by

$$J_{\varepsilon,\mu}(u) := \frac{1}{2}\int_{\mathbb{R}^3} (\varepsilon^2 |\nabla u|^2 + u^2) + \frac{1}{4}\int_{\mathbb{R}^3} \rho \phi_u u^2 - \mu \int_{\mathbb{R}^3} G(x,u), \quad \mu \in \left[\frac{1}{2}, 1\right]. \tag{7.15}$$

For each $\mu \in \left[\frac{1}{2}, 1\right]$, the associated perturbed limit problem is

$$\begin{cases} -\Delta u + u = \mu |u|^{p-1} u, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3. \end{cases} \tag{7.16}$$

Weak solutions of this limit problem are critical points of the functional $I_\mu : H^1(\mathbb{R}^3) \to$

$\mathbb{R}$ given by

$$I_\mu(u) := \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+u^2) - \frac{\mu}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}, \quad \mu \in \left[\frac{1}{2},1\right]. \tag{7.17}$$

7.2.4 Outline of the proposed method and associated difficulties

Inspired by the approach of del Pino and Felmer [30] and with the aim to obtain Conjecture 7.1, for $p\in[3,5)$, we first prove that for each $\varepsilon>0$ there exists a solution u_ε to the penalised problem (7.7) (Theorem 7.2). Our goal is to then study the asymptotic behaviour of u_ε as $\varepsilon\to 0$. The first key ingredient of such study is a comparison of the mountain pass level associated with the penalised problem (7.7) and the energy level associated with the limit problem (7.9) for small ε. Doing so gives an upper estimate of the mountain pass level. We obtain a preliminary estimate of this type in Lemma 7.18, however it seems that due to the nature of our problem we need higher order estimates. Namely, we obtain an estimate of the form

$$\varepsilon^{-3}J_\varepsilon(u_\varepsilon) \le I(v)+o(1), \tag{7.18}$$

where v solves the limit problem (7.9). However, following a personal communication of Denis Bonheure [17], the ideal type of estimate that we need to obtain in order to retain information on the minimum of ρ is

$$\frac{\varepsilon^{-3}J_\varepsilon(u_\varepsilon)-I(v)}{\varepsilon^2} \le \frac{1}{4}\rho^2(x_0)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{v^2(x)v^2(y)}{|x-y|}+o(1), \tag{7.19}$$

where v solves the limit problem (7.9) and $\rho(x_0)=\inf_{x\in\Lambda}\rho(x)$. Thus, we can see that we require the remainder terms on the right hand side of (7.18) to be of order ε^2. Rescaling u_ε as $u_\varepsilon(x_\varepsilon+\varepsilon\cdot)$ around a well-chosen family of points x_ε, the next step is to show that the rescaled sequences are relatively compact for the uniform $C^{1,\alpha}$ convergence over compact sets (Proposition 7.3). Using this, the aim is to then estimate from below the energy of solution u_ε as a function of the number and location of its local maxima. In the approach of del Pino and Felmer [30], estimates are obtained inside and outside small balls and then such estimates are combined. Here, we endeavour to obtain an estimate of the form

$$\frac{\varepsilon^{-3}J_\varepsilon(u_\varepsilon)-I(v)}{\varepsilon^2} \ge \frac{1}{4}\rho^2(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{v^2(x)v^2(y)}{|x-y|}+o(1), \tag{7.20}$$

where v solves the limit problem (7.9) and $x_\varepsilon \to \xi \in \bar{\Lambda}$. However, we once again face difficulties in mimicking the technique discussed for the nonlinear Schrödinger equation due to the lack of direct relation between our concentration function (right hand side of (7.20)) and the limit problem (7.9). Once obtained, the estimates from above and below would then be used in a contradiction argument to show that, in fact, $x_\varepsilon \to x_0$ where $\rho(x_0) = \inf_{x\in\Lambda} \rho(x)$. Namely, assuming by contradiction that $x_\varepsilon \to \xi$ where $\rho(x_0) = \inf_{x\in\Lambda} \rho(x) < \rho(\xi)$, then if (7.19) and (7.20) hold, it would follow that as $\varepsilon \to 0$,

$$\begin{aligned}\frac{1}{4}\rho^2(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{v^2(x)v^2(y)}{|x-y|} &\leq \frac{1}{4}\rho^2(x_0)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{v^2(x)v^2(y)}{|x-y|} \\ &< \frac{1}{4}\rho^2(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{v^2(x)v^2(y)}{|x-y|},\end{aligned}$$

a contradiction. This would give a first concentration result. Finally, the asymptotic estimates would then enable the application of the maximum/comparison principle to show that u_ε are small outside Λ and actually solve the original problem (7.1) for ε small.

7.3 Initial results for nonlinear Schrödinger-Poisson system

In this section, we discuss the results that we have obtained thus far which, based on the discussion in the previous section, should aid in proving Conjecture 7.1.

7.3.1 Existence of solutions to penalised problem

Using the properties of the penalisation, we first prove that the functional J_ε has the Mountain-Pass Geometry for each $\varepsilon > 0$.

Lemma 7.13. *[**Mountain-Pass Geometry for** J_ε] Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative and $p \in (2,5]$. Then, for each $\varepsilon > 0$, it holds:*

(i) $J_\varepsilon(0) = 0$ and there exists constants $r, a > 0$ such that $J_\varepsilon(u) \geq a$ if $||u||_{E_\varepsilon} = r$.

(ii) There exists $v \in E_\varepsilon$ with $||v||_{E_\varepsilon} > r$, such that $J_\varepsilon(v) \leq 0$.

Proof. To prove (i), we first note that by (g_3) and the Sobolev inequality, for all $u \in E_\varepsilon$, it holds that

$$\begin{aligned} J_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^3}(\varepsilon^2|\nabla u|^2+u^2)+\frac{1}{4}\int_{\mathbb{R}^3}\rho\phi_u u^2-\int_\Lambda G(x,u)-\int_{\mathbb{R}^3\setminus\Lambda}G(x,u) \\ &\geq \frac{1}{2}\int_{\mathbb{R}^3}(\varepsilon^2|\nabla u|^2+u^2)+\frac{1}{4}\int_{\mathbb{R}^3}\rho\phi_u u^2-\frac{1}{p+1}\int_\Lambda |u|^{p+1}-\frac{1}{2}\int_{\mathbb{R}^3\setminus\Lambda}\kappa u^2 \\ &\geq \left(\frac{1-\kappa}{2}\right)||u||^2_{H^1_\varepsilon}+\frac{1}{4}\int_{\mathbb{R}^3}\rho\phi_u u^2-\frac{1}{p+1}\int_\Lambda |u|^{p+1} \\ &\geq \left(\frac{1-\kappa}{2}\right)||u||^2_{H^1_\varepsilon}+\frac{1}{4}\int_{\mathbb{R}^3}\rho\phi_u u^2-C||u||^{p+1}_{H^1_\varepsilon}, \end{aligned} \tag{7.21}$$

for some $C>0$. Using the definition of the norm in E_ε and following the arguments of Lemma 4.1 we can show that the origin is a strict local minimum for J_ε in E_ε if $p \in [2,5]$.

To show (ii), we pick $u \in C^1(\mathbb{R}^3)\setminus\{0\}$, supported in Λ. Setting $v_t(x) := t^2u(tx)$, with $t>0$, we argue as in Lemma 4.1 to show $J_\varepsilon(v_t) \to -\infty$ as $t \to +\infty$ if $p \in (2,5]$. This concludes the proof. □

We now define the min-max level associated with J_ε as

$$c_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon}\max_{t\in[0,1]} J_\varepsilon(\gamma(t)), \tag{7.22}$$

where Γ_ε is the family of paths

$$\Gamma_\varepsilon := \{\gamma \in C([0,1],E_\varepsilon) : \gamma(0)=0, J_\varepsilon(\gamma(1))<0\}.$$

The previous lemma gives us the existence of a Palais-Smale sequence for J_ε at the level c_ε. That is, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset E_\varepsilon$ such that $J_\varepsilon(u_n) \to c_\varepsilon$ and $J'_\varepsilon(u_n) \to 0$. For large p, the boundedness of such Palais-Smale sequences follows relatively straightforwardly. Namely, we have the following result.

Lemma 7.14. *Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative, $p \in [3,5)$, and $(u_n)_{n\in\mathbb{N}} \subset E_\varepsilon(\mathbb{R}^3)$ is a Palais-Smale sequence for J_ε. Then, $(u_n)_{n\in\mathbb{N}}$ is bounded in $E_\varepsilon(\mathbb{R}^3)$.*

Proof. Assume $(u_n)_{n\in\mathbb{N}} \subset E_\varepsilon(\mathbb{R}^3)$ is a Palais-Smale sequence for J_ε. Then,

$$(p+1)J_\varepsilon(u_n) - J'_\varepsilon(u_n)(u_n) = \left(\frac{p-1}{2}\right)\int_{\mathbb{R}^3}(\varepsilon^2|\nabla u_n|^2 + u_n^2) + \left(\frac{p-3}{4}\right)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\rho$$
$$+\int_{\mathbb{R}^3}(g(x,u_n)u_n - (p+1)G(x,u_n)). \tag{7.23}$$

Using the definition of g and property (g_3), we find that

$$\begin{aligned}\int_{\mathbb{R}^3}(g(x,u_n)u_n-(p+1)G(x,u_n)) &\geq \int_{\mathbb{R}^3\setminus\Lambda}(g(x,u_n)u_n-(p+1)G(x,u_n))\\ &\geq (-p+1)\int_{\mathbb{R}^3\setminus\Lambda}G(x,u_n)\\ &\geq \frac{-p+1}{2}\kappa\int_{\mathbb{R}^3\setminus\Lambda}u_n^2\\ &\geq \frac{-p+1}{2}\kappa||u_n||^2_{H^1_\varepsilon}.\end{aligned} \tag{7.24}$$

Putting (7.23) and (7.24) together, and using the fact that $(u_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence, we obtain

$$\begin{aligned}c+o(1)||u_n||_{E_\varepsilon} &\geq (p+1)J_\varepsilon(u_n)-J'_\varepsilon(u_n)(u_n)\\ &\geq \left(\frac{p-1}{2}\right)(1-\kappa)||u_n||^2_{H^1_\varepsilon}+\left(\frac{p-3}{4}\right)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\rho.\end{aligned} \tag{7.25}$$

For convenience, we now set

$$a_n := ||u_n||_{H^1_\varepsilon},$$

and

$$b_n := \left(\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\rho\right)^{\frac{1}{2}}.$$

We then assume, by contradiction, $||u_n||_{E_\varepsilon} \to +\infty$, and note that this can occur in three ways; namely

(i) a_n, $b_n \to +\infty$,

(ii) a_n bounded and $b_n \to +\infty$,

(iii) $a_n \to +\infty$ and b_n bounded.

