On long range axion hairs for black holes

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Abstract

The physics of black holes can suggest new ways to test the existence of axions. Much work has been done so far to analyse the phenomenon of superradiance associated with axions in the ergoregion surrounding rotating black holes. In this work, we instead investigate how Chern-Simons axion couplings of the form $\phi\, F\, \tilde{F}$ and $\phi\, R\, \tilde{R}$, well motivated by particle physics and string theory, can induce long range profiles for light axion fields around charged black holes, with or without spin. We extend known solutions describing axion hairs around spherically symmetric, asymptotically flat dyonic black hole configurations, charged under $U(1)$ gauge symmetries, by including non-minimal couplings with gravity. The axion acquires a profile controlled by the black hole conserved charges, and we analytically determine how it influences the black hole horizon and its properties. We find a Smarr formula applying to our configurations. We then generalise known solutions describing axion hairs around slowly rotating black hole configurations with charge. To make contact with phenomenology, we briefly study how long range axion profiles induce polarised deflection of light rays, and the properties of ISCOs for the black hole configurations we investigate.

1 Introduction

Axions can be considered among the most well motivated candidates for physics beyond the Standard Model. Introduced in order to solve the strong CP problem [1–4] (see the review [5]), it was soon realised they have relevant implications for cosmology, as dark matter [6–8] or dark energy candidates [9] (see the review [10]). Moreover, axion fields arise naturally in string theory constructions (see e.g. [11]) with a broad variety of couplings and masses, motivating the string axiverse scenario of [12]. The quest of their experimental detection is currently an active research field, see e.g. [13] for a recent pedagogical review. Very light axion fields are specially interesting dark matter candidates [14–22], and their specific properties can lead to distinctive observational consequences.

In this work we investigate the relation between the physics of axions and black holes. Light axions, whose Compton wavelength is of the order of a black hole Schwarzschild radius, can deposit on the ergosphere of a rotating black hole, and cause instabilities associated with the phenomenon of black hole superradiance [23–25]. Such instabilities can have observable implications, since axions can extract rotational energy from spinning black holes [26–29] (see [30] for a comprehensive review), and lead to a distinctive emission of detectable gravitational waves (see e.g. [31,32]) [1]. Here we focus on the situation where the axion Compton wavelength is *much larger* than a black hole Schwarzschild radius. In this regime, the axion ϕ can be considered effectively massless, and the axion Lagrangian enjoys a shift symmetry $\phi \to \phi+\text{const}$. It is then natural to ask whether ϕ can develop an extended radial profile, leading to long range axion hairs around a black hole. There are several no-hair arguments to overcome, starting from Bekenstein results [36, 37]. On the other hand, a key property of axion fields we can exploit is the fact that they are characterised by Chern-Simons couplings to gauge fields and gravity,

$$\phi\, F\, \tilde{F} \quad , \quad \phi\, R\, \tilde{R} \tag{1}$$

with $\tilde{F}$ and $\tilde{R}$ respectively the Hodge dual of the gauge field strength and Riemann tensor. While the gauge Chern-Simons coupling naturally appears when dealing with anomalous symmetries of the QCD axion (see the reviews [5, 13]), the gravitational Chern-Simons coupling arises in explicit calculations in string theory constructions, see e.g. [11]. The linear Chern-Simons axion couplings in eq (1) source a non-vanishing asymptotic value for the axion fields for charged and/or rotating black holes, and allow one to avoid black hole no-hair theorems, somewhat analogously to what happens in Horndenski theories, where linear couplings with the Gauss-Bonnet invariant were found in [38,39] to be an important ingredient for overcoming the no-hair theorem of [40]. Our work proceeds as follows:

- In Section 2 we set the stage presenting the shift symmetric axion system we consider. We include additional derivative couplings of the axion field with curvature, allowed by the symmetries of the system, which can be motivated by high energy constructions. We explain in detail why the Chern-Simons couplings (1) are expected to induce regular axion hairs around regular spherically symmetric charged black holes, and around rotating black holes (with or without charge).

- In Section 3 we study spherically symmetric dyonic black holes, with electric and magnetic charges, equipped by a long range axion hair. We generalise known solutions [41–43] to cases where the axion has derivative couplings with the Ricci tensor, motivated in Section 2, showing that Chern-Simons couplings can be effective in generating axion hairs also in theories with non-minimal derivative couplings of axions to curvature. The axion profile has a 'secondary' charge, i.e. it depends on the black hole conserved charges. We work at leading order in an expansion of the relevant coupling constants, finding analytical solutions and discussing how the properties of the black hole horizon are modified by the axion hair.

- In Section 4 we consider slowly rotating charged black holes, finding new configurations with axion hairs, which generalize and interpolate between known solutions [44,45]. Inter-

[1] Possible ways to probe properties of axions with neutron stars are also being investigated, see e.g. [33–35].

estingly, we find that the gauge and gravitational Chern-Simons couplings symultaneously play a role in characterising the axion hair and the geometry, leading to mixed contributions depending on both these parameters.

- In Section 5 we study phenomenological applications of our findings, showing how long range black hole axion hairs can bend differently the two polarizations of light crossing in proximity of the black hole (elaborating on the methods of [46]), and by studying properties of the innermost stable circular orbits (ISCOs) for our configurations.

2 System under consideration

2.1 The action

We consider the following action describing gravity coupled with a $U(1)$ gauge field A_μ (with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ its field strength) and a pseudoscalar axion field ϕ

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{4} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \mathcal{L}_\phi (\phi, \partial\phi) - \frac{g_F}{4} \phi F_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{g_R}{4} \phi R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} \right] , \qquad (2)$$

where from now on we have set units such that $c = G_N = 1$. This action describes Einstein gravity coupled with a $U(1)$ Maxwell field, and a pseudoscalar axion described by the Lagrangian density $\mathcal{L}_\phi$ (to be specified in what follows). The axion is also non-minimally coupled with gauge and metric fields through Chern-Simons operators weighted by the constants g_F and g_R: the dual of the field strength and Riemann tensor are defined as $\tilde{F}^{\mu\nu} = \frac{1}{2}\, \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ and $\tilde{R}^{\mu\nu\rho\sigma} = \frac{1}{2}\, \epsilon^{\mu\nu\alpha\beta} R^{\rho\sigma}{}_{\alpha\beta}$. The axion can be identified with the Nambu-Goldstone boson of a spontaneously broken Peccei-Quinn symmetry, or with the Hodge dual of a three-form which appear by string theory scenarios. In both cases, Chern-Simons couplings with the structure above arise naturally. The dimensionful coupling constants of the Chern-Simons terms can be associated with the axion decay constants: for our purposes we appropriately weight those operators with powers of some fundamental mass scale (for example the Planck mass) and consider the quantities $g_{F,R}$ as dimensionless free constant parameters, influencing the geometry of the space-times we consider.

