First jump time in simulation of sampling trajectories of affine jump-diffusions driven by α-stable white noise

Jiao Song[1,2], Jiang-Lun Wu[2] and Fangzhou Huang[3]

[1] *Research and Evaluation Division, Public Health Wales, Cardiff, UK*

[2] *Department of Mathematics, Computational Foundry, Swansea University, Swansea, UK*

[3] *Accounting and Finance Department, School of Management, Swansea University, Swansea, UK*

Email: jiao.song@wales.nhs.uk; j.l.wu@swansea.ac.uk

Abstract

The aim of this paper is twofold. Firstly, we derive an explicit expression of the (theoretical) solutions of stochastic differential equations with affine coefficients driven by α-stable white noise. This is done by means of Itô formula. Secondly, we develop a detection algorithm for the first jump time in simulation of sampling trajectories which are described by the solutions. The algorithm is carried out through a multivariate Lagrange interpolation approach. To this end, we utilise a computer simulation algorithm in MATLAB to visualise the sampling trajectories of the jump-diffusions for two combinations of parameters arising in the modelling structure of stochastic differential equations with affine coefficients.

Key words: Stochastic differential equations, affine coefficients, α-stable processes, simulation, multivariate Lagrange interpolation.
AMS Subject Classification(2010): 65C99; 68U20; 60E07; 60G17

1 Introduction

Since the pioneer work (23), (22), and (6), affine processes are by now widely use in financial modeling. Under the assumption of an arbitrage-free market, affine jump-diffusions and their alike processes were proposed to model factors in various dynamic asset pricing models. The advantages of affine jump-diffusions over classic jump diffusion are well documented and attributed to their flexibility and avoidance of inducing additional volatility (21). With further development from pioneers (23) and (22), there is substantial literature on the application of affine jump-diffusions, particularly in three aspects: credit sensitive financial security modelling, asset pricing model estimation and option pricing. In more recent research, Wu and Yang utilised affine jump-diffusions to model the systematic and idiosyncratic risk in Collateralised Debt Obligation market (20). Jarrow et al. explored the mispricing in Credit Default Swap spread in the U.S. market and proposed a trading strategy to capture the abnormal return by using a affine jump-diffusion model (24). Furthermore, Hain et al. adopted affine jump-diffusion type model to detect the higher moment risk in crude oil market (25). Campbell et al. introduced the affine jump-diffusions to vector autoregressive model to rescue the classic Capital Asset Price Model. Moreover, Bardgett et al. (2019) applied a flexible affine model to estimate the risk prima of Chicago Board

Options Exchange Volatility Index option prices (26). Barletta et al. proposed an affine jump-diffusion based model to hedge European options (27). On the other hand, with the passage of time, modeling time evolution uncertainty by stochastic differential equations (SDEs) appears in many diverse areas such as studies of dynamical particle systems in physics, biological and medical studies, engineering and industrial studies, as well as most recently micro analytic studies in mathematical finance and social sciences. Modeling studies with different features of Lévy processes, like distributions with asymmetric, and/or heavy-tail property, and/or infinite moments, etc. have been appeared increasingly in the literature, see e.g. Barndorff-Nielsen (1997), Samorodnitsky and Taqqu (1994), Giacometti et al. (2007), Zopounidis and Pardalos (2013), Dror, L'Ecuyer, and Szidarovszky (2002) and Fiche et al. (2013), Campbell, Lo, and MacKinlay 1997, Mandelbrot (1960), Zolotarev (1986, Leland et al. (1993), Shlesinger, Zaslavsky, and Frisch (1995). In particular, we refer the reader to (Du, Wu, and Yang 2010) for interesting discussions of utilizing a-stable distributions to model the mechanism of Collateralised Debt Obligations (CDOs) in mathematical finance. As such, SDEs with affine coefficients driven by Lévy processes become a good modeling instrument for both theoretical and practical investigations.

On the other hand, from mathematics historical aspect, applicable probability distributions with infinite moments are encountered in the study of critical phenomena. For instance, at the critical point one finds clusters of all sizes while the mean of the distribution of clusters sizes diverges. Thus, analysis from the earlier intuition about moments had to be shifted to newer notions involving calculations of exponents, like e.g. Lyapunov, spectral, fractal etc., and topics such as strange kinetics and strange attractors have to be investigated. Although Lévy's ideas and algebra of random variables with infinite moments appeared in the 1920s and the 1930s (cf. Lévy 1925, 1937), it is only from the 1990s that the greatness of Lévy's theory became much more appreciated as a foundation for probabilistic aspects of chaotic dynamics with high entropy in statistical analysis in mathematical modeling (cf. Samorodnitsky and Taqqu 1994; Shlesinger, Zaslavsky, and Frisch 1995; see also Mandelbrot 1960; Zolotarev 1986). Indeed, in statistical analysis, systems with highly complexity and (non linear) chaotic dynamics became a vast area for the application of Lévy processes and the phenomenon of dynamical chaos became a real laboratory for developing generalizations of Lévy processes to create new tools to study non linear dynamics and kinetics. Following up this point, SDEs driven by Lévy type processes, in particular α-stable processes or α-stable white noise, and their influence on long time statistical asymptotic will be unavoidably encountered. Comparing to the continuity feature of trajectories of diffusions —- solutions of SDEs driven by Brownian motion or Gaussian white noise, jump-diffusions possess a feature that sampling trajectories are with jumps which seem to be more natural when volatile noise influence becomes extremely high. The scenario of sampling trajectories of jump-diffusions is that there are countable jump times and there are diffusion trajectories between any two adjacent jump times. For SDEs driven by a-stable noise, the solution trajectories enjoy certain self similar property. Therefore, from modeling aspect, to detect the first jump time for sampling trajectories in simulations is crucial, not only for the comparison of numerical errors, but also for the propose that one can treat the model as a diffusion model before that time. With self similar property, one can further infer the structure of sampling trajectories of jump-diffusions driven by $alpha$-stable noise. Due to high uncertainty, the first jump time is of course a random time (also called stopping times). Theoretically, it is not possible to get the first jump time analytically, due to infinite volumes of Lévy measures and the self similarity for α-stable processes, one can evan argue that the first jump time should be zero, see for instance Theorem 3.2 in (18) for an interesting discussion. However, as performing simulations, the first jump time does indeed appear in the sampling trajectories as shown in our work. Therefore, one could try to simulate sampling trajectories to get an algorithm towards statistical detection of the (random) first jump time. The motivation of this paper is to obtain a critical link among the parameters in the SDEs driven by a-stable white noises to develop a detection algorithm for the first jump time. Very interestingly, this can be further linked to sampling data analysis after model identifications (i.e., through certain specification of the parameters in the model equations). In the recent work (19), this problem was considered and focused on testing two simple SDEs in modeling, one class is the SDEs with linear drift coefficient and additive a-stable white noise and the solutions are called a-stable Ornstein-Uhlenbeck processes and the other class is the linear SDEs (i.e., SDEs with linear drift and diffusion coefficients or the linear SDEs with multiplicative a-stable noise) and the solutions are called a-stable geometric Lévy motion. Therein as the chaotic structure of sample trajectories of a-stable processes are varying for a in the different intervals (0, 1) and (1, 2) with $a = 1$ being critical (see, e.g., Janicki and Weron 1993), respectively, in (19) it was performed the simulations of the sample solution trajectories with the sample size of $2^9 = 512$, which yields a clear picture to identify successfully the first jump time for each simulated trajectory. Furthermore, it was used such sample data to find the critical link of the parameters arising in the coefficients of the SDEs.

