On the quantum flag manifold $SU_q(3)/\mathbb{T}^2$

Tomasz Brzeziński and Wojciech Szymański

Abstract. The structure of the C^*-algebra of functions on the quantum flag manifold $SU_q(3)/\mathbb{T}^2$ is investigated. Building on the representation theory of $C(SU_q(3))$, we analyze irreducible representations and the primitive ideal space of $C(SU_q(3)/\mathbb{T}^2)$, with a view towards unearthing the "quantum sphere bundle" $\mathbb{C}P^1_q \to SU_q(3)/\mathbb{T}^2 \to \mathbb{C}P^2_q$.

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1. Introduction

The theory of principal and associated fibre bundles lies at the heart of geometry and underpins important applications to physics. Due to combined effort of many researchers, see e.g. [2, 3, 6], this theory has been successfully incorporated into noncommutative geometry. In the noncommutative setting, spaces are replaced by (noncommutative) algebras of functions, typically C^*-algebras or their dense $*$-subalgebras, and quantum groups (or Hopf algebras) play the role of structure groups. By contrast, precious little is known about noncommutative analogs of more general fibre bundles, in which the fibre does not correspond to a group.

This short note is intended as a first step towards a case study of noncommutative sphere bundles. More specifically, the classical flag manifold $SU(3)/\mathbb{T}^2$ has a natural structure of the sphere bundle

$$\mathbb{C}P^1 \to SU(3)/\mathbb{T}^2 \to \mathbb{C}P^2.$$

We intend to analyze the structure of the quantum analog of this flag manifold, corresponding to the C^*-algebra $C(SU_q(3)/\mathbb{T}^2)$ playing the role of the total space. Here $SU_q(3)$ denotes the Woronowicz quantum $SU(3)$ group, and

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$C(SU_q(3)/\mathbb{T}^2)$ itself is the C^*-algebra of fixed points for the action of $\mathbb{T}^2$ on $C(SU_q(3))$ coming from its maximal torus.

In order to be able to attack this problem from the analytic point of view, it is necessary to obtain a detailed and explicit information about the internal structure of the C^*-algebra $C(SU_q(3)/\mathbb{T}^2)$. Most of this note is devoted to this task. In particular, we carefully describe the primitive ideal space of this AF-algebra, building on the explicit description of irreducible representations of $C(SU_q(3))$, calculated originally by K. Bragiel in his PhD dissertation [1].

In the final section of this note we show how to construct a faithful conditional expectation from $C(SU_q(3)/\mathbb{T}^2)$ onto its subalgebra $C(\mathbb{C}P_q^2)$, using integration over the quantum group $U_q(2)$ (realised as a quantum subgroup of $SU_q(3)$). More detailed, algebraic description of the noncommutative sphere bundle

$$\mathbb{C}P_q^1 \to SU(3)/\mathbb{T}^2 \to \mathbb{C}P_q^2$$

and its K-theory is deferred to the forthcoming paper [4].

2. The quantum flag manifold

2.1. The algebra of functions on the quantum $\mathbf{SU(3)}$ group

For $q \in (0,1)$, the C^*-algebra $C(SU_q(3))$ of 'continuous functions' on the quantum $SU(3)$ group is defined by Woronowicz [13, 14] as the universal C^*-algebra generated by elements $\{u_{ij} : i,j = 1,2,3\}$ such that the matrix $\mathbf{u} = (u_{ij})_{i,j=1}^3$ is unitary and

$$\sum_{i_1=1}^3 \sum_{i_2=1}^3 \sum_{i_3=1}^3 E_{i_1 i_2 i_3} u_{j_1 i_1} u_{j_2 i_2} u_{j_3 i_3} = E_{j_1 j_2 j_3}, \qquad \forall (j_1, j_2, j_3) \in \{1,2,3\},$$

where

$$E_{i_1 i_2 i_3} = \begin{cases} (-q)^{I(i_1,i_2,i_3)} & \text{if } i_r \neq i_s \text{ for } r \neq s, \\ 0 & \text{otherwise,} \end{cases}$$

and $I(i_1, i_2, i_3)$ denotes the number of inversed pairs in the sequence i_1, i_2, i_3. As pointed out by Bragiel [1], $\{u_{ij}\}$ are coordinate functions of a quantum matrix [5, 9, 10]. That is, the following relations are also satisfied

$$u_{ij}u_{ik} = q u_{ik}u_{ij}, \qquad j < k, \tag{1a}$$

$$u_{ji}u_{ki} = q u_{ki}u_{ji}, \qquad j < k, \tag{1b}$$

$$u_{ij}u_{km} = u_{km}u_{ij}, \qquad i < k,\ j > m, \tag{1c}$$

$$u_{ij}u_{km} - u_{km}u_{ij} = (q - q^{-1}) u_{im}u_{kj}, \qquad i < k,\ j < m, \tag{1d}$$

