Nash inequality for Diffusion Processes Associated with Dirichlet Distributions *

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Abstract

For any $N \geq 2$ and $\alpha = (\alpha_1, \cdots, \alpha_{N+1}) \in (0,\infty)^{N+1}$, let $\mu_\alpha^{(N)}$ be the Dirichlet distribution with parameter α on the set $\Delta^{(N)} := \{x \in [0,1]^N : \sum_{1\leq i\leq N} x_i \leq 1\}$. The multivariate Dirichlet diffusion is associated with the Dirichlet form

$$\mathscr{E}_\alpha^{(N)}(f,f) := \sum_{n=1}^N \int_{\Delta^{(N)}} \Big(1 - \sum_{1\leq i\leq N} x_i\Big) x_n (\partial_n f)^2(x)\, \mu_\alpha^{(N)}(\mathrm{d}x)$$

with Domain $\mathscr{D}(\mathscr{E}_\alpha^{(N)})$ being the closure of $C^1(\Delta^{(N)})$. We prove the Nash inequality

$$\mu_\alpha^{(N)}(f^2) \leq C \mathscr{E}_\alpha^{(N)}(f,f)^{\frac{p}{p+1}} \mu_\alpha^{(N)}(|f|)^{\frac{2}{p+1}}, \quad f \in \mathscr{D}(\mathscr{E}_\alpha^{(N)}), \mu_\alpha^{(N)}(f) = 0$$

for some constant $C > 0$ and $p = (\alpha_{N+1} - 1)^+ + \sum_{i=1}^N 1 \vee (2\alpha_i)$, where the constant p is sharp when $\max_{1\leq i\leq N} \alpha_i \leq \frac{1}{2}$ and $\alpha_{N+1} \geq 1$. This Nash inequality also holds for the corresponding Fleming-Viot process.

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1 Introduction

Let $N \geq 1$ be a natural number and $\alpha = (\alpha_1, \cdots, \alpha_{N+1}) \in (0, \infty)^{N+1}$. The Dirichlet distribution $\mu_\alpha^{(N)}$ with parameter α is a probability measure on the set

$$\Delta^{(N)} := \left\{ x = (x_i)_{1\leq i\leq N} \in [0,1]^N : \ |x|_1 := \sum_{i=1}^N x_i \leq 1 \right\}$$

with density function

$$\rho(x) := \frac{\Gamma(|\alpha|_1)}{\prod_{1\leq i\leq N+1} \Gamma(\alpha_i)} (1 - |x|_1)^{\alpha_{N+1}-1} \prod_{1\leq i\leq N} x_i^{\alpha_i - 1}, \quad x = (x_i)_{1\leq i\leq N} \in \Delta^{(N)}, \tag{1.1}$$

where $|\alpha|_1 := \sum_{i=1}^{N+1} \alpha_i$. This distribution arises naturally in Bayesian inference as conjugate prior for categorical distribution, and it describes the distribution of allelic frequencies in population genetics, see for instance [3, 11, 14].

To investigate stochastic dynamics converging to $\mu_\alpha^{(N)}$, different models of diffusion processes have been proposed. In this paper, we investigate functional inequalities of these diffusions.

In the following three subsections, we first briefly recall some facts on functional inequalities for Dirichlet forms, as well as known results for diffusion processes associated with the Dirichlet distribution, then propose problems in the direction and state the main result of the paper.

1.1 Functional inequalities

In general, let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a conservative symmetric Dirichlet form on $L^2(\mu)$ for some probability space $(E, \mathscr{F}, \mu)$, $(L, \mathscr{D}(L))$ be the associated Dirichlet operator, and $P_t := \mathrm{e}^{tL}, t \geq 0$, be the Markov semigroup. The following is a brief summary from [19] for the Poincaré, log-Sobolev, super Poincaré and Nash inequalities, see also [1, 9] and references within.

Firstly, we consider the spectral gap of L: $\mathrm{gap}(L)$ is the largest constant $C > 0$ such that the Poincaré inequality

$$\mu(f^2) \leq \frac{1}{C} \mathscr{E}(f, f), \quad f \in \mathscr{D}(\mathscr{E}), \mu(f) = 0 \tag{1.2}$$

holds. In case this inequality is not available, we say that L does not have spectral gap, and denote $\mathrm{gap}(L) = 0$. The Poincaré inequality (1.2) is equivalent to the L^2-exponential convergence of P_t:

$$\|P_t f\|_{L^2(\mu)} \leq \mathrm{e}^{-Ct} \|f\|_{L^2(\mu)}, \quad t \geq 0, f \in L^2(\mu), \mu(f) = 0.$$

Next, we consider the log-Sobolev constant $C_{LS}(L)$, which is the largest positive constant C such that the log-Sobolev inequality

$$\mu(f^2 \log f^2) \leq \frac{2}{C} \mathscr{E}(f, f), \quad f \in \mathscr{D}(\mathscr{E}), \mu(f^2) = 1 \tag{1.3}$$

holds. We have $C_{LS}(L) \le \mathrm{gap}(L)$. In general, (1.3) implies the exponential decay of P_t in entropy:

$$\mu((P_t f)\log P_t f) \le \mathrm{e}^{-Ct}\mu(f\log f), \quad t \ge 0, f \in \mathscr{B}^+(E), \mu(f) = 1,$$

and in the diffusion setting they are equivalent. Moreover, the log-Sobolev inequality (1.3) holds for some constant $C > 0$ if and only if P_t is hypercontractive, i.e. $\|P_t\|_{L^2(\mu)\to L^4(\mu)} = 1$ for large enough t.

Finally, we say that $(\mathscr{E}, \mu)$ satisfies the super Poincaré inequality with rate function $\beta : (0, \infty) \to (0, \infty)$, if

$$\mu(f^2) \le r\mathscr{E}(f, f) + \beta(r)\mu(|f|)^2, \quad r > 0, f \in \mathscr{D}(\mathscr{E}). \tag{1.4}$$

This inequality is equivalent to the uniform integrability of P_t, i.e. P_t has zero tail norm:

$$\|P_t\|_{tail} := \lim_{R\to\infty} \sup_{\mu(f^2)\le 1} \mu((P_t f)^2 1_{\{|P_t f|\ge R\}}) = 0, \quad t > 0.$$

When P_t has a heat kernel with respect to μ, it is also equivalent to the absence of the essential spectrum of L (i.e. the spectrum of L is purely discrete). The super Poincaré inequality generalizes the classical Sobolev/Nash type inequalities. For instance, when $\mathrm{gap}(L) > 0$, (1.4) with $\beta(r) = \mathrm{e}^{c(1+r^{-1})}$ for some $c > 0$ is equivalent to the log-Sobolev inequality (1.3) for some constant $C > 0$; while for a constant $p > 0$, (1.4) with $\beta(r) = c(1 + r^{-p})$ holds for some $c > 0$ if and only if the Nash inequality

$$\mu(f^2) \le C\mathscr{E}(f, f)^{\frac{p}{p+1}} \mu(|f|)^{\frac{2}{p+1}}, \quad f \in \mathscr{D}(\mathscr{E}), \mu(f) = 0 \tag{1.5}$$

holds for some constant $C > 0$. They are also equivalent to

$$\|P_t - \mu\|_{L^1(\mu)\to L^\infty(\mu)} \le \frac{c'}{(t \wedge 1)^p}\mathrm{e}^{-\mathrm{gap}(L)t}, \quad t > 0.$$

The later implies the hypercontractivity of P_t, and hence the log-Sobolev inequality (1.3) for some constant $C > 0$.

1.2 Diffusion processes associated with Dirichlet distributions

In this part, we recall existing results on functional inequalities for some diffusion processes on $\Delta^{(N)}$, which are reversible with respect to the Dirichlet distribution $\mu_\alpha^{(N)}$.

