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D. Wang , M.I. Friswell

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Highlights

- The attachment positions of intermediate elastic supports are optimally designed for raising the natural frequency of a plate structure.
- The minimum restraint stiffness is investigated for a more economic support design with consideration of the additional support mass.
- The natural frequency derivative formulation with regard to the support position variation is first derived with the inclusion of the support mass.
- A typical relation is proposed between the support mass and the stiffness to demonstrate effects of the additional support mass on the minimum restraint stiffness of the support.
- The researches in this work are more practical in engineering applications.

Support Position Optimization with Minimum Stiffness for Plate Structures Including Support Mass

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D. Wang*, M.I. Friswell[$]

*Department of Aeronautical Structural Engineering

Northwestern Polytechnical University

Xi'an, Shaanxi, 710072, P.R. China

[$]College of Engineering, Swansea University

Bay Campus, Fabian Way, Swansea SA1 8EN, UK

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* **Corresponding author:**

Prof. Dong Wang

ORCID: 0000-0002-6905-4651

Department of Aeronautical Structural Engineering

Northwestern Polytechnical University
Xi’an, Shaanxi, 710072, P.R. China
Tel: +86-15029930716
Email: dwang@nwpu.edu.cn

Abstract

The optimum position and minimum restraint stiffness of a flexible point support to raise a natural frequency of a thin bending plate is investigated, with the inclusion of the corresponding additional support mass. First the derivatives of the natural frequencies of the plate structure are derived with respect to the support movement using a finite element model. Second, the minimum support stiffness is analyzed to raise a plate's natural frequency to a target value by solving a characteristic eigenvalue problem. Then the optimal support design is studied to find the optimal attachment point and the associated minimum stiffness. Several typical examples of plate systems are analyzed with addition of the point supports with non-negligible mass. It appears that including the support mass in the plate vibration analysis can significantly increase the minimum support stiffness required to raise a given natural frequency to its target, whereas the optimal support position remains consistent with the massless support design case.

Keywords: Support additional mass; Optimal support position; Minimum support stiffness; Natural frequency increase; Plate structural system

1. Introduction

A flexural plate with intermediate simple or point supports is one of the most commonly used structural elements in civil, aerospace, marine, electronic and mechanical engineering applications. Usually, these supports are used to hold the plate structure statically. Often, they are also employed to improve the structural characteristics and performance by the optimal design of the supports' stiffnesses and attachment points, especially when other structural design modifications cannot be effectively performed in practical problems [1-3]. Thus far, a great number of publications are available in the literature investigating the dynamic properties of plates with various boundary conditions resting on fixed or movable point supports. Usually, an exact solution of the transverse vibration is not available even for a thin (Kirchhoff model) plates with general (elastic or rigid) point supports. Therefore, various numerical approaches, for example based on the finite element method (FEM) or the Rayleigh-Ritz method, have been developed in order to determine the dynamic behaviors, typically the natural frequencies, mode shapes of the plate system and its vibration response to a general excitation [1-7].

It is well known that both the restraint stiffness and the attachment location of an elastic support are very important in engineering applications. Small changes to either the stiffness or position of an intermediate support can dramatically affect the dynamic properties of a beam or plate structure [5, 8-11]. Thus, these parameters are often utilized on purpose to modify the vibration characteristics or the critical buckling load of the structure [1, 4, 8, 9]. Moreover, there exists an exact optimum position for a point support, at which a certain or critical value of the stiffness can essentially raise a natural frequency of interest to a preset target value or to its upper limit [1, 12]. Olhoff and Akesson [8] highlighted that attaining the minimum stiffness of a structure gives a much more efficient design of the support in practice, because both the economic and material costs of a flexible support are directly related to its restraint stiffness. Therefore, estimating the minimum stiffness of the flexible support enables designers and engineers to obtain the minimum weight design of a structural system in practical engineering. In addition, previous studies [1, 12] have shown that the optimal support position to maximize a specified natural frequency may be non-unique once the restraint stiffness of the additional support is beyond a critical or threshold value. Besides, above this minimum stiffness, the target natural frequency cannot be raised further by increasing the support restraint stiffness, but the associated mode shapes of the beam or plate structure are modified, primarily due to mode switching between two consecutive modes [9].

A survey of the early literature reveals that an elastic transverse support is typically modeled as a massless translational spring simply connected at a point with a finite or infinite stiffness [2, 4, 7-9]. Thus, the mass or inertia properties of the spring support are neglected or excluded in the dynamic analyses of the beam or plate structure. The massless support assumption also means that the support stiffness is not fully correlated with its material or economic expenditure, which is not realistic in engineering practice [8]. However, it is well recognized in general that the restraint stiffness of a spring support is closely associated with its material cost or mass. Moreover, from the structural vibration theory [13], it is commonly known that part of the elastic support mass does virtually participate in the transverse vibration of the structure, and therefore affects its dynamic properties, including its natural frequencies. In other words, the additional mass of a point support should be incorporated into the support position optimization to achieve the minimum stiffness required, or to maximize a natural frequency of interest. Such a problem has practical importance in structural designs, but to the authors' knowledge, has not been addressed as yet in the available literature.

The problem under investigation in this paper is to optimize the positions of elastic point supports in order to maximize a natural frequency, particularly the fundamental frequency, of a flexural plate structure. This is because in many cases of engineering applications, the structural dynamic behavior is highly dependent on the first few natural frequencies and the relevant mode shapes. Raising a natural frequency of a structure as far away as possible from the driving frequency of an external load can significantly reduce its vibration response. Damping is not considered in our analysis, even though the response amplitude of a structure near resonance is mainly determined by the modal damping. However, the concept here is to ensure that the natural frequencies and the excitation frequencies are well separated, and in this case the damping has little influence on the response. To obtain more realistic results, both the stiffness and mass of a simple support are considered simultaneously in the plate vibration analysis to obtain the corresponding minimum stiffness mainly due to its practical significance. To achieve this, the frequency sensitivity analysis with respect to an elastic support location is first conducted using the finite element (FE) approach [12]. Since the dynamic analysis most commonly uses FEM, such a derivation of the design sensitivity is fully consistent with the numerical modal analysis of a structure. Second, the minimum stiffness of the interior support required for a certain target natural frequency is estimated at a point of attachment to the plate. For the general bending vibration of a plate structure, the determination of the minimum stiffness of the flexible support can be simply formulated as a generalized eigenvalue problem [5, 14], and therefore the optimum stiffness may be obtained numerically as the lowest positive eigenvalue.

Afterwards, a heuristic optimization procedure, called the evolutionary shift method [12], is implemented to determine the optimal support location as well as the corresponding minimum restraint stiffness for maximization of the structural natural frequency of interest. Initially, the optimization of the support location assumes that the attachment occurs only at the nodes of the FE model, with the contribution of the support mass included. On the basis of the design sensitivity analysis, the support location will be shifted in the specified direction with a step size given by the element size, to gradually reach to the approximate optimal position for the design task. However, the optimal support position is unlikely to occur exactly at an FE node, and will usually occur within an element. To gain a more accurate estimate of the optimal position, without discretizing the local region near the solution with a very fine FE mesh, the stiffness matrix of an elastic point support located within an element is used to efficiently obtain the optimal position and make the design solution insensitive to the FE mesh [5, 15]. Finally, the feasibility and effectiveness of the proposed optimization algorithm is demonstrated by three benchmark examples of rectangular plates. The optimal results are compared to the traditional solutions that neglect the mass of the spring support [5, 12, 14] to demonstrate the effect of the support mass inclusion on the optimal design of the intermediate spring supports.

2. Derivative of Natural Frequency with Respect to Support Position

In structural dynamic analysis, the characteristic equation of an undamped system in the discrete form is [16]

$$([K] - \omega_i^2 [M])\{\phi\}_i = \{0\} \tag{1}$$

where $[K]$ and $[M]$ are the global stiffness and mass matrices, respectively. ω_i denotes the ith natural frequency in radians and $\{\phi\}_i$ is the associated vibration mode of the structure, which has been mass normalized. Notice that ω_i is an implicit function of the support parameters.

As is well known, design sensitivity analysis determines the effect of a design variable modification on the structural response of a vibrating system. It may play a vital role in design optimization algorithms. Generally, the derivative of the ith natural frequency with regard to a support position is given by [12, 15]

$$\frac{\mathrm{d}\omega_i^2}{\mathrm{d}s} = \{\phi\}_i^{\mathrm{T}} \left(\frac{\mathrm{d}[K]}{\mathrm{d}s} - \omega_i^2 \frac{\mathrm{d}[M]}{\mathrm{d}s} \right) \{\phi\}_i \tag{2}$$

where s denotes the position or coordinate of a spring support. Generally, the movement of an elastic support will redistribute the stiffness and inertia properties of the structure. Thus, the elastic support movement will affect both the global stiffness and mass matrices, and ultimately change both the natural frequencies and the mode shapes.

