SUPPORTS FOR DEGENERATE STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS AND APPLICATIONS*

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Abstract. In the paper, we are concerned with degenerate stochastic differential equations with jumps. We first establish two theorems about supports for the solution laws of the degenerate stochastic differential equations, under different (sufficient) conditions. We then apply one of our results to a class of degenerate stochastic evolution equations (that is, stochastic differential equations in infinite dimensions) with jumps to obtain a characterisation of path-independence for the densities of their Girsanov transformations.

1. Introduction

Stochastic differential equations (SDEs in short) with degenerate coefficients and jumps draw much attention in recent years. Thus, to study the support property for degenerate SDEs with jumps is interesting and potentially useful. Simon [11] was the first to establish support theorems for a class of SDEs driven by pure Poisson random measures, and Fournier [3] further showed that the densities of the distributions for their solutions at certain times are positive on the whole spaces. Since then, SDEs driven by Poisson random measures, viewed as degenerate SDEs with vanishing Brownian motion term, received certain attentions in the literature. However, there are few results concerning the support of degenerate SDEs driven by Brownian motions and compensated Poisson random measures. In the present paper, we are concerned with the supports of a class of degenerate SDEs driven by Brownian motions and compensated Poisson random measures for a fixed time. We aim to establish supports for the solution laws of these SDEs. We then utilise our result to derive a characterisation theorem for path-independent property of Girsanov transformation for degenerate stochastic evolution equations with jumps. We plan to consider support theorems for degenerate SDEs with jumps in our forthcoming work (cf. [4, 11] for the support theorems for SDEs driven by Brownian motion and driven by Poisson random measures, respectively).

More specifically, in the paper, we first prove a theorem on supports of solution distributions for SDEs under certain comparably general assumptions. Then, by strengthening

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the conditions such that SDEs have densities, and applying the fact that existence of the density for an SDE implies full support, we obtain the other support results. Furthermore, we apply one of our results to a problem on degenerate stochastic evolution equations with jumps on Hilbert spaces, in which path-independence for the densities of their Girsanov transformations is characterised.

It is worthwhile to mentioning our previous results. In [9, 10], assuming that the coefficients of their continuous diffusion terms are non-degenerate, we showed that these densities of Girsanov transformations for SDEs with jumps and stochastic evolution equations with jumps are path-independent, And then the second named author and Wu [12] only mentioned that these densities of Girsanov transformations for degenerate stochastic differential equations are path-independent. Here we allow their continuous diffusion coefficients to be degenerate, and we give some concrete conditions and provide detailed proofs.

This rest of the paper is organised as follows. In Section 2, we prove two results about supports for solution laws of SDEs with jumps under different (sufficient) conditions. Section 3 is devoted to applying one result to a problem on degenerate stochastic evolution equations with jumps. Finally, we obtain path-independence for the density of the associated Girsanov transformation.

The following convention will be used throughout the paper: the capital letter C with or without indices denote different positive constants whose values may change from one place to another.

2. Supports of SDEs with jumps

In this section, we will prove two results about supports for solution laws of SDEs with jumps which will be used in the next section.

Let $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t\geqslant 0})$ be a complete filtered probability space and let $\{B_t\}$ be a d-dimensional $(\mathscr{F}_t)_{t\geqslant 0}$-Brownian motion. Furhter, let $(\mathbb{U}, \|\cdot\|_{\mathbb{U}})$ be a finite dimensional normed space with topological σ-algebra $\mathscr{U}$, and let ν be a σ-finite measure defined on the measurable sapce $(\mathbb{U}, \mathscr{U})$. We fix $\mathbb{U}_0 \in \mathscr{U}, \mathbb{U}_0 \subset \mathbb{U} - \{0\}$ with $\nu(\mathbb{U} \setminus \mathbb{U}_0) < \infty$ and $\int_{\mathbb{U}_0} \|u\|_{\mathbb{U}}^2 \nu(\mathrm{d}u) < \infty$. We follow the line of [4] to construct an integer-valued $(\mathscr{F}_t)_{t\geqslant 0}$-Poisson random measure $N(\mathrm{d}t, \mathrm{d}u)$ with the intensity $\mathrm{d}t\nu(\mathrm{d}u)$, which is independent of $\{B_t\}$. Set

$$\tilde{N}(\mathrm{d}t, \mathrm{d}u) := N(\mathrm{d}t, \mathrm{d}u) - \mathrm{d}t\nu(\mathrm{d}u)$$

which is the $\tilde{N}(\mathrm{d}t, \mathrm{d}u)$ is the compensated predictable martingale measure of $N(\mathrm{d}t, \mathrm{d}u)$. Fix $T > 0$, we consider the following SDE with jumps on $\mathbb{R}^d$

$$\left\{\begin{array}{ll} \mathrm{d}Z_t = \xi(Z_t)\mathrm{d}t + \eta(Z_t)\mathrm{d}B_t + \int_{\mathbb{U}_0} \zeta(Z_{t-}, u)\tilde{N}(\mathrm{d}t, \mathrm{d}u), & t \in (0, T], \\ Z_0 = \gamma, & \end{array}\right. \tag{1}$$

where the coefficients $\xi : \mathbb{R}^d \mapsto \mathbb{R}^d$, $\eta : \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$ and $\zeta : \mathbb{R}^d \times \mathbb{U}_0 \mapsto \mathbb{R}^d$ are all Borel measurable, and the initial γ is an $\mathscr{F}_0$-measurable random variable with $\mathbb{E}|\gamma|^2 < \infty$.

