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An Upper Bound for the Proof Theoretic Strength of Martin-Löf Type Theory with W-type and one Universe

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Abstract

We present an upper bound for the proof theoretic strength of Martin-Löf's type theory with W-type and one universe. This proof, together with the well ordering proof carried out in [Set98b] shows that the proof theoretic strength of this theory is precisely $\psi_{\Omega_1}\Omega_{\mathrm{I}+\omega}$, which is slightly more than the strength of Feferman's theory T_0, the classical set theory KPI, and the subsystem of analysis $(\Delta^1_2 - \mathrm{CA}) + (\mathrm{BI})$. The strength of the intensional and extensional version, and of the version à la Tarski and à la Russell are shown to be the same. The proof is carried out by interpreting the type theories in question in an extension of Kripke-Platek set theory KPI^+. We show that validity of Π^1_1-sentences is preserved in this interpretation.

Dedicated to the master of proof theory, Kurt Schütte

1 Introduction

1.1 Proof theory and Type Theory

Proof theory and type theory are two answers of mathematical logic to the crisis of the foundations of mathematics at the beginning of the 20th century. Proof theory

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was established by Hilbert in order to prove the consistency of theories by using finitary methods. When Gödel showed that Hilbert's program cannot be carried out as originally intended, the focus of proof theory changed towards analysing theories and determining the minimum ordinal in a natural ordinal notation system such that transfinite induction up to this ordinal proves the consistency of the theory. That ordinal is called the proof theoretic strength, which turned out to be an excellent measure for the strength of theories.
On the other hand, type theories were designed to provide a new framework for mathematics, the consistency of which can be justified, as far as possible, by itself (see [Set15] for a discussion about why a justification of the consistency of mathematical theories is needed, what can be achieved, and what the limitations are).
Both directions of mathematical logic have become quite important because of their applicability to computer science. Proof theoretic methods can be used for instance to extract programs from proofs, to analyse term rewriting systems and for theoretical questions the area of logic programming.
On the other hand many machine assisted theorem provers, such as Agda, Coq, Idris, Lego, and Epigram, are based on type theory. One reason why type theory is an excellent basis theory is that in type theory algorithms and proofs are essentially the same. Therefore proving becomes very similar to programming, and techniques from software engineering can be used to develop proofs.
Therefore it seems to be interesting to apply proof theory to type theory. In particular, the question mainly answered in this article is: what is the precise proof theoretic strength of Martin-Löf's type theory? This is interesting because the answer determines the exact place of Martin-Löf's type theory on the proof theoretic scale. This allows to compare it with other theories, the strength of which is already known.
More precisely, in this article we are dealing with the strength of Martin-Löf's type theory with one universe and W-type. This work was first presented in our PhD thesis [Set93][1]. There are two directions to be proved. One is to determine a lower bound, which is presented in [Set98b]. This was done by carrying out a well ordering proof directly in Martin-Löf Type Theory. In this article we present a refined version of the upper bound. We embed type theory in a Kripke-Platek style set theory, KPI^+, the strength of which can then be determined easily using standard proof theoretic techniques. We show that validity of Π^1_1-sentences is preserved in this interpretation.

[1] In that thesis we were using the formulation of [TD88] of intensional type theory, which trivially embeds into extensional type theory for which an upper bound is obtained in the current article. In [Set98b] we transferred the well-ordering proof from [Set93] to the formulation of intensional type theory as it is used in the Martin-Löf Type Theory community. We introduce that theory and embed it into extensional type theory in the current article in Sect. 3. The transfer of the well-ordering proof was possible since the equality type was used in the well-ordering proof in [Set98b] only for natural numbers.

1.2 Related Work

An overview over the state of the art of proof theory of Martin-Löf Type Theory can be found in [Set04] and [Set08a]. As mentioned before [Set98b] contains the lower bound for the proof theoretic strength of the theories discussed in this article. A more easy readable introduction to well ordering proofs in Martin-Löf type theory can be found in [Set98a].

In [GR94] Griffor and Rathjen were, independent of the author and in parallel, following another approach towards determining the proof theoretic strength of Martin-Löf's type theory by embedding constructive set theory into type theory. [GR94] contains an excellent review of all the research carried out in the past in this area. We refer the interested reader to that article and only mention the main new results concerning type theory obtained in [GR94]. Griffor and Rathjen showed, that the theory $\mathrm{ML}_1\mathrm{V}$, Martin-Löf's type theory with one universe and Aczel's iterative set V or elimination rules for the universe or both has the strength of Kripke-Platek set theory $\mathrm{KP}\omega$. They showed, that type theory with one universe and the W-type restricted to elements of the universe only, which they called $\mathrm{ML}_1\mathrm{W}$, has strength $(\Delta^1_2 - \mathrm{CA}) + (\mathrm{BI})$. Adding elimination rules for the universe and/or Aczel's iterative set V is shown to yield the same strength. In [GR94] the obvious generalisation of these results to n universes and ω universes together with their strength is mentioned as well (no detailed proof is given).