Thus, we need to find a contradiction in each of these three cases.

We first restrict to $p > 3$. If $b_n \to +\infty$, then it follows that $b_n^2 \geq b_n$ for large n. Hence, for large n, (7.25) can be written as

$$\begin{aligned} c + o(1)||u_n||_{E_\varepsilon} &\geq \min\left\{\left(\frac{p-1}{2}\right)(1-\kappa), \left(\frac{p-3}{4}\right)\right\}\left(||u_n||^2_{H^1_\varepsilon} + \left(\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \rho\right)^{\frac{1}{2}}\right) \\ &= \min\left\{\left(\frac{p-1}{2}\right)(1-\kappa), \left(\frac{p-3}{4}\right)\right\}||u_n||^2_{E_\varepsilon}. \end{aligned}$$

So, dividing by $||u_n||_{E_\varepsilon}$ and letting $n \to +\infty$, we have reached a contradiction in case (i) and (ii). If $a_n \to +\infty$ and b_n is bounded, then, for large n, this implies

$$||u_n||_{E_\varepsilon} \sim a_n.$$

Combining this with (7.25), we find that, for large n,

$$\begin{aligned} c + o(1)a_n &\geq \min\left\{\left(\frac{p-1}{2}\right)(1-\kappa), \left(\frac{p-3}{4}\right)\right\}\left(||u_n||^2_{H^1_\varepsilon} + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \rho\right) \\ &\geq \min\left\{\left(\frac{p-1}{2}\right)(1-\kappa), \left(\frac{p-3}{4}\right)\right\}a_n^2. \end{aligned}$$

Thus, dividing by a_n and letting $n \to +\infty$, we have reached a contradiction in case (iii). Therefore, since we have reached a contradiction in each possible case, we have shown that if $p > 3$, then $(u_n)_{n\in\mathbb{N}}$ is bounded in E_ε.

We now consider $p = 3$. Once again, we need to find a contradiction in each of the three cases. We first notice that since u_n is a Palais-Smale sequence for J_ε and by property (g_3) and the Sobolev inequality, it holds that

$$\begin{aligned} c &\geq J_\varepsilon(u_n) \\ &= \frac{1}{2}a_n^2 + \frac{1}{4}b_n^2 - \int_{\mathbb{R}^3} G(x,u) \\ &\geq \left(\frac{1-\kappa}{2}\right) a_n^2 + \frac{1}{4}b_n^2 - Ca_n^{p+1}, \end{aligned}$$

for some $c, C > 0$. Hence, it follows that

$$\begin{aligned} c + Ca_n^{p+1} &\geq \left(\frac{1-\kappa}{2}\right) a_n^2 + \frac{1}{4}b_n^2 \\ &\geq \frac{1}{4}b_n^2, \end{aligned} \tag{7.26}$$

which immediately yields a contradiction in case (ii). In both remaining cases we assume $a_n \to +\infty$, and so, since $p = 3$, we can see from (7.26) that, for large n,

$$b_n \lesssim a_n^2. \tag{7.27}$$

We also notice that since $p = 3$, (7.25) becomes

$$c + o(1)||u_n||_{E_\varepsilon} \geq (1-\kappa)||u_n||_{H_\varepsilon^1}^2. \tag{7.28}$$

Now, dividing both sides of this inequality by $||u_n||_{E_\varepsilon} = \left(a_n^2 + b_n\right)^{\frac{1}{2}}$, we obtain

$$(1-\kappa) \cdot \frac{a_n^2}{(a_n^2 + b_n)^{\frac{1}{2}}} = o(1),$$

or equivalently

$$(1-\kappa)^2 \cdot \frac{a_n^4}{a_n^2 + b_n} = o(1).$$

Hence, we find that, for large n,

$$b_n \gtrsim a_n^4,$$

immediately yielding a contradiction in case (iii). Otherwise, by combining the previous inequality with (7.27), we obtain, for large n,

$$a_n^4 \lesssim a_n^2,$$

a contradiction in case (i). Therefore, we have reached a contradiction in each possible case, and so we have shown that if $p = 3$, then $(u_n)_{n\in\mathbb{N}}$ is bounded in E_ε. This completes the proof. □

We note that the previous result only holds for $p \in [3,5)$. When $p \in (2,3)$ we can recover the boundedness of Palais Smale sequences associated with the perturbed energy functional $J_{\varepsilon,\mu}$ for almost every $\mu \in \left[\frac{1}{2},1\right]$. We will first need the following preliminary result.

Lemma 7.15. *[**Mountain-Pass Geometry for** $J_{\varepsilon,\mu}$] Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative and $p \in (2,5]$. Then, for each $\varepsilon > 0$ and each $\mu \in \left[\frac{1}{2},1\right]$, it holds:*

(i) $J_{\varepsilon,\mu}(0) = 0$ and there exists constants $r, a > 0$ such that $J_{\varepsilon,\mu}(u) \geq a$ if $||u||_{E_\varepsilon} = r$.

(ii) There exists $v \in E_\varepsilon$ with $||v||_{E_\varepsilon} > r$, such that $J_{\varepsilon,\mu}(v) \leq 0$.

Proof. The proof follows similarly to that of Lemma 7.13. □

The previous lemma, as well as the monotonicity of $J_{\varepsilon,\mu}$ with respect to μ, imply that for each $\varepsilon > 0$, there exists $\bar{v} \in E_\varepsilon \setminus \{0\}$ such that

$$J_{\varepsilon,\mu}(\bar{v}) \leq J_{\varepsilon,\frac{1}{2}}(\bar{v}) \leq 0, \qquad \forall \mu \in \left[\frac{1}{2},1\right].$$

Thus, for each $\varepsilon > 0$, we can define the min-max level associated with $J_{\varepsilon,\mu}$ as

$$c_{\varepsilon,\mu} := \inf_{\gamma\in\tilde{\Gamma}_\varepsilon} \max_{t\in[0,1]} J_{\varepsilon,\mu}(\gamma(t)), \tag{7.29}$$

where $\tilde{\Gamma}_\varepsilon$ is the family of paths

$$\tilde{\Gamma}_\varepsilon := \{\gamma \in C([0,1], E_\varepsilon) : \gamma(0) = 0, \gamma(1) = \bar{v}\}.$$

Applying Lemma 4.2, it follows that for each $\varepsilon > 0$ and for almost every $\mu \in \left[\frac{1}{2}, 1\right]$, there exists a bounded Palais-Smale sequence for $J_{\varepsilon,\mu}$ at the level $c_{\varepsilon,\mu}$. That is, for every fixed $\varepsilon > 0$, we can define the following dense set of $\left[\frac{1}{2}, 1\right]$,

$$\mathscr{M}_\varepsilon := \left\{ \mu \in \left[\frac{1}{2}, 1\right] : \exists \text{ bounded Palais-Smale sequence} \right.$$
$$\left. \text{for } J_{\varepsilon,\mu} \text{ at the level } c_{\varepsilon,\mu} \right\}. \tag{7.30}$$

Using the results we have obtained thus far, we can now obtain existence to the penalised problem for each $\varepsilon > 0$ if $p \in [3,5)$ and existence to the perturbed penalised problem for each $\varepsilon > 0$ and each $\mu \in \mathscr{M}_\varepsilon$ if $p \in (2,3)$.

Theorem 7.2. *Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative. If $p \in (2,3)$, then for each $\varepsilon > 0$ and each $\mu \in \mathscr{M}_\varepsilon$, there exists a solution $(u_{\varepsilon,\mu}, \phi_{u_{\varepsilon,\mu}}) \in E_\varepsilon(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ of (7.13), whose components are nonnegative functions, such that $u_{\varepsilon,\mu}$ is a critical point of $J_{\varepsilon,\mu}$ at level $c_{\varepsilon,\mu}$. If $p \in [3,5)$, then for each $\varepsilon > 0$, there exists a solution $(u_\varepsilon, \phi_{u_\varepsilon}) \in E_\varepsilon(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ of (7.6), whose components are positive functions, such that u_ε is a critical point of J_ε at level c_ε.*

Before completing the proof of this theorem, we will need two preliminary lemmas. These results are known in the case $E(\mathbb{R}^3)$ is compactly embedded in $L^{p+1}(\mathbb{R}^3)$, so here we will only highlight the parts of the proof where the penalisation replaces the role of this embedding.

Lemma 7.16. *Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded Palais-Smale sequence for $J_{\varepsilon,\mu}$, with $\mu \in \left[\frac{1}{2}, 1\right]$, such that, up to a subsequence, $u_n \rightharpoonup u$ in $E_\varepsilon(\mathbb{R}^3)$. Then, for every $\delta > 0$, there exists a ball $B \subset \mathbb{R}^3$ such that*

$$(i) \quad \limsup_{n\to+\infty} \int_{\mathbb{R}^3\setminus B} \phi_{u_n} u_n^2 \rho < \delta,$$

$$(ii) \quad \limsup_{n\to+\infty} \int_{\mathbb{R}^3\setminus B} u_n^2 < \delta,$$

$$(iii) \quad \limsup_{n\to+\infty} \left| \int_{\mathbb{R}^3\setminus B} \phi_{u_n} u_n u \rho \right| < \delta.$$

Proof. Consider a smooth function $\xi(r)$ such that $\xi(r) = 1$ on $[2,+\infty)$ and $\xi(r) = 0$ on $[0,1]$. Define

$$\eta_R(x) := \xi\left(\frac{\log(1+|x|)}{R}\right).$$

Since $(u_n)_{n\in\mathbb{N}}$ is a bounded Palais-Smale sequence for $J_{\varepsilon,\mu}$, arguing as in Lemma 16 in [19], it holds that

$$\begin{aligned} o(1) &\geq J'_{\varepsilon,\mu}(u_n)(u_n\eta_R) \\ &\geq \int_{\mathbb{R}^3} \varepsilon^2 |\nabla u_n|^2 \eta_R + \int_{\mathbb{R}^3} u_n^2 \eta_R + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \rho \eta_R - \mu \int_{\mathbb{R}^3} g(x,u_n) u_n \eta_R + O\left(\frac{1}{R}\right). \end{aligned}$$