At the perturbative level, axion fields typically enjoy a shift symmetry $\phi \to \phi + \text{const}$, which can be broken by non-perturbative effects giving a mass to the axion. For cosmological purposes – as dark energy or dark matter candidates – axions are typically very light. For the aim of this work, we consider axions whose Compton wavelength is much larger than the black hole size. Being the Compton wavelength of an axion field of mass m_ϕ

$$\lambda_{\text{Compton}} \simeq \left(\frac{10^{-10}\,\text{eV}}{m_\phi} \right) \left(\frac{M_{BH}}{M_\odot} \right) r^{\text{sun}}_{\text{Schw}} \,, \qquad (3)$$

we can assume that the axion masses are much smaller than 10^{-10} eV when considering solar mass black holes, or much smaller 10^{-16} eV when investigating supermassive black holes sitting in the centre of galaxies, so that λ_{Compton} is well larger than the corresponding black hole

Schwarzschild radius. As a consequence, we will neglect shift symmetry-breaking effects, and consider the axion field as effectively massless. The shift-symmetric axion Lagrangian $\mathcal{L}_\phi$ we shall consider is quadratic in the axion field, so to respect the pseudoscalar properties of the axion. It contains the standard axion kinetic terms, plus non-minimal derivative couplings with gravity:

$$\mathcal{L}_\phi \,=\, \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi + \lambda\, G^{\mu\nu}\,\nabla_\mu\phi\nabla_\nu\phi, \tag{4}$$

with $G^{\mu\nu}$ being the Einstein tensor. This is the most general shift symmetric quadratic action for the pseudoscalar axion field, which is invariant under the parity symmetry properties of the pseudoscalar axion, and leads to second order equations of motion. The second contribution in (4) is a dimension-six operator belonging to the Horndeski action. We weight it with appropriate powers of the Planck mass, and consider the parameter λ as dimensionless.

We include the operator controlled by λ for the following reasons:

i) It is allowed by the symmetries of the axion system – shift symmetry and parity invariance – hence it can be expected from an effective field theory point of view, being induced by perturbative loop corrections to the tree-level action. Although derivative couplings of axions with curvature are rarely considered (see however [47] for an example), they can be motivated by string theory constructions. For example, [48] computed α' and loop corrections in heterotic string systems including the 3-form antisymmetric tensor field $H_{\mu\nu\rho}$, finding various couplings of the latter with Riemann and Ricci tensors. Once the three form field is Hodge-dualised to a pseudoscalar, such couplings can lead to derivative couplings of the axion to curvature tensors [2], as in Lagrangian (4).

ii) Non-minimal couplings of scalar fields with curvature have a long history in models of dark energy (see e.g. [49] for a review) and cosmological inflation (as in Starobinsky [50] or Higgs inflation [51, 52]). Although they are typically suppressed by some high energy scale, they can have important implications when the scalar acquires a non-trivial *vev*. Their consequences for black hole physics, and the existence of scalar hairs evading strong no-hair theorems, is a well studied subject. It is then natural to investigate this topic for the case of pseudoscalar axions, and to explore the role of Chern-Simons couplings in allowing for black hole axion hairs, including derivative couplings with gravity.

In the next subsection, we explain why the gauge and Chern-Simons couplings play an essential role in allowing axion hairs on charged and rotating black holes.

2.2 Long range axion hairs and black holes

Cosmological considerations motivate the investigation of light axion fields with Compton wavelengths well larger than the typical sizes of astrophysical black holes. In this context, the axion field can be considered as massless; it might acquire a long range profile, influencing spherically

[2]These are heuristic, preliminary considerations that would be interesting to further develop in a proper string theory setting; on the other hand, this goes beyond the scope of this work.

symmetric configurations. But can a regular, asymptotically flat black hole with extended axion hairs exist? There is a long list of no-hair theorems to overcome, starting from Bekenstein results [36, 37], which forbid configurations with a long range (pseudo)scalar ϕ; no-hair arguments are particularly powerful in the case of shift-symmetric Lagrangians [40] like ours. (See e.g. [53–55] for reviews.) It is then interesting, both theoretically and phenomenologically, to assess whether long range axion hairs exist around black holes.

The theorem discussed in [40] makes use of the properties of the conserved current J^μ associated with the shift symmetry: [40] shows that – for systems that do not contain linear terms in ϕ – regularity of the current at the horizon imposes a trivial radial profile for ϕ. On the other hand, in the context of Horndeski theories, a way out has been found in [38] by noticing that the quintic Horndeski Lagrangian contains contributions linear in ϕ (coupling the field with the Gauss-Bonnet combination), hence evading the theorem and allowing for black hole hairs in asymptotically flat configurations. (Exact solutions in related systems were found in [56].)

In our shift-symmetric context, action (2), we do as well have linear couplings of the axion field with gauge and gravitational Chern-Simons terms [3], hence we can in principle find black holes with long range axion hairs. The scope of our work is indeed to demonstrate that these couplings offer new opportunities to find regular, asymptotically flat black hole solutions with long-range pseudoscalar fields. Examples of black hole solutions with axion hairs, with and without charge and spin, have indeed been determined in the past [41–44, 57], and the subject has been particularly developed in the context of Chern-Simons gravity [58] (see e.g. [59] and the review [60]).

In our work, we further generalise these solutions at the light of these considerations, and extend them to the case of further derivative couplings of the axion with gravity, described by Lagrangian (4), and by considering the simultaneous presence of g_F, g_R. The presence of Chern-Simons terms leads to sources for the axion field which are non-vanishing at spatial infinity, inevitably inducing long-range axion profiles around black holes. In particular, the $F\tilde{F}$ coupling provides a source for spherically symmetric dyonic black holes, with electric (Q) and magnetic (P) charges, since it acquires the following profile asymptotically far from the black hole:

$$F\tilde{F} \sim \frac{P\,Q}{r^4}\,. \tag{5}$$

On the other hand, the gravitational $R\tilde{R}$ coupling sources an axion radial profile when the black hole geometry is axisymmetric, for example when it is rotating with rotation parameter $a = J/M$, (J being the angular momentum and M the black hole mass). In the uncharged case, and in the small rotation limit ($a \ll 1$), one finds

$$R\tilde{R} \sim \frac{a\,M^2}{r^7}\cos\theta\,. \tag{6}$$

The resulting axion field induced by the sources (5) and (6) backreacts on the geometry and influences the physical properties of the black hole at the position of the horizon.