with $i, j, k, m \in \{1,2,3\}$. The comultiplication

$$\Delta : C(SU_q(3)) \longrightarrow C(SU_q(3)) \otimes C(SU_q(3))$$

is a unital C^*-algebra homomorphism such that

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}.$$

We denote by $\mathcal{O}(SU_q(3))$ the $*$-subalgebra of $C(SU_q(3))$ generated by the u_{ij}, $i,j = 1,2,3$. Thus $\mathcal{O}(SU_q(3))$, the polynomial algebra of $SU_q(3)$, is a dense $*$-subalgebra of $C(SU_q(3))$.

In [1], Bragiel described explicitly all irreducible representations of the algebra $C(SU_q(3))$. There are six families of these representations, each indexed by elements (ϕ, ψ) of the 2-torus. We denote them by $\pi_0^{\phi,\psi}$, $\pi_{11}^{\phi,\psi}$, $\pi_{12}^{\phi,\psi}$, $\pi_{21}^{\phi,\psi}$, $\pi_{22}^{\phi,\psi}$ and $\pi_3^{\phi,\psi}$. Each of the representations $\pi_*^{\phi,\psi}$ acts on the Hilbert space $\mathcal{H}_*$, where

$$\mathcal{H}_0 = \mathbb{C},\ \mathcal{H}_{11} = \mathcal{H}_{12} = \ell^2(\mathbb{N}),\ \mathcal{H}_{21} = \mathcal{H}_{22} = \ell^2(\mathbb{N}^2) \text{ and } \mathcal{H}_3 = \ell^2(\mathbb{N}^3).$$

Each of the $\pi_*^{\phi,\psi}$ contains compact operators of $\mathcal{H}_*$ in its image [1], and thus $C(SU_q(3))$ is a type I algebra. The kernels of these irreducible representations are primitive ideals of $C(SU_q(3))$ with the following generators:

$$\ker(\pi_3^{\phi,\psi}) = \langle \overline{\phi} u_{31} - |u_{31}|,\ \overline{\psi} u_{13} - |u_{13}| \rangle, \tag{2a}$$

$$\ker(\pi_{21}^{\phi,\psi}) = \langle u_{31},\ \overline{\phi} u_{21} - |u_{21}|,\ \overline{\psi} u_{13} - |u_{13}| \rangle, \tag{2b}$$

$$\ker(\pi_{22}^{\phi,\psi}) = \langle u_{13},\ \overline{\phi} u_{31} - |u_{31}|,\ \overline{\psi} u_{12} - |u_{12}| \rangle, \tag{2c}$$

$$\ker(\pi_{11}^{\phi,\psi}) = \langle u_{13},\ u_{31},\ u_{23},\ \overline{\phi} u_{12} - |u_{12}|,\ \overline{\psi} u_{21} - |u_{21}| \rangle, \tag{2d}$$

$$\ker(\pi_{12}^{\phi,\psi}) = \langle u_{13},\ u_{31},\ u_{12},\ \phi\psi u_{32} - |u_{32}|,\ \overline{\psi} u_{23} - |u_{23}| \rangle, \tag{2e}$$

$$\ker(\pi_0^{\phi,\psi}) = \langle u_{13},\ u_{31},\ u_{12},\ u_{23},\ \overline{\phi} u_{11} - 1,\ \overline{\psi} u_{22} - 1 \rangle. \tag{2f}$$

2.2. The gauge action and its fixed point algebra

The family of 1-dimensional irreducible representations $\pi_0^{\phi,\psi}$ of $C(SU_q(3))$ produces a surjective morphism of compact quantum groups

$$\hat{\pi}_0 : C(SU_q(3)) \longrightarrow C(\mathbb{T}^2)$$

(the diagonal imbedding of $\mathbb{T}^2$ into $SU_q(3)$), which gives rise to a gauge coaction of coordinate algebras

$$\hat{\mu} : \mathcal{O}(SU_q(3)) \to \mathcal{O}(SU_q(3)) \otimes \mathcal{O}(\mathbb{T}^2), \qquad \hat{\mu} = (\mathrm{id} \otimes \hat{\pi}_0) \circ \Delta_{SU_q(3)}.$$