We note that we can take R large enough so that $\eta_R \equiv 0$ on Λ, and so, from the definition of g and the previous equation, it follows that, for R large,

$$o(1) \geq \int_{\mathbb{R}^3} \varepsilon^2 |\nabla u_n|^2 \eta_R + \int_{\mathbb{R}^3} (1-\mu\kappa) u_n^2 \eta_R + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \rho \eta_R + O\left(\frac{1}{R}\right).$$

Since all of the terms are nonnegative, taking $B := \{x \in \mathbb{R}^3 : |x| \leq e^{2R}\}$ with R large enough, the previous inequality yields both (i) and (ii). The proof of (iii) follows as in Lemma 16 (b) in [19]. $\square$

Lemma 7.17. *Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded Palais-Smale sequence for $J_{\varepsilon,\mu}$, with $\mu \in \left[\frac{1}{2}, 1\right]$, such that, up to a subsequence, $u_n \rightharpoonup u$ in $E_\varepsilon(\mathbb{R}^3)$. Then, passing if necessary to another subsequence, it holds that*

$$||u_n||^2_{H^1_\varepsilon} \to ||u||^2_{H^1_\varepsilon}.$$

Proof. We first note that since $(u_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence and $u_n \rightharpoonup u$ in $E_\varepsilon(\mathbb{R}^3)$, it follows that

$$\begin{aligned} o(1) &= J'_{\varepsilon,\mu}(u_n)(u_n - u) \\ &= ||u_n||^2_{H^1_\varepsilon} - ||u||^2_{H^1_\varepsilon} + o(1) \\ &\quad + \int_{\mathbb{R}^3} \phi_{u_n} u_n (u_n - u)\rho - \mu \int_{\mathbb{R}^3} g(x, u_n)(u_n - u). \end{aligned} \tag{7.31}$$

Using Lemma 7.16, we can argue as in Lemma 18 in [19], to show

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n (u_n - u)\rho = o(1), \tag{7.32}$$

and so it remains only to prove

$$\mu \int_{\mathbb{R}^3} g(x, u_n)(u_n - u) = o(1). \tag{7.33}$$

To do so, let $\delta > 0$ be arbitrary. By Lemma 7.16 (ii), we can choose $B \subset \mathbb{R}^3$ large enough so that it contains Λ and, for large enough n,

$$\begin{aligned} \left| \int_{\mathbb{R}^3 \setminus B} g(x, u_n)(u_n - u) \right| &\leq \int_{\mathbb{R}^3 \setminus B} \kappa u_n |u_n - u| \\ &\leq \kappa \left(\int_{\mathbb{R}^3 \setminus B} u_n^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3 \setminus B} |u_n - u|^2 \right)^{\frac{1}{2}} \\ &< \delta. \end{aligned} \tag{7.34}$$

Now, since $(u_n)_{n\in\mathbb{N}}$ is bounded, then by Rellich Theorem, up to a subsequence $u_n \to u$ in $L^{p+1}(\Lambda)$. So, on Λ, we have that $|g(x, u_n)| < g(x)$ for some $g \in L^{\frac{p+1}{p}}(\Lambda)$. Thus, using the Dominated Convergence Theorem, it follows that

$$\left|\int_{\Lambda} g(x,u_n)(u_n-u)\right| \leq \left(\int_{\Lambda} |g(x,u_n)|^{\frac{p+1}{p}}\right)^{\frac{p}{p+1}} \left(\int_{\Lambda} |u_n-u|^{p+1}\right)^{\frac{1}{p+1}} \to 0. \tag{7.35}$$

Similarly, by Rellich Theorem, up to a subsequence $u_n \to u$ in $L^2(B\setminus\Lambda)$, and so again using the Dominated Convergence Theorem, we have

$$\left|\int_{B\setminus\Lambda} g(x,u_n)(u_n-u)\right| \leq \left(\int_{B\setminus\Lambda} |g(x,u_n)|^2\right)^{\frac{1}{2}} \left(\int_{B\setminus\Lambda} |u_n-u|^2\right)^{\frac{1}{2}} \to 0. \tag{7.36}$$

Since $\delta > 0$ was arbitrary, putting together (7.34), (7.35) and (7.36), we have proved (7.33), and therefore, we have shown $||u_n||^2_{H^1_\varepsilon} \to ||u||^2_{H^1_\varepsilon}$. □

We are now in position to complete the proof of Theorem 7.2.

Proof of Theorem 7.2. For $p \in (2,3)$, we follow the ideas of Theorem 4.2 Claim 1. First, fix $\varepsilon > 0$ and $\mu \in \mathscr{M}_\varepsilon$. By definition, there exists a bounded sequence $(u_n)_{n\in\mathbb{N}} \subset E_\varepsilon(\mathbb{R}^3)$ such that $J_{\varepsilon,\mu}(u_n) \to c_{\varepsilon,\mu}$ and $J'_{\varepsilon,\mu}(u_n) \to 0$ as $n \to +\infty$. Since $(u_n)_{n\in\mathbb{N}}$ is bounded, there exists $u \in E_\varepsilon(\mathbb{R}^3)$ such that, up to a subsequence $u_n \rightharpoonup u$ in $E_\varepsilon(\mathbb{R}^3)$ as $n \to +\infty$. We will show

$$\begin{aligned} (i) &\quad J_{\varepsilon,\mu}(u_n) \to J_{\varepsilon,\mu}(u), \\ (ii) &\quad J'_{\varepsilon,\mu}(u) = 0. \end{aligned}$$

To prove (i), we first note that by Lemma 7.17, it holds that

$$||u_n||^2_{H^1_\varepsilon} \to ||u||^2_{H^1_\varepsilon}.$$

Moreover, using Lemma 7.16 and the boundedness of $(u_n)_{n\in\mathbb{N}}$, we can argue as in the proof of Theorem 1 in [19], to show that

$$\int_{\mathbb{R}^3} \rho \phi_{u_n} u_n^2 \rho \to \int_{\mathbb{R}^3} \rho \phi_u u^2.$$

It therefore remains to show that

$$\mu \int_{\mathbb{R}^3} G(x, u_n) \to \mu \int_{\mathbb{R}^3} G(x, u). \tag{7.37}$$

To do so, we let $\delta > 0$ be arbitrary. Then, we can choose $B \subset \mathbb{R}^3$ containing Λ and large enough so that, arguing as in the proof of the previous lemma and using property (g_3), the compact embeddings $E_\varepsilon \hookrightarrow\hookrightarrow L^q(\Lambda)$ for all $q \in [1, 6)$ and $E_\varepsilon \hookrightarrow\hookrightarrow L^2(B \setminus \Lambda)$, the Dominated Convergence Theorem and Lemma 7.16 (ii), we can show that

$$\begin{aligned}\left| \int_{\mathbb{R}^3} G(x, u_n) - G(x, u) \right| = \left| \int_{\Lambda} G(x, u_n) - G(x, u) \right| &+ \left| \int_{B \setminus \Lambda} G(x, u_n) - G(x, u) \right| \\ &+ \left| \int_{\mathbb{R}^3 \setminus B} G(x, u_n) - G(x, u) \right| \\ &< \delta,\end{aligned}$$

for large n. Since $\delta > 0$ was arbitrary, it follows that (7.37) holds and so we have proved (i). The proof of (ii) is standard using the weak convergence and similar splitting arguments as we have used above and in the proof of the previous lemma. Therefore, by (i) and (ii), for each $\varepsilon > 0$ and $\mu \in \mathscr{M}_\varepsilon$, we have shown the existence of a solution to (7.13), which we denote by $(u_{\varepsilon,\mu}, \phi_{u_{\varepsilon,\mu}}) \in E_\varepsilon(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ to represent the dependence on ε and μ. By the strong maximum principle, $u_{\varepsilon,\mu} > 0$.

When $p \in [3, 5)$, we first fix $\varepsilon > 0$. By Lemma 7.13, there exists a Palais-Smale sequence for J_ε at the level c_ε. That is, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset E_\varepsilon$ such that $J_\varepsilon(u_n) \to c_\varepsilon$ and $J'_\varepsilon(u_n) \to 0$. By Lemma 7.14, it follows that $(u_n)_{n\in\mathbb{N}}$ is bounded in $E_\varepsilon(\mathbb{R}^3)$. Thus, up to a subsequence, $u_n \rightharpoonup u$ in $E_\varepsilon(\mathbb{R}^3)$. Now, setting $\mu = 1$, we notice that $J_{\varepsilon,1} = J_\varepsilon$, and so both Lemma 7.16 and Lemma 7.17 hold with $J_{\varepsilon,1}$ replaced by J_ε. Therefore, we can argue exactly as in the case $p \in (2, 3)$, to show

$$(i) \quad J_\varepsilon(u_n) \to J_\varepsilon(u),$$

$$(ii) \quad J'_{\varepsilon}(u) = 0,$$

yielding a solution to (7.6), which we denote by $(u_\varepsilon, \phi_{u_\varepsilon}) \in E_\varepsilon(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ to represent the dependence on ε. By the strong maximum principle, $u_\varepsilon > 0$. This concludes the proof. $\square$

Remark 7.3. *For $p \in (2,3)$, the idea will be to prove that each of the results that we obtain are independent of μ, with the ultimate goal to show that, for sufficiently small ε, there exists a family of solutions $(u_{\varepsilon,\mu}, \phi_{u_{\varepsilon,\mu}})$ of the perturbed Schrödinger-Poisson system*

$$\begin{cases} -\varepsilon^2 \Delta u + u + \rho(x)\phi u = \mu |u|^{p-1} u, & x \in \mathbb{R}^3,\ \mu \in \left[\frac{1}{2}, 1\right], \\ -\Delta \phi = \rho(x) u^2, & x \in \mathbb{R}^3. \end{cases}$$

For each $\varepsilon > 0$ small enough, we would then take $(\mu_n)_{n\in\mathbb{N}}$ an increasing sequence in $\mathscr{M}_\varepsilon$ such that $\mu_n \to 1$, and follow the ideas of Theorem 4.2 Claim 2 to show that $u_{\varepsilon,\mu_n} \to u_\varepsilon$ where $(u_\varepsilon, \phi_{u_\varepsilon})$ is a solution of (7.1). In what is to follow, however, we only consider the case $p \in [3,5)$.

