Precise Limit in Wasserstein Distance for Conditional Empirical Measures of Dirichlet Diffusion Processes*

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Abstract

Let M be a d-dimensional connected compact Riemannian manifold with boundary ∂M, let $V \in C^2(M)$ such that $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure, and let X_t be the diffusion process generated by $L := \Delta + \nabla V$ with $\tau := \inf\{t \geq 0 : X_t \in \partial M\}$. Consider the conditional empirical measure $\mu_t^\nu := \mathbb{E}^\nu\big(\frac{1}{t}\int_0^t \delta_{X_s}\mathrm{d}s\big|t < \tau\big)$ for the diffusion process with initial distribution ν such that $\nu(\partial M) < 1$. Then

$$\lim_{t\to\infty} \big\{t\mathbb{W}_2(\mu_t^\nu, \mu_0)\big\}^2 = \frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2} \sum_{m=1}^{\infty} \frac{\{\nu(\phi_0)\mu(\phi_m) + \mu(\phi_0)\nu(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3},$$

where $\nu(f) := \int_M f\mathrm{d}\nu$ for a measure ν and $f \in L^1(\nu)$, $\mu_0 := \phi_0^2\mu$, $\{\phi_m\}_{m\geq 0}$ is the eigenbasis of $-L$ in $L^2(\mu)$ with the Dirichlet boundary, $\{\lambda_m\}_{m\geq 0}$ are the corresponding Dirichlet eigenvalues, and $\mathbb{W}_2$ is the L^2-Wasserstein distance induced by the Riemannian metric.

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1 Introduction

Let M be a d-dimensional connected compact Riemannian manifold with a smooth boundary ∂M. Let $V \in C^2(M)$ such that $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure on M, where $\mathrm{d}x$ is the Riemannian volume measure. Let X_t be the diffusion process generated by $L := \Delta + \nabla V$ with hitting time

$$\tau := \inf\{t \geq 0 : X_t \in \partial M\}.$$

Here, according to the convention in Riemannian geometry, the vector field ∇V is regarded as a first-order differential operator with $(\nabla V)f := \langle \nabla V, \nabla f \rangle$ for differentiable functions f. Denote by $\mathscr{P}$ the set of all probability measures on M, and let $\mathbb{E}^\nu$ be the expectation taken for the diffusion process with initial distribution $\nu \in \mathscr{P}$. Consider the conditional empirical measure

$$\mu_t^\nu := \mathbb{E}^\nu\bigg(\frac{1}{t}\int_0^t \delta_{X_s}\mathrm{d}s\bigg|t < \tau\bigg), \quad t > 0, \nu \in \mathscr{P}.$$

Since $\tau = 0$ when $X_0 \in \partial M$, to ensure $\mathbb{P}^\nu(\tau > t) > 0$ we only consider

$$\nu \in \mathscr{P}_0 := \big\{\nu \in \mathscr{P} : \ \nu(M^\circ) > 0\big\}, \quad M^\circ := M \setminus \partial M.$$

Let $\{\phi_m\}_{m\geq 0}$ be the eigenbasis in $L^2(\mu)$ of $-L$ with the Dirichlet boundary such that $\phi_0 > 0$ in M°, and let $\{\lambda_m\}_{m\geq 0}$ be the associated eigenvalues listed in the increasing order counting multiplicities; that is, $\{\phi_m\}_{m\geq 0}$ is an orthonormal basis of $L^2(\mu)$ such that

$$L\phi_m = -\lambda_m \phi_m, \quad m \geq 0.$$

Then $\mu_0 := \phi_0^2\mu$ is a probability measure on M. It is easy to see from [5, Theorem 2.1] that for any probability measure ν supported on M°, we have

$$\lim_{t\to\infty} \|\mu_t^\nu - \mu_0\|_{var} = 0,$$

where $\|\cdot\|_{var}$ is the total variational norm.

In this paper, we investigate the convergence of μ_t^ν to μ_0 under the Wasserstein distance $\mathbb{W}_2$:

$$\mathbb{W}_2(\mu_1, \mu_2) := \inf_{\pi \in \mathscr{C}(\mu_1,\mu_2)} \bigg(\int_{M\times M} \rho(x,y)^2 \pi(\mathrm{d}x, \mathrm{d}y)\bigg)^{\frac{1}{2}}, \quad \mu_1, \mu_2 \in \mathscr{P},$$

where $\mathscr{C}(\mu_1, \mu_2)$ is the set of all probability measures on $M \times M$ with marginal distributions μ_1 and μ_2, and $\rho(x, y)$ is the Riemannian distance between x and y, i.e. the length of the shortest curve on M linking x and y.

Recently, the convergence rate under $\mathbb{W}_2$ has been characterized in [18] for the empirical measures of the L-diffusion processes without boundary (i.e. $\partial M = \emptyset$) or with a reflecting boundary. Since in the present setting the diffusion process is killed at time τ, it is reasonable to consider the conditional empirical measure μ_t^ν given $t < \tau$. This is a counterpart to the quasi-ergodicity for the convergence of the conditional distribution $\tilde{\mu}_t$ of X_t given $t < \tau$. Unlike in the case without boundary or with a reflecting boundary where both the distribution and the empirical measure of X_t converge to the unique invariant probability measure, in the present case the conditional distribution $\tilde{\mu}_t$ of X_t given $t < \tau$ converges to $\tilde{\mu}_0 := \frac{\phi_0}{\mu(\phi_0)}\mu$ rather than $\mu_0 := \phi_0^2\mu$, and this convergence is called the quasi-ergodicity in the literature, see for instance [6] and references within.

Let $\nu(f) := \int_M f d\nu$ for $\nu \in \mathscr{P}$ and $f \in L^1(\nu)$. The main result of this paper is the following.

Theorem 1.1. *For any $\nu \in \mathscr{P}_0$,*

$$\lim_{t\to\infty} \left\{t^2 \mathbb{W}_2(\mu_t^\nu, \mu_0)^2\right\} = I := \frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2} \sum_{m=1}^{\infty} \frac{\{\nu(\phi_0)\mu(\phi_m) + \mu(\phi_0)\nu(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3} > 0.$$

If either $d \le 6$ or $d \ge 7$ but $\nu = h\mu$ with $h \in L^{\frac{2d}{d+6}}(\mu)$, then $I < \infty$.

Remark 1.1. (1) Let X_t be the (reflecting) diffusion process generated by L on M where ∂M may be empty. We consider the mean empirical measure $\hat{\mu}_t^\nu := \mathbb{E}(\frac{1}{t}\int_0^t \delta_{X_s} ds)$, where ν is the initial distribution of X_t. Then

$$\lim_{t\to\infty} \left\{t^2 \mathbb{W}_2(\hat{\mu}_t^\nu, \mu_0)^2\right\} = \sum_{m=1}^{\infty} \frac{\{\nu(\phi_m)\}^2}{\lambda_m^3} < \infty, \tag{1.1}$$

where $\{\phi_m\}_{m\ge 1}$ is the eigenbasis of $-L$ in $L^2(\mu)$ with the Neumann boundary condition if ∂M exists, $\{\lambda_m\}_{m\ge1}$ are the corresponding non-trivial (Neumann) eigenvalues, and the limit is zero if and only if $\nu = \mu$. This can be confirmed by the proof of Theorem 1.1 with $\phi_0 = 1, \lambda_0 = 0$ and $\mu(\phi_m) = 0$ for $m \ge 1$. In this case, μ is the unique invariant probability measure of X_t, so that $\hat{\mu}_t^\mu = \mu$ for $t \ge 0$ and hence the limit in (1.1) is zero for $\nu = \mu$. However, in the Dirichlet diffusion case, the conditional distribution of $(X_s)_{0\le s\le t}$ given $t < \tau$ is no longer stationary, so that even starting from the limit distribution μ_0 we **do not have** $\mu_t^{\mu_0} = \mu_0$ for $t > 0$. This leads to a non-zero limit in Theorem 1.1 even for $\nu = \mu_0$.

(2) It is also interesting to investigate the convergence of $\mathbb{E}^\nu(\mathbb{W}_2(\mu_t, \mu_0)^2 | t < \tau)$ for $\mu_t := \frac{1}{t}\int_0^t \delta_{X_s} ds$, which is the counterpart to the study of [18] where the case without boundary or with a reflecting boundary is considered. According to [18], the convergence rate of $\mathbb{E}^\nu(\mathbb{W}_2(\mu_t, \mu_0)^2 | t < \tau)$ will be at most t^{-1}, which is slower than the rate t^{-2} for $\mathbb{W}_2(\mu_t^\nu, \mu_0)^2$ as shown in Theorem 1.1, see [15] for details, see also [16, 17] for extensions to diffusion processes on non-compact manifolds and SPDEs.

(3) Let $\nu = h\mu$. It is easy to see that $I < \infty$ is equivalent to $h \in \mathscr{D}((-L)^{-\frac{3}{2}})$. By the Sobolev inequality, for any $p \in [1, \frac{d}{3})$, there exists a constant $K > 0$ such that

$$\|(-L)^{-\frac{3}{2}} f\|_{L^{\frac{dp}{d-3p}}(\mu)} \le K\|f\|_{L^p(\mu)}, \quad f \in L^p(\mu). \tag{1.2}$$

Taking $p = \frac{2d}{d+6}$ which is large than 1 when $d \ge 7$, we see that $h \in L^p(\mu)$ implies $h \in \mathscr{D}((-L)^{-\frac{3}{2}})$ and hence $I < \infty$. So, the sharpness of the Sobolev inequality implies that of the condition $h \in L^{\frac{2d}{d+6}}(\mu)$.

