Algebraic Localization from Power-Law Couplings in Disordered Quantum Wires

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We analyze the effects of disorder on the correlation functions of one-dimensional quantum models of fermions and spins with long-range interactions that decay with distance ℓ as a power-law $1/\ell^\alpha$. Using a combination of analytical and numerical results, we demonstrate that power-law interactions imply a long-distance algebraic decay of correlations within disordered-localized phases, for all exponents α. The exponent of algebraic decay depends only on α, and not, e.g., on the strength of disorder. We find a similar algebraic localization for wave-functions. These results are in contrast to expectations from short-range models and are of direct relevance for a variety of quantum mechanical systems in atomic, molecular and solid-state physics.

I. INTRODUCTION

Quantum waves are generally localized exponentially by disorder. Following the seminal work by Anderson with spin-polarized electrons[1] much experimental[2–8] and theoretical interest has been devoted to the study of localized phases and to the localization-delocalization transition for non-interacting and interacting quantum models[9–30].

While most works have focused on short-range couplings, long-range hopping and interactions that decay with distance ℓ as a power-law $1/\ell^\alpha$ have recently attracted significant interest[31–45] as they provide novel physical effects and can be now engineered in a variety of atomic, molecular and optical systems. For example, the Ising model with power-law spin interactions with tunable exponent $0 < \alpha < 3$ has been first realized in breakthrough experiments with arrays of laser-driven cold ions[46–50] and could also be obtained with atoms trapped in a photonic crystal waveguide[51–54]; dipolar-type $1/\ell^3$ or van-der-Waals-type $1/\ell^6$ couplings have been recently demonstrated experimentally with ground-state neutral atoms[55–58], Rydberg atoms[59–73], polar molecules[74–76] and nuclear spins[77]. In solid state materials, power-law hopping is of interest for, e.g., excitonic materials[78–90]; long-range $1/\ell$ coupling is found in helical Shiba chains[91,92], made of magnetic impurities on an s-wave superconductor, while planar arrays of Josephson Junctions[93,94] can effectively realize long-range extensions of the Kitaev chain for spinless fermions[95]. In the absence of disorder, theory and experiments have provided evidence for novel enticing static and dynamic phenomena in these systems, such as, e.g., the non-local propagation of correlations[33–35,96], time crystal phases[97], novel topological effects[98–103], and exotic behaviors of equal-time correlations, such as hybrid exponential and power-law decays within gapped phases, related to the violation of the area law for the entanglement entropy[104–107]. However, in many of these systems, disorder - in particles' positions, local energies, or coupling strengths - is an intrinsic feature. Understanding its effects on the above phenomena and in the context of single-particle and many-body localization remains a fundamental open question.

For non-interacting models, it is generally expected that long-range hopping induces delocalization in the presence of disorder for $\alpha < d$, while for $\alpha > d$ all wave-functions are exponentially localized[1,108–114]. However, recent theoretical works with positional[115] and diagonal[114] disorder have demonstrated that localization can survive even for $\alpha < d$. Surprisingly, wave-functions were found to be localized only algebraically in these models, in contrast to the usual Anderson-type exponential localization expected from short-range models. How these finding translate to the behavior of wave-functions and, crucially, correlation functions in many-particle systems is not known.

In this work, we investigate the effects of disorder on the decay of correlation functions and wave-functions in long-range quantum wires of fermions and spins. These are extensions of the Kitaev chain with long-range pairing[93,104,116] and the Ising model in a transverse field[117], providing both analytical insight and immediate experimental interest. While in the absence of disorder and for short-range interactions they are identical, for long-range interactions they correspond to integrable and non-integrable chains, respectively. For fermions, we determine the regimes of localization for all α for the cases of disordered hopping or pairing. For the Ising chain, we focus on the regime $\alpha > 1$, where the disordered phase diagram has been shown to display many-body localization theoretically[118] and experimentally[7]. For all models we compute the one-body and two-body connected correlation functions, finding several novel features: (i) The connected correlation functions decay algebraically at long distance within all localized phases, (ii) with an exponent that depends exclusively on α, and not, e.g., on the disorder strength. (iii) For the fermionic models, we derive analytic results for the long-distance decay of the correlations that explain the found algebraic decay, in excellent agreement with the numerics. (iv) The same analytical predictions are found to hold also for the correlations of the interacting Ising chain. (v) For any α, the localized wave-functions of the fermionic models display a long-distance algebraic decay with exponent α, different from recent predictions for long-range hopping models. These results should be of direct relevance to many experiments in cold atomic, molecular and solid-state physics with fermions and spins.

II. THE MODELS

We consider the following Hamiltonians for one-dimensional long-range fermionic models

$$H_{I,II} = H_0 + V_{I,II} \qquad (1)$$

where H_0 is a homogeneous Hamiltonian given by

$$
\begin{aligned}
H_0 = & -t \sum_{j=1}^{L} \left(a_j^\dagger a_{j+1} + \text{H.c.} \right) + \mu \sum_{j=1}^{L} n_j \\
& + \sum_{j,\ell} \frac{\Delta}{\ell^\alpha} \left(a_j a_{j+\ell} + \text{H.c.} \right)
\end{aligned} \qquad (2)
$$

that describes a *p*-wave superconductor with a long-range pairing, and the indices I, II refer to the two different types of Hamiltonians we consider, namely

$$V_I = \sum_{j=1}^{L} W_j \left(a_j^\dagger a_{j+1} + \text{H.c.} \right) \qquad (3)$$

that corresponds to a random hopping and

$$V_{II} = \sum_{j,\ell} \frac{W_j}{\ell^\alpha} \left(a_j a_{j+\ell} + \text{H.c.} \right) \qquad (4)$$

that corresponds to a random long-range pairing. In the previous equations, $a_j^\dagger$ (a_j) is a fermionic creation (annihilation) operator on site j, μ is the chemical potential, $n_j = a_j^\dagger a_j$ and W_j are i.i.d random variables drawn from a uniform distribution of width $2W$ and zero mean value. We fix the energy scale by letting $\Delta = 2t = 1$ and we choose $\mu = 2.5$, corresponding to a gapped paramagnetic phase[104] for $W_j = 0$. Different values of μ do not change the results we find in the following. The random Hamiltonians (1) can be written in diagonal form as $H_{I,II} = \sum_{q=0}^{L-1} \Lambda_q \eta_q^\dagger \eta_q$ by a generalized Bogoliubov transformation[119] defined by $\eta_q = \sum_j (g_{q,j} a_j + h_{q,j} a_j^\dagger)$, with Λ_q the energies of the single-particle states labelled by q. The ground state $|\Omega\rangle$ is then the vacuum of all quasi-particles η_q and the matrix elements $g_{q,j}$ and $h_{q,j}$ can be identified with the wave functions of the two fermionic modes $\eta_q^\dagger$ and η_q, respectively.