2.1 Modeling an Elastic Support

Previously, the optimization of the stiffness or position of the additional supports has generally modeled a pinned point support as a massless linear spring acting on the translational displacement at the attachment point [1-3, 5, 12, 15]. Therefore, most of the support optimization approaches were based on neglecting the effects of the support mass. Consequently, the mass matrix of the structural system was not affected as a spring support changes its position. However, in practical engineering structures, both the stiffness and mass of an elastic support are closely related [4, 8]. Often, the additional support mass is comparable to the mass of the structure around the attachment point. Its effect becomes an important issue in the support design, and hence it should be taken into consideration in the vibration analysis. Zhou and Ji [4] investigated the coupled free vibration of a plate-support system for gaining the exact solution of the dynamic properties of the plate structure with the support mass included.

For example, Figure 1 shows a schematic diagram of a discrete spring-mass system with non-negligible spring mass m. For an accurate estimation of the vibration response, it is well recognized that part of the spring mass or inertia should be included to appropriately evaluate the natural frequency ω_n of the system [13]

(Figure 1)

$$\omega_n = \sqrt{\frac{k}{M + m/3}} \tag{3}$$

This expression means that the effective spring mass m_s, that participates in the primary system vibration, is one-third of the total mass of the spring support:

$$m_s = \frac{m}{3} \tag{4}$$

In general, the effective mass (extra transverse inertia) of a spring is approximately proportional to the support stiffness. There will also be additional mass required to connect the support spring to the plate structure. However, within this study, to demonstrate the inclusion of the support mass, we assume that

$$m_s = rk \tag{5}$$

where r is a ratio factor between the support mass and the stiffness. Note the factor r has the dimensions (units) kg·m/N = s^2, and is usually a very small positive value in many practical applications. A typical value of $r = 10^{-6}$ (s^2) is employed in this paper. Other functions relating the effective support mass and the stiffness may be easily incorporated into the analysis. Evidently, the location variation of a spring point support will simultaneously affect both the global stiffness and mass matrices of the structural system in Eq. (2).

2.2 The Plate Element with Elastic Support

In Fig. 2, a four-node flexural uniform thin rectangular element based on the classical Kirchhoff hypothesis is illustrated with a grounded elastic support attached at a point within the element. According to the FEM theory [16], the transverse displacement at the support point $w(s_a, s_b)$ along the z-axis can be approximated in terms of the nodal displacements and slopes of the plate element as

(Figure 2)

$$w(s_a, s_b) = [N]_{(s_a, s_b)} \cdot \{u\}_e \tag{6}$$

where $[N]$ is a row vector of the shape (or interpolation) functions of a rectangular plate element, and $\{u\}_e$ is a column vector of element nodal degrees of freedom, given by

$$[N] = [N_1 \quad N_{x1} \quad N_{y1} \quad N_2 \quad N_{x2} \quad N_{y2} \quad N_3 \quad N_{x3} \quad N_{y3} \quad N_4 \quad N_{x4} \quad N_{y4}] \tag{7}$$

$$\{u\}_e = [w_1 \quad \theta_{x1} \quad \theta_{y1} \quad w_2 \quad \theta_{x2} \quad \theta_{y2} \quad w_3 \quad \theta_{x3} \quad \theta_{y3} \quad w_4 \quad \theta_{x4} \quad \theta_{y4}]^{\mathrm{T}} \tag{8}$$

The following standard shape functions are regularly adopted for the transverse components w_i, ϑ_{xi} and ϑ_{yi} of the plate element, respectively, [16]:

$$\begin{cases} N_i = (1+\xi_i\xi)(1+\eta_i\eta)(2+\xi_i\xi+\eta_i\eta-\xi^2-\eta^2)/8 \\ N_{xi} = -b\,\eta_i(1+\xi_i\xi)(1+\eta_i\eta)(1-\eta^2)/8 \\ N_{yi} = a\,\xi_i(1+\xi_i\xi)(1+\eta_i\eta)(1-\xi^2)/8 \end{cases}, \quad i = 1, 2, 3, 4 \tag{9a}$$

where

$$\begin{cases} \xi = \dfrac{x-a}{a} \in [-1,\ 1], \quad \xi_i = \dfrac{x_i}{a} - 1 \\ \eta = \dfrac{y-b}{b} \in [-1,\ 1], \quad \eta_i = \dfrac{y_i}{b} - 1 \end{cases} \tag{9b}$$

and *a* and *b* are half of the element size along the *x*- and *y*-axis, respectively, as illustrated in Fig. 2. Therefore, the total energy (both potential and kinetic), *E*, of the spring support due to its transverse deflection is

$$E = \frac{1}{2}kw^2(s_a, s_b) + \frac{1}{2}\omega^2 m_s w^2(s_a, s_b) \tag{10}$$

Substituting the displacement expression in Eq. (6) into the above energy formulation, the total energy can be expressed in the quadratic form in terms of the associated element nodal displacements as

$$E = \frac{1}{2}\{u\}_e^{\mathrm{T}}[K]_S\{u\}_e + \frac{1}{2}\omega^2\{u\}_e^{\mathrm{T}}[M]_S\{u\}_e \tag{11}$$

where

$$[K]_S = k\begin{bmatrix} N_1^2 & N_1N_{x1} & N_1N_{y1} & \cdots & N_1N_{x4} & N_1N_{y4} \\ & N_{x1}^2 & N_{x1}N_{y1} & \cdots & N_{x1}N_{x4} & N_{x1}N_{y4} \\ & & N_{y1}^2 & \cdots & N_{y1}N_{x4} & N_{y1}N_{y4} \\ & & & \vdots & \vdots & \vdots \\ & \text{Sym.} & & & & N_{y4}^2 \end{bmatrix}_{(s_a, s_b)} = k[S_t]_{(s_a, s_b)} \tag{12a}$$

and

$$[M]_S = m_s \begin{bmatrix} N_1^2 & N_1 N_{x1} & N_1 N_{y1} & \cdots & N_1 N_{x4} & N_1 N_{y4} \\ & N_{x1}^2 & N_{x1} N_{y1} & \cdots & N_{x1} N_{x4} & N_{x1} N_{y4} \\ & & N_{y1}^2 & \cdots & N_{y1} N_{x4} & N_{y1} N_{y4} \\ & & & \vdots & \vdots & \vdots \\ & \text{Sym.} & & & & N_{y4}^2 \end{bmatrix}_{(s_a, s_b)} = m_s [S_t]_{(s_a, s_b)} \tag{12b}$$

are the equivalent stiffness and mass matrices of the point support when it is attached at Point (s_a, s_b) in the element. [S_t] is henceforth referred to as the nominal support matrix, which is an explicit function of the support location. By using the support equivalent stiffness and mass matrices, the support model is now continuous with respect to its location in FEM [15].

The developed formulation in Eq. (12) allows the support to be located anywhere on the plate, and not just at the FE nodes. If an elastic support is attached at a node of the FE mesh, e.g. at Node 1 in Fig. 2, then according to the shape functions given in Eq. (9), the corresponding stiffness matrix of the spring support is:

$$[K]_S = k \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ & 0 & 0 & \cdots & 0 & 0 \\ & & 0 & \cdots & 0 & 0 \\ & & & \vdots & \vdots & \vdots \\ & \text{Sym.} & & & & 0 \end{bmatrix}_{12 \times 12} \tag{13a}$$

and its mass matrix is

$$[M]_S = m_s \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ & 0 & 0 & \cdots & 0 & 0 \\ & & 0 & \cdots & 0 & 0 \\ & & & \vdots & \vdots & \vdots \\ & \text{Sym.} & & & & 0 \end{bmatrix}_{12 \times 12} \tag{13b}$$

These are the typical results for a spring support without rotational stiffnesses that we usually apply in FEM.

2.3 Natural Frequency Derivative with Respect to Support Location

Since an elastic support movement does not affect the stiffness and mass matrices of the plate structure itself, the derivative of the *i*th natural frequency (eigenvalue) can be readily obtained from Eq. (2) as

$$\frac{\partial \omega_i^2}{\partial s_a} = \{\phi_e\}_i^{\mathrm{T}} \left(\frac{\partial [K]_S}{\partial s_a} - \omega_i^2 \frac{\partial [M]_S}{\partial s_a} \right) \{\phi_e\}_i \tag{14a}$$

$$\frac{\partial \omega_i^2}{\partial s_b} = \{\phi_e\}_i^{\mathrm{T}} \left(\frac{\partial [K]_S}{\partial s_b} - \omega_i^2 \frac{\partial [M]_S}{\partial s_b} \right) \{\phi_e\}_i \tag{14b}$$

where $\{\phi_e\}_i$ is the *i*th mode shape at the degrees of freedom of the plate element in which the spring support is located.