7.3.2 Asymptotics of solutions to penalised problem

Now that we have established the existence of solutions to the penalised problem, the next step is to study the asymptotic behaviour of such solutions as $\varepsilon \to 0$. The following lemma gives a preliminary upper estimate of the mountain pass level c_ε.

Lemma 7.18. *Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative and $p \in [3,5)$. Then, for ε sufficiently small, it holds that*

$$c_\varepsilon \le \varepsilon^3 (\bar{c} + o(1)),$$

where c_ε is defined in (7.22) and $\bar{c}$ is defined in (7.12).

Proof. We first note that, by definition of infimum, for all $\delta > 0$, we can choose a continuous path $\gamma_\delta \in \bar{\Gamma}$ such that

$$\bar{c} = \inf_{\gamma \in \bar{\Gamma}} \max_{t \in [0,1]} I(\gamma(t)) \leq \max_{t \in [0,1]} I(\gamma_\delta(t)) \leq \bar{c} + \delta.$$

Let $x_0 \in \Lambda$ be such that $\rho(x_0) = \inf_{x \in \Lambda} \rho(x)$. Define a cut-off function $\eta \in C(\mathbb{R}^3, [0,1])$ such that

$$\begin{aligned}
&\eta(x) = 1 \text{ for all } x \text{ in a neighborhood of } x_0, \\
&\eta(x) = 0 \text{ for all } x \in \mathbb{R}^3 \setminus \Lambda, \\
&||\nabla \eta||_{L^\infty(\mathbb{R}^3)} \leq C \text{ for some } C > 0,
\end{aligned}$$

and consider the path

$$\bar{\gamma}_\delta(t) : x \to \eta(x) \left(\gamma_\delta(t) \left(\frac{x - x_0}{\varepsilon} \right) \right).$$

By a change of variables, and using the fact that $\gamma_\delta(t) \in H^1(\mathbb{R}^3)$, we find that

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\varepsilon^2 |\nabla \bar{\gamma}_\delta(t)(x)|^2 + |\bar{\gamma}_\delta(t)(x)|^2) \, \mathrm{d}x \\
&\quad = \varepsilon^5 \int_{\mathbb{R}^3} \left| (\gamma_\delta(t)(y)) \nabla \eta(\varepsilon y + x_0) + \frac{1}{\varepsilon} (\nabla \gamma_\delta(t)(y)) \eta(\varepsilon y + x_0) \right|^2 \mathrm{d}y \\
&\qquad + \varepsilon^3 \int_{\mathbb{R}^3} |(\gamma_\delta(t)(y)) \eta(\varepsilon y + x_0)|^2 \, \mathrm{d}y \\
&\quad = \varepsilon^5 \int_{\mathbb{R}^3} |\gamma_\delta(t)(y)|^2 |\nabla \eta(\varepsilon y + x_0)|^2 \, \mathrm{d}y \\
&\qquad + 2\varepsilon^4 \int_{\mathbb{R}^3} (\gamma_\delta(t)(y)) \nabla \eta(\varepsilon y + x_0) (\nabla \gamma_\delta(t)(y)) \eta(\varepsilon y + x_0) \, \mathrm{d}y \\
&\qquad\quad + \varepsilon^3 \int_{\mathbb{R}^3} |\nabla \gamma_\delta(t)(y)|^2 |\eta(\varepsilon y + x_0)|^2 \, \mathrm{d}y \\
&\qquad\qquad + \varepsilon^3 \int_{\mathbb{R}^3} |\gamma_\delta(t)(y)|^2 |\eta(\varepsilon y + x_0)|^2 \, \mathrm{d}y \\
&\quad = \varepsilon^3 \int_{\mathbb{R}^3} (|\nabla \gamma_\delta(t)(y)|^2 + |\gamma_\delta(t)(y)|^2)(|\eta(\varepsilon y + x_0)|^2 - 1) \, \mathrm{d}y \\
&\qquad + \varepsilon^3 \int_{\mathbb{R}^3} (|\nabla \gamma_\delta(t)(y)|^2 + |\gamma_\delta(t)(y)|^2) \, \mathrm{d}y + o(\varepsilon^3).
\end{aligned} \tag{7.38}$$

Now, let $\bar{\delta} > 0$ be arbitrary. Since $\gamma_\delta(t) \in H^1(\mathbb{R}^3)$, there exists a compact set containing x_0 such that

$$\int_{\mathbb{R}^3 \setminus K} (|\nabla \gamma_\delta(t)(y)|^2 + |\gamma_\delta(t)(y)|^2)(|\eta(\varepsilon y + x_0)|^2 - 1)\,\mathrm{d}y \leq \frac{\bar{\delta}}{2}.$$

Moreover, since $|\eta(\varepsilon y + x_0)|^2 - 1$ converges uniformly to 0 on K as $\varepsilon \to 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ it holds that

$$\int_K (|\nabla \gamma_\delta(t)(y)|^2 + |\gamma_\delta(t)(y)|^2)(|\eta(\varepsilon y + x_0)|^2 - 1)\,\mathrm{d}y \leq \frac{\bar{\delta}}{2}.$$

Therefore, combining the previous two inequalities and (7.38), we have shown that for all $\varepsilon < \varepsilon_0$,

$$\int_{\mathbb{R}^3} (\varepsilon^2 |\nabla \bar{\gamma}_\delta(t)(x)|^2 + |\bar{\gamma}_\delta(t)(x)|^2)\,\mathrm{d}x = \varepsilon^3 \int_{\mathbb{R}^3} (|\nabla \gamma_\delta(t)(y)|^2 + |\gamma_\delta(t)(y)|^2)\,\mathrm{d}y + o(\varepsilon^3).$$

We now focus on the Poisson term. Making the change of variables $x \to \varepsilon x' + x_0$ and $y \to \varepsilon y' + x_0$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(\bar{\gamma}_\delta(t)(x))^2 \rho(x) (\bar{\gamma}_\delta(t)(y))^2 \rho(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \leq \varepsilon^5 ||\rho||^2_{L^\infty(\Lambda)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(\gamma_\delta(t)(x'))^2 (\eta(\varepsilon x' + x_0))^2 (\gamma_\delta(t)(y'))^2 (\eta(\varepsilon y' + x_0))^2}{|x'-y'|}\,\mathrm{d}x'\,\mathrm{d}y' \\
&\quad = o(\varepsilon^3),
\end{aligned}$$

by the Hardy-Littlewood-Sobolev inequality since $\gamma_\delta(t) \in H^1(\mathbb{R}^3)$. Finally, arguing similarly, since $\eta = 0$ on $\mathbb{R}^3 \setminus \Lambda$ we see that

$$\begin{aligned}
\int_{\mathbb{R}^3} G(x, \bar{\gamma}_\delta(t)(x))\,\mathrm{d}x &= \frac{1}{p+1} \int_{\mathbb{R}^3} \left[\eta(x)\left(\gamma_\delta(t)\left(\frac{x-x_0}{\varepsilon}\right)\right)\right]^{p+1} \mathrm{d}x \\
&= \frac{\varepsilon^3}{p+1} \int_{\mathbb{R}^3} [\eta(\varepsilon y + x_0)(\gamma_\delta(t)(y))]^{p+1}\,\mathrm{d}y \\
&= \frac{\varepsilon^3}{p+1} \int_{\mathbb{R}^3} [\gamma_\delta(t)(y)]^{p+1}\,\mathrm{d}y + o(\varepsilon^3).
\end{aligned}$$

Thus, for $\varepsilon > 0$ small enough, we have

$$\begin{aligned} J_\varepsilon(\bar{\gamma}_\delta(t)) &\le \varepsilon^3 \int_{\mathbb{R}^3} (|\nabla \gamma_\delta(t)|^2 + |\gamma_\delta(t)|^2)\,\mathrm{d}y - \frac{\varepsilon^3}{p+1} \int_{\mathbb{R}^3} [\gamma_\delta(t)]^{p+1}\,\mathrm{d}y + o(\varepsilon^3) \\ &= \varepsilon^3 I(\gamma_\delta(t)) + o(\varepsilon^3). \end{aligned}$$

It follows that for $\varepsilon > 0$ small enough, $\bar{\gamma}_\delta$ belongs to the class of paths Γ_ε. Therefore,

$$\begin{aligned} c_\varepsilon &= \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} J_\varepsilon(\gamma(t)) \\ &\le \max_{t \in [0,1]} J_\varepsilon(\bar{\gamma}_\delta(t)) \\ &\le \max_{t \in [0,1]} \varepsilon^3 I(\gamma_\delta(t)) + o(\varepsilon^3) \\ &\le \varepsilon^3(\bar{c} + \delta) + o(\varepsilon^3), \end{aligned}$$

and so, since $\delta > 0$ is arbitrary, we have

$$c_\varepsilon \le \varepsilon^3(\bar{c} + o(1)) \quad \text{as } \varepsilon \to 0.$$

□

As a consequence of the previous result, the following estimates on the H^1_ε norm of the solutions and on the Poisson term can be established.

