CANCELLATION THEOREM FOR FRAMED MOTIVES OF ALGEBRAIC VARIETIES

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ABSTRACT. The machinery of framed (pre)sheaves was developed by Voevodsky [V1]. Based on the theory, framed motives of algebraic varieties are introduced and studied in [GP1]. An analog of Voevodsky's Cancellation Theorem [V2] is proved in this paper for framed motives stating that a natural map of framed S^1-spectra

$$M_{fr}(X)(n) \to \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1)), \quad n \geqslant 0,$$

is a schemewise stable equivalence, where $M_{fr}(X)(n)$ is the nth twisted framed motive of X. This result is also necessary for the proof of the main theorem of [GP1] computing fibrant resolutions of suspension $\mathbb{P}^1$-spectra $\Sigma^\infty_{\mathbb{P}^1} X_+$ with X a smooth algebraic variety.

The Cancellation Theorem for framed motives is reduced to the Cancellation Theorem for linear framed motives stating that the natural map of complexes of abelian groups

$$\mathbb{Z}\mathrm{F}(\Delta^\bullet \times X, Y) \to \mathbb{Z}\mathrm{F}((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1)), \quad X, Y \in Sm/k,$$

is a quasi-isomorphism, where $\mathbb{Z}\mathrm{F}(X,Y)$ is the group of stable linear framed correspondences in the sense of [GP1].

1. INTRODUCTION

The main goal of the Voevodsky theory on framed correspondences (see [V1, Introduction]) is to suggest a new approach to the stable motivic homotopy theory $SH(k)$ over a field k. This approach is more amenable to explicit calculations. Recall that Voevodsky [V1, Section 2] invented a category of framed correspondences $Fr_*(k)$ whose objects are those of Sm/k and morphisms sets $Fr_*(X,Y) = \sqcup_{n\geqslant 0} Fr_n(X,Y)$ are defined by means of certain geometric data. The elements of $Fr_n(X,Y)$ are called *framed correspondences of level n*. The definition of $Fr_*(k)$ is recalled in Section 2 below. For every $Y \in Sm/k$ there is a distinguished morphism $\sigma_Y = (Y \times 0, Y \times \mathbb{A}^1, t, pr_Y) \in Fr_1(Y,Y)$. Following Voevodsky [V1], we denote by

$$Fr(X,Y) := \mathrm{colim}(Fr_0(X,Y) \xrightarrow{\sigma_Y} Fr_1(X,Y) \xrightarrow{\sigma_Y} \dots \xrightarrow{\sigma_Y} Fr_n(X,Y) \xrightarrow{\sigma_Y} \dots)$$

and refer to it as the *set of stable framed correspondences*. Replacing Y by a simplicial object $Y^\bullet$ in Sm/k, we get a simplicial set $Fr(X,Y^\bullet)$. Finally, one can take the diagonal of the pointed bisimplicial set $Fr(\Delta^\bullet \times X, Y^\bullet)$. Voevodsky conjectured that if the motivic space $Fr(\Delta^\bullet \times -, Y^\bullet)$ is locally connected in the Nisnevich topology, then it is isomorphic in $H_{\mathbb{A}^1}(k)$ to the motivic space $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(Y^\bullet_+)$. In particular, the theory of framed correspondences gives a machinery for computing motivic infinite loop spaces.

Inspired by the Voevodsky theory [V1], the theory of framed motives of algebraic varieties is introduced and developed in [GP1]. As an application, the above Voevodsky conjecture is solved in [GP1, Section 10] in the affirmative. Moreover, under the above assumption on $Y^\bullet$ the motivic space $Fr(\Delta^\bullet \times -, Y^\bullet)$ is $\mathbb{A}^1$-local. This result can be regarded as a motivic counterpart of the Segal

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theorem. Also, an alternative approach to the classical Morel–Voevodsky [MV] stable homotopy theory $SH(k)$ is suggested in [GP3], which is based on the machinery of framed bispectra. One of the key steps in the computations of [GP1, GP3] is Theorem A proved in this paper. Theorem A is the main result of the present paper. In order to state it, we have to recall some definitions and constructions from [GP1].

The framed motive of $X \in Sm/k$ is an explicitly constructed S^1-spectrum $M_{fr}(X)$, which is connected and an Ω-spectrum in positive degrees (see [GP1] for details). Following the notation of [GP1, Section 8] let $\mathbb{G}$ be the cone $(\mathbb{G}_m)_+//pt_+$ of the embedding $pt_+ \overset{1}{\hookrightarrow} (\mathbb{G}_m)_+$ in the category of pointed simplicial presheaves $sPre_\bullet(Sm/k)$. Its sheafification is represented in the category $\Delta^{\mathrm{op}}(Fr_0(k))$ by the object $\mathbb{G}_m^{\wedge 1}$ (see [GP1, Notation 8.1]). For any integer $n \geqslant 1$ let $\mathbb{G}_m^{\wedge n}$ be the nth monoidal power of $\mathbb{G}_m^{\wedge 1}$ in the symmetric monoidal category $\Delta^{\mathrm{op}}(Fr_0(k))$ (see [GP1, Notation 8.1]). For a variety $X \in Sm/k$ let $M_{fr}(X \times \mathbb{G}_m^{\wedge n})$ be the framed motive of the simplicial object $X \times \mathbb{G}_m^{\wedge n} \in \Delta^{\mathrm{op}}(Fr_0(k))$. It is an explicitly constructed S^1-spectrum which is connected and an Ω-spectrum in positive degrees (see [GP1, Sections 5 and 6] for details). For brevity we also write $M_{fr}(X)(n)$ to denote $M_{fr}(X \times \mathbb{G}_m^{\wedge n})$ and call $M_{fr}(X)(n)$ the *n-twisted framed motive of X* (see [GP1, Section 11]). The main object of [GP1] is the bispectrum

$$M_{fr}^{\mathbb{G}}(X) = (M_{fr}(X), M_{fr}(X)(1), M_{fr}(X)(2), \ldots),$$

each term of which is a twisted framed motive of X and structure maps of the bispectra

$$M_{fr}(X)(n) \to \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1)),$$

are defined in [GP1, Section 11] (we use [GP1, "General Framework" of Section 5]).

The major property of the bispectrum $M_{fr}^{\mathbb{G}}(X)$ is that its levelwise Nisnevich local stable replacement $M_{fr}^{\mathbb{G}}(X)_f$ is a fibrant replacement of the suspension bispectrum $\Sigma_{\mathbb{G}}^\infty \Sigma_{S^1}^\infty X_+$. This may also be viewed as a motivic version of the Barratt, Priddy, and Quillen theorem. The proof of this major property is given in [GP1] and is heavily based on the Cancellation Theorem.

The main purpose of the paper is to prove the following (cf. Voevodsky [V2])

Theorem A (Cancellation)**.** *Let k be an infinite perfect field, $X \in Sm/k$ and $n \geqslant 0$. Then the following statements are true:*

(1) *the natural map of S^1-spectra*

$$M_{fr}(X)(n) \to \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1))$$

is a schemewise stable equivalence;

(2) *the induced map of S^1-spectra*

$$M_{fr}(X)(n)_f \to \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1)_f)$$

is a schemewise stable equivalence. Here $M_{fr}(X)(n)_f$ and $M_{fr}(X)(n+1)_f$ are Nisnevich local stable fibrant replacements of $M_{fr}(X)(n)$ and $M_{fr}(X)(n+1)$ in the injective local stable model structure of S^1-spectra.

As an application of Theorem A we prove the following

Theorem B. *Let k be an infinite perfect field, $X \in Sm/k$ and $n \geqslant 0$. Then the bispectrum*

$$M_{fr}^{\mathbb{G}}(X)_f = (M_{fr}(X)_f, M_{fr}(X)(1)_f, M_{fr}(X)(2)_f, \ldots)$$

obtained from $M_{fr}^{\mathbb{G}}(X)$ by taking levelwise Nisnevich local stable fibrant replacements with structure maps those of Theorem A(2) is a motivically fibrant $(S^1, \mathbb{G})$-bispectrum.

The main strategy of proving Theorem A is to reduce it to the "Linear Cancellation Theorem". In order to formulate it, recall from [GP1, Definition 8.3] that the category $\mathbb{Z}\mathrm{F}_*(k)$ is the additive category whose objects are those of Sm/k with Hom-groups described in Definition 2.4. Briefly speaking, for every $n \geqslant 0$ and $X,Y \in Sm/k$ we set

$$\mathbb{Z}\mathrm{F}_n(X,Y) := \mathbb{Z}\mathrm{Fr}_n(X,Y)/\langle Z_1 \sqcup Z_2 - Z_1 - Z_2\rangle,$$

where Z_1, Z_2 are supports of framed correspondences level n in the sense of Voevodsky [V1] (see Definition 2.4 as well). In other words, $\mathbb{Z}\mathrm{F}_n(X,Y)$ is the free abelian group generated by the framed correspondences of level n with connected supports. We then set

$$\mathrm{Hom}_{\mathbb{Z}\mathrm{F}_*(k)}(X,Y) := \bigoplus_{n\geqslant 0} \mathbb{Z}\mathrm{F}_n(X,Y).$$

Given smooth varieties $X,Y \in Sm/k$ and $n \geqslant 0$, there is a canonical suspension morphism $\Sigma : \mathbb{Z}\mathrm{F}_n(X,Y) \to \mathbb{Z}\mathrm{F}_{n+1}(X,Y)$. We can stabilise in the Σ-direction to get an abelian group (see Definition 2.6)

$$\mathbb{Z}\mathrm{F}(X,Y) := \operatorname{colim}(\mathbb{Z}\mathrm{F}_0(X,Y) \xrightarrow{\Sigma} \mathbb{Z}\mathrm{F}_1(X,Y) \xrightarrow{\Sigma} \cdots).$$

The presheaf $\mathbb{Z}\mathrm{F}(Y) := \mathbb{Z}\mathrm{F}(-,Y)$ has a canonical structure of a $\mathbb{Z}\mathrm{F}_*(k)$-presheaf. For each scheme $Y \in Sm/k$ and each scheme $S \in Sm/k$ pointed at a k-rational point $s \in S$, the natural functor

$$\boxtimes : Pre_{Ab}(\mathbb{Z}\mathrm{F}_*(k)) \times Pre_{Ab}(\mathbb{Z}\mathrm{F}_0(k)) \to Pre_{Ab}(\mathbb{Z}\mathrm{F}_*(k))$$

defined on p. 6 takes the pair $(\mathbb{Z}\mathrm{F}(Y),(S,s))$ to the $\mathbb{Z}\mathrm{F}_*(k)$-presheaf $\mathbb{Z}\mathrm{F}(Y)\boxtimes(S,s)$ which we also denote by $\mathbb{Z}\mathrm{F}(Y \wedge (S,s))$. By the General Framework of [GP1, Section 5] (also see p. 6) one has a $\mathbb{Z}\mathrm{F}_*(k)$-presheaf $\underline{\mathrm{Hom}}((S,s),\mathbb{Z}\mathrm{F}(Y \wedge (S,s)))$ together with a morphism of $\mathbb{Z}\mathrm{F}_*(k)$-presheaves

$$\mathbb{Z}\mathrm{F}(Y) \xrightarrow{-\boxtimes(S,s)} \underline{\mathrm{Hom}}((S,s),\mathbb{Z}\mathrm{F}(Y \wedge (S,s)).$$

The Linear Cancellation Theorem is formulated as follows (see Section 2 for details).

Theorem C (Linear Cancellation)**.** *Let k be an infinite perfect field and let Y be a k-smooth scheme. Then*

$$-\boxtimes(\mathbb{G}_m,1)\colon\ \mathbb{Z}\mathrm{F}(\Delta^\bullet \times -,Y) \to \underline{\mathrm{Hom}}((\mathbb{G}_m,1),\mathbb{Z}\mathrm{F}(\Delta^\bullet \times -,Y \wedge (\mathbb{G}_m,1))$$

is a quasi-isomorphism of complexes of $\mathbb{Z}\mathrm{F}_*(k)$-presheaves of abelian groups. Here $\Delta^\bullet$ is the standard cosimplicial object in Sm/k.

One of the main computational results of [GNP] says that schemewise homology of the complex $\mathbb{Z}\mathrm{F}(\Delta^\bullet \times -,Y)$ computes homology of the framed motive $M_{fr}(Y)$ of $Y \in Sm/k$. Moreover, the complex represents the "linear framed motive" of Y (see [GNP] for details).

Throughout the paper the base field k is supposed to be infinite. We also employ the following notation:

- all schemes are separated Noetherian k-schemes, all morphisms of schemes are k-morphisms; write pt for the scheme $\mathrm{Spec}(k)$.
- Sm/k is the category of smooth k-schemes of finite type;
- we refer to the objects of Sm/k as k-smooth schemes or smooth k-schemes;
- Following [GrD], by an essentially smooth k-scheme we mean a Noetherian k-scheme X which is the inverse limit of a left filtering system $(X_i)_{i\in I}$ with each transition morphism $X_i \to X_j$ being an étale affine morphism between smooth k-schemes.

2. PRELIMINARIES

In this section we collect basic facts for framed correspondences. We start with preparations.

Let V be a scheme and Z be a closed subscheme. Recall that an *étale neighborhood of Z in V* is a triple $(W', \pi' : W' \to V, s' : Z \to W')$ satisfying the following conditions:

(i) π' is an étale morphism;

(ii) $\pi' \circ s'$ coincides with the inclusion $Z \hookrightarrow V$ (thus s' is a closed embedding);

(iii) $(\pi')^{-1}(Z) = s'(Z)$.

A morphism between two étale neighborhoods $(W', \pi', s') \to (W'', \pi'', s'')$ of Z in V is a morphism $\rho : W' \to W''$ such that $\pi'' \circ \rho = \pi'$ and $\rho \circ s' = s''$. Note that such ρ is automatically étale by [EGA4, VI.4.7].

Definition 2.1 (Voevodsky [V1])**.** For k-smooth schemes X, Y and $n \geqslant 0$ an *explicit framed correspondence* Φ of level n consists of the following data:

1. a closed subset Z in $\mathbb{A}^n_X$ which is finite over X;
2. an etale neighborhood $p : U \to \mathbb{A}^n_X$ of Z in $\mathbb{A}^n_X$;
3. a collection of regular functions $\varphi = (\varphi_1, \dots, \varphi_n)$ on U such that $\cap_{i=1}^n \{\varphi_i = 0\} = Z$;
4. a morphism $g : U \to Y$.

The subset Z will be referred to as the *support* of the correspondence. We shall also write triples $\Phi = (U, \varphi, g)$ or quadruples $\Phi = (Z, U, \varphi, g)$ to denote explicit framed correspondences.

Two explicit framed correspondences Φ and Φ' of level n are said to be *equivalent* if they have the same support and there exists an etale neighborhood V of Z in $U \times_{\mathbb{A}^n_X} U'$ such that the morphism $g \circ pr$ agrees with $g' \circ pr'$ and $\varphi \circ pr$ agrees with $\varphi' \circ pr'$ on V. A *framed correspondence of level n* is an equivalence class of explicit framed correspondences of level n.

We let $\mathrm{Fr}_n(X, Y)$ denote the set of framed correspondences from X to Y. It is a pointed set with the distinguished point being the class 0_n of the explicit correspondence with $U = \emptyset$.

As an example, the set $\mathrm{Fr}_0(X, Y)$ coincides with the set of pointed morphisms $X_+ \to Y_+$. In particular, for a connected scheme X one has

$$\mathrm{Fr}_0(X, Y) = \mathrm{Hom}_{Sm/k}(X, Y) \sqcup \{0_0\}.$$

If $f : X' \to X$ is a morphism of schemes and $\Phi = (U, \varphi, g)$ an explicit correspondence from X to Y then

$$f^*(\Phi) := (U' = U \times_X X', \varphi \circ pr, g \circ pr)$$

is an explicit correspondence from X' to Y.

The following definition is to describe compositions of framed correspondences.

Definition 2.2. Let X, Y and S be k-smooth schemes and let $a = (Z, U, (\varphi_1, \varphi_2, \dots, \varphi_n), g)$ be an explicit correspondence of level n from X to Y and let $b = (Z', U', (\psi_1, \psi_2, \dots, \psi_m), g')$ be an explicit correspondence of level m from Y to S. We define their composition as an explicit correspondence of level $n+m$ from X to S by

$$(Z \times_Y Z', U \times_Y U', (\varphi_1, \varphi_2, \dots, \varphi_n, \psi_1, \psi_2, \dots, \psi_m), g').$$

Clearly, the composition of explicit correspondences respects the equivalence relation on them and defines associative pairings

$$\mathrm{Fr}_n(X, Y) \times \mathrm{Fr}_m(Y, S) \to \mathrm{Fr}_{n+m}(X, S).$$

Given $X, Y \in Sm/k$, denote by $\mathrm{Fr}_*(X, Y)$ the set $\bigsqcup_n \mathrm{Fr}_n(X, Y)$. The composition of framed correspondences defined above gives a category $\mathrm{Fr}_*(k)$. Its objects are those of Sm/k and the morphisms are given by the sets $\mathrm{Fr}_*(X,Y)$, $X,Y \in Sm/k$. Since the naive morphisms of schemes can be identified with certain framed correspondences of level zero, we get a canonical functor

$$Sm/k \to \mathrm{Fr}_*(k).$$

One can easily see that for a framed correspondence $\Phi : X \to Y$ and a morphism $f : X' \to X$, one has $f^*(\Phi) = \Phi \circ f$.

Definition 2.3. Let X,Y,S and T be smooth schemes. There is an *external product*

$$\mathrm{Fr}_n(X,Y) \times \mathrm{Fr}_m(S,T) \xrightarrow{-\boxtimes-} \mathrm{Fr}_{n+m}(X \times S, Y \times T)$$

given by

$$(Z,U,(\varphi_1,\varphi_2,\dots,\varphi_n),g) \boxtimes (Z',U',(\psi_1,\psi_2,\dots,\psi_m),g') =$$
$$(Z \times Z', U \times U', (\varphi_1,\varphi_2,\dots,\varphi_n,\psi_1,\psi_2,\dots,\psi_m), g \times g').$$

For the constant morphism $c\colon \mathbb{A}^1 \to \mathrm{pt}$, we set (following Voevodsky [V1])

$$\Sigma = - \boxtimes (t,c,\{0\},\mathbb{A}^1,t,c)\colon \ \mathrm{Fr}_n(X,Y) \to \mathrm{Fr}_{n+1}(X,Y)$$

and refer to it as the *suspension*. If there is no likelihood of confusion, we shall also write Σ to denote the element $1 \cdot (t,c,\{0\},\mathbb{A}^1,t,c)$ in $\mathbb{Z}\mathrm{F}_1(\mathrm{pt},\mathrm{pt})$ and Σ^n for $\Sigma \boxtimes \dots \boxtimes \Sigma$ in $\mathbb{Z}\mathrm{F}_n(\mathrm{pt},\mathrm{pt})$. It will always be clear from the context which of the meanings for Σ is used (either as the suspension or as the element in $\mathbb{Z}\mathrm{F}_1(\mathrm{pt},\mathrm{pt})$).