In Section 2, we first recall some well known facts on the Dirichlet semigroup, then present an upper bound estimate on $\|\nabla(\phi_m\phi_0^{-1})\|_\infty$. The latter is non-trivial when ∂M is non-convex, and should be interesting by itself. With these preparations, we prove upper and lower bound estimates in Sections 3 and 4 respectively.

2 Some preparations

We first recall some well known facts on the Dirichlet semigroup, see for instances [4, 7, 8, 13]. Let $\{\phi_m\}_{m\geq 0}$ be the eigenbasis of the Dirichlet operator L in $L^2(\mu)$, with Dirichlet eigenvalues $\{\lambda_m\}_{m\geq 0}$ of $-L$ listed in the increasing order counting multiplicities; that is, $\{\phi_m\}_{m\geq 0}$ is an orthonormal basis of $L^2(\mu)$ such that

$$L\phi_m = -\lambda_m\phi_m, \quad m\geq 0.$$

For simplicity, we denote $a\preceq b$ for two positive functions a and b if $a\leq cb$ holds for some constant $c>0$. Then $\lambda_0>0$ and

$$\|\phi_m\|_\infty \preceq \sqrt{m}, \quad m^{\frac{2}{d}}\preceq \lambda_m-\lambda_0\preceq m^{\frac{2}{d}}, \quad m\geq 1. \tag{2.1}$$

Let ρ_∂ be the Riemannian distance function to the boundary ∂M. Then $\phi_0^{-1}\rho_\partial$ is bounded such that

$$\|\phi_0^{-1}\|_{L^p(\mu_0)}<\infty, \quad p\in[1,3). \tag{2.2}$$

The Dirichlet heat kernel has the representation

$$p_t^D(x,y)=\sum_{m=0}^\infty \mathrm{e}^{-\lambda_m t}\phi_m(x)\phi_m(y), \quad t>0, x,y\in M.$$

Let $\mathbb{E}^x$ denote the expectation for the L-diffusion process starting at point x. Then Dirichlet diffusion semigroup generated by L is given by

$$\begin{aligned} P_t^D f(x) &:= \mathbb{E}^x[f(X_t)1_{\{t<\tau\}}] = \int_M p_t^D(x,y)f(y)\mu(\mathrm{d}y)\\ &= \sum_{m=0}^\infty \mathrm{e}^{-\lambda_m t}\mu(\phi_m f)\phi_m(x), \quad t>0, f\in L^2(\mu). \end{aligned} \tag{2.3}$$

We have

$$\|P_t^D\|_{L^p(\mu)\to L^q(\mu)} := \sup_{\mu(|f|^p)\leq 1}\|P_t^D f\|_{L^q(\mu)} \preceq \mathrm{e}^{-\lambda_0 t}(1\wedge t)^{-\frac{d(q-p)}{2pq}}, \quad t>0, q\geq p\geq 1. \tag{2.4}$$

Next, let $L_0=L+2\nabla\log\phi_0$. Then L_0 is a self-adjoint operator in $L^2(\mu_0)$ with semigroup $P_t^0:=\mathrm{e}^{tL_0}$ satisfying

$$P_t^0 f = \mathrm{e}^{\lambda_0 t}\phi_0^{-1}P_t^D(f\phi_0), \quad f\in L^2(\mu_0), \quad t\geq 0. \tag{2.5}$$

So, $\{\phi_0^{-1}\phi_m\}_{m\geq 0}$ is an eigenbasis of L_0 in $L^2(\mu_0)$ with

$$L_0(\phi_m\phi_0^{-1}) = -(\lambda_m - \lambda_0)\phi_m\phi_0^{-1}, \quad P_t^0(\phi_m\phi_0^{-1}) = \mathrm{e}^{-(\lambda_m-\lambda_0)t}\phi_m\phi_0^{-1}, \quad m \geq 0, t \geq 0. \tag{2.6}$$

Consequently,

$$P_t^0 f = \sum_{m=0}^{\infty} \mu_0(f\phi_m\phi_0^{-1})\mathrm{e}^{-(\lambda_m-\lambda_0)t}\phi_m\phi_0^{-1}, \quad f \in L^2(\mu_0), \tag{2.7}$$

and the heat kernel of P_t^0 with respect to μ_0 is given by

$$p_t^0(x,y) = \sum_{m=0}^{\infty}(\phi_m\phi_0^{-1})(x)(\phi_m\phi_0^{-1})(y)\mathrm{e}^{-(\lambda_m-\lambda_0)t}, \quad x, y \in M, t > 0. \tag{2.8}$$

By the intrinsic ultracontractivity, see for instance [9], we have

$$\|P_t^0 - \mu_0\|_{L^1(\mu_0)\to L^\infty(\mu_0)} := \sup_{\mu_0(|f|)\leq 1} \|P_t^0 f - \mu_0(f)\|_\infty \preceq \frac{\mathrm{e}^{-(\lambda_1-\lambda_0)t}}{(1\wedge t)^{\frac{d+2}{2}}}, \quad t > 0. \tag{2.9}$$

Combining this with the semigroup property and the contraction of P_t^0 in $L^p(\mu)$ for any $p \geq 1$, we obtain

$$\|P_t^0 - \mu_0\|_{L^p(\mu_0)} := \sup_{\mu_0(|f|^p)\leq 1} \|P_t^0 f - \mu_0(f)\|_{L^p(\mu_0)} \preceq \mathrm{e}^{-(\lambda_1-\lambda_0)t}, \quad t \geq 0, p \geq 1. \tag{2.10}$$

By the interpolation theorem, (2.9) and (2.10) yield

$$\|P_t^0 - \mu_0\|_{L^p(\mu_0)\to L^q(\mu_0)} \preceq \mathrm{e}^{-(\lambda_1-\lambda_0)t}\{1\wedge t\}^{-\frac{(d+2)(q-p)}{2pq}}, \quad t > 0, \infty \geq q > p \geq 1. \tag{2.11}$$

Since $\mu_0(\phi_m^2\phi_0^{-2}) = 1$, (2.11) for $p = 2$ implies

$$\|\phi_m\phi_0^{-1}\|_\infty = \mathrm{e}^{(\lambda_m-\lambda_0)t}\|P_t^0(\phi_m\phi_0^{-1})\|_\infty \preceq \frac{\mathrm{e}^{(\lambda_m-\lambda_0)t}}{(1\wedge t)^{\frac{d+2}{4}}}, \quad t > 0.$$

Taking $t = (\lambda_m - \lambda_0)^{-1}$ and applying (2.1), we derive

$$\|\phi_m\phi_0^{-1}\|_\infty \preceq m^{\frac{d+2}{2d}}, \quad m \geq 1. \tag{2.12}$$

In the remainder of this section, we investigate gradient estimates on P_t^0 and $\phi_m\phi_0^{-1}$, which will be used in Section 4 for the study of the lower bound estimate on $\mathbb{W}_2(\mu_t^\nu, \mu_0)$. To this end, we need to estimate the Hessian tensor of $\log\phi_0$.

Let N be the inward unit normal vector field of ∂M. We call M (or ∂M) convex if

$$\langle\nabla_u N, u\rangle = \mathrm{Hess}_{\rho_\partial}(u,u) \leq 0, \quad u \in T\partial M, \tag{2.13}$$

where ρ_∂ is the distance function to the boundary ∂M, and $T\partial M$ is the tangent bundle of the $(d-2)$-dimensional manifold ∂M. When $d = 1$, the boundary ∂M degenerates to a set of two end points, such that $\partial M = \emptyset$ and the condition (2.13) trivially holds; that is, M is convex for $d = 1$. Recall that $M^\circ := M \setminus \partial M$ is the interior of M.

Lemma 2.1. *If ∂M is convex, then there exists a constant $K_0 \geq 0$ such that*

$$\mathrm{Hess}_{\log \phi_0}(u,u) \leq K_0|u|^2, \quad u \in TM^\circ.$$

Proof. Since M is compact with smooth boundary, there exists a constant $r_0 > 0$ such that ρ_∂ is smooth on the set

$$\partial_0 M := \{x \in M : \rho_\partial(x) \leq r_0\}.$$

Since ϕ_0 is smooth and satisfies $\phi_0 \geq c\rho_\partial$ for some constant $c > 0$, we have $\log(\phi_0\rho_\partial^{-1}) \in C_b^2(\partial_0 M)$. So, it suffices to find a constant $c > 0$ such that

$$\mathrm{Hess}_{\log \rho_\partial}(u,u) \leq c|u|^2, \quad u \in TM^\circ. \tag{2.14}$$

To this end, we fisrt estimate $\mathrm{Hess}_{\rho_\partial}$ on the boundary ∂M. For any $x \in \partial M$ and $u \in T_xM$, consider the orthogonal decomposition $u = u_1 + u_2$, where

$$u_1 = \langle N, u\rangle N, \quad u_2 := u - u_1 \in T\partial M.$$

Since $|\nabla \rho_\partial| = 1$ on $\partial_0 M$, we have

$$\mathrm{Hess}_{\rho_\partial}(X,N) = \mathrm{Hess}_{\rho_\partial}(X,\nabla\rho_\partial) = \frac{1}{2}\langle X, \nabla|\nabla\rho_\partial|^2\rangle = 0, \quad X \in T_xM. \tag{2.15}$$