Assumption 1.

(i) There exists a constant $L_1 > 0$ such that for any $z, z_1, z_2 \in \mathbb{R}^d$ and $u, u_1, u_2 \in \mathbb{U}_0$,

$$|\xi(z_1) - \xi(z_2)| + \|\eta(z_1) - \eta(z_2)\| \leqslant L_1|z_1 - z_2|,$$

and

$$|\zeta(z_1, u) - \zeta(z_2, u)| \leqslant L_1|z_1 - z_2|\|u\|_{\mathbb{U}} \text{ and } |\zeta(z, u_1) - \zeta(z, u_2)| \leqslant L_1(1 + |z|)\|u_1 - u_2\|_{\mathbb{U}}$$

where $\|\cdot\|$ stands for the Hilbert-Schmidt norm of a matrix.

(ii) There exists a constant $L_2 > 0$ such that for any $z \in \mathbb{R}^d$

$$|\xi(z)|^2 + \|\eta(z)\|^2 + \int_{\mathbb{U}_0} |\zeta(z,u)|^2 \nu(\mathrm{d}u) \leqslant L_2(1+|z|^2).$$

Under **Assumption 1**, it is known that there exists a unique strong solution $\{Z_t\}_{t\in[0,T]}$ to Eq.(1) which is a Markov process with càdlàg paths, see, e.g., [1, Theorem 6.2.3 and Theorem 6.4.5]. Next, we define the support for a random variable and then study the support of Z_t for $t \in [0,T]$.

Definition 2.1. *Let $\mathbb{V}$ be a metric space with the metric ρ. The support of a $\mathbb{V}$-valued random variable V is defined as*

$$\mathrm{supp}(V) := \{z \in \mathbb{V} | (\mathbb{P}\circ V^{-1})(B(z,r)) > 0, \ \textit{for all } r > 0\},$$

where $B(z,r) := \{y \in \mathbb{V} | \rho(z,y) < r\}$.

Next, we use $\nu^{\mathbb{U}_0}$ to denote the restriction of ν to $\mathbb{U}_0$ and we let $\mathrm{supp}(\nu^{\mathbb{U}_0})$ stand for the support of $\nu^{\mathbb{U}_0}$, that is, $\mathrm{supp}(\nu^{\mathbb{U}_0}) \in \mathscr{U} \cap \mathbb{U}_0$ and $\nu\Big((\mathrm{supp}(\nu^{\mathbb{U}_0}))^c\Big) = 0$.

Assumption 2. For any $z \in \mathbb{R}^d$ and for any open ball $B \subset \mathbb{R}^d$, there exists a point $u \in \mathrm{supp}(\nu^{\mathbb{U}_0})$ such that $\zeta(z,u) \in B$.

We now state and prove the first main result of this section.

Theorem 2.2. *Suppose that* $\mathrm{supp}(\gamma) = \mathbb{R}^d$ *and that ξ, η, ζ satisfy* **Assumption 1** *and* **Assumption 2**. *Then* $\mathrm{supp}(Z_t) = \mathbb{R}^d$ *for $t \in [0,T]$.*

Proof. By Definition 2.1, it suffices to show that for any $t \in [0,T], a \in \mathbb{R}^d$ and $r > 0$

$$\mathbb{P}(Z_t \in B(a,r)) > 0.$$

To this end, we arbitrarily fix t, a, r and shall prove the above inequality with the help of the following two auxiliary equations. We divide our proof into four steps.

Step 1. For any subset $U \subset \mathbb{U}_0$ with $U \in \mathscr{U}, \nu(U) < \infty$ and $L_1\|u\|_{\mathbb{U}} < 1$ for any $u \in U$, we introduce the first auxiliary equation

$$Z_t^U = \gamma + \int_0^t \xi(Z_s^U)\mathrm{d}s + \int_0^t \int_U \zeta(Z_{s-}^U, u)\tilde{N}(\mathrm{d}s, \mathrm{d}u). \tag{2}$$

Note that there is no continuous diffusion term in the above equation. This is because we allow that Eq.(1) is degenerate so that the continuous diffusion term is not needed. Then, under **Assumption 1**, it follows from [1, Theorem 6.2.3] that Eq.(2) has a unique solution, which is denoted as Z^U. Furhter, by Lemma 2.3 below we obtain that for a.s. $\omega \in \Omega$ and any $\varepsilon \in (0,r)$, there exists a $t_0 > 0$ such that

$$\sup_{0\leqslant s\leqslant t_0} |Z_s - Z_s^U| < \varepsilon/2. \tag{3}$$

We fix such t_0 and ε.