In the present paper, we are concerned with the simulations of SDEs with affine coefficients driven by α-stable noise. We aim to establish detecting algorithm for the first jump time for the sampling trajectories. Due to the affine coefficients in the SDEs, the procedure for trajectories of explicit solutions of linear SDEs carried out in (19) does not work here. One needs to have certain explicit expression analytically. To this end, we utilise Itô formula to get an explicit formulation of our affine SDEs. With this in hand, we are able to realise our aim to derive an algorithm for detecting the first jump of the trajectories of solutions of SDEs with affine coefficients, by virtue of multivariate Lagrange interpolation approach. We hope that our results obtained in this paper would lead to further investigations for more general models, such as those determined by SDEs with periodic coefficients (treated as bounded coefficients over the whole spaces), as well as higher order representations of the first jump time in terms of the parameters and rigorous estimates of the first jump time. We will carry out these studies in our forthcoming papers. To the best of our knowledge, there is not any work in the literature addressing such problem.

To the best of our knowledge, there is not any work in the literature addressing such problem. To end up our introduction, we would like to mention that the study of stochastic differential equations driven by Lévy processes is well presented in the monograph (1). Numerical solutions and simulations of α-stable stochastic processes were carried out in (9).

The rest of the paper is organised as follows. In the next section, Section2, we set up preliminaries on analytic framework of the jump SDEs with affine coefficients and then we briefly discuss solutions for α-stable driven SDEs with affine coefficients. In Section 3, we perform simulations of the trajectories of solutions of the concerned SDEs and we further give examples to illustrate our results. Finally, in Appendix we show trajectories of solutions of the SDEs considered with different indices.

2 SDEs with affine coefficients driven by α-stable noise

Given a complete probability space $(\Omega, \mathcal{F}, P)$.Recall that a stochastic process $\{X_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$ is a Lévy process, if X_0=0 almost surely, X has independent and stationary increments, and X is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$

$$\lim_{t\to s} P(|X_t - X_s| > a) = 0.$$

Let $\{\mathcal{F}_t\}_{t\geq 0}$ denote the natural filtration of X. Every Lévy process has a *cádlág* (i.e., right continuous with left limits) modification. The associated jump process $\{\Delta X_t\}_{t\geq 0}$ is defined as $\Delta X_t = X_t - X_{t-}$, where X_{t-} stands for the left limit of X_t at the point t. Fix $t \in [0, \infty)$ and a Borel measurable set $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$, set

$$N(t, A) = \#\{0 \leq s \leq t; \Delta X_s \in A\} = \sum_{0\leq s\leq t} \chi_A(\Delta X_s)$$

where $\#\{...\}$ stands for the cardinal number of set $\{...\}$ and χ_A denotes the indicator function of A. If A is bounded Borel set, then $N(t, A) < \infty$ almost surely for all $t \geq 0$. N is a Poisson random measure with intensity measure $\nu(A) = \mathbb{E}(N(1, A))$ and $\tilde{N}$ is the associated compensated martingale measure

$$\tilde{N}(t, A) = N(t, A) - t\nu(A).$$

Lévy processes enjoy the celebrated Lévy-Itô decomposition, see e.g. (1), which we state as follows. For any (real-valued) Lévy process X, there exist a constant $b \in \mathbb{R}$, a Brownian motion B and a Poisson random measure N on $[0, \infty) \times (\mathbb{R} \setminus \{0\})$ which is independent of B such that, for each $t \geq 0$,

$$X_t = bt + B_t + \int_{0<|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx).$$

Next, recall that A random variable X is said to have a stable distribution if there are parameters $\alpha \in (0,2], \sigma \geq 0, \beta \in [-1,1]$, and $\mu \in \mathbb{R}$ such that its characteristic function is given by

$$E \exp i\theta X = \begin{cases} \exp\Big\{ -\sigma^\alpha |\theta|^\alpha \big(1 - i\beta(\mathrm{sign}\theta)\tan\frac{\pi\alpha}{2}\big) + i\mu\theta \Big\} & \text{if } \alpha \neq 1 \\ \exp\Big\{ -\sigma|\theta|\big(1 + i\beta\frac{\pi}{2}(\mathrm{sign}\theta)\ln|\theta| + i\mu\theta \Big\} & \text{if } \alpha = 1 \end{cases}$$

where signθ is the sign function. Such a random variable X is denoted as $X \sim S_\alpha(\sigma, \beta, \mu)$. The parameter α is the index of stability, β is the skewness parameter, σ is the scale parameter and μ is shift. β is irrelevant when α =2. When $\beta = \mu = 0$, X is a symmetric α-stable random variable and is denoted by $X \sim S\alpha S$. We will focus our attention on symmetric case in this paper.

Recall further that a Lévy process $\{L_t\}_{t\geq 0}$ is an α-stable Lévy motion, if $L_t - L_s \sim S_\alpha((t-s)^{1/\alpha}, \beta, 0)$ for any $0 \leq s < t < \infty$.