Although these natural frequency derivatives may be calculated for any point within the plate element, here we would evaluate the derivatives only at the nodes of the FE mesh in compliance with the structural FE computation. From the standard shape functions of the thin plate element in Eq. (9), evaluation of the functions and their derivatives at Vertex 1 shows

$$N_1 = \frac{\partial N_{x1}}{\partial y} = -\frac{\partial N_{y1}}{\partial x} = 1 \tag{15a}$$

$$N_{x1} = N_{y1} = \frac{\partial N_1}{\partial x} = \frac{\partial N_1}{\partial y} = \frac{\partial N_{x1}}{\partial x} = \frac{\partial N_{y1}}{\partial y} = 0 \tag{15b}$$

and the other shape functions and their first-order derivatives are all zero. Substituting Eqs. (12a) and (12b) into Eq. (14), we can then achieve the natural frequency derivative when a point support is attached at Vertex 1 as

$$\left.\frac{\partial \omega_i^2}{\partial s_a}\right|_{\substack{s_a=0 \\ s_b=0}} = -2k w_{1i} \theta_{y1i} + 2 m_s \omega_i^2 w_{1i} \theta_{y1i} \tag{16a}$$

$$\left.\frac{\partial \omega_i^2}{\partial s_b}\right|_{\substack{s_a=0\\ s_b=0}} = 2kw_{1i}\theta_{y1i} - 2m_s\omega_i^2 w_{1i}\theta_{x1i} \tag{16b}$$

where w_{1i}, ϑ_{x1i} and ϑ_{y1i} indicate the transverse displacement and slopes along *x*- and *y*-axes, respectively, for the *i*th vibration mode at Vertex 1 of the element as shown in Fig. 2.

The design derivatives of the *i*th natural frequency with respect to the support location attached at other corner vertices of the rectangular plate element can be readily derived in a similar way, and the obtained results are consistent with Eqs. (16). It is worth noting that only the nodal generalized displacements for the *i*th mode at the spring support location appear in the expression. Therefore, the subscripts indicating the element vertex in Eqs. (16) will be omitted subsequently, and for simplicity, the generalized displacements in the derivative formulations are just those of the point support location:

$$\frac{\partial \omega_i^2}{\partial x_s} = -2(k - m_s\omega_i^2)w_i(x_s, y_s)\theta_{yi}(x_s, y_s) \tag{17a}$$

$$\frac{\partial \omega_i^2}{\partial y_s} = 2(k - m_s\omega_i^2)w_i(x_s, y_s)\theta_{xi}(x_s, y_s) \tag{17b}$$

where (x_s, y_s) indicates the grid node of the structural FE mesh. Those terms are immediately available from computational results of FEM.

Furthermore, the support reaction force, R_i, for the *i*th mode shape can be calculated as

$$R_i = -k\,w_i(x_s, y_s) \tag{18}$$

and the vertical inertial force due to the effective mass is:

$$P_i = m_s\omega_i^2 w_i(x_s, y_s) \tag{19}$$

Thus, substituting these expressions into Eq. (17), the natural frequency derivatives become

$$\frac{\partial \omega_i^2}{\partial x_s} = 2(R_i + P_i)\theta_{yi}(x_s, y_s) = 2F_i\theta_{yi}(x_s, y_s) \tag{20a}$$

$$\frac{\partial \omega_i^2}{\partial y_s} = -2(R_i + P_i)\theta_{xi}(x_s, y_s) = -2F_i\theta_{xi}(x_s, y_s) \tag{20b}$$

where F_i is the resultant modal force due to the elastic support attached to the plate structure. Clearly, the derivatives are proportional to the internal force and slope of the mode shape along the direction of motion of the support position.

So far, the support has assumed to move parallel to the *x* or *y*-axes of the coordinate system. In certain cases, the support may move along a specific direction, and then the directional derivative of the *i*th natural frequency can be calculated as

$$\frac{\mathrm{d}\,\omega_i^2}{\mathrm{d}\,s} = \operatorname{grad}\ (\omega_i^2)\cdot \mathrm{d}\boldsymbol{s} = \frac{\partial\,\omega_i^2}{\partial\,x_s}\cos\ \alpha + \frac{\partial\,\omega_i^2}{\partial\,y_s}\sin\ \alpha \qquad (21)$$

where α is the orientation angle of the specific direction $\boldsymbol{s}$ with regard to the *x*-axis.

3. Evolutionary Procedure for Support Position Optimization

Often, a structural design optimization is performed based on size, shape and topology. However, when the structural design parameters cannot be altered due to some design limitations, changes in the restraint conditions of a structure can also be utilized to effectively improve the structural static or dynamic behaviors [1]. At present, the positions of the spring point supports in the plate structure are optimally designed to raise a natural frequency of interest to a target value or to its upper limit through the economic design of the support stiffnesses as is widely accepted. The optimization problem is defined as

$$\text{Find}\quad \{s\} = [s_1,\, s_2,\, \ldots\,,\, s_n]^{\mathrm{T}} \qquad (22)$$

$$\text{Minimize}\quad \sum_{j=1}^{n} k_j \qquad (23)$$

$$\text{Subject to:}\quad \begin{cases} \omega_i(\{s\}) \geq \omega_i^* \\ s_j \in D_j \quad (j = 1, \ldots, n) \end{cases} \qquad (24)$$

where k_j is the translational stiffness coefficient of the *j*th point support, and *n* the total number of available supports. ω_i is the *i*th natural frequency of a plate system that has to be increased, which is a function of the interior support positions {*s*}, and ω_i^* is the associated prescribed value. If $i = 1$, then ω_1 denotes the fundamental natural frequency of the system.

s_j indicates the design variable, representing the jth independent support position, and D_j indicates the preset region in which the jth support position can change.

Obviously, the design optimization problem described in Eqs. (22)-(24) is highly nonlinear. Consequently, an iterative procedure is required to achieve the final optimal support positions to increase the natural frequency of interest. The optimization procedure [12] has two stages: at the first stage the support locations are constrained to the FE nodes, and at the second stage the optimum locations within the element is obtained evolutionarily. Starting from an initial set of the support position variables, for the first step, the direction to move the support locations to increase the ith natural frequency ω_i of the plate system can be determined from

$$\text{sign}\ (\Delta s_j) = \text{sign}\ \left(\frac{\partial \omega_i^2}{\partial s_j}\right) \quad (j = 1, \ldots, n) \tag{25}$$

where Δs_j is the step length in the support position change, which is taken as the associated elementary size so that the support is located at a node of the FE mesh [12]. sign (·) is the sign function.

Once the location of an elastic support is specified, the governing characteristic eigenvalue equation presented in Eq. (1) for the plate with the spring support attachments is recast into

$$\left[[K]_P + [K]_S - \omega_i^2 ([M]_P + [M]_S)\right]\{\phi\}_i = \{0\} \tag{26}$$

where the $[K]_P$ and $[M]_P$ denote the plate stiffness and mass matrices, respectively. By using Eqs. (5) and (12), the generalized eigenvalue problem of the global plate system is written as

$$\left[[K]_P - \omega_i^2 [M]_P + k(1 - r\omega_i^2)[S_t]\right]\{\phi\}_i = \{0\} \tag{27}$$

For the purpose of raising the ith natural frequency ω_i to the prescribed value ω_i^*, a standard approach to calculate the support stiffness threshold is to solve the general eigenvalue problem

$$\left[[K]_P - (\omega_i^*)^2 [M]_P\right]\{\phi\}_i = -k(1 - r(\omega_i^*)^2)[S_t]\{\phi\}_i \tag{28}$$

It has been shown that the minimum positive eigenvalue for k in Eq. (28) is virtually the critical or minimum support stiffness required to increase the plate natural frequency to the target value [5]. Moreover, the associated eigenvector $\{\phi\}_i$ is the corresponding mode shape of the supported plate, which is also mass normalized.

If ω_i^* is set as a natural frequency of the unsupported plate, which is commonly adopted in many previous studies [1, 3, 5, 8], then the dynamic stiffness matrix of the plate $[D]_P = [K]_P - (\omega_i^*)^2 [M]_P$ is singular. In this case, there will usually be one zero eigenvalue in Eq. (28). Furthermore, if ω_i^* is set too high, there will be no eigenvalue solution which means that even the rigid support cannot raise the ith natural frequency to ω_i^*. According the Courant's maximum-minimum principle [17], n additional supports can only increase the ith structural natural frequency, ω_i, to between the ith and the (i+n)th natural frequencies of the originally unsupported structure. For instance, one additional flexible or rigid support can only increase the ith structural natural frequency, to between the ith and the next larger (i+1)th natural frequencies of the original structure.