When $N = 1$, the Wright-Fisher diffusion on the interval $[0, 1]$ generated by

$$L_\alpha^{(1)} := x(1 - x)\partial_x^2 + \{\alpha_1(1 - x) - \alpha_2 x\}\partial_x.$$

The associated Dirichlet form is the closure of $(\mathscr{E}_\alpha^{(1)}, C^1([0, 1]))$ given by

$$\mathscr{E}_\alpha^{(1)}(f, g) = \int_0^1 x(1 - x)f'(x)g'(x)\mu_\alpha^{(1)}(\mathrm{d}x), \quad f, g \in C^1([0, 1]).$$

Due to [15], we have $\mathrm{gap}(L_\alpha^{(1)}) = \alpha_1 + \alpha_2$, and [16, Lemma 2.7] shows that $C_{LS}(L) \ge \frac{\alpha_1\wedge\alpha_2}{160}$. Moreover, according to [8, Theorem 2.2], $(\mathscr{E}_\alpha^{(1)}, \mu_\alpha^{(1)})$ satisfies the super Poincaré inequality with $\beta(r) = c(1+r^{-(\frac{1}{2}\vee\alpha_1\vee\alpha_2)})$ for some constant $c > 0$, and hence, the Nash inequality holds for $p = \frac{1}{2}\vee\alpha_1\vee\alpha_2$, which is sharp in the sense that the super Poincaré inequality does not hold if $\lim_{r\to 0}\beta(r)r^{\frac{1}{2}\vee\alpha_1\vee\alpha_2} = 0$.

When $N \geq 2$, we consider the following three different generalizations of the Wright-Fisher diffusion arising from population genetics, see e.g. [7, 8, 12, 13, 16].

A. Wright-Fisher diffusion with mutation. Let $|\alpha|_1 = \sum_{i=1}^{N+1}\alpha_i$ and denote $\partial_i = \partial_{x_i}, 1 \leq i \leq N$. Consider the diffusion process on $\Delta^{(N)}$ generated by

$$L_{FV}^{\alpha,N} := \sum_{i,j=1}^{N} x_i(\delta_{ij} - x_j)\partial_i\partial_j + \sum_{i=1}^{N}(\alpha_i - |\alpha|_1 x_i)\partial_i,$$

where $\delta_{ij} = 1$ if $i = j$; $= 0$ otherwise. The associated Dirichlet form is the closure of $(\mathscr{E}_{FV}^{\alpha,N}, C^1(\Delta^{(N)}))$ given by

$$\mathscr{E}_{FV}^{\alpha,N}(f,g) = \int_{\Delta^{(N)}}\sum_{i,j=1}^{N} x_i(\delta_{ij} - x_j)\{(\partial_i f)(\partial_j g)\}(x)\mu_\alpha^{(N)}(\mathrm{d}x), \quad f,g \in C^1(\Delta^{(N)}).$$

Again due to [15, 16] we have

$$\mathrm{gap}(L_{FV}^{\alpha,N}) = |\alpha|_1, \quad C_{LS}(L) \geq \frac{1}{160}\min_{1\leq i\leq N+1}\alpha_i.$$

However, **the Nash inequality is unknown.**

B. GEM process. Let $\beta_i = \sum_{j=i+1}^{N+1}\alpha_j$, $1 \leq i \leq N$. Then $\mu_\alpha^{(N)} = \Pi_{\alpha,\beta}$, the GEM distribution with parameter (α,β), see e.g. [7]. For $x \in \Delta^{(N)}$ and $1 \leq i,j \leq N$, let

$$a_{ij}(x) = x_i x_j \sum_{k=1}^{i\wedge j}\frac{\big\{\delta_{ki}\big(1-\sum_{1\leq l\leq k-1}x_l\big) - x_k\big\}\cdot\big\{\delta_{kj}\big(1-\sum_{1\leq l\leq k-1}x_l\big) - x_k\big\}}{x_k\big(1-\sum_{1\leq l\leq k}x_l\big)},$$

$$b_i(x) = x_i\sum_{k=1}^{i}\frac{\big\{\delta_{ki}\big(1-\sum_{1\leq l\leq k-1}x_l\big) - x_k\big\}\cdot\big\{\alpha_k\big(1-\sum_{1\leq l\leq k-1}x_l\big) - \beta_i x_k\big\}}{x_k\big(1-\sum_{1\leq l\leq k}x_l\big)}.$$

The corresponding GEM process introduced in [7] is the diffusion process on $\Delta^{(N)}$ generated by

$$L_{GEM}^{\alpha,N} := \sum_{i,j=1}^{N} a_{ij}\partial_i\partial_j + \sum_{i=1}^{N} b_i\partial_i,$$

and the associated Dirichlet form is the closure of $(\mathscr{E}_{GEM}^{\alpha,N}, C^1(\Delta^{(N)}))$:

$$\mathscr{E}_{GEM}^{\alpha,N}(f,g) = \int_{\Delta^{(N)}}\sum_{i,j=1}^{N} a_{ij}(x)\{(\partial_i f)(\partial_j g)\}(x)\mu_\alpha^{(N)}(\mathrm{d}x), \quad f,g \in C^1(\Delta^{(N)}).$$

According to [7, Theorem 3.1], we have

$$\text{gap}(L^{\alpha,N}_{GEM}) = \alpha_N + \alpha_{N+1}, \quad C_{LS}(L^{\alpha,N}_{GEM}) \geq \frac{1}{160} \min_{1\leq i\leq N+1} \alpha_i.$$

Moreover, applying [8, (1.4)] for $a_i = \alpha_i, b_i = \beta_i := \sum_{j=i+1}^{N+1} \alpha_j$, and using [8, (2.24)], we see that the heat kernel $p_t^{GEM}(x,y)$ of the present GEM process with respect to $\mu_\alpha^{(N)}$ satisfies

$$c_1 t^{-\sum_{i=1}^N (\frac{1}{2}\vee\alpha_i\vee\beta_i)} \leq \sup_{x,y\in\Delta^{(N)}} p_t^{GEM}(x,y) \leq c_2 t^{-\sum_{i=1}^N (\frac{1}{2}\vee\alpha_i\vee\beta_i)}, \quad t\in(0,1]$$

for some constants $c_2 > c_1 > 0$. So, there exists a constant $C > 0$ such that the Nash inequality (1.5) holds for $(\mathscr{E}^{\alpha,N}_{GEM}, \mu_\alpha^{(N)})$ replacing $(\mathscr{E}, \mu)$ with

$$p = \sum_{i=1}^N \max\Big\{\frac{1}{2}, \alpha_i, \sum_{i+1\leq j\leq N+1} \alpha_j\Big\},$$

which is sharp in the sense that the Nash inequality fails when this p is replaced by any smaller constant.

C. Multivariate Dirichlet diffusion. This process was introduced in [10], and was used in [2] to describe a fluctuating ensemble of N variables subject to a conservation principle. It can be constructed as the unique solution to the following SDE on $\Delta^{(N)}$:

(1.6) $$\mathrm{d}X_i(t) = \big\{\alpha_i(1-|X(t)|_1) - \alpha_{N+1}X_i(t)\big\}\mathrm{d}t + \sqrt{2(1-|X(t)|_1)X_i(t)}\,\mathrm{d}B_i(t), \quad 1\leq i\leq N,$$

where $B(t) := (B_1(t), \cdots, B_N(t))$ is the N-dimensional Brownian motion. The infinitesimal generator of the diffusion is

$$L_\alpha^{(N)} := \sum_{1\leq n\leq N} \Big(x_n(1-|x|_1)\partial_n^2 + \big\{\alpha_n(1-|x|_1) - \alpha_{N+1}x_n\big\}\partial_n\Big),$$

and the associated Dirichelt form is the closure of $(\mathscr{E}_\alpha^{(N)}, C^1(\Delta^{(N)}))$:

$$\mathscr{E}_\alpha^{(N)}(f,g) = \int_{\Delta^{(N)}} \sum_{i=1}^N x_i(1-|x|_1)\{(\partial_i f)(\partial_i g)\}(x)\mu_\alpha^{(N)}(\mathrm{d}x).$$

According to [6, Theorem 1.1], we have

$$\text{gap}(L_\alpha^N) = \alpha_{N+1}.$$

Not that when $N=1$, $\text{gap}(L_\alpha^{(1)}) = \alpha_1 + \alpha_2 > \alpha_2$.

Moreover, the whole spectrum of $L_\alpha^{(N)}$ has been characterized in [6]. In particular, the essential spectrum is empty, so that the super Poincaré inequality

(1.7) $$\mu_\alpha^{(N)}(f^2) \leq r\mathscr{E}_\alpha^{(N)}(f,f) + \beta(r)\mu_\alpha^{(N)}(|f|)^2, \quad r>0, f\in C^1(\Delta^{(N)})$$

holds for some function $\beta : (0,\infty) \to (0,\infty)$. However, there is no any explicit estimate on $\beta(r)$ and hence, both **the log-Soblev and the Nash inequalities are unknown**.

1.3 Questions and the Main result

According to the last subsection, the following two things remain unknown.