Also, following Voevodsky [V1], one puts

$$\mathrm{Fr}(X,Y) = \mathrm{colim}(\mathrm{Fr}_0(X,Y) \xrightarrow{\Sigma} \mathrm{Fr}_1(X,Y) \xrightarrow{\Sigma} \dots \xrightarrow{\Sigma} \mathrm{Fr}_n(X,Y) \xrightarrow{\Sigma} \dots)$$

and refer to it as the *set of stable framed correspondences*. The above external product induces external products

$$\mathrm{Fr}_n(X,Y) \times \mathrm{Fr}(S,T) \xrightarrow{-\boxtimes-} \mathrm{Fr}(X \times S, Y \times T),$$
$$\mathrm{Fr}(X,Y) \times \mathrm{Fr}_0(S,T) \xrightarrow{-\boxtimes-} \mathrm{Fr}(X \times S, Y \times T).$$

Recall now the definition of the *category of linear framed correspondences* $\mathbb{Z}\mathrm{F}_*(k)$.

Definition 2.4. (see [GP1]) Let X and Y be smooth schemes. Denote by

- ◇ $\mathbb{Z}\mathrm{Fr}_n(X,Y) := \widetilde{\mathbb{Z}}[\mathrm{Fr}_n(X,Y)] = \mathbb{Z}[\mathrm{Fr}_n(X,Y)]/\mathbb{Z}\cdot 0_n$, i.e the free abelian group generated by the set $\mathrm{Fr}_n(X,Y)$ modulo $\mathbb{Z}\cdot 0_n$;
- ◇ $\mathbb{Z}\mathrm{F}_n(X,Y) := \mathbb{Z}\mathrm{Fr}_n(X,Y)/A$, where A is a subgroup generated by the elementts

$$(Z \sqcup Z', U, (\varphi_1,\varphi_2,\dots,\varphi_n), g) -$$
$$- (Z, U \setminus Z', (\varphi_1,\varphi_2,\dots,\varphi_n)|_{U\setminus Z'}, g|_{U\setminus Z'}) - (Z', U \setminus Z, (\varphi_1,\varphi_2,\dots,\varphi_n)|_{U\setminus Z}, g|_{U\setminus Z}).$$

We shall also refer to the latter relation as the *additivity property for supports*. In other words, it says that a framed correspondence in $\mathbb{Z}\mathrm{F}_n(X,Y)$ whose support is a disjoint union $Z \sqcup Z'$ equals the sum of the framed correspondences with supports Z and Z' respectively. Note that $\mathbb{Z}\mathrm{F}_n(X,Y)$ is $\mathbb{Z}[\mathrm{Fr}_n(X,Y)]$ modulo the subgroup generated by the elements as above, because $0_n = 0_n + 0_n$ in this quotient group, hence 0_n equals zero. Indeed, it is enough to observe that the support of 0_n equals $\emptyset \sqcup \emptyset$ and then apply the above relation to this support.

The elements of $\mathbb{Z}\mathrm{F}_n(X,Y)$ are called *linear framed correspondences of level n* or just *linear framed correspondences*.

Denote by $\mathbb{Z}\mathrm{F}_*(k)$ the additive category whose objects are those of Sm/k with Hom-groups defined as

$$\mathrm{Hom}_{\mathbb{Z}\mathrm{F}_*(k)}(X,Y) = \bigoplus_{n\geqslant 0} \mathbb{Z}\mathrm{F}_n(X,Y).$$

The composition is induced by the composition in the category $\mathrm{Fr}_*(k)$. Denote by $Pre_{Ab}(\mathbb{Z}\mathrm{F}_*(k))$ the Grothendieck category of additive presheaves of abelian groups on the additive category $\mathbb{Z}\mathrm{F}_*(k)$.

Denote by $\mathbb{Z}\mathrm{F}_0(k)$ the additive category whose objects are those of Sm/k with Hom-groups defined as $\mathrm{Hom}_{\mathbb{Z}\mathrm{F}_0(k)}(X,Y) = \mathbb{Z}\mathrm{F}_0(X,Y)$. Clearly, $\mathbb{Z}\mathrm{F}_0(k)$ is an additive subcategory of the additive category $\mathbb{Z}\mathrm{F}_*(k)$. Finally, denote by $Pre_{Ab}(\mathbb{Z}\mathrm{F}_0(k))$ the category of additive presheaves of abelian groups on the additive category $\mathbb{Z}\mathrm{F}_0(k)$.

There is a natural functor from Sm/k to $\mathbb{Z}\mathrm{F}_0(k)$. It is the identity on objects and takes a regular morphism $f: X \to Y$ to the linear framed correspondence $1 \cdot (X, X \times \mathbb{A}^0, pr_{\mathbb{A}^0}, f \circ pr_X) \in \mathbb{Z}\mathrm{F}_0(k)$.

Definition 2.5. Let X,Y,S and T be schemes. The external product from Definition 2.3 induces a unique external product

$$\mathbb{Z}\mathrm{F}_n(X,Y) \times \mathbb{Z}\mathrm{F}_m(S,T) \xrightarrow{-\boxtimes-} \mathbb{Z}\mathrm{F}_{n+m}(X \times S, Y \times T)$$

such that for any elements $a \in \mathrm{Fr}_n(X,Y)$ and $b \in \mathrm{Fr}_m(S,T)$ one has $1 \cdot a \boxtimes 1 \cdot b = 1 \cdot (a \boxtimes b) \in \mathbb{Z}\mathrm{F}_{n+m}(X \times S, Y \times T)$.

Definition 2.6. For any k-smooth variety Y, the presheaf represented by Y is denoted by $\mathbb{Z}\mathrm{F}_*(-,Y)$. One of the main $\mathbb{Z}\mathrm{F}_*(k)$-presheaves of this paper is defined as

$$\mathbb{Z}\mathrm{F}(-,Y) = \mathrm{colim}(\mathbb{Z}\mathrm{F}_0(-,Y) \xrightarrow{\Sigma} \mathbb{Z}\mathrm{F}_1(-,Y) \xrightarrow{\Sigma} \ldots \xrightarrow{\Sigma} \mathbb{Z}\mathrm{F}_n(-,Y) \xrightarrow{\Sigma} \ldots).$$

For a k-smooth variety X, the elements of $\mathbb{Z}\mathrm{F}(X,Y)$ are also called *stable linear framed correspondences*. Notice that stable linear framed correspondences do not form morphisms of a category.

General Framework. The pairing $\boxtimes$ of Definition 2.5 gives rise to a functor

$$\mathbb{Z}\mathrm{F}_*(k) \times \mathbb{Z}\mathrm{F}_0(k) \xrightarrow{\boxtimes} \mathbb{Z}\mathrm{F}_*(k)$$

taking a pair of schemes (X,S) to $X \times S$ and taking a pair of morphisms (a,b) to the morphism $a \boxtimes b$. It is naturally extended to a functor

$$Pre_{Ab}(\mathbb{Z}\mathrm{F}_*(k)) \times Pre_{Ab}(\mathbb{Z}\mathrm{F}_0(k)) \xrightarrow{\boxtimes} Pre_{Ab}(\mathbb{Z}\mathrm{F}_*(k)).$$

Given schemes $Y,S \in \mathbb{Z}\mathrm{F}_0(k)$, consider the presheaf $\mathbb{Z}\mathrm{F}(S)$ in $Pre_{Ab}(\mathbb{Z}\mathrm{F}_0(k))$ and the presheaf $\mathbb{Z}\mathrm{F}(Y)$ in $Pre_{Ab}(\mathbb{Z}\mathrm{F}_*(k))$. Similarly to [GP1, General Framework, Section 5] there are defined $\mathbb{Z}\mathrm{F}_*(k)$-presheaves $\mathbb{Z}\mathrm{F}(Y) \boxtimes S$ and $\underline{\mathrm{Hom}}(S, \mathbb{Z}\mathrm{F}(Y) \boxtimes S)$ as well as a natural $\mathbb{Z}\mathrm{F}_*(k)$-morphism $\mathbb{Z}\mathrm{F}(Y) \xrightarrow{-\boxtimes S} \underline{\mathrm{Hom}}(S, \mathbb{Z}\mathrm{F}(Y) \boxtimes S)$. By construction, $\mathbb{Z}\mathrm{F}(Y) \boxtimes S = \mathbb{Z}\mathrm{F}(Y \times S)$. Thus one has the following morphism of $\mathbb{Z}\mathrm{F}_*(k)$-presheaves

$$- \boxtimes id_S : \mathbb{Z}\mathrm{F}(Y) \to \underline{\mathrm{Hom}}(S, \mathbb{Z}\mathrm{F}(Y \times S))$$

taking $a \in \mathbb{Z}\mathrm{F}(X,Y)$ to $a \boxtimes id_S \in \mathbb{Z}\mathrm{F}(X \times S, Y \times S)$.

Definition 2.7. Let (S,s) be a k-smooth pointed scheme. Then the morphism $e_s \colon S \to \mathrm{pt} \xrightarrow{s} S$ defines an idempotent $\underline{\mathrm{Hom}}(S,e_s) : \underline{\mathrm{Hom}}(S, \mathbb{Z}\mathrm{F}(Y \times S)) \to \underline{\mathrm{Hom}}(S, \mathbb{Z}\mathrm{F}(Y \times S))$ in the category of $\mathbb{Z}\mathrm{F}_*$-presheaves. Set,

$$\underline{\mathrm{Hom}}(S, \mathbb{Z}\mathrm{F}(Y \wedge (S,s))) := \mathrm{Ker}[\underline{\mathrm{Hom}}(S,e_s)].$$

Consider the idempotent $\underline{\mathrm{Hom}}(e_s, \underline{\mathrm{Hom}}(S, \mathbb{Z}\mathrm{F}(Y \wedge (S,s))))$ of $\underline{\mathrm{Hom}}(S, \mathbb{Z}\mathrm{F}(Y \wedge (S,s)))$ in the category of $\mathbb{Z}\mathrm{F}_*(k)$-presheaves. Set,

$$\underline{\mathrm{Hom}}((S,s), \mathbb{Z}\mathrm{F}(Y \wedge (S,s))) := \mathrm{Ker}[\underline{\mathrm{Hom}}(e_s, \underline{\mathrm{Hom}}(S, \mathbb{Z}\mathrm{F}(Y \wedge (S,s))))].$$

For any $X \in Sm/k$ denote by $\mathbb{Z}\mathrm{F}(X \wedge (S,s), Y \wedge (S,s))$ the value of $\underline{\mathrm{Hom}}((S,s), \mathbb{Z}\mathrm{F}(Y \wedge (S,s)))$ on X. There is a natural morphism of $\mathbb{Z}\mathrm{F}_*(k)$-presheaves

$$- \boxtimes id_{(S,s)} : \mathbb{Z}\mathrm{F}(Y) \to \underline{\mathrm{Hom}}((S,s), \mathbb{Z}\mathrm{F}(Y \wedge (S,s))).$$

Definition 2.8. Let X and Y be k-smooth schemes and let (S,s) be a k-smooth pointed scheme.

- ◇ Denote by $e_s\colon S \to \mathrm{pt} \xrightarrow{s} S$ the idempotent in $\mathrm{End}_{\mathbb{Z}\mathrm{F}_0(k)}(S) = \mathbb{Z}\mathrm{F}_0(S,S)$ given by the composition of the constant map and the embedding of s into S.
- ◇ For each integer $m \geqslant 0$ define $\mathbb{Z}\mathrm{F}_m(X \wedge (S,s), Y \wedge (S,s))$ as a subgroup of the group $\mathbb{Z}\mathrm{F}_m(X \times S, Y \times S)$ consisting of all a such that $a \circ (\mathrm{id}_X \boxtimes e_s) = (\mathrm{id}_Y \boxtimes e_s) \circ a = 0$. Note that the suspension map $\Sigma : \mathbb{Z}\mathrm{F}_m(X \times S, Y \times S) \to \mathbb{Z}\mathrm{F}_{m+1}(X \times S, Y \times S)$ takes the subgroup $\mathbb{Z}\mathrm{F}_m(X \wedge (S,s), Y \wedge (S,s)))$ to the subgroup $\mathbb{Z}\mathrm{F}_{m+1}(X \wedge (S,s), Y \wedge (S,s))$. Set,

$$\mathbb{Z}\mathrm{F}(X \wedge (S,s), Y \wedge (S,s)) := \mathrm{colim}[\mathbb{Z}\mathrm{F}_0(X \wedge (S,s), Y \wedge (S,s)) \xrightarrow{\Sigma} \mathbb{Z}\mathrm{F}_1(X \wedge (S,s), Y \wedge (S,s)) \xrightarrow{\Sigma} \dots],$$

It is easy to see that the morphisms $id_X \boxtimes (\mathrm{id}_{\mathbb{G}_m} - e_s) : \mathbb{Z}\mathrm{F}_m(X,Y) \to \mathbb{Z}\mathrm{F}_m(X \times S, Y \times S)$ take values in $\mathbb{Z}\mathrm{F}_m(X \wedge (S,s), Y \wedge (S,s))$. They are compatible with the suspension Σ and we define a morphism

$$id_X \boxtimes \mathrm{id}_{(S,s)}\colon\ \mathbb{Z}\mathrm{F}(X,Y) \to \mathbb{Z}\mathrm{F}(X \wedge (S,s), Y \wedge (S,s)).$$

Lemma 2.9. *Let Y be k-smooth scheme and let (S,s) be a k-smooth pointed scheme. Then one has a commutative diagram of $\mathbb{Z}\mathrm{F}_*(k)$-presheaves*

$$\begin{array}{ccc}
\mathbb{Z}\mathrm{F}(-,Y) & \xrightarrow{-\boxtimes(\mathrm{id}_{\mathbb{G}_m} - e_s)} & \mathbb{Z}\mathrm{F}(- \wedge (S,s), Y \wedge (S,s)) \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle can} \\
\mathbb{Z}\mathrm{F}(-,Y) & \xrightarrow{-\boxtimes \mathrm{id}_{(\mathbb{G}_m,1)}} & \underline{\mathrm{Hom}}((S,s), \mathbb{Z}\mathrm{F}(-, Y \wedge (S,s))),
\end{array}$$

where can is the canonical isomorphism.

Theorem C. *Let X and Y be k-smooth schemes and let $(\mathbb{G}_m, 1)$ be the scheme $\mathbb{G}_m$ pointed by the point 1. Then the morphisms*

$$- \boxtimes (\mathrm{id}_{\mathbb{G}_m} - e_1)\colon\ \mathbb{Z}\mathrm{F}(\Delta^\bullet \times -, Y) \to \mathbb{Z}\mathrm{F}((\Delta^\bullet \times -) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1)) \quad (1)$$

$$- \boxtimes \mathrm{id}_{(\mathbb{G}_m,1)}\colon\ \mathbb{Z}\mathrm{F}(\Delta^\bullet \times -, Y) \to \underline{\mathrm{Hom}}((\mathbb{G}_m, 1), \mathbb{Z}\mathrm{F}(\Delta^\bullet \times -, Y \wedge (\mathbb{G}_m, 1))) \quad (2)$$

are sectionwise quasi-isomorphisms of complexes of $\mathbb{Z}\mathrm{F}_*(k)$-presheaves of abelian groups.

Remark 2.10. By Lemma 2.9 the morphism (1) is a quasi-isomorphism if and only if the morphism (2) is a quasi-isomorphism. Sometimes it is convenient to work with the morphism (1) and sometimes it is convenient to work with the morphism (2).

Theorem C will be proved at the end of the paper.

3. Theorem A and Theorem B

Before proving Theorem A we recall some definitions and constructions for framed motives for the convenience of the reader. We adhere to [GP1]. Let $\mathrm{Fr}_0(k)$ be the category whose objects are those of Sm/k and whose morphism set between X and Y is given by the set of framed correspondences of level zero [V1, Example 2.1], [GP1, Definition 2.1]. As it is shown in [GP1, Section 5], the category of framed correspondences of level zero $\mathrm{Fr}_0(k)$ has an action by finite pointed sets $Y\otimes K := \bigsqcup_{K\setminus *} Y$ with $Y\in Sm/k$ and K a finite pointed set. Let $U,X\in \mathrm{Fr}_0(k)$. By the Additivity Theorem of [GP1] the Γ-space in the sense of Segal [Seg]

$$K\in\Gamma^{\mathrm{op}}\mapsto C_*\mathrm{Fr}(U,X\otimes K):=\mathrm{Fr}(U\times\Delta^\bullet,X\otimes K)$$

is special.

Definition 3.1 (see [GP1])**.** The *framed motive $M_{fr}(X)$ of a smooth algebraic variety $X\in\mathrm{Fr}_0(k)$* is the Segal S^1-spectrum $(C_*\mathrm{Fr}(-,X),C_*\mathrm{Fr}(-,X\otimes S^1),C_*\mathrm{Fr}(-,X\otimes S^2),\ldots)$ associated with the special Γ-space $K\in\Gamma^{\mathrm{op}}\mapsto C_*\mathrm{Fr}(-,X\otimes K)$. The framed motive $M_{fr}(X)\in Sp_{S^1}(k)$ is covariantly functorial in framed correspondences of level zero.

Let $\mathrm{Fr}_0(k)$ be the category whose objects are those of Sm/k and whose morphism set between X and Y is given by the set of framed correspondences of level zero [V1, Example 2.1], [GP1, Definition 2.1]. As it is shown in [GP1, Section 5], the category of framed correspondences of level zero $\mathrm{Fr}_0(k)$ has an action by finite pointed sets $Y\otimes K := \bigsqcup_{K\setminus *} Y$ with $Y\in Sm/k$ and K a finite pointed set. The cone of Y is the simplicial object $Y\otimes I$ in $\mathrm{Fr}_0(k)$, where $(I,1)$ is the pointed simplicial set $\Delta[1]$ with basepoint 1. There is a natural morphism $i_0: Y\to Y\otimes I$ in $\Delta^{\mathrm{op}}\mathrm{Fr}_0(k)$. Given a closed inclusion of smooth schemes $j:Y\hookrightarrow X$, denote by $X//Y$ a simplicial object in $\mathrm{Fr}_0(k)$ which is obtained by taking the pushout of the diagram $X\hookleftarrow Y\overset{i_0}{\hookrightarrow} Y\otimes I$ in $\Delta^{\mathrm{op}}\mathrm{Fr}_0(k)$. The simplicial object $X//Y$ termwise equals $X, X\sqcup Y, X\sqcup Y\sqcup Y,\ldots$. By $\mathbb{G}_m^{\wedge 1}$ we mean the simplicial object $\mathbb{G}_m//\{1\}$ in $\mathrm{Fr}_0(k)$. It looks termwise as

$$\mathbb{G}_m,\mathbb{G}_m\sqcup pt,\mathbb{G}_m\sqcup pt\sqcup pt,\ldots$$

Applying $M_{fr}(X\times -)$ to $\mathbb{G}_m^{\wedge 1}$ and realizing by taking diagonals, one gets a framed S^1-spectrum $M_{fr}(X\times\mathbb{G}_m^{\wedge 1})$. We shall also denote it by $M_{fr}(X)(1)$. The nth iteration gives the spectrum $M_{fr}(X\times\mathbb{G}_m^{\wedge n})$, also denoted by $M_{fr}(X)(n)$.