In [Set00] the author has introduced the Mahlo universe, which is significantly stronger than the type theory with W and finitely many universes, and determined a lower bound for its proof theoretic strength. In [Set08b] the basic model construction for type theory with one universe and W-type and with one Mahlo universe and the W-type was given. That article concentrated mainly on the universe constructions and presented the main constructions, whereas the current article contains full details for the upper bound of Martin-Löf Type Theory with W-type and one universe. In the current article we show as well that all Π^1_1-sentences provable in the type theory in question can be shown in KPI^+.

The author has developed an autonomous Mahlo universe [Set11], a Π_3-reflecting universe, and a Π_n-reflecting universe, but these constructions have not been published yet.

1.3 Overview

The article is organised as follows: In Sect. 2 we introduce the versions of extensional Martin-Löf's type theory for which we prove the upper bound. To make a precise definition of the substitution, we introduce sets of b-objects, g-terms, g-types, R-terms and R-types, which should contain all the terms and types occurring in Martin-Löf's type theory (g-terms and g-types correspond to the Tarski-formalisation, R-terms and R-types to the Russel-formalisation). These concepts will be needed afterwards for the interpretation of the theory $\mathrm{ML}^{\mathrm{e}}_1\mathrm{W}_{\mathrm{T,U}}$ in KPI^+. Then we define the rules for the extensional Martin-Löf Type Theories with a universe à la Tarski ($\mathrm{ML}^{\mathrm{e}}_1\mathrm{W}_{\mathrm{T}}$,

$\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T,U}}$) and à la Russell ($\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R}}$, $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R,U}}$) considered in this paper. In Sect. 3 we introduce intensional variants $\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{T}}$, $\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{T,U}}$, $\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{R}}$, $\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{R,U}}$ of those theories. We carry out the well-known result that they embed in a straightforward way into the extensional versions of those theories. In Sect. 4 we compare the Tarski and the Russel-versions and show that all of them can be embedded into the strongest Tarski version $\mathrm{ML} := \mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T,U}}$. In Sect. 5 we introduce Kripke-Platek set theory and the extension KPI^+ used for modelling the type theories in question in this article.
Having introduced the relevant theories we now develop an interpretation of ML in the extension of Kripke-Platek style called KPI^+. This embedding is a quite general and flexible method, which can be adopted to variations of Martin-Löf's type theory. In Sect. 6, we show, how to interpret terms and types in KPI^+. Types will be introduced as sets of pairs of terms, which are considered to be equal in that type. We see, how much strength is needed in order to interpret the W-type and the universe, which correspond to the proof theoretic strength added by those constructions. In Sect. 7 we prove basic properties of this interpretation, such as monotonicity of operators, that one obtains equivalence relations, and substitution lemmas.
In Sect. 8 we prove that the interpretation is correct, that is

$$\text{If } \mathrm{ML} \vdash r : A \text{ then } \mathrm{KPI}^+ \vdash \langle r, r\rangle \in A^*$$

Furthermore, we conclude, that the extended version $|\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T,U}}|$ can be interpreted as well.
In Sect. 9 we interpret sentences of Arithmetic A as $\widehat{A}$ in $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T}}$ and prove, that for Π_1^1-sentences A we get that $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T}} \vdash a \in \widehat{A}$ implies $\mathrm{KPI}^+ \vdash A$. We conclude in Sect. 10 that $|\mathrm{ML}| \leq |\mathrm{KPI}^+|$. To complete the proof, we show $|\mathrm{KPI}^+| \leq \psi_{\Omega_1}(\Omega_{\mathrm{I}+\omega})$, which implies because of the embedding of the Martin-Löf type theories used into ML, that $|\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T,U}}|$, $|\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T}}|$, $|\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R,U}}|$, $|\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R}}|$, $|\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{T,U}}|$, $|\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{T}}|$, $|\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{R,U}}|$, $|\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{R}}| \leq \psi_{\Omega_1}(\Omega_{\mathrm{I}+\omega})$. By [Set98b], this bound is sharp.

2 Definition of the Formal System of Extensional Martin-Löf's type Theory

In this section we will introduce the formulations of extensional Martin-Löf Type Theory, both a là Tarski and a là Russell, which we will embed into KPI^+. In Sect. 3 we will introduce intensional type theory and show that it embeds into extensional type theory. Therefore the upper bound for extensional type theory is an upper bound for intensional type theory as well.