As an interacting model, we consider the following random long-range Ising Hamiltonian[117] in transverse field

$$H_{LRI} = \sum_{j,\ell} (\sin\theta + B_{j,j+\ell}) \frac{\sigma_j^x \sigma_{j+\ell}^x}{\ell^\alpha} + \sum_{j=1}^{L} (\cos\theta + W_j)\sigma_j^z, \quad (5)$$

where σ_j^ν ($\nu = x,z$) are Pauli matrices for a spin-1/2 at site j and $B_{j,j+\ell}$ are i.i.d random variables drawn from a uniform distribution of width $2B$ and zero mean value. We choose $\theta = \pi/5$, corresponding to a paramagnetic phase[116] for $B_{j,j+\ell} = W_j = 0$. Different values of θ will not change the results we find in the following. For any finite disorder strength, the model Eq. (5) displays a many-body localized (MBL) phase[120–124] for $\alpha > 1$.

In the following, we first determine the regimes of localization for the fermionic models Eqs. (1) and then compute
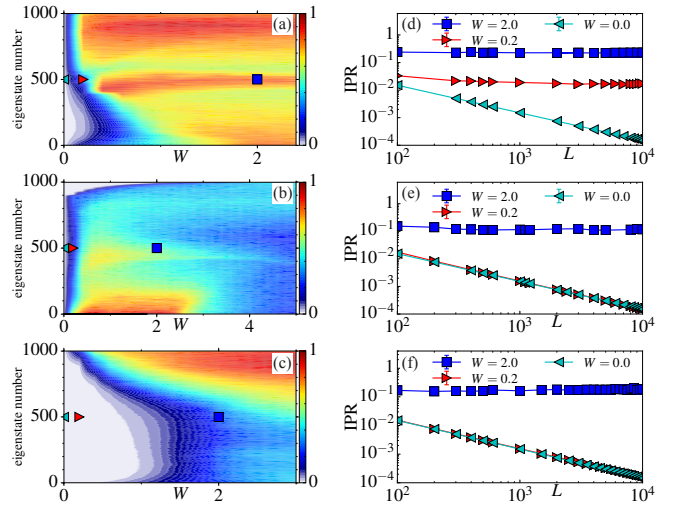

FIG. 1. Left panels: IPR for a system of $L = 1000$ sites as a function of the disorder strength W for (a) $\alpha = 3$ and (b) $\alpha = 0.8$ for the model (I) with random hopping and (c) for $\alpha = 3.0$ for the model (II) with random pairing. Only in these panels, for drawing purpose, the IPR has been rescaled to 1 in correspondence of its maximum value. Right panels: Scaling of the IPR for the states at the centre of the band as a function of the system size L and different W for (d) $\alpha = 3$ and (e) $\alpha = 0.8$ for the model (I) and (f) for $\alpha = 3.0$ for the model (II). In panels (a-c) the symbols indicate the values of W that we choose to plot the IPR in panels (d-f).

the single and the two-body correlation functions, as well as the wave-functions, within the localized phases using analytical and numerical techniques. For the long-range Ising model Eq. (5) we compute the spin-spin connected correlation functions numerically. Our goal is to demonstrate that all these quantities decay algebraically at large distances both for non-interacting and interacting MBL localized models and to characterize their decay exponents.

III. LOCALIZED PHASES OF DISORDERED FERMIONS

We determine the localized phases for Hamiltonians Eqs. (1) by combining information from the inverse participation ratio (IPR). The IPR gives information about the spatial extension of single-particle states and is defined as $\text{IPR}_q = \sum_{j=1}^{L} [|g_{q,j}|^4 + |h_{q,j}|^4]$ for a normalized state with energy Λ_q. The IPR tends to zero for increasing L for extended states, while it remains finite for localized states.

Figure 1 shows the IPR as a function of W for all the single particle eigenstates of a system of $L = 1000$ for model Eqs. (1) (I) [for $\alpha = 3$ and 0.8 in panels (a) and (b), respectively], and (II) [for $\alpha = 3$ in panel (c)] together with examples of finite size scaling [panels (d-e)].

For model (I) with disordered hopping and $\alpha > 1$ [panel (a)] essentially all states are localized. For $\alpha < 1$ [panel (b)] we find that, at W fixed, there exists an energy below (above) which all the states are localized (delocalized). However, the fraction of delocalized states that are found at high energy

scales to zero as $\sim 1/L^{1/(4-2\alpha)}$ and those states approach the band edge for $L \to \infty$ [see Appendix B]. For model (II) with disordered pairing when $\alpha > 1$ localized states are present at all energies if $W \gtrsim 2$ [panel (c)].

IV. CORRELATION FUNCTIONS

We consider the single-particle correlator $C(j,\ell) = \langle a_j^\dagger a_{j+\ell} \rangle_W$ for the two free-fermionic models of Eqs. (1) as well as the spin-spin correlation function $S_\nu(j,\ell) = \left[\langle \sigma_j^\nu \sigma_{j+\ell}^\nu \rangle - \langle \sigma_j^\nu \rangle \langle \sigma_{j+\ell}^\nu \rangle \right]_W$ (for $\nu = x, z$) for the interacting long-range Ising model of Eq. (5) in the localized phases. In the definitions of $C(j,\ell)$ and $S_\nu(j,\ell)$ the subscript W indicates averaging over the disorder distribution. For models with short-range interactions, all the correlation functions decay exponentially with ℓ. For models with-long range coupling, it has been shown that correlation functions can display a power-tail at long distance [see Appendix A]. Here we are interested in the effects that can originate from the interplay between long-range interactions and disorder.

Figures 2(a) and (b) show the correlator $C(\ell) := C(j_0, \ell)$ for models H_I and H_{II}, respectively, for different values of α. We choose $j_0 = L/4$ far from the edges in order to avoid boundary effects. We find numerically that the long-distance decay of $C(\ell)$ is always of power-law type $C(\ell) \sim \ell^{-\gamma}$ for all α within localized phases. In particular, for model H_I [panel (a)] and $\alpha < 1$ the decay is essentially algebraic at all distances with $\gamma \sim 2 - \alpha$, while for $\alpha > 1$ we find for both models a hybrid decay that is exponential at short distances and power-law at large distances, with $\gamma \sim \alpha$ [panels (a) and (b)]. Surprisingly, we find that the values of the decay exponents of the power-law tails do not depend on the disorder strength [see Appendix C]. Remarkably, we observe that this behaviour is typical also of states far from the bottom of the energy band. This is shown in Appendix C, where we compute the one-body correlation function of a localized single particle state at the centre of the energy band. This is reminiscent of recent results for fermions at finite temperature, in the absence of disorder[43].