(Q_1) Nash inequality for the Wright-Fisher diffusion with mutation and multivariate Dirichlet diffusion processes.

(Q_2) Estimate on the log-Sobolev constant for the multivariate Dirichlet diffusion, and the sharp log-Sobolev constant for the Wright-Fisher/Wright-Fisher diffusion with mutation/GEM processes.

In this paper, we only investigate (Q_1), and the main result is the following.

Theorem 1.1. *Let $N \geq 2$.*

(1) *There exists a constant $C > 0$ such that the Nash inequality*

$$\mu_\alpha^{(N)}(f^2) \leq C\mathscr{E}_\alpha^{(N)}(f,f)^{\frac{p}{p+1}}\mu_\alpha^{(N)}(|f|)^{\frac{2}{p+1}}, \quad f \in \mathscr{D}(\mathscr{E}_\alpha^{(N)}), \mu_\alpha^{(N)}(f) = 0 \tag{1.8}$$

holds for $p = p_\alpha := \sum_{i=1}^N 1 \vee (2\alpha_i) + (\alpha_{N+1} - 1)^+$, and the inequality remains true for $\mathscr{E}_{FV}^{(\alpha,N)}$ replacing $\mathscr{E}_\alpha^{(N)}$.

(2) *If* (1.8) *holds for some constant $C > 0$, then*

$$p \geq \tilde{p}_\alpha := \max\Big\{\max_{1\leq i\leq N+1} \sum_{j\neq i, 1\leq j\leq N+1} \alpha_j,\ \alpha_{N+1} + \max_{1\leq i\leq N} \sum_{j\neq i, 1\leq j\leq N} (1 \vee \alpha_j)\Big\}.$$

(3) *If* (1.8) *with $\mathscr{E}_{FV}^{\alpha,N}$ replacing $\mathscr{E}_\alpha^{(N)}$ holds for some constant $C > 0$, then*

$$p \geq p'_\alpha := \max\Big\{\sum_{1\leq j\leq N} \alpha_i,\ \frac{1}{2}\alpha_{N+1} + \frac{1}{2}\max_{1\leq i\leq N} \sum_{j\neq i, 1\leq j\leq N} (1 \vee \alpha_j)\Big\}.$$

Remark 1.2. (1) Let p_c be the smallest positive constant $p > 0$ such that (1.8) holds for some constant $C > 0$, then assertions (1)-(2) in Theorem 1.1 imply $p_c \in [\tilde{p}_\alpha, p_\alpha]$. In particular, when $\max_{1\leq i\leq N} \alpha_i \leq \frac{1}{2}$ and $\alpha_{N+1} \geq 1$, we have $p_c = N + \alpha_{N+1} - 1$; that is, in this case the Nash inequality presented in Theorem 1.1(1) is sharp for $\mathscr{E}_\alpha^{(N)}$. But the sharpness for $\mathscr{E}_{FV}^{\alpha,N}$ is unknown.

(2) As mentioned in the end of Subsection 1.1 that the Nash inequality (1.8) implies the log-Sobolev inequality

$$\mu_\alpha^{(N)}(f^2 \log f^2) \leq C\mathscr{E}_\alpha^{(N)}(f,f), \quad f \in \mathscr{D}(\mathscr{E}_\alpha^{(N)}), \mu_\alpha^{(N)}(f^2) = 1$$

for some constant $C > 0$. However, in the moment we do not have any explicit estimate on the log-Sobolev constant $C_{LS}(L_\alpha^{(N)})$.

(3) Consider the infinite-dimensional setting where $N = \infty$ and $\alpha = (\alpha_i)_{i\in\bar{\mathbb{N}}}$ with $|\alpha|_1 := \sum_{i\in\bar{\mathbb{N}}} \alpha_i < \infty$. According to [16, 6], we have

$$\text{gap}(L_{FV}^{\alpha,\infty}) = |\alpha|_1, \quad \text{gap}(L_\alpha^{(\infty)}) = \alpha_\infty.$$

Next, [16, Theorem 3.5] shows that the set

$$D_0 := \left\{ f \in \mathscr{D}(\mathscr{E}_{FV}^{\alpha,\infty}) : \ \mu_\alpha^{(\infty)}(f^2) + \mathscr{E}_{FV}^{\alpha,\infty}(f,f) \le 1 \right\}$$

is not uniform integrable in $L^2(\mu_\alpha^{(\infty)})$, so that the super Poincaré inequality is not available for $(\mathscr{E}_{FV}^{\alpha,\infty}, \mu_\alpha^{(\infty)})$. Indeed, by [18, Theorem 1.2] (see also [17, 19]), if there exists $\beta : (0,\infty) \to (0,\infty)$ such that

$$\mu_\alpha^{(\infty)}(f^2) \le r\mathscr{E}_{FV}^{\alpha,\infty}(f,f) + \beta(r)\mu_\alpha^{(\infty)}(|f|)^2, \quad r > 0, f \in \mathscr{D}(\mathscr{E}_{FV}^{\alpha,\infty}),$$

then there exists a positive increasing function F on $[0,\infty)$ with $F(r) \uparrow \infty$ as $r \uparrow \infty$ such that

$$\mu_\alpha^{(\infty)}(f^2 F(f^2)) \le \mathscr{E}_{FV}^{\alpha,\infty}(f,f), \quad f \in \mathscr{D}(\mathscr{E}_{FV}^{\alpha,\infty}), \ \mu_\alpha^{(\infty)}(f^2) \le 1,$$

and hence D_0 is uniform integrable in $L^2(\mu_\alpha^{(\infty)})$. Since $\mathscr{E}_{FV}^{\alpha,\infty} \ge \mathscr{E}_\alpha^{(\infty)}$, see the beginning of Section 3 for finite N, the super Poincaré inequality is invalid for $\mathscr{E}_\alpha^{(\infty)}$.

To prove Theorem 1.1, we will present a localization theorem in Section 2, which enables one to establish the super Poincaré inequality (1.7) by using local inequalities. A complete proof of Theorem 1.1 will be addressed in Section 3 and Section 4.

2 Preparations

To establish (1.7) with an explicit rate function β, the main difficulty comes from the singularity of the density $\rho(x)$ as well as the degeneracy of the diffusion coefficient on the boundary

$$\partial\Delta^{(N)} = \Big\{ x = (x_i)_{1\le i\le N} \in \Delta^{(N)} : \ \min\{x_i : 1 \le i \le N+1\} = 0 \Big\}, \quad x_{N+1} := 1 - \sum_{i=1}^N x_i.$$

To overcome such difficulties, a localization result has been presented in [19, Theorem 3.4.6]. However, this result is less sharp and inconvenient for application to the present model. So, in this section we give a new version of this result. We will also present an additivity property of the super Poincaré inequality, which is more or less trivial but will be used to establish local super Poincaré inequalities in the proof of Theorem 1.1(1).

2.1 A localization result

Let $(E, \mathscr{F}, \mu)$ be a separable complete probability space, and $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a conservative symmetric local Dirichlet form on $L^2(\mu)$ as the closure of

$$\mathscr{E}(f,g) = \mu(\Gamma(f,g)), \quad f, g \in \mathscr{D}_0(\Gamma),$$

where $\Gamma : \mathscr{D}(\Gamma) \times \mathscr{D}(\Gamma) \to \mathscr{B}(E)$ is a positive definite symmetric bilinear mapping, $\mathscr{B}(E)$ is the set of all μ-a.e. finite measurable real functions on E, $\mathscr{D}(\Gamma)$ is a sub-algebra of $\mathscr{B}(E)$, and $\mathscr{D}_0(\Gamma) := \{f \in \mathscr{D}(\Gamma) : f^2, \Gamma(f,f) \in L^1(\mu)\}$ such that

(a) $\mathscr{D}_0(\Gamma)$ is dense in $L^2(\mu)$.

(b) $\mathscr{D}(\Gamma)$ is closed under combinations with $\psi \in C([-\infty,\infty])$ such that ψ is C^1 in $\mathbb{R}$ and ψ' has compact support, and $\Gamma(\psi\circ f, g) = \psi'(f)\Gamma(f,g)$ μ-a.e. for $f,g\in\mathscr{D}(\Gamma)$.

(c) $\Gamma(fg,h) = g\Gamma(f,h) + f\Gamma(g,h)$ μ-a.e. for $f,g,h\in\mathscr{D}(\Gamma)$.

We aim to establish the super Poincaré inequality (1.4) with an explicit $\beta:(0,\infty)\to(0,\infty)$.