In most practical problems, however, the optimal position of a point support with the minimum or critical stiffness may not be exactly at the grid node of the FE model of the plate structure, and is often located within an element. In this case, a refined FE mesh in the neighborhood of the optimal solution could be employed to find the more accurate support position. Herein, an alternative approach [5, 15] is considered to facilitate the convergence of the optimization process. According to the sign change of the frequency derivative, the particular element containing the optimal support solution can be essentially identified. Then, by using the equivalent stiffness and mass matrices of a flexible support within an element, represented in Eq. (12), the optimal position of a spring support can be estimated with ease by finding the zero value of the natural frequency derivative [18].

4. Illustrative Examples

The validity of the formulation for the frequency derivative calculation and the effectiveness of the proposed optimization approach to obtain both the minimum stiffness and the optimal position of internal point supports will be demonstrated with several examples. In this section, different boundary conditions of the rectangular plate structures are explored and the optimal results are compared with those obtained in the literature that ignore the support mass [5, 14] to illustrate the effects of including the support mass. In the following numerical examples, the plate thickness is set as $h = 3.0$ mm uniformly. The Young's modulus of elasticity is $E = 70.0$ GPa, Poisson's ratio $\nu = 0.3$ and the mass density of material ρ = 2800 kg/m^3. For comparison purposes, the obtained characteristic results will be presented in terms of the non-dimensional parameters of the natural frequency $\lambda = \omega L^2 \sqrt{\rho h / D}$, support stiffness $\gamma_s = k L^2 / D$, where $D = Eh^3 / 12(1 - \nu^2)$ is the constant flexural rigidity of the plate, and the optimal coordinate $\eta_s = x_s / L$ along the *x*-axis, where *L* is the length of the plate in the global *x*-axis. The ratio of the effective support mass to the plate mass, $\beta = m_s / (\rho LWh)$, is also given for illustration of the support mass. In these examples we assume $m_s = rk$, as given in Eq. (5), where *r* is fixed.

In the solution process, the support optimal position and minimum stiffness are obtained using the evolutionary method presented in Section 3. Representative vibration mode shapes of the plate supported by the additional spring support are then plotted to verify the optimal results. It will be seen that the proposed method is very effective in obtaining an efficient design for the flexible supports to increase a natural frequency of a plate structure.

4.1 A rectangular plate with one edge restrained

A flat rectangular plate, having one edge conventionally constrained (either simply supported or clamped) and the other edges free, together with one additional elastic support, is demonstrated schematically in Fig. 3. This is a typical model for the dynamic analysis of plate behaviors when designing a support, such as a column of a slab in civil engineering or for a printed circuit board in electrical engineering [1, 4, 5]. It is of particular interest to know exactly the optimal position and the minimum stiffness to achieve a target fundamental frequency of the whole structural system.

(Figure 3)

Two geometrical shapes of the rectangular plate with aspect ratios $\alpha = L/W$ of 1.0 (square) or 1.5 (rectangle) are investigated, and clamped and simply supported boundary conditions are modeled, respectively. Table 1 gives the dimensions and masses of the plates, the FE meshes for different aspects and the first three natural frequency parameters for the unsupported plates. The corresponding mode shapes of the square plate with the clamped edge are illustrated in Fig. 4.

(Table 1 and Figure 4)

4.1.1 One elastic support on the free edge

In this example, a single flexible support is attached along the free edge opposite to the restrained boundary to raise the fundamental natural frequency of the plate as high as possible to improve the structure's dynamic behavior [1]. Due to the structural symmetry about the horizontal center line ($y = 0$), the additional support should be located at the mid-point of the free edge, as shown in Fig. 3, where the requirement of zero slope of the fundamental mode shape in the *y*-direction is readily satisfied [5]. Since the support is located on the nodal line of the second mode shape (corresponding to the first torsional mode) of the unsupported plate, given in Fig. 4b, the fundamental natural frequency (corresponding to the first bending mode) can only be raised extremely to the second natural frequency of the unsupported structure [17]. Increasing the support stiffness further above the minimum value cannot raise the fundamental frequency of the supported plate anymore due to mode switching of the lowest two frequencies [1].

Attached at this particular spot, the minimum stiffnesses of the point support can be directly estimated by Eq. (28), and listed in Table 2 are the optimal results, which are highly dependent on the aspect ratio and the boundary constraints of the rectangular plate. Also listed are the earlier results based on the assumption of a massless support model for comparison. As expected, by using an elastic point support, the fundamental natural frequency can be effectively raised to its upper limit and then becomes a bimodal (doubly repeated) frequency with two basis modes of purely bending and torsional deflections.

Additionally, with the inclusion of the effective support mass, which is relatively small compared to the plate mass (3.9-10.3%) on the prescribed ratio factor $r = 10^{-6}$, a larger minimum stiffness of the support (increased by 7.3-22.6%) is required. This will certainly increase the cost of the support to raise the lowest natural frequency to the target value. Similar to the situation of the massless support model [5], with a single flexible support there is no solution for the simply supported boundary of the rectangular plate ($\alpha = 1.5$), which means that the maximum attainable increment of the fundamental natural frequency is limited by adding a point support at the free edge the plate.

(Table 2)

The effect of the support mass on the third natural frequency (corresponding to the second bending mode) of the supported plate is also shown in Table 2. Although the required support stiffness has been obviously increased due to the inclusion of the additional support mass, all of the third natural frequencies are noticeably lower than their counterparts for the massless supports. Moreover, the third natural frequency for the square plate with a clamped edge is less than the original value for the unsupported plate, given in Table 1. This fact highlights that sometimes, the additional increment of the support stiffness cannot compensate for the negative effect of the support mass involvement on particular structural natural frequencies, although the additional support mass is very small compared to the plate mass, as shown in Table 2. In other words, the support influence on the dynamic characteristics of the plate is no longer monotonous when the extra support mass is considered in the vibration analysis.

Furthermore, in order to move the support position along the center line and determine the moving direction, it is very crucial to compute the frequency derivative with respect to the support position. However, at present, the first two natural frequencies are repeated, and the fundamental natural frequency becomes bimodal, as seen in Table 2. Consequently, the corresponding modes are tightly coupled with each other such that the basic modes are not readily attainable. It is clear that a repeated frequency is generally not differentiable in the common sense (i.e. the Fréchet derivative does not exist). Only directional derivatives can be obtained [19-21]. In order to evaluate the directional derivatives of the fundamental frequency of the plate system, a sub-eigenvalue problem has to be solved. For complete understanding, a brief outline of the computational procedure is presented herein. First, the Eigenspace Directional Derivative Matrix [*G*] [21] is formulated at the element level:

$$[G] = [\{\phi\}_1, \{\phi\}_2]_e^{\mathrm{T}} \left[\frac{\mathrm{d}[K]_S}{\mathrm{d}x_s} - \omega_1^2 \frac{\mathrm{d}[M]_S}{\mathrm{d}x_s} \right] [\{\phi\}_1, \{\phi\}_2]_e \tag{29}$$

where $\{\phi\}_1$ and $\{\phi\}_2$ are the fundamental natural modes of the supported plate with the critical support stiffness. It is worth mentioning that these two modes may not be the two basis mode shapes, but should be orthogonal to each other.

From the derivation of the equivalent stiffness and mass matrices for a spring support, presented in Eq. (12), the derivative expressions of the equivalent matrices can be simply obtained using the shape functions of the relevant element with the point support attachment, see the dark element in Fig. 3. Thus

$$\frac{\mathrm{d}[K]_S}{\mathrm{d}x_s} = k \begin{bmatrix} [0]_{3\times3} & [0] & [0] \\ & \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} & [0] \\ \text{Sym.} & & [0]_{6\times6} \end{bmatrix} \tag{30a}$$

$$\frac{\mathrm{d}[M]_S}{\mathrm{d}x_s} = m_s \begin{bmatrix} [0]_{3\times3} & [0] & [0] \\ & \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} & [0] \\ \text{Sym.} & & [0]_{6\times6} \end{bmatrix} \tag{30b}$$

Consequently, [*G*] in Eq. (29) can be estimated with the related mode shapes and then, the associated eigenvalues of [*G*] can be straightforwardly evaluated, which are just the directional derivatives of the bimodal fundamental frequency [19, 21].