This surprising long-distance behavior of correlations can be understood by computing the correlations analytically treating disorder as a perturbation. Here, we focus on model (I) with perturbation V_I. The homogeneous Hamiltonian H_0 can be diagonalised via Fourier and Bogoliubov transformations as $H_0 = \sum_k \lambda_\alpha(k) \xi_k^\dagger \xi_k$, where $\lambda_\alpha(k) = [(\cos k - \mu)^2 + 4f_\alpha^2(k)]^{1/2}$ and ξ_k are extended Bogolioubov quasi-particles related to the unperturbed fermionic operators in momentum space via $\tilde{a}_k = v_k \xi_k - u_k \xi_{-k}^\dagger$ with $v_k = \cos\varphi(k)$ and $u_k = \mathrm{i}\sin\varphi(k)$, with $\tan 2\varphi(k) = f_\alpha(k)/[\mu - \cos k]$ and $f_\alpha(k) = \sum_{\ell=1}^{L-1} \sin(k\ell)/\ell^\alpha$. At first order in W_j the ground state $|\Omega_0\rangle$ of the unperturbed Hamiltonian H_0 is modified by V_I as

$$|\Omega\rangle = |\Omega_0\rangle + |\delta\Omega_0\rangle = |\Omega_0\rangle - \sum_{kk'} J_{k,k'} A(k,k') \xi_k^\dagger \xi_{k'}^\dagger |\Omega_0\rangle, \quad (6)$$

where we define $J_{k,k'} = -\sum_j e^{\mathrm{i}(k-k')j} W_j/L$ and $A(k,k') =$

$2(e^{\mathrm{i}k} + e^{-\mathrm{i}k'}) v_k u_{k'}^* / [\lambda(k) + \lambda(k')]$. We find that $\langle \Omega | H | \Omega \rangle$ provides an excellent approximation of the exact ground-state energy for $0 < W < 3$ [see Appendix C], suggesting that $|\Omega\rangle$ is a good approximate ground-state in that range. For correlations, since $\langle J_{k,k'} \rangle_W = 0$, we find that $\langle \delta\Omega_0 | a_j^\dagger a_{j+\ell} | \Omega_0 \rangle_W$ and $\langle \Omega_0 | a_j^\dagger a_{j+\ell} | \delta\Omega_0 \rangle_W$ vanish due to averaging over the disorder distribution. We obtain the following expression for $C(\ell)$

$$\langle \Omega | a_j^\dagger a_{j+\ell} | \Omega \rangle_W = \langle \Omega_0 | a_j^\dagger a_{j+\ell} | \Omega_0 \rangle + \langle \delta\Omega_0 | a_j^\dagger a_{j+\ell} | \delta\Omega_0 \rangle_W. \quad (7)$$

The first term in the r.h.s. of Eq. (7) corresponds to the correlator of the homogeneous system[116] that is $\langle \Omega_0 | a_j^\dagger a_{j+\ell} | \Omega_0 \rangle = \int_0^{2\pi} dk\, e^{\mathrm{i}k\ell} R_0(k)$, with $R_0(k) = |u_k|^2$. The second term arises instead because of the random part of the Hamiltonian and reads

$$\langle \delta\Omega_0 | a_j^\dagger a_{j+\ell} | \delta\Omega_0 \rangle_W = \frac{2W^2}{3} \int_0^{2\pi} dk\, e^{\mathrm{i}k\ell} R_1(k), \quad (8)$$

where we have defined $R_1(k) = [c - U(k)]|u_k|^2 - V(k)|v_k|^2$, with c that does not depend on k, $V(k) = \sum_p A(p,k)A(k,p) \sim f_\alpha(k)/\lambda_\alpha(k)$ and $U(k) = V(-k)$. The behaviour of both integrals for $\ell \to \infty$ can be extracted by integrating $R_0(k)$ and $R_1(k)$ for $k \to 0$. In this limit, $f_\alpha(k)$, and thus the single-particle energy $\lambda_\alpha(k)$, display a non-analytical scaling $f_\alpha(k) \sim |k|^{\alpha-1}$. For the first term in the r.h.s. of Eq. (7) the latter behavior results in (details in Appendix C)

$$\langle \Omega_0 | a_j^\dagger a_{j+\ell} | \Omega_0 \rangle \sim \begin{cases} 1/\ell^{2-\alpha} & \text{for } \alpha < 1 \\ 1/\ell^{2\alpha-1} & \text{for } 1 < \alpha < 2 \\ 1/\ell^{\alpha+1} & \text{for } \alpha > 2 \end{cases} \quad (9)$$

which corresponds to the expected long-distance power-law decay of correlation functions for the homogeneous gapped superconductor with long-range pairing[104,116,125–127]. Instead, for $R_1(k)$ the scaling of $f_\alpha(k)$ near $k \to 0$ implies

$$R_1(k) \sim \begin{cases} k^{1-\alpha} & \text{for } \alpha < 1 \\ k^{\alpha-1} & \text{for } \alpha > 1 \end{cases} \quad (10)$$

which entails the following form of the disordered part of $C(\ell)$

$$\langle \delta\Omega_0 | a_j^\dagger a_{j+\ell} | \delta\Omega_0 \rangle_W \sim \begin{cases} W^2/\ell^{2-\alpha} & \text{for } \alpha < 1 \\ W^2/\ell^{\alpha} & \text{for } \alpha > 1 \end{cases} \quad (11)$$

after the integration of $R_1(k)$ in Eq. (8)

The discussion above demonstrates the following surprising results: (i) For $\alpha < 1$ disorder does not modify the power of the algebraic decay of correlations, rather it affects its strength. (ii) For $\alpha > 1$, the decay of correlations due to disorder is always algebraic, with an exponent that is smaller than for the homogeneous case with $W_j = 0$. This implies that disorder enhances algebraic localization in these gapped models. (iii) For $\alpha \leq 2$ we find the duality relation $\gamma(\alpha) = \gamma(2 - \alpha)$ in the exponents of the algebraic decay. This is reminiscent of the duality recently found for the decay exponent of the wave functions of long-range non-interacting spin models with positional disorder[115]. We come back to this point below.
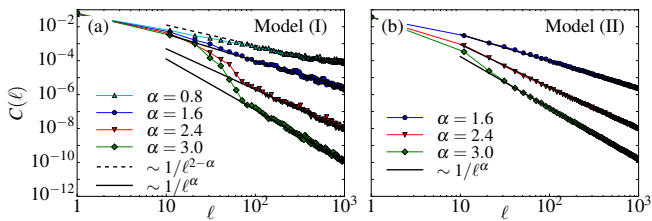
FIG. 2. (a) Correlation function $C(\ell)$ for the model (I) as a function of the lattice site ℓ for different values of α and for $W = 5$, $L = 2000$ and 400 disorder realizations. The power-law tails are fit by the black lines scaling as $1/\ell^{2-\alpha}$ (dashed) and $1/\ell^\alpha$ (solid) in agreement with the analytical results in Eq. (11). (b) Same as panel (a) but for the model (II).
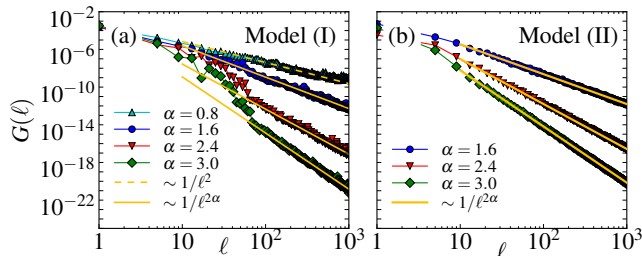

FIG. 3. (a) Density-density correlation function $G(\ell)$ for the model (I) as a function of the lattice site ℓ for different values of α and for $W = 5$, $L = 2000$ and 400 disorder realizations. The power-law tails are fit by the yellow lines scaling as $1/\ell^2$ (dashed) and $1/\ell^{2\alpha}$ (solid). (b) Same as panel (a) but for the model (II).