Step 2. Let $\{s_i\}$ be a positive sequence such that $s_i \uparrow \infty$, and $\{u_i\}$ be a sequence in the support of ν^U, the restriction of ν to U. And let $\mathscr{G}^U$ be the collection of the above sequence pairs $\{s_i\}, \{u_i\}$. For any $g \in \mathscr{G}^U$, we introduce the second auxiliary equation

$$Z_t^{g,U} = \gamma + \int_0^t \left[\xi(Z_s^{g,U}) - \int_U \zeta(Z_s^{g,U}, u)\nu(\mathrm{d}u)\right] \mathrm{d}s + \sum_{i:s_i \leqslant t} \zeta(Z_{s_i-}^{g,U}, u_i). \tag{4}$$

Then, under **Assumption 1**, it holds that Eq.(4) has a unique solution, denoted as $Z^{g,U}$. Next, by **Assumption 2**, we know that for the open ball $B(a, r-\varepsilon)$, there exist $s_1 > 0, s_i > t_0, i = 2, 3, \dots$ and $u_1 \in \mathrm{supp}(\nu^U)$ such that

$$|Z_{t_0}^{g,U} - a| < r - \varepsilon. \tag{5}$$

This may be possible if s_1 is taken enough small so that $s_1 \leqslant t_0$ and $Z_{t_0}^{g,U} \in B(a, r-\varepsilon)$. Fix such $\{s_i\}$ and u_1.

Step 3. Let us now study the relationship between Z^U and $Z^{g,U}$. Set

$$\chi_t := \int_0^t \int_U u\tilde{N}(\mathrm{d}s, \mathrm{d}u), \quad \Delta\chi_t := \chi_t - \chi_{t-}, \quad D := \{t \in [0, \infty), \Delta\chi_t \in U\},$$

and then D is a discrete set in $[0, \infty)$ a.s.. Let $0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$ be the enumeration of all elements in D. Besides, we take any $u_2 \in \mathrm{supp}(\nu^U)$. For any $\varepsilon' > 0$, set

$$\begin{aligned} A_1 &:= \{\omega \in \Omega : 0 < s_1 - \tau_1 < \varepsilon', \|u_1 - \Delta\chi_{\tau_1}\|_{\mathbb{U}} < \varepsilon'\}, \\ A_2 &:= \{\omega \in \Omega : 0 < s_2 - s_1 - (\tau_2 - \tau_1) < \varepsilon', \|u_2 - \Delta\chi_{\tau_2}\|_{\mathbb{U}} < \varepsilon'\}, \end{aligned}$$

and then it follows from independence for increments of $N(\mathrm{d}t, \mathrm{d}u)$ that $\mathbb{P}(A_1 \cap A_2) > 0$. Thus, by Lemma 2.4 below it holds that for the above ε, there exists an $\varepsilon' > 0$ such that

$$\sup_{0 \leqslant s \leqslant t_0} |Z_s^U - Z_s^{g,U}| < \varepsilon/2, \tag{6}$$

on $A_1 \cap A_2$.

Step 4. Combining (3) (5) with (6), we obtain that

$$|Z_s - a| \leqslant |Z_s - Z_s^U| + |Z_s^U - Z_s^{g,U}| + |Z_s^{g,U} - a| < \varepsilon/2 + \varepsilon/2 + r - \varepsilon = r, \quad s \in (0, t_0],$$

on $A_1 \cap A_2$. Thus $\mathbb{P}(|Z_s - a| < r) > 0$ for $s \in (0, t_0]$. If $t \leqslant t_0$, the proof is over; if $t > t_0$, by the Markov property, we still can obtain $\mathbb{P}(|Z_t - a| < r) > 0$. The proof is complete. □

Lemma 2.3. *Under* **Assumption 1**, *it holds that*

$$\lim_{t \to 0} \sup_{0 \leqslant s \leqslant t} |Z_s - Z_s^U| = 0, \quad \mathbb{P} - a.s..$$

Proof. Firstly, we compute $Z_t - Z_t^U$ for $t \in [0, T]$. By (1) and (2), it holds that

$$\begin{aligned} Z_t - Z_t^U &= \int_0^t \left(\xi(Z_s) - \xi(Z_s^U)\right) \mathrm{d}s + \int_0^t \eta(Z_s) \mathrm{d}B_s \\ &\quad + \int_0^t \int_U \left(\zeta(Z_{s-}, u) - \zeta(Z_{s-}^U, u)\right) \tilde{N}(\mathrm{d}s, \mathrm{d}u) \\ &\quad + \int_0^t \int_{\mathbb{U}_0 \setminus U} \zeta(Z_{s-}, u)\tilde{N}(\mathrm{d}s, \mathrm{d}u). \end{aligned}$$

By the Burkholder-Davis-Gundy inequality and the Hölder inequality, one can have that

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}|Z_s-Z_s^U|^2\right) &\leqslant 4t\mathbb{E}\int_0^t\left|\xi(Z_s)-\xi(Z_s^U)\right|^2\mathrm{d}s+16\mathbb{E}\int_0^t|\eta(Z_s)|^2\mathrm{d}s\\ &+16\mathbb{E}\int_0^t\int_U\left|\zeta(Z_{s-},u)-\zeta(Z_{s-}^U,u)\right|^2\nu(\mathrm{d}u)\mathrm{d}s\\ &+16\mathbb{E}\int_0^t\int_{\mathbb{U}_0\setminus U}|\zeta(Z_{s-},u)|^2\nu(\mathrm{d}u)\mathrm{d}s.\end{aligned}$$