Of the two directional derivatives of the fundamental natural frequency, one is presented in Table 2 for different boundary constraints. The other is just equal to zero, which means that moving the point support in the *x*-direction along the plate's symmetric line cannot change the fundamental natural frequency with the first torsional mode shape.

This can be understood physically because the horizontal center line is just the nodal line of the torsional vibration mode of the plate [1].

It is worth noting that the fundamental frequency derivatives, listed in Table 2, are all negative, which implies that the mid-point of the free edge is not the most suitable position for an efficient support design. According to Eq. (25), shifting the support location inwards along the center (symmetric) line could raise the fundamental natural frequency of the first bending mode of the plate structure. In other words, the support movement toward the restrained edge of the plate can certainly reduce the support stiffnesses required so as to lead to an even more economic support design. Therefore, the optimum support location along the center line, as well as the corresponding minimum stiffness, will be investigated in the next section while preserving the lowest frequency at its extremum, which is the second natural frequency of the unsupported plate.

4.1.2 One elastic support on the center line

In this situation, a single elastic support is allowed to move along the axis of symmetry ($y = 0$) of the plate while the fundamental natural frequency is constrained to remain at its upper limit. At the same time, the optimal solution for the rectangular plate of the aspect ratio 1.5 with a simply supported edge is also explored; even a rigid point support located at the free edge is not able to sufficiently raise the fundamental frequency to the second frequency of the unsupported plate structure [5, 14]. Figure 5 shows the minimum support stiffness as well as the derivative of the first natural frequency to the support position in the x-direction, versus the change in the support location to keep the maximum fundamental natural frequency achieved by adding a single support. It is clearly demonstrated that the minimum support stiffness heavily depends on the support position. Furthermore, the fundamental frequency derivative reaches to zero at a point close to 0.90 of the rectangular plate length for the clamped plate or 0.79 for the simply-supported-edge rectangular plate. At these optimum locations the support stiffness arrives at the minimum threshold. Table 3 gives the optimum solutions of the minimum support stiffness and its location for the rectangular and square plates. The optimum results with no concern of the support mass are also presented in Table 3 for comparison.

(Figure 5 and Table 3)

From the optimum results in Table 3 it can be observed that the fundamental natural frequency of the plate has been successfully raised to the respective second natural frequency in all of the boundary restraint cases. Moreover, the minimum support stiffnesses are all reduced more or less from their corresponding values with a spring support located at the free end. These are the certain outcome of the negative frequency derivatives at the free boundary. Furthermore, the optimal support positions are all identical to those obtained when the effective support mass is neglected. This is not a surprise since at the given optimal support position with the minimum support stiffness, the frequency derivative should be zero along the *x*-direction. This can be simply achieved by the zero slope ϑ_y of the first bending mode shape, as seen in Eq. (17), which is the same criterion used to determine the optimal support position in reference [5]. Figure 6 shows respectively the fundamental bending mode shape of the rectangular plate supported by a single support with the minimum stiffness at the optimum location for illustration.

(Figure 6)

4.2 A rectangular plate with two spring supports

A rectangular plate with one long edge simply supported and the others free is considered, as shown in the schematic diagram in Fig. 7. The plate is discretized with a regular mesh of 10×16 elements, and the first three natural frequency parameters of the unsupported plate are listed in Table 4. Clearly, the plate system has a zero fundamental frequency. Suppose two identical elastic supports, which are only allowed to move synchronously and symmetrically along the specified orthogonal lines, are employed to increase one of the natural frequencies of the plate structure. The representative position of the upper support is (x_s, y_s).

(Figure 7 and Table 4)

In this case, although there are four position coordinates for the two elastic supports, only one, say x_s of the upper support, is the independent design variable due to symmetry, and y_s, is virtually a dependent coordinate when the support moves along the specified diagonal directions in the present coordinate system:

$$y_s = \pm(x_s - 0.06) \tag{31}$$

So, the frequency derivative is the algebraic sum from the two support position movements. In addition, by using the two grounded supports, it is theoretically possible to increase the fundamental frequency of a plate to its third natural frequency [1, 17]. But for this plate model with the elastic supports, the fundamental frequency cannot be increased up to the third natural frequency of the original plate (corresponding to the second torsional mode). Thus, we only perform the optimization support design to increase the fundamental frequency to the second natural frequency of the unsupported plate. By using Eq. (21), the fundamental frequency derivatives with regard to the locations of the supports in the specific directions can be readily calculated, and the values at different FE nodes on the specific lines are plotted in Fig. 8a together with the corresponding minimum stiffness. It is clear that the optimal support position occurs between 0.80 and 0.90 typically due to the opposite signs of the frequency derivatives. The optimal support design for achieving the target fundamental frequency is then found by means of the support equivalent stiffness and mass matrices in Eq. (12), and the results are listed in Table 4. The optimal support designs on the massless assumption are also calculated in order to make a comparison. Evidently, the minimum support stiffness required is larger than the corresponding stiffness neglecting the support mass, but the support optimal positions are identical with the two support models.

(Figure 8)

Alternatively, with the two elastic supports, we can also raise the second natural frequency up to the third natural frequency of the unsupported plate, and the corresponding minimum stiffness variation is illustrated in Fig. 8b, together with the frequency derivatives. The optimal support designs are also listed in Table 4. In this situation, the obtained minimum support stiffness increases significantly by 4.98 times in comparison to the first natural frequency increment, while the corresponding increment with the exclusion of the effective support mass is only 3.02 times, although the additional support masses in the two cases are all very small compared to the plate mass. In addition, with the greater support stiffness, all of the first three natural frequencies of the plate are larger than the corresponding natural frequencies in the previous case.

4.3 A free-free square plate with four elastic supports

A fully free square plate [12] supported on four identical elastic supports is shown schematically in Fig. 9. The plate size is $L = 0.3$ m and the plate is discretized with a regular mesh of 20×20 flexural elements. Suppose the four identical elastic supports are located symmetrically on the particular lines so as to maximize the fundamental frequency of the plate system. The first three flexural natural frequency parameters for the unsupported plate are listed in Table 5. From the previous study [12, 14], it is well-known that the fundamental frequency can be raised to the first or ultimately the second flexural frequency of the unsupported plate with four grounded point supports.

(Figure 9 and Table 5)

4.3.1 Elastic supports along the diagonals

In this case, the four point supports are located symmetrically along the plate orthogonal diagonals, that is, the elastic supports move synchronously along the plate diagonal directions, as shown by the solid points in Fig. 9. The representative position of the upper-right support is (x_s, y_s), and all of the additional support positions are linearly correlated with x_s, which is the only independent position coordinate due to symmetries of the specified moving directions. First, we perform the optimization of the support design to raise the fundamental frequency to the first flexural frequency of the free-free plate. The minimum support stiffness and the fundamental frequency derivative with regard to the support movement at different FE nodes on the diagonals of the plate are plotted in Fig. 10a. It is clear that the support optimal position occurs between 0.25 and 0.30. Then the optimal design of the support position and the associated minimum stiffness are determined and listed in Table 5. The optimal position in the present scenario is also identical to that obtained with the massless supports [14] estimated by the Rayleigh-Ritz method. Nevertheless, the minimum support stiffness is much larger when the additional support mass is included, which is now no longer small in comparison to the plate mass. This result shows apparently that consideration of the support mass in the structural vibration analysis can significantly affect the required minimum stiffness of the additional support to raise the natural frequencies of a plate.

(Figure 10)

Furthermore, we can employ the four elastic supports to raise the fundamental frequency to the second flexural frequency of the free-free plate. In this extreme case, the required minimum stiffness of the elastic supports increases significantly to 5175, more than 44 times the required stiffness threshold for a massless support, and the additional contribution of the support mass is more than 13 times the plate mass for each of the supports, as listed in Table 5. This result means that a very big lumped mass is incorporated into the plate vibration at each of the attachment points. This is clearly a challenging optimal design problem for the plate supports in engineering practice. It is therefore understood that the addition of the support mass makes the increment of the fundamental frequency of the supported plate to the original second flexural natural frequency much more difficult in practical applications.