From the single-particle correlators $\langle a_j^\dagger a_{j+\ell} \rangle$ and $\langle a_j^\dagger a_{j+\ell}^\dagger \rangle$, by means of the Wick theorem, we computed also the density-density correlation functions

$$G(j, \ell) = \left[\langle n_j n_{j+\ell} \rangle - \langle n_j \rangle \langle n_{j+\ell} \rangle \right]_W$$
$$= \left[|\langle a_j a_{j+\ell} \rangle|^2 - |\langle a_j^\dagger a_{j+\ell} \rangle|^2 \right]_W. \quad (12)$$

Examples of $G(\ell) = G(j_0, \ell)$ with $j_0 = L/4$ are shown in Fig. 3 for a system of $L = 2000$ sites and for a disorder strength $W = 5$. Numerically we find that in the localized phases for model (I) when $\alpha < 1$, $G(\ell) \sim 1/\ell^2$ while for both models $G(\ell) \sim 1/\ell^{2\alpha}$ when $\alpha > 1$. The first behaviour with a decay exponent that does not depend on α has been already observed in Refs.[104,116], while the second can be explained by looking at the $\ell \to \infty$ scaling of $|C(\ell)|^2 \sim 1/\ell^{2\alpha}$ in Eq. (C26) in Appendix C.

For the random interacting long-range Ising model, we compute the spin-spin correlation functions $S_v(\ell) := S_v(j_0, \ell)$ ($v = x, z$) within the MBL phase with $\alpha > 1$, by using a DMRG alghoritm[128]. Here we choose $j_0 = L/10$. For the simulations, we use up to 400 local DMRG states, 16 sweeps and we average $S_v(\ell)$ over 100 disorder realizations. Strikingly, we find that $S_v(\ell)$ decays algebraically with ℓ as $S_v(\ell) \sim \ell^{-\gamma}$ with an exponent that is consistent with $\gamma = \alpha$, in complete agreement with the discussion above for non-interacting theories. As an example, Fig. 4(a) shows $S_x(\ell)$ for different values of α, $W = 5\sin(\pi/5) \approx 2.93$ and $B = 0$, while Fig. 4(b) shows
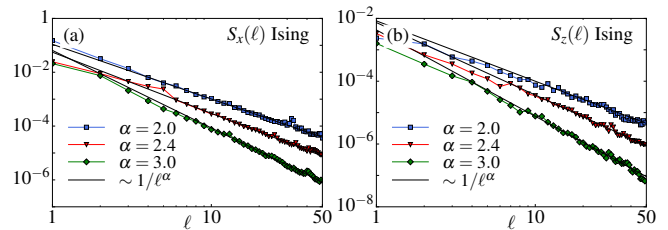

FIG. 4. (a) Correlation function $S_x(\ell)$ for the long-range Ising model with a random transverse field $[W = 5\sin(\pi/5)]$ and a constant interaction term ($B = 0$) for a system of $L = 100$ spins and 50 disorder realizations. (b) Correlation function $S_z(\ell)$ for the long-range Ising model with a random interaction $[B = 5\sin(\pi/5)]$ and a constant magnetic field ($W = 0$). In both panels, the power-law tails are fit by the black lines scaling as $1/\ell^\alpha$.

$S_z(\ell)$ for different values of α, $W = 0$ and $B = 5\sin(\pi/5)$. The corresponding fits (continuous lines) with $1/\ell^\alpha$ perfectly match the numerical results.

The demonstration of algebraic localization found in long-range couplings in the presence of disorder is a central result of this work. We argue that the fact that these results are found both for non-interacting and interacting models strongly suggests the existence of a universal behavior due to long-range coupling.

V. LOCALIZATION OF WAVE FUNCTIONS

Numerical results on the decay of the single-particle wave functions are obtained by considering the mean value $\Phi(\ell) = \sum_{q=1}^N |g_{q,\ell-j_M}|/N$ where we average $N = L/4$ wave functions $g_{q,\ell}$ with lowest energies, shifted by the quantity j_M that corresponds to the lattice site where $|g_{q,\ell}|$ shows its maximum value. We average $\Phi(\ell)$ also over several disorder realizations (of the order of 500).

Figure 5 shows typical results of the decay of $\Phi(\ell)$ as a function of the distance ℓ within the localized phases of models (I) and (II) of Eqs. (1) [panels (a,b) and (c,d), respectively].

Remarkably, we find that the wave functions decay algebraically at long distances regardless of the strength W of the disorder, mimicking the scaling of the correlation functions discussed above. However, for all α, i.e. both $\alpha > 1$ and $\alpha < 1$, $\Phi(\ell)$ decays at large distances as $\Phi(\ell) \sim \ell^{-\gamma_{\rm wf}}$, with an exponent $\gamma_{\rm wf}$ consistent with $\gamma_{\rm wf} \sim \alpha$. This is different from the results of Ref.[115] with positional disorder, where for $\alpha < 1$ one gets $\gamma_{\rm wf} \sim 2 - \alpha$. For sufficiently large $\alpha > 1$ this algebraic decay is preceded by an exponential decay at short distances, reminiscent of the exponentially localized states of short-range random Hamiltonians.

VI. CONCLUSIONS

In this work we have demonstrated that couplings that decay as a power-law with distance induce an algebraic decay of correlation functions and wave functions both in non-
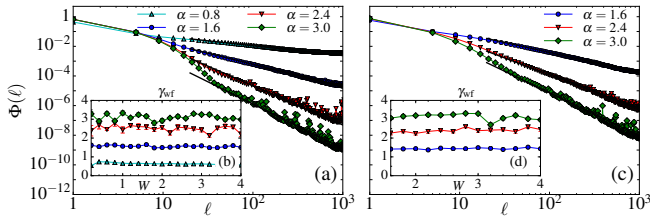
FIG. 5. (a) Decay of the averaged wave function $\Phi(\ell)$ (absolute value, see text) of localized states for the model (I): If $\alpha > 1$ we find an hybrid exponential and power-law behaviour. If $\alpha < 1$ the exponential part is suppressed and only the power-law tail is visible. The black lines correspond to fit of the data scaling as $1/\ell^{\gamma_{wf}}$. (b) Decay exponent γ_{wf} for the model (I) of the long-distance tail of $\Phi(\ell)$ as a function of W for different values of α. The decay exponent satisfies $\gamma_{wf} \sim \alpha$ and does not show significance dependence on W. (c-d) Same as (a-b) but for the model (II) with random long-range pairing.

interacting and interacting models in the presence of disorder. This is in stark contrast to results expected from short-range models, and generalises recent results for the decay of wavefunctions in quadratic models. These results are of immediate interest for experiments with cold ions, molecule, Rydberg atoms and quantum emitters in cavity fields, to name a few. It is an exciting prospect to explore the properties of many-body quantum phases in the search of exotic transport phenomena with long-range interactions.

ACKNOWLEDGMENTS

Appendix A: Correlation in the long-range models with no disorder

In this Appendix we summarize some results on the decay of the correlation functions for both the long-range Kitaev and Ising models in the absence of disorder.