Theorem 2.1. *Let $\phi\in\mathscr{D}(\Gamma)$ be an unbounded nonnegative function such that*

$$h(s) := \sup_{D_s}\Gamma(\phi,\phi) < \infty,\quad D_s := \{\phi\le s\},\quad s\ge 0, \tag{2.1}$$

where $\sup_\emptyset = 0$ by convention. If there exists $s_0\ge 1$ such that for every $s\ge s_0$, the local super Poincaré inequality

$$\mu(f^2)\le r\mathscr{E}(f,f) + \beta_s(r)\mu(|f|)^2,\quad r>0, f\in\mathscr{D}(\mathscr{E}), f|_{D_s^c} = 0 \tag{2.2}$$

holds for some decreasing function $\beta_s:(0,\infty)\to(0,\infty)$, and for $s\ge s_0$

$$0<\lambda(s) := \inf\{\mathscr{E}(f,f):\mu(f^2)=1, f|_{D_s}=0\}\uparrow\infty \text{ as } s\uparrow\infty. \tag{2.3}$$

Then

$$s_r := \inf\{s\ge s_0:\lambda(s)\ge 8r^{-1}\}\in(0,\infty),\quad r>0, \tag{2.4}$$

and there exists a constant $c>0$ such that the super Poincaré inequality (1.4) holds for

$$\beta(r) := c + \Big(2+\frac{rh(2s_r)}{s_r^2}\Big)\beta_{3s_r}\Big(\frac{r}{8+2rh(2s_r)s_r^{-2}}\Big),\quad r>0. \tag{2.5}$$

Proof. By condition (a), it suffices to consider $f\in\mathscr{D}_0(\Gamma)$. For any $s\ge s_0$ and small $\varepsilon\in(0,1)$, let $\varphi_i\in C^1([0,\infty])$ with $0\le\varphi_i\le 1, |\varphi_i'(s)|\le(1+\varepsilon)s^{-1}, i=1,2$ such that

$$\varphi_1|_{[0,s]} = 0,\quad \varphi_1|_{[2s,\infty]} = 1;\quad \varphi_2|_{[0,2s]} = 1,\quad \varphi_2|_{[3s,\infty]} = 0.$$

Let $f_i = f\cdot\varphi_i\circ\phi, 1\le i\le 2$. Then $f^2\le f_1^2+f_2^2$ and by conditions (b) and (c),

$$\begin{aligned}\Gamma(f_1,f_1) &\le 2\Gamma(f,f) + 2(1+\varepsilon)^2 f^2 s^{-2}1_{\{\phi\le 2s\}}h(2s)\\ &\le 2\Gamma(f,f) + \frac{2(1+\varepsilon)^2h(2s)}{s^2}f_2^2,\\ \Gamma(f_2,f_2) &\le 2\Gamma(f,f) + \frac{2}{s^2}(1+\varepsilon)^2f^2.\end{aligned}$$

In particular, $f_1,f_2\in\mathscr{D}_0(\Gamma)\subset\mathscr{D}(\mathscr{E})$. Combining these with (2.2) and (2.3), we obtain

$$\begin{aligned}\mu(f^2)&\le\mu(f_1^2)+\mu(f_2^2)\le\frac{2}{\lambda(s)}\mathscr{E}(f,f) + \Big(1+\frac{2(1+\varepsilon)^2h(2s)}{s^2\lambda(s)}\Big)\mu(f_2^2)\\ &\le\Big\{\frac{2}{\lambda(s)} + 2t\Big(1+\frac{2(1+\varepsilon)^2h(2s)}{s^2\lambda(s)}\Big)\Big\}\mathscr{E}(f,f) + \frac{2(1+\varepsilon)^2t}{s^2}\Big(1+\frac{2(1+\varepsilon)^2h(2s)}{s^2\lambda(s)}\Big)\mu(f^2)\\ &\quad+\Big(1+\frac{2(1+\varepsilon)^2h(2s)}{s^2\lambda(s)}\Big)\beta_{3s}(t)\mu(|f|)^2,\quad t>0.\end{aligned}\tag{2.6}$$

Let $r \in (0,1]$, and s_r be in (2.4). We have $\lambda(s_r) \ge 8r^{-1}$, so that

$$\frac{2}{\lambda(s_r)} \le \frac{r}{4}, \quad t_r := \frac{r}{8 + 16h(2s_r)/[s_r^2\lambda(s_r)]} \ge \frac{r}{8 + 2rs_r^{-2}h(2s_r)},$$

and when $\varepsilon > 0$ is small enough,

$$\begin{aligned}
&(1+\varepsilon)^2 \le 2, \quad \frac{2(1+\varepsilon)^2 h(2s_r)}{s_r^2\lambda(s_r)} \le \frac{rh(2s_r)}{2s_r^2},\\
&2(1+\varepsilon)^2 t_r\Big(1 + \frac{2(1+\varepsilon)^2 h(2s_r)}{s_r^2\lambda(s_r)}\Big) \le \frac{3r}{8},\\
&\frac{2(1+\varepsilon)^2 t_r}{s_r^2}\Big(1 + \frac{2(1+\varepsilon)^2 h(2s_r)}{s_r^2\lambda(s_r)}\Big) \le \frac{3r}{8} \le \frac{3}{8}, \quad r \in (0,1].
\end{aligned}$$

Combining these with (2.6) we arrive at

$$\mu(f^2) \le \frac{5r}{8}\mathscr{E}(f,f) + \frac{3}{8}\mu(f^2) + \Big(1 + \frac{rh(2s_r)}{2s_r^2}\Big)\beta_{3s}\Big(\frac{r}{8 + 2rs_r^{-2}h(2s_r)}\Big)\mu(|f|)^2, \quad r \in (0,1].$$

Therefore,

$$\mu(f^2) \le r\mathscr{E}(f,f) + \Big(2 + \frac{rh(2s_r)}{s_r^2}\Big)\beta_{3s}\Big(\frac{r}{8 + 2rs_r^{-2}h(2s_r)}\Big)\mu(|f|)^2, \quad r \in (0,1].$$

Since for the super Poincaré inequality we may take decreasing β, this finishes the proof.

□

2.2 Additivity of super Poincaré inequality

For every $1 \le i \le N$, let $(\mathscr{E}_i, \mathscr{D}(\mathscr{E}_i))$ be a symmetric Dirichlet form on $L^2(\mu_i)$ over a σ-finite measure space $(E_i, \mathscr{F}_i, \mu_i)$. Let $\mu = \prod_{i=1}^N \mu_i$, and $\mathscr{D}(\mathscr{E})$ be the class of $f \in L^2(\mu)$ such that for any $1 \le i \le N$ and $\big(\prod_{j\ne i}\mu_j\big)$-a.e. x, we have $f(x,\cdot) \in \mathscr{D}(\mathscr{E}_i)$ and

$$\mathscr{E}(f,f) := \sum_{i=1}^N \int_{\prod_{j\ne i} E_j} \mathscr{E}_i(f(x,\cdot), f(x,\cdot))\Big(\prod_{j\ne i}\mu_j\Big)(\mathrm{d}x) < \infty.$$

Consider the following Dirichlet form on $L^2(\mu)$:

$$\mathscr{E}(f,g) := \sum_{i=1}^N \int_{\prod_{j\ne i} E_j} \mathscr{E}_i(f(x,\cdot), g(x,\cdot))\Big(\prod_{j\ne i}\mu_j\Big)(\mathrm{d}x), \quad f,g \in \mathscr{D}(\mathscr{E}).$$

The following additivity property is a simple consequence of the equivalence between the heat kernel upper bound and the super Poincaré inequality.