[3] Shift symmetry is ensured by the fact that $F\tilde{F}$ and $R\tilde{R}$ are by themselves total derivatives.

In the next Sections, we first study charged spherically symmetric black hole configurations for this system; then we study slowly rotating charged solutions, and afterwards we analyse phenomenological consequences of our findings.

3 Spherically symmetric configurations

In this Section, we discuss asymptotically flat, spherically symmetric configurations with axion hairs around dyonic black holes, characterised by both magnetic and electric charges associated with the $U(1)$ gauge field A_μ. This gauge field can be associated with some gauge symmetry beyond the Standard Model – for example associated with dark gauge bosons for dark matter interactions [61–63], or some additional gauge group motivated by string theory – or with standard electromagnetism: our discussion applies to both cases [4]. The Chern-Simons gauge coupling $F\tilde{F}$ sources an axion profile that scales as $1/r$ asymptotically far from the object (while the gravitational Chern-Simons term $R\tilde{R}$ vanishes for spherically symmetric solutions). Its backreaction on the geometry and the gauge field can be analytically computed in a perturbative expansion in the dimensionless coupling constants g_F and λ, appearing in action (2) and scalar Lagrangian (4). We work at leading order in a perturbative expansion on these parameters to maintain our expressions relatively simple, and since interesting physical phenomena already appear at leading order in such expansion [5].

The axion profile affects the electric field, but not the magnetic one. Our solution generalises the spherically symmetric configurations studied by Campbell et al [41, 42] and by Lee and Weinberg [43] (see also [57]) by including non-minimal couplings between axion and metric, described by the Horndeski coupling λ in the Lagrangian density (4). As we shall discuss, the properties of our configurations are different from other black hole solutions in theories with non-minimal couplings with gravity.

The axion profile At leading order in an expansion in both g_F and λ, the unique solution for the pseudoscalar equation of motion that is regular at the black hole horizon is

$$\phi(r) \,=\, -g_F\, P\, Q \left[\frac{\log\left(1-\frac{r_-}{r}\right)}{r_- r_+} - \frac{\lambda}{r_-^4} \left(\frac{2\, r_-}{r} + \frac{r_-^2}{r^2} + \frac{2\, r_-^3}{3 r^3} + \frac{r_-^4}{2 r^4} + 2 \log\left(1-\frac{r_-}{r}\right) \right) \right] \,, \tag{7}$$

where $r_\pm$ correspond to the outer/inner horizons of Reissner-Nördstrom (RN) black holes of mass M with electric and magnetic charges, respectively Q and P (recall that we are choosing units so that $M_{\rm Pl}^2 \,=\, 1/2$):

$$r_\pm = M \pm \sqrt{M^2 - P^2 - Q^2}\,. \tag{8}$$

In writing the solution (7), we fix integration constants such to impose regularity of this configuration at asymptotic infinity, and at the external RN horizon r_+; we could add to this

[4]The configurations we consider differ from string theory configurations found in dilaton–Maxwell-gravity systems and dilaton-axion-Maxwell-gravity systems (see e.g. respectively [64], [65] and the comprehensive review [66]), since in our case we do not have any scalar dilaton in our system, but only a pseudoscalar axion.

[5]For completeness, in Appendix A we present an exact solution for a specific, potentially large value of the coupling constant λ.

expression a constant ϕ_0 which however has no physical consequences due to shift-symmetry. The axion profile corresponds to a 'secondary' hair, since there is no independent pseudoscalar charge but it is fully controlled by the black hole charges (M, P, Q); on the other hand, as we shall see, the axion affects properties of the black hole horizon. Notice that the axion profile is regular at the position r_+ of the outer horizon, while it diverges at the inner horizon r_- (more on this later). Expanding for large values of r, one finds [6]

$$\phi \sim \left(\frac{g_F\, P\, Q}{r_+}\right)\frac{1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) . \tag{9}$$

Interestingly, we find that the pseudoscalar axion profile scales as $1/r$, so charged black holes with gauge couplings $\phi F\tilde{F}$ activate a monopolar axion configurations. This is a difference with respect to what happens with gravitational Chern-Simons couplings $\phi R\tilde{R}$, as we shall learn in Section 4, where the axion radial profile scales at least as $1/r^2$ [44]. This axion profile has interesting consequences for the physics of the horizon and for the phenomenology of the system. Notice that although the axion scales with the radial coordinate as $1/r$, it can more easily circumvent fifth force constraints [9, 67], since being a pseudoscalar it does not directly couple to the trace of matter energy momentum tensor. Moreover, one can check that the corrections weighted by λ, associated with derivative couplings with the metric, influence the axion profile only starting at order $1/r^4$: this is probably a manifestation of Vainshtein screening in this context (see e.g. [68] for a review).

Gauge field solution The vector potential for this configuration has both electric and magnetic components

$$A_\mu = \{A_t(r),\, 0,\, 0,\, A_\varphi(\theta)\} \tag{10}$$

where, at leading order in a g_F and λ expansion, we find the solution

$$\begin{aligned}
A_t(r) =& -\frac{Q}{r} + \frac{g_F^2\, P^2 Q}{r}\left[\frac{r_-\left(r_-r_+ - 2Q^2\right) + \left[Q^2(r_- - 2r) + r_+r_-(r-r_-)\right]\log\left(1-\frac{r_-}{r}\right)}{r_-^3 r_+^2} + \lambda\, a_\lambda(r)\right] \\
a_\lambda(r) =& \frac{1}{30r^5r_-^2}\Bigg\{r_-r_+\Big[60r^4(r_-+4r_+) + 30r^3r_-(r_- - 4r_+) + 20r^2r_-^2(r_- - 2r_+) + 5rr_-^3(3r_- - 4r_+) \\
& - 12r_-^4 r_+\Big] - 5P^2\Big(12r^4(r_- + 5r_+) + 6r^3r_-(r_- - 5r_+) + 2r^2r_-^2(2r_- - 5r_+) + rr_-^3(3r_- - 5r_+) \\
& - 3r_-^4 r_+\Big) - 60\frac{r^4}{r_-}\log\left(1 - \frac{r_-}{r}\right)\left[Q^2\Big(-rr_- - 5r_+r + 5r_-r_+\Big) + r_-r_+(rr_+ - r_-r_+)\right]\Bigg\} \\
A_\varphi(\theta) =& -P\,\cos\theta.
\end{aligned} \tag{11}$$

[6]It is straightforward to verify that this asymptotic, large r profile for the scalar field remains valid also beyond a leading order expansion in the coupling constants g_F, λ. We will use this fact in Section 5.1.