Moreover, by **Assumption 1**, we obtain that

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}|Z_s-Z_s^U|^2\right) &\leqslant 4L_1^2\Big(t+4\int_U\|u\|_{\mathbb{U}}^2\nu(\mathrm{d}u)\Big)\int_0^t\mathbb{E}\left(\sup_{0\leqslant s\leqslant r}|Z_s-Z_s^U|^2\right)\mathrm{d}r\\ &+16\mathbb{E}\int_0^t L_2(1+|Z_s|^2)\mathrm{d}s. \qquad (7)\end{aligned}$$

To estimate the last term in (7), we look back to Eq.(1). By the similar deduction to above, one can get that

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}|Z_s|^2\right) &\leqslant 4\mathbb{E}|\gamma|^2+4t\mathbb{E}\int_0^t|\xi(Z_s)|^2\mathrm{d}s+16\mathbb{E}\int_0^t|\eta(Z_s)|^2\mathrm{d}s\\ &+16\mathbb{E}\int_0^t\int_{\mathbb{U}_0}|\zeta(Z_{s-},u)|^2\nu(\mathrm{d}u)\mathrm{d}s,\end{aligned}$$

and furthermore by **Assumption 1**

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}|Z_s|^2\right) &\leqslant 4\mathbb{E}|\gamma|^2+4(t+4)L_2\int_0^t\mathbb{E}\left(1+|Z_s|^2\right)\mathrm{d}s\\ &\leqslant 4\mathbb{E}|\gamma|^2+4(t+4)tL_2+4(t+4)L_2\int_0^t\mathbb{E}\left(\sup_{0\leqslant s\leqslant r}|Z_s|^2\right)\mathrm{d}r.\end{aligned}$$

Thus, by the Gronwall inequality we have that

$$\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}|Z_s|^2\right)\leqslant C, \qquad (8)$$

where the constant $C>0$ depends on $\mathbb{E}|\gamma|^2, T, L_2$.

Next, combining (7) with (8), we get that

$$\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}|Z_s-Z_s^U|^2\right)\leq 16L_2(C+1)t+4L_1^2\Big(T+4\int_U\|u\|_{\mathbb{U}}^2\nu(\mathrm{d}u)\Big)\int_0^t\mathbb{E}\left(\sup_{0\leqslant s\leqslant r}|Z_s-Z_s^U|^2\right)\mathrm{d}r.$$

Based on the Gronwall inequality, it holds that

$$\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}|Z_s-Z_s^U|^2\right)\leqslant C(e^{Ct}-1),$$

where the constant $C>0$ depends on $\mathbb{E}|\gamma|^2, T, L_1, L_2$. Thus, we have that

$$\lim_{t\to 0}\sup_{0\leqslant s\leqslant t}|Z_s-Z_s^U|=0,\quad \mathbb{P}-a.s..$$

The proof is complete. □

Lemma 2.4. *Under* **Assumption 1**, *for the above ε, there exist a constant $C > 0$ and an $\varepsilon' > 0$ such that on $A_1 \cap A_2$*

$$\sup_{0\leqslant s\leqslant t_0} |Z_s^U - Z_s^{g,U}| < C\varepsilon'/2$$

and further

$$\sup_{0\leqslant s\leqslant t_0} |Z_s^U - Z_s^{g,U}| < \varepsilon/2.$$

Proof. By (2) and (4), it holds that for $0 \leqslant t \leqslant t_0$

$$\begin{aligned} Z_t^U - Z_t^{g,U} &= \int_0^t \left[\xi(Z_s^U) - \xi(Z_s^{g,U})\right] \mathrm{d}s - \int_0^t \int_U \left[\zeta(Z_s^U, u) - \zeta(Z_s^{g,U}, u)\right] \nu(\mathrm{d}u)\mathrm{d}s \\ &\quad + \zeta(Z_{\tau_1-}^U, \Delta\chi_{\tau_1}) - \zeta(Z_{s_1-}^{g,U}, u_1), \end{aligned}$$

and

$$\begin{aligned} |Z_t^U - Z_t^{g,U}| &\leqslant \int_0^t \left|\xi(Z_s^U) - \xi(Z_s^{g,U})\right| \mathrm{d}s + \int_0^t \int_U \left|\zeta(Z_s^U, u) - \zeta(Z_s^{g,U}, u)\right| \nu(\mathrm{d}u)\mathrm{d}s \\ &\quad + |\zeta(Z_{\tau_1-}^U, \Delta\chi_{\tau_1}) - \zeta(Z_{s_1-}^U, \Delta\chi_{\tau_1})| + |\zeta(Z_{s_1-}^U, \Delta\chi_{\tau_1}) - \zeta(Z_{s_1-}^U, u_1)| \\ &\quad + |\zeta(Z_{s_1-}^U, u_1) - \zeta(Z_{s_1-}^{g,U}, u_1)|. \end{aligned}$$