Explicitly, on the polynomial algebra $\mathcal{O}(SU_q(3))$, $\hat{\pi}_0$ is a Hopf $*$-algebra epimorphism,

$$\hat{\pi}_0 : \mathcal{O}(SU_q(3)) \longrightarrow \mathcal{O}(\mathbb{T}^2), \qquad \mathbf{u} \mapsto \begin{pmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & U_1^* U_2^* \end{pmatrix},$$

where U_1, U_2 are unitary, group-like generators of the Hopf algebra $\mathcal{O}(\mathbb{T}^2)$ of polynomials on $\mathbb{T}^2$ (the algebra of Laurent polynomials in two indeterminates). Hence the coaction comes out as

$$\hat{\mu} : \mathcal{O}(SU_q(3)) \to \mathcal{O}(SU_q(3)) \otimes \mathcal{O}(\mathbb{T}^2), \quad u_{ij} \mapsto \begin{cases} u_{ij} \otimes U_j & \text{if } j = 1,2, \\ u_{ij} \otimes (U_1 U_2)^{-1} & \text{if } j = 3. \end{cases}$$

Equivalently, $\mu : \mathbb{T}^2 \longrightarrow \mathrm{Aut}(C(SU_q(3)))$ is given by

$$z \longmapsto \mu_z, \qquad \mu_z(u_{ij}) = \begin{cases} z_j u_{ij} & \text{if } j = 1, 2, \\ (z_1 z_2)^{-1} u_{ij} & \text{if } j = 3. \end{cases}$$

Here $z = (z_1, z_2) \in \mathbb{T}^2$ and each z_i is a complex number of modulus 1. Let $C(SU_q(3)/\mathbb{T}^2)$ be the fixed point algebra of this gauge action, and let $\mathcal{O}(SU_q(3)/\mathbb{T}^2) = \mathcal{O}(SU_q(3)) \cap C(SU_q(3)/\mathbb{T}^2)$ be its polynomial $*$-subalgebra, i.e. the subalgebra of coinvariants of $\hat{\mu}$,

$$\mathcal{O}(SU_q(3)/\mathbb{T}^2) = \mathcal{O}(SU_q(3))^{\mathrm{co}\mathcal{O}(\mathbb{T}^2)} = \{f \in \mathcal{O}(SU_q(3)) \mid \hat{\mu}(f) = f \otimes 1\}.$$

Integration with respect to the Haar measure over $\mathbb{T}^2$ gives rise to a faithful conditional expectation $\Phi : C(SU_q(3)) \to C(SU_q(3)/\mathbb{T}^2)$, namely

$$\Phi(x) = \int_{z \in \mathbb{T}^2} \mu_z(x) dz.$$

If w is a monomial in $\{u_{ij}\}$ then $\Phi(w)$ is either 0 or w. Thus we have $\Phi(\mathcal{O}(SU_q(3))) = \mathcal{O}(SU_q(3)/\mathbb{T}^2)$, and whence $\mathcal{O}(SU_q(3)/\mathbb{T}^2)$ is a dense $*$-subalgebra of $C(SU_q(3)/\mathbb{T}^2)$.

There is a third equivalent way of understanding the gauge action, which is particularly useful in determining the freeness of the action (alas we will not employ this point of view in this note): $\mathcal{O}(SU_q(3))$ is a $\mathbb{Z}^2$-graded algebra with the degrees of the generators given by

$$\deg(u_{i1}) = (1, 0),\ \deg(u_{i2}) = (0, 1),\ \deg(u_{i3}) = (-1, -1), \quad i = 1, 2, 3.$$

From this point of view, $\mathcal{O}(SU_q(3)/\mathbb{T}^2)$ is the $(0, 0)$-degree part of $\mathcal{O}(SU_q(3))$.

In what follows, we denote

$$w_{ijk} = u_{i1} u_{j2} u_{k3}, \qquad i, j, k = 1, 2, 3. \tag{3}$$

Clearly, elements w_{ijk} are contained in the polynomial algebra $\mathcal{O}(SU_q(3)/\mathbb{T}^2)$.

Let $\rho_*^{\phi,\psi}$ be the restriction to $C(SU_q(3)/\mathbb{T}^2)$ of the representation $\pi_*^{\phi,\psi}$ of $C(SU_q(3))$.

Lemma 1. *For each $(\phi, \psi) \in \mathbb{T}^2$, the representation $\rho_*^{\phi,\psi}$ is unitarily equivalent to $\rho_*^{1,1}$.*

Proof. It follows immediately from formulae (2a)–(2f) that the gauge action μ on the primitive ideal space is transitive on each of the six families. Since $C(SU_q(3))$ is of type I, irreducible representations with identical kernels are unitarily equivalent. Thus, for each (ϕ, ψ) there exist (z_1, z_2) such that $\pi_*^{\phi,\psi}$ is unitarily equivalent to $\pi_*^{1,1} \circ \mu_{z_1,z_2}$. But $\rho_*^{1,1} \circ \mu_{z_1,z_2} = \rho_*^{1,1}$, and whence $\rho_*^{\phi,\psi}$ is unitarily equivalent to $\rho_*^{1,1}$. □

In what follows we use the simplified notation $\rho_* = \rho_*^{1,1}$.