In this paper we are concerned with the following SDE with affine coefficients driven by α-stable Lévy motion

$$dX_t = [b_1(t)X_t + b_2(t)]dt + [\sigma_1(t)X_{t-} + \sigma_2(t)]dL_t, \tag{1}$$

where b_1, b_2, σ_1 and σ_2 are bounded functions, and $\{L_t\}_{t\geq 0}$ is an α-stable Lévy process with the following Lévy-Ito representation

$$L_t = \int_0^{t+} dL_s = \int_0^{t+} \int_{0<|z|<1} z\tilde{N}(ds, dz) + \int_0^{t+} \int_{|z|\geq 1} zN(ds, dz)$$

with $N : \mathcal{B}([0,\infty) \times \mathbb{R}\setminus\{0\}) \to \mathbb{N} \cup \{0\}$ being the Poisson random (counting) measure on $(\Omega, \mathcal{F}, P)$ and

$$\tilde{N}(dt, dz) := N(dt, dz) - \frac{dtdz}{|z|^{1+\alpha}}$$

the associated compensated martingale measure with density $\mathbb{E}N(dtdz) = \frac{dtdz}{|z|^{1+\alpha}}$, where $\alpha \in (0,2)$ is fixed. Clearly, the affine coefficients of SDE (1) fulfil the linear growth and local Lipschitz conditions, thus there is a unique solution to the above SDE with initial data X_0. One can further derive a closed formula for solutions of SDE (1), which we state as follows

Theorem 2.1. *Assume that* $b_1(t)+1 > 0$ *for* $t \geq 0$. *Then the unique solution of SDE (1) is given explicitly by*

$$\begin{aligned} X_t =& U_t\Big\{X_0 + \int_0^t \frac{b_2(s)}{U_s}ds + \int_0^{t+}\int_{0<|z|<1} \frac{\sigma_2(s)}{U_s + U_s\sigma_1(s)} z\tilde{N}(ds,dz) \\ &+ \int_0^{t+}\int_{|z|\geq 1} \frac{\sigma_2(s)}{U_s + U_s\sigma_1(s)} zN(ds,dz)\Big\},\ t \geq 0 \end{aligned} \tag{2}$$

where

$$\begin{aligned} U_t =& U_0 \exp\Big\{b_1(t) + \int_0^{t+}\int_{0<|z|<1} ln\big(1+b_1(s)\big)z\tilde{N}(ds,dz) \\ &+ \int_0^{t+}\int_{|z|\geq 1} ln\big(1+b_1(s)\big)zN(ds,dz) \\ &+ \int_0^{t}\int_{0<|z|<1} [ln\big(1+b_1(s)\big) - \sigma_1(s)]\frac{dsdz}{|z|^{1+\alpha}}\Big\},\ t \geq 0. \end{aligned} \tag{3}$$

Proof. Let first $b_2(t) = \sigma_2(t) = 0, t \geq 0$, then we have the linear SDE

$$dU_t = b_1(t)U_t dt + \sigma_1(t)U_{t-}dL_t\,. \tag{4}$$

Under the assumption that $b_1(t)+1 > 0$ for $t \geq 0$, the solution of SDE (4) is a geometric α-stable *Lévy* motion

$$\begin{aligned} U_t =& U_0 \exp\Big\{b_1(t) + \int_0^{t+}\int_{0<|z|<1} ln\big(1+b_1(s)\big)z\tilde{N}(ds,dz) \\ &+ \int_0^{t+}\int_{|z|\geq 1} ln\big(1+b_1(s)\big)zN(ds,dz) \\ &+ \int_0^{t}\int_{0<|z|<1} [ln\big(1+b_1(s)\big) - \sigma_1(s)]\frac{dsdz}{|z|^{1+\alpha}}\Big\}. \end{aligned}$$

Clearly, $U_t \neq 0$ provided $U_0 \neq 0$ and $U_t > 0$ if $U_0 > 0$. For general $b_2(t)$ and $\sigma_2(t)$, one could seek a solution in the form of $X_t = U_t V_t$, with

$$dU_t = b_1(t)U_t dt + \sigma_1(t)U_{t-}dL_t$$

and

$$dV_t = m(t)dt + n(t)dL_t$$

for m and n being specified in the sequel. By Itô's product rule

$$\begin{aligned}
dX_t =& U_t dV_t + V_t dU_t + dU_t dV_t \\
=& U_t[m(t)dt + n(t)dL_t] + V_t[b_1(t)U_t dt + \sigma_1(t)U_{t-}dL_t] \\
&+ [b_1(t)U_t dt + \sigma_1(t)U_{t-}dL_t][m(t)dt + n(t)dL_t] \\
=& U_t[m(t)dt + n(t)dL_t] + X_t[b_1(t)dt + \sigma_1(t)dL_t] \\
&+ U_t[b_1(t)dt + \sigma_1(t)dL_t][m(t)dt + n(t)dL_t] \,.
\end{aligned}$$

On the other hand, noticing that by Itô formula with vanishing higher terms over dt

$$\begin{aligned}
dU_t dV_t =& U_t[b_1(t)dt + \sigma_1(t)dL_t][m(t)dt + n(t)dL_t] \\
=& U_t\Big\{\sigma_1(t)n(t)\int_{0<|z|<1} z\tilde{N}(dt,dz) + \sigma_1(t)n(t)\int_{|z|\geq 1} z\tilde{N}(dt,dz)\Big\} \\
=& U_t\sigma_1(t)n(t)dL_t \,,
\end{aligned}$$

we then get

$$\begin{aligned}
dX_t =& U_t[m(t)dt + n(t)dL_t] + X_t[b_1(t)dt + \sigma_1(t)dL_t] + U_t\sigma_1(t)n(t)dL_t \\
=& [U_t m(t) + X_t b_1(t)]dt + [U_t n(t) + X_t\sigma_1(t) + U_t\sigma_1(t)n(t)]dL_t \,.
\end{aligned}$$

By virtue of the uniqueness of Lévy-Itô decomposition for semi-martingales and by comparing the coefficients, we get

$$m(t) = \frac{b_2(t)}{U_t}, \qquad n(t) = \frac{\sigma_2(t)}{U_t + U_t\sigma_1(t)}.$$

Next, from

$$dV_t = m(t)dt + n(t)dL_t,$$

$$
\begin{aligned}
V_t - V_0 &= \int_0^t m(s)ds + \int_0^{t+}\int_{0<|z|<1} n(s)z\tilde{N}(ds,dz) + \int_0^{t+}\int_{|z|\geq 1} n(s)zN(ds,dz) \\
&= \int_0^t \frac{b_2(s)}{U_s}ds + \int_0^{t+}\int_{1<|z|<1} \frac{\sigma_2(s)}{U_s + U_s\sigma_1(s)} z\tilde{N}(ds,dz) \\
&\quad + \int_0^{t+}\int_{|z|\geq 1} \frac{\sigma_2(s)}{U_s + U_s\sigma_1(s)} zN(ds,dz).
\end{aligned}
$$