For the long-range Kitaev model it has been shown in Refs.[104,116,125] that the one-body correlation function $C(j,\ell) = \langle a_j^\dagger a_\ell \rangle$ shows a different behaviour depending on the value α of the decay exponent of the pairing term. In the limit of a infinite system the one-body correlator takes the form of an integral on the Brillouin zone:

$$C(j,\ell) = \langle a_j^\dagger a_\ell \rangle = \frac{1}{2\pi} \int_0^{2\pi} dk \, \frac{\mu + \cos k}{\lambda_\alpha(k)} e^{ik(j-\ell)} \quad (A1)$$

where $\lambda_\alpha(k) = [(\cos k - \mu)^2 + 4f_\alpha^2(k)]^{1/2}$ and $f_\alpha(k) =$

$\mathrm{Im}\,\mathrm{Li}_\alpha(e^{ik})$. An explicit calculation of the previous integral gives

$$C(R) \equiv C(R,0) \sim e^{-\xi R} + A(\alpha, \mu) \cdot \begin{cases} \dfrac{1}{R^{\alpha+1}} & \text{if } \alpha > 2; \\[2mm] \dfrac{1}{R^{2\alpha-1}} & \text{if } 1 < \alpha < 2; \\[2mm] \dfrac{1}{R^{2-\alpha}} & \text{if } 0 < \alpha < 1. \end{cases} \quad (A2)$$

The parameter ξ is the smallest real solution of $\lambda_\alpha(i\pi + \xi) = 0$ and depend on μ and α in an implicit way and the coefficient $A(\alpha, \mu)$ can be found in Ref.[125]. It is possible to see that the correlation function shows a hybrid decay, i.e. exponential at short distances, followed by an algebraic tail whose decaying exponent depends on α.

For the long-range Ising model, Ref.[116] showed that the connected part of the correlation functions

$$S_v(j,\ell) = \langle \sigma_j^v \sigma_{j+\ell}^v \rangle - \langle \sigma_j^v \rangle \langle \sigma_{j+\ell}^v \rangle \quad (A3)$$

for $v = x, z$, decay with distance with a hybrid behaviour that is exponential at short distances and algebraic at long ones.

An example is shown in Fig. 6(a) for $S_x(R) \equiv S_x(R,0)$. The exponent γ_x of the long-distance decay displays three difference behaviours: (i) for $\alpha > 2$ it fulfils $\gamma_x = \alpha$; (ii) for $1 < \alpha < 2$ a hybrid decay is observed and the algebraic tail decays with an exponent γ that depends linearly on α with a slope consistent with ~ 0.55 and (iii) for $\alpha \lesssim 1$, $\gamma_x \sim 0.25\alpha$. The correlator $S_z(R)$ is shown in Fig. 6(b) and it also displays an algebraic tails that decays as $1/R^{\gamma_z}$ where $\gamma_z \sim 2\alpha$ for $\alpha > 1$.
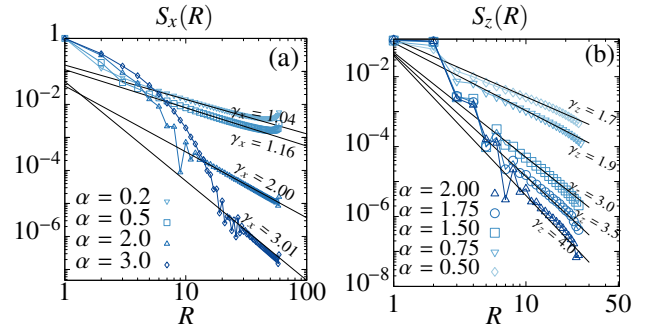

FIG. 6. (a) $S_x(R)$ correlation for the long-range Ising model (Eq. (5) of the main text) for $B_{ij} = W_{ij} = 0$ ($\theta = 0.2\pi$ and $L = 60$), showing the hybrid exponential and power-law behavior for $\alpha \gtrsim 1$ and a purely power-law for $\alpha \lesssim 1$. (b) $S_z(R)$ correlation for the long-range Ising model (Eq. (5) of the main text) for $B_{ij} = W_j = 0$ and $\theta = 0.207\pi$, $L = 100$ and different α.

Appendix B: Energy scaling analysis

In this Appendix we give an analytical insight (based essentially on Ref.[112]) on the different behaviours of the IPR (plot-

ted in Fig.1(a) and (b) of the main text) of the single-particle states for the Hamiltonians (I) when $\alpha > 1$ and $\alpha < 1$.

For the Hamiltonian (I), the random term V_I, which reads $V_I = \sum_{kk'} J_{k,k'}(e^{ik} + e^{-ik'})a_k^\dagger a_{k'}$ with $J_{k,k'} = -\sum_j e^{i(k-k')j}W_j/L$ after a Fourier transform, couples the extended eigenmodes of H_0 to each others and it can lead to localization if the mean fluctuation $\sigma_I^2 := \langle J_{k,k'}^2 \rangle = \delta_{kk'}W^2/(3L)$ is much larger than the level spacing $\delta\lambda_\alpha$ of the energies $\lambda_\alpha(k)$ of H_0. Here, we will restrict our attention to those states lying either at the minimum or at the maximun of $\lambda_\alpha(k)$. The level spacing $\delta\lambda_\alpha$ can be computed analytically from $\lambda_\alpha(k)$ and it strongly depends on α: (i) If $\alpha > 1$, $\lambda_\alpha(k)$ is finite for all k and we obtain $\lambda_\alpha(k) \sim k^2$ both at the minimum and at the maximum of the band. The corresponding level spacing thus scales with L as $\delta\lambda_{\alpha>1} \sim 1/L^2$ and it decays faster than σ_I. In this case the extended states of H_0 are coupled by the random part of V_I and they will be localized. No mobility edge is then expected for $\alpha > 1$. The IPR computed for this case is shown in Fig.1(a).

On the contrary, (ii) if $\alpha < 1$ the energy diverges as $\lambda_\alpha(k) \sim 1/|k|^{1-\alpha}$ and the level spacing for the high-energy states grows as $\delta\lambda_{\alpha<1} \sim L^{1-\alpha}$. The fluctuation of the random couplings $\sigma_I \sim 1/\sqrt{L}$ is thus suppressed by $\delta\lambda_{\alpha<1}$ and the high-energy states of H_0 remain extended. As the lowest-energy states can be localised, a single-particle mobility edge can be then expected for all $\alpha < 1$. By the equality $\delta\lambda_{\alpha<1} \sim \sigma_I$ that defines the single particle mobility edge, it is possible to show that the number of extended states increases with L as $N_{ext} \sim L^{(3/2-\alpha)/(2-\alpha)}$ but their fraction N_{ext}/L vanishes in the thermodynamic limit[113]. The IPR for this case is shown in Fig.1(b).

Appendix C: Decay of correlation functions

In this Appendix we show how to compute the correlation function $C(j,i) = \langle a_j^\dagger a_i \rangle$ of the model with random hopping by perturbation theory.

1. Correlation functions - Perturbation theory

We recall that the Hamiltonian H_I in Eq. (1) is formed by two parts:

$$H_I = H_0 + V_I. \tag{C1}$$

In order to compute the correlation function $C(j,i) = \langle\Omega|a_j^\dagger a_i|\Omega\rangle$ on the ground state $|\Omega\rangle$ of H_I in Eq. (1), we first find the first-order correction $|\delta\Omega_0\rangle$ to the ground state $|\Omega_0\rangle$ of H_0 by treating V_I as a perturbation.