Lemma 2. *The image of ρ_* contains all the compact operators $\mathcal{K}(\mathcal{H}_*)$ on its space $\mathcal{H}_*$, and thus each ρ_* is irreducible.*

Proof. Representation ρ_0 is 1-dimensional and there is nothing to prove in this case.

Considering ρ_{12}, given by formulae (14) of [1], we have

$$\rho_{12}(w_{132})|N\rangle = -q^{2N+1}|N\rangle.$$

Thus the image of ρ_{12} contains one-dimensional projections corresponding to the basis $\{|N\rangle : N \in \mathbb{N}\}$ of $\mathcal{H}_{12}$. Since

$$\rho_{12}(w_{133})|N\rangle = \text{scalar}|N+1\rangle$$

(in the course of the proof of this lemma we denote by ‘scalar’ a non-zero constant which may depend on N, M, L), it follows that the image of ρ_{12} contains all the compact operators on $\mathcal{H}_{12}$. In the case of ρ_{11} the same argument works, since $\rho_{11}(w_{213}) = \rho_{12}(w_{132})$ and $\rho_{11}(w_{223})|N\rangle = \text{scalar}|N+1\rangle$.

By formulae (12) of [1],

$$\rho_{22}(w_{312})|N, M\rangle = q^{2(N+M+1)}|N, M\rangle$$

and

$$\rho_{22}(w_{132})|N, M\rangle = -q^{2M+1}(1 - q^{2(N+1)})|N, M\rangle.$$

It follows that the image of ρ_{22} contains all one-dimensional projections corresponding to the basis $\{|N, M\rangle : N, M \in \mathbb{N}\}$ of $\mathcal{H}_{22}$. We also find that

$$\rho_{22}(w_{112})|N, M\rangle = \text{scalar}|N-1, M\rangle$$

and

$$\rho_{22}(w_{212})|N, M\rangle = \text{scalar}|N, M-1\rangle,$$

and it follows that the image of ρ_{22} contains all the compact operators on $\mathcal{H}_{22}$. The argument for ρ_{21} is similar and based on the identities:

$$\rho_{21}(w_{231}) = \rho_{22}(w_{312}), \qquad \rho_{21}(w_{132}) = \rho_{22}(w_{132}).$$

$$\rho_{21}(w_{131})|N, M\rangle = \text{scalar}|N-1, M\rangle$$

and

$$\rho_{21}(w_{211})|N, M\rangle = \text{scalar}|N, M-1\rangle.$$

Finally, considering ρ_3, given by formulae (10) of [1], we have

$$\rho_3(|w_{311}|^2)|N, M, L\rangle = q^{2(3N+M+L+3)}(1 - q^{2M})|N, M, L\rangle, \tag{4a}$$

$$\rho_3(w_{111})|N, M, L\rangle = \text{scalar}|N-1, M-1, L\rangle, \tag{4b}$$

$$\rho_3(w_{211})|N, M, L\rangle = \text{scalar}|N, M-1, L-1\rangle, \tag{4c}$$

$$\rho_3(w_{311})|N, M, L\rangle = \text{scalar}|N, M-1, L\rangle. \tag{4d}$$

By (4a), the operator $\rho_3(|w_{311}|^2)$ is compact and its spectral subspace corresponding to the maximal eigenvalue is spanned by vectors $|0, M, 0\rangle$ for which M is a positive integer such that $q^{2M}(1 - q^{2M})$ is maximal. This space is either one or two-dimensional. In the former case, the image of ρ_3 contains the one-dimensional projection onto $|0, M_0, 0\rangle$, and formulae (4b)–(4d) imply that it contains all the compact operators on $\mathcal{H}_3$. In the latter case, the image of ρ_3 contains the two-dimensional projection Q onto the span of $|0, M_0, 0\rangle$ and $|0, M_0 + 1, 0\rangle$. Then $Q\rho_3(w_{311})Q$ is a rank one operator, and just as above it follows from formulae (4b)–(4d) that the image of ρ_3 contains all the compact operators on $\mathcal{H}_3$. □

We define $J_* = \rho_*^{-1}(\mathcal{K}(\mathcal{H}_*))$, closed ideals of $C(SU_q(3)/\mathbb{T}^2)$.