Similarly to the General Framework on p. 6 there is a natural pairing

$$\boxtimes: sPre^{fr}_\bullet(k)\times sPre_\bullet(\mathrm{Fr}_0(k))\to sPre^{fr}_\bullet(k),$$

where $sPre^{fr}_\bullet(k)$ (respectively $sPre_\bullet(\mathrm{Fr}_0(k))$) is the category of pointed simplicial presheaves with framed correspondences (respectivley the category of pointed simplicial presheaves on $\mathrm{Fr}_0(k)$). It is extended from the pairing $\mathrm{Fr}_*(k)\times\mathrm{Fr}_0(k)\xrightarrow{\boxtimes}\mathrm{Fr}_*(k)$ that takes (X,Y) to $X\times Y$ and $a\in\mathrm{Fr}_m(X,X')$, $b\in\mathrm{Fr}_0(Y,Y')$ to $a\boxtimes b\in\mathrm{Fr}_m(X\times X',Y\times Y')$.

We will also write $\wedge$ for the monoidal product in $\mathrm{Fr}_0(k)$ and in $\Delta^{\mathrm{op}}\mathrm{Fr}_0(k)$. The Yoneda embedding identifies $\Delta^{\mathrm{op}}\mathrm{Fr}_0(k)$ with a full subcategory of $sPre_\bullet(\mathrm{Fr}_0(k))$. For each integer $n\geqslant 0$ there is a natural map of spectra

$$a_n: M_{fr}(X\times\mathbb{G}_m^{\wedge n})\xrightarrow{-\boxtimes\mathbb{G}_m^{\wedge 1}}\underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1},M_{fr}(X\times\mathbb{G}_m^{\wedge n+1}))\to\underline{\mathrm{Hom}}(\mathbb{G},M_{fr}(X\times\mathbb{G}_m^{\wedge n+1})),$$

where the right arrow is induced by the adjunction unit $\mathrm{adj}:\mathbb{G}\to(\mathbb{G}_m^{\wedge 1}|_{Sm/k})$. Note that a_n respects framed correspondences of level zero and coincides with the morphism described in [GP1, p. 297]).

Definition 3.2. The $(S^1,\mathbb{G})$-bispectrum $M_{fr}^{\mathbb{G}}(X)$ is defined as

$$(M_{fr}(X), M_{fr}(X\times\mathbb{G}_m^{\wedge 1}), M_{fr}(X\times\mathbb{G}_m^{\wedge 2}),\dots)$$

together with the structure morphisms a_n-s.

We shall prove below (see the proof of Theorem A) that each a_n is a schemewise stable equivalence of spectra, but first let us discuss further useful spectra. Denote by $\mathbb{Z}M_{fr}(X)$, $X\in Sm/k$, the Segal S^1-spectrum $(\mathbb{Z}\mathrm{Fr}(\Delta^\bullet\times -,X),\mathbb{Z}\mathrm{Fr}(\Delta^\bullet\times -,X\otimes S^1),\dots)$. Denote by $LM_{fr}(X)$ the Segal S^1-spectrum $EM(\mathbb{Z}\mathrm{F}(\Delta^\bullet\times -,X))=(\mathbb{Z}\mathrm{F}(\Delta^\bullet\times -,X),\mathbb{Z}\mathrm{F}(\Delta^\bullet\times -,X\otimes S^1),\dots)$.

The equalities $\mathbb{Z}\mathrm{F}(-,X\sqcup X')=\mathbb{Z}\mathrm{F}(-,X)\oplus\mathbb{Z}\mathrm{F}(-,X')$ show that the Γ-space $(K,*)\mapsto\mathbb{Z}\mathrm{F}(\Delta^\bullet\times U,X\otimes K)$ corresponds to the complex of abelian groups $\mathbb{Z}\mathrm{F}(\Delta^\bullet\times U,X)$. Hence $LM_{fr}(X)$ is the Eilenberg–Mac Lane spectrum for the complex $\mathbb{Z}\mathrm{F}(\Delta^\bullet\times -,X)$. The Γ-space morphism

$$[(K,*)\mapsto\mathbb{Z}\mathrm{Fr}(\Delta^\bullet\times -,X\otimes K)]\to[(K,*)\mapsto\mathbb{Z}\mathrm{F}(\Delta^\bullet\times -,X\otimes K)]$$

induces a morphism of S^1-spectra $l_X:\mathbb{Z}M_{fr}(X)\to EM(\mathbb{Z}\mathrm{F}(-,X))$.

Note that homotopy groups of $LM_{fr}(X)=EM(\mathbb{Z}\mathrm{F}(\Delta^\bullet\times -,X))$ are equal to homology groups of the complex $\mathbb{Z}\mathrm{F}(\Delta^\bullet\times -,X)$. By [Sch, §II.6.2] the homotopy groups $\pi_*(\mathbb{Z}M_{fr}(X)(U))$ of $\mathbb{Z}M_{fr}(X)$ evaluated at $U\in Sm/k$ are the homology groups $H_*(M_{fr}(X)(U))$ of $M_{fr}(X)(U)$.

The following result, referred to as the Linearisation Theorem in [GNP, Theorem 1.2], is true:

Theorem 3.3 (see [GNP])**.** *The morphism of S^1-spectra*

$$l_X:\mathbb{Z}M_{fr}(X)\to LM_{fr}(X)$$

is a schemewise stable equivalence. In particular, if U is smooth, then

$$H_*(M_{fr}(X)(U))=\pi_*(\mathbb{Z}M_{fr}(X)(U))=\pi_*(LM_{fr}(X)(U))=H_*(\mathbb{Z}\mathrm{F}(\Delta^\bullet\times U,X)).$$

Replacing simplicial framed sheaves C_*Fr in Definition 3.2 by simplicial abelian framed presheaves $C_*\mathbb{Z}\mathrm{F}$, we define Segal S^1-spectra $LM_{fr}(X\times\mathbb{G}_m^{\wedge n})$-s. Following the General Framework on p. 6, there is a natural morphism of S^1-spectra for each integer $n\geqslant 0$

$$c_n: LM_{fr}(X\times\mathbb{G}_m^{\wedge n})\xrightarrow{-\boxtimes\mathbb{G}_m^{\wedge 1}}\underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1},M_{fr}(X\times\mathbb{G}_m^{\wedge n+1}))\to\underline{\mathrm{Hom}}(\mathbb{G},LM_{fr}(X\times\mathbb{G}_m^{\wedge n+1})),\qquad(3)$$

where the right arrow is induced by the adjunction unit $\mathrm{adj}:\mathbb{G}\to(\mathbb{G}_m^{\wedge 1}|_{Sm/k})$.

Definition 3.4. The $(S^1,\mathbb{G})$-bispectrum $LM_{fr}^{\mathbb{G}}(X)$ is defined as

$$(LM_{fr}(X), LM_{fr}(X\times\mathbb{G}_m^{\wedge 1}), LM_{fr}(X\times\mathbb{G}_m^{\wedge 2}),\dots)$$

together with the structure morphisms c_n-s.

We are now in a position to prove Theorem A.

Proof of Theorem A. (1). We claim that for every $n>0$ the sequence

$$M_{fr}(X)(n-1)\to M_{fr}(X\times\mathbb{G}_m)(n-1)\to M_{fr}(X)(n)$$

is a homotopy cofiber sequence of S^1-spectra. Since all spectra are connected, it is enough to show that

$$\mathbb{Z}M_{fr}(X)(n-1)\to\mathbb{Z}M_{fr}(X\times\mathbb{G}_m)(n-1)\to\mathbb{Z}M_{fr}(X)(n)$$

is a homotopy cofiber sequence of S^1-spectra. By Theorem 3.3 the latter is equivalent to showing that

$$LM_{fr}(X)(n-1)\to LM_{fr}(X\times\mathbb{G}_m)(n-1)\to LM_{fr}(X)(n)$$

is a homotopy cofiber sequence of S^1-spectra. This sequence is a homotopy cofiber sequence if and only if

$$\mathbb{Z}F(\Delta^\bullet \times -, X \times \mathbb{G}_m^{\wedge(n-1)}) \to \mathbb{Z}F(\Delta^\bullet \times -, X \times \mathbb{G}_m^{\wedge(n-1)} \times \mathbb{G}_m) \to \mathbb{Z}F(\Delta^\bullet \times -, X \times \mathbb{G}_m^{\wedge n})$$

is a homotopy cofiber sequence of complexes of abelian presheaves. But this is obvious because $\mathbb{Z}F(\Delta^\bullet \times -, X \times \mathbb{G}_m^{\wedge n})$ is the mapping cone of the left arrow, and hence the desired claim follows. We have used here the fact that $\mathbb{Z}F(-, X \sqcup Y) = \mathbb{Z}F(-,X) \oplus \mathbb{Z}F(-,Y)$.

Next, it is enough to prove that

$$a_0 : M_{fr}(X) \to \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(1))$$

is a schemewise equivalence of spectra. Indeed, consider a commutative diagram of homotopy cofiber sequences in $Sp_{S^1}(k)$

$$\begin{array}{ccccc}
M_{fr}(X)(n-1) & \longrightarrow & M_{fr}(X \times \mathbb{G}_m)(n-1) & \longrightarrow & M_{fr}(X)(n) \\
\downarrow{\scriptstyle a_{n-1}} & & \downarrow{\scriptstyle a_{n-1}} & & \downarrow{\scriptstyle a_n} \\
\underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n)) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m)(n)) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1))
\end{array}$$

with $n \geqslant 1$. If a_{n-1} is a schemewise equivalence of spectra, then so is a_n by [Hir, 13.5.10]. Thus using induction in n, it suffices to verify that a_0 is a schemewise equivalence of spectra.

By the stable Whitehead theorem [Sch, II.6.30] a_0 is a stable equivalence if and only if so is

$$a_0 : \mathbb{Z}M_{fr}(X) \to \mathbb{Z}[\underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m^{\wedge 1}))].$$

Consider a commutative diagram of homotopy fiber sequences in $Sp_{S^1}(k)$

$$\begin{array}{ccccc}
\underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m^{\wedge 1})) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{G}_m, M_{fr}(X \times \mathbb{G}_m^{\wedge 1})) & \longrightarrow & M_{fr}(X \times \mathbb{G}_m^{\wedge 1}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}[\underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m^{\wedge 1}))] & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{G}_m, \mathbb{Z}M_{fr}(X \times \mathbb{G}_m^{\wedge 1})) & \longrightarrow & \mathbb{Z}M_{fr}(X \times \mathbb{G}_m^{\wedge 1}) \\
\downarrow{\scriptstyle \ell_X} & & \downarrow & & \downarrow{\scriptstyle l_X} \\
\underline{\mathrm{Hom}}(\mathbb{G}, LM_{fr}(X \times \mathbb{G}_m^{\wedge 1})) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{G}_m, LM_{fr}(X \times \mathbb{G}_m^{\wedge 1})) & \longrightarrow & LM_{fr}(X \times \mathbb{G}_m^{\wedge 1})
\end{array}$$

The arrow l_X and the middle lower arrow are a stable weak equivalences of spectra by Theorem 3.3. It follows that ℓ_X is a stable weak equivalence. Consider a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}M_{fr}(X) & \xrightarrow{a_0} & \mathbb{Z}[\underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m^{\wedge 1}))] \\
\downarrow{\scriptstyle l_X} & & \downarrow{\scriptstyle \ell_X} \\
LM_{fr}(X) & \xrightarrow{c_0} & \underline{\mathrm{Hom}}(\mathbb{G}, LM_{fr}(X \times \mathbb{G}_m^{\wedge 1})).
\end{array}$$

Since l_X, ℓ_X are stable weak equivalences, it follows that a_0 is a stable local equivalence if and only if so is c_0. By Theorem D from Appendix A the morphism c_0 is a sectionwise stable weak equivalence. The proof of the first part of the theorem is completed.

(2). Since each spectrum $M_{fr}(X)(n)_f$ is fibrant in the injective local stable model structure of S^1-spectra, it is enough to show that each map

$$b_n : M_{fr}(X)(n)_f \to \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1)_f), \quad n \geqslant 0,$$

is a Nisnevich local stable equivalence of spectra. Using the same argument as in the proof of the first statement, it suffices to verify that b_0 is a local stable equivalence.

There is a commutative diagram

$$\begin{array}{ccc} M_{fr}(X) & \xrightarrow{a_0} & \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(1)) \\ \alpha \downarrow & & \downarrow d_1 \\ M_{fr}(X)_f & \xrightarrow{b_0} & \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(1)_f) \end{array}$$

in which the left vertical arrow is a local stable equivalence and a_0 is a schemewise stable equivalence by the first statement. It follows that b_0 is a local stable equivalence if and only if so is $d_1 = \underline{\mathrm{Hom}}(\mathbb{G}, \alpha)$.

The presheaves of stable homotopy groups of $\underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(1))$ equal $(\pi_n(M_{fr}(X)(1)))_{-1}$. These presheaves are $\mathbb{A}^1$-invariant quasi-stable radditive with framed correspondences (see [GP2, Introduction] for the definition of such presheaves). It follows from [GP2, Theorem 1.1] (complemented by [DP] in characteristic 2) that each Nisnevich sheaf $((\pi_n(M_{fr}(X)(1)))_{-1})^{\mathrm{nis}}$ is strictly $\mathbb{A}^1$-invariant quasi-stable radditive with framed correspondences.

Each spectrum $M_{fr}(X)(n)$ has homotopy invariant, quasi-stable radditive presheaves with framed correspondences of stable homotopy groups $\pi_*(M_{fr}(X)(n))$. By [GP2, Theorem 1.1] (complemented by [DP] in characteristic 2) the Nisnevich sheaves $\pi_*^{\mathrm{nis}}(M_{fr}(X)(n))$ are strictly homotopy invariant. It follows from [GP1, Proposition 7.1] that $M_{fr}(X)(n)_f$ is motivically fibrant in the injective stable motivic model structure of S^1-spectra.

In order to compute the Nisnevich sheaf $\pi_n^{\mathrm{nis}}(\underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(1)_f))$, consider the Brown–Gersten convergent spectral sequence

$$H^p_{\mathrm{nis}}(V \times \mathbb{G}_m, \pi_q^{\mathrm{nis}}(M_{fr}(X)(1))) \Rightarrow \pi_{q-p}(M_{fr}(X)(1)_f(V \times \mathbb{G}_m)), \quad V \in Sm/k.$$

It follows from [GP2, Corollary 16.8, Theorems 17.15-16] that each presheaf

$$V \mapsto H^p_{\mathrm{nis}}(V \times \mathbb{G}_m, \pi_q^{\mathrm{nis}}(M_{fr}(X)(1)))$$

is $\mathbb{A}^1$-invariant quasi-stable radditive with framed correspondences.

Let $V \in Sm/k$ be irreducible, $u \in V$ be a point, $U = \mathrm{Spec}(\mathscr{O}_{V,u})$. Let U^h_u be the henselization of U at u and let $k(U^h_u)$ be the function field on U^h_u. Consider the above spectral sequence and replace V by U^h_u in it. We claim that in this case the spectral sequence degenerates and $H^0_{\mathrm{nis}}(U^h_u \times \mathbb{G}_m, \pi_n^{\mathrm{nis}}(M_{fr}(X)(1))) = \pi_n(M_{fr}(X)(1)_f(U^h_u \times \mathbb{G}_m))$. For this notice that by [GP2, 3.15(3')] the map $H^p_{\mathrm{nis}}(\mathbb{G}_m \times U, \pi_q^{\mathrm{nis}}(M_{fr}(X)(1))) \hookrightarrow H^p_{\mathrm{nis}}(\mathbb{G}_{m,k(U^h_u)}, \pi_q^{\mathrm{nis}}(M_{fr}(X)(1)))$ is injective, where $\eta_h : \mathrm{Spec}(k(U^h_u)) \to U^h_u$ is the canonical morphism. In turn, by [GP2, 3.15(1)] the canonical homomorphism

$$H^p_{\mathrm{nis}}(\mathbb{G}_{m,k(U^h_u)}, \pi_q^{\mathrm{nis}}(M_{fr}(X)(1))) \hookrightarrow H^p_{\mathrm{nis}}(\mathrm{Spec}(k(U^h_u)(t)), \pi_q^{\mathrm{nis}}(M_{fr}(X)(1)))$$

is injective. Since $0 = H^p_{\mathrm{nis}}(\mathrm{Spec}(k(U^h_u)(t)), \pi_q^{\mathrm{nis}}(M_{fr}(X)(1)))$ for $p > 0$, the group $H^p_{\mathrm{nis}}(U^h_u \times \mathbb{G}_m, \pi_q^{\mathrm{nis}}(M_{fr}(X)(1)))$ vanishes for $p > 0$. Thus we have checked the equality

$$H^0_{\mathrm{nis}}(U^h_u \times \mathbb{G}_m, \pi_n^{\mathrm{nis}}(M_{fr}(X)(1))) = \pi_n(M_{fr}(X)(1)_f(U^h_u \times \mathbb{G}_m)).$$

We can conclude that $\pi_n^{\mathrm{nis}}(\underline{\mathrm{Hom}}(\mathbb{G}_m, M_{fr}(X)(1)_f)) = \pi_n^{\mathrm{nis}}(M_{fr}(X)(1)_f)(\mathbb{G}_m \times -)$. It follows that

$$\pi_n^{\mathrm{nis}}(\underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(1)_f)) = (\pi_n^{\mathrm{nis}}(M_{fr}(X)(1)_f))_{-1} = (\pi_n^{\mathrm{nis}}(M_{fr}(X)(1)))_{-1}.$$

It remains to show that the morphism of $\mathbb{A}^1$-invariant radditive quasi-stable framed sheaves

$$((\pi_n(M_{fr}(X)(1)))_{-1})^{\mathrm{nis}} \to (\pi_n^{\mathrm{nis}}(M_{fr}(X)(1)))_{-1}$$

is an isomorphism. Using [GP2, 3.15(3')] it suffices to check that it is an isomorphism for every field extension K/k. The homomorphism of abelian groups

$$((\pi_n(M_{fr}(X)(1)))_{-1})^{\mathrm{nis}}(K) = (\pi_n(M_{fr}(X)(1)))_{-1}(K) \to (\pi_n^{\mathrm{nis}}(M_{fr}(X)(1)))_{-1}(K)$$

is an isomorphism, because for every homotopy invariant radditive quasi-stable framed presheaf of abelian groups $\mathscr{F}$ and every open $V \subset \mathbb{A}^1_K$, one has $\mathscr{F}(V) = \mathscr{F}^{\mathrm{nis}}(V)$ (see the proof of [GP2, 3.1]). This completes the proof of Theorem A. □

Proof of Theorem B. The proof of Theorem A(2) shows that $M_{fr}(X)(n)_f$ is motivically fibrant in the injective stable motivic model structure of S^1-spectra. By Theorem A each structure map b_n is a schemewise equivalence. We conclude that the bispectrum $M_{fr}^{\mathbb{G}}(X)_f$ is a motivically fibrant $(S^1, \mathbb{G})$-bispectrum in the sense of Jardine [Jar]. □

4. Useful lemmas

In this section we discuss several useful $\mathbb{A}^1$-homotopies and collect a number of facts used in the following sections. We start with some definitions and notation.