Proposition 2.2. *Let $\{p_i\}_{1\le i\le N}\subset(0,\infty)$ such that for any $1\le i\le N$, the super Poincaré inequality*

$$\mu_i(f^2)\le r\mathscr{E}_i(f,f)+c_i(1+r^{-p_i})\mu_i(|f|)^2,\quad r>0, f\in\mathscr{D}(\mathscr{E}_i) \tag{2.7}$$

holds for some constant $c_i>0$. Then there exists a constant $c>0$ such that

$$\mu(f^2)\le r\mathscr{E}(f,f)+c\big(1+r^{-\sum_{i=1}^N p_i}\big)\mu(|f|)^2,\quad r>0, f\in\mathscr{D}(\mathscr{E}). \tag{2.8}$$

Proof. Let P_t^i be the (sub) Markov semigroup associated with $(\mathscr{E}_i,\mathscr{D}(\mathscr{E}_i))$. By [19, Theorem 3.3.15(2)], (2.7) implies that P_t^i has a density $p_t^i(x_i,y_i)$ with respect to μ_i such that

$$\mathrm{ess}_{\mu_i\times\mu_i}\sup p_t^i\le C_i(1+t^{-p_i}),\quad t>0$$

holds for some constant $C_i>0$. Then the semigroup P_t associated with $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ has the density

$$p_t(x,y):=\prod_{i=1}^N p_t^i(x_i,y_i),\quad x=(x_1,\cdots,x_N), y=(y_1,\cdots,y_N)\in E:=\prod_{i=1}^N E_i$$

with respect to μ, and

$$\mathrm{ess}_{\mu\times\mu}\sup p_t\le C\big(1+t^{-\sum_{i=1}^N p_i}\big),\quad t>0$$

holds for some constant $C>0$. By [19, Theorem 3.3.15(2)] again, this implies that (2.8) holds for some constant $c>0$. □

3 Proof of Theorem 1.1(1)

We first observe that for any $f\in C^1(\Delta^{(N)})$,

$$\begin{aligned}\sum_{i,j=1}^N x_i(\delta_{ij}-x_j)(\partial_i f)\partial_j f&=\sum_{i=1}^N x_i(\partial_i f)^2-\sum_{i,j=1}^N x_ix_j(\partial_i f)\partial_j f\\ &\ge\sum_{i=1}^N x_i(\partial_i f)^2-\sum_{i,j=1}^N x_ix_j\cdot\frac{(\partial_i f)^2+(\partial_j f)^2}{2}=\sum_{i=1}^N x_i(1-|x|_1)(\partial_i f)^2.\end{aligned}$$

So, $\mathscr{E}_\alpha^{(N)}(f,f)\le\mathscr{E}_{FV}^{\alpha,N}(f,f)$, and we only need to prove the desired Nash inequality for $(\mathscr{E}_\alpha^{(N)},\mu_\alpha^{(N)})$. To this end, it suffices to prove

$$\mu_\alpha^{(N)}(f^2)\le r\mathscr{E}_\alpha^{(N)}(f,f)+cr^{-p_\alpha}\mu_\alpha^{(N)}(|f|)^2,\quad r\in(0,r_1],\ f\in C^1(\Delta^{(N)}) \tag{3.1}$$

for some constants $c,r_1>0$. Indeed, this inequality is equivalent to

$$\mu_\alpha^{(N)}(f^2)\le r\mathscr{E}_\alpha^{(N)}(f,f)+c(r\wedge r_1)^{-p_\alpha}\mu_\alpha^{(N)}(|f|)^2,\quad r>0,\ f\in C^1(\Delta^{(N)}). \tag{3.2}$$

Since by [6, Theorem 1.1] the generator $L_\alpha^{(N)}$ has spectral gap $\alpha_{N+1} > 0$, there holds

$$\mu_\alpha^{(N)}(f^2) \le \frac{1}{\alpha_{N+1}} \mathscr{E}_\alpha^{(N)}(f,f), \quad r>0,\ f \in C^1(\Delta^{(N)}), \mu_\alpha^{(N)}(f) = 0. \tag{3.3}$$

Noting that for some constant $c(r_1, \alpha_{N+1}) > 0$ we have

$$(r \wedge r_1)^{-p_\alpha} \le c(r_1, \alpha_{N+1}) r^{-p_\alpha}, \quad r \in (0, \alpha_{N+1}^{-1}),$$

so that (3.2) and (3.3) yield

$$\mu_\alpha^{(N)}(f^2) \le r\mathscr{E}_\alpha^{(N)}(f,f) + c' r^{-p_\alpha} \mu_\alpha^{(N)}(|f|)^2, \quad r>0,\ f \in C^1(\Delta^{(N)}), \mu_\alpha^{(N)}(f) = 0$$

for some constant $c' > 0$. Minimizing the upper bound in $r > 0$, we prove (1.8) for some constant $C > 0$ and $p = p_\alpha$.

To prove (3.1) using Theorem 2.1, we denote $x_{N+1} = 1 - |x|_1 = 1 - \sum_{i=1}^N x_i$ and take

$$\phi(x) = x_{N+1}^{-1}, \quad x = (x_i)_{1 \le i \le N} \in \Delta^{(N)}. \tag{3.4}$$

Then

$$D_s := \{\phi \le s\} = \big\{x \in \Delta^{(N)} :\ x_{N+1} \ge s^{-1}\big\}, \quad s > 1 \tag{3.5}$$

holds. For the present model we have

$$\Gamma(\phi,\phi)(x) = \sum_{i=1}^N x_i x_{N+1} (\partial_i \phi)^2(x) = \frac{1 - x_{N+1}}{x_{N+1}^3} \le s^3, \quad x \in D_s, s > 0,$$

so that

$$h(s) := \sup_{D_s} \Gamma(\phi,\phi) \le s^3, \quad s > 0. \tag{3.6}$$

To apply Theorem 2.1, we take

$$\mathscr{D}(\Gamma) = \Big\{f \in C(\Delta^{(N)}; [-\infty, \infty]) : f \text{ is finite and } C^1 \text{ in } \Delta^{(N)} \setminus \{x_{N+1} = 0\}\Big\},$$

and let

$$\Gamma(f,g)(x) = 1_{\{x_{N+1} > 0\}} \sum_{i=1}^N x_i x_{N+1} (\partial_i f)(x)(\partial_i g)(x), \quad f, g \in \mathscr{D}(\Gamma).$$

Obviously, conditions (a)-(c) hold, and the function ϕ in (3.4) meets the requirement of Theorem 2.1. In the following two subsections, we estimate $\lambda(s)$ and β_s respectively.

3.1 Estimate on $\lambda(s)$

Let $\lambda(s) = \inf\{\mathscr{E}_\alpha^{(N)}(f,f) : \ f \in C^1(\Delta^{(N)}), \mu_\alpha^{(N)}(f^2) = 1, f|_{D_s} = 0\}$. We will adopt the following Cheeger type estimate for $\lambda(s)$. Let

$$\partial D_s = \{x \in \Delta^{(N)} : x_{N+1} = s^{-1}\};$$

$$D_s^c = \Delta^{(N)} \backslash D_s, \ s \geq 1.$$

Lemma 3.1. *If there exists a function $\psi \in C^2(\Delta^{(N)} \setminus \{x_{N+1} = 0\})$ such that*

$$\Gamma(\psi,\psi)(x) := \sum_{i=1}^N x_i x_{N+1} (\partial_i \psi)^2(x) \leq a_1, \quad |L_\alpha^{(N)}\psi|(x) \geq a_2, \quad x \in D_s^c \tag{3.7}$$

holds for some constants $a_1, a_2 > 0$, and that

$$\lim_{r\to\infty} \sup_{x\in\partial D_r} x_{N+1}^{\alpha_{N+1}} \sum_{i=1}^N x_i |\partial_i \psi(x)| = 0, \tag{3.8}$$

then

$$\lambda(s) \geq \frac{a_2^2}{4a_1}.$$

Proof. By (3.7), since D_s^c is connected and $L_\alpha^{(N)}\psi$ is continuous, we may assume that $L_\alpha^{(N)}\psi|_{D_s^c} \geq a_2$, otherwise simply use $-\psi$ replacing ψ. Let $\sigma(x) = \mathrm{diag}\{\sqrt{x_i x_{N+1}}\}_{1\leq i\leq N}$. For any nonnegative $f \in C^1(\Delta^{(N)})$ with $f|_{D_s} = 0$, we have $f|_{\partial D_s} = 0$. So, by integration by parts formula,

$$\begin{aligned} a_2 \mu_\alpha^{(N)}(f) &\leq \mu_\alpha^{(N)}(f L_\alpha^{(N)}\psi) = \lim_{r\to\infty} \int_{D_r \backslash D_s} (\rho f L_\alpha^{(N)}\psi)(x)\mathrm{d}x \\ &\leq -\mu_\alpha^{(N)}\big(\langle \sigma\nabla f, \sigma\nabla\psi\rangle\big) + \|f\|_\infty \limsup_{r\to\infty} \int_{\partial D_r} \sum_{i=1}^N x_i x_{N+1} \rho(x) |\partial_i \psi|(x)\mathrm{d}A, \end{aligned} \tag{3.9}$$

where A is the area measure on ∂D_r induced by the Lebesgue measure. We have

$$\partial D_r = \Big\{x \in \Delta^{(N)} : \sum_{i=1}^N x_i = 1 - r^{-1}\Big\}, \quad r \geq 2,$$

and

$$\int_{\partial D_r} \prod_{i=1}^N x_i^{\alpha_i - 1}\mathrm{d}A = (1-r^{-1})^{\sum_{i=1}^N \alpha_i} \int_{\Delta^{(N-1)}} \Big(1 - \sum_{1\leq i\leq N-1} x_i\Big)^{\alpha_N - 1} \prod_{i=1}^{N-1} x_i^{\alpha_i - 1}\mathrm{d}x$$

is bounded in $r \geq 2$. Combining this with (1.1), (3.7), (3.8) and (3.9), we obtain

$$a_2 \mu_\alpha^{(N)}(f) \leq |\mu_\alpha^{(N)}\big(\langle \sigma\nabla f, \sigma\nabla\psi\rangle\big)| \leq \sqrt{a_1}\mu_\alpha^{(N)}(|\sigma\nabla f|).$$