Lemma 3. *The following properties hold:*

(i) *Representation ρ_3 of $C(SU_q(3)/\mathbb{T}^2)$ is faithful.*

(ii) $J_3 = \ker(\rho_{21}) \cap \ker(\rho_{22})$.

(iii) $J_{21} = J_{22} = \ker(\rho_{21}) + \ker(\rho_{22}) = \ker(\rho_{11}) \cap \ker(\rho_{12})$.

(iv) $J_{11} = J_{12} = \ker(\rho_{11}) + \ker(\rho_{12}) = \ker(\rho_0)$.

Proof. We have

$$\ker(\rho_*) = C(SU_q(3)/\mathbb{T}^2) \cap \bigcap_{\phi,\psi} \ker(\pi_*^{\phi,\psi}).$$

Using formulae (2a)–(2f) we see that $\cap_{\phi,\psi} \ker(\pi_*^{\phi,\psi})$ are ideals of $C(SU_q(3))$ with the following sets of generators:

$$\bigcap_{\phi,\psi} \ker(\pi_3^{\phi,\psi}) = \langle 0 \rangle, \tag{5a}$$

$$\bigcap_{\phi,\psi} \ker(\pi_{21}^{\phi,\psi}) = \langle u_{31} \rangle, \qquad \bigcap_{\phi,\psi} \ker(\pi_{22}^{\phi,\psi}) = \langle u_{13} \rangle, \tag{5b}$$

$$\bigcap_{\phi,\psi} \ker(\pi_{11}^{\phi,\psi}) = \langle u_{13},\ u_{31},\ u_{23} \rangle, \tag{5c}$$

$$\bigcap_{\phi,\psi} \ker(\pi_{12}^{\phi,\psi}) = \langle u_{13},\ u_{31},\ u_{12} \rangle, \tag{5d}$$

On the other hand, $J_* = \ker(\rho_*) \cap \langle x_* \rangle$, where $\langle x_* \rangle$ is the ideal of $C(SU_q(3))$ generated by x_* such that $\pi_*^{\phi,\psi}(x_*)$ is a non-zero element of $\mathcal{K}(\mathcal{H}_*)$ for all ϕ, ψ. For example, we can take $x_3 = u_{31}u_{13}$, $x_{21} = u_{13}$, $x_{22} = u_{31}$, $x_{11} = u_{12}$ and $x_{12} = u_{23}$.

Now (i) follows from formula (5a). Claim (ii) follows from (5b) and the identity $\langle u_{31} \rangle \cap \langle u_{13} \rangle = \langle u_{31}u_{13} \rangle$. The latter follows from the fact that both u_{13} and u_{31} either commute or q-commute with every generator of $C(SU_q(3))$.

The identity $J_{21} = J_{22} = \ker(\rho_{21}) + \ker(\rho_{22})$ follows from (5b). For the remaining part of claim (iii), it suffices to show that

$$\langle u_{13},\ u_{31} \rangle = \langle u_{13},\ u_{31},\ u_{12} \rangle \cap \langle u_{13},\ u_{31},\ u_{23} \rangle.$$

To this end, we first note that modulo the ideal $\langle u_{13},\ u_{31} \rangle$ both u_{12} and u_{23} either commute or q-commute with every generator of $C(SU_q(3))$. Thus

$$\langle u_{13},\ u_{31},\ u_{12} \rangle \cap \langle u_{13},\ u_{31},\ u_{23} \rangle = \langle u_{13},\ u_{31},\ u_{12}u_{23} \rangle,$$

and it suffices to verify that $u_{12}u_{23} \in \langle u_{13},\ u_{31} \rangle$. By formulae (11) of [1], we have $\pi_{21}^{\phi,\psi}(u_{12}u_{23}) \in \mathcal{K}(\mathcal{H}_{21})$ for all ϕ, ψ, and the claim follows.

The identity $J_{11} = J_{12} = \ker(\rho_{11}) + \ker(\rho_{12})$ follows from (5c) and (5d). For the remaining part of claim (iv), we must prove that

$$\ker(\rho_0) = C(SU_q(3)/\mathbb{T}^2) \cap \langle u_{13},\ u_{31},\ u_{23},\ u_{12} \rangle.$$

However, as shown in [1], $\oplus_{\phi,\psi}\pi_0^{\phi,\psi}$ is faithful on the quotient of $C(SU_q(3)$ by $\langle u_{13},\ u_{31},\ u_{23},\ u_{12}\rangle$, and the claim follows. □

In the following corollary we summarise properties of the algebra of continuous functions on the quantum flag manifold $SU_q(3)/\mathbb{T}^2$.