Definition 4.1. Let $\mathscr{F} : Sm/k \to Sets$ be a presheaf of sets. Let $X \in Sm/k$ be a smooth variety and $a, b \in \mathscr{F}(X)$ be two sections. We write $a \sim b$ if a and b are in the same connected component of the simplicial set $\mathscr{F}(\Delta^\bullet \times X)$. If $h \in \mathscr{F}(\Delta^1 \times X)$ is such that $\partial_0(h) = a$ and $\partial_1(h) = b$, then we will write $a \overset{h}{\sim} b$. In this case $a \sim b$.

Let $\mathscr{A} : Sm/k \to Ab$ be a presheaf of abelian groups. Let $X \in Sm/k$ be a smooth variety and $a, b \in \mathscr{A}(X)$ be two sections. We will write $a \sim b$ if the classes of a and b in $H_0(\mathscr{A}(\Delta^\bullet \times X))$ coincide. This is equivalent to saying that there is $h \in \mathscr{A}(\Delta^1 \times X)$ such that $\partial_0(h) = a$ and $\partial_1(h) = b$. For such an h we will write $a \overset{h}{\sim} b$.

Definition 4.2. Let $\mathscr{F}$ and $\mathscr{G}$ be two presheaves of sets on the category of k-smooth schemes and let $\varphi_0, \varphi_1 : \mathscr{F} \rightrightarrows \mathscr{G}$ be two morphisms. An $\mathbb{A}^1$-*homotopy* between φ_0 and φ_1 is a morphism $H : \mathscr{F} \to \underline{\mathrm{Hom}}(\mathbb{A}^1, \mathscr{G})$ such that $H_0 = \varphi_0$ and $H_1 = \varphi_1$. We will write $\varphi_0 \sim \varphi_1$ if there is an $\mathbb{A}^1$-homotopy between φ_0 and φ_1.

Let $\mathscr{A}$ and $\mathscr{B}$ be two presheaves of abelian groups on the category of k-smooth schemes and let $\varphi_0, \varphi_1 : \mathscr{A} \rightrightarrows \mathscr{B}$ be two morphisms. An $\mathbb{A}^1$-*homotopy* between φ_0 and φ_1 is a morphism $H : \mathscr{A} \to \underline{\mathrm{Hom}}(\mathbb{A}^1, \mathscr{B})$ of presheaves of abelian groups such that $H_0 = \varphi_0$ and $H_1 = \varphi_1$. If H is an $\mathbb{A}^1$-homotopy between φ_0 and φ_1, then we will write $\varphi_0 \overset{H}{\sim} \varphi_1$. If we do not specify an $\mathbb{A}^1$-homotopy between φ_0 and φ_1, then we will write $\varphi_0 \sim \varphi_1$.

If $\varphi : \mathscr{A} \to \mathscr{B}$ is a morphism of presheaves of abelian groups, then there is a constant $\mathbb{A}^1$-homotopy H_φ between φ and φ defined as follows. Given $a \in \mathscr{A}(X)$ set $H_\varphi(a) = pr_X^*(\varphi(a)) \in \mathscr{B}(X \times \mathbb{A}^1)$.

Lemma 4.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be two presheaves of abelian groups on the category of k-smooth schemes and let $\varphi_0, \varphi_1 : \mathscr{A} \rightrightarrows \mathscr{B}$ be two morphisms such that $\varphi_0 \sim \varphi_1$. Then the induced morphisms*

$$\varphi_0, \varphi_1 : \mathscr{A}(\Delta^\bullet) \rightrightarrows \mathscr{B}(\Delta^\bullet)$$

between two simplicial abelian groups give the same morphisms on the homology of the associated Moore complexes.

Lemma 4.4. *Let $\varphi_0, \varphi_1, \varphi_2 : \mathscr{A} \to \mathscr{B}$ be morphisms of presheaves of abelian groups and let $\varphi_0 \overset{H'}{\text{---}} \varphi_1$ and $\varphi_1 \overset{H''}{\text{---}} \varphi_2$. Then*

$$\varphi_0 \overset{H' + H'' - H_{\varphi_1}}{\text{-----------}} \varphi_2$$

Lemma 4.5. *Let $\mathscr{A}$ and $\mathscr{B}$ be two presheaves of abelian groups on the category of k-smooth schemes and let $\varphi_0 \overset{H}{\text{---}} \varphi_1$. Let $\rho : \mathscr{A}' \to \mathscr{A}$ be a morphism. Then $\varphi_0 \circ \rho \overset{H \circ \rho}{\text{---}} \varphi_1 \circ \rho$. Moreover, let $\eta : \mathscr{B} \to \mathscr{B}'$ be a morphism, then $\psi \circ \varphi_0 \overset{\psi \circ H}{\text{---}} \psi \circ \varphi_1$ with $\psi = \underline{\mathrm{Hom}}(\mathbb{A}^1, \eta) : \underline{\mathrm{Hom}}(\mathbb{A}^1, \mathscr{B}) \to \underline{\mathrm{Hom}}(\mathbb{A}^1, \mathscr{B}')$.*

We now want to discuss actions of matrices on framed correspondences and associated homotopies. Let X and Y be k-smooth schemes and $A \in GL_n(k)$ be a matrix. Then A defines an automorphism

$$\varphi_A : \mathrm{Fr}_n(- \times X, Y) \to \mathrm{Fr}_n(- \times X, Y)$$

of the presheaf $\mathrm{Fr}_n(- \times X, Y)$ in the following way. Given $W \in Sm/k$ and $a = (Z, U, (\varphi_1, \varphi_2, \dots, \varphi_n), g) \in \mathrm{Fr}_n(W \times X, Y)$, set

$$\varphi_A(Z, U, (\varphi_1, \varphi_2, \dots, \varphi_n), g)) := (Z, U, A \circ (\varphi_1, \varphi_2, \dots, \varphi_n), g),$$

where A is regarded as a linear automorphism of $\mathbb{A}^n_k$.

The automorphism φ_A of the presheaf $\mathrm{Fr}_n(- \times X, Y)$ induces an automorphism of the free abelian presheaf $\mathbb{Z}[\mathrm{Fr}_n(- \times X, Y)]$ and an automorphism φ_A of the presheaf of abelian groups $\mathbb{Z}\mathrm{F}_n(- \times X, Y)$.

Definition 4.6. Let $A \in SL_n(k)$. Choose a matrix $A_s \in SL_n(k[s])$ such that $A_0 = id$ and $A_1 = A$. The matrix A_s, regarded as a morphism $\mathbb{A}^n \times \mathbb{A}^1 \to \mathbb{A}^n$, gives rise to an $\mathbb{A}^1$-homotopy h between id and φ_A as follows. Given $a = (\alpha, f, Z, U, \varphi, g)) = ((\alpha_1, \alpha_2, \dots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, ..., \varphi_n), g) \in \mathrm{Fr}_n(W \times X, Y)$, one sets

$$h(a) = (\alpha, f \times id_{\mathbb{A}^1}, Z \times \mathbb{A}^1, U \times \mathbb{A}^1, A_s \circ (\varphi \times id_{\mathbb{A}^1}), g \circ pr_U) \in \mathrm{Fr}_n(W \times X \times \mathbb{A}^1, Y).$$

Clearly, $h_0(a) = a$ and $h_1(a) = \varphi_A(a)$. By linearity the homotopy h induces an $\mathbb{A}^1$-homotopy H_{A_s}

$$id \overset{H_{A_s}}{\text{---}} \varphi_A : \mathbb{Z}\mathrm{F}_n(- \times X, Y) \rightrightarrows \mathbb{Z}\mathrm{F}_n(- \times X, Y)$$

between the identity id and the morphism φ_A.

Lemma 4.7. *Let $\rho : \mathbb{Z}\mathrm{F}_m(- \times X, Y) \to \mathbb{Z}\mathrm{F}_n(- \times X, Y)$ be a presheaf morphism. Let $A \in SL_n(k)$, $A_s \in SL_n(k[s])$ and H_{A_s} be as in Definition 4.6. Then one has*

$$\rho \overset{H_{A_s} \circ \rho}{\text{-----}} \varphi_A \circ \rho : \mathbb{Z}\mathrm{F}_m(- \times X, Y) \rightrightarrows \mathbb{Z}\mathrm{F}_n(- \times X, Y).$$

For $b \in \mathbb{Z}\mathrm{F}_m(Y, S)$ define a presheaf morphism

$$\varphi_b : \mathbb{Z}\mathrm{F}_n(- \times X, Y) \to \mathbb{Z}\mathrm{F}_{n+m}(- \times X, S)$$

sending $a \in \mathbb{Z}\mathrm{F}_n(W \times X, Y)$ to $b \circ a \in \mathbb{Z}\mathrm{F}_{n+m}(W \times X, S)$. Also, any $b \in \mathbb{Z}\mathrm{F}_m(pt, pt)$ defines a morphism of presheaves

$$- \boxtimes b : \mathbb{Z}\mathrm{F}_n(- \times X, Y) \to \mathbb{Z}\mathrm{F}_{n+m}(- \times X, Y)$$

sending $a \in \mathbb{Z}\mathrm{F}_n(W \times X, Y)$ to $a \boxtimes b \in \mathbb{Z}\mathrm{F}_{n+m}(W \times X, Y)$.

The next three lemmas are straightforward.

Lemma 4.8. *Let $b_1, b_2 \in \mathbb{Z}\mathrm{F}_m(Y,S)$ be such that $b_1 \sim b_2$, then*

$$\varphi_{b_1} \sim \varphi_{b_2} : \mathbb{Z}\mathrm{F}_n(-\times X, Y) \rightrightarrows \mathbb{Z}\mathrm{F}_{n+m}(-\times X, S).$$

Lemma 4.9. *Let $b_1, b_2 \in \mathbb{Z}\mathrm{F}_m(pt,pt)$ and $h \in \mathbb{Z}\mathrm{F}_m(\mathbb{A}^1, pt)$ be such that $b_1 \overset{h}{-} b_2$, then*

$$(-\boxtimes b_1) \xrightarrow{-\boxtimes h} (-\boxtimes b_2) : \mathbb{Z}\mathrm{F}_n(-\times X, Y) \rightrightarrows \mathbb{Z}\mathrm{F}_{n+m}(-\times X, Y).$$

The following lemma is proved in Appendix B.

Lemma 4.10. *Let $z \in \mathbb{A}^m$ be a k-rational point. Set $U' = (\mathbb{A}^m)^h_z$ to be the henzelization of $\mathbb{A}^m$ at the point z. Let $i_z : \mathrm{pt} \hookrightarrow U'$ be the closed point of U'. Let $U'_s := (\mathbb{A}^1 \times \mathbb{A}^m)^h_{\mathbb{A}^1 \times z}$ be the henzelization of $\mathbb{A}^1 \times \mathbb{A}^m$ at $\mathbb{A}^1 \times z$. Then the morphism $f_s : \mathbb{A}^1 \times \mathbb{A}^m \to \mathbb{A}^m$ mapping (s,y) to $s \cdot (y-x) + x$ induces a morphism $H_s := f^h_s : U'_s \to U'$ such that:*

(a) *$H_1 := (f^h_s)|_{(1\times X)^h_{(1,x)}} : U' \to U'$ is the identity morphism;*

(b) *$H_0 := (f^h_s)|_{(0\times X)^h_{(0,x)}} : U' \to U'$ coincides with the composite morphism $U' \xrightarrow{p^h} \mathrm{pt} \xrightarrow{s_z} U'$, where $p^h : U' \to \mathrm{pt} = \mathrm{Spec}(k)$ is the structure morphism and $s_z : \mathrm{pt} \hookrightarrow U'$ is the closed point of U'.*

Let $z \in \mathbb{A}^m$ be a k-rational point. The projection $pr : \mathbb{A}^1 \times \mathbb{A}^m \to \mathbb{A}^m$ induces a morphism $can_s := pr^h : U'_s \to U'$ such that $can_0 = can_1 = \mathrm{id}_{U'}$ (see Appendix B). The preceding lemma gives the following

Corollary 4.11. *Let $z \in \mathbb{A}^m$ be a k-rational point and let $(z, U', \psi; \mathrm{id}_{U'}) \in \mathrm{Fr}_m(\mathrm{pt}, U')$ with U' as in Lemma 4.10. Suppose U'_s is as in Lemma 4.10 and let $h_s = (\mathbb{A}^1 \times z, U'_s, can^*_s(\psi); H_s) \in \mathrm{Fr}_m(\mathbb{A}^1, U')$. Then one has:*

(a) *$h_1 = (z, U', \psi; \mathrm{id}_{U'}) \in \mathrm{Fr}_m(\mathrm{pt}, U')$;*
(b) *$h_0 = (z, U', \psi; s_z \circ p^h) = s_z \circ (\{z\}, U', \psi; p^h) \in \mathrm{Fr}_m(\mathrm{pt}, U')$, where $p^h : U' \to \mathrm{pt} = \mathrm{Spec}(k)$ is the structure morphism and $s_z : \mathrm{pt} \hookrightarrow U'$ is the closed point of U'.*

Lemma 4.12. *Let $z \in \mathbb{A}^m$ be a k-rational point. Let Y be a k-smooth scheme and let $(z, U, (\varphi_1, \varphi_2, \dots, \varphi_m), g) \in \mathrm{Fr}_m(\mathrm{pt}, Y)$ be a framed correspondence. Then*

$$(z, U, (\varphi_1, \varphi_2, \dots, \varphi_m), g) \sim (z, U, (\varphi_1, \varphi_2, \dots, \varphi_m), c_{g(z)}),$$

where $c_{g(z)} = g(z) \circ p : U \xrightarrow{p} \mathrm{pt} \xrightarrow{g(z)} Y$.

Proof. Let U', U'_s, i_z and h_s be as in Corollary 4.11. Let $\pi : U' \to U$ be the canonical morphism. Set $h'_s = g \circ \pi \circ h_s \in \mathrm{Fr}_m(\mathrm{pt}, Y)$. We want to check that $h'_1 = (z, U, \varphi, g)$ and $h'_0 = (z, U, \varphi, c_{g(z)})$. This will prove our statement. One has,

$$h'_1 = (g \circ \pi) \circ h_1 = (g \circ \pi) \circ (z, U', \varphi \circ \pi; id_{U'}) = (z, U', \varphi \circ \pi; g \circ \pi) = (z, U, \varphi; g),$$

$$h'_0 = (g \circ \pi) \circ h_0 = (g \circ \pi) \circ (z, U', \varphi \circ \pi; s_z \circ p^h) = (z, U', \varphi \circ \pi; g \circ \pi \circ s_z \circ p^h) =$$
$$= (z, U', \varphi \circ \pi; c_{g(z)} \circ \pi) = (z, U, \varphi; c_{g(z)})$$

as required. □

Lemma 4.13. *Let Y be a k-smooth scheme and let $(Z, U, \varphi, g) \in \mathrm{Fr}_1(\mathrm{pt}, Y)$ be a framed correspondence. Suppose that $U \subset \mathbb{A}^1$ and $\varphi = p(t) \in k[t]$ is a polynomial, where t is the coordinate function on $\mathbb{A}^1$. Let $g : U \to Y$ be a morphism.*

(1) *Then for every $a \in k$ we have*

$$(Z,U,p(t),g(t)) \sim (m_a^{-1}(Z), m_a^{-1}(U), p(t-a), g(t-a)) \in \mathrm{Fr}_1(\mathrm{pt},Y),$$

where $m_a\colon \mathbb{A}^1 \to \mathbb{A}^1$ is given by $m_a(t) = t-a$.

(2) *If $Z = \{x_0\}$ for some $x_0 \in k$ and $p(t) = (t-x_0)^n r(t)$ with $r(t)$ invertible on U, then*

$$(Z,U,p(t),g) \sim (\{0\}, \mathbb{A}^1, r(x_0)t^n, c_{g(x_0)}) \in \mathrm{Fr}_1(\mathrm{pt},Y),$$

where $c_{g(x_0)}\colon \mathbb{A}^1 \to \mathrm{pt} \xrightarrow{g(x_0)} Y$ is the constant map taking $\mathbb{A}^1$ to the point $g(x_0) \in Y$.

Proof. (1) The homotopy is given by

$$(m_{sa}^{-1}(Z), m_{sa}^{-1}(U), p(t-sa), g(t-sa)) \in \mathrm{Fr}_1(\mathbb{A}^1,Y),$$

where s is the homotopy parameter and $m_{sa}\colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is the morphism $m_{sa}(t) = t - sa$.