On the other hand, since $u_2 \in T\partial M$ and $\nabla\rho_\partial = N$ on ∂M, (2.13) implies

$$\mathrm{Hess}_{\rho_\partial}(u_2,u_2) = \langle \nabla_{u_2}N, u_2\rangle \leq 0.$$

Combining this with (2.15) we obtain

$$\mathrm{Hess}_{\rho_\partial}(u,u) = \langle N,u\rangle^2\mathrm{Hess}_{\rho_\partial}(N,N) + 2\langle N,u\rangle\mathrm{Hess}_{\rho_\partial}(u_2,N) + \mathrm{Hess}_{\rho_\partial}(u_2,u_2) \leq 0$$

for $u \in \cup_{x\in\partial M}T_xM$. Since $\mathrm{Hess}_{\rho_\partial}$ is smooth on the compact set $\partial_0 M$, this implies

$$\mathrm{Hess}_{\rho_\partial}(u,u) \leq c|u|^2\rho_\partial(x), \quad x \in M, u \in T_xM$$

for some constant $c > 0$. Then the desired estimate (2.14) follows from

$$\mathrm{Hess}_{\log\rho_\partial}(u,u) = \rho_\partial^{-1}\mathrm{Hess}_{\rho_\partial}(u,u) - \rho_\partial^{-2}\langle\nabla\rho_\partial, u\rangle^2 \leq c|u|^2, \quad u \in TM^\circ.$$

□

By Lemma 2.1, when ∂M is convex, there exists a constant $K \geq 0$ such that

$$\mathrm{Ric} - \mathrm{Hess}_{V+2\log\phi_0} \geq -K. \tag{2.16}$$

Since the diffusion process generated by $L_0 := \Delta + \nabla(V + 2\log\phi_0)$ is non-explosive in M°, by (2.16) and Bakry-Emery's semigroup calculus, (see for instance [3] or [13, Theorem 2.3.3]), we have

$$|\nabla P_t^0 g| \leq \mathrm{e}^{Kt}P_t^0|\nabla g|, \quad t \geq 0, g \in C_b^1(M) \tag{2.17}$$

and for any $p > 1$, there exists a constant $c(p) > 0$ such that

$$
\begin{aligned}
|\nabla P_t^0 g|^2 &\le \frac{2K\{(P_t^0|g|^{p\wedge 2})(P_t^0|g|)^{(2-p)^+} - (P_t^0|g|)^2\}}{(p\wedge 2)(p\wedge 2-1)(1-\mathrm{e}^{-2Kt})} \\
&\le \frac{c(p)}{1\wedge t}(P_t^0|g|^p)^{\frac{2}{p}}, \quad t>0, g\in \mathscr{B}_b(M).
\end{aligned}
\tag{2.18}
$$

When ∂M is non-convex, we take as in [12] a conformal change of metric to make it convex under the new metric. More precisely, we have the following result.

Lemma 2.2. *There exists a function $1 \le \phi \in C_b^\infty(M)$ such that ∂M is convex under the metric $\langle\cdot,\cdot\rangle_\phi := \phi^{-2}\langle\cdot,\cdot\rangle$. Moreover, there exists a smooth vector field Z_ϕ on M such that*

$$
L_0 = \phi^{-2}\Delta^\phi + Z_\phi + 2\phi^{-1}\nabla^\phi \log\phi_0, \tag{2.19}
$$

where ∇^ϕ and Δ^ϕ are the gradient and Lapalce-Beltrami operators induced by $\langle\cdot,\cdot\rangle_\phi$ respectively.

Proof. let $\delta > 0$ such that the second fundamental form of ∂M is bounded below by $-\delta$. Take $1 \le \phi \in C_b^\infty(M)$ such that $\phi = 1 + \delta\rho_\partial$ in a neighborhood of ∂M in which the distance function ρ_∂ to ∂M is smooth. By [14, Lemma 2.1](see also [12]), ∂M is convex under the metric $\langle\cdot,\cdot\rangle_\phi := \phi^{-2}\langle\cdot,\cdot\rangle$. Next, according to the proof of [14, Lemma 2.2], there exists a smooth vector field Z_ϕ on M such that (2.19) holds. □

Let $1 \le \phi \in C_b^\infty(M)$ be as in Lemma 2.2, and let P_t^ϕ be the diffusion semigroup generated by

$$
L^\phi := \phi L_0 = \phi^{-1}\Delta^\phi + \phi Z_\phi + 2\nabla^\phi \log\phi_0.
$$

We have the following result.

Lemma 2.3. *Let $1 \le \phi \in C_b^\infty(M)$ be as in Lemma 2.2.*

(1) *For any $q \in (1,\infty]$, there exists a constant $c(q) > 0$ such that*

$$
|\nabla^\phi P_t^\phi f|_\phi \le \frac{c(q)}{\sqrt{t}}(P_t^\phi |f|^q)^{\frac{1}{q}}, \quad t>0, f\in C_b^1(M). \tag{2.20}
$$

Moreover, there exists a constant $K > 0$ such that

$$
|\nabla^\phi P_t^\phi f|_\phi \le \mathrm{e}^{Kt} P_t^\phi |\nabla^\phi f|_\phi, \quad t>0, f\in C_b^1(M). \tag{2.21}
$$

(2) *There exists a constant $c > 0$ such that*

$$
\|P_t^\phi\|_{L^p(\mu_0)\to L^\infty(\mu_0)} \le c(1\wedge t)^{-\frac{d+2}{2p}}, \quad t>0, p\in[1,\infty]. \tag{2.22}
$$

Proof. (1) Since ∂M is convex under the metric $\langle\cdot,\cdot\rangle_\phi$, by Lemma 2.1, we find a constant $K_0^\phi > 0$ such that

$$2\mathrm{Hess}^\phi_{\log\phi_0}(u,u) \le K_0^\phi |u|_\phi^2, \quad u \in TM^\circ, \tag{2.23}$$

where Hess^ϕ is the Hessian tensor induced by the metric $\langle\cdot,\cdot\rangle_\phi$. Since the operator $A^\phi := \phi^{-1}\Delta^\phi + \phi Z_\phi$ is a C^2-smooth strictly elliptic second order differential operator on the compact manifold M, it has bounded below Bakry-Emery curvature; that is, there exists a constant $K_1^\phi > 0$ such that

$$A^\phi|\nabla^\phi f|_\phi^2 - 2\langle \nabla^\phi A^\phi f, \nabla^\phi f\rangle_\phi \ge -K_1^\phi |\nabla^\phi f|_\phi^2, \quad f \in C^\infty(M).$$

Combining this with (2.23) we obtain

$$L^\phi|\nabla^\phi f|_\phi^2 - 2\langle \nabla^\phi L^\phi f, \nabla^\phi f\rangle_\phi \ge -(K_0^\phi + K_1^\phi)|\nabla^\phi f|_\phi^2 =: -K^\phi |\nabla^\phi f|_\phi^2, \quad f \in C^\infty(M^\circ),$$

which means that the Bakry-Emery curvature of L^ϕ is bounded below by $-K^\phi$. By the same reason leading to (2.17) and (2.18), this implies (2.20) and (2.21).

(2) To estimate $\|P_t^\phi\|_{L^p(\mu_0)\to L^\infty(\mu_0)}$, we make use of [10, Theorem 4.5(b)] or [11, Theorem 3.3.15(2)], which says that (2.9) implies the super Poincaré inequality

$$\mu_0(f^2) \le r\mu_0(|\nabla f|^2) + \beta(1 + r^{-\frac{d+2}{2}})\mu_0(|f|)^2, \quad r > 0, f \in C_b^1(M)$$

for some constant $\beta > 0$. Let $\mu^\phi = \frac{\phi^{-1}\mu_0}{\mu_0(\phi^{-1})}$. By $L^\phi = \phi L_0$ we obtain

$$\mathscr{E}^\phi(f,g) := -\int_M fL^\phi g \mathrm{d}\mu^\phi = -\frac{1}{\mu_0(\phi^{-1})}\int_M fL_0 g\mathrm{d}\mu_0 = \frac{1}{\mu(\phi^{-1})}\mu_0(\langle\nabla f, \nabla g\rangle), \;\; f, g \in C_b^2(M).$$

Then the above super Poincaré inequality implies

$$\mu^\phi(f^2) \le r\mathscr{E}^\phi(f,f) + \beta'(1 + r^{-\frac{d+2}{2}})\mu^\phi(|f|)^2, \quad f \in C_b^1(M)$$

for some constant $\beta' > 0$. Using [10, Theorem 4.5(b)] or [11, Theorem 3.3.15(2)] again, this implies

$$\|P_t^\phi\|_{L^p(\mu^\phi)\to L^\infty(\mu^\phi)} \le \kappa(1\wedge t)^{-\frac{d+2}{2p}}, \quad t > 0, p \in [1,\infty]$$

for some constant $\kappa > 0$. Noting that

$$\|\phi\|_\infty^{-1}\mu_0 \le \mu^\phi \le \|\phi\|_\infty \mu_0,$$

we find a constant $c > 0$ such that (2.22) holds. $\square$

Lemma 2.4. *For any $p \in (1,\infty]$ and $q \in (1,p)$, there exists a constant $c > 0$ such that for any $f \in D(L_0)$,*

$$\|\nabla P_t^0 f\|_\infty \le c\mathrm{e}^{-\lambda_0 t}\big\{(1\wedge t)^{-\frac{1}{2}-\frac{d+2}{2p}}\|f\|_{L^p(\mu_0)} + (1\wedge t)^{\frac{1}{2}-\frac{q(d+2)}{2p}}\|L_0 f\|_{L^p(\mu_0)}\big\}, \quad t > 0. \tag{2.24}$$