Finally, we get the formula (2), that is

$$
\begin{aligned}
X_t =& U_t V_t \\
=& U_t \Big\{ X_0 + \int_0^t \frac{b_2(s)}{U_s}ds + \int_0^{t+}\int_{0<|z|<1} \frac{\sigma_2(s)}{U_s + U_s\sigma_1(s)} z\tilde{N}(ds,dz) \\
&+ \int_0^{t+}\int_{|z|\geq 1} \frac{\sigma_2(s)}{U_s + U_s\sigma_1(s)} zN(ds,dz) \Big\}
\end{aligned}
$$

where

$$
\begin{aligned}
U_t =& U_0 \exp \Big\{ b_1(t) + \int_0^{t+}\int_{0<|z|<1} ln\big(1 + b_1(s)\big) z\tilde{N}(ds,dz) \\
&+ \int_0^{t+}\int_{|z|>1} ln\big(1 + b_1(s)\big) zN(ds,dz) \\
&+ \int_0^t \int_{0<|z|<1} [ln\big(1 + b_1(s)\big) - \sigma_1(s)] \frac{dsdz}{|z|^{1+\alpha}} \Big\}.
\end{aligned}
$$

This completes the proof. □

3 Simulations and examples

We generate sample trajectories of SDE with affine coefficients driven by α-stable white noises by applying simulation methods in MATLAB. A multivariate Lagrange interpolation method is utilised to derive links among coefficients in SDE towards the first jump time (19). Simulations of the sample trajectories are with the sample size of $2^9 = 512$, which yields a clear picture to identify successfully the first jump time for each simulated trajectory. Two cases of the following SDE are considered in this paper ($\alpha_1 < 1$

and $\alpha_2 > 1$, see Figure 1, 2 and 3; $\alpha_1 > 1$ and $\alpha_2 < 1$, see Figure 4, 5 and 6).

$$dX_t = \lambda X_t dt + \mu_1 X_{t-} dL_t^{\alpha_1} + \mu_2 dL_t^{\alpha_2} \tag{5}$$

where $\lambda > 0$, $\mu_1 > 0$, $\mu_2 > 0$ and $t \geq 0$.

Remark 3.1. If $\mu_1 \gg \mu_2$ or $\mu_2 \gg \mu_1$, the trajectories will perform as geometric α-stable Lévy motion and α-stable Ornstein-Uhlenbeck process respectively, presented in (19).

We could clarify the model into different perspectives by observations and general characteristics of trajectories are summarised as follows,

1. Fix λ, μ_1, μ_2 and α_1, sample trajectories $\{X_t\}_{t\geq 0}$ become more tempered as the stability index α_2 increases, but the jump size becomes smaller and smaller so that the trajectories become less and less volatile. If fix λ, μ_1, μ_2 and α_2, instead, the trajectories obtain similar properties.

2. Fix μ_1, μ_2, α_1 and α_2, trajectories tend to show deterministic exponential paths along with the increase of λ. The trajectories are more tempered for bigger α_1 or α_2.

3. Fix λ, α_1, α_2 and μ_2, increasing the volatility parameter μ_1 indicates higher chaoticity.

3.1 Case when $\alpha_1 < 1$ and $\alpha_2 > 1$

For the set $(\lambda, \mu_1, \mu_2, \alpha_1, \alpha_2)$, there is a link among the five parameters λ, μ_1, μ_2, α_1 and α_2 towards first jump point detection of the sample trajectories. By substantial amount of simulations, we randomly choose the situations and keep records of values of the parameters λ, μ_1, μ_2, α_1 and α_2 when the first jump appears (see Table 1). Especially, the degree 1 linear relationship among these five parameters is useful in data modelling for uncertainty targeted problems in reality.

Table 1: Data processed for sample trajectories when $\alpha_1 < 1$ and $\alpha_2 > 1$

λ	μ_1	μ_2	α_1	α_2	t	X_t^α
10	1	100	0.5	1.5	0.08203	-63.86
1	0.25	100	0.75	1.25	0.1855	-303.2
1	100	1	0.75	1.75	0.1035	122.5
1	100	0.25	0.75	1.5	0.207	-896.1
10	100	0.25	0.5	1.25	0.3301	252.1
100	100	1	0.25	1.75	0.1934	-3028000
10	1	0.25	0.25	1.25	0.5762	533.7

We have degrees $n = 1$, variables $m = 6$, so terms= $\begin{pmatrix} 1+6 \\ 1 \end{pmatrix} = 7$. If we have $g = f(h, j, k, l, m, n)$ which is a degree 1 function with 6 parameters, and

$$g_i = \beta_1 h_i + \beta_2 j_i + \beta_3 k_i + \beta_4 l_i + \beta_5 m_i + \beta_6 n_i + \beta_7$$

where $\beta_1, \beta_2, \cdots, \beta_7$ are coefficients, $1 \leq i \leq 7$.

$$
\begin{aligned}
-63.86 &= 10\beta_1 + 1\beta_2 + 100\beta_3 + 0.5\beta_4 + 1.5\beta_5 + 0.08203\beta_6 + \beta_7 \\
-303.2 &= \beta_1 + 0.25\beta_2 + 100\beta_3 + 0.75\beta_4 + 1.25\beta_5 + 0.1855\beta_6 + \beta_7 \\
122.5 &= \beta_1 + 100\beta_2 + \beta_3 + 0.75\beta_4 + 1.75\beta_5 + 0.1035\beta_6 + \beta_7 \\
-896.1 &= \beta_1 + 100\beta_2 + 0.25\beta_3 + 0.75\beta_4 + 1.5\beta_5 + 0.207\beta_6 + \beta_7 \\
252.1 &= 10\beta_1 + 100\beta_2 + 0.25\beta_3 + 0.5\beta_4 + 1.25\beta_5 + 0.3301\beta_6 + \beta_7 \\
-3028000 &= 100\beta_1 + 100\beta_2 + \beta_3 + 0.25\beta_4 + 1.75\beta_5 + 0.1934\beta_6 + \beta_7 \\
533.7 &= 10\beta_1 + \beta_2 + 0.25\beta_3 + 0.25\beta_4 + 1.25\beta_5 + 0.5762\beta_6 + \beta_7
\end{aligned}
$$