The first-order correction $|\delta\Omega_0\rangle$ to the ground state $|\Omega_0\rangle$ of the Hamiltonian H_0 due to the perturbation V_I is given by[130]

$$|\delta\Omega_0\rangle = \sum_{\mathbf{n}_0} \frac{\langle\mathbf{n}_0|V_I|\Omega_0\rangle}{E(\mathbf{n}_0) - E_0}|\mathbf{n}_0\rangle \tag{C2}$$

where the quantities $E(\mathbf{n}_0)$ and E_0 are the energy of the states $|\mathbf{n}_0\rangle$ and of $|\Omega_0\rangle$, respectively and $|\mathbf{n}_0\rangle$ indicates an excited

state of the homogeneous Hamiltonian H_0 that can be diagonalized via Fourier and Bogoliubov transformations as

$$H_0 = \sum_k \lambda_\alpha(k)\xi_k^\dagger\xi_k. \tag{C3}$$

The ground state $|\Omega_0\rangle$ of H_0 is then the vacuum of all quasi-particles ξ_k.

In Eq. (C3) we have defined the single-particle energy

$$\lambda_\alpha(k) = [(\cos k - \mu)^2 + 4f_\alpha^2(k)]^{1/2} \tag{C4}$$

and the Bogolioubov quasi-particles ξ_k that are related to the original fermionic operators $\tilde{a}_k$ in momentum space via

$$\tilde{a}_k = v_k\xi_k - u_k\xi_{-k}^\dagger \tag{C5}$$

with $v_k = \cos\varphi(k)$ and $u_k = i\sin\varphi(k)$ where $\tan 2\varphi(k) = f_\alpha(k)/[\mu - \cos k]$ and $f_\alpha(k) = \sum_{\ell=1}^{L-1} \sin(k\ell)/\ell^\alpha$. We notice that the functions $f_\alpha(k)$ when $L \to \infty$ become $f_\alpha(k) = \left[\mathrm{Li}_\alpha(e^{ik}) - \mathrm{Li}_\alpha(e^{-ik})\right]/(2i)$, with $\mathrm{Li}_\alpha(z) = \sum_j z^j/j^\alpha$ a polylogarithm of order α.

The excited states $|\mathbf{n}_0\rangle$ are defined by assigning a set of occupied modes $\mathbf{n}_0 = \{n_1, n_2, \ldots, n_L\}$ with $n_q = 0, 1$ and then creating single quasi-particles $\xi_q^\dagger$ on the ground state $|\Omega_0\rangle$ if the mode q is occupied

$$|\mathbf{n}_0\rangle = \prod_{q=0}^{L-1} [\xi_q^\dagger]^{n_q}|\Omega_0\rangle. \tag{C6}$$

The first-order correction $|\delta\Omega_0\rangle$ can now be obtained from Eq. (C2) and the true ground state $|\Omega\rangle$ becomes

$$\begin{aligned}
|\Omega\rangle &= |\tilde{\Omega}\rangle + \mathcal{O}(W^2) \\
&= |\Omega_0\rangle + |\delta\Omega_0\rangle + \mathcal{O}(W^2) \\
&= |\Omega_0\rangle - \sum_{kk'} J_{k,k'}A(k,k')\xi_k^\dagger\xi_{k'}^\dagger|\Omega_0\rangle + \mathcal{O}(W^2),
\end{aligned} \tag{C7}$$

where we have defined $J_{k,k'} = -\sum_j e^{i(k-k')j}W_j/L$ and $A(k,k') = 2(e^{ik} + e^{-ik'})v_k u_{k'}^*/[\lambda(k) + \lambda(k')]$.

On a single disorder realization the correlation function $\langle\Omega|a_j^\dagger a_i|\Omega\rangle$ takes the form

$$\begin{aligned}
\langle\Omega|a_j^\dagger a_i|\Omega\rangle &= \langle\Omega_0|a_j^\dagger a_i|\Omega_0\rangle + \langle\delta\Omega_0|a_j^\dagger a_i|\Omega_0\rangle \\
&\quad + \langle\Omega_0|a_j^\dagger a_i|\delta\Omega_0\rangle + \langle\delta\Omega_0|a_j^\dagger a_i|\delta\Omega_0\rangle.
\end{aligned} \tag{C8}$$

If we now average Eq. (C8) over many disorder realizations, the cross terms $\langle\delta\Omega_0|a_j^\dagger a_i|\Omega_0\rangle$ and $\langle\Omega_0|a_j^\dagger a_i|\delta\Omega_0\rangle$ vanish as, due to the correction $|\delta\Omega_0\rangle$, only one random term W_j (that has mean value zero) appears in them. Therefore we get

$$\langle\Omega|a_j^\dagger a_i|\Omega\rangle_W = \langle\Omega_0|a_j^\dagger a_i|\Omega_0\rangle + \langle\delta\Omega_0|a_j^\dagger a_i|\delta\Omega_0\rangle_W. \tag{C9}$$

The first term of the r.h.s. of Eq. (C9) corresponds to the correlator for a homogenous translationally-invariant system. By rewriting $a_j^\dagger$ and a_i in momentum space and by using Eq. (C5) recalling that $\xi_k|\Omega_0\rangle = 0$ we obtain

$$C_0(\ell) := \langle\Omega_0|a_j^\dagger a_i|\Omega_0\rangle = \frac{1}{L}\sum_k e^{ik\ell}R_0(k) \tag{C10}$$

where $\ell = j - i$ and $R_0(k) = |u_k|^2$.

In the second term of the r.h.s. of Eq. (C9), as we are averaging on the disorder configurations, we can expect that the disorder average $\langle \delta\Omega_0 | a_j^\dagger a_i | \delta\Omega_0 \rangle_W$ will be translationally invariant, i.e. it will depend on the relative distance $\ell = j - i$ while the terms that depend on i and j separately will average out to zero (see §6.5 in Ref.[131] or §12.3 in Ref.[132]). By keeping only the terms that depend on ℓ, after rewriting $a_j^\dagger$ and a_i in momentum space and using again Eq. (C5) recalling that $\xi_k | \Omega_0 \rangle = 0$, the second term becomes

$$C_1(\ell) := \langle \delta\Omega_0 | a_j^\dagger a_i | \delta\Omega_0 \rangle_W = \frac{W^2}{3L} \sum_k e^{ik\ell} R_1(k) \qquad \text{(C11)}$$

where

$$R_1(k) = c|u_k|^2 + U(k)|u_k|^2 - V(k)|v_k|^2, \qquad \text{(C12)}$$

$$c = \sum_p A(p,p)^2 - \sum_{p_1 p_2} A(p_1,p_2)A(p_2,p_1), \qquad \text{(C13)}$$

$$U(k) = 2\sum_p A(p,-k)A(-k,p) \qquad \text{(C14)}$$

$$= -\frac{f_\alpha(k)}{\lambda_\alpha(k)} \sum_p \frac{2+2\cos(p-k)}{(\lambda_\alpha(k)+\lambda_\alpha(p))^2} \frac{f_\alpha(p)}{\lambda_\alpha(p)},$$

$$V(k) = U(-k) \qquad \text{(C15)}$$

We note that the quantity c does not depend on k.

2. Correlation functions - Asymptotic behavior

In this Section we show how the two correlators $C_0(\ell)$ and $C_1(\ell)$ behave asymptotically for $\ell \to \infty$.