4.3.2 Elastic supports along the two axes parallel to the plate edges

Alternatively, the four flexible supports may be located symmetrically along the two orthogonal axes to increase the fundamental natural frequency of the plate, as shown by the hollow points in Fig. 9. In the present support array, the representative position of the right support is (x_s, 0), and all of the support positions are linearly correlated with x_s due to symmetries of the moving directions. Likewise, there is only the positive solution for Eq. (28) to raise the fundamental frequency to the first flexural natural frequency, but no positive solution to the second one of the free-free plate [14]. Figure 10b shows the required minimum support stiffness and the fundamental frequency derivative with regard to the support location at different FE nodes on the center lines. Clearly, the support optimal position can only occur between 0.40 and 0.45 due to the opposite signs of the derivatives of the fundamental frequency. Therefore, the minimum stiffness of the optimally located supports can be evaluated, and the computational results are tabulated in Table 5. Once again, the corresponding minimum support stiffness is much greater than the corresponding stiffness of the massless support design, while the optimal position is consistent. Moreover, the minimum support stiffness is also larger than the counterpart of the support on the plate diagonals. Figure 11 shows one of the first representative vibration mode shapes of the plate resting on the four optimally designed flexible supports on the diagonals or the axes, respectively. Note that the two first mode shapes are remarkably different, even though they all correspond to the same value of the fundamental natural frequency of the supported plate structure. Comparatively, the flexural deformation of the plate in Fig 11b is larger than that in Fig 11a for a similar modal mass.

(Figure 11)

5. Conclusions

In this work, simple translational supports are optimally designed to raise a natural frequency of the rectangular flexural plate structure to a target value or to its upper limit. For engineering applications, the effective mass of the spring support should be included in the vibration analysis of the plate structure to achieve a more practical design of the additional support. First, the frequency derivative formulation with respect to the support movement is developed using FE analysis, which is consistent with the numerical calculation of the structural dynamic characteristics by FEM. Then, an evolutionary procedure is proposed to determine the optimal position with the minimum stiffness of the support so as to produce a more economic design of the spring support.

From the numerical results of typical examples, it is evident that the additional support mass has a significant influence on the minimum stiffness of the elastic point support, even though the mass addition is sometimes very small compared to the plate mass. The minimum support stiffness is usually increases due to the inclusion of the support mass. But the optimal location remains unchanged to that obtained from the massless support model.

Credit Author Statement:

Dong Wang: Methodology, Software, Writing-Original, draft preparation.

Michael I. Friswell: Conceptualization, Data curation, Visualization, Validation Writing-Reviewing and Editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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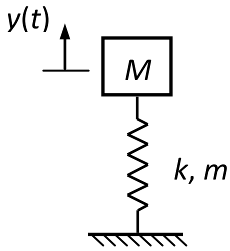


Figure 1 Schematic diagram of a single-spring-mass system with the non-negligible spring mass, m, which is included in the vibration analysis.

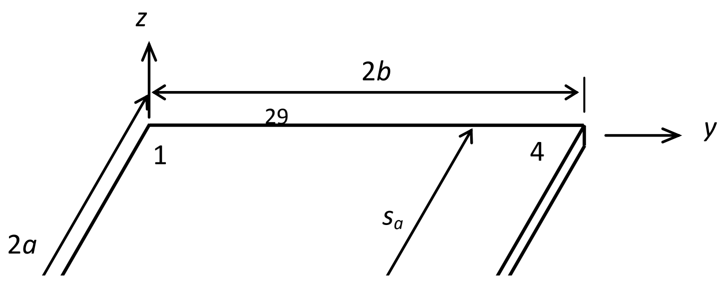


Figure 2 Schematic diagram of a thin flexural plate element with a grounded elastic support with effective mass m_s.

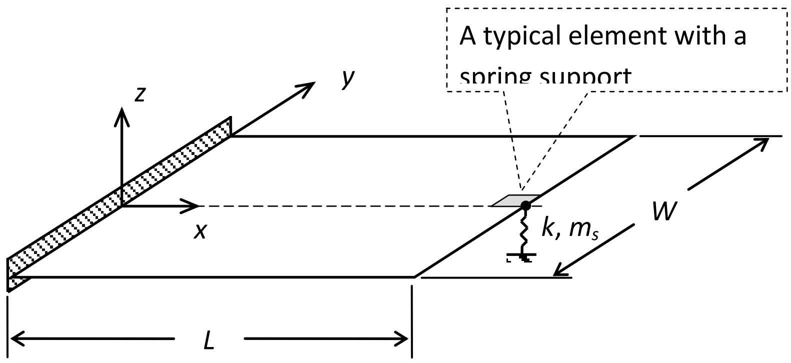
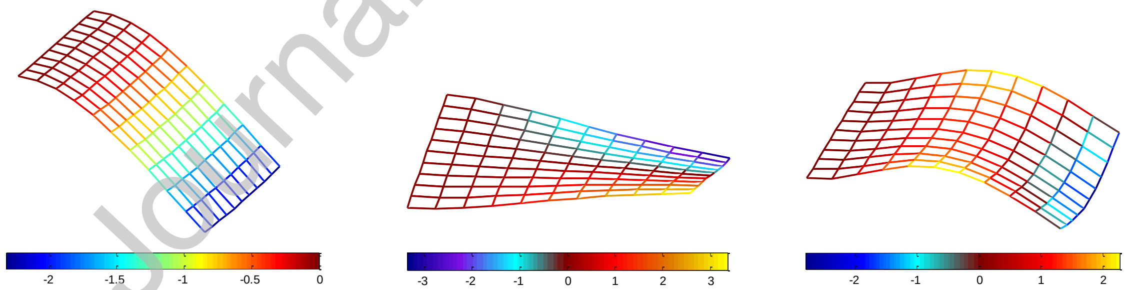
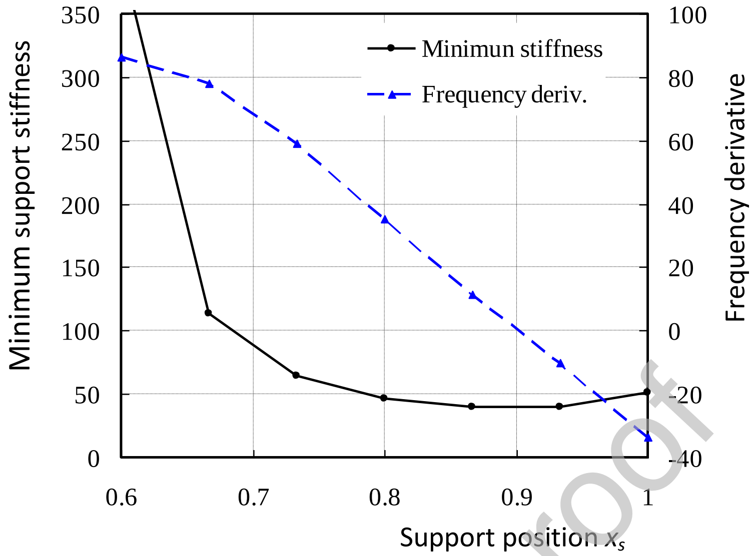


Figure 3 A flat rectangular plate restrained on one edge is additionally supported with an elastic point support positioned at the mid-point of the free edge opposite to the restrained boundary. The dark element shows that a point support is attached at one of its four nodes.

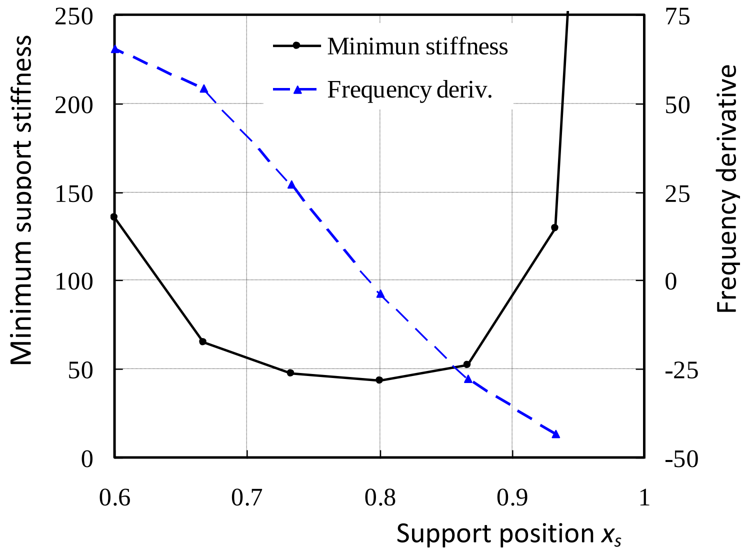


(a) (b) (c)

Figure 4 The first three mode shapes of the clamped and unsupported square plate: (a) first bending (b) first torsional (c) second bending.