By calculation

$$g = 3432.78j - 37385.2h + 3445.58k - 1342799l + 1189.85m + 18000.5n + 693929.$$

Then

$$X_t^\alpha = 3433\lambda - 37385\mu_1 + 3446\mu_2 - 1342799\alpha_1 + 1190\alpha_2 + 18001t + 693929$$

If we take the average value of t, we have

$$\bar{t} = 0.23968$$

and average value of X_t^α, we have

$$\overline{X_t^\alpha} = -432622$$

Therefore

$$3433\lambda - 37385\mu_1 + 3446\mu_2 - 1342799\alpha_1 + 1190\alpha_2 = -1130865$$

We summarise our derivation as

Proposition 3.1. *The link among parameters for first jump point detection of the sample trajectories of SDE with affine coefficients driven by α-stable noise ($\alpha_1 < 1$ and $\alpha_2 > 1$) is given by the following liner equation*

$$3433\lambda - 37385\mu_1 + 3446\mu_2 - 1342799\alpha_1 + 1190\alpha_2 = -1130865$$

3.2 Case when $\alpha_1 > 1$ and $\alpha_2 < 1$

Similarly, for he set $(\lambda, \mu_1, \mu_2, \alpha_1, \alpha_2)$ described in Equation (5), we are working on determining a link among these three parameters towards first jump time detection. The data (see Table 2) and calculations have been processed to obtain the degree 1 linear relationship are as follows.

Table 2: Data processed for sample trajectories when $\alpha_1 > 1$ and $\alpha_2 < 1$

λ	μ_1	μ_2	α_1	α_2	t	X_t^α
10	0.25	1	1.5	0.5	0.1973	-91.87
10	1	100	1.5	0.25	0.01953	95.72
1	100	1	1.25	0.5	0.04492	305.9
100	1	0.25	1.5	0.25	0.1211	346
1	1	100	1.75	0.5	0.09766	311.1
100	1	100	1.5	0.5	0.05273	-242.9
10	1	0.25	1.75	0.75	0.5742	-105.1

We have degrees $n = 1$, variables $m = 6$, so terms= $\begin{pmatrix} 1+6 \\ 1 \end{pmatrix} = 7$. If we have $g = f(h, j, k, l, m, n)$ which is a degree 1 function with 6 parameters, and

$$g_i = \beta_1 h_i + \beta_2 j_i + \beta_3 k_i + \beta_4 l_i + \beta_5 m_i + \beta_6 n_i + \beta_7$$

where $\beta_1, \beta_2, \cdots, \beta_7$ are coefficients, $1 \leq i \leq 7$.

$$\begin{aligned}
-91.87 &= 10\beta_1 + 0.25\beta_2 + \beta_3 + 1.5\beta_4 + 0.5\beta_5 + 0.1973\beta_6 + \beta_7 \\
95.72 &= 10\beta_1 + \beta_2 + 100\beta_3 + 1.5\beta_4 + 0.25\beta_5 + 0.0.01953\beta_6 + \beta_7 \\
305.9 &= \beta_1 + 100\beta_2 + \beta_3 + 1.25\beta_4 + 0.5\beta_5 + 0.04492\beta_6 + \beta_7 \\
346 &= 100\beta_1 + \beta_2 + 0.25\beta_3 + 1.5\beta_4 + 0.25\beta_5 + 0.1211\beta_6 + \beta_7 \\
311.1 &= \beta_1 + \beta_2 + 100\beta_3 + 1.75\beta_4 + 0.5\beta_5 + 0.09766\beta_6 + \beta_7 \\
-242.9 &= 100\beta_1 + \beta_2 + 100\beta_3 + 1.5\beta_4 + 0.5\beta_5 + 0.05273\beta_6 + \beta_7 \\
-105.1 &= 10\beta_1 + \beta_2 + 0.25\beta_3 + 1.75\beta_4 + 0.75\beta_5 + 0.5742\beta_6 + \beta_7
\end{aligned}$$

By calculation

$$g = 0.3089h + 9.00913j - 3.0498k + 2482.95l - 1358.81m - 804.747n - 2980.41$$

Then

$$X_t^\alpha = 0.3089\lambda + 9.00913\mu_1 - 3.0498\mu_2 + 2482.95\alpha_1 - 1358.81\alpha_2 - 804.747t - 2980.41$$

If we take the average value of t, we have

$$\bar{t} = 0.158206$$

and average value of X_t^α, we have

$$\overline{X_t^\alpha} = 88.407143$$

Therefore

$$0.31\lambda - 9.01\mu_1 + 3.05\mu_2 + 2482.95\alpha_1 - 1358.81\alpha_2 = 3196.13$$

We summarise our derivation as

Proposition 3.2. *The link among parameters for first jump point detection of the sample trajectories of SDE with affine coefficients driven by α-stable noise ($\alpha_1 > 1$ and $\alpha_2 < 1$) is given by the following liner equation*

$$-0.0017124\lambda - 0.066287\mu + 0.15752\alpha = -0.3949911.$$

Remark 3.2. Here we only consider linear Lagrange interpolation. One can extend to higher order polynomial interpolation in which more computation is needed. Our consideration gives a simple yet efficient calculation.