This corresponds to a magnetic monopole configuration with magnetic charge P, additionally charged under an electric field $E_r = \partial_r A_t(r)$ whose profile is modulated by the axion configuration, and by the presence of the magnetic charge P. This fact resembles, in curved space-times, a phenomenon called Witten effect [69,70], where an axion configuration switches on an electric field in a magnetic monopole background in the presence of a Chern-Simons coupling $\phi F\tilde{F}$.

Metric solution At leading order in a g_F and λ expansion, the metric line element reads as follows:

$$ds^2 = -F(r)\,dt^2 + \frac{dr^2}{h(r)\,F(r)} + r^2\,d\theta^2 + r^2\,\sin^2\theta\,d\varphi^2\,, \tag{12}$$

with

$$\begin{aligned}F(r) =& \left(1-\frac{r_-}{r}\right)\left(1-\frac{r_+}{r}\right) + g_F^2\left(\frac{P\,Q}{r_-r_+}\right)^2\Bigg\{-\frac{2r_-}{r}+\frac{r_+r_-}{r^2}-\frac{2r-r_+-r_-}{r}\log\left(1-\frac{r_-}{r}\right)\\ &+\frac{\lambda}{30r_-^4r_+}\Bigg[\frac{600}{r}-\frac{60(4r_-+9r_+)}{r^2}-\frac{70r_-(r_--3r_+)}{r^3}-\frac{30r_-^2(r_--2r_+)}{r^4}\\ &-\frac{5r_-^3(3r_--5r_+)}{r^5}+\frac{12r_-^4r_+}{r^6}+\frac{60\left(10r^2-9r(r_-+r_+)+8r_-r_+\right)\log\left(1-\frac{r_-}{r}\right)}{r^2r_-}\Bigg]\Bigg\}\\ h(r) =& 1+\frac{2\,g_F^2P^2Q^2}{r_-^2r_+^2}\Bigg\{\frac{r_-}{r-r_-}+\log\left(1-\frac{r_-}{r}\right)\\ &-\lambda\,\frac{10\,r_+}{r_-^2\,(r-r_-)}\Bigg[1-\frac{r_-}{2\,r}-\frac{r_-^2}{6\,r^2}-\frac{r_-^3\,(5r_+-12r_-)}{60\,r_+\,r^3}-\frac{r_-^4}{4\,r^4}-(1-\frac{r}{r_-})\log\left(1-\frac{r_-}{r}\right)\Bigg]\Bigg\}\end{aligned} \tag{13}$$

Setting $g_F = 0$, one finds the standard Reissner-Nördstrom solution with magnetic and electric charges. Setting $\lambda = 0$ (and leaving $g_F = 0$ switched on) one finds the solution of [43]. The axion profile backreacts on the metric, with corrections starting only at order $1/r^3$ in an $1/r$ expansion. Interestingly, the axion profile changes the position of the outer horizon of the charged black hole: at leading order in an expansion in the dimensionless couplings g_F, λ, the position of the outer horizon is

$$r_H = r_+ + g_F^2\,r_S, \tag{14}$$

with

$$\begin{aligned}r_S =& \frac{P^2Q^2}{r_+r_-^2}\Bigg[\log\left(1-\frac{r_-}{r_+}\right)+\frac{r_-}{r_+-r_-}\\ &+\lambda\,\frac{r_-\left(3r_-^4+5r_-^3r_++10r_-^2r_+^2+30r_-r_+^3-60r_+^4\right)-60r_+^4(r_+-r_-)\log\left(1-\frac{r_-}{r_+}\right)}{30\,r_-^3r_+^3\,(r_+-r_-)}\Bigg].\end{aligned} \tag{15}$$

The geometry is regular at the outer horizon, and we find (in our approximations of small g_F, λ) that the presence of the axion hair increases the size of the outer horizon. Curvature invariants *diverge* at the position of the inner horizon r_-: the axion profile makes singular the inner horizon of the charged black hole. This fact was already found in [41, 43] for the case with minimal couplings with gravity, and remains true in our set-up with derivative couplings to gravity described by Lagrangian (4).

So, as anticipated in Subsection 2.2, Chern-Simons gauge couplings $\phi F\tilde{F}$ provide qualitatively new opportunities to find new black hole solutions with axion hairs, also in the case of non-minimal couplings of the axion with gravity. Our analytic formulas – valid for small values of the coupling constants – describe how the black hole geometry is affected by the long range axion configurations.

To conclude this subsection, we qualitatively compare some aspects of our findings with some of the regular black hole solutions in theories of Horndeski. In [38, 39], asymptotically flat, hairy solutions have been found in a system with a linear coupling between the scalar field and the Gauss-Bonnet combination (hence avoiding the no-hair theorem [40], see our Section 2.2), weighted by a dimensionless quantity that we call g_{GB}. Moreover, [39] determines analytical solutions for the system in a perturbative expansion for small g_{GB}, finding that the geometry is affected only at next to leading order in such expansion, requiring a control of the theory up to next-to-leading level in the small parameter g_{GB}. Instead, in our case, differences with respect to the standard Reissner-Nördstrom configuration start already at leading order in our parameter expansion. The works [71–73] determine regular black hole solutions with non-minimal couplings to gravity like ours, but in order to avoid the no-hair theorem [40] the solutions are not asymptotically flat. A more comprehensive discussion of black holes in theories non-minimally coupled with gravity can be found in the review [74].

3.1 A Smarr formula for our configurations

Since we analytically determined – at leading order in a perturbative expansion in the parameters g_F, λ – how the axion profile modifies the location of black hole horizon, we can enquire whether classical black hole thermodynamic formulas remain valid in our system. (See also [75] for a general discussion on black hole thermodynamics in theories containing non-minimal couplings with gravity and higher derivative interactions.)