(2) Using the preceding statement, we may assume that $x_0 = 0$. Consider a polynomial

$$h(s,t) = sr(t)t^n + (1-s)r(0)t^n \in k[s,t].$$

If $r_1(t)$ is such that $r(t) = r(0) + t \cdot r_1(t)$, then one has $h(s,t) = t^n \cdot (r(0) + t \cdot r_1(t) \cdot s)$. If S is the vanishing locus of $r(0) + t \cdot r_1(t) \cdot s$, then $S \cap \mathbb{A}^1 \times 0 = \emptyset$. Hence for the zero locus $Z(h)$ of h one has $Z(h) = (\mathbb{A}^1 \times 0) \sqcup S$. The framed correspondence

$$(\mathbb{A}^1 \times \{0\}, (\mathbb{A}^1 \times U) \setminus S, sr(t)t^n + (1-s)r(0)t^n, g \circ pr_U) \in \mathrm{Fr}_1(\mathbb{A}^1,Y)$$

yields the relation $(\{0\},U,r(t)t^n,g) \sim (\{0\},U,r(0)t^n,g)$ in $\mathrm{Fr}_1(\mathrm{pt},Y)$. Lemma 4.12 shows that

$$(\{0\},U,r(0)t^n,g) \sim (\{0\},U,r(0)t^n,g(0)) = (\{0\},\mathbb{A}^1,r(0)t^n,g(0)) \in \mathrm{Fr}_1(\mathrm{pt},Y)$$

and our lemma follows. □

Lemma 4.14. *Let $a \in k^\times$. Let $p(t), q(t) \in k[t]$ be two polynomials of degree n with the leading coefficient a. Let $(Z(p), \mathbb{A}^1, p(t), c) \in \mathrm{Fr}_1(\mathrm{pt},\mathrm{pt})$, $(Z(q), \mathbb{A}^1, q(t), c) \in \mathrm{Fr}_1(\mathrm{pt},\mathrm{pt})$ be two framed correspondences. Here $c\colon \mathbb{A}^1 \to \mathrm{pt}$ is the structure morphism. Then*

$$(Z(p), \mathbb{A}^1, p(t), c) \sim (Z(q), \mathbb{A}^1, q(t), c) \in \mathrm{Fr}_1(\mathrm{pt},\mathrm{pt}).$$

Proof. As a polinomial in t the leading coefficient of the polinomial $p(t) + s(q(t) - p(t))$ is $a \in k^\times$. Hence the $k[s]$-module $k[s,t]/(p(t) + s(q(t) - p(t)))$ is a free module rank n. Let $Z_s \subset \mathbb{A}^1 \times \mathbb{A}^1$ be the vanishing locus of $p(t) + s(q(t) - p(t))$. The desired homotopy is given by the framed correspondence

$$(Z_s, \mathbb{A}^1 \times \mathbb{A}^1, p(t) + s(q(t) - p(t)), c'),$$

where s is the homotopy parameter and $c'\colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathrm{pt}$ is the canonical projection. □

5. Homotopies for swapping coordinates of $\mathbb{G}_m \times \mathbb{G}_m$

In this section we follow notation of Section 2. Denote by $\varepsilon = (\{0\}, \mathbb{A}^1, -t, c) \in \mathrm{Fr}_1(\mathrm{pt},\mathrm{pt})$, where $c\colon \mathbb{A}^1 \to \mathrm{pt}$ is the canonical projection. We work in this Section with the elements $\Sigma^n \in \mathbb{Z}\mathrm{F}_n(\mathrm{pt},\mathrm{pt})$ as in Definitition 2.3.

Proposition 5.1. *Let Y be a k-smooth scheme. Then the canonical homomorphism*

$$H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet \times \mathbb{G}_m \times \mathbb{G}_m, Y)) \to H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet_{\mathrm{Spec}\, k(t,u)}, Y))$$

is injective.

Proof. By [GP2, 3.15(1)] the canonical homomorhisms

$$H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet \times \mathbb{G}_m \times \mathbb{G}_m, Y)) \to H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet \times \mathbb{G}_{m,k(u)}, Y))$$

and

$$H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet \times \mathbb{G}_{m,k(u)}, Y)) \to H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet_{\mathrm{Spec}\,k(t,u)}, Y))$$

are injective, hence the lemma. □

Let Y be a k-smooth variety and F/k be a field extension. There is a map of pointed sets

$$adj : \mathrm{Fr}_n(\mathrm{Spec}(F), Y) \to \mathrm{Fr}_n^F(\mathrm{Spec}(F), Y_F)$$

given by the assignment $(Z,W,\varphi,g) \mapsto (Z,W,\varphi^F,g^F)$. Here for a k-morphism $g : W \to Y$ we write g^F to denote the F-morphism $(g, pr_{\mathrm{Spec}(F)}) : W \to Y_F$ and $pr_{\mathrm{Spec}(F)} : W \to \mathrm{Spec}(F)$ is the structure morphism. In particular, for $Y = \mathbb{A}^n$ and $\varphi : W \to \mathbb{A}^n_k$ we write φ^F for $(\varphi, pr_{\mathrm{Spec}(F)}) : W \to \mathbb{A}^n_F$. It is easy to see that the map adj is a bijection. Moreover, it induces bijections

$$adj : \mathbb{Z}\mathrm{F}_n(\mathrm{Spec}(F), Y) \to \mathbb{Z}\mathrm{F}_n^F(\mathrm{Spec}(F), Y_F) \text{ and } \mathbb{Z}\mathrm{F}(\mathrm{Spec}(F), Y) \to \mathbb{Z}\mathrm{F}^F(\mathrm{Spec}(F), Y_F).$$

Lemma 5.2. *Let F/k be a field extension, choose $x,y \in F^\times$ such that $x \neq y^{\pm 1}$ and let u_1,u_2 be coordinates on $\mathbb{G}_m \times \mathbb{G}_m$. Consider morphisms $f,g\colon \mathrm{Spec}\,F \to \mathbb{G}_m \times \mathbb{G}_m$ given by $u_1 \mapsto x, u_2 \mapsto y$ and $u_1 \mapsto y, u_2 \mapsto x$ respectively. Then for $p = (\mathrm{id} - e_1) \boxtimes (\mathrm{id} - e_1)$ we have*

$$p \circ (f \boxtimes \Sigma) \sim p \circ (g \boxtimes (-\varepsilon))$$

in $\mathbb{Z}\mathrm{F}(\mathrm{Spec}\,F, \mathbb{G}_m \times \mathbb{G}_m)$.

Proof. The above adjunction isomorphism

$$adj : \mathbb{Z}\mathrm{F}(\mathrm{Spec}\,F, \mathbb{G}_m \times \mathbb{G}_m) \cong \mathbb{Z}\mathrm{F}^F(\mathrm{Spec}\,F, \mathbb{G}_{m,F} \times \mathbb{G}_{m,F})$$

implies it is sufficient to verify the case $F = k$. So we have morphisms $f,g\colon \mathrm{pt} \to \mathbb{G}_m \times \mathbb{G}_m$, $\mathrm{pt} \mapsto (x,y)$ and $\mathrm{pt} \mapsto (y,x)$ respectively. Taking suspensions, we obtain framed correspondences

$$(\{0\}, \mathbb{A}^1, t, c_{(x,y)}),\ (\{0\}, \mathbb{A}^1, t, c_{(y,x)}) \in \mathrm{Fr}_1(\mathrm{pt}, \mathbb{G}_m \times \mathbb{G}_m),$$

where $c_{(x,y)}$ and $c_{(y,x)}$ are morphisms on $\mathbb{A}^1$ sending it to the points (x,y) and (y,x) respectively.

Consider $h(s,t) = \frac{1}{x-y}(t^2 - (s(x+y) + (1-s)(xy+1))t + xy) \in k[s,t,t^{-1}] = k[\mathbb{A}^1 \times \mathbb{G}_m]$ and a framed correspondence

$$H_s := (Z(h), \mathbb{A}^1 \times \mathbb{G}_m, h(s,t), (t, xyt^{-1}) \circ pr_{\mathbb{G}_m}) \in \mathrm{Fr}_1(\mathbb{A}^1, \mathbb{G}_m \times \mathbb{G}_m). \tag{4}$$

We have $h(0,t) = \frac{1}{x-y}(t-xy)(t-1)$ and $h(1,t) = \frac{1}{x-y}(t-x)(t-y)$. Using the additivity property for supports in $\mathbb{Z}\mathrm{F}_1(\mathrm{pt}, \mathbb{G}_m \times \mathbb{G}_m)$ (see Definition 2.4) and Lemma 4.13 we will check below that

$$(\{0\}, \mathbb{A}^1, t, c_{(x,y)}) + (\{0\}, \mathbb{A}^1, -t, c_{(y,x)}) \sim (\{0\}, \mathbb{A}^1, \frac{1-xy}{x-y}t, c_{(1,xy)}) + (\{0\}, \mathbb{A}^1, \frac{xy-1}{x-y}t, c_{(xy,1)}) \tag{5}$$

in $\mathbb{Z}\mathrm{F}_1(\mathrm{pt}, \mathbb{G}_m \times \mathbb{G}_m)$. The composition with the idempotent p annihilates all extra summands and proves the lemma.

In order to prove the relation (5), consider the framed correspondence (4) in $\mathbb{Z}\mathrm{F}_1(\mathbb{A}^1, \mathbb{G}_m \times \mathbb{G}_m)$. Observe that in $\mathbb{Z}\mathrm{F}_1(\mathrm{pt}, \mathbb{G}_m \times \mathbb{G}_m)$

$$H_1 = (Z(h(t,1)), \mathbb{G}_m, h(1,t), (t, xyt^{-1})) =$$

$$= (\{x\}, \mathbb{G}_m - \{y\}, \frac{1}{x-y}(t-x)(t-y), (t, xyt^{-1})) + (\{y\}, \mathbb{G}_m - \{x\}, \frac{1}{x-y}(t-x)(t-y), (t, xyt^{-1})).$$

By Lemma 4.13 one has in $\mathbb{Z}\mathrm{F}_1(\mathrm{pt},\mathbb{G}_m\times\mathbb{G}_m)$

$$(\{x\},\mathbb{G}_m-\{y\},\frac{1}{x-y}(t-x)(t-y),(t,xyt^{-1}))\sim(\{0\},\mathbb{A}^1,\frac{x-y}{x-y}t,c_{(x,y)})=(\{0\},\mathbb{A}^1,t,c_{(x,y)}),$$

$$(\{y\},\mathbb{G}_m-\{x\},\frac{1}{x-y}(t-x)(t-y),(t,xyt^{-1}))\sim(\{0\},\mathbb{A}^1,\frac{y-x}{x-y}t,c_{(x,y)})=(\{0\},\mathbb{A}^1,-t,c_{(x,y)}).$$

Thus $H_1\sim(\{0\},\mathbb{A}^1,t,c_{(x,y)})+(\{0\},\mathbb{A}^1,-t,c_{(y,x)})$ in $\mathbb{Z}\mathrm{F}_1(\mathrm{pt},\mathbb{G}_m\times\mathbb{G}_m)$. Similar computations show that $H_0\sim(\{0\},\mathbb{A}^1,\frac{1-xy}{x-y}t,c_{(1,xy)})+(\{0\},\mathbb{A}^1,\frac{xy-1}{x-y}t,c_{(xy,1)})$ in $\mathbb{Z}\mathrm{F}_1(\mathrm{pt},\mathbb{G}_m\times\mathbb{G}_m)$. The equality (5) is proved. Since the right hand side of the equality (5) is annihilated by the idempotent p, our lemma follows. □

Proposition 5.3. *Let $\tau\colon\mathbb{G}_m\times\mathbb{G}_m\to\mathbb{G}_m\times\mathbb{G}_m$ be the permutation of coordinates morphism. Denote $p=(\mathrm{id}-e_1)\boxtimes(\mathrm{id}-e_1)$. Then $p\circ(\mathrm{id}_{\mathbb{G}_m\times\mathbb{G}_m}\boxtimes(-\varepsilon))\sim p\circ(\tau\boxtimes\Sigma)$ in $\mathbb{Z}\mathrm{F}(\mathbb{G}_m\times\mathbb{G}_m,\mathbb{G}_m\times\mathbb{G}_m)$.*

Proof. Note that $\tau\boxtimes(-\varepsilon)=(\mathrm{id}_{\mathbb{G}_m\times\mathbb{G}_m}\boxtimes(-\varepsilon))\circ\tau$. Hence

$$p\circ(\tau\boxtimes(-\varepsilon))\circ\tau=p\circ(\mathrm{id}_{\mathbb{G}_m\times\mathbb{G}_m}\boxtimes(-\varepsilon))\in\mathbb{Z}\mathrm{F}_1(\mathbb{G}_m\times\mathbb{G}_m,\mathbb{G}_m\times\mathbb{G}_m)).$$

Similarly, $p\circ(\mathrm{id}_{\mathbb{G}_m\times\mathbb{G}_m}\boxtimes\Sigma)\circ\tau=p\circ(\tau\boxtimes\Sigma)$ in $\mathbb{Z}\mathrm{F}_1(\mathbb{G}_m\times\mathbb{G}_m,\mathbb{G}_m\times\mathbb{G}_m))$. It remains to check that

$$p\circ(\mathrm{id}_{\mathbb{G}_m\times\mathbb{G}_m}\boxtimes\Sigma)=p\circ(\tau\boxtimes(-\varepsilon)) \tag{6}$$

in $H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet\times\mathbb{G}_m\times\mathbb{G}_m,\mathbb{G}_m\times\mathbb{G}_m))$.

Let u_1 and u_2 be coordinate functions on $\mathbb{G}_m\times\mathbb{G}_m$. Let $f\colon\operatorname{Spec}k(u_1,u_2)\to\operatorname{Spec}k[u_1,u_2]$ be the canonical embedding and $g\colon\operatorname{Spec}k(u_1,u_2)\to\operatorname{Spec}k[u_1,u_2]$ be given by $g^*(u_1)=u_2,g^*(u_2)=u_1$. By Proposition 5.2 we know that $p\circ(f\boxtimes\Sigma)\sim p\circ(g\boxtimes(-\varepsilon))$ in $H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet_{k(u_1,u_2)},\mathbb{G}_m\times\mathbb{G}_m))$. Proposition 5.1 yields the desired equality (6) in $H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet\times\mathbb{G}_m\times\mathbb{G}_m,\mathbb{G}_m\times\mathbb{G}_m))$. □

Recall that $\Sigma=(\{0\},\mathbb{A}^1,t)\in\mathbb{Z}\mathrm{F}_1(\mathrm{pt},\mathrm{pt})$. For every $k>0$ we write Σ^k to denote $\Sigma\boxtimes\overset{k}{\cdots}\boxtimes\Sigma\in\mathbb{Z}\mathrm{F}_k(\mathrm{pt},\mathrm{pt})$.

Let $\tau\colon\mathbb{G}_m\times\mathbb{G}_m\to\mathbb{G}_m\times\mathbb{G}_m$ be the permutation of coordinates morphism. For each even integer $m\geqslant0$ and each integer $n\geqslant0$ consider two presheaf morphisms

$$(-\boxtimes\Sigma^{2n}):\mathbb{Z}\mathrm{F}_m(-\times X\times\mathbb{G}_m\times\mathbb{G}_m,Y\times\mathbb{G}_m\times\mathbb{G}_m)\to\mathbb{Z}\mathrm{F}_{m+2n}(-\times X\times\mathbb{G}_m\times\mathbb{G}_m,Y\times\mathbb{G}_m\times\mathbb{G}_m),$$

$$(-\boxtimes\Sigma^{2n})\circ sw:\mathbb{Z}\mathrm{F}_m(-\times X\times\mathbb{G}_m\times\mathbb{G}_m,Y\times\mathbb{G}_m\times\mathbb{G}_m)\to\mathbb{Z}\mathrm{F}_{m+2n}(-\times X\times\mathbb{G}_m\times\mathbb{G}_m,Y\times\mathbb{G}_m\times\mathbb{G}_m),$$

where $sw(a)=(\mathrm{id}_Y\times\tau)\circ a\circ(\mathrm{id}_X\times\tau)$.

Lemma 5.4. *Let X,Y be k-smooth schemes. Given an even integer $m\geqslant0$, there exists a large enough n and a homotopy*

$$H:\mathbb{Z}\mathrm{F}_m(-\times X\times\mathbb{G}_m\times\mathbb{G}_m,Y\times\mathbb{G}_m\times\mathbb{G}_m)\to\mathbb{Z}\mathrm{F}_{m+2n}(-\times X\times\mathbb{G}_m\times\mathbb{G}_m\times\mathbb{A}^1,Y\times\mathbb{G}_m\times\mathbb{G}_m)$$

such that for any $a\in\mathbb{Z}\mathrm{F}_m(W\times X\wedge(\mathbb{G}_m,1)\wedge(\mathbb{G}_m,1),Y\wedge(\mathbb{G}_m,1)\wedge(\mathbb{G}_m,1))$ one has

$$a\boxtimes\Sigma^{2n}=H_0(a)\ \text{ and }\ H_1(a)=\Sigma^{2n}([(\mathrm{id}_Y\times\tau)\circ a\circ(\mathrm{id}_X\times\tau)]).$$

And both $H_0(a)$ and $H_1(a)$ are in $\mathbb{Z}\mathrm{F}_{m+2n}(W\times X\wedge(\mathbb{G}_m,1)\wedge(\mathbb{G}_m,1),Y\wedge(\mathbb{G}_m,1)\wedge(\mathbb{G}_m,1))$.

Proof. It follows from Proposition 5.3 that there exists a large enough integer n and a homotopy $\Psi\in\mathbb{Z}\mathrm{F}_n(\mathbb{G}_m\times\mathbb{G}_m\times\mathbb{A}^1,\mathbb{G}_m\times\mathbb{G}_m)$ such that $i_0^*(\Psi)=p\circ(-\varepsilon\boxtimes(\Sigma^{n-1}\mathrm{id}_{\mathbb{G}_m\times\mathbb{G}_m}))$ and $i_1^*(\Psi)=p\circ\Sigma^n\tau$, where $p=(\mathrm{id}-e_1)\boxtimes(\mathrm{id}-e_1)$.