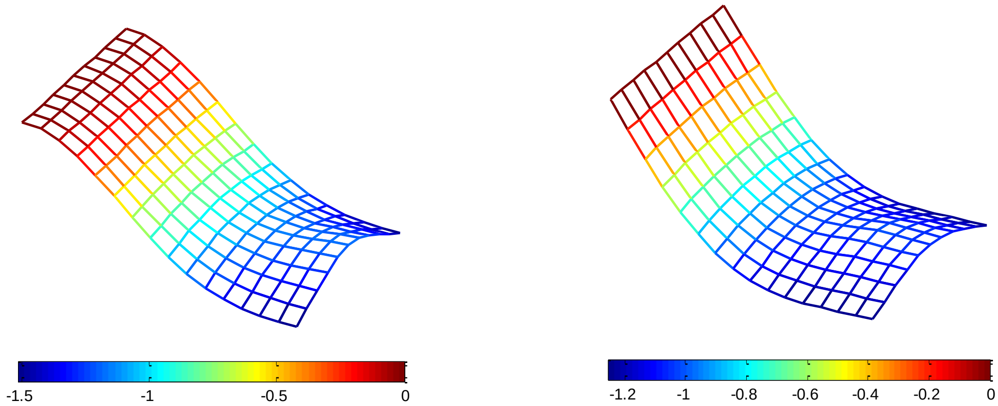

350
300
250
200
150
100
50
0
100
80
60
40
20
0
-20
-40
0.6
0.7
0.8
0.9
1
Minimum support stiffness
Frequency derivative
Support position x_s
Minimun stiffness
Frequency deriv.

(a)


250
200
150
100
50
0
75
50
25
0
-25
-50
0.6
0.7
0.8
0.9
1
Minimum support stiffness
Frequency derivative
Support position x_s
Minimun stiffness
Frequency deriv.

(b)

Figure 5 The minimum support stiffness (solid-line) and the fundamental frequency derivative (dashed-line) with respect to the support movement in the x-direction at different positions on the symmetric line of the rectangular plate ($\alpha = 1.5$): (a) clamped boundary, (b) simply supported boundary.



(a) (b)

Figure 6 The first bending mode shape of the rectangular plate ($\alpha = 1.5$) with a single elastic support of the minimum stiffness at the optimum position on the center line: (a) for the clamped (b) for the simply supported boundary.

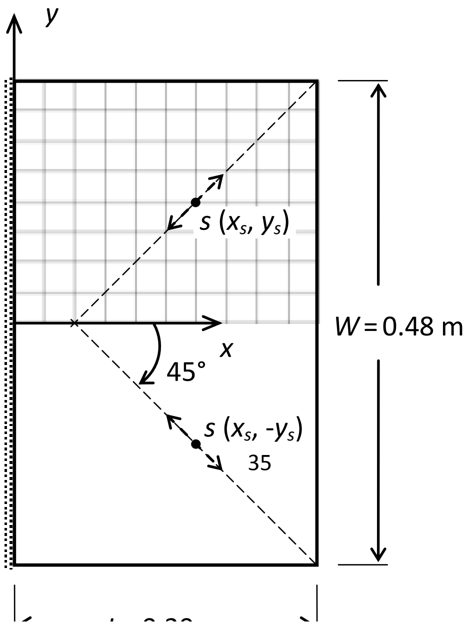
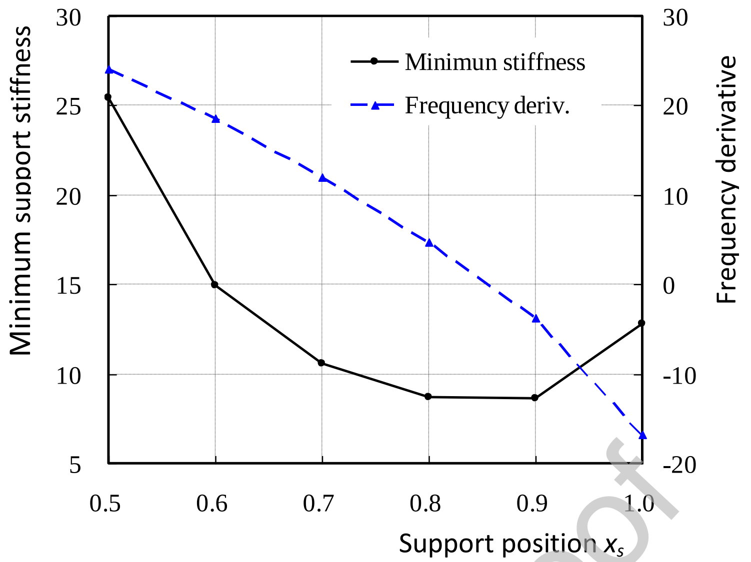
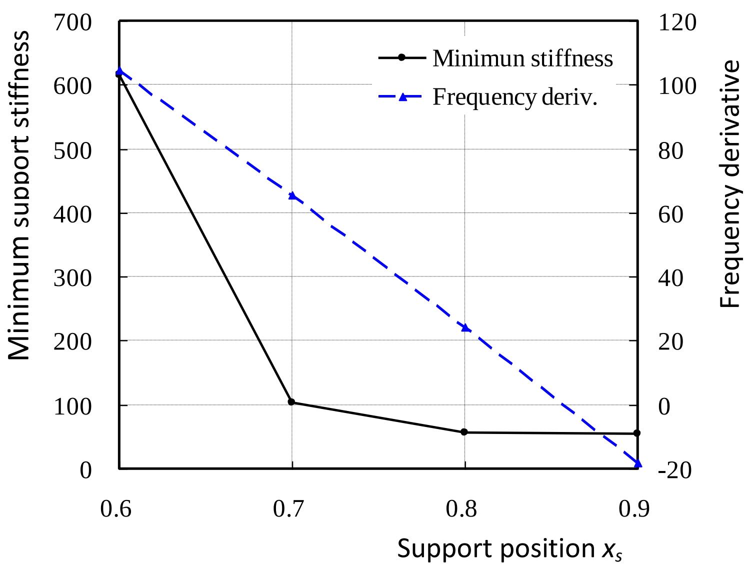


Figure 7 A rectangular plate simply supported in one long edge is additionally supported with two elastic point supports located along the orthogonal lines departing from the vertices of the free edge opposite to the constrained boundary



30
25
20
15
10
5
Minimum support stiffness
30
20
10
0
-10
-20
Frequency derivative
0.5
0.6
0.7
0.8
0.9
1.0
Support position x_s
Minimun stiffness
Frequency deriv.

(a)



700
600
500
400
300
200
100
0
Minimum support stiffness
120
100
80
60
40
20
0
-20
Frequency derivative
0.6
0.7
0.8
0.9
Support position x_s
Minimun stiffness
Frequency deriv.

(b)

Figure 8 The minimum support stiffness (solid-line) and the frequency derivative (dashed-line) with respect to the synchronous movement of the two elastic supports along the specified directions on the rectangular plate: (a) for the first natural frequency (b) for the second natural frequency.

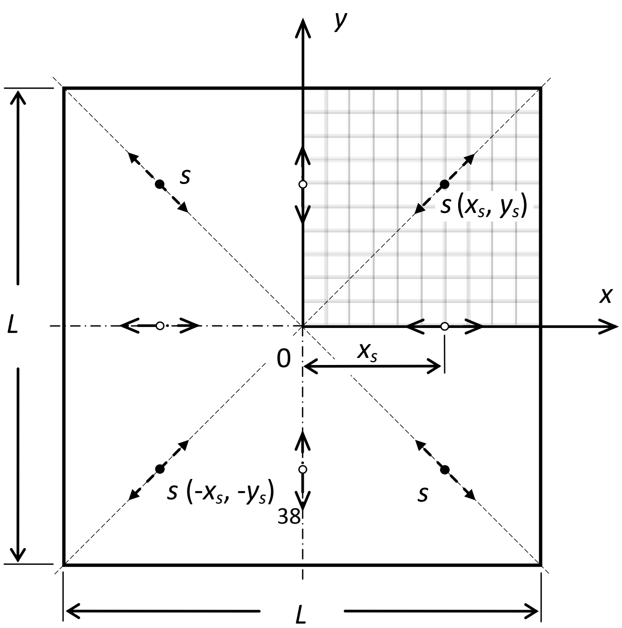
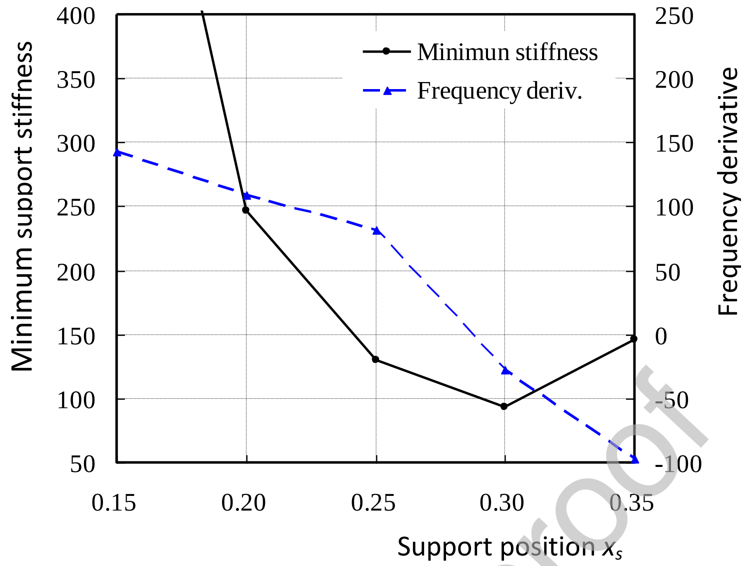
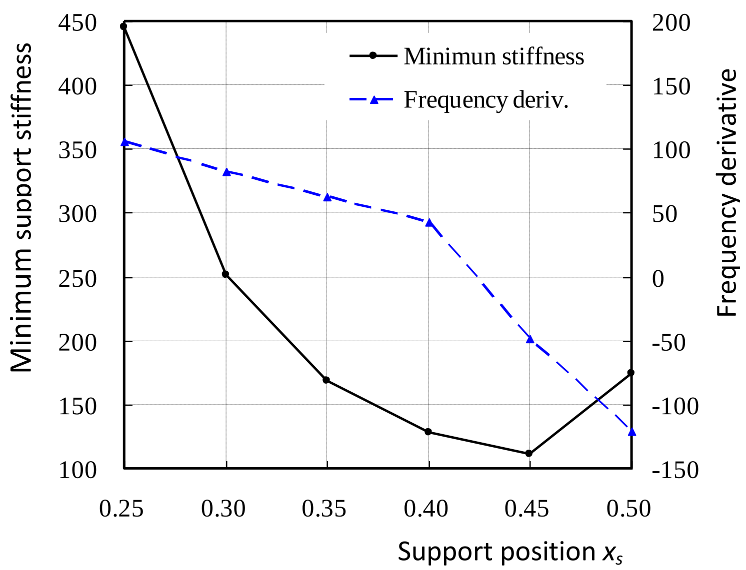