Corollary 4. *The C^*-algebra $C(SU_q(3)/\mathbb{T}^2)$ has the following properties.*

1. *It has a composition series with factors: $\mathcal{K}$, $\mathcal{K}\oplus\mathcal{K}$, $\mathcal{K}\oplus\mathcal{K}$, $\mathbb{C}$.*
2. *It is AF and of type I.*
3. *Its K-groups are $K_0 \cong \mathbb{Z}^6$ and $K_1 = 0$.*
4. *$\{\rho_*\}$ is a complete set of representatives (up to unitary equivalence) of its irreducible representations.*
5. *Each irreducible representation of $C(SU_q(3)/\mathbb{T}^2)$ extends to an irreducible representation of $C(SU_q(3))$ acting on the same Hilbert space.*
6. *Its primitive ideal space consists of six elements $\{\ker(\rho_*)\}$, with topology determined by the following closure operation.*
   (a) *The point $\ker(\rho_3)$ is dense in the entire space.*
   (b) *The closures of $\ker(\rho_{21})$ and $\ker(\rho_{22})$, respectively, consist of the union of itself and $\{\ker(\rho_0), \ker(\rho_{11}), \ker(\rho_{12})\}$.*
   (c) *The closures of $\ker(\rho_{11})$ and $\ker(\rho_{12})$, respectively, consist of the union of itself and $\ker(\rho_0)$.*
   (d) *The point $\ker(\rho_0)$ is closed.*

3. Towards a noncommutative sphere bundle

The classical flag manifold $SU(3)/\mathbb{T}^2$ has the structure of a fibre bundle with the base space $\mathbb{C}P^2$ and the fibre $\mathbb{C}P^1 \cong S^2$. Therefore, it is natural to expect that the quantum flag manifold $SU_q(3)/\mathbb{T}^2$ should have an analogous structure of a noncommutative 'fibre bundle'

$$\mathbb{C}P_q^1 \longrightarrow SU_q(3)/\mathbb{T}^2 \longrightarrow \mathbb{C}P_q^2. \tag{6}$$

It is not entirely clear how to reinterpret the "bundle" from (6) in the noncommutative setting. However, as a minimum, we should have a projection (conditional expectation) from the algebra of "functions on the total space" $C(SU_q(3)/\mathbb{T}^2)$ onto the algebra of "functions on the base space" $C(\mathbb{C}P_q^2)$. So we begin by constructing such a conditional expectation.

The algebra $C(\mathbb{C}P_q^2)$ is a C^*-subalgebra of $C(SU_q(3)/\mathbb{T}^2)$ in a natural way as follows (cf. [11]). The C^*-subalgebra of $C(SU_q(3))$ generated by the first column matrix elements of $\mathbf{u}$, i.e. u_{11}, u_{21} and u_{31}, may be identified with the C^*-algebra $C(S_q^5)$ of continuous functions on the quantum 5-sphere. This C^*-subalgebra is invariant under the gauge action μ of $\mathbb{T}^2$ on $C(SU_q(3))$. When restricted to $C(S_q^5)$, μ reduces to the generator-rescaling circle action $u_{j1} \mapsto zu_{j1}$, $z \in \mathbb{T}$, whose fixed point algebra is $C(\mathbb{C}P_q^2)$ (cf. [7,11]). Thus, in the setting of the present article, we have

$$C(\mathbb{C}P_q^2) = C(SU_q(3)/\mathbb{T}^2) \cap C^*(u_{11}, u_{21}, u_{31}).$$

In order to construct the desired conditional expectation

$$E : C(SU_q(3)/\mathbb{T}^2) \to C(\mathbb{C}P_q^2),$$

we will use integration over a quantum subgroup of $SU_q(3)$ isomorphic to the quantum unitary group $U_q(2)$. Indeed, recall from [9] or [6] that $U_q(2)$ is a compact matrix quantum group with the C^*-algebra of continuous functions $C(U_q(2))$ generated densely by three elements u, α, γ, organised into a fundamental unitary matrix

$$\mathbf{v} = \begin{pmatrix} u & 0 & 0 \\ 0 & \alpha & -q\gamma^* u^* \\ 0 & \gamma & \alpha^* u^* \end{pmatrix}.$$

The generator u is central, while

$$\alpha\gamma = q\gamma\alpha, \qquad \gamma\gamma^* = \gamma^*\gamma.$$

The unitarity of $\mathbf{v}$ implies that u is unitary, while α and γ satisfy the remaining $SU_q(2)$ (cf. [12]) q-commutation rules

$$\alpha\gamma^* = q\gamma^*\alpha, \qquad \alpha^*\alpha + \gamma\gamma^* = 1, \qquad \alpha\alpha^* + q^2\gamma\gamma^* = 1.$$