Given any element $a \in \mathbb{Z}\mathrm{F}_m(W \times X \times \mathbb{G}_m \times \mathbb{G}_m, Y \times \mathbb{G}_m \times \mathbb{G}_m)$, set

$$H'(a) = (id_Y \times \Psi) \circ (a \times id_{\mathbb{A}^1}) \circ (id_{W \times X} \times \Psi \times id_{\mathbb{A}^1}) \circ (id_{W \times X \times \mathbb{G}_m \times \mathbb{G}_m} \times \Delta) \in$$
$$\in \mathbb{Z}\mathrm{F}_{m+2n}(W \times X \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m),$$

where $\Delta : \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1$ is the diagonal morphism. Then for any element $a \in \mathbb{Z}\mathrm{F}_m(W \times X \wedge (\mathbb{G}_m, 1) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1) \wedge (\mathbb{G}_m, 1))$ one has

$$H'(a)_0 = [id_Y \times \Sigma^{n-1}(-\varepsilon)] \circ a \circ [id_X \times \Sigma^{n-1}(-\varepsilon)] \text{ and } H'(a)_1 = [\mathrm{id}_Y \times \Sigma^n(\tau)] \circ a \circ [\mathrm{id}_X \times \Sigma^n(\tau)].$$

It is easy to see that there are matrices $A, B \in SL_{m+2n}(k)$ such that for any element a in $\mathbb{Z}\mathrm{F}_m(W \times X \wedge (\mathbb{G}_m, 1) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1) \wedge (\mathbb{G}_m, 1))$ one has

$$\varphi_A([id_Y \times \Sigma^{n-1}(-\varepsilon)] \circ a \circ [id_X \times \Sigma^{n-1}(-\varepsilon)]) = a \boxtimes \Sigma^{2n} = \Sigma^{2n}(a),$$
$$\varphi_B([\mathrm{id}_Y \times \Sigma^n(\tau)] \circ a \circ [\mathrm{id}_X \times \Sigma^n(\tau)]) = ([\mathrm{id}_Y \times \tau] \circ a \circ [\mathrm{id}_X \times \tau)) \boxtimes \Sigma^{2n} = \Sigma^{2n}([\mathrm{id}_Y \times \tau] \circ a \circ [\mathrm{id}_X \times \tau]).$$

Choose matrices $A_s, B_s \in SL_{m+2n}(k[s])$ such that $A_0 = id$, $A_1 = A$, $B_0 = id$, $B_1 = B$. Then for the matrix $C_s = B_s \circ A_{1-s} \in SL_{m+2n}(k[s])$ one has $C_0 = A$, $C_1 = B$. Set $H = \varphi_{C_s} \circ H'$. Then for the chosen element $a \in \mathbb{Z}\mathrm{F}_m(W \times X \wedge (\mathbb{G}_m, 1) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1) \wedge (\mathbb{G}_m, 1))$, one has

$$H_0(a) = \varphi_A(H'(a)_0) = \Sigma^{2n}(a) \text{ and } H_1(a) = \varphi_B(H'(a)_1) = \Sigma^{2n}([\mathrm{id}_Y \times \tau] \circ a \circ [\mathrm{id}_X \times \tau)),$$

as was to be proved. □

6. THE INVERSE MORPHISM

The main aim of this section is to define for any integers $n, m \geqslant 0$ a subpresheaf $\mathbb{Z}\mathrm{F}_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ of the presheaf $\mathbb{Z}\mathrm{F}_m(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ and define a morphism of abelian presheaves

$$\rho_n : \mathbb{Z}\mathrm{F}_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m) \to \mathbb{Z}\mathrm{F}_m(-, Y).$$

We also prove certain properties of morphisms ρ_n and of presheaves $\mathbb{Z}\mathrm{F}_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ which are used in the proof of the Linear Cancelation Theorem (Theorem C).

We begin with some general remarks. Let X and Y be k-smooth schemes. Consider a framed correspondence

$$a = (Z, U, (\varphi_1, \varphi_2, \ldots, \varphi_m), g) \in \mathrm{Fr}_m(X \times \mathbb{G}_m, Y \times \mathbb{G}_m).$$

Let $(U, p : U \to \mathbb{A}^m \times (X \times \mathbb{G}_m), s : Z \to U)$ be the étale neighborhood of Z in $\mathbb{A}^m \times (X \times \mathbb{G}_m)$ from the definition of the framed correspondence a. Let t be the invertible function on $X \times \mathbb{G}_m$ corresponding to the projection on $\mathbb{G}_m$ and u be invertible function on $Y \times \mathbb{G}_m$ corresponding to the projection on $\mathbb{G}_m$. Let $f_2 = g^*(u)$ and $f_1 = p^*_{X \times \mathbb{G}_m}(t)$ be two invertible functions on U, where $p_{X \times \mathbb{G}_m} = pr_{X \times \mathbb{G}_m} \circ p : U \to X \times \mathbb{G}_m$. Set $g = (g_1, g_2)$, where $g_1 = pr_Y \circ g$ and $g_2 = pr_{\mathbb{G}_m} \circ g$.

Since Z is finite over $X \times \mathbb{G}_m$, the $\mathscr{O}_{X \times \mathbb{G}_m \times Y \times \mathbb{G}_m}$-sheaf $P_a := \mathscr{O}_U / (\varphi_1, \varphi_2, \ldots, \varphi_m)$ is finite over $X \times \mathbb{G}_m$. Since the sheaf P_a is finite over $X \times \mathbb{G}_m$, it is flat over $X \times \mathbb{G}_m$ by [OP, Lemma 7.3].

Let Z_n^+ be the closed subset of Z defined by the equation $(f_1^{n+1} - 1)|_Z = 0$. Let Z_n^- be the closed subset of Z defined by the equation $(f_1^{n+1} - f_2)|_Z = 0$. Note that Z_n^+ is finite over X if and only if $\mathscr{O}_U / (f_1^{n+1} - 1, \varphi_1, \varphi_2, \ldots, \varphi_m)$ is finite over X. By [S, 4.1] the latter $\mathscr{O}_X$-module is always finite and even flat. Note that Z_n^- is finite over X if and only if $\mathscr{O}_U / (f_1^{n+1} - f_2, \varphi_1, \varphi_2, \ldots, \varphi_m)$ is finite over X. We have mentioned above that the $\mathscr{O}_{X \times \mathbb{G}_m}$-module $P_a = \mathscr{O}_U / (\varphi_1, \varphi_2, \ldots, \varphi_m)$ is finite and flat over $\mathscr{O}_{X \times \mathbb{G}_m}$. Thus by [S, 4.1.b] there exists an integer N such that for any $n \geqslant N$ the $\mathscr{O}_X$-module $\mathscr{O}_U / (f_1^{n+1} - f_2, \varphi_1, \varphi_2, \ldots, \varphi_m)$ is finite and even flat over X. In particular, Z_n^- is finite over X for any $n \geqslant N$.

The following definition is inspired by [S, Section 4].

Definition 6.1. Let X and Y be k-smooth schemes. Consider a framed correspondence $a = (Z,U,(\varphi_1,\varphi_2,\dots,\varphi_m),g) \in \mathrm{Fr}_m(X\times\mathbb{G}_m, Y\times\mathbb{G}_m)$. Set

$$\rho^+_{n,fr}(a) := (Z^+_n, U, (f_1^{n+1}-1,\varphi_1,\varphi_2,\dots,\varphi_m), g_1)$$

and

$$\rho^-_{n,fr}(a) := (Z^-_n, U, (f_1^{n+1}-f_2,\varphi_1,\varphi_2,\dots,\varphi_m), g_1).$$

As we have mentioned above, Z^+_n is finite over X for all $n \geqslant 0$, hence $\rho^+_{n,fr}(a) \in \mathbb{Z}\mathrm{F}_{m+1}(X,Y)$. We say that $\rho^-_{n,fr}(a)$ is *defined* if Z^-_n is finite over X, which is equivalent to saying that the $\mathscr{O}_X$-module $P_a/(f_1^{n+1}-f_2)P_a$ is finite and flat over X. If $\rho^-_{n,fr}(a)$ is defined, then we set

$$\rho_{n,fr}(a) = \rho^+_{n,fr}(a) - \rho^-_{n,fr}(a) \in \mathbb{Z}\mathrm{F}_{m+1}(X,Y)$$

and say that $\rho_{n,fr}(a)$ is *defined*.

Given integers $m,n \geqslant 0$, denote by $\mathrm{Fr}^{(n)}_m(X\times\mathbb{G}_m, Y\times\mathbb{G}_m)$ the subset of those framed correspondences $a \in \mathrm{Fr}_m(X\times\mathbb{G}_m, Y\times\mathbb{G}_m)$ for which the $\mathscr{O}_X$-module $P_a/(f_1^{n+1}-f_2)P_a$ is finite over X (that is $\rho_{n,fr}(a)$ is defined). It follows from [S, 4.4] that the assignment $X' \mapsto \mathrm{Fr}^{(n)}_m(X'\times\mathbb{G}_m, Y\times\mathbb{G}_m)$ is a subpresheaf of $\mathrm{Fr}_m(-\times\mathbb{G}_m, Y\times\mathbb{G}_m)$.

Definition 6.2. Define a presheaf of abelian groups $\mathbb{Z}\mathrm{F}^{(n)}_m(-\times\mathbb{G}_m, Y\times\mathbb{G}_m)$ as follows. Its sections on X is the abelian group $\mathbb{Z}[\mathrm{Fr}^{(n)}_m(X\times\mathbb{G}_m, Y\times\mathbb{G}_m)]$ modulo a subgroup generated by all elements of the form

$$(Z_1\sqcup Z_2, U_1\sqcup U_2, \varphi_1\sqcup\varphi_2, g_1\sqcup g_2) - (Z_1,U_1,\varphi_1,g_1) - (Z_2,U_2,\varphi_2,g_2).$$

It is straightforward to check that $\mathbb{Z}\mathrm{F}^{(n)}_m(X\times\mathbb{G}_m, Y\times\mathbb{G}_m)$ is a free abelian group with a free basis consisting of the elements of the form $a = (Z,U,\varphi,g)$, where Z is connected and the $\mathscr{O}_X$-module $P_a/(f_1^{n+1}-f_2)P_a$ is finite and flat over X. Moreover, the group $\mathbb{Z}\mathrm{F}^{(n)}_m(X\times\mathbb{G}_m, Y\times\mathbb{G}_m)$ is a subgroup of the group $\mathbb{Z}\mathrm{F}_m(X\times\mathbb{G}_m, Y\times\mathbb{G}_m)$, and $\mathbb{Z}\mathrm{F}^{(n)}_m(-\times\mathbb{G}_m, Y\times\mathbb{G}_m)$ is a subpresheaf of the presheaf $\mathbb{Z}\mathrm{F}_m(-\times\mathbb{G}_m, Y\times\mathbb{G}_m)$.

It follows from [S, 4.4] that for any morphism $f : X' \to X$ of smooth varieties the following diagram is commutative

$$\begin{array}{ccc}
\mathrm{Fr}^{(n)}_m(X\times\mathbb{G}_m, Y\times\mathbb{G}_m) & \xrightarrow{(f\times id)^*} & \mathrm{Fr}^{(n)}_m(X'\times\mathbb{G}_m, Y\times\mathbb{G}_m) \\
\downarrow{\rho_{n,fr}} & & \downarrow{\rho_{n,fr}} \\
\mathbb{Z}\mathrm{F}_{m+1}(X,Y) & \xrightarrow{f^*} & \mathbb{Z}\mathrm{F}_{m+1}(X',Y).
\end{array}$$

We see that $\rho_{n,fr} : \mathrm{Fr}^{(n)}_m(-\times\mathbb{G}_m, Y\times\mathbb{G}_m) \to \mathbb{Z}\mathrm{F}_{m+1}(-,Y)$ is a morphism of pointed presheaves. We can extend it to get a morphism of presheaves of abelian groups $\mathbb{Z}[\mathrm{Fr}^{(n)}_m(-\times\mathbb{G}_m, Y\times\mathbb{G}_m)] \to \mathbb{Z}\mathrm{F}_{m+1}(-,Y)$. This morphism annihilates the elements of the form

$$(Z_1\sqcup Z_2, U_1\sqcup U_2, \varphi_1\sqcup\varphi_2, g_1\sqcup g_2) - (Z_1,U_1,\varphi_1,g_1) - (Z_2,U_2,\varphi_2,g_2).$$

Definition 6.3. The above arguments show that the presheaf morphism $\rho_{n,fr}$ induces a unique presheaf of abelian groups morphism

$$\rho_n : \mathbb{Z}\mathrm{F}^{(n)}_m(-\times\mathbb{G}_m, Y\times\mathbb{G}_m) \to \mathbb{Z}\mathrm{F}_{m+1}(-,Y)$$

such that for any $a \in Fr_m^{(n)}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ one has $\rho_n(a) = \rho_{n,fr}(a)$. We also call ρ_n the *inverse morphism*.

Lemma 6.4. *The following relations are true:*

$$\mathrm{Fr}_m(-\times \mathbb{G}_m, Y \times \mathbb{G}_m) = \mathrm{colim}_n \mathrm{Fr}_m^{(n)}(-\times \mathbb{G}_m, Y \times \mathbb{G}_m),$$

$$\mathbb{Z}\mathrm{F}_m(-\times \mathbb{G}_m, Y \times \mathbb{G}_m) = \mathrm{colim}_n \mathbb{Z}\mathrm{F}_m^{(n)}(-\times \mathbb{G}_m, Y \times \mathbb{G}_m).$$

This lemma follows from the following

Proposition 6.5. *([S, 4.1]) For any framed correspondence $a \in \mathrm{Fr}_m(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ one has:*

(a) *for any $n \geqslant 0$, the sheaf $P_a/(f_1^{n+1}-1)P_a$ is finite and flat over X;*
(b) *there exists an integer N such that, for any $n \geqslant N$, the sheaf $P_a/(f_1^{n+1}-f_2)P_a$ is finite and flat over X.*

We shall need the following obvious property of ρ_n.

Lemma 6.6. *For any integers $m,n,r \geqslant 0$, the following diagram commutes*

$$\begin{array}{ccc} \mathbb{Z}\mathrm{F}_m^{(n)}(-\times \mathbb{G}_m, Y \times \mathbb{G}_m) & \xrightarrow{\Sigma^r} & \mathbb{Z}\mathrm{F}_{m+r}^{(n)}(-\times \mathbb{G}_m, Y \times \mathbb{G}_m) \\ \downarrow \rho_n & & \downarrow \rho_n \\ \mathbb{Z}\mathrm{F}_{m+1}(-,Y) & \xrightarrow{\Sigma^r} & \mathbb{Z}\mathrm{F}_{m+1+r}(-,Y). \end{array}$$

Lemma 6.7. *Let X and Y be k-smooth schemes. Then for any integers m and n and any $a \in \mathbb{Z}\mathrm{F}_m(X,Y)$, one has $a \boxtimes (\mathrm{id} - e_1) \in \mathbb{Z}\mathrm{F}_m^{(n)}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$. In particular, for any integers m and n there is a well defined composite morphism*

$$\rho_n \circ (-\boxtimes(\mathrm{id}-e_1)) : \mathbb{Z}\mathrm{F}_m(-\times X, Y) \to \mathbb{Z}\mathrm{F}_m^{(n)}(-\times X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \to \mathbb{Z}\mathrm{F}_{m+1}(-\times X, Y).$$

Moreover, for an element $a \in \mathbb{Z}\mathrm{F}_m(W \times X, Y)$ of the form $(Z, U, (\varphi_1, \varphi_2, \dots, \varphi_m), g)$ one has

$$\rho_n(a \boxtimes (\mathrm{id}-e_1)) = -(Z \times Z(t^{n+1}-t), U \times \mathbb{G}_m, (t^{n+1}-t, \varphi_1, \varphi_2, \dots, \varphi_m), g) + \\ + (Z \times Z(t^{n+1}-1), U \times \mathbb{G}_m, (t^{n+1}-1, \varphi_1, \varphi_2, \dots, \varphi_m), g) \in \mathbb{Z}\mathrm{F}_{m+1}(W \times X, Y).$$

Proof. Let $a \in \mathbb{Z}\mathrm{F}_m(W \times X, Y)$ be the image of $(Z, U, (\varphi_1, \varphi_2, \dots, \varphi_m), g) \in \mathrm{Fr}_m(W \times X, Y)$. Then

$$a \boxtimes (\mathrm{id}-e_1) = (Z \times \mathbb{G}_m, U \times \mathbb{G}_m, (\varphi_1, \varphi_2, \dots, \varphi_m), (g,t)) - \\ - (Z \times \mathbb{G}_m, U \times \mathbb{G}_m, (\varphi_1, \varphi_2, \dots, \varphi_m), (g, e_1)) \in \mathbb{Z}\mathrm{F}_m(W \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m),$$

where t is the coordinate function on $\mathbb{G}_m$. Clearly, $Z_n^+ = Z \times Z(t^{n+1}-1) \subset Z \times \mathbb{G}_m$ and $Z_n^- = Z \times Z(t^{n+1}-t) \subset Z \times \mathbb{G}_m$. Both sets are finite over X. Hence $a \boxtimes (\mathrm{id}-e_1) \in \mathbb{Z}\mathrm{F}_m^{(n)}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ in this case. Any element of $\mathbb{Z}\mathrm{F}_m(W \times X, Y)$ is a linear combination of elements of the form $(Z, U, (\varphi_1, \varphi_2, \dots, \varphi_m), g)$. This proves the first assertion of the lemma.

Computing $\rho_n(a \boxtimes (\mathrm{id}-e_1))$ for $a = (Z, U, (\varphi_1, \varphi_2, \dots, \varphi_m), g)$ we obtain

$$\rho_n(a \boxtimes (\mathrm{id}-e_1)) = -(Z \times Z(t^{n+1}-t), U \times \mathbb{G}_m, (t^{n+1}-t, \varphi_1, \varphi_2, \dots, \varphi_m), g) + \\ + (Z \times Z(t^{n+1}-1), U \times \mathbb{G}_m, (t^{n+1}-1, \varphi_1, \varphi_2, \dots, \varphi_m), g) \in \mathbb{Z}\mathrm{F}_{m+1}(W \times X, Y),$$

as was to be shown. □

Lemma 6.8. *Let X and Y be k-smooth schemes. Then for every even integer m and any n one has*

$$\rho_n \circ (-\boxtimes(\mathrm{id}-e_1)) \sim (-\boxtimes \varepsilon) : \mathbb{Z}\mathrm{F}_m(-\times X,Y) \rightrightarrows \mathbb{Z}\mathrm{F}_{m+1}(-\times X,Y),$$

where $\varepsilon = (\{0\},\mathbb{A}^1,-t,c') \in \mathbb{Z}\mathrm{F}_1(\mathrm{pt},\mathrm{pt})$.

Proof. Set $\eta_n = \rho_n \circ (-\boxtimes(\mathrm{id}-e_1))$. Take the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \in SL_{m+1}(k)$$

and let $A_s \in SL_{m+1}(k[s])$ be such that $A_0 = id$, $A_1 = A$. Let H_{A_s} be the $\mathbb{A}^1$-homotopy from Definition 4.6 between the identity and φ_A. By Definition 4.6 one has

$$\eta_n = \rho_n \circ (-\boxtimes(\mathrm{id}-e_1)) \overset{H_{A_s}\circ\eta_n}{\overline{\qquad\qquad}} \varphi_A \circ \rho_n \circ (-\boxtimes(\mathrm{id}-e_1)) = \varphi_A \circ \eta_n.$$

Set $H' = H_{A_s} \circ \eta_n$. By Lemma 4.4 it remains to find an H'' such that $\varphi_A \circ \eta_n \overset{H''}{\overline{\quad}} (-\boxtimes \varepsilon)$ and set $H = H' + H'' - H_{\varphi_A\circ\eta_n}$. In this case by Lemma 4.4 one gets $\rho_n \circ (-\boxtimes(\mathrm{id}-e_1)) = \eta_n \overset{H}{\overline{\quad}} (-\boxtimes\varepsilon)$.