Consequently, there exists a constant $c > 0$ such that

$$\|\nabla(\phi_m\phi_0^{-1})\|_\infty \le cm^{\frac{d+4}{2d}}, \quad m \ge 1. \tag{2.25}$$

Proof. (a) By the semigroup property and the $L^p(\mu_0)$ contraction of P_t^0, for the proof of (2.24) it suffices to consider $t\in(0,1]$. Since $1\le\phi\in C_b^\infty(M)$, we have $\mathscr{D}(L_0)=\mathscr{D}(L^\phi)$ and

$$P_t^0 f = P_t^\phi f - \int_0^t P_s^\phi\{(\phi-1)P_{t-s}^0 L_0 f\}\mathrm{d}s,\quad t\ge 0, f\in\mathscr{D}(L_0). \tag{2.26}$$

Next, by (2.20) and (2.22), we obtain

$$\begin{aligned}\|\nabla P_t^\phi f\|_\infty &= \|\nabla P_{t/2}^\phi(P_{t/2}^\phi f)\|_\infty\\ &\preceq t^{-\frac12}\|P_{t/2}^\phi f\|_\infty \preceq t^{-\frac12-\frac{d+2}{2p}}\|f\|_{L^p(\mu_0)},\quad t\in(0,1].\end{aligned} \tag{2.27}$$

Combining this with (2.11) and (2.20) leads to

$$\begin{aligned}&\int_0^t \|\nabla P_s^\phi\{(\phi-1)P_{t-s}^0 L_0 f\}\|_\infty\mathrm{d}s \preceq \int_0^t s^{-\frac12}\big\|\{P_s^\phi|P_{t-s}^0L_0f|^q\}^{\frac1q}\big\|_\infty\mathrm{d}s\\
&\preceq \int_0^{\frac t2} s^{-\frac12}\|P_{t-s}^0L_0f\|_\infty\mathrm{d}s + \int_{\frac t2}^t s^{-\frac12}\big\|\{P_s^\phi|P_{t-s}^0L_0f|^q\}^{\frac1q}\big\|_\infty\mathrm{d}s\\
&\preceq \int_0^{\frac t2} s^{-\frac12}\|P_{t-s}^0\|_{L^p(\mu_0)\to L^\infty(\mu_0)}\|L_0f\|_{L^p(\mu_0)}\mathrm{d}s + \int_{\frac t2}^t s^{-\frac12}\|P_s^\phi\|_{L^{\frac pq}(\mu_0)\to L^\infty(\mu_0)}\|L_0f\|_{L^p(\mu_0)}\\
&\preceq t^{\frac12-\frac{q(d+2)}{2p}}\|L_0f\|_{L^p(\mu_0)}.\end{aligned}$$

Substituting this and (2.27) into (2.26), we prove (2.24).

(b) Applying (2.24) to $p=\infty$, $f=\phi_m\phi_0^{-1}, t=(\lambda_m-\lambda_0)^{-1}$ and using (2.6), we obtain

$$\mathrm{e}^{-1}\|\nabla(\phi_m\phi_0^{-1})\|_\infty \preceq (\lambda_m-\lambda_0)^{\frac12}\|\phi_m\phi_0^{-1}\|_\infty,\quad m\ge 1.$$

This together with (2.1) and (2.12) implies (2.25) for some constant $c>0$. □

3 Upper bound estimate

According to [18, Lemma 2.3], we have

$$\mathbb{W}_2(\mu_t^\nu,\mu_0)^2 \le \int_M \frac{|\nabla L_0^{-1}(h_t^\nu-1)|^2}{\mathscr{M}(h_t^\nu,1)}\mathrm{d}\mu_0 a, \tag{3.1}$$

where

$$h_t^\nu := \frac{\mathrm{d}\mu_t^\nu}{\mathrm{d}\mu_0},\quad \mathscr{M}(a,b) := 1_{\{a\wedge b>0\}}\frac{a-b}{\log a-\log b}.$$

So, to investigate the upper bound estimate, we first calculate h_t^ν.

By (2.8), we have

$$\psi_s^\nu := \int_M \phi_0(x)p_s^0(x,\cdot)\nu(\mathrm{d}x) = \nu(\phi_0) + \sum_{m=1}^\infty \nu(\phi_m)\mathrm{e}^{-(\lambda_m-\lambda_0)s}\phi_m\phi_0^{-1},\quad s>0. \tag{3.2}$$

Next, (2.5) and (2.8) imply

$$\nu(P_s^D f) = \mathrm{e}^{-\lambda_0 s}\nu(\phi_0 P_s^0(f\phi_0^{-1})) = \mathrm{e}^{-\lambda_0 s}\int_M \psi_s^\nu \phi_0^{-1} f \mathrm{d}\mu_0, \quad f \in \mathscr{B}^+(M), \tag{3.3}$$

where $\mathscr{B}^+(M)$ is the class of nonnegative measurable functions on M. Moreover, for any $t \geq s > 0$, by the Markov property, (2.3), (2.5) and (3.3), we obtain

$$\begin{aligned}&\int_M f \mathrm{d}\mathbb{E}^\nu[\delta_{X_s} 1_{\{t<\tau\}}] = \mathbb{E}^\nu\big[f(X_s)1_{\{s<\tau\}}(P_{t-s}^D 1)(X_s)\big] = \nu\big(P_s^D\{fP_{t-s}^D 1\}\big)\\ &= \mathrm{e}^{-\lambda_0 t}\int_M (\psi_s^v P_{t-s}^0 \phi_0^{-1}) f \mathrm{d}\mu_0, \quad f \in \mathscr{B}^+(M).\end{aligned}$$

Then

$$\frac{\mathrm{d}\mathbb{E}^\nu[\delta_{X_s}1_{\{t<\tau\}}]}{\mathrm{d}\mu_0} = \mathrm{e}^{-\lambda_0 t}\psi_s^v P_{t-s}^0\phi_0^{-1}.$$

Noting that (3.3) implies

$$\mathbb{E}^\nu[1_{\{t<\tau\}}] = \nu(P_t^D 1) = \mathrm{e}^{-\lambda_0 t}\mu_0(\psi_t^\nu \phi_0^{-1}) = \mathrm{e}^{-\lambda_0 t}\nu(\phi_0 P_t^0 \phi_0^{-1}),$$

we arrive at

$$\begin{aligned}&h_t^\nu := \frac{\mathrm{d}\mu_t^\nu}{\mathrm{d}\mu_0} = \frac{1}{t\mathbb{E}^\nu 1_{\{t<\tau\}}}\int_0^t \frac{\mathrm{d}\mathbb{E}^\nu[\delta_{X_s}1_{\{t<\tau\}}]}{\mathrm{d}\mu_0}\mathrm{d}s = 1 + \rho_t^\nu,\\ &\rho_t^\nu := \frac{1}{t\nu(\phi_0 P_t^0\phi_0^{-1})}\int_0^t \big\{\psi_s^\nu P_{t-s}^0\phi_0^{-1} - \nu(\phi_0 P_t^0 \phi_0^{-1})\big\}\mathrm{d}s.\end{aligned} \tag{3.4}$$

By (2.11), $\|\phi_0\|_\infty < \infty$ and $\|\phi_0^{-1}\|_{L^2(\mu_0)} = 1$, we find a constant $c > 0$ such that

$$\begin{aligned}&|\nu(\phi_0 P_t^0 \phi_0^{-1}) - \nu(\phi_0)\mu(\phi_0)| \leq \nu(\phi_0)\|P_t^0\phi_0^{-1} - \mu_0(\phi_0^{-1})\|_\infty\\ &\leq c\mathrm{e}^{-(\lambda_1-\lambda_0)t}, \quad t \geq 1, \nu \in \mathscr{P}_0.\end{aligned} \tag{3.5}$$

Due to the lack of simple representation of the product $\psi_s^\nu P_{t-s}^0\phi_0^{-1}$ in terms of the eigenbasis $\{\phi_m\phi_0^{-1}\}_{m\geq 0}$, it is inconvenient to estimate the upper bound in (3.1). To this end, below we reduce this product to a linear combination of ψ_s^ν and $P_{t-s}^0\phi_0^{-1}$, for which the spectral representation works. Write

$$\begin{aligned}&\psi_s^\nu P_{t-s}^0\phi_0^{-1} - \nu(\phi_0 P_t^0\phi_0^{-1}) = I_1(s) + I_2(s),\\ &I_1(s) := \{\psi_s^\nu - \nu(\phi_0)\}\cdot\{P_{t-s}^0\phi_0^{-1} - \mu(\phi_0)\} + \nu(\phi_0\{\mu(\phi_0) - P_t^0\phi_0^{-1}\}),\\ &I_2(s) := \mu(\phi_0)\{\psi_s^\nu - \nu(\phi_0)\} + \nu(\phi_0)\{P_{t-s}^0\phi_0^{-1} - \mu(\phi_0)\}.\end{aligned} \tag{3.6}$$