Corollary 7.2. *Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative. Let $u_\varepsilon \subset E_\varepsilon$ be the sequence of positive solutions to (7.7) found in Theorem 7.2. For all $p \in [3,5)$, it holds*

$$||u_\varepsilon||^2_{H^1_\varepsilon} \le C\varepsilon^3,$$

Furthermore, for all $p \in (3,5)$, it holds

$$\int_{\mathbb{R}^3} |\nabla \phi_{u_\varepsilon}|^2\,\mathrm{d}x \le C\varepsilon^3,$$

for some $C > 0$*.*

Proof. Using the result of the previous theorem, we see that

$$\begin{aligned}\varepsilon^3(p+1)(\bar{c}+o(1)) &\geq (p+1)c_\varepsilon \\ &= (p+1)J_\varepsilon(u_\varepsilon) - J'_\varepsilon(u_\varepsilon)(u_\varepsilon) \\ &= \frac{p-1}{2}||u_\varepsilon||_{H^1_\varepsilon} + \frac{p-3}{4}\int_{\mathbb{R}^3}|\nabla\phi_{u_\varepsilon}|^2 \\ &\quad + \int_{\mathbb{R}^3}(g(x,u_\varepsilon)u_\varepsilon - (p+1)G(x,u_\varepsilon)). \end{aligned} \tag{7.39}$$

By (g_3), we have that

$$\begin{aligned}\int_{\mathbb{R}^3}(g(x,u_\varepsilon)u_\varepsilon - (p+1)G(x,u_\varepsilon)) &\geq -(p-1)\int_{\mathbb{R}^3\setminus\Lambda}G(x,u_\varepsilon) \\ &\geq \frac{-(p-1)}{2}\int_{\mathbb{R}^3\setminus\Lambda}\kappa u_\varepsilon^2 \\ &\geq \frac{-(p-1)}{4}||u_\varepsilon||^2_{H^1_\varepsilon}. \end{aligned} \tag{7.40}$$

Hence, putting (7.39) and (7.40) together, we have

$$\varepsilon^3(p+1)(\bar{c}+o(1)) \geq \frac{p-1}{4}||u_\varepsilon||^2_{H^1_\varepsilon} + \frac{p-3}{4}\int_{\mathbb{R}^3}|\nabla\phi_{u_\varepsilon}|^2,$$

and so we have shown that for $\varepsilon > 0$ small enough, it holds

$$||u_\varepsilon||^2_{H^1_\varepsilon} \leq \frac{4(p+1)\bar{c}}{p-1}\varepsilon^3,$$

and

$$^{2}\int_{\mathbb{R}^3}|\nabla\phi_{u_\varepsilon}|^2\,\mathrm{d}x \leq \frac{4(p+1)\bar{c}}{p-3}\varepsilon^3.$$

[2]This is the only point in which we use the restriction $p > 3$.

This concludes the proof. □

As in the case of the nonlinear Schrödinger equation, it is natural to rescale the solutions u_ε as $u_\varepsilon(x_\varepsilon+\varepsilon\cdot)$ around a well chosen family of points x_ε. A crucial step is to observe that such sequences are relatively compact for the uniform $C^{1,\alpha}$-convergence of compact sets. This requires much more delicate regularity and bootstrap arguments in the case of the nonlinear Schrödinger-Poisson system, provided in the proof of the following proposition.

Proposition 7.3. *Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative and $p \in [3,5)$. Let $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^3$ be sequences such that $\varepsilon_n \to 0$ and $x_n \to x^* \in \bar{\Lambda}$ as $n \to +\infty$. Assume $u_{\varepsilon_n} \subset E_{\varepsilon_n}$ is the sequence of positive solutions to (7.7) found in Theorem 7.2 and define $v_n(x) := u_{\varepsilon_n}(x_n+\varepsilon_n x)$. Then, v_n is positive and there exists $v \in H^1(\mathbb{R}^3)$ such that, up to a subsequence,*

$$v_n \to v \text{ in } C^{1,\alpha}_{loc}(\mathbb{R}^3).$$

Proof. Since u_{ε_n} solves (7.7), it follows that v_n solves

$$-\Delta v_n = -v_n - \rho(x_n+\varepsilon_n x)\bar{\phi}_n(x)v_n + g(x_n+\varepsilon_n x, v_n), \tag{7.41}$$

where

$$\bar{\phi}_n(x) := \phi_{u_{\varepsilon_n}}(x_n+\varepsilon_n x) = \varepsilon_n^2\left(\int_{\mathbb{R}^3}\frac{v_n^2(y)\rho(x_n+\varepsilon_n y)}{4\pi|x-y|}\right),$$

and

$$g(x_n+\varepsilon_n x, v_n) = \chi_\Lambda(x_n+\varepsilon_n x)v_n^p + (1-\chi_\Lambda(x_n+\varepsilon_n x))\min\{\kappa v_n, v_n^p\}.$$

We first note that by Corollary 7.2, we have that v_n is uniformly bounded in $H^1(\mathbb{R}^3)$. We then notice that since $x_n \to x^* \in \bar{\Lambda}$, then for any $x \in K$, where $K \subset \mathbb{R}^3$ is some fixed compact set, and for n large enough, it holds that $x_n+\varepsilon_n x \in B$, where B is some ball such that $\bar{\Lambda} \subset B$. So, for n large,

$$\varepsilon_n^2 \left[\int_K v_n^6 \rho^3(x_n+\varepsilon_n x)\right]^{\frac{1}{3}} \leq \varepsilon_n^2 ||\rho||_{L^\infty(B)} ||v_n||^2_{L^6(\mathbb{R}^3)}$$
$$\leq C\varepsilon_n^2 ||\rho||_{L^\infty(B)} ||\nabla v_n||^2_{L^2(\mathbb{R}^3)}$$
$$\leq C\varepsilon_n^2 ||\rho||_{L^\infty(B)} ||v_n||^2_{H^1(\mathbb{R}^3)}$$
$$\leq C_K \varepsilon_n^2,$$

and thus $\varepsilon_n^2 v_n^2 \rho(x_n+\varepsilon_n x)$ is uniformly bounded in $L^3_{loc}(\mathbb{R}^3)$. It follows that $\bar{\phi}_n$ is uniformly bounded in $C^{0,\alpha}_{loc}(\mathbb{R}^3)$ and consequently, is uniformly bounded in $L^\infty_{loc}(\mathbb{R}^3)$ (see e.g. [42, p. 260]; [1, p. 11]). Hence, $\rho(x_n+\varepsilon_n x)\bar{\phi}_n(x)$ is uniformly bounded in $L^\infty_{loc}(\mathbb{R}^3)$ since by assumption $\rho \in L^\infty_{loc}(\mathbb{R}^3)$ and $x_n \to x^* \in \bar{\Lambda}$. So, looking at the right-hand side of (7.41), on any compact set K, we have that

$$|-v_n - \rho(x_n+\varepsilon_n x)\bar{\phi}_n(x) v_n + g(x_n+\varepsilon_n x, v_n)| \leq (1+\rho(x_n+\varepsilon_n x)\bar{\phi}_n(x))v_n + v_n^p$$
$$\leq C_K(v_n + v_n^p). \tag{7.42}$$

We now use a bootstrap argument. From Corollary 7.2 and the Sobolev inequality it holds that v_n is uniformly bounded in $L^q(B(0,R))$ for all $q \in [1,6]$, and so from (7.42) we have that the right-hand side of (7.41) is uniformly bounded in $L^{\frac{6}{p}}(B(0,R))$ for any $R>0$. It then follows that v_n is uniformly bounded in $W^{2,\frac{6}{p}}(B(0,R))$ (see e.g. p. 235 in [35]). We have three cases:

(i) If $\frac{12}{p} > 3$ (namely, $p \in [3,4)$), then

$$W^{2,\frac{6}{p}}(B(0,R)) \subset C^{0,\alpha}(\overline{B(0,R)}), \quad \alpha \leq 2 - \frac{p}{2},$$

and so v_n is is uniformly bounded in $L^\infty(\overline{B(0,R)})$. Therefore, using this and (7.42) it follows that the right-hand side of (7.41) is uniformly bounded in $L^\infty(\overline{B(0,R)})$, which implies that v_n is uniformly bounded in $C^{1,\alpha}(\overline{B(0,R)})$.

(ii) If $\frac{12}{p} = 3$ (namely, $p=4$), then

$$W^{2,\frac{6}{p}}(B(0,R)) \subset L^q(B(0,R)), \quad \forall q \in [1,+\infty).$$

Using this and (7.42) we have that the right-hand side of (7.41) is uniformly bounded in $L^{q'}(B(0,R))$, where $q' = \frac{q}{p} \in [1,+\infty)$. It follows that v_n is uniformly bounded in $W^{2,q'}(B(0,R))$ for all $q' \in [1,+\infty)$.

(iii) If $p \in (4,5)$, then

$$W^{2,\frac{6}{p}}(B(0,R)) \subset L^q(B(0,R)), \quad \forall q \in \left[1, \frac{6}{p-4}\right].$$

Using this and (7.42) we have that the right-hand side of (7.41) is uniformly bounded in $L^{\frac{6}{p(p-4)}}(B(0,R))$. It follows that v_n is uniformly bounded in $W^{2,\frac{6}{p(p-4)}}(B(0,R))$.

In cases (i) and (ii), namely for $p \in [3,4]$, we are done. In case (iii), namely if $p \in (4,5)$, we go through a second iteration of this process and again find three cases depending on the value of p:

(i') If $\frac{12}{p(p-4)} > 3$ (namely, $p \in (4, 2+\sqrt{8} \approx 4.83)$, then

$$W^{2,\frac{6}{p(p-4)}}(B(0,R)) \subset C^{0,\alpha}(\overline{B(0,R)}), \quad \alpha \leq 2 - \frac{p(p-4)}{2}.$$

As in (i), we can show v_n is uniformly bounded in $C^{1,\alpha}(\overline{B(0,R)})$.

(ii') If $\frac{12}{p(p-4)} = 3$ (namely, $p = 2+\sqrt{8} \approx 4.83$), then

$$W^{2,\frac{6}{p(p-4)}}(B(0,R)) \subset L^q(B(0,R)), \quad \forall q \in [1,+\infty).$$

As in (ii), it follows that v_n is uniformly bounded in $W^{2,q'}(B(0,R))$ for all $q' \in [1,+\infty)$.

(iii') If $\frac{12}{p(p-4)} < 3$ (namely, $p \in (2+\sqrt{8} \approx 4.83, 5)$), then

$$W^{2,\frac{6}{p(p-4)}}(B(0,R)) \subset L^q(B(0,R)), \quad \forall q \in \left[1, \frac{6}{p(p-4)-4}\right].$$

Using this and (7.42) we have that the right-hand side of (7.41) is uniformly bounded in $L^{\frac{6}{p[p(p-4)-4]}}(B(0,R))$. It follows that v_n is uniformly bounded in $W^{2,\frac{6}{p[p(p-4)-4]}}(B(0,R))$.