The Smarr formula is associated with the first law of thermodynamics, and relates the black hole entropy with conserved asymptotic charges. It reads, in the context of spherically symmetric geometries [76],

$$M = \frac{\kappa}{2\pi} S + Q\,\Phi_H^E + P\,\Phi_H^M\,, \tag{16}$$

where M is the ADM mass, Q and P are the electric and the magnetic charges, κ is the surface gravity, S is the black hole entropy and $\Phi_H^{E,\,M}$ respectively are the electric and magnetic potentials evaluated at the horizon. Using Hawking relation between the black hole entropy and its event horizon area

$$S = \frac{A_H}{4}, \tag{17}$$

we can compute the various quantities appearing in eq (16). At leading order in g_F, λ, we find

$$\begin{aligned}\kappa = \frac{r_+ - r_-}{2r_+^2} - \frac{g_F^2 P^2 Q^2}{r_-^2 r_+^4}\Bigg[&\frac{r_-(r_+ - 2r_-) + (r_+ - r_-)^2 \log\left(1 - \frac{r_-}{r_+}\right)}{r_+ - r_-} \\ &- \lambda\frac{\left(r_-^5 - 2r_-^4 r_+ - 5r_-^3 r_+^2 - 20r_-^2 r_+^3 + 90 r_- r_+^4 - 60 r_+^5\right) - 60\, r_+^4\, r_- (1 - \frac{r_+}{r_-})^2 \log\left(1 - \frac{r_-}{r_+}\right)}{10\, r_-^2\, r_+^3\,(r_+ - r_-)}\Bigg],\end{aligned} \tag{18}$$

$$A_H = 4\pi\left(r_+^2 + 2\,g_F^2\,r_S\,r_+\right),$$

$$\Phi_H^E = -A_t(r), \tag{19}$$

$$\Phi_H^M = \Phi_H^E(P \leftrightarrow Q),$$

where A_t is given in eq (11), r_S in (15), while the magnetic potential can be computed using the procedure explained in [77], and in our case is simply related with the electric potential interchanging P with Q. Substituting these quantities in (16), we find that the Smarr formula is satisfied for our black hole solutions: hence the first law of thermodynamics applies also to dyonic black holes with axion fields non-minimally coupled with gravity. Notice that formula (16) does not receive 'corrections' associated with the pseudoscalar axion field: a possible reason is that the axion profile does not modify the asymptotic conserved charges of our configuration (M, P, Q), since it only affects metric and gauge fields with higher order corrections in an $1/r$ expansion. In fact, the asymptotic black hole mass and charges do not receive any contribution from the axion, which consequently does not influence the overall energetic balance controlled by the Smarr formula.

4 Slowly rotating charged configurations

When turning on spin, besides the gauge, also the gravitational Chern-Simons coupling $\phi R\tilde{R}$ acquires a non-zero value at asymptotic infinity – see eq (6) – and can source a non-trivial axion profile. This fact is related with the well-developed topic of rotating black hole solutions in Chern-Simons gravity, see e.g. [59, 78–84] and the review [60].

In this section, we investigate for the first time how the simultaneous presence of gauge and gravitational Chern-Simons couplings affects the geometry of rotating charged black holes with a long range axion hair. To stress the role of the gravitational Chern-Simons couplings, we consider only black holes with a magnetic monopole charge (no electric charge), and we work in the limit of small rotation, $a = J/M \ll 1$. For simplicity, we also turn off the non-minimal couplings with gravity, $\lambda = 0$ in Lagrangian (4). Notice that this situation is different from Wald configuration of a black hole immersed in an external magnetic field [85], since in our case it is the black hole itself that is magnetically charged.

The Ansatz we adopt for the metric is

$$ds^2 \,=\, -F(r)\,dt^2 \,+\, \frac{dr^2}{F(r)} + \,r^2\,d\theta^2 +\, r^2\,\sin^2\theta\,d\varphi^2 + 2\,a\,w(r)\,r^2\,\sin^2\theta\,dt\,d\varphi, \tag{20}$$

where the quantity $w(r)$ in the off-diagonal component controls the effect of black hole rotation. The gauge vector has the structure

$$A_\mu = \Big\{A_t(r,\theta)\;0\,,\,0\,,\,A_\varphi(r,\theta)\Big\}. \tag{21}$$

At leading order in the coupling constants g_F and g_R, we find the solution

$$\begin{aligned} F(r) &= 1 - \frac{2\,M}{r} + \frac{P^2}{r^2} \\ w(r) &= -\frac{2M}{r^3} + \frac{P^2}{r^4} - g_F\, g_R\, P^2 \left(\frac{3}{16M^2r^4} + \frac{1}{10Mr^5} + \frac{21}{20r^6}\right) \\ &\quad + g_R^2 \left[\frac{5}{2r^6} + \frac{30M}{7r^7} + \frac{27M^2}{4r^8} - P^2\left(\frac{3}{16M^4r^4} + \frac{9}{40M^3r^5} + \frac{3}{10M^2r^6} + \frac{41M}{4r^9} + \frac{18}{7Mr^7} + \frac{21}{4r^8}\right)\right] \end{aligned} \tag{22}$$

and

$$\begin{aligned} A_t(r,\theta) &= \frac{a\,P\,\cos\theta}{r^2}\Big[1 + g_F\, g_R \left(-\frac{3}{16M^2} + \frac{1}{8Mr} + \frac{11}{40r^2} + \frac{9M}{20r^3}\right) \\ &\qquad - g_R^2\left(\frac{3}{16M^4} + \frac{3}{16M^3r} + \frac{9}{40M^2r^2} + \frac{3}{10Mr^3} + \frac{1}{4r^4} + \frac{9M}{56r^5}\right)\Big], \\ A_\varphi(r,\theta) &= -P\,\cos\theta. \end{aligned} \tag{23}$$

The associated profile for the pseudoscalar axion field has a dipolar structure and results

$$\begin{aligned} \phi(r,\theta) &= a\,g_F \frac{P^6}{(r_+r_-)^4}\left[2(M-r)\log\left(\frac{r-r_-}{r}\right) + \frac{r_+r_-}{r} - 2\,r_-\right]\cos\theta \\ &\quad - \frac{8\,a\,g_R}{5}\Bigg\{\frac{10M^4 - 21M^2\,(r_+r_-) + 6\,(r_+r_-)^2}{2\,r\,(r_+r_-)^3} - \frac{29M\,(r_+r_-) - 15M^3}{12\,r^2\,(r_+r_-)^2} - \frac{18\,(r_+r_-) - 5M^2}{12\,r^3\,(r_+r_-)} \\ &\quad - \frac{5M}{4\,r^4} + \frac{(r_+r_-)}{r^5} + \frac{10M^4 - 21M^2\,(r_+r_-) + 6\,(r_+r_-)^2}{(r_+r_-)^4}\left[(M-r)\log\left(\frac{r-r_-}{r}\right) - r_-\right]\Bigg\}\cos\theta. \end{aligned} \tag{24}$$

Interestingly, it includes contributions from both the gauge and gravitational Chern-Simons terms, weighted by the (small) rotation parameter a.