Figure 9 A free square plate is supported symmetrically by four flexible supports, along the diagonals (solid points) or the axes parallel to the edges (hollow points), alternatively.



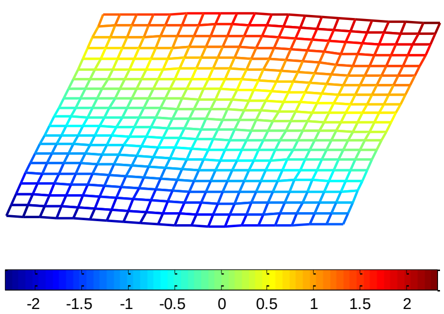
Minimum support stiffness
Frequency derivative
Support position x_s
Minimun stiffness
Frequency deriv.
400
350
300
250
200
150
100
50
250
200
150
100
50
0
-50
-100
0.15
0.20
0.25
0.30
0.35

(a)

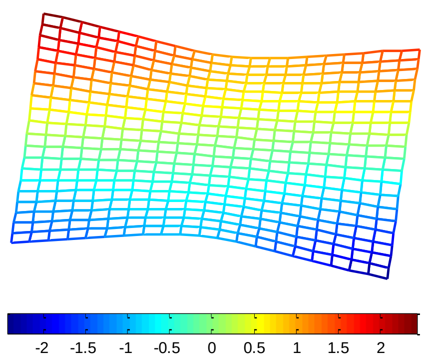


Minimum support stiffness
Frequency derivative
Support position x_s
Minimun stiffness
Frequency deriv.
450
400
350
300
250
200
150
100
200
150
100
50
0
-50
-100
-150
0.25
0.30
0.35
0.40
0.45
0.50

Figure 10 The minimum support stiffness (solid-line) and the fundamental frequency derivative (dashed-line) with respect to the support synchronous movement of the four elastic supports: (a) along the diagonals (b) along the axes of the square plate.



(a)



(b)

Figure 11 The fundamental mode shapes of the plate supported by four elastic supports with the minimum stiffnesses: (a) along the orthogonal diagonals (b) along the axes of the square plate.

Table 1 The geometry dimensions and first three natural frequency parameters of an unsupported rectangular plate with one boundary edge clamped or simply supported.

Boundary edge restrained		Clamped		Simply supported	
Aspect ratio α		1.0	1.5	1.0	1.5
Length, L (m)		0.3	0.45	0.3	0.45
Width, W (m)		0.3	0.3	0.3	0.3
Plate mass (kg)		0.756	1.134	0.756	1.134
Element mesh		10×10	15×10	10×10	15×10
Natural frequency parameters λ_i	First bending (1B)	3.4710	3.4535	0	0
	First torsional (1T)	8.5088	11.6573	6.6457	9.8461
	Second bending (2B)	21.3307	21.4889	14.9213	14.8989

Table 2 The optimal position with the minimum stiffness and corresponding natural frequency parameters for the rectangular plate with a point support at the mid-point of the free edge.

Boundary edge restrained		Clamped		Simply supported	Results without considering support mass [5]		
					Clamped		Simply supported
Aspect ratio α		1.0	1.5	1.0	1.0	1.5	1.0
Natural frequency parameters λ_i	1 (1B)	8.5088	11.6573	6.6457	8.5088	11.6573	6.6457
	2 (1T)	8.5088	11.6573	6.6457	8.5088	11.6573	6.6457
	3 (2B)	20.8733	26.1718	16.6347	23.7338	27.6186	18.7203*
Support minimum stiffness γ_s		29.3695	51.3106	40.2909	23.9606	47.8070	35.7646
Mass ratio β		0.07471	0.03867	0.1025			
Frequency derivatives to support position along x-direction $d\lambda_1/dx_s$		-7.3801	-33.4806	-26.9908			

*This value was originally presented as 16.1827 due to a typographical error.

Table 3 The optimal position with the minimum stiffness of a single elastic support on the plate centre line and the corresponding natural frequencies for a rectangular plate with a conventional restrained edge

Boundary edge restrained		Clamped		Simply supported		Results without considering support mass [5]			
						Clamped		Simply supported	
Aspect ratio α		1.0	1.5	1.0	1.5	1.0	1.5	1.0	1.5
Natural frequency parameters λ_i	1 (1B)	8.5088	11.6573	6.6457	9.8461	8.5088	11.6573	6.6457	9.8461
	2 (1T)	8.5088	11.6573	6.6457	9.8461	8.5088	11.6573	6.6457	9.8461
	3 (2B)	20.9481	22.9976	15.4586	15.4959	23.3674	23.4554	16.1148	15.5690
Support minimum stiffness γ_s		28.9659	38.6401	29.5316	43.4123	23.6313	36.0017	26.2139	41.2976
Support optimum position η_s		0.9734	0.9017	0.8711	0.7917	0.9734	0.9017	0.8711	0.7917
Mass ratio β		0.07368	0.02912	0.07512	0.03272				

Table 4 The optimal positions with the minimum stiffnesses for two identical supports in the specified directions with the corresponding natural frequency parameters for the rectangular plate with one long edge simply supported

Support condition		Unsupported	With considering support mass		Without considering support mass	
Frequency raised			First	Second	First	Second
Natural frequency parameters λ_i	1 (1B)	0	4.2029	6.6950	4.2029	6.3408
	2 (1T)	4.2029	7.2799	11.8692	7.5106	11.8692
	3 (2T)	11.8692	12.3616	13.4369	12.5974	13.5324
Minimum stiffness γ_s (×2)			8.4295	50.4458	8.0508	32.3680
Optimal position η_s			0.8575	0.8584	0.8575	0.8584
Mass ratio β (×2)			0.0134	0.0802		

Table 5 The optimal positions with the minimum stiffnesses for four identical supports on the diagonals or axes with the corresponding natural frequency parameters for a square plate.

Support layout		Unsupported	Supports on the diagonals		Supports on the axes	Results by Rayleigh-Ritz [14]		
						Supports on the diagonals		Supports on the axes
Objective frequency			First	Second	First	First	Second	First
Flexural natural frequency parameters λ_i	1	13.4715	13.4715	19.5997	13.4715	13.4682	13.4682	13.4682
	2	19.5997	13.4715	19.5997	13.4715	13.4682	19.5961	13.4682
	3	24.2777	13.4744	19.5997	13.4715	13.4722	24.2702	13.4682
Minimum stiffness γ_s (×4)			91.1643	5174.7401	108.7936	48.8639	116.9779	58.3136
Optimal position η_s			0.2901	0.2892	0.4446	0.2901	0.2892	0.4447
Mass ratio β (×4)			0.2319	13.1633	0.2767			