Let us consider $C_0(\ell)$ in Eq. (C10) first. In the limit $L \to \infty$ we can replace the summation with an integral

$$C_0(\ell) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik\ell} R_0(k). \qquad \text{(C16)}$$

The asymptotic behavior of $C_0(\ell)$ for $\ell \to \infty$ can be computed by considering the integrals I_0^+ and I_0^- on the complex plane in Fig. 7 that are

$$I_0^\pm = \frac{1}{2\pi} \int_{s_\pm} dz\, e^{iz\ell} R_0(z) + \frac{1}{2\pi} \int_{\Gamma_\pm} dz\, e^{iz\ell} R_0(z) + \frac{1}{2\pi} \int_0^\infty dk\, e^{ik\ell} R_0(k) \qquad \text{(C17)}$$

where we have chosen to put the branch cut of the complex logarithm [see the expansion of the polylogarithm in Eq. (C19)] on the imaginary positive axis.

By sending the radius r of the circles $\Gamma_\pm$ to infinity and by neglecting possible residues inside the integration contour that will contribute only with exponential decaying terms we have

$$C_0(\ell) = -\frac{1}{2\pi} \int_{s_+} dz\, e^{iz\ell} R_0(z) - \frac{1}{2\pi} \int_{s_-} dz\, e^{iz\ell} R_0(z)$$

$$= \frac{i}{2\pi} \int_0^\infty dy\, e^{-y\ell} R_0(\varepsilon+iy) - \frac{i}{2\pi} \int_{-\infty}^0 dy\, e^{-y\ell} R_0(-\varepsilon+iy)$$

$$= \frac{1}{\pi} \int_0^\infty dy\, e^{-y\ell} \operatorname{Im} R_0(iy)$$

$$\text{(C18)}$$

where on the lines $s_\pm$ the complex variable is $z = \pm\varepsilon + iy$ with ε a small positive parameter that we send to zero.

We are able now to evaluate the asymptotic behavior of $C_0(\ell)$ by computing the $y \to 0$ part of $\operatorname{Im}[R_0(iy)]$ and then integrating the last equality in Eq. (C18). This is done by recalling that the polylogarithm admits the series expansion[133,134] for a general complex number z as

$$\operatorname{Li}_\alpha(z) = \Gamma(1-\alpha)\left(\ln\frac{1}{z}\right)^{\alpha-1} + \sum_{n=0}^{\infty} \zeta(\alpha-n)\frac{(\ln z)^n}{n!} \quad \text{(C19)}$$

that makes them non-analytical due to the presence of the complex logarithm and the power-law. In Eq. (C19), $\Gamma(x)$ and $\zeta(x)$ are the Euler gamma function and the Riemann zeta function, respectively.

By using the series expansion of the polylogarithms from Eq. (C19) we can obtain the function $R_0(iy)$ on the imaginary axis:

$$R_0(iy) = \frac{\mu - \cosh y}{2\lambda_\alpha(iy)} \sim \frac{\mu - 1}{2\sqrt{(\mu-1)^2 - \Gamma^2(1-\alpha)(e^{i\pi\alpha}+1)^2 y^{2\alpha-2} - 4\Gamma(1-\alpha)(e^{i\pi\alpha}+1)\zeta(\alpha-1)y^\alpha}}. \qquad \text{(C20)}$$

The previous equation in the limit $y \to 0$ gives

$$\mathrm{Im}\, R_0(iy) = \begin{cases} y^{1-\alpha} & \text{for } \alpha < 1 \\ y^{2\alpha-2} & \text{for } 1 < \alpha < 2 \\ y^{\alpha} & \text{for } \alpha > 2 \end{cases} \qquad \text{(C21)}$$

and, after performing the last integral in Eq. (C18), the asymptotic behavior of C_0 turns out to be

$$C_0(\ell) \sim \begin{cases} 1/\ell^{2-\alpha} & \text{for } \alpha < 1 \\ 1/\ell^{2\alpha-1} & \text{for } 1 < \alpha < 2 \\ 1/\ell^{\alpha+1} & \text{for } \alpha > 2. \end{cases} \qquad \text{(C22)}$$

For the correlator $C_1(\ell)$ in Eq. (C11) we can use the same contour in Fig. 7 and get

$$C_1(\ell) = \frac{W^2}{3\pi} \int_0^\infty \mathrm{d}y\, e^{-y\ell}\, \mathrm{Im}\, R_1(iy). \qquad \text{(C23)}$$

For the asymptotic behaviour of $C_1(\ell)$, we need again the $y \to 0$ part of $R_1(iy)$. Let us start by noting that from Eqs. (C14) and (C15) the $y \to 0$ part of both $U(iy)$ and $V(iy)$ is given by

$$\mathrm{Im}[U(iy)|u_{iy}|^2] \sim \mathrm{Im}[V(iy)|v_{iy}|^2]$$
$$\sim \mathrm{Im}\, \frac{f_\alpha(iy)(\mu - \cosh y)}{\lambda_\alpha^2(iy)} \qquad \text{(C24)}$$
$$\sim \begin{cases} y^{1-\alpha} & \text{for } \alpha < 1 \\ y^{\alpha-1} & \text{for } \alpha > 1. \end{cases}$$

The previous equation, by considering also the contribution coming from $c|u_{iy}|^2$ [see Eq. (C12)], gives

$$\mathrm{Im}\, R_1(iy) \sim \begin{cases} y^{1-\alpha} & \text{for } \alpha < 1 \\ y^{\alpha-1} & \text{for } \alpha > 1 \end{cases} \qquad \text{(C25)}$$

and after integrating Eq. (C23), we finally get the correlator

$$C_1(\ell) = \begin{cases} W^2/\ell^{2-\alpha} & \text{for } \alpha < 1 \\ W^2/\ell^{\alpha} & \text{for } \alpha > 1. \end{cases} \qquad \text{(C26)}$$

The asymptotic behavior coming from Eqs. (C22), (C26) can be checked by computing the correlator $C(\ell)$ numerically as reported in Fig. 2(a) of the main text. Remarkably, the values of the decay exponents of the power-law tails do not depend on the disorder strength W as shown in Fig. 8 where we plot the decay exponents of $C(\ell)$ as a function of W for different values of α. For completeness we show also the decay exponent of the correlation function $C(\ell)$ for the model (II) with random long-range pairing.