By (2.7), (2.8) and (3.2), we have

$$\begin{aligned}&P_{t-s}^0\phi_0^{-1} - \mu(\phi_0) = \sum_{m=1}^\infty \mu(\phi_m)\mathrm{e}^{-(\lambda_m-\lambda_0)(t-s)}\phi_m\phi_0^{-1},\\ &\psi_s^\nu - \nu(\phi_0) = \sum_{m=1}^\infty \nu(\phi_m)\mathrm{e}^{-(\lambda_m-\lambda_0)s}\phi_m\phi_0^{-1}, \quad t > s > 0.\end{aligned} \tag{3.7}$$

Then

$$\begin{aligned}
\rho_t^\nu &= \tilde\rho_t^\nu + \frac{1}{t\nu(\phi_0 P_t^0\phi_0^{-1})}\int_0^t I_1(s)\mathrm{d}s - A_t,\\
\tilde\rho_t^\nu &:= \frac{1}{t\nu(\phi_0 P_t^0\phi_0^{-1})}\sum_{m=1}^\infty \frac{\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)}{\lambda_m-\lambda_0}\phi_m\phi_0^{-1}\\
A_t &:= \frac{1}{t\nu(\phi_0 P_t^0\phi_0^{-1})}\sum_{m=1}^\infty \frac{\{\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)\}\mathrm{e}^{-(\lambda_m-\lambda_0)t}}{\lambda_m-\lambda_0}\phi_m\phi_0^{-1}.
\end{aligned}\tag{3.8}$$

Since $\rho_t^\nu\in L^1(\mu_0)$, the following lemma implies $\tilde\rho_t^\nu\in L^1(\mu_0)$ for $t>0$.

Lemma 3.1. *For any $t_0>0$, there exists a constant $c>0$ such that*

$$\mu_0(|\rho_t^\nu-\tilde\rho_t^\nu|)\le c\|h\|_{L^2(\mu)}\mathrm{e}^{-(\lambda_1-\lambda_0)t},\quad t\ge t_0, \nu=h\mu\in\mathscr{P}_0.\tag{3.9}$$

Proof. By (2.1) and (2.12), for any $t_0>0$ we have

$$\sum_{m=1}^\infty \|\phi_m\|_\infty \mathrm{e}^{-(\lambda_m-\lambda_0)t}\preceq \mathrm{e}^{-(\lambda_1-\lambda_0)t},\quad t\ge t_0.\tag{3.10}$$

Combining this with (3.8) and (3.5), and noting that $\|h\phi_0^{-1}\|_{L^2(\mu_0)}=\|h\|_{L^2(\mu)}$, it suffices to show that

$$B:=\frac1t\int_0^t \big\|\{\psi_s^\nu-\nu(\phi_0)\}\cdot\{P_{t-s}^0\phi_0^{-1}-\mu(\phi_0)\}\big\|_{L^1(\mu_0)}\mathrm{d}s\preceq\|h\|_{L^2(\mu)}\mathrm{e}^{-(\lambda_1-\lambda_0)t},\quad t\ge t_0.\tag{3.11}$$

Since $\|\phi_0^{-1}\|_{L^2(\mu_0)}=1$ and $\psi_s^\nu=P_0^s(h\phi_0^{-1})$ for $\nu=h\mu$, (2.10) yields that

$$\begin{aligned}
B&\le\frac1t\int_0^t\|P_{t-s}^0\phi_0^{-1}-\mu_0(\phi_0^{-1})\|_{L^2(\mu_0)}\|P_s^0(h\phi_0^{-1})-\mu_0(h\phi_0^{-1})\|_{L^2(\mu_0)}\mathrm{d}s\\
&\le\frac1t\int_0^t\|P_{t-s}^0-\mu_0\|_{L^2(\mu_0)}\|P_s^0-\mu_0\|_{L^2(\mu_0)}\|h\|_{L^2(\mu)}\mathrm{d}s\\
&\preceq\|h\|_{L^2(\mu)}\mathrm{e}^{-(\lambda_1-\lambda_0)t},\quad t\ge t_0.
\end{aligned}$$

□

Lemma 3.2. *For any $\alpha>0$, there exist constants $c_0,t_0>0$ such that*

$$\tilde\rho_t^\nu\ge-\frac{c_0}{\nu(\phi_0)t},\quad t\ge t_0,\quad \nu\in\mathscr{P}_0,\nu\in\mathscr{P}_0.\tag{3.12}$$

Consequently, if $\nu=h\mu$ with $h\in L^2(\mu)$, then $\tilde\mu_t^\nu:=(1+\tilde\rho_t^\nu)\mu_0$ is a probability measure for $t>t_0(1+c_0)$.

Proof. By Lemma 3.1, if $\nu = h\mu$ with $h \in L^2(\mu)$, we have $\tilde\rho_t^\nu \in L^1(\mu_0)$ for $t > 0$, and it is easy to see that $\mu_0(\tilde\rho_t^\nu) = 0$. Since (3.12) implies $1 + \tilde\rho_t^\nu > 0$ for $t > t_0(1 + c_0)$, $\tilde\mu_t^\nu$ is a probability measure. It remains to prove (3.12).

By (3.5) and (3.8), it suffices to find a constant $c_1 > 0$ such that

$$g := \sum_{m=1}^\infty \frac{\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)}{\lambda_m - \lambda_0}\phi_m\phi_0^{-1} \ge -c_1. \tag{3.13}$$

By (2.1) and (2.12), we have

$$\|P_1^0 g\|_\infty \le c_2 := \sum_{m=1}^\infty \frac{2\|\phi_0\|_\infty\|\phi_m\|_\infty\|\phi_m\phi_0^{-1}\|_\infty}{(\lambda_m - \lambda_0)\mathrm{e}^{\lambda_m-\lambda_0}} < \infty. \tag{3.14}$$

Next, by (3.7) and the same formula for $\mu = \nu$, we obtain

$$P_s^0 g = (-L_0)^{-1}\big\{\mu(\phi_0)(\psi_s^\nu - \nu(\phi_0)) + \nu(\phi_0)(\psi_s^\mu - \mu(\phi_0))\big\} = (-L_0)^{-1}g_s, \quad s > 0, \tag{3.15}$$

where by $\phi_0, \psi_s^\nu, \psi_s^\mu \ge 0$,

$$g_s := \mu(\phi_0)(\psi_s^\nu - \nu(\phi_0)) + \nu(\phi_0)(\psi_s^\mu - \mu(\phi_0)) \ge -2\mu(\phi_0)\nu(\phi_0) \ge -2\nu(\phi_0), \quad s > 0.$$

This together with (3.15) yields

$$-L_0P_s^0 g \ge -2\nu(\phi_0), \quad s > 0.$$

Therefore, it follows from (3.14) that

$$g = P_1^0 g - \int_0^1 L_0P_r^0 g\mathrm{d}r \ge -c_2 - 2\nu(\phi_0) \ge -c_2 - 2\|\phi_0\|_\infty.$$

So, (3.13) holds for $c_1 = c_2 + 2\|\phi_0\|_\infty$. □

Lemma 3.3. *There exist constants $c, t_0 > 0$ such that for any $t \ge t_0$, and any $\nu \in \mathscr{P}_0$ with $\nu = h\mu$ such that $h \in L^2(\mu)$, we have $\tilde\mu_t^\nu \in \mathscr{P}_0$ and*

$$t^2\mathbb{W}_2(\tilde\mu_t^\nu, \mu_0)^2 \le \frac{1 + ct^{-1}}{\{\mu(\phi_0)\nu(\phi_0)\}^2}\sum_{m=1}^\infty \frac{\{\nu(\phi_0)\mu_0(\phi_m) + \mu(\phi_0)\nu(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3}. \tag{3.16}$$

Proof. By Lemma 3.2, there exist constants $c, t_0 > 0$ such that $\tilde\mu_t^\nu \in \mathscr{P}_0$ for $t \ge t_0$, and

$$\mathscr{M}(1 + \tilde\rho_t^\nu, 1) \ge 1 \wedge (1 + \tilde\rho_t^\nu) \ge \frac{1}{1 + ct^{-1}}, \quad t \ge t_0.$$

So, [18, Lemma 2.3] implies

$$\mathbb{W}_2(\tilde\mu_t^\nu, \mu_0)^2 \le \int_M \frac{|\nabla L_0^{-1}\tilde\rho_t^\nu|^2}{\mathscr{M}(1 + \tilde\rho_t^\nu, 1)}\mathrm{d}\mu_0 \le (1 + ct^{-1})\mu_0(|\nabla L_0^{-1}\tilde\rho_t^\nu|^2), \quad t \ge t_0. \tag{3.17}$$

Next, (2.6) and (3.8) yield

$$t^2\mu_0(|\nabla L_0^{-1}\tilde\rho_t^\nu|^2) = \frac{1}{\{\nu(\phi_0P_t^0\phi_0^{-1})\}^2}\sum_{m=1}^\infty \frac{\{\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3}.$$

Combining this with (3.5) and (3.17), we finish the proof. □

We are now ready to prove the following result.