Again, in cases (i') and (ii'), namely for $p \in [3, 2+\sqrt{8} \approx 4.83]$, we are done. In case (iii'), namely if $p \in (2+\sqrt{8} \approx 4.83, 5)$, a third iteration of this process reveals, once again, three cases depending on the value of p. Continuing to iterate this process, we find that at the n^{th} iteration, the third case does not occur for all p such that

$$\begin{cases} \frac{12}{p} \geq 3, & \text{if } n = 1 \\ \frac{12}{p^n - 4\sum_{i=1}^{n-1} p^i} \geq 3, & \text{if } n \geq 2. \end{cases}.$$

For each $n \geq 1$, this can be rewritten as

$$p^n - 4\sum_{i=0}^{n-1} p^i \leq 0,$$

namely,

$$p^n - 4\left(\frac{1-p^n}{1-p}\right) \leq 0.$$

Solving this inequality for n, we find that this condition is equivalent to

$$n \geq \frac{\ln\left(\frac{4}{5-p}\right)}{\ln(p)}.$$

So, for fixed $p \in [3,5)$ we find that the third case does not occur after the n^{th} iteration, where

$$n = \left\lceil \frac{\ln\left(\frac{4}{5-p}\right)}{\ln(p)} \right\rceil \in \mathbb{N}.$$

Therefore, we have shown that for any $p \in [3,5)$, we are done after a finite number of steps depending on p. Now, since v_n in uniformly bounded in $H^1(\mathbb{R}^3)$ by Corollary 7.2, then up to a subsequence $v_n \rightharpoonup v$, for some $v \in H^1(\mathbb{R}^3)$. Hence, using the result of our bootstrap argument and Morrey estimates, we have that $v_n \to v$ in $C^{1,\alpha}(\overline{B(0,R)})$

for every $R > 0$. By a diagonal argument, it follows that, up to a subsequence, $v_n \to v$ in $C^{1,\alpha}_{loc}(\mathbb{R}^3)$. Finally, applying the strong maximum principle, we have that v_n is positive. □

Remark 7.4. *An alternative way to conclude the bootstrap argument is as follows. After the first iteration, if we are in case (iii), it holds that v_n^{p-1} is uniformly bounded in $L^{\frac{3}{2}+\delta}(B(0,R))$ for some $\delta > 0$, where $(p-1)\left(\frac{3}{2}+\delta\right) < \frac{6}{p-4}$. Note that it is possible to choose such a δ since*

$$\frac{6}{p-4} - \frac{3(p-1)}{2} = \frac{-3p(p-5)}{2(p-4)} > 0,$$

for $p \in (4,5)$. Multiplying and dividing by $1+v_n$, we can rewrite (7.41) as

$$-\Delta v_n = a_n + a_n v_n,$$

where

$$a_n := \frac{-v_n - \rho(x_n+\varepsilon_n x)\bar{\phi}_n(x)v_n + g(x_n+\varepsilon_n x, v_n)}{1+v_n}.$$

Using (7.42), we can see that

$$|a_n| \leq C(1+v_n^{p-1}),$$

and so a_n is uniformly bounded in $L^{\frac{3}{2}+\delta}(B(0,R))$. From the Harnack inequality (see e.g. [54, p. 163]), it follows that

$$\sup_{B(0,R)} v_n \leq C\left(\inf_{B(0,2R)} v_n + K(R)\right),$$

with C and $K(R)$ depending only on δ, R, and $||a_n||_{L^{\frac{3}{2}+\delta}(B(0,4R))}$. Using this and Corollary 7.2 it holds that

$$\sup_{B(0,R)} v_n^6 \leq C_1 \int_B v_n^6 \leq C_2,$$

with $B(0,R) \subset B$. Therefore, it follows that v_n is uniformly bounded in $L^\infty(B(0,R))$, *and we are done.*

Corollary 7.4. *Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative and $p \in [3,5)$. Let $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^3$ be sequences such that $\varepsilon_n \to 0$, $x_n \to x^* \in \bar{\Lambda}$ as $n \to +\infty$, and*

$$\liminf_{n\to+\infty} u_{\varepsilon_n}(x_n) > 0,$$

where $u_{\varepsilon_n} \subset E_{\varepsilon_n}$ is the sequence of positive solutions to (7.7) found in Theorem 7.2. Define $v_n(x) := u_{\varepsilon_n}(x_n + \varepsilon_n x)$ and let $v \in H^1(\mathbb{R}^3)$ be the strong $C^{1,\alpha}_{loc}(\mathbb{R}^3)$ limit found in Proposition 7.3. Then, v is a positive weak solution to the equation

$$-\Delta v + v = \chi(x)v^p + (1-\chi(x))\min\{\kappa v, v^p\}, \tag{7.43}$$

where χ is a measurable function satisfying $0 \le \chi(x) \le 1$.

Proof. By Proposition 7.3, we have that, up to a subsequence,

$$v_n \to v \text{ in } C^{1,\alpha}_{loc}(\mathbb{R}^3).$$

Moreover, since Λ is smooth, up to a subsequence, $\chi_\Lambda(x_n + \varepsilon_n x)$ converges almost everywhere to a measurable function χ satisfying $0 \le \chi(x) \le 1$. Thus, taking the limit $n \to +\infty$ in the weak formulation of (7.41), it follows that v is a weak solution to the equation

$$-\Delta v + v = \chi(x)v^p + (1-\chi(x))\min\{\kappa v, v^p\}.$$

By assumption, we have that

$$v(0) = \lim_{n\to\infty} v_n(0) = \lim_{n\to\infty} u_{\varepsilon_n}(x_n) > 0,$$

and so it follows that v is nontrivial, and, in fact, positive by the strong maximum principle. □

In the next result, we show that there does, in fact, exist a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^3$ satisfying the assumptions of the previous corollary.

Lemma 7.19. *Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative and $p \in [3,5)$. Let $u_\varepsilon \subset E_\varepsilon$ be the sequence of positive solutions to (7.7) found in Theorem 7.2. Then, it holds that*

$$||u_\varepsilon||_{L^\infty(\Lambda)} > \kappa^{\frac{1}{p-1}},$$

where $\kappa > 0$ is as in (7.5).

Proof. Assume by contradiction, that $u_\varepsilon \leq \kappa^{\frac{1}{p-1}}$ in Λ. Then, for all $x \in \Lambda$,

$$g(x, u_\varepsilon) = u_\varepsilon^{p-1} \cdot u_\varepsilon \leq \kappa u_\varepsilon,$$

and, moreover, for all $x \notin \Lambda$

$$g(x, u_\varepsilon) = \min\left\{\kappa u_\varepsilon, u_\varepsilon^p\right\} \leq \kappa u_\varepsilon.$$

Thus, for all $x \in \mathbb{R}^3$, it holds that

$$\begin{aligned} 0 &= -\varepsilon^2 \Delta u_\varepsilon + u_\varepsilon + \rho(x)\phi_{u_\varepsilon} u_\varepsilon - g(x, u_\varepsilon) \\ &\geq -\varepsilon^2 \Delta u_\varepsilon + u_\varepsilon - g(x, u_\varepsilon) \\ &\geq -\varepsilon^2 \Delta u_\varepsilon + (1-\kappa) u_\varepsilon \end{aligned}$$

Multiplying the previous inequality by u_ε and integrating, we obtain

$$\int_{\mathbb{R}^3} \left(\varepsilon^2 |\nabla u_\varepsilon|^2 + (1-\kappa) u_\varepsilon^2\right) \leq 0,$$

contradicting the positivity of u_ε. This completes the proof. □

7.3.3 Further properties of rescaled solutions

In this section we establish some properties of the rescaled solutions v_n. Since we cannot directly apply the technique we discussed for the nonlinear Schrödinger equation in Section 7.1 in order to obtain a lower estimate of $\varepsilon_n^{-3} J(u_\varepsilon)$, we expect that having further information on the rescaled solutions will aid us in adapting the method to our problem. Going forward, we would like to use these properties to obtain a lower estimate similar to (7.20). We first state a result on the pointwise decay of the rescaled solutions, which we will need for the subsequent theorem.

Lemma 7.20. *Suppose $\rho \in C(\mathbb{R}^3)$ is nonnegative and $p \in [3,5)$. Let $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^3$ be sequences such that $\varepsilon_n \to 0$ and $x_n \to x^* \in \bar{\Lambda}$ as $n \to +\infty$. Assume $u_{\varepsilon_n} \subset E_{\varepsilon_n}$ is the sequence of positive solutions to (7.7) found in Theorem 7.2 and define $v_n(x) := u_{\varepsilon_n}(x_n + \varepsilon_n x)$. Then, for every $\lambda > 0$ and $\delta \in (0,1)$, there exists $C > 0$, possibly depending on n, such that*

$$v_n(x) \leq C e^{-\lambda(1+|x|)^\delta}.$$

Proof. We note that v_n satisfies (7.41), which we can rewrite as

$$-\Delta v_n + \left[1 + \varepsilon_n^2 \rho(x_n + \varepsilon_n x) \left(\int_{\mathbb{R}^3} \frac{v_n^2(y)\rho(x_n + \varepsilon_n y)}{4\pi|x-y|} \right) \right] v_n = \frac{g(x_n + \varepsilon_n x, v_n)}{v_n^p} v_n^p.$$

By definition, it holds that $g(x_n + \varepsilon_n x, v_n) \leq v_n^p$, and so

$$\frac{g(x_n + \varepsilon_n x, v_n)}{v_n^p} \leq 1.$$

Moreover,

$$\liminf_{|x|\to+\infty} \left[1 + \varepsilon_n^2 \rho(x_n + \varepsilon_n x) \left(\int_{\mathbb{R}^3} \frac{v_n^2(y)\rho(x_n + \varepsilon_n y)}{4\pi|x-y|} \right) \right] |x|^{2-2\delta} \geq |x|^{2-2\delta} > \lambda^2,$$

for any $\lambda > 0$ and $\delta \in (0,1)$. Thus, the conclusion follows from Theorem 8 (*ii*) in [21]. □

Next, using the previous result and the bounds established in Corollary 7.2, we find a uniform L^∞ bound on the rescaled solutions. We note that, at this point, we can only prove the following theorem under the additional assumption that $\rho \in L^\infty(\mathbb{R}^3)$, and so, in what follows, the space E_{ε_n} coincides with the larger space $H^1_{\varepsilon_n}$.