Moreover, **Assumption 1** admits us to obtain the following

$$\begin{aligned} \sup_{0\leqslant s\leqslant t_0} |Z_s^U - Z_s^{g,U}| &\leqslant L_1 \int_0^{t_0} \sup_{0\leqslant s\leqslant r} |Z_s^U - Z_s^{g,U}|\mathrm{d}r \\ &\quad + L_1 \left(\int_U \|u\|_{\mathbb{U}} \nu(\mathrm{d}u)\right) \int_0^{t_0} \sup_{0\leqslant s\leqslant r} |Z_s^U - Z_s^{g,U}|\mathrm{d}r \\ &\quad + L_1 \|\Delta\chi_{\tau_1}\|_{\mathbb{U}} |Z_{\tau_1-}^U - Z_{s_1-}^U| + L_1(1 + |Z_{s_1-}^U|)\|\Delta\chi_{\tau_1} - u_1\|_{\mathbb{U}} \\ &\quad + L_1 \|u_1\|_{\mathbb{U}} |Z_{s_1-}^U - Z_{s_1-}^{g,U}| \\ &\leqslant L_1 \left(1 + \int_U \|u\|_{\mathbb{U}} \nu(\mathrm{d}u)\right) \int_0^{t_0} \sup_{0\leqslant s\leqslant r} |Z_s^U - Z_s^{g,U}|\mathrm{d}r \\ &\quad + L_1 \|\Delta\chi_{\tau_1}\|_{\mathbb{U}} |Z_{\tau_1-}^U - Z_{s_1-}^U| + L_1(1 + |Z_{s_1-}^U|)\|\Delta\chi_{\tau_1} - u_1\|_{\mathbb{U}} \\ &\quad + L_1 \|u_1\|_{\mathbb{U}} \sup_{0\leqslant s\leqslant t_0} |Z_s^U - Z_s^{g,U}|. \end{aligned}$$

Further by the definition of A_1, A_2 and the Gronwall inequality, we know that there exists a constant $C > 0$ such that

$$|Z_{s_1-}^U| < C, \quad |Z_{\tau_1-}^U - Z_{s_1-}^U| < C\varepsilon'.$$

Thus, on A_1 one have

$$\begin{aligned} \sup_{0\leqslant s\leqslant t_0} |Z_s^U - Z_s^{g,U}| &\leqslant \frac{L_1\left(1 + \int_U \|u\|_{\mathbb{U}} \nu(\mathrm{d}u)\right)}{1 - L_1\|u_1\|_{\mathbb{U}}} \int_0^{t_0} \sup_{0\leqslant s\leqslant r} |Z_s^U - Z_s^{g,U}|\mathrm{d}r \\ &\quad + \frac{L_1(\varepsilon' + \|u_1\|_{\mathbb{U}})C}{1 - L_1\|u_1\|_{\mathbb{U}}}\varepsilon' + \frac{L_1(1 + C)}{1 - L_1\|u_1\|_{\mathbb{U}}}\varepsilon'. \end{aligned}$$

The Gronwall inequality admits us to obtain that

$$\sup_{0\leqslant s\leqslant t_0} |Z_s^U - Z_s^{g,U}| \leqslant C\varepsilon'.$$

Taking $C\varepsilon' = \varepsilon/2$, one can attain that

$$\sup_{0\leqslant s\leqslant t_0} |Z_s^U - Z_s^{g,U}| < \varepsilon/2,$$

on $A_1 \cap A_2$. The proof is complete. □

Next, we would like to strengthen the conditions in **Assumptions 1** and **Assumptions 2** and derive a further result. Let $\mathbb{U} \in \mathscr{B}(\mathbb{R}^k)$ for $k \in \mathbb{N}$.

Assumption 3.

- **(i)** The distribution of γ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.
- **(ii)** The measure ν is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^k$, and ρ^ν denotes its density.
- **(iii)** ξ and η are 3-times differentiable with bounded derivatives of all order between 1 and 3.
- **(iv)** For any $u \in \mathbb{U}_0$, $\zeta(\cdot, u)$ is 3-times differentiable, and

$$\begin{aligned} \zeta(0,\cdot) &\in \bigcap_{2\leqslant q<\infty} L^q(\mathbb{U}_0,\nu) \\ \sup_x |\partial_x^r \zeta(x,\cdot)| &\in \bigcap_{2\leqslant q<\infty} L^q(\mathbb{U}_0,\nu), \quad 1\leqslant r\leqslant 3, \end{aligned}$$

where $\partial_x^r\zeta(x,\cdot)$ denotes the r-th order partial derivative of $\zeta(x,\cdot)$ with respect to x.

Under **Assumption 3**, by [2, Theorem 2-14, p.11], it holds that Eq.(1) has a unique solution, which is denoted by (Z_t^γ), and the distribution of Z_t^γ possesses a density $\rho_t(\gamma,\cdot)$. Our second main result of this section states as follows.

Theorem 2.5. *Suppose that* **Assumption 3** *is satisfied. Then* $\operatorname{supp}(Z_t) = \mathbb{R}^d$ *for* $t \in [0,T]$.