Proposition 3.4. *For any $\nu \in \mathscr{P}_0$,*

$$\limsup_{t\to\infty} \left\{ t^2 \mathbb{W}_2(\mu_t^\nu, \mu_0)^2 \right\} \le I. \tag{3.18}$$

Proof. (1) We first consider $\nu = h\mu$ with $h \in L^2(\mu)$. Let D be the diameter of M. By Lemma 3.1, there exist constants $c_1, t_0 > 0$ such that $\tilde{\mu}_t^\nu$ is probability measure for $t \ge t_0$ and

$$\mathbb{W}_2(\mu_t^\nu, \tilde{\mu}_t^\nu)^2 \le D^2 \|\mu_t^\nu - \tilde{\mu}_t^\nu\|_{var} = D^2 \mu_0(|\rho_t^\nu - \tilde{\rho}_t^\nu|) \le c_1 \|h\|_{L^2(\mu)} \mathrm{e}^{-(\lambda_1-\lambda_0)t}, \quad t \ge t_0. \tag{3.19}$$

Combining this with Lemma 3.3 and the triangle inequality of $\mathbb{W}_2$, we obtain

$$t^2 \mathbb{W}_2(\mu_t^\nu, \mu_0)^2 \le (1+\delta^{-1}) c_1 t^2 \mathrm{e}^{-(\lambda_1-\lambda_0)t} \|h\|_{L^2(\mu)} + (1+\delta)(1+ct^{-1}) I, \quad \delta > 0. \tag{3.20}$$

(2) In general, we may go back to the first situation by shifting a small time $\varepsilon > 0$. More precisely, by the Markov property, (2.3), (2.5) and (3.2), for any $f \in \mathscr{B}_b(M)$ and $t \ge s \ge \varepsilon > 0$, we have

$$\begin{aligned}
&\mathbb{E}^\nu[f(X_s) 1_{\{t<\tau\}}] = \mathbb{E}^\nu\big[1_{\{\varepsilon<\tau\}} \mathbb{E}^{X_\varepsilon}(f(X_{s-\varepsilon}) 1_{\{t-\varepsilon<\tau\}})\big] \\
&= \int_M p_\varepsilon^D(x,y) \mathbb{E}^y[f(X_{s-\varepsilon}) 1_{\{t-\varepsilon<\tau\}}] \nu(\mathrm{d}x)\mu(\mathrm{d}y) \\
&= \mathrm{e}^{-\lambda_0\varepsilon} \int_M (\psi_\varepsilon^\nu \phi_0)(y) \mathbb{E}^y[f(X_{s-\varepsilon}) 1_{\{t-\varepsilon<\tau\}}] \nu(\mathrm{d}x)\mu(\mathrm{d}y).
\end{aligned}$$

With $f = 1$ this implies

$$\mathbb{P}^\nu(t<\tau) = \mathrm{e}^{-\lambda_0\varepsilon} \int_M (\psi_\varepsilon^\nu \phi_0)(y) \mathbb{P}^y(t-\varepsilon<\tau) \mu(\mathrm{d}y)\mu(\mathrm{d}y).$$

So, letting

$$\nu_\varepsilon = \frac{\psi_\varepsilon^\nu \phi_0}{\mu(\psi_\varepsilon^\nu \phi_0)} =: h_\varepsilon \mu,$$

we arrive at

$$\mathbb{E}^\nu[f(X_s)|t<\tau] = \frac{\mathbb{E}^\nu[f(X_s) 1_{\{t<\tau\}}]}{\mathbb{P}^\nu(t<\tau)} = \frac{\mathbb{E}^{\nu_\varepsilon}[f(X_{s-\varepsilon}) 1_{\{t-\varepsilon<\tau\}}]}{\mathbb{P}^{\nu_\varepsilon}(t-\varepsilon<\tau)} = \mathbb{E}^{\nu_\varepsilon}[f(X_{s-\varepsilon})|t-\varepsilon<\tau].$$

Therefore,

$$\mu_{t,\varepsilon}^\nu := \frac{1}{t-\varepsilon} \int_\varepsilon^t \mathbb{E}^\nu(\delta_{X_s}|t<\tau) \mathrm{d}s = \mu_{t-\varepsilon}^{\nu_\varepsilon}, \quad t > \varepsilon. \tag{3.21}$$

Since

$$\mu(\psi_\varepsilon^\nu \phi_0) = \int_M p_\varepsilon^0(x,y) \phi_0(x)\phi_0(y) \nu(\mathrm{d}x)\mu(\mathrm{d}y) = \nu(\phi_0 P_\varepsilon^0 \phi_0^{-1}) \ge \nu(\phi_0) \|\phi_0\|_\infty^{-1} =: \alpha > 0,$$

by (2.9) we find a constant $c_2 > 0$ such that

$$\|h_\varepsilon \phi_0^{-1}\|_{L^2(\mu_0)} \le \alpha^{-1}\|\psi_\varepsilon^\nu\|_{L^2(\mu_0)} \le \alpha^{-1}\|\phi_0\|_\infty \|p_\varepsilon^0\|_{L^\infty(\mu_0)} \le c_2 \varepsilon^{-\frac{d+2}{2}}, \quad \varepsilon \in (0,1). \tag{3.22}$$

Then (3.20) and (3.21) yield

$$\begin{aligned} &t^2 \mathbb{W}_2(\mu_{t,\varepsilon}^\nu, \mu_0)^2 \\ &\le (1+\delta^{-1}) c_1 c_2 \alpha^{-1} t^2 \mathrm{e}^{-(\lambda_1-\lambda_0)t} \varepsilon^{-\frac{d+2}{2}} + (1+\delta)(1+ct^{-1}) I_\varepsilon, \quad \delta > 0, \varepsilon \in (0,1), \end{aligned} \tag{3.23}$$

where

$$I_\varepsilon := \frac{1}{\{\mu(\phi_0)\nu_\varepsilon(\phi_0)\}^2} \sum_{m=1}^\infty \frac{\{\nu_\varepsilon(\phi_0)\mu(\phi_m) + \mu(\phi_0)\nu_\varepsilon(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3}.$$

By (2.5), (2.6) and (3.2), we have

$$\begin{aligned} \mu(\psi_\varepsilon^\nu \phi_0) &= \nu(\phi_0 P_\varepsilon^{-1} \phi_0^{-1}) = \mathrm{e}^{\lambda_0 \varepsilon} \nu(P_\varepsilon^D 1), \\ \mu(\psi_\nu \phi_0) &= \nu(\phi_0 P_\varepsilon^0(\phi_m \phi_0^{-1})) = \mathrm{e}^{-(\lambda_m - \lambda_0)\varepsilon} \nu(\phi_m), \end{aligned}$$

so that

$$\nu_\varepsilon(\phi_m) = \frac{\mathrm{e}^{-\lambda_m \varepsilon} \nu(\phi_m)}{\nu(P_\varepsilon^D 1)}, \quad m \ge 0.$$

Thus, $\lim_{\varepsilon \to 0} \nu_\varepsilon(\phi_0) = \nu(\phi_0)$ and there exists a constant $C > 1$ such that

$$C^{-1} \mathrm{e}^{-\lambda_m \varepsilon} |\nu(\phi_m)| \le |\nu_\varepsilon(\phi_m)| \le C|\nu(\phi_m)|, \quad m \ge 1, \varepsilon \in (0,1). \tag{3.24}$$

Therefore, if $I < \infty$, by this and

$$\sum_{m=1}^\infty \mu(\phi_m)^2 \le \mu(1) = 1, \tag{3.25}$$

we may apply the dominated convergence theorem to derive $\lim_{\varepsilon \to 0} I_\varepsilon = I$. On the other hand, if $I = \infty$, which is equivalent to

$$\sum_{m=1}^\infty \frac{\nu(\phi_m)^2}{(\lambda_m - \lambda_0)^3} = \infty,$$

then by (3.24) and the monotone convergence theorem we get

$$\liminf_{\varepsilon \to 0} \sum_{m=1}^\infty \frac{\nu_\varepsilon(\phi_m)^2}{(\lambda_m - \lambda_0)^3} \ge C^{-2} \liminf_{\varepsilon \to 0} \sum_{m=1}^\infty \frac{\mathrm{e}^{-2\lambda_m \varepsilon} \nu(\phi_m)^2}{(\lambda_m - \lambda_0)^3} = \infty,$$

which together with (3.25) and $\nu_\varepsilon(\phi_0) \to \nu(\phi_0)$ implies

$$\liminf_{\varepsilon \to 0} I_\varepsilon = \frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2} \liminf_{\varepsilon \to 0} \sum_{m=1}^\infty \frac{\{\nu_\varepsilon(\phi_0)\mu(\phi_m) + \mu(\phi_0)\nu_\varepsilon(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3}$$

$$\geq \frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2}\liminf_{\varepsilon\to 0}\frac{\frac{1}{2}\{\mu(\phi_0)\nu_\varepsilon(\phi_m)\}^2 - \|\phi_0\|_\infty^2\mu(\phi_m)^2}{(\lambda_m-\lambda_0)^3} = \infty.$$

In conclusion, we have

$$\lim_{\varepsilon\to 0} I_\varepsilon = I. \tag{3.26}$$

This together with (3.23) for $\varepsilon = t^{-2}$ gives

$$\limsup_{t\to\infty}\left\{t^2\mathbb{W}_2(\mu^\nu_{t,t^{-2}},\mu_0)^2\right\} \leq I. \tag{3.27}$$

On the other hand, it is easy to see that

$$\|\mu^\nu_{t,\varepsilon} - \mu^\nu_t\|_{var} \leq \frac{2\varepsilon}{t},\quad 0<\varepsilon<t,$$

so that

$$\mathbb{W}_2(\mu^\nu_t,\mu^\nu_{t,t^{-2}})^2 \leq D^2\|\mu^\nu_{t,t^{-2}} - \mu^\nu_t\|_{var} \leq 2D^2t^{-3},\quad t>1. \tag{3.28}$$

Combining this with (3.27), we prove (3.18). □

4 Lower bound estimate and the finiteness of the limit

We will follow the idea of [1, 18], for which we need to modify $\tilde\mu^\nu_t$ as follows. For any $\beta>0$, consider

$$\tilde\mu^\nu_{t,\beta} := (1+\tilde\rho^\nu_{t,\beta})\mu_0,\quad \tilde\rho^\nu_{t,\beta} := P^0_{t^{-\beta}}\tilde\rho^\nu_t,\quad t>0.$$

According to Lemma 3.2, there exists $t_0>0$ such that

$$\tilde h^\nu_t := 1+\tilde\rho^\nu_t \geq \frac{1}{2},\quad \tilde h^\nu_{t,\beta} := 1+\tilde\rho^\nu_{t,\beta} \geq \frac{1}{2},\quad \beta>0, t\geq t_0. \tag{4.1}$$

Consequently, $\tilde\mu^\nu_{t,\beta}$ and $\tilde\mu^\nu_t$ are probability measures for any $\beta>0, t\geq t_0$.