To construct H'', note that by the last statement of Lemma 6.7 one has

$$\varphi_A \circ \eta_n = -\boxtimes[(Z(t^{n+1}-1),\mathbb{G}_m,t^{n+1}-1,c) - (Z(t^{n+1}-t),\mathbb{G}_m,t^{n+1}-t,c)]$$

and $(-\boxtimes\varepsilon) = -\boxtimes(\{0\},\mathbb{A}^1,-t,c')$, where $c\colon \mathbb{G}_m \to \mathrm{pt}$ is the canonical projection. By Lemma 4.9 one can take H'' to be an $\mathbb{A}^1$-homotopy of the form $H'' = (-\boxtimes h'')$, where $h'' \in \mathbb{Z}\mathrm{F}_1(\mathbb{A}^1,pt)$ is such that

$$(Z(t^{n+1}-1),\mathbb{G}_m,t^{n+1}-1,c) - (Z(t^{n+1}-t),\mathbb{G}_m,t^{n+1}-t,c) = h''_0$$

and

$$h''_1 = (\{0\},\mathbb{A}^1,-t,c') \in \mathbb{Z}\mathrm{F}_1(\mathrm{pt},\mathrm{pt}),$$

where $c'\colon \mathbb{A}^1 \to \mathrm{pt}$ is the canonical projection. Now let us find the desired element h''. Since $t^{n+1}-1$ does not vanish at $t=0$, we can extend the neighborhood from $\mathbb{G}_m$ to $\mathbb{A}^1$ to get an equality,

$$(Z(t^{n+1}-1),\mathbb{G}_m,t^{n+1}-1,c) = (Z(t^{n+1}-1),\mathbb{A}^1,t^{n+1}-1),c') \in \mathbb{Z}\mathrm{F}_1(pt,pt).$$

By Lemma 4.14 there is $h''' \in \mathbb{Z}\mathrm{F}_1(\mathbb{A}^1,pt)$ such that

$$(Z(t^{n+1}-1),\mathbb{A}^1,t^{n+1}-1,c') = h'''_0 \quad \text{and} \quad h'''_1 = (Z(t^{n+1}-t),\mathbb{A}^1,t^{n+1}-t,c') \in \mathbb{Z}\mathrm{F}_1(pt,pt),$$

because polynomials $t^{n+1}-t$ and $t^{n+1}-1$ have the same degree and the same leading coefficient. Using the additivity property for supports in $\mathbb{Z}\mathrm{F}_1(pt,pt)$ and the second statement of Lemma 4.13, we can find an element $h^{iv} \in \mathbb{Z}\mathrm{F}_1(\mathbb{A}^1,pt)$ such that

$$(Z(t^{n+1}-t),\mathbb{G}_m,t^{n+1}-t,c) = h^{iv}_0 \text{ and } h^{iv}_1 = (Z(t^{n+1}-t),\mathbb{A}^1,t^{n+1}-t,c') - (\{0\},\mathbb{A}^1,-t,c') \in \mathbb{Z}\mathrm{F}_1(pt,pt)$$

Set $h'' := h''' - h^{iv} \in \mathbb{Z}\mathrm{F}_1(\mathbb{A}^1,pt)$. Then h'' is the desired element.

Set $H'' = (-\boxtimes h'')$ and $H = H' + H'' - H_{\varphi_A\circ\eta_n}$. Then H is the desired $\mathbb{A}^1$-homotopy. That is

$$\rho_n \circ (-\boxtimes(\mathrm{id}-e_1)) \overset{H}{\overline{\quad}} (-\boxtimes\varepsilon)$$

and our statement follows. □

7. THEOREM C

The main purpose of this section is to prove Theorem C. We sometimes identify simplicial abelian groups with chain complexes concentrated in non-negative degrees by using the Dold–Kan correspondence.

Lemma 7.1. *Let X and Y be k-smooth schemes and $m,r,N \geqslant 0$ be integers. Then for any Moore cycle $a \in \mathbb{Z}\mathrm{F}_m(\Delta^r \times X, Y)$ of the simplicial abelian group $\mathbb{Z}\mathrm{F}_m(\Delta^\bullet \times X, Y)$, one has $a \boxtimes (id - e_1) \in \mathbb{Z}\mathrm{F}_m^{(N)}(\Delta^r \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$. Moreover, $\rho_N(a \boxtimes (id - e_1))$ is a Moore cycle. The homology classes of Moore cycles*

$$a \boxtimes \varepsilon \ \ and \ \ \rho_N(a \boxtimes (id - e_1))$$

coincide in $\mathbb{Z}\mathrm{F}_{m+1}(\Delta^\bullet \times X, Y)$.

Proof. The element $a \boxtimes (id - e_1)$ is in $\mathbb{Z}\mathrm{F}_m^{(N)}(\Delta^r \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ by Lemma 6.7. Since $\mathbb{Z}\mathrm{F}_m^{(N)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m))$ is a presheaf, then $\partial_i(a \boxtimes (id - e_1)) \in \mathbb{Z}\mathrm{F}_m^{(N)}(\Delta^r \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$. Since the morphism ρ_N is a morphism of presheaves, then

$$\partial_i(\rho_N(a \boxtimes (id - e_1))) = \rho_N(\partial_i(a \boxtimes (id - e_1)) = \rho_N(\partial_i(a) \boxtimes (id - e_1)) = 0.$$

This proves the first assertion of the lemma.

By Lemma 6.8 the morphism

$$a' \mapsto \rho_N(a' \boxtimes (\mathrm{id}_{\mathbb{G}_m} - e_1)) : \mathbb{Z}\mathrm{F}_m(- \times X, Y) \to \mathbb{Z}\mathrm{F}_m^{(N)}(- \times X, Y) \to \mathbb{Z}\mathrm{F}_{m+1}(- \times X, Y)$$

is $\mathbb{A}^1$-homotopic to the morphism $a' \mapsto a' \boxtimes \varepsilon$. Thus the corresponding morphisms of the simplicial abelian groups $\mathbb{Z}\mathrm{F}_m(\Delta^\bullet \times X, Y) \rightrightarrows \mathbb{Z}\mathrm{F}_{m+1}(\Delta^\bullet \times X, Y)$ induce the same morphisms on homology. Hence the homology class of the Moore cycle $\rho_N(a \boxtimes (\mathrm{id}_{\mathbb{G}_m} - e_1))$ coincides with the homology class of the Moore cycle $a \boxtimes \varepsilon$. $\square$

Lemma 7.2. *One has $\varepsilon \boxtimes \varepsilon \sim \Sigma^2$ in $\mathbb{Z}\mathrm{F}_2(\mathrm{pt}, \mathrm{pt})$. Moreover, for any integer $r \geqslant 0$ one has $\varepsilon \boxtimes \varepsilon \boxtimes \Sigma^r \sim \Sigma^{2+r}$ in $\mathbb{Z}\mathrm{F}_{2+r}(\mathrm{pt}, \mathrm{pt})$.*

Proof. Let $c : \mathbb{A}^1 \times \mathbb{A}^2 \to \mathrm{pt}$ be the structure morphism. Take the matrix

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \in SL_2(k).$$

There is an $A_s \in SL_2(k[s])$ such that $A_0 = id$, $A_1 = A$. Take

$$h_s = (\mathbb{A}^1 \times 0, \mathbb{A}^1 \times \mathbb{A}^2, A_s \circ (t_1, t_2), c) \in \mathbb{Z}\mathrm{F}_2(\mathbb{A}^1, \mathrm{pt}).$$

Clearly, $h_0 = \Sigma^2$ and $h_1 = \varepsilon \boxtimes \varepsilon$. The first assertion is proved. To prove the second one take the element $h_s \boxtimes \Sigma^r \in \mathbb{Z}\mathrm{F}_{2+r}(\mathbb{A}^1, \mathrm{pt})$. Then $h_0 \boxtimes \Sigma^r = \Sigma^{2+r}$ and $h_1 \boxtimes \Sigma^r = \varepsilon \boxtimes \varepsilon \boxtimes \Sigma^r$. $\square$

Corollary 7.3. *Let X and Y be k-smooth schemes and $m \geqslant 0$ be an integer. Then,*

$$(- \boxtimes \varepsilon^2) \sim (- \boxtimes \Sigma^2) : \mathbb{Z}\mathrm{F}_m(- \times X, Y) \rightrightarrows \mathbb{Z}\mathrm{F}_{m+2}(- \times X, Y)$$

and

$$(- \boxtimes \varepsilon^2 \boxtimes \Sigma^r) \sim (- \boxtimes \Sigma^{2+r}) : \mathbb{Z}\mathrm{F}_m(- \times X, Y) \rightrightarrows \mathbb{Z}\mathrm{F}_{m+2+r}(- \times X, Y).$$

Therefore the first pair of maps produces the same maps on homology

$$H_*(\mathbb{Z}\mathrm{F}_m(\Delta^\bullet \times X, Y)) \rightrightarrows H_*(\mathbb{Z}\mathrm{F}_{m+2}(\Delta^\bullet \times X, Y)).$$

Similarly, the second pair of maps gives the same maps on homology

$$H_*(\mathbb{Z}\mathrm{F}_m(\Delta^\bullet \times X, Y)) \rightrightarrows H_*(\mathbb{Z}\mathrm{F}_{m+2+r}(\Delta^\bullet \times X, Y)).$$

Lemma 7.4. *Let X and Y be k-smooth schemes and $m \geqslant 0$ be an integer. Then for any integer $r \geqslant 0$ one has*

$$Ker[-\boxtimes(\mathrm{id}_{\mathbb{G}_m} - e_1) : H_r(\mathbb{Z}\mathrm{F}_m(\Delta^\bullet \times X, Y)) \to H_r(\mathbb{Z}\mathrm{F}_m((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1)))] \subseteq$$
$$\subseteq Ker[(-\boxtimes \Sigma^2) : H_r(\mathbb{Z}\mathrm{F}_m(\Delta^\bullet \times X, Y)) \to H_r(\mathbb{Z}\mathrm{F}_{m+2}(\Delta^\bullet \times X, Y))].$$

Proof. Consider the associated Moore complexes. Assume that

$$a \in \mathbb{Z}\mathrm{F}_m(\Delta^r \times X, Y)$$

is a Moore cycle for which $a \boxtimes (\mathrm{id}_{\mathbb{G}_m} - e_1)$ is a boundary, i.e., there exists $b \in \mathbb{Z}\mathrm{F}_m((\Delta^{r+1} \times X) \times \mathbb{G}_m, Y \times \mathbb{G}_m))$ such that $\partial_i(b) = 0$ for $i = 0, 1, \dots, r$ and $\partial_{r+1}(b) = a \boxtimes (\mathrm{id}_{\mathbb{G}_m} - e_1)$. By Lemma 6.4 there exists an N such that $b \in \mathbb{Z}\mathrm{F}_m^{(N)}(\Delta^{r+1} \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$. Since $\mathbb{Z}\mathrm{F}_m^{(N)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ is a presheaf, then $\partial_i(b) \in \mathbb{Z}\mathrm{F}_m^{(N)}(\Delta^r \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$. Since ρ_N is a presheaf morphism $\mathbb{Z}\mathrm{F}_m^{(N)}(- \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \to \mathbb{Z}\mathrm{F}_{m+1}(- \times X, Y)$, one has $\partial_i(\rho_N(b)) = \rho_N(\partial_i(b))$. Thus,

$$\partial_i(\rho_N(b)) = \rho_N(\partial_i(b)) = 0 \;\text{ for }\; 0 \leqslant i \leqslant r,$$
$$\partial_{r+1}(\rho_N(b)) = \rho_N(\partial_{n+1}(b)) = \rho_N(a \boxtimes (\mathrm{id}_{\mathbb{G}_m} - e_1)).$$

We see that the homology class of the Moore cycle $\rho_N(a \boxtimes (\mathrm{id}_{\mathbb{G}_m} - e_1))$ vanishes. By Lemma 7.1 the homology class of the Moore cycle $a \boxtimes \varepsilon$ vanishes in $H_r(\mathbb{Z}\mathrm{F}_{m+1}(\Delta^\bullet \times X, Y))$. Thus the homology class of the Moore cycle $a \boxtimes \varepsilon \boxtimes \varepsilon$ vanishes in $H_r(\mathbb{Z}\mathrm{F}_{m+2}(\Delta^\bullet \times X, Y))$. By Corollary 7.3 the homology class of $a \boxtimes \Sigma^2$ vanishes in $H_r(\mathbb{Z}\mathrm{F}_{m+2}(\Delta^\bullet \times X, Y))$, too. □

Lemma 7.5. *Let X and Y be k-smooth schemes and $m, r \geqslant 0$ be integers. Let n be the integer from Lemma 5.4. Then for any Moore cycle $a \in \mathbb{Z}\mathrm{F}_m((\Delta^r \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))$ there exists an integer N such that the element $\rho_N(a)$ is defined and the homology class of the Moore cycle*

$$\Sigma^{2n}(\rho_N(a)) \boxtimes (id - e_1) \in \mathbb{Z}\mathrm{F}_{m+2n+1}((\Delta^r \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))$$

coincides with the homology class of the Moore cycle $\Sigma^{2n}(a \boxtimes \varepsilon)$.

Proof. Set $a' = a \boxtimes (id - e_1)$. Let H be the $\mathbb{A}^1$-homotopy from Lemma 5.4. Consider the element

$$H(a') \in \mathbb{Z}\mathrm{F}_{m+2n}((\Delta^r \times X) \times \mathbb{G}_m \times \mathbb{G}_m, Y \times \mathbb{G}_m \times \mathbb{G}_m).$$

By Lemma 6.4 there is an integer N such that

$$a \in \mathbb{Z}\mathrm{F}_m^{(N)}((\Delta^r \times X) \times \mathbb{G}_m, Y \times \mathbb{G}_m)$$

and

$$H(a') \in \mathbb{Z}\mathrm{F}_{m+2n}^{(N)}((\Delta^r \times X) \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m).$$

Since a' is a Moore cycle and H is a presheaf morphism, the element $H(a')$ is a Moore cycle in $\mathbb{Z}\mathrm{F}_{m+2n}((\Delta^\bullet \times X) \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m)$. Since

$$\mathbb{Z}\mathrm{F}_{m+2n}^{(N)}((- \times X) \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m)$$

is a subpresheaf of $\mathbb{Z}\mathrm{F}_m((- \times X) \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m)$, it follows that $H(a')$ is a Moore cycle in $\mathbb{Z}\mathrm{F}_{m+2n}^{(N)}((\Delta^\bullet \times X) \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m)$.

Applying the presheaf morphism

$$\rho_N : \mathbb{Z}\mathrm{F}_{m+2n}^{(N)}((- \times X) \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m) \to \mathbb{Z}\mathrm{F}_{m+2n+1}((- \times X) \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m)$$

to the Moore cycle $H(a')$, we get a Moore cycle

$$\rho_N(H(a')) \in \mathbb{Z}\mathrm{F}_{m+2n+1}((\Delta^r \times X) \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m).$$

Hence $i_0^*(\rho_N(H(a'))) \in \mathbb{Z}\mathrm{F}_{m+2n+1}((\Delta^r \times X) \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ and $i_1^*(\rho_N(H(a'))) \in \mathbb{Z}\mathrm{F}_{m+2n+1}((\Delta^r \times X) \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ are Moore cycles, too. Furthermore,

$$i_0^*(\rho_N(H(a'))) = \rho_N(i_0^*(H(a'))) = \rho_N(\Sigma^{2n}(a')) = \Sigma^{2n}(\rho_N(a'))$$

and

$$i_1^*(\rho_N(H(a'))) = \rho_N(i_1^*(H(a'))) = \rho_N(\Sigma^{2n}[(\mathrm{id}_Y \times \tau) \circ a' \circ (\mathrm{id}_X \times \tau)]).$$

The two morphisms

$$i_0^*, i_1^* : \mathbb{Z}\mathrm{F}_{n+2m+1}((\Delta^\bullet \times X) \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m)) \rightrightarrows \mathbb{Z}\mathrm{F}_{n+2m+1}((\Delta^\bullet \times X) \times \mathbb{G}_m, Y \times \mathbb{G}_m))$$

of simplicial abelian groups induce the same morphisms on homology. The element $\rho_N((H(a'))$ is a Moore cycle. Thus the homological classes of the Moore cycles $i_0^*(\rho_N(H(a')))$ and $i_1^*(\rho_N(H(a')))$ coincide in $H_r(\mathbb{Z}\mathrm{F}_{n+2m+1}((\Delta^\bullet \times X) \times \mathbb{G}_m, Y \times \mathbb{G}_m)))$.

By Lemma 6.6 one has $\rho_N(\Sigma^{2n}(a')) = \Sigma^{2n}(\rho_N(a'))$. Thus the first homological class is the class of $\Sigma^{2n}(\rho_N(a')) = \Sigma^{2n}(\rho_N(a \boxtimes (id - e_1)))$. By Lemma 7.1 the latter homological class coincides with the class of the element $\Sigma^{2n}(a \boxtimes \varepsilon)$.

The element $i_1^*(\rho_N(H(a')))$ coincides with $\rho_N(\Sigma^{2n}[(\mathrm{id}_Y \times \tau) \circ (a \boxtimes (id - e_1)) \circ (\mathrm{id}_X \times \tau)])$. By Lemma 6.6 the latter element coincides with

$$\Sigma^{2n}(\rho_N[(\mathrm{id}_Y \times \tau) \circ (a \boxtimes (id - e_1)) \circ (\mathrm{id}_X \times \tau)]) = \Sigma^{2n}[\rho_N(a) \boxtimes (id - e_1)].$$

Hence the homological classes $\Sigma^{2n}(a \boxtimes \varepsilon)$ and $[\Sigma^{2n}[\rho_N(a) \boxtimes (id - e_1)]]$ coincide in $H_r(\mathbb{Z}\mathrm{F}_{n+2m+1}((\Delta^\bullet \times X) \times \mathbb{G}_m, Y \times \mathbb{G}_m))$. Finally, the complex $\mathbb{Z}\mathrm{F}_{n+2m+1}((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))$ is a direct summand in $\mathbb{Z}\mathrm{F}_{n+2m+1}((\Delta^\bullet \times X) \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ and the elements $\Sigma^{2n}(a \boxtimes \varepsilon), \Sigma^{2n}(\rho_N(a) \boxtimes (id - e_1))$ are in $\mathbb{Z}\mathrm{F}_{n+2m+1}((\Delta^r \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))$. Hence the homological classes $[\Sigma^{2n}[\rho_N(a) \boxtimes (id - e_1)]]$ and $[\Sigma^{2n}(a \boxtimes \varepsilon)]$ coincide in $H_r(\mathbb{Z}\mathrm{F}_{n+2m+1}((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1)))$. □

Lemma 7.6. *Let X and Y be k-smooth schemes and $m, r \geqslant 0$ be integers. Let n be the integer from Lemma 5.4. Then*

$$Im[(- \boxtimes \Sigma^{2n+2}) : H_r(\mathbb{Z}\mathrm{F}_m((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))) \to \to H_r(\mathbb{Z}\mathrm{F}_{m+2n+2}((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1)))] \subseteq$$
$$Im[(- \boxtimes (\mathrm{id}_{\mathbb{G}_m} - e_1)) : H_r(\mathbb{Z}\mathrm{F}_{m+2n+2}(\Delta^\bullet \times X, Y)) \to H_r(\mathbb{Z}\mathrm{F}_{m+2n+2}((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1)))].$$

Proof. Take a Moore cycle $a' \in \mathbb{Z}\mathrm{F}_m((\Delta^r \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))$. Then the element $a := a' \boxtimes \varepsilon$ is a Moore cycle in $\mathbb{Z}\mathrm{F}_{m+1}((\Delta^r \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))$. By Lemma 7.5 the homology classes of $\Sigma^{2n}(a \boxtimes \varepsilon)$ and $\Sigma^{2n}(\rho_N(a)) \boxtimes (id - e_1)$ coincide in

$$H_r(\mathbb{Z}\mathrm{F}_{m+2+2n}((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))).$$

By Corollary 7.3 the homology classes of $\Sigma^{2n}(a \boxtimes \varepsilon) = \Sigma^{2n}(a' \boxtimes \varepsilon \boxtimes \varepsilon)$ and $\Sigma^{2n+2}(a')$ coincide. Hence the homology classes of $\Sigma^{2n+2}(a')$ and $\Sigma^{2n}(\rho_N(a' \boxtimes \varepsilon)) \boxtimes (id - e_1)$ coincide in $H_r(\mathbb{Z}\mathrm{F}_{m+2+2n}((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1)))$. □

We are now in a position to prove Theorem C.