Proof. By [2, Theorem 2-14, p.11], we know that the distribution of Z_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ and clearly, the support of the Lebesgue measure is the whole $\mathbb{R}^d$, the support for the distribution of Z_t is also $\mathbb{R}^d$. The proof is complete. □

Remark 2.6. *Note that if* $\gamma = z \in \mathbb{R}^d$ *is determinitstic,* $\rho^\nu > 0$*,* $\xi \neq 0$*,* $\zeta \neq 0$ *and* $\eta = 0$*, Fournier proved* $\{y \in \mathbb{R}^d : \rho_t(z,y) > 0\} = \mathbb{R}^d$ *for* $t \in (0,T]$ *in* [3, Theorem 2.3].

3. Application to stochastic evolution equations with jumps

In the section, we will apply Theorem 2.2 to a problem on stochastic evolution equations with jumps on (separable) Hilbert spaces.

We start with some notions. Let $\mathbb{H}$ be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ and the norm $\|\cdot\|_{\mathbb{H}}$. Let $L(\mathbb{H})$ be the set of all bounded linear operators $L : \mathbb{H} \to \mathbb{H}$ and $L_{HS}(\mathbb{H})$ be the collection of all Hilbert-Schmidt operators $L : \mathbb{H} \to \mathbb{H}$ equipped with the Hilbert-Schmidt norm $\|\cdot\|_{HS}$.

Let A be a linear, unbounded, negative definite and self-adjoint operator on $\mathbb{H}$ and $D(A)$ be the domain of the operator A. Let $\{e^{tA}\}_{t\geq 0}$ be the contraction C_0-semigroup generated by A. Let $L_A(\mathbb{H})$ be the collection of all densely defined closed linear operators $(L, D(L))$ on $\mathbb{H}$ so that $e^{tA}L$ can extend uniquely to a Hilbert-Schmidt operator still denoted by $e^{tA}L$ for any $t>0$. And then $L_A(\mathbb{H})$, endowed with the σ-algebra induced by $\{L\to\langle e^{tA}Lx,y\rangle_{\mathbb{H}}\mid t>0,x,y\in\mathbb{H}\}$, becomes a measurable space.

Let $\{\beta^i, i\in\mathbb{N}\}$ be a family of mutually independent one-dimensional Brownian motions on $(\Omega,\mathscr{F},\mathbb{P};(\mathscr{F}_t)_{t\geqslant 0})$. So, we can construct a cylindrical Brownian motion on $\mathbb{H}$ by

$$W_t:=\sum_{i=1}^{\infty}\beta_t^i e_i,\qquad t\in[0,\infty),$$

where $\{e_i, i\in\mathbb{N}\}$ is a complete orthonormal basis for $\mathbb{H}$ which will be specified later. It can be shown that the covariance operator of the cylindrical Brownian motion W is the identity operator I on $\mathbb{H}$. It is worthwhile to mention that W is not a process on $\mathbb{H}$. However, W can be realized as a continuous process on an enlarged Hilbert space $\tilde{\mathbb{H}}$, the completion of $\mathbb{H}$ under the inner product

$$\langle x,y\rangle_{\tilde{\mathbb{H}}}:=\sum_{i=1}^{\infty}2^{-i}\langle x,e_i\rangle_{\mathbb{H}}\langle y,e_i\rangle_{\mathbb{H}},\quad x,y\in\mathbb{H}.$$

Next, we introduce a type of jump measures. Let $\lambda:\mathbb{U}\to(0,1)$ be a measurable function. Then, by Theorem I.8.1 of [4], we can construct an integer-valued random measure on $[0,\infty)\times\mathbb{U}$

$$N_\lambda:\mathscr{B}([0,\infty))\times\mathscr{U}\times\Omega\to\mathbb{N}_0:=\mathbb{N}\cup\{0\}$$

with the predictable compensator $\lambda(u)\mathrm{d}t\nu(\mathrm{d}u)$:

$$\mathbb{E}N_\lambda(\mathrm{d}t,\mathrm{d}u,\cdot)=\lambda(u)\mathrm{d}t\nu(\mathrm{d}u).$$

Set

$$\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u):=N_\lambda(\mathrm{d}t,\mathrm{d}u)-\lambda(u)\mathrm{d}t\nu(\mathrm{d}u),$$

and then $\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u)$ is the associated compensated martingale measure of $N_\lambda(\mathrm{d}t,\mathrm{d}u)$. Moreover, we assume that $W_t, N_\lambda(\mathrm{d}t,\mathrm{d}u)$ are mutually independent.

Now consider the following stochastic evolution equation with jumps on $\mathbb{H}$

$$\begin{cases} dX_t=\{AX_t+b(X_t)\}dt+\sigma(X_t)dW_t+\displaystyle\int_{\mathbb{U}_0}f(X_{t-},u)\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u),\quad 0<t\leqslant T,\\ X_0=\Gamma,\end{cases}\tag{9}$$

where $b:\mathbb{H}\to\mathbb{H}$, $\sigma:\mathbb{H}\to L_A(\mathbb{H})$ and $f:\mathbb{H}\times\mathbb{U}_0\to\mathbb{H}$ are all Borel measurable mappings, and Γ is a $\mathscr{F}_0$-measurable $\mathbb{H}$-valued random variable with $\mathbb{E}|\Gamma|^2<\infty$ and $\mathrm{supp}(\Gamma)=\mathbb{H}$. Set $\|x\|_{\mathbb{H}}=\infty, x\notin\mathbb{H}$. For b,σ,f, we make the following assumption.