Appendices

A Case when $\alpha_1 < 1$ and $\alpha_2 > 1$

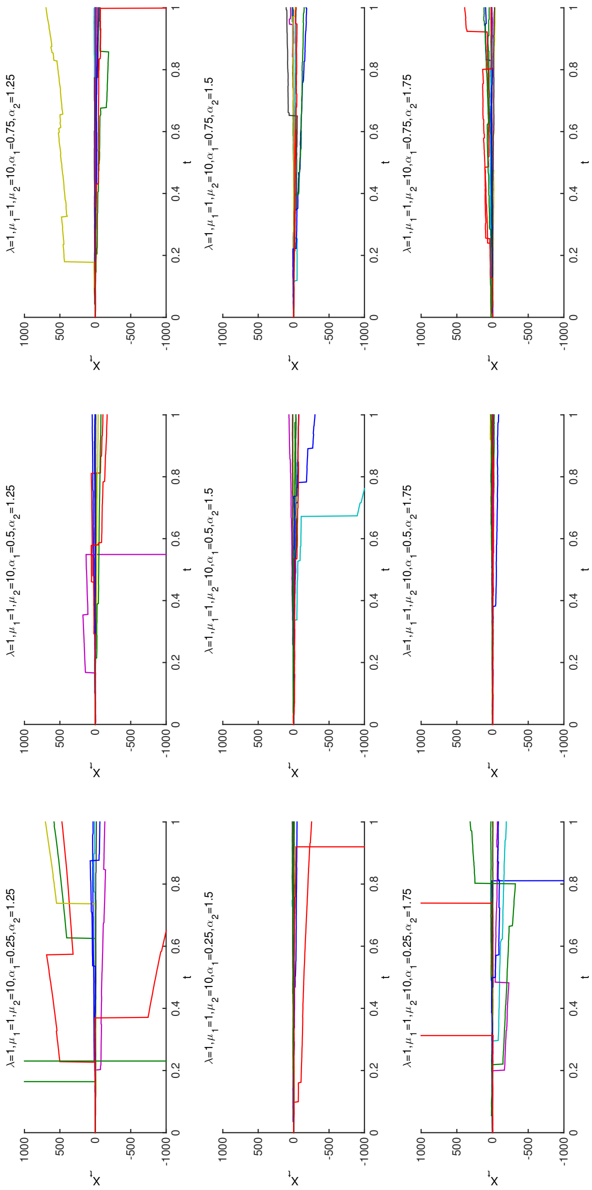

Figure 1: Fix λ=1, μ_1=1 and μ_2=10

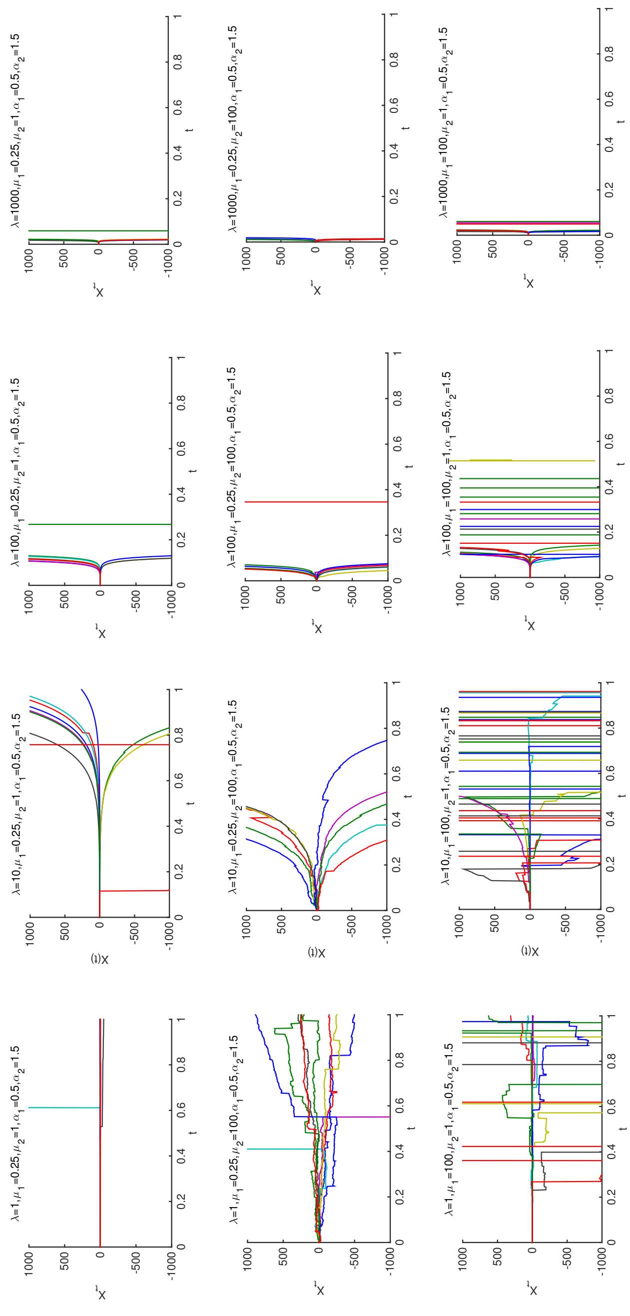

Figure 2: λ changes when α_1=0.5 and α_2=1.5

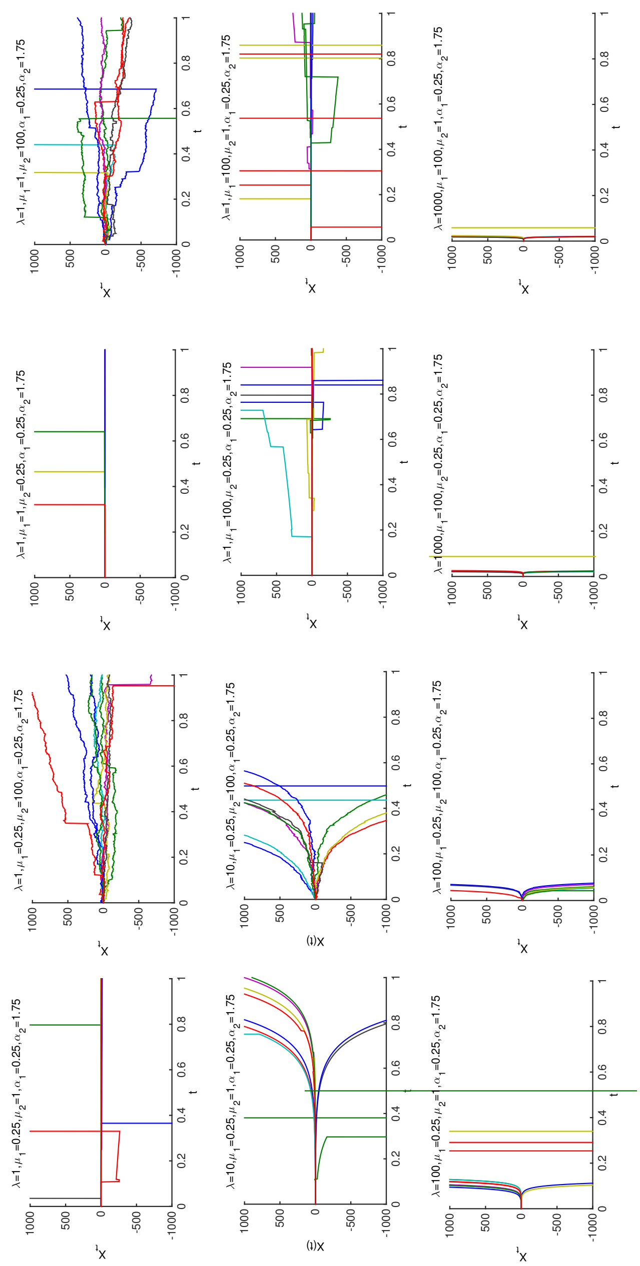

Figure 3: μ_2 changes when α_1=0.25 and α_2=1.75

B Case when $\alpha_1 > 1$ and $\alpha_2 < 1$

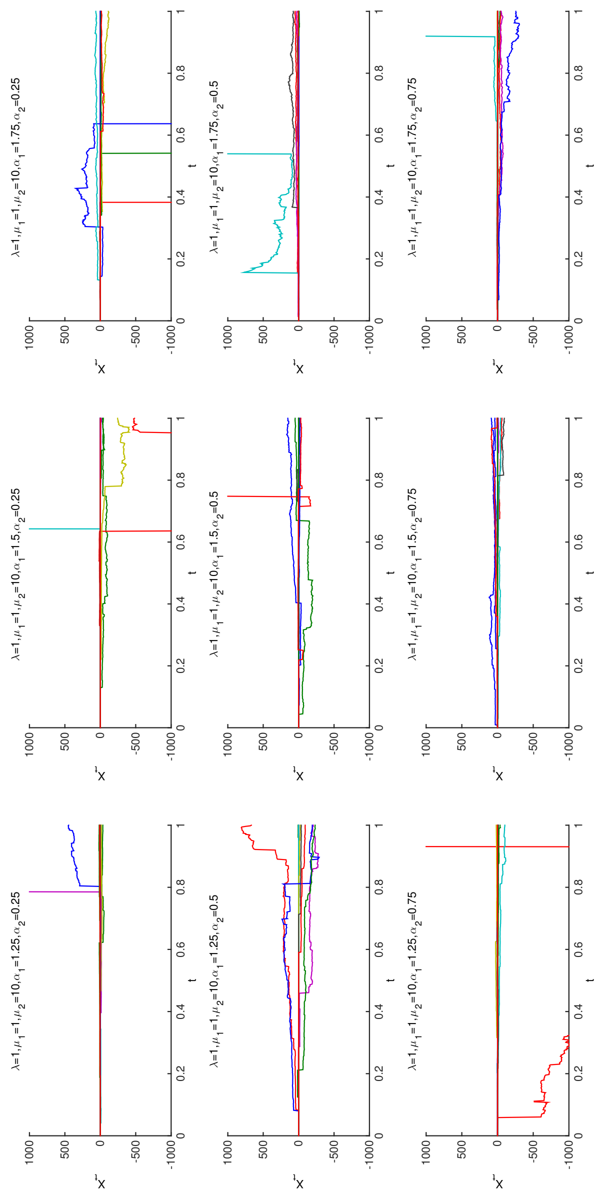

Figure 4: Fix λ=1, μ_1=1 and μ_2=10

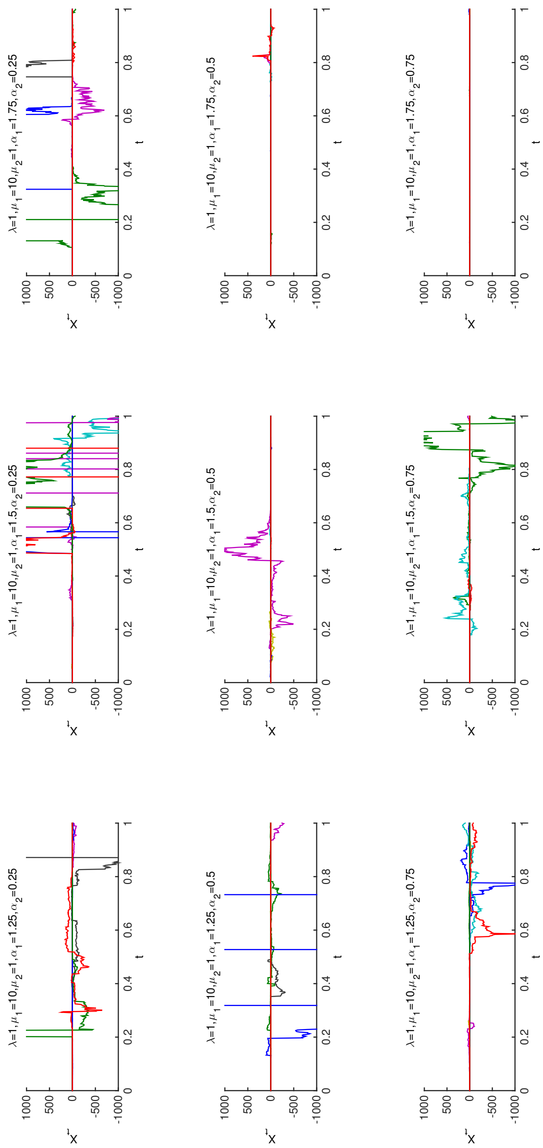

Figure 5: Fix λ=1, μ_1=10 and μ_2=1

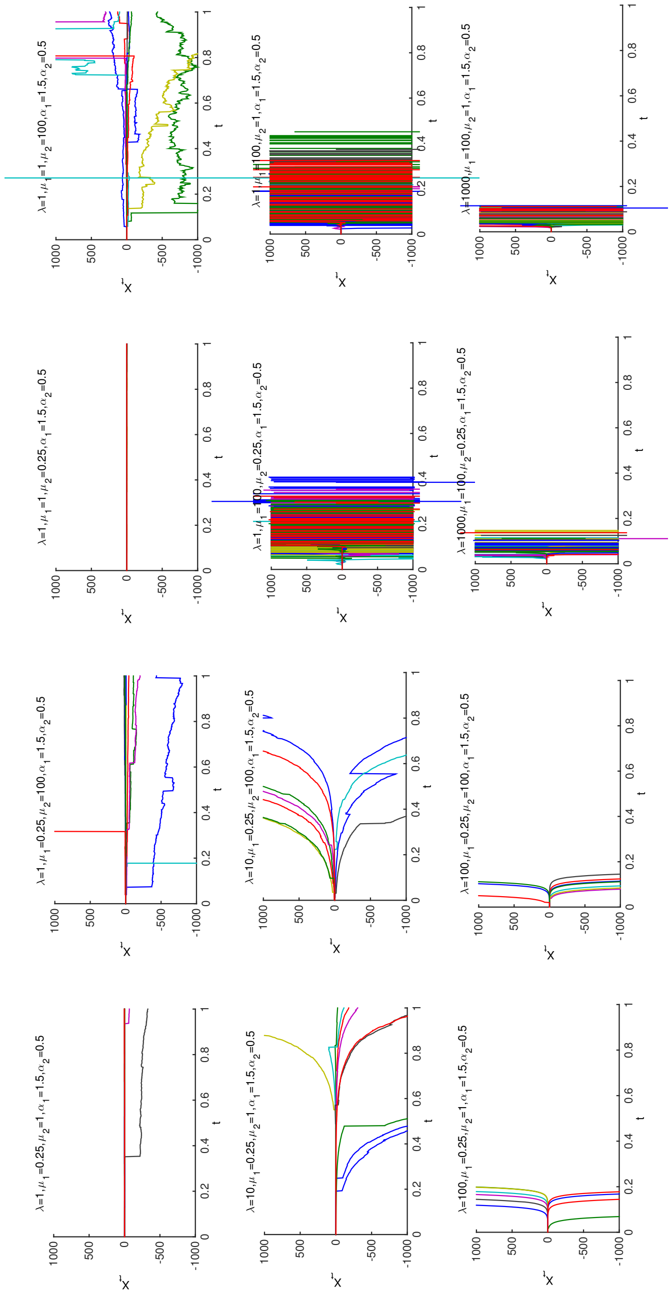

Figure 6: μ_2 changes when α_1=1.5 and α_2=0.5

References

[1] Applebaum, D. (2009). *Lévy processes and stochastic calculus*. Cambridge University Press.

[2] Barndorff-Nielsen, O. (1997). Normal inverse Gaussian distributions and stochastic volatility modelling. *Scandinavian Journal of Statistics* 24 (1), 1-13.

[3] Campbell, J. Y., Lo, A. W. C., & MacKinlay, A. C. (1997). *The econometrics of financial markets*. Princeton University Press.

[4] Dror, M., L'ecuyer, P., & Szidarovszky, F. (Eds.). (2002). *Modeling Uncertainty: An Examination of Stochastic Theory, Methods, and Applications* (Vol. 46). Springer Science & Business Media.

[5] Du, H., Wu, J., & Yang, W. (2010). On the mechanism of CDOs behind the current financial crisis and mathematical modeling with Lévy distributions. *Intelligent Information Management* 2(2), 149-158.

[6] Duffie, D., Filipović, D., & Schachermayer, W. (2003). Affine processes and applications in finance. *Annals of Applied Probability* 13(3), 984-1053.