Therefore, for any $f \in C^1(\Delta^{(N)})$ with $f|_{D_s} = 0$,

$$\mu_\alpha^{(N)}(f^2) \leq \frac{\sqrt{a_1}}{a_2}\mu_\alpha^{(N)}(|\sigma\nabla f^2|) \leq \frac{2\sqrt{a_1}}{a_2}\sqrt{\mu_\alpha^{(N)}(f^2)\mu_\alpha^{(N)}(|\sigma\nabla f|^2)}.$$

Noting that $\mu_\alpha^{(N)}(|\sigma\nabla f|^2) = \mathscr{E}_\alpha^{(N)}(f,f)$, we arrive at

$$\mu_\alpha^{(N)}(f^2) \leq \frac{4a_1}{a_2^2}\mathscr{E}_\alpha^{(N)}(f,f), \quad f \in C_b^1(\Delta^{(N)}), f|_{D_s} = 0,$$

which finishes the proof. □

Lemma 3.2. *There exist constants $s_0, c_0 > 0$ such that*

$$\lambda(s) \geq c_0 s, \quad s \geq s_0.$$

Proof. Let $\gamma \in [\frac{1}{2}, 1) \cap (1 - \alpha_{N+1}, 1)$. Take

$$\psi(x) = x_{N+1}^\gamma, \quad x \in \Delta^{(N)}.$$

Then

$$\begin{aligned}\Gamma(\psi,\psi)(x) &= \sum_{i=1}^N x_i x_{N+1}(\partial_i\psi)^2(x) = \gamma^2(1 - x_{N+1})x_{N+1}^{2\gamma-1} \\ &\leq \gamma^2 s^{1-2\gamma}, \quad x \in D_s^c,\end{aligned} \tag{3.10}$$

and

$$\lim_{s\to\infty}\sup_{x\in D_s} x_{N+1}^{\alpha_{N+1}}\sum_{i=1}^N x_i|\partial_i\psi(x)| \leq \lim_{s\to\infty}\gamma s^{1-\alpha_{N+1}-\gamma} = 0. \tag{3.11}$$

Let $s_0 \geq 1$ such that

$$(1 + \alpha_{N+1} - \gamma)(1 - s^{-1}) \geq 1 - \gamma + s^{-1}\sum_{i=1}^N \alpha_i, \quad s \geq s_0.$$

Then for $x \in D_s^c$ and $s \geq s_0$,

$$\begin{aligned}L_\alpha^{(N)}\psi(x) &= \sum_{i=1}^N \big\{x_i x_{N+1}\partial_i^2\psi(x) + (\alpha_i x_{N+1} - \alpha_{N+1}x_i)\partial_i\psi(x)\\ &= \gamma(1 + \alpha_{N+1} - \gamma)(1 - x_{N+1})x_{N+1}^{\gamma-1} - \gamma x_{N+1}^\gamma\sum_{i=1}^N \alpha_i \\ &\geq \gamma(1-\gamma)s^{1-\gamma}.\end{aligned}$$

Combining this with (3.10) and (3.11), we derive from Lemma 3.1 that

$$\lambda(s) \geq \frac{\gamma^2(1-\gamma)^2 s^{2(1-\gamma)}}{4\gamma^2 s^{1-2\gamma}} = \frac{(1-\gamma)^2}{4}s, \quad s \geq s_0.$$

□

3.2 Estimate on $\beta_s(r)$

We first present a sharp super Poincaré inequality for a product probability measure, then estimate $\beta_s(r)$ using a perturbation argument. Consider the following probability measures on $[0,1]$:

$$\mu_i(\mathrm{d}s) = \alpha_i s^{\alpha_i - 1}\mathrm{d}s, \quad 1 \le i \le N,$$

and let $\mu = \prod_{i=1}^N \mu_i$ on $[0,1]^N$. We have the following result.

Lemma 3.3. *Let $p(\alpha) = \sum_{i=1}^N (\frac{1}{2} \vee \alpha_i)$. There exists a constant $c > 0$ such that*

$$\mu(f^2) \le r \int_{[0,1]^N} \sum_{i=1}^N x_i (\partial_i f)^2(x)\mu(\mathrm{d}x) + c\big(r^{-p(\alpha)} + 1\big)\mu(|f|)^2, \quad r > 0, f \in C^1([0,1]^N).$$

Proof. By Proposition 2.2, it suffices to prove that for every $1 \le i \le N$ there exists a constant $c_i > 0$ such that

$$\mu_i(f^2) \le r \int_0^1 s f'(s)^2 \mathrm{d}s + c_i(1 + r^{-(\frac{1}{2}\vee\alpha_i)})\mu_i(|f|)^2, \quad r > 0, f \in C^1([0,1]). \tag{3.12}$$

For fixed $1 \le i \le N$, we will prove this inequality using isoperimetric constants

$$\kappa(r) := \inf_{I \subset [0,1], 0 < \mu(I) \le r} \frac{A_i((\partial I) \setminus \{0,1\})}{\mu_i(I)}, \quad r \in (0, 1/2),$$

where A_i is the boundary measure induced by μ_i and the intrinsic metric of the square field $\Gamma_0(f,f)(s) := s(f')^2(s)$ on $[0,1]$. Let $r \in (0, \frac{1}{2})$, for any measurable set $I \subset [0,1]$ with $\mu_i(I) = r$, we may find out $a \in (\partial I) \setminus \{0,1\}$ such that $a \ge r^{\frac{1}{\alpha_i}}$. Otherwise, $[r^{\frac{1}{\alpha_i}}, 1)$ is either in the interior of I or in that of I^c. For the first case we have

$$r = \mu_i(I) \ge \alpha_i \int_{r^{\frac{1}{\alpha_i}}}^1 s^{\alpha_i - 1}\mathrm{d}s = 1 - r > \frac{1}{2} > r$$

which is a contraction; while in the second case we may find a small constant $\varepsilon > 0$ such that $[r^{\frac{1}{\alpha_i}} - \varepsilon, 1) \subset I^c$, so that

$$r = \mu_i(I) \le \alpha_i \int_0^{r^{\frac{1}{\alpha_i}} - \varepsilon} s^{\alpha_i - 1}\mathrm{d}s = (r^{\frac{1}{\alpha_i}} - \varepsilon)^{\alpha_i} < r$$

which is again impossible. Since the intrinsic metric induced by Γ_0 is

$$\mathrm{d}(s,t) := 2|\sqrt{s} - \sqrt{t}|, \quad s,t \in [0,1],$$

the corresponding boundary measure of $\{a\}$ is given by

$$A_i(\{a\}) := \lim_{\varepsilon \downarrow 0} \frac{\mu_i([a-\varepsilon, a])}{2(\sqrt{a} - \sqrt{a-\varepsilon})} = \sqrt{a}a^{\alpha_i - 1} = a^{\alpha_i - \frac{1}{2}}.$$

Therefore,

$$\frac{A_i((\partial I)\setminus\{0,1\})}{\mu_i(I)} \ge \frac{A_i(\{a\})}{r} \ge r^{-(1\wedge\frac{1}{2\alpha_i})}.$$

Hence,

$$\kappa(r) \ge r^{-(1\wedge\frac{1}{2\alpha_i})}, \quad r\in(0,1/2).$$

According to [19, Theorem 3.4.16(1)], this implies (3.12) for some constant $c_i>0$. □