As shown in [4], the $*$-homomorphism

$$\pi : \mathcal{O}(SU_q(3)) \longrightarrow \mathcal{O}(U_q(2)), \qquad \mathbf{u} \mapsto \mathbf{v},$$

is an epimorphism of compact quantum groups, and thus we obtain a right coaction

$$\begin{aligned} \varrho_{SU_q(3)} &: C(SU_q(3)) \longrightarrow C(SU_q(3)) \otimes C(U_q(2)), \\ \varrho_{SU_q(3)} &= (\mathrm{id} \otimes \pi) \circ \Delta_{SU_q(3)}. \end{aligned} \tag{7}$$

One immediately checks that

$$\varrho_{SU_q(3)} \circ \mu_z = (\mu_z \otimes \mathrm{id}) \circ \varrho_{SU_q(3)},$$

for all $z \in \mathbb{T}^2$, and this implies that the restriction of $\varrho_{SU_q(3)}$ to $C(SU_q(3)/\mathbb{T}^2)$ yields the coaction

$$\varrho_{SU_q(3)/\mathbb{T}^2} : C(SU_q(3)/\mathbb{T}^2) \longrightarrow C(SU_q(3)/\mathbb{T}^2) \otimes C(U_q(2)).$$

Consequently,

$$(\mathrm{id} \otimes \mathfrak{h}) \circ \varrho_{SU_q(3)/\mathbb{T}^2} : C(SU_q(3)/\mathbb{T}^2) \to C(SU_q(3)/\mathbb{T}^2)^{co\, U_q(2)}$$

is a faithful conditional expectation. Here $\mathfrak{h}$ denotes the Haar state on $C(U_q(2))$ and

$$C(SU_q(3)/\mathbb{T}^2)^{co\, U_q(2)} = \{a \in C(SU_q(3)/\mathbb{T}^2) \mid \varrho_{SU_q(3)/\mathbb{T}^2}(a) = a \otimes 1\}$$

is the C^*-subalgebra of coinvariants. It is shown in [4] that

$$C(SU_q(3)/\mathbb{T}^2)^{co\, U_q(2)} = C(SU_q(3)) \cap C^*(u_{11}, u_{21}, u_{31}),$$

and thus

$$E = (\mathrm{id} \otimes \mathfrak{h}) \circ \varrho_{SU_q(3)/\mathbb{T}^2} \tag{8}$$

is the desired faithful conditional expectation from the algebra $C(SU_q(3)/\mathbb{T}^2)$ onto $C(\mathbb{C}P_q^2)$.

In order to compute the conditional expectation value (8) it is useful or, indeed necessary, to have an explicit description of the Haar state on $C(U_q(2))$. In fact, it is sufficient to have such a description on the dense subalgebra $\mathcal{O}(U_q(2))$ of $C(U_q(2))$. One way of obtaining the Haar measure is first to realise that $\mathcal{O}(U_q(2))$ is a right $\mathcal{O}(SU_q(2))$-comodule algebra (i.e. the quantum group $SU_q(2)$ acts on $U_q(2)$) with fixed points equal to $\mathcal{O}(U(1))$ and then to compose the Haar integrals on $\mathcal{O}(SU_q(2))$ and $\mathcal{O}(U(1))$ (both well-known, the first one described by Woronowicz in [13]).

The coaction $\varrho_{U_q(2)}$ of $\mathcal{O}(SU_q(2))$ on $\mathcal{O}(U_q(2))$ is induced from the Hopf-algebra projection

$$\mathcal{O}(U_q(2)) \xrightarrow{\quad p \quad} \mathcal{O}(SU_q(2))\,,$$

$$\begin{pmatrix} u & 0 & 0 \\ 0 & \alpha & -q\gamma^* u^* \\ 0 & \gamma & \alpha^* u^* \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & -q\gamma^* \\ 0 & \gamma & \alpha^* \end{pmatrix},$$

by

$$\varrho_{U_q(2)} = (\mathrm{id} \otimes p) \circ \Delta_{U_q(2)} : \mathcal{O}(U_q(2)) \longrightarrow \mathcal{O}(U_q(2)) \otimes \mathcal{O}(SU_q(2)). \tag{9}$$