At our level of approximations, the position of the black hole horizon and consequently the black hole thermodynamics are not modified with respect to the Reissner-Nördstrom magnetised solution. On the other hand, the geometry and the gauge field receive non-trivial corrections in the metric coefficient $w(r)$ and in the gauge component $A_t(r,\theta)$, which acquire new 'mixed' contributions proportional to $g_F\, g_R$ due to the simultaneous presence of gauge and gravitational Chern-Simons terms. As far as we are aware, this is the first time that such mixed contributions have been found in this context, showing there is an interplay between gauge and gravitational Chern-Simons terms for determining the black hole geometry with axion hairs. Thanks to this contribution, our solution generalises to a rotating, charged setting similar configurations discussed in the works [44], [45].

5 Phenomenological considerations

5.1 Black hole axion hairs and light polarisation

In the previous Sections, we have seen that charged black holes can develop long range axion hairs, thanks to the gauge and gravitational Chern-Simons couplings contained in action (2). The Abelian charges carried by the black holes are not necessarily electromagnetic, and they can be associated with some additional gauge group motivated by string theory constructions, or dark forces associated with dark matter interactions.

Even if the black hole configurations are charged under extra Abelian gauge groups that *do not correspond* to the electromagnetic $U(1)$ symmetry, we make the hypothesis that axions additionally couple also with electromagnetic photons, through a coupling

$$g_{\rm EM}\,\phi\,\mathcal{F}_{\mu\nu}\,\tilde{\mathcal{F}}^{\mu\nu}\,, \tag{25}$$

with $\mathcal{F}_{\mu\nu}$ being the electromagnetic $U(1)$ field strength and $g_{\rm EM}$ the coupling constant controlling the axion-photon interaction. The previous formula denotes a dimension five operator, that we weight with appropriate powers of the Planck mass (as in the previous Sections) and regard $g_{\rm EM}$ as a dimensionless coupling. (In case the black holes are electromagnetically charged, then $g_{\rm EM} = g_F$ with g_F the coupling we analysed in the previous sections.)

A possible way to observationally detect a pseudoscalar black hole hair is to measure polarised bending of light travelling in proximity of the black hole. In fact, the parity violating coupling (25) bends by a different amount left-handed and right-handed photon trajectories crossing the pseudoscalar profile, by a magnitude depending also on the values of g_{EM}, g_F, g_R: so, quantifying this effect could help in probing the magnitude of the Chern-Simons couplings characterising the theory.

This effect has been studied in detail in the recent work [46] for the case of photons travelling through the cloud of light axions depositing on a rotating black hole ergosphere. Instead, here we study the same effect for light travelling through the long range axion profile studied in the previous Sections. We use the same approach and methods of [46], hence we refer to the reader to [46] for more details on the derivations. We study two cases, spherically symmetric and slowly rotating black holes, which can probe different sets of parameters.

- **Spherically symmetric black hole configurations**

  We study the bending of light passing in proximity of spherically symmetric, dyonic black hole configurations with long range axion hairs, discussed in Section 3. At large distances from the black hole horizon, the black hole geometry can be described by a Schwarzschild configuration (the corrections associated with the axion hair backreaction starts only at order $1/r^3$ in a large r expansion), with a long range axion hair profile, that scales with radius as in eq (9):

$$\Phi_{\rm dyon}(r)\,=\,\frac{g_F\,P\,Q}{r_+}\,\frac{1}{r}+\mathcal{O}\left(\frac{1}{r^2}\right)\,, \tag{26}$$

  with r_+ the position of the external horizon in a dyonic Reissner-Nördstrom configuration (see eq (8)). We assume that light passes sufficiently far from the black hole, so that a Schwarzschild approximation for the geometry is valid and for simplicity we assume that the photon trajectory lies in the equatorial plane of the black hole. Calling E_γ and L the conserved photon energy and angular momentum, and r and φ respectively the radial and angular polar coordinates controlling the photon path on the black hole equatorial plane (the black hole being located at $r = 0$), we find the orbit equation

$$\frac{d\varphi_\pm}{dr}\,=\,\frac{1}{r^2}\left[\frac{\pm g_{\rm EM}}{2L}\left(1-\frac{2M}{r}\right)\Phi'_{\rm dyon}(r)+\sqrt{\frac{E_\gamma^2}{L^2}+\frac{2M-r}{r^3}+\frac{g_{\rm EM}^2}{4L^2}\frac{(2M-r)^2}{r^2}\Phi'^2_{\rm dyon}(r)}\right]^{-1}\,, \tag{27}$$

where $\pm$ refer to the two photon polarisations, while $\Phi_{\rm dyon}$ is the axion profile given in eq (26). The contributions proportional to $g_{\rm EM}$ are associated with the axion-photon interactions: remarkably, in the presence of a long range axion profile they can 'distinguish' among the two different photon polarisations. The closest radial approach r_0 of the light orbit to the black hole configuration is given by the ratio of the energy and the angular momentum:

$$\frac{E_\gamma}{L} = \frac{1}{r_0}\sqrt{\frac{r_0 - 2M}{r_0}}\,. \tag{28}$$

Starting from these formulas, we can straightforwardly determine the photon deflection angle as travelling from spatial infinity to the distance to closest approach to the black hole. We are especially interested on the total difference $\Delta\varphi_\pm$ in the angular deflection between right and left polarisation. Calling $x_0 = r_0/M$, we find that this quantity is described by a simple formula

$$|\Delta\varphi_\pm^{\rm charged}| = \frac{(2x_0 - 3)}{x_0^{5/2}\sqrt{x_0 - 2}}\;\frac{g_F\, g_{\rm EM}}{6E_\gamma\, M^2}\;\frac{P\,Q}{r_+}. \tag{29}$$

A measurement of this quantity can then probe the electromagnetic coupling of the axion to photons, and the amount of charge of a black hole configuration.