Lemma 4.1. *For any $\beta>0$, there exists a constant $c>0$ such that $f_{t,\beta} := L_0^{-1}\tilde\rho^\nu_{t,\beta}$ satisfies*

$$\|f_{t,\beta}\|_\infty + \|L_0 f_{t,\beta}\|_\infty + \|\nabla f_{t,\beta}\|_\infty \leq ct^{\frac{5\beta d}{4}-1},\quad t\geq 1.$$

Proof. By (2.6) and (3.8), we have

$$f_{t,\beta} = -\sum_{m=1}^\infty \frac{\{\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)\}\mathrm{e}^{-(\lambda_m-\lambda_0)t^{-\beta}}}{t(\lambda_m-\lambda_0)^2\nu(\phi_0P^0_t\phi_0^{-1})}\big(\phi_m\phi_0^{-1}\big),$$
$$L_0 f_{t,\beta} = \sum_{m=1}^\infty \frac{\{\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)\}\mathrm{e}^{-(\lambda_m-\lambda_0)t^{-\beta}}}{t(\lambda_m-\lambda_0)\nu(\phi_0P^0_t\phi_0^{-1})}\big(\phi_m\phi_0^{-1}\big).$$

Combining these with (2.1), (2.12), (3.5), and

$$|\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)|\le \|\phi_0\|_\infty+\|\phi_m\|_\infty\preceq m,\quad m\ge 1,$$

we find a constant $c_1>0$ such that

$$\begin{aligned}t\{\|f_{t,\beta}\|_\infty+\|L_0f_{t,\beta}\|_\infty\}&\preceq\sum_{m=1}^\infty\frac{\mathrm{e}^{-(\lambda_m-\lambda_0)t^{-\beta}}m^{\frac{3d+2}{2d}}}{\lambda_m-\lambda_0}\\ \preceq\sum_{m=1}^\infty \mathrm{e}^{-c_1m^{\frac{2}{d}}t^{-\beta}}m^{\frac{3d-2}{2d}}&\preceq\int_0^\infty \mathrm{e}^{-c_1s^{\frac{2}{d}}t^{-\beta}}s^{\frac{3d-2}{2d}}\mathrm{d}s\preceq t^{\frac{\beta(5d-2)}{4}},\quad t\ge 1.\end{aligned}$$

Similarly, (2.25) implies

$$\begin{aligned}t\|\nabla f_{t,\beta}\|_\infty&\preceq\sum_{m=1}^\infty\frac{\mathrm{e}^{-(\lambda_m-\lambda_0)t^{-\beta}}m^{\frac{3d+4}{2d}}}{(\lambda_m-\lambda_0)^2}\\ &\preceq\sum_{m=1}^\infty \mathrm{e}^{-c_1m^{\frac{2}{d}}t^{-\beta}}m^{\frac{3d-4}{2d}}\preceq t^{\frac{\beta(5d-4)}{4}},\quad t\ge 1.\end{aligned}$$

Then the proof is finished. □

Lemma 4.2. *For any $\beta\in(0,\frac{1}{20d}]$, there exits a constant $c>0$ such that*

$$t^2\mathbb{W}_2(\tilde\mu^\nu_{t,\beta},\mu_0)^2\ge\frac{1-ct^{-1}}{\{\mu(\phi_0)\nu(\phi_0)\}^2}\sum_{m=1}^\infty\frac{\{\mu(h\phi_0)\mu_0(\phi_m)+\mu(\phi_0)\nu(\phi_m)\}^2}{(\lambda_m-\lambda_0)^3}-ct^{-\frac{1}{4}}.$$

Proof. To estimate $\mathbb{W}_2(\tilde\mu^\nu_{t,\beta},\mu_0)$ from below by using the argument in [1, 18], we take

$$\varphi^\varepsilon_\theta:=-\varepsilon\log P^0_{\frac{\varepsilon\theta}{2}}\mathrm{e}^{-\varepsilon^{-1}f_{t,\beta}},\quad \theta\in[0,1],\varepsilon>0.$$

We have $\varphi^\varepsilon_0=f_{t,\beta}$, $\|\varphi^\varepsilon_\theta\|_\infty\le\|f_{t,\beta}\|_\infty$, and by [18, Lemma 2.9], there exists a constant $c_1>0$ such that for any $\varepsilon\in(0,1)$,

$$\begin{aligned}&\varphi^\varepsilon_1(y)-\varphi^\varepsilon_0(x)\le\frac{1}{2}\big\{\rho(x,y)^2+\varepsilon\|(L_0f_{t,\beta})^+\|_\infty+c_1\sqrt{\varepsilon}\|\nabla f_{t,\beta}\|^2_\infty\big\},\quad x,y\in M,\\ &\int_M(\varphi^\varepsilon_0-\varphi^\varepsilon_1)\mathrm{d}\mu_0\le\frac{1}{2}\int_M|\nabla f_{t,\beta}|^2\mathrm{d}\mu_0+c_1\varepsilon^{-1}\|\nabla f_{t,\beta}\|^4_\infty.\end{aligned}$$

Therefore, by the Kantorovich dual formula, $\varphi^\varepsilon_0=f_{t,\beta}$ and the integration by parts formula

$$\int_M f_{t,\beta}\tilde\rho^\nu_{t,\beta}\mathrm{d}\mu_0=\int_M f_{t,\beta}L_0f_{t,\beta}\mathrm{d}\mu_0=-\int_M|\nabla f_{t,\beta}|^2\mathrm{d}\mu_0,$$

we find a constant $c>0$ such that

$$\begin{aligned}&c\big(\varepsilon\|L_0f_{t,\beta}\|_\infty+\varepsilon^{\frac{1}{2}}\|\nabla f_{t,\beta}\|^2_\infty\big)+\frac{1}{2}\mathbb{W}_2(\tilde\mu^\nu_{t,\beta},\mu_0)^2\ge\int_M\varphi^\varepsilon_1\mathrm{d}\mu_0-\int_M\varphi^\varepsilon_0\mathrm{d}\tilde\mu^\nu_{t,\beta}\\ &=\int_M(\varphi^\varepsilon_1-\varphi^\varepsilon_0)\mathrm{d}\mu_0-\int_M f_{t,\beta}\tilde\rho^\nu_{t,\beta}\mathrm{d}\mu_0=\int_M(\varphi^\varepsilon_1-\varphi^\varepsilon_0)\mathrm{d}\mu_0-\int_M f_{t,\beta}L_0f_{t,\beta}\mathrm{d}\mu_0\\ &\ge\frac{1}{2}\int_M|\nabla f_{t,\beta}|^2\mathrm{d}\mu_0-c\varepsilon^{-1}\|\nabla f_{t,\beta}\|^4_\infty.\end{aligned}\tag{4.2}$$

Taking $\varepsilon = t^{-\frac{3}{2}}$ and applying Lemma 4.1, when $\beta \le \frac{1}{20d}$ we find a constant $c' > 0$ such that

$$t^2 \mathbb{W}_2(\tilde{\mu}^\nu_{t,\beta}, \mu_0)^2 \ge t^2 \mu_0(|\nabla f_{t,\beta}|^2) - c' t^{-\frac{1}{4}}, \quad t \ge t_0. \tag{4.3}$$

Combining this with (3.5) and (4.3), we complete the proof. □

Lemma 4.3. *There exist constants $c, t_0 > 0$ such that for any $\nu = h\mu \in \mathscr{P}_0$ with $h \in L^2(\mu)$, $\tilde{\mu}^\nu_{t,\beta}, \tilde{\mu}^\nu_t \in \mathscr{P}_0$ for $t \ge t_0$ and*

$$t \mathbb{W}_2(\tilde{\mu}^\nu_{t,\beta}, \tilde{\mu}^\nu_t) \le c \|h\|_{L^2(\mu)} t^{-\beta}, \quad t \ge t_0.$$