[7] Fiche, A., Cexus, J.C., Martin, A., & Khenchaf, A. (2013) Features modeling with an α-stable distribution: Application to pattern recognition based on continuous belief functions. *Information Fusion* 14(4), 504-520.

[8] Giacometti, R., Bertocchi, M., Rachev, S. T., & Fabozzi, F. J. (2007). Stable distributions in the Black-Litterman approach to asset allocation. *Quantitative Finance* 7(4), 423-433.

[9] Janicki, A., & Weron, A. (1993) *Simulation and Chaotic Behavior of α-Stable Stochastic Processes* (Vol.178). CRC Press.

[10] Leland, W. E., Taqqu, M. S., Willinger, W., & Wilson, D. W. (1993) On the self-similar nature of Ethernet traffic. *ACM SIGCOMM Computer Communication Review* 23(4), 183-193.

[11] Lévy P. (1925) *Calcul des probabilités. Gauther-Villars.*

[12] Lévy P. (1937) Théorie de l'addition des variables aléatoires. Gauther-Villars.

[13] Mandelbrot, B. (1960). The Pareto-Lévy Law and the Distribution of Income. *International Economic Review* 1(2), 79-106.

[14] Samorodnitsky, G., & Taqqu, M.S. (1994) *Stable Random Processes: Stochastic Models with Infinite Variance*. CRC Press.

[15] Shlesinger, M. F., Zaslavsky, G. M., & Frisch, U. (1995). Lévy flights and related topics in physics. *Lecture notes in physics* 450, Springer-Verlag.

[16] Zolotarev, V. M. (1986). *One-dimensional stable distributions* (Vol. 65). American Mathematical Society.

[17] Zopounidis, C., & Pardalos, P. M. (2013). *Managing in Uncertainty: Theory and Practice* (Vol. 19). Springer Science & Business Media.