Theorem 7.3. *Suppose $\rho \in C(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is nonnegative and $p \in [3,5)$. Let $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^3$ be sequences such that $\varepsilon_n \to 0$ and $x_n \to x^* \in \bar{\Lambda}$ as $n \to +\infty$. Assume $u_{\varepsilon_n} \subset H^1_{\varepsilon_n}$ is the sequence of positive solutions to (7.7) found in Theorem 7.2 and define $v_n(x) := u_{\varepsilon_n}(x_n + \varepsilon_n x)$. Then, it holds that $\sup_{n\in\mathbb{N}} ||v_n||_{L^\infty(\mathbb{R}^3)} < +\infty$.*

Proof. We will argue by contradiction. Assume, to the contrary, that there exists a sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ such that $\varepsilon_m \to 0$ as $m \to +\infty$, u_{ε_m} solves (7.7) for each m, and it holds

$$||v_m||_{L^\infty(\mathbb{R}^3)} \to +\infty \text{ as } m \to +\infty,$$

where $v_m(x) := u_{\varepsilon_m}(x_m + \varepsilon_m x)$. Let

$$\alpha_m := \max v_m, \qquad (\alpha_m \to +\infty \text{ as } m \to +\infty),$$
$$\beta_m := \alpha_m^{-(p-1)/2}, \qquad (\beta_m \to 0 \text{ as } m \to +\infty).$$

Define

$$\bar{v}_m(x) := \frac{1}{\alpha_m} v_m(\bar{x}_m + \beta_m x),$$

where $\bar{x}_m$ is a global maximum point of v_m. We note that such a point exists because, by regularity theory, v_m are solutions in the classical sense and, moreover, by Lemma 7.20, v_m decays to zero for each m. Now, since u_{ε_m} solves (7.7), then v_m solves (7.41). Multiplying this equation by $\frac{\beta_m^2}{\alpha_m}$, we obtain

$$\begin{aligned}
-\frac{\beta_m^2}{\alpha_m}&\Delta v_m(\bar{x}_m + \beta_m x) \\
&= -\frac{\beta_m^2}{\alpha_m} v_m(\bar{x}_m + \beta_m x) \\
&\quad - \frac{\beta_m^2}{\alpha_m} \rho(x_m + \varepsilon_m \bar{x}_m + \varepsilon_m \beta_m x) \phi_{u_{\varepsilon_m}}(x_m + \varepsilon_m \bar{x}_m + \varepsilon_m \beta_m x) v_m(\bar{x}_m + \beta_m x)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\beta_m^2}{\alpha_m}\Big[\chi_\Lambda(x_m+\varepsilon_m\bar{x}_m+\varepsilon_m\beta_m x)v_m^p(\bar{x}_m+\beta_m x)\\
&+(1-\chi_\Lambda(x_m+\varepsilon_m\bar{x}_m+\varepsilon_m\beta_m x))\min\{\kappa v_m(\bar{x}_m+\beta_m x),v_m^p(\bar{x}_m+\beta_m x)\}\Big].
\end{aligned}
$$

Noting that $\Delta\bar{v}_m(x)=\beta_m^2\Delta v_m(\bar{x}_m+\beta_m x)/\alpha_m$ and $\beta_m^2\alpha_m^{p-1}=1$, we see that $\bar{v}_m$ satisfies

$$
\begin{aligned}
-\Delta\bar{v}_m=&-\beta_m^2\bar{v}_m\\
&-\beta_m^2\rho(x_m+\varepsilon_m\bar{x}_m+\varepsilon_m\beta_m x)\phi_{u_{\varepsilon_m}}(x_m+\varepsilon_m\bar{x}_m+\varepsilon_m\beta_m x)\bar{v}_m\\
&+\chi_\Lambda(x_m+\varepsilon_m\bar{x}_m+\varepsilon_m\beta_m x)\bar{v}_m^p\\
&+(1-\chi_\Lambda(x_m+\varepsilon_m\bar{x}_m+\varepsilon_m\beta_m x))\min\left\{\kappa\frac{\bar{v}_m}{\alpha_m^{p-1}},\bar{v}_m^p\right\}. \qquad (7.44)
\end{aligned}
$$

Now, since

$$
\begin{aligned}
\phi_{u_{\varepsilon_m}}(x_m+\varepsilon_m\bar{x}_m+\varepsilon_m\beta_m x)&=\int_{\mathbb{R}^3}\frac{u_{\varepsilon_m}^2(y)\rho(y)}{|x_m+\varepsilon_m\bar{x}_m+\varepsilon_m\beta_m x-y|}\,\mathrm{d}y\\
&=\alpha_m^2\varepsilon_m^2\beta_m^2\int_{\mathbb{R}^3}\frac{\bar{v}_m^2(y)\rho(x_m+\varepsilon_m\bar{x}_m+\varepsilon_m\beta_m y)}{|x-y|}\,\mathrm{d}y,
\end{aligned}
$$

then (7.44) reads as

$$
\begin{aligned}
-\Delta\bar{v}_m=&-\beta_m^2\bar{v}_m\\
&-\alpha_m^2\varepsilon_m^2\beta_m^4\rho(x_m+\varepsilon_m\bar{x}_m+\varepsilon_m\beta_m x)\left(\int_{\mathbb{R}^3}\frac{\bar{v}_m^2(y)\rho(x_m+\varepsilon_m\bar{x}_m+\varepsilon_m\beta_m y)}{|x-y|}\,\mathrm{d}y\right)\bar{v}_m\\
&+\chi_\Lambda(x_m+\varepsilon_m\bar{x}_m+\varepsilon_m\beta_m x)\bar{v}_m^p\\
&+(1-\chi_\Lambda(x_m+\varepsilon_m\bar{x}_m+\varepsilon_m\beta_m x))\min\left\{\kappa\frac{\bar{v}_m}{\alpha_m^{p-1}},\bar{v}_m^p\right\}. \qquad (7.45)
\end{aligned}
$$

It is worth noting here that since $\alpha_m\to+\infty$ as $m\to+\infty$, then, using the assumption $p\geq 3$, it follows that $\alpha_m^2\varepsilon_m^2\beta_m^4=\alpha_m^{-2(p-1)}\varepsilon_m^2\alpha_m^2=\alpha_m^{4-2p}\varepsilon_m^2\to 0$ as $m\to+\infty$. Hence, we can use the same regularity arguments as in the proof of Theorem 6.2. Namely, we notice that, by construction, $||\bar{v}_m||_{L^\infty(\mathbb{R}^3)} = 1$ for all m, and, by assumption, ρ is in $L^\infty(\mathbb{R}^3)$. So, since $\bar{v}_m^2\rho$ is uniformly bounded in $L^\infty(\mathbb{R}^3)$, then $\int_{\mathbb{R}^3} \frac{\bar{v}_m^2(y)\rho(x_m+\varepsilon_m\bar{x}_m+\varepsilon_m\beta_m y)}{4\pi|x-y|}$ is uniformly bounded in $C^{0,\alpha}_{loc}(\mathbb{R}^3)$ and consequently, is uniformly bounded in $L^\infty_{loc}(\mathbb{R}^3)$ (see e.g. [42, p. 260]; [1, p. 11]). Thus, the entire right-hand side of (7.45) is uniformly bounded in $L^\infty_{loc}(\mathbb{R}^3)$ which implies $\bar{v}_m$ is uniformly bounded in $C^{1,\alpha}_{loc}(\mathbb{R}^3)$ (see e.g. [35]). It then follows that uniformly on compact sets and for some $\bar{v} \in C^1(\mathbb{R}^3)$

$$\partial^\beta \bar{v}_m \to \partial^\beta \bar{v} \text{ as } m \to +\infty, \quad |\beta| \leq 1.^3 \tag{7.46}$$

We now notice that Corollary 7.2 implies that $||v_m||_{H^1} \leq C$ for some $C > 0$, and so we have

$$\begin{aligned} ||\bar{v}_m||^{p+1}_{L^{p+1}} &= \frac{1}{\alpha_m^{p+1}\beta_m^3}||v_m||^{p+1}_{L^{p+1}} \\ &= \alpha_m^{(p-5)/2}||v_m||^{p+1}_{L^{p+1}} \\ &\leq C\alpha_m^{(p-5)/2}||v_m||^{p+1}_{H^1} \\ &\leq \bar{C}\alpha_m^{(p-5)/2}, \end{aligned} \tag{7.47}$$

Since $\alpha_m^{(p-5)/2} \to 0$ as $m \to +\infty$, it follows from (7.46) and (7.47), that $\bar{v} \equiv 0$ by Fatou's Lemma. On the other hand, by definition $\bar{v}_m(0) = v_m(\bar{x}_m)/\alpha_m = \alpha_m/\alpha_m = 1$ for all m, and so in particular $\bar{v}(0) = 1$. We have therefore reached a contradiction and have proved that

$$\sup_{n\in\mathbb{N}} ||v_n||_{L^\infty(\mathbb{R}^3)} < +\infty.$$

□

Remark 7.5. *We note that this proof is very similar to the proof of Theorem 6.2, however in the previous proof we used (7.47) to arrive at a contradiction rather than obtaining a contradiction from the limiting problem. We are not able to simplify the proof of Theorem 6.2 in this way because we have no information about the variational characterisation of the solutions in that theorem. Indeed, to obtain (7.47) in the previous theorem, we use Corollary 7.2, which relies in a vital way on the fact that the solutions we deal with are Mountain Pass solutions.*

[3] In fact, we only require uniform convergence at this stage.