Lemma 3.4. *Let* $p(\alpha)=\sum_{i=1}^N(\frac{1}{2}\vee\alpha_i)$. *There exist constants* $c_0, s_0>0$ *such that for any* $s\ge s_0$,

$$\mu_\alpha^{(N)}(f^2)\le r\mathscr{E}_\alpha^{(N)}(f,f)+\beta_s(r)\mu_\alpha^{(N)}(|f|)^2,\quad r>0, f\in C^1(\Delta^{(N)}), f|_{D_s^c}=0$$

holds for

$$\beta_s(r)=c_0 s^{p(\alpha)+(\alpha_{N+1}-1)^+}\big(r^{-p(\alpha)}+s^{p(\alpha)}\big),\quad r>0. \tag{3.13}$$

Proof. Let $f\in C^1(\Delta^{(N)}),\ f|_{D_s^c}=0$. For simplicity, we will regard x_i as the function mapping $x\in\Delta^{(N)}$ into $x_i, 1\le i\le N+1$. Recall that $x_{N+1}:=1-\sum_{i=1}^N x_i$. Applying Lemma 3.3 to $g:=x_{N+1}^{(\alpha_{N+1}-1)/2}f$ replacing f, which is supported on D_s, we may find out constants $c_1,c_2,c_3,c_4>0$ such that for any $t>0$ and $s\ge 1$,

$$\begin{aligned}
\mu_\alpha^{(N)}(f^2)&=c_1\mu(g^2)\le c_1 t\mu_\alpha^{(N)}\Big(\sum_{i=1}^N x_i(\partial_i g)^2\Big)+c_2(1+t^{-p(\alpha)})\mu_\alpha^{(N)}(|g|)^2\\
&\le tc_3\mu_\alpha^{(N)}\Big(\sum_{i=1}^N x_i\{(\partial_i f)^2+x_{N+1}^{-2}f^2\}\Big)+c_2(1+t^{-p(\alpha)})\mu_\alpha^{(N)}\big(x_{N+1}^{-\frac{\alpha_{N+1}-1}{2}}|f|\big)^2\\
&\le c_3ts\mu_\alpha^{(N)}\Big(\sum_{i=1}^N x_ix_{N+1}(\partial_i f)^2\Big)+c_3ts^2\mu_\alpha^{(N)}(f^2)+c_2(1+t^{-p(\alpha)})s^{(\alpha_{N+1}-1)^+}\mu_\alpha^{(N)}(|f|)^2\\
&\le c_3ts\mathscr{E}_\alpha^{(N)}(f,f)+c_3ts^2\mu_\alpha^{(N)}(f^2)+c_2(1+t^{-p(\alpha)})s^{(\alpha_{N+1}-1)^+}\mu_\alpha^{(N)}(|f|)^2.
\end{aligned}$$

For any $r>0$, take

$$t=\frac{r}{2c_3s}\wedge\frac{1}{2c_3s^2}.$$

We may find out a constant $c>0$ such that the above gives

$$\mu_\alpha^{(N)}(f^2)\le r\mathscr{E}_\alpha^{(N)}(f,f)+c\big(r^{-p(\alpha)}+s^{p(\alpha)}\big)s^{p(\alpha)+(\alpha_{N+1}-1)^+}\mu_\alpha^{(N)}(|f|)^2,\quad r>0.$$

Therefore, the proof is finished.

□

3.3 Proof of (3.1)

By (2.4) and Lemma 3.2, there exist constants $r_0, c_1 > 0$ such that

$$s_r \le c_1 r^{-1}, \quad r \in (0, r_0]. \tag{3.14}$$

Combining this with (3.6), we obtain

$$\frac{rh(2s_r)}{s_r^2} \le 8rs_r \le 8c_1.$$

So, there exist constants $r_1 \in (0, r_0]$ and $c_2 > 1$ such that for any $r \in (0, r_1]$,

$$\frac{4s_r^2 + h(2s_r)r}{2s_r^2} \le c_2, \quad \frac{r}{8 + 2rh(2s_r)s_r^{-2}} \ge \frac{r}{c_2}.$$

Combining these with (3.13) and (3.14), we may find out constants $c_3, c_4 > 0$ such that $\beta(r)$ in (2.5) satisfies

$$\begin{aligned}\beta(r) &\le c + 2\beta_{3s_r}(r/c_2) \le c_3 s_r^{p(\alpha)+(\alpha_{N+1}-1)^+}\big(r^{-p(\alpha)} + s_r^{p(\alpha)}\big)\\ &\le c + c_4 r^{-\{2p(\alpha)+(\alpha_{N+1}-1)^+\}}, \quad r \in (0, r_1].\end{aligned}$$

This completes the proof since

$$2p(\alpha) + (\alpha_{N+1} - 1)^+ = \sum_{i=1}^N 1 \vee (2\alpha_i) + (\alpha_{N+1} - 1)^+ = p_\alpha.$$

4 Proof of Theorem 1.1(2)-(3)

Proof of Theorem 1.1(2). Let (1.8) hold. We aim to prove $p \ge \tilde{p}_\alpha^{(1)}$ and $p \ge \tilde{p}_\alpha^{(2)}$ respectively, where

$$\begin{aligned}\tilde{p}_\alpha^{(1)} &:= \alpha_{N+1} + \max_{1\le i\le N} \sum_{1\le j\le N, j\ne i} (1 \vee \alpha_j),\\ \tilde{p}_\alpha^{(2)} &:= \max_{1\le i\le N+1} \sum_{1\le j\le N+1, j\ne i} \alpha_j.\end{aligned}$$

(a) Let $1 \le i_0 \le N$ be such that $\alpha_{i_0} = \min_{1\le i\le N} \alpha_i$. Let

$$I_1 = \{i_0\} \cup \{1 \le i \le N : \alpha_i \le 1\}, \quad I_2 = \{1, \cdots, N\} \setminus I_1.$$

We have $n_1 := \#I_1 \ge 1$, $\#I_2 = N - n_1$, and

$$\sum_{i\in I_2} (\alpha_i - 1) = \max_{1\le i\le N} \sum_{1\le j\le N, j\ne i} (\alpha_j - 1)^+ = \max_{1\le i\le N} \sum_{1\le j\le N, j\ne i} (1 \vee \alpha_j) + 1 - N. \tag{4.1}$$

Take $h \in C^\infty(\mathbb{R})$ such that $0 \le h \le 1, |h'| \le 2$ and

$$h|_{(-\infty,1]} = h|_{[4,\infty)} = 0, \quad h|_{[2,3]} = 1.$$

Let $\varepsilon_N = \frac{1}{32N^2}$ and take

$$f_\varepsilon(x) = \bigg(\prod_{i\in I_1} h\Big(\frac{n_1^{-1} - x_i}{4N\varepsilon}\Big)\bigg) \cdot \prod_{i\in I_2} \Big(1 - \frac{x_i}{2\varepsilon}\Big)^+, \quad x \in \Delta^{(N)}, \varepsilon \in (0, \varepsilon_N]. \tag{4.2}$$

It is easy to see that $A_\varepsilon := \mathrm{supp} f_\varepsilon$ satisfies

$$\begin{aligned} A_\varepsilon^{(1)} &:= [n_1^{-1} - 12N\varepsilon, n_1^{-1} - 8N\varepsilon]^{I_1} \times [\varepsilon, 2\varepsilon]^{I_2} \subset A_\varepsilon \\ &\subset A_\varepsilon^{(2)} := [n_1^{-1} - 16N\varepsilon, n_1^{-1} - 4N\varepsilon]^{I_1} \times [0, 2\varepsilon]^{I_2}. \end{aligned}$$

So, for $x \in A_\varepsilon$ we have

$$\begin{aligned} 2N\varepsilon &\le 1 - \sum_{i\in I_1} (n_1^{-1} - 4N\varepsilon) - 2\varepsilon(N - n_1) \\ &\le 1 - \sum_{i=1}^N x_i = x_{N+1} \le 1 - \sum_{i\in I_1} (n_1^{-1} - 16N\varepsilon) \le 16N^2\varepsilon, \end{aligned}$$

and there exist constants $c_2 > c_1 > 0$ such that

$$\begin{aligned} & c_1 1_{A_\varepsilon^{(1)}}(x) \varepsilon^{\sum_{i\in I_2}(\alpha_i - 1) + \alpha_{N+1} - 1} \le (1_{A_\varepsilon}\rho)(x) \le c_2 1_{A_\varepsilon^{(2)}}(x) \varepsilon^{\sum_{i\in I_2}(\alpha_i - 1) + \alpha_{N+1} - 1}, \\ & 1_{A_\varepsilon^{(1)}}(x) \le f_\varepsilon(x) \le 1_{A_\varepsilon^{(2)}}(x), \\ & \sum_{i=1}^N x_i x_{N+1} (\partial_i f_\varepsilon)^2(x) \le c_2 \varepsilon^{-1} 1_{A_\varepsilon^{(2)}}(x), \quad x \in \Delta^{(N)}, \varepsilon \in (0, \varepsilon_N]. \end{aligned}$$