[18] Janson, S. (2011). Stable distributions. Preprint (http://www2.math.uu.se/ svante/papers/sjN12.pdf).

[19] Song, J., & Wu, J. (2019). A detection algorithm for the first jump time in sample trajectories of jump-diffusions driven by α-stable white noise. *Communications in Statistics-Theory and Methods*, 48:19, 4888-4902.

[20] Wu, J.,& Yang, W. (2013). Valuation of synthetic CDOs with affine jump-diffusion processes involving Lévy stable distributions. *Mathematical and Computer Modelling*, 57 (3-4), 570-583.

[21] Duffie, D., Pan, J. & Singleton, K. (2000). Transform Analysis and Asset Pricing for Affine Jump-diffusions. *Econometrica*, 68, 1343-1376.

[22] Cox, J. C., J. E. Ingersoll, & S. A. Ross. (1985). A Theory of the Term Structure of Interest Rates. *Econometrica*, 53, 385-407.

[23] Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, 52, 177-188.

[24] Jarrow, R. (2018). Exploring mispricing in the term structure of CDS spreads. *Review of Finance*, 23, 161-198.

[25] Hain, M., Uhrig-Homburg, M., & Unger, N. (2018). Risk factors and their associated risk premia: An empirical analysis of the crude oil market. *Journal of Banking & Finance*, 95, 44-63.

[26] Campbell, J., Giglio, S., Polk, C., & Turley, R. (2018). An intertemporal CAPM with stochastic volatility. *Journal of Financial Economics*, 128, 207-233.

[27] Barletta, A., Magistris, P., & Sloth, D. (2019). It only takes a few moments to hedge options. *Journal of Economic Dynamics and Control*, 100, 251-269.