- **Slowly rotating black hole configurations**

We now study polarised light deflection around the slowly rotating configurations examined in Section 4. We would like to focus on the effects of purely gravitational Chern-Simons couplings, hence we switch off the black hole magnetic and electric charges. At large distances from the black hole horizon, the geometry (at leading order in a $1/r$ expansion) is still well described by a Schwarzschild solution, and the axionic profile corresponds to a dipolar configuration given by

$$\Phi_{\rm rot}(r,\theta) \;=\; \frac{5}{8}\frac{a}{M}\frac{g_R}{r^2}\cos\theta + \mathcal{O}\left(\frac{1}{r^2}\right)\;. \tag{30}$$

Also in this case we study photon trajectories as done above, but – since the axion configuration is a dipole – we consider trajectories lying not on the equatorial plane, but on the more convenient plane $\theta = \pi/6$. Proceeding as we explained in the spherically symmetric case, we find that the angular deflection of polarised photons is governed by the following equation

$$\frac{d\varphi_\pm}{dr} = \frac{2}{r_0^2}\left[\pm\alpha\left(1-\frac{2M}{r}\right)\sqrt{\frac{1-\frac{2M}{r_0}}{r_0^2}} + \sqrt{-\frac{1-\frac{2M}{r}}{r^2} + \frac{1-\frac{2M}{r_0}}{r_0^2}\left(1+\frac{\alpha^2(r-2M)^2}{r^2}\right)}\,\right]^{-1}. \tag{31}$$

where r_0 is the closest approach distance of the photon to the black hole – see eq. (28) – and

$$\alpha = \frac{g_{\rm EM}}{2\,E_\gamma}\frac{\partial\Phi_{\rm rot}(r,\pi/6)}{\partial r}. \tag{32}$$

We learn that also in this case the axion profile bends differently the two photon polarisations. Integrating over all the trajectory, the difference $\Delta\varphi_\pm$ in the angular deflection between right and left polarisations now reads

$$|\Delta\varphi_\pm^{\rm rot}| \;=\; \frac{\sqrt{3}}{16}\,\frac{(5x_0-8)}{x_0^{7/2}\sqrt{x_0-2}}\,\frac{a\,g_R\,g_{\rm EM}}{E_\gamma\,M^4}. \tag{33}$$

Interestingly, eq (33) can probe a different set of parameters in this case, with respect to spherically symmetric charged configurations (compare with eq (29)).

To summarise, we find compact formulas, eqs (29) and (33), for the difference in light bending for the two polarisations of a photon travelling within a region containing black hole axion hairs, as described in Sections 3 and 4. These formulas can probe different parameters: while in the spherically symmetric case the polarised light bending can probe gauge couplings with Chern-Simons terms, in the slowly rotating case we can also probe gravitational Chern-Simons interactions – a coupling that as far as we are aware is not probed by direct axion experiments. Future astronomical observations based on radio astronomy can achieve an angular resolution of order $\Delta\phi \sim 10^{-4}$ [46]: it would be interesting to quantitatively estimate at what extent these precise measurements can test the existence and properties of black hole axionic hairs, and their couplings with electromagnetic photons. We plan to return on these topics soon.

5.2 ISCO

As a second application of our findings, we compute the ISCO (Innermost Stable Circular Orbit) of the dyonic black hole configuration with axion hairs as discussed in Section 3, including the effects of non-minimal couplings of axions with gravity, Lagrangian (4). The properties of ISCOs can be interesting for a phenomenological characterization of the geometrical features of the system under investigation. Since they depend on the properties of the configuration relatively near the black hole horizon, they can be sensitive to the effects of Chern-Simons couplings, or of the derivative non-minimal couplings with gravity. By defining

$$f(r) = \log[F(r)], \tag{34}$$

with $F(r)$ being the coefficient of the time-time metric component, see eq (13), the ISCO radius is given by the root [39, 86] of the following equation

$$3\, f'(r) - r\, f'(r)^2 + r\, f''(r)^2 = 0\,. \tag{35}$$

We can solve the previous condition analitically by expanding for small values of magnetic and electric charges P and Q. We push our perturbative expansion up to order $\mathcal{O}(\text{charge}^6)$ for catching the effects of the Chern-Simons coupling g_F, and non-minimal couplings with gravity λ. In these approximations, the ISCO radius results

$$\begin{aligned} r_{ISCO} \;&=\; 6M - \frac{\left(P^2+Q^2\right)}{2M} - \frac{19\,\left(P^2+Q^2\right)^2}{72M^3} - \frac{5\left(P^2+Q^2\right)^3}{48M^5} \\ &+\; g_F^2 P^2\, Q^2 \left[\frac{19}{144} + \left(P^2+Q^2\right)\, \frac{(49410M^2 - 193\lambda)}{466560M^7}\right]. \end{aligned} \tag{36}$$

The first line contains the General Relativity expression for the ISCO radius: the Schwarzschild result $(6M)$ and the first corrections in small values of the charges associated with a dyonic Reissner-Nördstrom configuration. The second line contains instead the contributions associated with the presence of black hole axion hair, the Chern-Simons terms, and non-minimal coupling with gravity. For small values of the charges, these contributions start only at order $\mathcal{O}(\text{charge}^4)$, hence it can be difficult to use the properties of ISCO to reveal the existence of axion hairs. To conclude, we report the result for the angular frequency associated with the ISCO trajectory

$$\omega_{ISCO} = + \frac{1}{6\sqrt{6}M} + \frac{7\left(P^2+Q^2\right)^2}{144\sqrt{6}M^3} + \frac{49\left(P^2+Q^2\right)^4}{2304\sqrt{6}M^5} + \frac{5489\left(P^2+Q^2\right)^6}{497664\sqrt{6}M^7} + g_F^2 P^2 Q^2\left[-\frac{1}{216\sqrt{6}M^5} + \left(P^2+Q^2\right)^2\left(\frac{11\,\lambda}{699840\sqrt{6}M^9} - \frac{47}{7776\sqrt{6}M^7}\right)\right]. \tag{37}$$

Again, the first term correspond to the Schwarzschild result, while the contributions from the scalar field start to appear at order $\mathcal{O}(\text{charge}^4)$.

6 Conclusions

We studied spherically symmetric and slowly rotating charged black hole configurations with long range axion hairs. We focussed on Einstein-Maxwell theories equipped with an axion Lagrangian that preserves a shift symmetry for the axion field. Gauge and gravitational Chern-Simons couplings, eq (1), are essential for evading no-hair theorems, and lead to long range axion profiles. We extended known black hole solutions to cases where additional derivative couplings of axion to curvature are switched on, and to situations in which both gauge and gravitational Chern-Simons couplings are present simultaneously. In all cases, we determined analytical solutions at leading order in the coupling constants involved, determining how the axion profile backreacts on the metric and the gauge field. The metric remains regular outside the outer horizon, and the position of such horizon is increased with respect to dyonic solutions in Einstein-Maxwell gravity with the same asymptotic conserved charges. Moreover, the solution for the electric potential is modified with respect to the standard case due to the effect of the gauge Chern-Simons couplings. To make contact with phenomenology, we studied two possible consequences of our findings. First, we studied how axion hairs can induced polarised bending of photons travelling within the range of the axion profile around a black hole. Then, we investigated the properties of ISCOs around the spherically symmetric configurations we analysed. It would be interesting, in the future, to study the stability of the systems considered under small perturbations of the fields involved, and investigate possibly parity breaking effects – induced by the Chern-Simons couplings – in the dynamics of fluctuations around these geometries.