Combining these together we may find out constants $c_3, c_4 > 0$ such that for any $\varepsilon \in (0, \varepsilon_N]$,

$$\begin{aligned} & \mu_\alpha^{(N)}(f_\varepsilon^2) \ge \mu_\alpha^{(N)}(A_\varepsilon^{(1)}) = \int_{A_\varepsilon^{(1)}} \rho(x) \mathrm{d}x \ge c_3 \varepsilon^{\sum_{i\in I_2}(\alpha_i - 1) + N + \alpha_{N+1} - 1}, \\ & \mu_\alpha^{(N)}(f_\varepsilon)^2 \le \mu_\alpha^{(N)}(A_\varepsilon^{(2)})^2 \le c_4 \varepsilon^{2N + 2(\alpha_{N+1} - 1) + 2\sum_{i\in I_2}(\alpha_i - 1)}, \\ & \mu_\alpha^{(N)}\Big(\sum_{i=1}^N x_i x_{N+1} (\partial_i f_\varepsilon)^2\Big) \le c_2 \varepsilon^{-1} \mu_\alpha^{(N)}(A_\varepsilon^{(2)}) \le c_4 \varepsilon^{\sum_{i\in I_2}(\alpha_i - 1) + N + \alpha_{N+1} - 2}. \end{aligned} \tag{4.3}$$

Therefore, if (1.7) holds then

$$\begin{aligned} & c_3 \varepsilon^{\sum_{i\in I_2}(\alpha_i - 1) + N + \alpha_{N+1} - 1} \\ & \le r c_4 \varepsilon^{\sum_{i\in I_2}(\alpha_i - 1) + N + \alpha_{N+1} - 2} + c_4 \beta(r) \varepsilon^{2N + 2(\alpha_{N+1} - 1) + 2\sum_{i\in I_2}(\alpha_i - 1)}, \quad r > 0, \varepsilon \in (0, \varepsilon_N]. \end{aligned}$$

This is equivalent to

$$1 - \frac{c_4}{c_3} r \varepsilon^{-1} \le \frac{c_4}{c_3} \beta(r) \varepsilon^{\sum_{i\in I_2}(\alpha_i - 1) + N + \alpha_{N+1} - 1}, \quad r > 0, \varepsilon \in (0, \varepsilon_N].$$

Let $r_N = \frac{c_3}{2c_4}\varepsilon_N$. For any $r \in (0, r_N]$, we take $\varepsilon = \frac{2c_4}{c_3} r \in (0, \varepsilon_N]$ in the above inequality to derive from (4.1) that

$$\beta(r) \geq \frac{c_3}{2c_4}\varepsilon^{1-\sum_{i\in I_2}(\alpha_i-1)-N-\alpha_{N+1}} = cr^{-\tilde{p}_\alpha^{(1)}}, \quad r \in (0, r_N]$$

for some constant $c > 0$. Since (1.8) implies (1.7) for $\beta(r) = c(1 + r^{-p})$ for some constant $c > 0$, this implies $p \geq \tilde{p}_\alpha^{(1)}$.

(b) On the other hand, $1 \leq i_0 \leq N + 1$ be such that $\alpha_{i_0} = \min_{1\leq i\leq N+1} \alpha_i$. Let

$$I = \{i : i \neq i_0, 1 \leq i \leq N + 1\}.$$

For any $\varepsilon \in (0, 1)$, we take

$$f_\varepsilon(x) = \prod_{i\in I} \big(\varepsilon - x_i\big)^+, \quad x \in \Delta^{(N)}.$$

Then on the support of f_ε we have

$$x_i x_{N+1} \leq \min\{x_i, x_{N+1}\} \leq \varepsilon, \quad 1 \leq i \leq N.$$

So, as shown in (a) we may find out a constant $a \in (0, 1)$ such that for all $\varepsilon \in (0, 1)$,

$$\begin{aligned}
&\mu_\alpha^{(N)}(f_\varepsilon^2) \geq a\varepsilon^{3N+\sum_{i\in I}(\alpha_i-1)} = a\varepsilon^{2N+\sum_{i\in I}\alpha_i},\\
&\mu_\alpha^{(N)}(f_\varepsilon)^2 \leq a^{-1}\varepsilon^{4N+2\sum_{i\in I}(\alpha_i-1)} = a^{-1}\varepsilon^{2N+2\sum_{i\in I}\alpha_i},\\
&\mu_\alpha^{(N)}(|\sigma\nabla f_\varepsilon|^2) \leq a^{-1}\varepsilon^{3N+\sum_{i\in I}(\alpha_i-1)-1} = a^{-1}\varepsilon^{2N+\sum_{i\in I}\alpha_i-1},
\end{aligned}$$

and these together with (1.8) imply

$$p \geq \sum_{i\in I}\alpha_i = \max_{1\leq i\leq N+1} \sum_{j\neq i, 1\leq j\leq N+1} \alpha_j = \tilde{p}_\alpha^{(2)}.$$

□

Proof of Theorem 1.1(3). Let f_ε be as in (4.2). We have

$$\mathscr{E}_{FV}^{\alpha,N}(f_\varepsilon, f_\varepsilon) \leq \mu_\alpha^{(N)}\left(\sum_{i=1}^N x_i(\partial_i f_\varepsilon)^2\right) \leq c_4\varepsilon^{\sum_{i\in I_2}(\alpha_i-1)+N+\alpha_{N+1}-1}$$

for some constant $c_4 > 0$. Combining this with the first two lines in (4.3), we derive from (1.7) with $\mathscr{E}_{FV}^{\alpha,N}$ replacing $\mathscr{E}_\alpha^{(N)}$ that

$$c_3\varepsilon^{\sum_{i\in I_2}(\alpha_i-1)+N+\alpha_{N+1}-1} \leq c_4\varepsilon^{\sum_{i\in I_2}(\alpha_i-1)+N+\alpha_{N+1}-3} + c_4\varepsilon^{2\sum_{i\in I_2}(\alpha_i-1)+2N+2\alpha_{N+1}-2}\beta(r),$$

thus,

$$1 - \frac{c_4 r}{c_3\varepsilon^2} \leq \frac{c_4}{c_3}\beta(r)\varepsilon^{\sum_{i\in I_2}(\alpha_i-1)+N+\alpha_{N+1}-1}.$$

Taking $\varepsilon = \left(\frac{2c_3 r}{c_4}\right)^{\frac{1}{2}}$ for small $r > 0$ we arrive at

$$\beta(r) \geq c r^{-\frac{1}{2}(N+\alpha_{N+1}-1+\sum_{i\in I_2}(\alpha_i-1))}.$$

Combining this with (1.8) implies $p \geq \frac{1}{2}\alpha_{N+1} + \frac{1}{2}\max_{1\leq i\leq N}\sum_{j\neq i, 1\leq j\leq N}(1\vee\alpha_j)$.

On the other hand, take

$$f_\varepsilon(x) = \prod_{1\leq i\leq N}(\varepsilon - x_i)^+$$

for small $\varepsilon > 0$. Then there exists a constant $a \in (0,1)$ such that for small $\varepsilon > 0$ we have

$$\begin{aligned}
&\mu_\alpha^{(N)}(f_\varepsilon^2) \geq a\varepsilon^{2N+\sum_{1\leq i\leq N}\alpha_i},\\
&\mu_\alpha^{(N)}(f_\varepsilon)^2 \leq a^{-1}\varepsilon^{2N+2\sum_{1\leq i\leq N}\alpha_i},\\
&\mathscr{E}_{FV}^{\alpha,N}(f_\varepsilon, f_\varepsilon) \leq \mu_\alpha^{(N)}\left(\sum_{i=1}^N x_i(\partial_i f_\varepsilon)^2\right) \leq a^{-1}\varepsilon^{2N+\sum_{1\leq i\leq N}\alpha_i - 1},
\end{aligned}$$

so that (1.8) for $\mathscr{E}_{FV}^{\alpha,N}$ implies $p \geq \sum_{1\leq i\leq N}\alpha_i$. □

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