Proof. $\tilde{\mu}^\nu_{t,\beta}, \tilde{\mu}^\nu_t \in \mathscr{P}_0$ for large t is implied by Lemma 3.2. Next, by (4.1), we have

$$\mathscr{M}(\tilde{h}^\nu_t, \tilde{h}^\nu_{t,\beta}) \ge \frac{1}{2},$$

so that [18, Lemma 2.3] implies

$$\mathbb{W}_2(\tilde{\mu}^\nu_{t,\beta}, \tilde{\mu}^\nu_t)^2 \le \int_M \frac{|\nabla L_0^{-1}(\tilde{h}^\nu_t - \tilde{h}^\nu_{t,\beta})|^2}{\mathscr{M}(\tilde{h}^\nu_t, \tilde{h}^\nu_{t,\beta})} \mathrm{d}\mu_0 \le 2\mu_0(|\nabla L_0^{-1}(\tilde{\rho}^\nu_t - \tilde{\rho}^\nu_{t,\beta})|^2). \tag{4.4}$$

To estimate the upper bound in this inequality, we first observe that by (3.7) and (3.8), when $\nu = h\mu$ we have

$$\begin{aligned} L_0^{-1}(\tilde{\rho}^\nu_{t,\beta} - \tilde{\rho}^\nu_t) &= L_0^{-1}(P^0_{t^{-\beta}} \tilde{\rho}^\nu_t - \tilde{\rho}^\nu_t) = \int_0^{t^{-\beta}} P^0_r \tilde{\rho}^\nu_t \mathrm{d}r \\ &= \frac{1}{t\nu(\phi_0 P^0_t \phi_0^{-1})} \int_0^{t^{-\beta}} (-L_0)^{-1}(P^0_r - \mu_0) g \, \mathrm{d}r, \end{aligned} \tag{4.5}$$

where

$$g := \mu(\phi_0) h \phi_0^{-1} + \nu(\phi_0)\phi_0^{-1}.$$

Since $\|h\|_{L^2(\mu)} \ge \mu(h) = 1$,

$$\|g\|_{L^2(\mu_0)} \le \|\phi_0\|_\infty (1 + \|h\|_{L^2(\mu)}) \le 2\|\phi_0\|_\infty \|h\|_{L^2(\mu)}. \tag{4.6}$$

By (2.10), (4.6) and the fact that $(-L_0)^{-\frac{1}{2}} = c\int_0^\infty P^0_{s^2} \mathrm{d}s$ for some constant $c > 0$, we find a constants $c_1, c_2 > 0$ such that

$$\begin{aligned} \big\|\nabla L_0^{-1}(P^0_r - \mu_0) g\|_{L^2(\mu_0)} &= \big\|L_0^{-\frac{1}{2}}(P^0_r - \mu_0) g\|_{L^2(\mu_0)} \le \int_0^\infty \|(P^0_{r+s^2} - \mu_0) g\|_{L^2(\mu_0)} \mathrm{d}s \\ &\le c_1 \|h\|_{L^2(\mu)} \int_1^\infty \mathrm{e}^{-(\lambda_1 - \lambda_0)(s^2 + r)} \mathrm{d}s \le c_2 \|h\|_{L^2(\mu)}, \quad r \in [0,1]. \end{aligned}$$

Therefore, by (3.5) and (4.5), we obtain

$$\|\nabla L_0^{-1}(\tilde{\rho}^\nu_{t,\beta} - \tilde{\rho}^\nu_t)\|_{L^2(\mu_0)} \preceq \frac{1}{t} \int_0^{t^{-\beta}} \|\nabla L_0^{-1}(P^0_r - \mu_0) g\|_{L^2(\mu_0)} \mathrm{d}r \preceq t^{-(1+\beta)} \|h\|_{L^2(\mu)}, \quad t \ge t_0.$$

Combining this with (4.4) we finish the proof. □

We are now ready to prove the following result.

Proposition 4.4. *For any $\nu \in \mathscr{P}_0$,*

$$\liminf_{t\to\infty} \left\{ t^2 \mathbb{W}_2(\mu_t^\nu, \mu_0)^2 \right\} \ge I > 0, \tag{4.7}$$

and $I < \infty$ provided either $d \le 7$, or $d \ge 7$ but $\nu = h\mu$ with $h \in L^{\frac{2d}{d+6}}$.

Proof. Let $\beta \in (0, \frac{1}{20d}]$. By (3.19), Lemma 4.2 and Lemma 4.3, there exist constants $c, t_0 > 0$ such that for $\nu = h\mu \in \mathscr{P}_0$ and $t \ge t_0$,

$$\begin{aligned} &t\mathbb{W}_2(\mu_t^\nu, \tilde\mu_t^\nu) \le c\|h\|_{L^2(\mu)} t^{-\beta t},\\ &t\mathbb{W}_2(\tilde\mu_{t,\beta}^\nu, \mu_0) \ge \left(\{(1-ct^{-1})I - ct^{-\frac14})^+\right\}^{\frac12},\\ &t\mathbb{W}_2(\mu_t^\nu, \tilde\mu_t^\nu) \le ct\mathrm{e}^{-(\lambda_1-\lambda_0)t/2}\|h\|_{L^2(\mu)}^{\frac12}. \end{aligned}$$

Then

$$t\mathbb{W}_2(\mu_t^\nu, \mu_0) \ge \left(\{(1-ct^{-1})I - ct^{-\frac14})^+\right\}^{\frac12} - c\|h\|_{L^2(\mu)} t^{-\beta t} - ct\mathrm{e}^{-(\lambda_1-\lambda_0)t/2}\|h\|_{L^2(\mu)}^{\frac12},\ \ t \ge t_0. \tag{4.8}$$

In general, let $\mu_{t,\varepsilon}^\nu = \mu_{t-\varepsilon}^{\nu_\varepsilon}$ be as in the proof of Proposition 3.4. Applying (4.8) to $\mu_{t,t^{-2}}^\nu$ replacing μ_y^ν and using (3.22), (3.26), we obtain

$$\liminf_{t\to\infty} \left\{ t\mathbb{W}_2(\mu_{t,t^{-2}}^\nu, \mu_0) \right\} \ge \sqrt{I},$$

which together with (3.28) proves (4.7).

It remains to prove $I > 0$ and $I < \infty$ the under given conditions, where due to (3.25), $I < \infty$ is equivalent to

$$I' := \sum_{m=1}^\infty \frac{\nu(\phi_m)^2}{\lambda_m^3} < \infty. \tag{4.9}$$

Below we first prove $I > 0$ then show $I' < \infty$ under the given conditions.

(a) $I > 0$. If this is not true, then

$$\mu(h\phi_0)\mu(\phi_m) = -\mu(\phi_0)\mu(h\phi_m),\ \ m \ge 1.$$

Combining this with the representation in $L^2(\mu)$

$$f = \sum_{m=0}^\infty \mu(f\phi_m)\phi_m,\ \ f \in L^2(\mu),$$

where the equation holds point-wisely if $f \in C_b(M)$ by the continuity, we obtain

$$\mu(\phi_0)\nu(f) = \sum_{m=0}^\infty \mu(f\phi_m)\mu(\phi_0)\nu(\phi_m) = 2\mu(f\phi_0)\nu(\phi_0)\mu(\phi_0) - \sum_{m=0}^\infty \mu(f\phi_m)\mu(\phi_m)\nu(\phi_0)$$

$$= 2\mu(f\phi_0)\nu(\phi_0)\mu(\phi_0) - \nu(\phi_0)\mu(f), \quad f \in C_b(M).$$

Consequently,

$$0 \le \mu(\phi_0)\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 2\phi_0\nu(\phi_0)\mu(\phi_0) - \nu(\phi_0),$$

which is however impossible since the upper bound is negative in a neighborhood of ∂M, because $\nu(M^\circ) > 0$ implies $\nu(\phi_0) > 0$ for $\phi_0 > 0$ in M°, and ϕ_0 is continuous with $\phi_0|_{\partial M} = 0$. Therefore, we must have $I > 0$.

(b) $I' < \infty$. Let $\{h_n\}_{n\ge1}$ be a sequence of probability density functions with respect to μ such that

$$\nu_n := h_n\mu \to \nu \ \text{ weakly as } \ n \to \infty. \tag{4.10}$$

By the spectral representation for $(-L)^{-\frac{3}{2}}$, and applying the Sobolev inequality (1.2) with $p = \frac{2d}{d+6} \vee 1$, we obtain

$$I_n' := \sum_{m=1}^{\infty} \frac{\nu_n(\phi_m)^2}{\lambda_m^3} \le \|(-L)^{-\frac{3}{2}} h_n\|_{L^2(\mu)}^2 \le K^2 \|h_n\|^2_{L^{\frac{2d}{d+6}\vee 1}(\mu)}, \quad n \ge 1. \tag{4.11}$$

It is easy to see that for $d \le 6$ we have $\frac{2d}{d+6} \le 1$, so that $\|h_n\|_{L^{\frac{2d}{d+6}\vee1}(\mu)} = \mu(h_n) = 1$. Combining this with (4.10), (4.11) and applying Fatou's lemma, we derive

$$I' = \sum_{m=1}^{\infty} \liminf_{n\to\infty} \frac{\nu_n(\phi_m)^2}{\lambda_m^3} \le \liminf_{n\to\infty} I_n' \le K^2 < \infty, \quad d \le 6.$$

Finally, when $d \ge 7$ and $\nu = h\mu$ with $h \in L^{\frac{2d}{d+6}}(\mu)$, by applying (4.11) to $h_n = h$ we prove $I' < \infty$. □

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