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A An exact solution for large values of λ

In the main text, we presented a solution for a spherically symmetric dyonic black hole configuration at leading order in a perturbative expansion in the coupling constant λ, characterising derivative couplings of the axion to the curvature, introducted eq (4). Solutions for arbitrary values of λ can be found in general, but their expressions are very cumbersome. On the other hand, for special values of this parameter, their expression simplifies. For example, with the particular choice

$$\lambda = -\frac{r_-^3}{2\,r_+} \tag{38}$$

the solution for the axion configuration is relatively simple, and reads

$$\phi(r) = \frac{g_F\, P\, Q}{8 r_- r_+}\left[\frac{2\, r_-}{r - r_-} + \log\left(r^2 + r_-^2\right) + \log(r + r_-) - 3\log(r - r_-) + 2\tan^{-1}\left(\frac{r_-}{r}\right)\right]. \tag{39}$$

Notice that the choice (38) allows for tuning large values of λ, by choosing r_+ and r_- appropriately. The metric line element is

$$ds^2 = -F(r)\, dt^2 + \frac{dr^2}{h(r)F(r)} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2. \tag{40}$$

Choosing λ as in eq (38), at leading order in the coupling g_F we find

$$\begin{aligned} F(r) = &\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right) + \frac{g_F^2\, P^2\, Q^2}{8 r_-^2\, r_+^2}\left[\frac{2r_-\,(4r_+ - 3r)}{r^2} + \frac{(r r_- + r r_+ - 2 r_- r_+)}{r^2}\log\left(r^2 + r_-^2\right)\right. \\ &\left. + \frac{\left(r^2 - 2 r r_+ - r_- r_+\right)}{r^2}\log(r + r_-) - \frac{\left(r^2 + 2 r r_- - 5 r_- r_+\right)}{r^2}\log(r - r_-) + \frac{2(2r - r_- - 3r_+)}{r}\tan^{-1}\left(\frac{r_-}{r}\right)\right], \\ h(r) = &1 - \frac{g_F^2\, P^2\, Q^2}{2 r_+^2}\left[\frac{3r^8 r_+ + r^7 r_- r_+ + r^6 r_-^2 r_+ + r^5 r_-^3 (8r_- + r_+) - 13 r^4 r_-^4 r_+ - r^3 r_-^5 r_+ - r^2 r_-^6 r_+ - r r_-^7 r_+ + 2 r_-^8 r_+}{2 r_- r_+ (r - r_-)^3 (r + r_-)^2 \left(r^2 + r_-^2\right)^2}\right. \\ &\left. + \frac{2\tan^{-1}\left(\frac{r_-}{r}\right) + \tanh^{-1}\left(\frac{r_-}{r}\right)}{r_-^2}\right]. \end{aligned} \tag{41}$$

As in the small coupling approximation $\lambda \ll 1$ (see section 2.2), the Reissner-Nördstrom case is recovered setting $g_F = 0$, while metric corrections due to axion's backreaction start only at order $1/r^3$ in an $1/r$ expansion. At leading order in g_F expansion, the position of the outer horizon becomes

$$r_H = r_+ + g_F^2\, r_S \tag{42}$$

with

$$\begin{aligned} r_S = \frac{P^2 Q^2}{8 r_-^2 r_+}\Bigg[&- \frac{2r_-}{(r_+ - r_-)} - \log\left(r_-^2 + r_+^2\right) + \frac{(r_+ - 3r_-)}{(r_+ - r_-)}\log(r_+ - r_-) + \frac{(r_- + r_+)}{(r_+ - r_-)}\log(r_- + r_+) \\ &+ \frac{2(r_- + r_+)}{(r_+ - r_-)}\tan^{-1}\left(\frac{r_-}{r_+}\right)\Bigg]. \end{aligned} \tag{43}$$

With the particular choice (38), compared with the Reissner-Nördstrom case the axion's backreaction increases the size of the outer horizon. It would be interesting to understand if there is any choice of the coupling λ which could lead to a reduction of the size of the outer horizon, but we leave it to future works. The geometry is regular on the outer horizon r_H and everywhere outside it, while there is a singularity located on the inner horizon r_-, which cannot be removed by any choice of the coupling parameter λ.

Moreover, the axion's backreaction also modifies the gauge potential

$$A_\mu = \{A_t(r),\, 0,\, 0,\, A_\varphi(\theta)\}, \tag{44}$$

and at leading order in a g_F expansion we find

$$
\begin{aligned}
A_t(r) =& -\frac{Q}{r} + g_F^2\, Q \Bigg\{ \frac{P^2 Q^2}{8 r_-^2 r_+^2} \Bigg[\frac{r^5(3r_- + r_+) + r^4 r_-(r_+ - r_-) + r^3 r_-^2 (r_+ - r_-) + r^2 r_-^3 (r_+ - r_-) + 4 r r_-^5 - 8 r_-^5 r_+}{4 r r_+ (r-r_-)^2 (r+r_-)\left(r^2 + r_-^2\right)} \\
& - \frac{(r_- - 5r_+)\log\left(r^2 + r_-^2\right)}{4 r_- r_+} + \frac{(5 r r_- - 13 r r_+ + 4 r_- r_+)\log(r - r_-)}{8 r r_- r_+} - \frac{(r r_- + 7 r r_+ + 4 r_- r_+)\log(r + r_-)}{8 r r_- r_+} \\
& + \frac{\tan^{-1}\left(\frac{r}{r_-}\right)}{r_-} \Bigg] + \frac{P^2}{4\, r_-^2 r_+} \Bigg[\frac{(r_- - r)\log\left(r^2 + r_-^2\right)}{2r} + \frac{(r - 3r_-)\log(r - r_-)}{2r} + \frac{(r + r_-)\log(r + r_-)}{2r} \\
& - \frac{r \tan^{-1}\left(\frac{r}{r_-}\right) + r_-}{r} + \frac{P^2}{r_+} \frac{\tan^{-1}\left(\frac{r_-}{r}\right)}{r} \Bigg] \Bigg\}, \\
A_\varphi(\theta) =& - P\, \cos\theta .
\end{aligned} \tag{45}
$$

As for the perturbative solution in the main text, the magnetic potential is the same as in Einstein-Maxwell theory, while the electric potential is modified by the presence of the axion field.

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