As $*$-algebra the coinvariants of this coaction are generated by the unitary u, and hence are isomorphic to $\mathcal{O}(U(1))$. The Haar functional $\mathfrak{h}$ on $\mathcal{O}(U_q(2))$ (and, consequently, on $C(U_q(2))$) is the composite

$$\mathfrak{h}:\ \mathcal{O}(U_q(2)) \xrightarrow{\varrho_{U_q(2)}} \mathcal{O}(U_q(2)) \otimes \mathcal{O}(SU_q(2)) \xrightarrow{\mathrm{id}\otimes \mathfrak{h}_{SU_q(2)}} \mathcal{O}(U(1)) \xrightarrow{\mathfrak{h}_{U(1)}} \mathbb{C}.$$

Here $\mathfrak{h}_{SU_q(2)}$ is the Haar measure on the quantum group $SU_q(2)$ given on the standard basis of $\mathcal{O}(U_q(2))$ as

$$\mathfrak{h}_{SU_q(2)}\left(\alpha^k \gamma^m \gamma^{*n}\right) = \delta_{k0}\,\delta_{mn}\,\frac{q^2-1}{q^{2n+2}-1}, \quad \text{for all } k \in \mathbb{Z},\ m,n \in \mathbb{N}, \tag{10}$$

where we use the convention that, for $k<0$, $x^k = (x^*)^{-k}$; see [13, Appendix A1]. The Haar functional $\mathfrak{h}_{U(1)}$ on the standard basis of $\mathcal{O}(U(1))$ is given by

$$\mathfrak{h}_{U(1)}\left(u^k\right) = \delta_{k0}, \quad \text{for all } k \in \mathbb{Z}. \tag{11}$$

Combining formulae (9)–(11) we thus obtain an explicit expression for the Haar functional on $\mathcal{O}(U_q(2))$,

$$\mathfrak{h}\left(\alpha^k u^l \gamma^m \gamma^{*n}\right) = \delta_{k0}\,\delta_{l0}\,\delta_{mn}\,\frac{q^2-1}{q^{2n+2}-1}, \quad \text{for all } k,l \in \mathbb{Z},\ m,n \in \mathbb{N}. \tag{12}$$

With the explicit formula (12) at hand we can now compute the value of the conditional expectation (8) on the elements (3) of the quantum flag variety algebra $\mathcal{O}(SU_q(3)/\mathbb{T}^2)$ densely included in the C^*-algebra of continuous functions $C(SU_q(3)/\mathbb{T}^2)$. In view of the fact that the coaction $\varrho_{SU_q(3)/\mathbb{T}^2}$

is the restriction of the map (7) one easily finds that

$$\begin{aligned}\varrho_{SU_q(3)}(w_{ijk}) = w_{ijk} \otimes 1 + u_{i1}u_{j2}u_{k2} \otimes uv_{22}v_{23} \\ + u_{i1}(u_{j3}u_{k2} + qu_{j2}u_{k3}) \otimes uv_{32}v_{23} + u_{i1}u_{j3}u_{k3} \otimes uv_{32}v_{33} \\ = w_{ijk} \otimes 1 - qu_{i1}u_{j2}u_{k2} \otimes \alpha\gamma^* \\ - qu_{i1}(u_{j3}u_{k2} + qu_{j2}u_{k3}) \otimes \gamma\gamma^* + u_{i1}u_{j3}u_{k3} \otimes \gamma\alpha^*.\end{aligned}$$

Now, the application of $\mathrm{id} \otimes \mathfrak{h}$ together with the commutation rules (1) yield,

$$E(w_{ijk}) = \frac{w_{ijk} - w_{ikj}}{1+q^2}.$$

4. Conclusions

In this short note we have studied representations and the structure of the algebra of continuous functions on the quantum flag manifold $SU_q(3)/\mathbb{T}^2$ obtained as the fixed points of the gauge action of the classical two-torus on the quantum $SU(3)$-group. We have also indicated that the quantum flag manifold $SU_q(3)/\mathbb{T}^2$ can be interpreted as the total space of a quantum sphere bundle over the quantum projective space $\mathbb{C}P^2_q$, and we have presented an explicit formula for a faithful conditional expectation from $C(SU_q(3)/\mathbb{T}^2)$ onto $C(\mathbb{C}P^2_q)$. The detailed analysis of this bundle is presented in [4].

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Tomasz Brzeziński
Department of Mathematics, Swansea University
Swansea University Bay Campus
Fabian Way, Swansea SA1 8EN, U.K.
and
Department of Mathematics
University of Białystok
K. Ciołkowskiego 1M
15–245 Białystok
Poland
e-mail: `T.Brzezinski@swansea.ac.uk`

Wojciech Szymański
Department of Mathematics and Computer Science
University of Southern Denmark
Campusvej 55
5230 Odense M, Denmark
e-mail: `szymanski@imada.sdu.dk`