3. Validity of the perturbation theory

In this Section we give some details on the validity of the perturbation theory in approximating the true ground state of
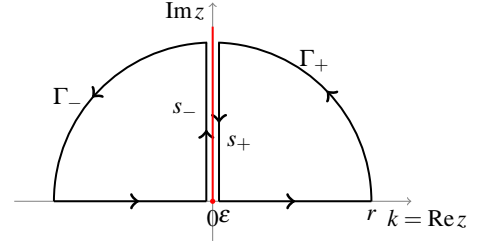

FIG. 7. Integration contour for evaluating the asymptotic behaviors of the correlators $C_0(\ell)$ in Eq. (C18) and $C_1(\ell)$ in Eq. (C23).
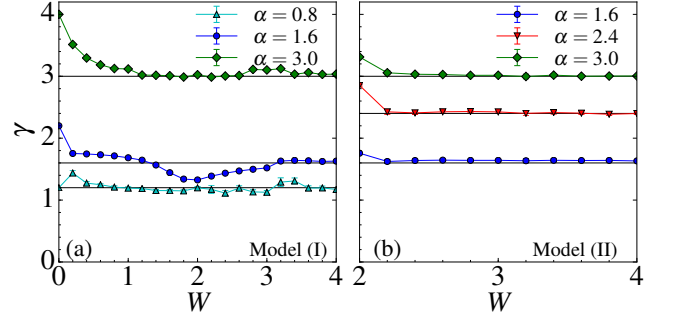

FIG. 8. (a) Decay exponent γ of the long-distance tail of the correlation function $C(\ell)$ for the model (I) as a function of W and for $\alpha = 0.8$ (cyan triangles), $\alpha = 1.6$ (blue circles), $\alpha = 3.0$ (green diamonds). If $W > 0$, the decay exponent satisfies $\gamma \sim \alpha$ for $\alpha > 1$ and $\gamma \sim 2 - \alpha$ for $\alpha < 1$ and it does not show significance dependence on W. These data are obtained by computing the correlation function $C(\ell)$ numerically from the full random Hamiltonian in Eq. (1) and then by fitting the long-range decaying tail of $C(\ell)$ with $1/\ell^\gamma$. The black lines represent the expected exponents: $\gamma = 1.2$ for $\alpha = 0.8$, $\gamma = 1.6$ for $\alpha = 1.6$, $\gamma = 3.0$ for $\alpha = 3.0$. (b) Same as panel (a) but for the localized phase (for $W \gtrsim 2$) of model (II).

H_I by the state $|\widetilde{\Omega}\rangle$ in Eq. (C7). To this end, we will compare the energy $E_{|\Omega\rangle}$ of the ground state of the Hamiltonian H_I computed numerically with the energy $E_{|\widetilde{\Omega}\rangle}$ of the state $|\widetilde{\Omega}\rangle = |\Omega_0\rangle + |\delta\Omega_0\rangle$ coming from the first order correction given in Eq. (C7).

The energy $E_{|\widetilde{\Omega}\rangle}$ can be obtained by considering that the first order correction to a wave function gives a second order correction to the energy[135]. Therefore, the energy for the perturbed state $|\widetilde{\Omega}\rangle$ is given by

$$E_{|\widetilde{\Omega}\rangle} = E_{|\Omega_0\rangle} + \sum_{\mathbf{n}_0} \frac{|\langle \Omega_0 | V_I | \mathbf{n}_0 \rangle|^2}{E_{|\Omega_0\rangle} - E_{|\mathbf{n}_0\rangle}} \qquad \text{(C27)}$$

where $E_{|\Omega_0\rangle}$ is the ground state energy of the Hamiltonian H_0, $|\mathbf{n}_0\rangle$ denotes the excited states of H_0 from Eq. (C6) and $E_{|\mathbf{n}_0\rangle}$ denotes their energy.

By computing the scalar product $\langle \mathbf{n}_0 | V_I | \Omega_0 \rangle$ we get

$$E_{|\widetilde{\Omega}\rangle} = E_{|\Omega_0\rangle} - \sum_{q_1, q_2} \frac{|\mathscr{E}_{q_1, q_2}|^2}{\lambda_\alpha(q_1) + \lambda_\alpha(q_2)} \qquad \text{(C28)}$$

where $\mathscr{E}_{q_1, q_2} = 2J_{q_1, q_2} e^{iq_1} v_{q_1} u_{q_2}^* + \text{h.c.}$ where v_{q_1} and u_{q_2} are
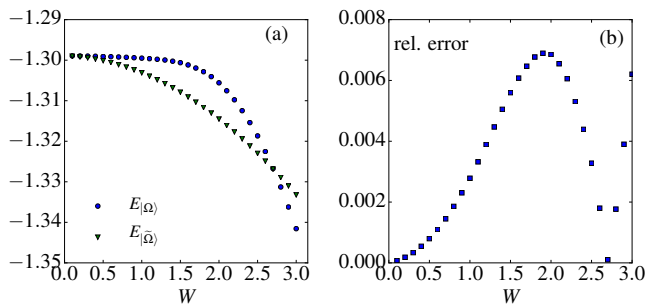
FIG. 9. (a) Energy density $E_{|\Omega\rangle}/L$ of the ground state of the Hamiltonian H_I computed numerically (blue circles) and the energy $E_{|\widetilde{\Omega}_0\rangle}/L$ (green triangles) computed by perturbation theory. (b) Relative error defined as $|1 - E_{|\widetilde{\Omega}\rangle}/E_{|\Omega\rangle}|$. It is possible to see that the difference between the true ground state energy and the perturbed one is bounded by $8 \cdot 10^{-3}$ when the disorder strength satisfies $0 \leq W \leq 3$. For both panels we consider $\alpha = 3.0$ and a system of $L = 400$ sites and after averaging 200 disorder realizations.
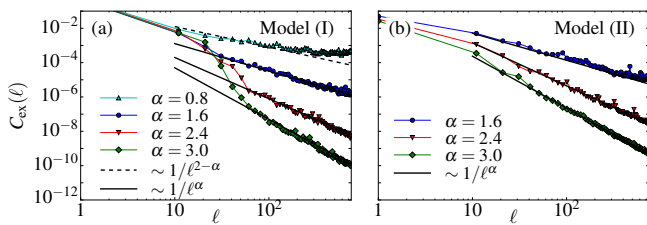

FIG. 10. Correlation function $C_{\mathrm{ex}}(\ell)$ computed on the an excited state at the middle of the energy band for the model (I) as a function of the lattice site ℓ for different values of α and for $W = 5$, $L = 1000$ and 200 disorder realizations. The continuous lines are guides to the eye and show that the power-law tails scales as $1/\ell^{2-\alpha}$ (dashed) and $1/\ell^{2\alpha}$ (solid) also for that excite state. (b) Same as panel (a) but for the model (II).

defined after Eq. (C5) and J_{q_1,q_2} after Eq. (C7). The left panel of Fig. 9 shows the values of the energy density $E_{|\Omega\rangle}/L$ of the ground state of the Hamiltonian H_I computed numerically (blue circles) with the energy $E_{|\widetilde{\Omega}_0\rangle}/L$ (green triangles), while the right panel shows the relative error defined as $|1 - E_{|\widetilde{\Omega}\rangle}/E_{|\Omega\rangle}|$ for a system of $L = 400$ sites and after averaging 200 disorder realizations for $\alpha = 3.0$. It is possible to see that the difference between the true ground state energy and the perturbed one is bounded by $8 \cdot 10^{-3}$ when the disorder strength satisfies $0 \leq W \leq 3$.

4. Correlators on excited states

In order to show that the algebraic tail of the correlators $a_j^\dagger a_\ell$ is not peculiar only to the ground state, but it is typical also for the excited states, we computed the correlation $C_{\mathrm{ex}}(j,\ell) = \langle\Omega|\eta_q\, a_j^\dagger a_\ell\, \eta_q^\dagger|\Omega\rangle$ for the excited state $\eta_q^\dagger|\Omega\rangle$. In Fig. 10 we plot $C_{\mathrm{ex}}(j,\ell)$ for an excited state that lies in the middle of the energy band for both the models (I) and (II) and for different values of α. It is possible to see that the algebraic tail is present also in this case.

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