Theorem C. *Let X and Y be k-smooth schemes. Then*

$$- \boxtimes (\mathrm{id}_{\mathbb{G}_m} - e_1) : \ \mathbb{Z}\mathrm{F}(\Delta^\bullet \times X, Y) \to \mathbb{Z}\mathrm{F}((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))$$

is a quasi-isomorphism of complexes of abelian groups.

Proof. The theorem follows from Lemmas 7.4 and 7.6.

In more detail, first prove that the morphism $-\boxtimes(\mathrm{id}_{\mathbb{G}_m}-e_1)$ induces an epimophism on homology groups. For this take an integer $r\geqslant 0$ and an element $a\in H_r(\mathbb{Z}\mathrm{F}((\Delta^\bullet\times X)\wedge(\mathbb{G}_m,1),Y\wedge(\mathbb{G}_m,1)))$. We will find an element $b\in H_r(\mathbb{Z}\mathrm{F}(\Delta^\bullet\times X,Y))$ such that $b\boxtimes(\mathrm{id}_{\mathbb{G}_m}-e_1)=a$. Note that there exist an integer $m\geqslant 0$ and an element $a_m\in H_r(\mathbb{Z}\mathrm{F}_m((\Delta^\bullet\times X)\wedge(\mathbb{G}_m,1),Y\wedge(\mathbb{G}_m,1)))$ which is a lift of the element a. Let n be the integer from Lemma 5.4. By Lemma 7.6 there exists an element $b_{m+2n+2}\in H_r(\mathbb{Z}\mathrm{F}_{m+2n+2}(\Delta^\bullet\times X,Y))$ such that

$$b_{m+2n+2}\boxtimes(\mathrm{id}_{\mathbb{G}_m}-e_1)=a_m\boxtimes\Sigma^{2n+2}\in H_r(\mathbb{Z}\mathrm{F}_{m+2n+2}((\Delta^\bullet\times X)\wedge(\mathbb{G}_m,1),Y\wedge(\mathbb{G}_m,1))).$$

Let b be the image of b_{m+2n+2} in $H_r(\mathbb{Z}\mathrm{F}(\Delta^\bullet\times X,Y))$. Clearly, $b\boxtimes(\mathrm{id}_{\mathbb{G}_m}-e_1)=a$ in $H_r(\mathbb{Z}\mathrm{F}((\Delta^\bullet\times X)\wedge(\mathbb{G}_m,1),Y\wedge(\mathbb{G}_m,1)))$. Thus the morphism $-\boxtimes(\mathrm{id}_{\mathbb{G}_m}-e_1)$ induces an epimophism on homology. The fact that the morphism $-\boxtimes(\mathrm{id}_{\mathbb{G}_m}-e_1)$ induces a monomophism on homology is proved in a similar fashion. Theorem C is proved. □

APPENDIX A.

The main goal of this section is to prove Theorem D.

Let $(\mathscr{V},\otimes)$ be a closed symmetric monoidal category and $\mathscr{C}$ is a bicomplete category which is tensored and cotensored over $\mathscr{V}$. Then for every $V\in\mathscr{V}$ and $C\in\mathscr{C}$ there are defined objects $V\otimes C$, $C\otimes V$, $\underline{\mathrm{Hom}}(V,C)$ of $\mathscr{C}$. They are all functorial in V and C. Moreover, for every morphism $r:V\to V'$ in $\mathscr{V}$ the square in $\mathscr{C}$

$$\begin{array}{ccc} C & \xrightarrow{-\otimes V} & \underline{\mathrm{Hom}}(V,C\otimes V) \\ {\scriptstyle -\otimes V'}\downarrow & & \downarrow{\scriptstyle r_*} \\ \underline{\mathrm{Hom}}(V',C\otimes V') & \xrightarrow{r^*} & \underline{\mathrm{Hom}}(V,C\otimes V') \end{array} \qquad (7)$$

is commutative.

As an important example, $\mathscr{V}$ is the category of simplicial objects $sPre(\mathbb{Z}\mathrm{F}_0(k))$ in the category $Pre(\mathbb{Z}\mathrm{F}_0(k))$ and $\mathscr{C}$ is the category $sPre_{Ab}(\mathbb{Z}\mathrm{F}_*(k))$ of simplicial objects in $Pre_{Ab}(\mathbb{Z}\mathrm{F}_*(k))$. The General Framework of p. 6 is then immediately extended to this couple $(\mathscr{V},\mathscr{C})$. Recall that the functor $\mathbb{Z}\mathrm{F}_*(k)\times\mathbb{Z}\mathrm{F}_0(k)\xrightarrow{\boxtimes}\mathbb{Z}\mathrm{F}_*(k)$ takes (X,Y) to $X\times Y$. As usual, the Yoneda embedding identifies the category simplicial objects in $\mathbb{Z}\mathrm{F}_0(k)$ with a full subcategory of $sPre_{Ab}(\mathbb{Z}\mathrm{F}_0(k))$.

The following lemma is obvious.

Lemma A.1. *Suppose in the diagram (7) the morphisms r_*, r^* and $-\otimes V'$ are sectionwise weak equivalences, then the morphism $-\otimes V$ is a sectionwise weak equivalence.*

As it is shown in [GP1, Section 5], the category of framed correspondences of level zero $\mathrm{Fr}_0(k)$ has an action by finite pointed sets $S\otimes K:=\bigsqcup_{K\setminus *}S$ with $S\in Sm/k$ and K a finite pointed set. The cone of S is the simplicial object $S\otimes I$ in $\mathrm{Fr}_0(k)$, where $(I,1)$ is the pointed simplicial set $\Delta[1]$ with basepoint 1. There is a natural morphism $i_0:S\to S\otimes I$ in $\Delta^{\mathrm{op}}\mathrm{Fr}_0(k)$. Let $\mathrm{pt}\xrightarrow{e_1}\mathbb{G}_m$ be the point 1 in $\mathbb{G}_m(k)$. Then $\mathbb{G}_m^{\wedge 1}$ is the simplicial object in $\mathrm{Fr}_0(k)$ which is obtained by taking the pushout of the diagram $\mathbb{G}_m\xleftarrow{e_1}\mathrm{pt}\xhookrightarrow{i_0}\mathrm{pt}\otimes I$ in $\Delta^{\mathrm{op}}\mathrm{Fr}_0(k)$.

Let $L:\mathrm{Fr}_0\to\mathbb{Z}\mathrm{F}_0$ be the canonical functor which is the identity on objects and which takes a morphism $\varphi\in\mathrm{Fr}_0(Y,X)$ to the class $1\cdot\varphi\in\mathbb{Z}\mathrm{F}_0(Y,X)$. If we apply the functor L to $\mathbb{G}_m^{\wedge 1}$, we get an object in $sPre_{Ab}(\mathbb{Z}\mathrm{F}_0(k))$ denoted by $\mathbb{Z}\mathrm{F}_0(\mathbb{G}_m^{\wedge 1})$.

Put $\mathbb{Z}F_0(\mathbb{G}_m,1)=\mathbb{Z}F_0(\mathbb{G}_m)/Im(e_{1,*})=\mathrm{Ker}(e_1^*)$. There is a unique morphism $r:\mathbb{Z}F_0(\mathbb{G}_m^{\wedge 1})\to \mathbb{Z}F_0(\mathbb{G}_m,1)$ which restricts to the quotient map $q:\mathbb{Z}F_0(\mathbb{G}_m)\to\mathbb{Z}F_0(\mathbb{G}_m)/Im(e_{1,*})$ on $\mathbb{Z}F_0(\mathbb{G}_m)$ and which restricts to the zero map on $\mathrm{pt}\otimes I$.

The following lemma is straightforward and left to the reader.

Lemma A.2. $\mathbb{Z}F(X)\boxtimes\mathbb{Z}F_0(\mathbb{G}_m,1)=\mathbb{Z}F(X\wedge(\mathbb{G}_m,1))$, $\mathbb{Z}F(X)\boxtimes\mathbb{Z}F_0(\mathbb{G}_m^{\wedge 1})=\mathbb{Z}F(X\times\mathbb{G}_m^{\wedge 1})$.

Lemma A.3. *The morphisms*

$$r_*:\underline{\mathrm{Hom}}(\mathbb{Z}F_0(\mathbb{G}_m^{\wedge 1}),C_*(\mathbb{Z}F(X)\boxtimes\mathbb{Z}F_0(\mathbb{G}_m^{\wedge 1})))\to\underline{\mathrm{Hom}}(\mathbb{Z}F_0(\mathbb{G}_m^{\wedge 1}),C_*(\mathbb{Z}F(X)\boxtimes\mathbb{Z}F_0(\mathbb{G}_m,1)))$$

$$r^*:\underline{\mathrm{Hom}}(\mathbb{Z}F_0(\mathbb{G}_m,1),C_*(\mathbb{Z}F(X)\boxtimes\mathbb{Z}F_0(\mathbb{G}_m,1)))\to\underline{\mathrm{Hom}}(\mathbb{Z}F_0(\mathbb{G}_m^{\wedge 1}),C_*(\mathbb{Z}F(X)\boxtimes\mathbb{Z}F_0(\mathbb{G}_m,1)))$$

are sectionwise weak equivalences.

Proof. It is easy to see that the morphisms

$$r:\mathbb{Z}F_0(\mathbb{G}_m^{\wedge 1})\to\mathbb{Z}F_0(\mathbb{G}_m,1),$$
$$id\boxtimes r:\mathbb{Z}F(X)\boxtimes\mathbb{Z}F_0(\mathbb{G}_m^{\wedge 1})\to\mathbb{Z}F(X)\boxtimes\mathbb{Z}F_0(\mathbb{G}_m,1),$$
$$id\boxtimes r:C_*(\mathbb{Z}F(X)\boxtimes\mathbb{Z}F_0(\mathbb{G}_m^{\wedge 1}))\to C_*(\mathbb{Z}F(X)\boxtimes\mathbb{Z}F_0(\mathbb{G}_m,1))$$

are sectionwise weak equivalences. The lemma now follows. □

Theorem C, Lemma A.1 and Lemma A.3 imply the following

Corollary A.4. *The morphism*

$$-\boxtimes\mathbb{G}_m^{\wedge 1}:C_*\mathbb{Z}F(X)\to\underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1},C_*\mathbb{Z}F(X\times\mathbb{G}_m^{\wedge 1}))$$

is a sectionwise weak equivalence in $sPre_{Ab}(\mathbb{Z}F_*(k))$.

We are now in a position to prove the following

Theorem D. *The morphism $c_0:LM_{fr}(X)\to\underline{\mathrm{Hom}}(\mathbb{G},LM_{fr}(X\times\mathbb{G}_m^{\wedge 1}))$ is a sectionwise stable weak equivalence of presheaves of S^1-spectra.*

Proof. First, the adjunction unit $\mathbb{G}\xrightarrow{\text{adj}}(\mathbb{G}_m^{\wedge 1}|_{Sm/k})$ in $sPre_\bullet(Sm/k)$ induces an isomorphism $\underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1},LM_{fr}(X\times\mathbb{G}_m^{\wedge 1}))\xrightarrow{\text{adj}^*}\underline{\mathrm{Hom}}(\mathbb{G},LM_{fr}(X\times\mathbb{G}_m^{\wedge 1}))$ of S^1-spectra. Second, the morphism $\text{adj}^*\circ(-\boxtimes\mathbb{G}_m^{\wedge 1})$ coincides with the morphism

$$c_0:LM_{fr}(X)\to\underline{\mathrm{Hom}}(\mathbb{G},LM_{fr}(X\times\mathbb{G}_m^{\wedge 1})).$$

It is the morphism (3). The theorem now follows from Corollary A.4. □

Appendix B. Some facts on henzelization

We refer the reader to [Gab] or [FP] for the definition of henzelization of an affine scheme along a closed subscheme.

Let X,X_1 be k-smooth affine varieties, $Z\subset X$, $Z_1\subset X_1$ be closed subsets. Let $f:X_1\to X$ be a k-morphism such that $Z_1\subset f^{-1}(Z)$. For an étale neighborhood $(W,\pi:W\to X,s:Z\to W)$ of Z in X set $W_1=X_1\times_X W$. Let $\pi_1:W_1\to X_1$ be the projection and let $s_1=(i_1,f|_{Z_1}):Z_1\to W_1$, where $i_1:Z_1\hookrightarrow X_1$ be the inclusion. Then (W_1,π_1,s_1) is an étale neighborhood of Z_1 in X_1. Denote by $f_W:W_1\to W$ the projection. Then one has a morphism $\lim(f_W):\lim_{(W,\pi,s)}W_1\to\lim_{(W,\pi,s)}W=X_Z^h$. Set,

$$f^h=\lim(f_W)\circ can_f:(X_1)_{Z_1}^h\to X_Z^h, \tag{8}$$

where $can_f : (X_1)^h_{Z_1} \to \lim_{(W,\pi,s)} W_1$ is the canonical morphism. Clearly, $\rho \circ f^h = f \circ \rho_1$, where $\rho : X^h_Z \to X$ and $\rho_1 : (X_1)^h_{Z_1} \to X_1$ are the canonical morphisms.

The following properties of the morphism f^h are straightforward:

(1) For any affine k-smooth variety X one has $\mathrm{id}^h_X = \mathrm{id}_{X^h_Z}$. If $p : X \to \mathrm{pt}$ is the structure map, then for any closed Z in X the morphism $p^h : X^h_Z \to (\mathrm{pt})^h_{\mathrm{pt}} = \mathrm{pt}$ is the structure morphism.

(2) Given a k-morphism $f_1 : X_2 \to X_1$ of affine k-smooth varieties and a closed subset $Z_2 \subset X_2$ with $Z_2 \subset f_1^{-1}(Z_1)$ one has $(f \circ f_1)^h = f^h \circ f^h_1$.

(3) If $i : Z \hookrightarrow X$ is the closed inclusion, $Z_1 = Z$, then $Z^h_Z = Z$ and $i^h : Z = Z^h_Z \to X^h_Z$ coincides with the canonical closed inclusion $s : Z \to X^h_Z$.

The last two properties imply the following property. Let X be an affine k-smooth variety and $x \in X$ be a k-rational point. Suppose $s : \mathrm{pt} \to X^h_x$ is the closed point of X^h_x and $i_x : \mathrm{pt} \to X$ is the point x. Let $p : X \to \mathrm{pt}$ be the structure map. Then one has equalities

$$(i_x \circ p)^h = i^h_x \circ p^h = s \circ p^h : X^h_x \to X^h_x.$$

These observations imply the following

Lemma B.1. *Let X be an affine k-smooth variety and $x \in X$ be its k-rational point. Let $f_s : \mathbb{A}^1 \times X \to X$ be a morphism such that $f_1 : X \to X$ is the identity, $f_0 : X \to X$ coincides with the morphism $X \xrightarrow{p} \mathrm{pt} \xrightarrow{i_x} X$ and $f_s(\mathbb{A}^1 \times \{x\}) = \{x\}$. Then the morphism $f^h_s : (\mathbb{A}^1 \times X)^h_{\mathbb{A}^1 \times x} \to X^h_x$ defined by (8) has the following properties:*

(1) *$(f^h_s)|_{(1 \times X)^h_{(1,x)}} : X^h_x \to X^h_x$ is the identity;*

(2) *$(f^h_s)|_{(0 \times X)^h_{(0,x)}} : X^h_x \to X^h_x$ coincides with the morphism $X^h_x \xrightarrow{p^h} \mathrm{pt} \xrightarrow{s_x} X^h_x$, where $s_x : \mathrm{pt} \hookrightarrow X^h_x$ is the closed point of X^h_x.*

Proof. The first assertion follows from the equalities

$$\mathrm{id}_{X^h_Z} = \mathrm{id}^h_X = (f_1)^h = (f_s \circ i_1)^h = f^h_s \circ i^h_1 = (f^h_s)|_{(1 \times X)^h_{(1,x)}}.$$

Let $s_x : \mathrm{pt} \hookrightarrow X^h_x$ be the closed point of X^h_x. As mentioned above, $s_x = i^h_x$, where $i_x : \mathrm{pt} \to X$ is the closed point x of X. The equalities

$$s_x \circ p^h = i^h_x \circ p^h = (i_x \circ p)^h = f^h_0 = (f_s \circ i_0)^h = f^h_s \circ i^h_0 = (f^h_s)|_{(0 \times X)^h_{(0,x)}}$$

imply the second assertion. □

If we take $X = \mathbb{A}^m$, a k-rational point $x \in \mathbb{A}^m$ and the morphism $f_s : \mathbb{A}^1 \times \mathbb{A}^m \to \mathbb{A}^m$ mapping (s, y) to $s \cdot (y - x) + x$, then $f_s : \mathbb{A}^1 \times X \to X$ satisfies the hypotheses of Lemma B.1. Thus Lemma B.1 implies the following statement, which is in fact Lemma 4.10.

Corollary B.2. *The morphism $H_s := f^h_s : U'_s \to U'$ has the following properties:*

(a) *$H_1 := (f^h_s)|_{(1 \times X)^h_{(1,x)}} : U' \to U'$ is the identity morphism;*

(b) *$H_0 := (f^h_s)|_{(0 \times X)^h_{(0,x)}} : U' \to U'$ coincides with the composite morphism $U' \xrightarrow{p^h} pt \xrightarrow{s_x} U'$, where $p^h : U' \to pt = \mathrm{Spec}(k)$ is the structure morphism and $s_x : \mathrm{pt} \hookrightarrow X^h_x$ is the closed point of X^h_x.*

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