Assumption 4.

(i) There exists an integrable function $L_b:(0,T]\to(0,\infty)$ such that

$$\|e^{sA}(b(x)-b(y))\|_{\mathbb{H}}^2\leqslant L_b(s)\|x-y\|_{\mathbb{H}}^2,\quad s\in(0,T],\quad x,y\in\mathbb{H},$$

and

$$\int_0^T\|e^{sA}b(0)\|_{\mathbb{H}}^2\mathrm{d}s<\infty.$$

(ii) There exists an integrable function $L_\sigma:(0,T]\to(0,\infty)$ such that for $\forall s\in(0,T]$ and $\forall x,y\in\mathbb{H}$

$$\|e^{sA}\left(\sigma(x)-\sigma(y)\right)\|_{HS}^2\leqslant L_\sigma(s)\|x-y\|_{\mathbb{H}}^2$$

and

$$\int_0^T \|e^{sA}\sigma(0)\|_{HS}^2 \mathrm{d}s < \infty.$$

(iii) There exists an integrable function $L_f : [0,T] \to (0,\infty)$ such that

$$\|e^{sA}(f(x,u) - f(y,u))\|_{\mathbb{H}}^2 \leqslant L_f(s)\|u\|_{\mathbb{U}}^2\|x-y\|_{\mathbb{H}}^2, \quad s \in [0,T], u \in \mathbb{U}_0, x, y \in \mathbb{H},$$
$$\|e^{sA}(f(x,u_1) - f(x,u_2))\|_{\mathbb{H}}^2 \leqslant L_f(s)(1+\|x\|_{\mathbb{H}})^2\|u_1-u_2\|_{\mathbb{U}}^2, \quad u_1, u_2 \in \mathbb{U}_0,$$

and

$$\int_{\mathbb{U}_0} \|e^{sA}f(x,u)\|_{\mathbb{H}}^2 \lambda(u)\nu(\mathrm{d}u) \leqslant L_f(s)(1+\|x\|_{\mathbb{H}})^2.$$

Under **Assumption 4**, [10, Theorem 3.2] ensures that Eq.(9) has a unique mild solution, denoted by X_t.

Assumption 5. The operator $-A$ has the following eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \ldots$$

counting multiplicities.

The complete orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ of $\mathbb{H}$ is taken as the eigen-basis of $-A$ throughout the rest of the paper. Let $\mathbb{H}_n$ be the space spanned by $e_1, \cdots, e_n$ for $n \in \mathbb{N}$. Define

$$\pi_n : \mathbb{H} \to \mathbb{H}_n, \qquad \pi_n x := \sum_{i=1}^n \langle x, e_i\rangle_{\mathbb{H}} e_i, \quad x \in \mathbb{H},$$

and then π_n is the orthogonal projection operator from $\mathbb{H}$ to $\mathbb{H}_n$.

Assumption 6. For any $n \in \mathbb{N}, z \in \mathbb{H}_n$ and any open ball $B \subset \mathbb{H}_n$, there exists a point $u \in \mathrm{supp}(\nu^{\mathbb{U}_0})$ such that $\pi_n f(z,u) \in B$.

We have the following support result under the above assumptions, while its proof is similar to that of [10, Lemma 4.2], so we omit it here.

Lemma 3.1. *Under* **Assumptions 4-6***, it holds that* $\mathrm{supp}(X_t) = \mathbb{H}$ *for* $t \in [0,T]$.

In order to present our main result in this section, we need to introduce the following

Assumption 7.

(i) There exists a Borel measurable mapping $\varrho : \mathbb{H} \to \mathbb{H}$ such that

$$b(x) = \sigma(x)\varrho(x),$$

(ii)

$$\mathbb{E}\Big[\exp\Big\{\frac{1}{2}\int_0^T \|\varrho(X_s)\|_{\mathbb{H}}^2\,\mathrm{d}s\Big\}\Big] < \infty,$$

(iii)

$$\int_0^T \int_{\mathbb{U}_0} \left(\frac{1-\lambda(u)}{\lambda(u)}\right)^2 \lambda(u)\nu(\mathrm{d}u)\mathrm{d}s < \infty.$$

Taking

$$\begin{aligned}\Lambda_t : \;&=\; \exp\bigg\{-\int_0^t \langle \varrho(X_s), \mathrm{d}W_s\rangle_{\tilde{\mathbb{H}}} - \frac{1}{2}\int_0^t \|\varrho(X_s)\|^2_{\mathbb{H}}\,\mathrm{d}s \\ &\qquad -\int_0^t\int_{\mathbb{U}_0} \log\lambda(u) N_\lambda(\mathrm{d}s,\mathrm{d}u) - \int_0^t\int_{\mathbb{U}_0}(1-\lambda(u))\nu(\mathrm{d}u)\mathrm{d}s\bigg\},\end{aligned}$$

by [7, Theorem 6], we know that Λ_t is an exponential martingale under **Assumption 7. (ii)-(iii)**. Define a new probability measure $\hat{\mathbb{P}}$ by

$$\frac{\mathrm{d}\hat{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \Lambda_T.$$

Thus, by [10, Theorem 2.1], we obtain that on the new filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \hat{\mathbb{P}})$, $\hat{W}_t := W_t + \int_0^t \varrho(X_s)\mathrm{d}s$ is a cylindrical Brownian motion, and the predictable compensator of $N_\lambda(\mathrm{d}t,\mathrm{d}u)$ is $\mathrm{d}t\nu(\mathrm{d}u)$.