Remark 7.6. *An interesting consideration is how far we can get in the previous proof if we do not have the variational characterisation of the solutions, namely if we do not have* (7.47)*. If this is the case, we must go to the limiting problem associated with* (7.45)*. We first recall that* $\alpha_m^2\varepsilon_m^2\beta_m^4 = \alpha_m^{4-2p}\varepsilon_m^2 \to 0$ *as* $m \to +\infty$ *since* $p \geq 3$*. Moreover, since* Λ *is smooth, it follows that, up to a subsequence,* $\chi_\Lambda(x_m + \varepsilon_m\bar{x}_m + \varepsilon_m\beta_m x)$ *converges almost everywhere to a measurable function* $\bar{\chi}$ *which satisfies* $\bar{\chi} \in [0,1]$*. Thus, taking the limit* $m \to +\infty$ *in* (7.45)*, we get*

$$\begin{cases} -\Delta\bar{v} = \bar{\chi}(x)\bar{v}^p, & x \in \mathbb{R}^3 \\ \bar{v}(0) = 1. \end{cases} \tag{7.48}$$

We now look at three possible cases. If

$$\limsup_{m\to+\infty} \frac{dist(x_m + \varepsilon_m\bar{x}_m, \mathbb{R}^3 \setminus \Lambda)}{\varepsilon_m\beta_m} = +\infty, \tag{7.49}$$

then this implies $\bar{\chi}(x) \equiv 1$*, and if*

$$\limsup_{m\to+\infty} \frac{dist(x_m + \varepsilon_m\bar{x}_m, \partial\Lambda)}{\varepsilon_m\beta_m} < +\infty, \tag{7.50}$$

then this would imply that $\bar{\chi}(x) = \chi_E(x)$*, where* E *is a half-space. In both cases, one can evoke some Liouville type theorems, as in [34], to show that* $\bar{v} \equiv 0$*, in contradiction with* $\bar{v}(0) = 1$*. If neither* (7.49) *nor* (7.50) *hold, then it follows that*

$$\limsup_{m\to+\infty} \frac{dist(x_m + \varepsilon_m\bar{x}_m, \Lambda)}{\varepsilon_m\beta_m} = +\infty, \tag{7.51}$$

and so $\bar{\chi}(x) \equiv 0$*. In order to arrive at a contradiction in this case, one needs to show that* $\bar{v}$ *has bounded energy. Thus, the conclusion in this case seems to rely on an* L^{p+1} *estimate such as* (7.47)*, and so we have the impression that this is precisely the point where the variational characterisation of the solutions is needed.*

7.3.3.1 Partial conclusions

We observe that by Proposition 7.3 we are able to prove the strong $C^{1,\alpha}_{loc}(\mathbb{R}^3)$ convergence of the rescaled solutions v_n. We will show that this strong convergence also holds in $H^1(\mathbb{R}^3)$, provided one proves the following uniform decay estimate on the rescaled solutions.

(H_1) For every $\lambda > 0$ and $\delta \in (0,1)$, there exists a uniform constant $C > 0$ such that

$$v_n(x) \le Ce^{-\lambda(1+|x|)^{\delta}},$$

where $v_n(x) := u_{\varepsilon_n}(x_n + \varepsilon_n x)$, u_{ε_n} is the sequence of nonnegative solutions to (7.7) found in Theorem 7.2, and $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^3$ are sequences such that $\varepsilon_n \to 0$ and $x_n \to x^* \in \bar{\Lambda}$ as $n \to +\infty$.

Assuming that $\rho \in C(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is nonnegative and $p \in [3,5)$, we believe that it is very plausible that (H_1) can be proved by combining the previous two results we have obtained, namely, the pointwise decay estimate found in Lemma 7.20 and the uniform L^∞ bound found in Theorem 7.3. Hypothesis (H_1) will be required for the following result.

Lemma 7.21. *Suppose $\rho \in C(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is nonnegative and $p \in (3,5)$. Let $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^3$ be sequences such that $\varepsilon_n \to 0$ and $x_n \to x^* \in \bar{\Lambda}$ as $n \to +\infty$. Assume $u_{\varepsilon_n} \subset H^1_{\varepsilon_n}$ is the sequence of nonnegative solutions to (7.7) found in Theorem 7.2 and define $v_n(x) := u_{\varepsilon_n}(x_n + \varepsilon_n x)$. Then, if (H_1) holds, it follows that up to a subsequence,*

$$v_n \to v \text{ in } H^1(\mathbb{R}^3),$$

where $v \in H^1(\mathbb{R}^3)$ is a positive weak solution to equation (7.43).

Proof. By Corollary 7.2 we have that v_n is uniformly bounded in $H^1(\mathbb{R}^3)$. By Proposition 7.3 and Corollary 7.4, we have that up to a subsequence $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$, where $v \in H^1(\mathbb{R}^3)$ is a positive weak solution to equation (7.43), which can be rewritten as

$$-\Delta v + v = \left[\frac{\chi(x)v^p + (1-\chi(x))\min\{\kappa v, v^p\}}{v^p}\right] v^p.$$

Since

$$\frac{\chi(x)v^p + (1-\chi(x))\min\{\kappa v, v^p\}}{v^p} \le 1,$$

we can follow the same arguments as in the proof of Lemma 7.20, to show that for every $\lambda > 0$ and $\delta \in (0,1)$, there exists a constant $\bar{C} > 0$ such that

$$v(x) \leq \bar{C}e^{-\lambda(1+|x|)^{\delta}},$$

by Theorem 8 (ii) in [21]. Combining this with (H_1), we have

$$|v_n - v| \leq 2\max\{C,\bar{C}\}e^{-\lambda(1+|x|)^{\delta}} \lesssim e^{-\lambda(1+|x|)^{\delta}}.$$

So, for any $\varepsilon > 0$, there exists an $R > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^3\setminus B_R} |v_n - v|^q &= \int_{\mathbb{R}^3\setminus B_R} |v_n - v|^{q-2}|v_n - v|^2 \\
&\lesssim e^{-\lambda(q-2)(1+|R|)^{\delta}}||v_n - v||^2_{H^1(\mathbb{R}^3)} \\
&< ce^{-\lambda(q-2)(1+|R|)^{\delta}} \\
&< c\varepsilon,
\end{aligned}$$

for some uniform constant $c > 0$ and for all $q > 2$. This and the classic Rellich Theorem implies that, passing if necessary to a subsequence,

$$\int_{\mathbb{R}^3} |v_n - v|^q \to 0,$$

for all $q \in (2,6)$. We now note that since v_n solves (7.41), it follows that

$$\int_{\mathbb{R}^3} (\nabla v_n \nabla(v_n - v) + v_n(v_n - v)) + \int_{\mathbb{R}^3} \rho(x_n + \varepsilon_n x)\bar{\phi}_n(x)v_n(v_n - v) \\
- \int_{\mathbb{R}^3} g(x_n + \varepsilon_n x, v_n)(v_n - v) \to 0, \tag{7.52}$$

where

$$\bar{\phi}_n(x) := \phi_{u_{\varepsilon_n}}(x_n + \varepsilon_n x) = \varepsilon_n^2 \left(\int_{\mathbb{R}^3} \frac{v_n^2(y)\rho(x_n + \varepsilon_n y)}{4\pi|x - y|} \right),$$

and

$$g(x_n+\varepsilon_n x, v_n) = \chi_\Lambda(x_n+\varepsilon_n x)v_n^p + (1-\chi_\Lambda(x_n+\varepsilon_n x))\min\{\kappa v_n, v_n^p\}.$$

Since $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$, it holds that

$$\int_{\mathbb{R}^3} (\nabla v_n \nabla(v_n - v) + v_n(v_n - v)) \to ||v_n||^2_{H^1(\mathbb{R}^3)} - ||v||^2_{H^1(\mathbb{R}^3)}, \tag{7.53}$$

and since we showed that $v_n \to v$ in $L^q(\mathbb{R}^3)$ for all $q \in (2,6)$, it also holds that

$$\int_{\mathbb{R}^3} g(x_n+\varepsilon_n x, v_n)(v_n - v) \to 0. \tag{7.54}$$

Moreover, since $\rho \in L^\infty(\mathbb{R}^3)$ by assumption and $\nabla\bar{\phi}_n$ is uniformly bounded in $L^2(\mathbb{R}^3)$ by Corollary 7.2, we have that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} \rho(x_n+\varepsilon_n x)\bar{\phi}_n(x)v_n(v_n-v)\right| &\leq ||\rho||_{L^\infty(\mathbb{R}^3)}||\bar{\phi}_n||_{L^6(\mathbb{R}^3)}||v_n(v_n-v)||_{L^{\frac{6}{5}}(\mathbb{R}^3)} \\ &\leq C||\rho||_{L^\infty(\mathbb{R}^3)}||\nabla\bar{\phi}_n||_{L^2(\mathbb{R}^3)}||v_n(v_n-v)||_{L^{\frac{6}{5}}(\mathbb{R}^3)} \\ &\leq C_1||v_n(v_n-v)||_{L^{\frac{6}{5}}(\mathbb{R}^3)} \\ &\leq C_1||v_n||_{L^{\frac{12}{5}}(\mathbb{R}^3)}||(v_n-v)||_{L^{\frac{12}{5}}(\mathbb{R}^3)} \to 0, \end{aligned} \tag{7.55}$$

where the final convergence holds since v_n is uniformly bounded in $L^{\frac{12}{5}}(\mathbb{R}^3)$ as a consequence of Corollary 7.2 and we showed $v_n \to v$ in $L^q(\mathbb{R}^3)$ for all $q \in (2,6)$. Therefore, combining (7.52), (7.53), (7.54), and (7.55), implies that

$$||v_n||_{H^1(\mathbb{R}^3)} \to ||v||_{H^1(\mathbb{R}^3)}.$$

Using this and the fact that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$, we obtain

$$||v_n - v||^2_{H^1(\mathbb{R}^3)} = ||v_n||^2_{H^1(\mathbb{R}^3)} - 2(v_n, v)_{H^1(\mathbb{R}^3)} + ||v||^2_{H^1(\mathbb{R}^3)} \to 0,$$

and so we have shown that

$$v_n \to v \text{ in } H^1(\mathbb{R}^3),$$

as required. $\square$

8 Related questions

In addition to accomplishing the sufficient conditions programme discussed in the previous chapter, there are a number of interesting questions related to our work in this thesis which, in our opinion, are worth studying in future projects.

A. Radial versus non-radial solutions. In the case ρ is a radial function one can restrict on functions having the same symmetry to find radial solutions, using Palais criticality principle, in all of the scenarios we have discussed (coercive/non-coercive cases, for low/large p). It is not clear how to compare the energy levels nor the symmetry of the solutions with those that one finds using the non-radial approaches employed in this thesis.

B. Variational characterisation. When $p \in (2,3]$, it is not obvious whether the mountain pass critical points for I, are least energy solutions. Namely, for $p \in (2,3]$, there is no clear relation between the solutions found in Theorem 4.1 (for $p = 3$) and Theorem 4.2 with those found in Corollary 4.3, as well as between the solutions found in Theorem 4.4 (for $p = 3$) and Theorem 4.3 with those found in Corollary 4.6.

C. Multiplicity. We suspect that the constrained minimisation approach in [61] may help refining the approach in [7], to obtain a multiplicity result for coercive ρ and $p \leq 3$, with a more relaxed relation between ρ and $\nabla\rho$ than is assumed in Theorem 5.3

D. 'Sharp' necessary conditions for point concentration. Is it possible to allow a faster growth for ρ in the necessary conditions for point concentration? The proof we provide is based on the uniform exponential decay of solutions, which is essentially due to the L^2 setting.

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