Now, we state the main result of this section.

Theorem 3.2. *Suppose that* **Assumptions 4-7** *are satisfied. Let $v : \mathbb{H} \to \mathbb{R}$ be a C^2 scalar function such that $[\nabla v(x)] \in D(A)$ for any $x \in \mathbb{H}$ and $\|A\nabla v(\cdot)\|_{\mathbb{H}}$ is locally bounded, and $\|A\nabla v(\cdot)\|_{\mathbb{H}} : \mathbb{H} \to [0,\infty)$ is continuous. Then the Girsanov density Λ_t for Eq.(9) has the following path-independent property:*

$$\Lambda_t = \exp\{v(\Gamma) - v(X_t)\}, \quad t \in [0,T],$$

if and only if

$$\begin{aligned}\varrho(x) &= (\sigma^*\nabla v)(x), && x \in \mathbb{H}, && (10)\\ \lambda(u) &= \exp\{v(x+f(x,u)) - v(x)\}, && (x,u) \in \mathbb{H}\times\mathbb{U}_0, && (11)\end{aligned}$$

and v satisfies the following (infinite-dimensional) integro-differential equation,

$$\begin{aligned}&\frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](x) + \frac{1}{2}\|\varrho(x)\|^2_{\mathbb{H}} + \langle x, A\nabla v(x)\rangle_{\mathbb{H}} \\ &+ \int_{\mathbb{U}_0}\Big[e^{v(x+f(x,u))-v(x)} - 1 - \langle f(x,u), \nabla v(x)\rangle_{\tilde{\mathbb{H}}} e^{v(x+f(x,u))-v(x)}\Big]\nu(\mathrm{d}u) = 0, \quad (12)\end{aligned}$$

where $\sigma^*(x)$ stands for the conjugate of $\sigma(x)$, ∇ and ∇^2 stand for the first and second Fréchet differential operators, respectively.

Since the proof of the above theorem is similar to [10, Theorem 4.3], we omit it here.

Remark 3.3. *Comparing Theorem 3.2 with* [10, Theorem 4.3]*, one can find that, the $\sigma(x)$ here might be degenerate or even could be zero.*

The above theorem gives a necessary and sufficient condition, and hence it is a characterisation of path-independence for the density Λ_t of the Girsanov transformation for a stochastic evolution equation with jumps in terms of an infinite-dimensional integro-differential equation. Namely, we establish a bridge from Eq.(9) to an infinite-dimensional integro-differential equation.

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References

[1] D. Applebaum: *Lévy Processes and Stochastic Calculus.* Second Edition, Cambridge Univ. Press, Cambridge, 2009.

[2] K. Bichteler, J. B. Gravereaux and J. Jacod: *Malliavin Calculus for Processes with Jumps,* Stochastic Monographs Volume 2, Gordon and Breach Science Publishers, 1987.

[3] N. Fournier: Strict positivity of the density for simple jump processes using the tools of support theorems. Application to the Kac equation without cutoff, *Ann. Probab.*, 30(2002)135-170.

[4] N. Ikeda and S. Watanabe: *Stochastic Differential Equations and Diffusion Processes*, 2nd ed., North-Holland/Kodanska, Amsterdam/Tokyo, 1989.

[5] J. Jacod and A.N. Shiryaev: *Limit Theorems for Stochastic Processes.* Springer-Verlag, Berlin, 1987.

[6] K. R. Parthasarathy: *Probability measures on metric spaces.* AMS Chelsea Publishing, 2005.

[7] P. E. Protter and K. Shimbo: No arbitrage and general semimartingales. *Markov Processes and related Topics: A Festschrift for Thomas G. Kurtz*, 4(2008)267-283.

[8] H. J. Qiao: Exponential ergodicity for SDEs with jumps and non-Lipschitz coefficients, *J. Theor. Probab.*, 27(2014)137-152.

[9] H. J. Qiao and J.-L. Wu: Characterising the path-independence of the Girsanov transformation for non-Lipschnitz SDEs with jumps, *Statistics and Probability Letters,* 119(2016)326-333.

[10] H. J. Qiao and J.-L. Wu: On the path-independence of the Girsanov transformation for stochastic evolution equations with jumps in Hilbert spaces, *Discrete and Continuous Dynamical Systems-B,* 24(2019)1449-1467.

[11] T. Simon: Support theorem for jump processes. *Stochastic Processes and their Applications,* 89(2000)1-30.

[12] B. Wu and J.-L. Wu: Characterising the path-independent property of the Girsanov density for degenerated stochastic differential equations, *Statistics and Probability Letters,* 133 (2018) 71-79.