Definition 2.1 *(a) The symbols of extensional Martin-Löf's type theory, are infinitely many variables z_i ($i \in \omega$); the symbols $\Rightarrow$, :, ,, (,), =, $\in$, λ; the term constructors (with their arity in parenthesis)* 0 *(0),* r *(0),* $\widehat{\mathbb{N}}$ *(0),* A_n^k *(for each $n < k$, with arity 0),* $\widehat{\mathbb{N}}_k$ *(for each $k \in \omega$, with arity 0),* S *(1),* i *(1),* j *(1),* p_0

(1), p_1 *(1),* p *(2),* sup *(2),* R *(2),* Ap *(2),* $\widehat{+}$ *(2),* $\widehat{\Pi}$ *(2),* $\widehat{\Sigma}$ *(2),* $\widehat{\mathrm{W}}$ *(2),* D *(3),* P *(3),* $\widehat{\mathrm{I}}$ *(3),* C_n *($n \in \omega$, arity $n+1$); the type constructors with their arity* $\mathbb{N}_k$ *(for each $k \in \omega$, arity 0),* $\mathbb{N}$ *(0),* U *(0),* T *(1),* + *(2),* Π *(2),* Σ *(2),* W *(2) and* I *(3).*
To make it easier to remember the meaning of the symbols, we give the following hints: r *is the (unique) element of an identity type* I*;* A^k_n *is the nth element of the finite type* $\mathbb{N}_k$ *with k elements,* C_k *the* **C***asedistinction for* $\mathbb{N}_k$*; O is the zero,* S *the* **S***uccessor,* P **P***rimitive recursion or induction over the natural numbers* $\mathbb{N}$*;* i *stands for left* **i***nclusion,* j *for right inclusion,* D *is the case distinction for a type $A + B$, which is the disjoint union of A and B;* p_0 *and* p_1 *are the projections,* p *the pairing for the* Σ*-type;* R *the* **R***ecursion operator over a* W*-type;* Ap *the application of a function (as an element of a* Π*-type) to an argument;* $\widehat{\mathbb{N}}$, $\widehat{\mathbb{N}}_k$, $\widehat{\mathrm{I}}$, $\widehat{+}$, $\widehat{\Sigma}$, $\widehat{\Pi}$, $\widehat{\mathrm{W}}$ *are codes for the types* $\mathbb{N}$, $\mathbb{N}_k$, I, +, Σ, Π, W*, respectively, as elements of the universe* U*, which become, if the* **T***arski-operator* T *is applied to them, the corresponding type.*

(b) *We usually write "symbols of Martin-Löf Type Theory" for "symbols of extensional Martin-Löf Type Theory", when there is no confusion.*

(c) *The b-objects are variables, $\lambda x.b$ and $C(b_1, \ldots, b_n)$, if C is an n-ary term or type constructor and b, $b_1, \ldots, b_n$ are b-objects.*
The set of free variables of a b-object $\mathrm{FV}(b)$ *are defined as usual. We write* +, $\widehat{+}$ *infix (that is $(a+b)$ for $+(a,b)$) $(x)t$ for $\lambda x.t$, $(x,y)t$ for $(x)(y)t$, $(x,y,z)t$ for $(x)(y,z)t$, and if $S \in \{\Sigma, \Pi, \mathrm{W}, \widehat{\Sigma}, \widehat{\Pi}, \widehat{\mathrm{W}}\}$, $Sx \in s.t := S(s,(x)t)$. Furthermore, we write sometimes $r\ s$ or rs for* $\mathrm{Ap}(r,s)$.
We have the usual conventions about omitting brackets, especially the scope of $\lambda x.$ is as long as possible, for instance $\lambda x.s\ t$ should be read as $\lambda x.(s\ t)$. We define for b-objects $b_1, \ldots, b_n, b$ and variables $x_1, \ldots, x_n$ the simultaneous substitution $b[x_1/b_1, \ldots, x_n/b_n]$ as usual (using the convention, that if $x_i = x_j$ then the first substitution applies) and "$b[x_1/b_1, \ldots, x_n/b_n]$ is an allowed substitution".
α-equality ($=_\alpha$) is defined as usual.

(d) *The set of g-terms (for generalised terms) is inductively defined as: variables x are g-terms; if $n < k$, $n, k \in \mathbb{N}$, then A^k_n is a g-term; and if $k \in \mathbb{N}$, then $\widehat{\mathbb{N}}_k$ is a g-term; if r, s, t are g-terms, $x, y, z, x' \in \mathrm{Var}_{\mathrm{ML}}$, $x \neq y \neq z \neq x$, then* 0, r, $\widehat{\mathbb{N}}$, $\mathrm{S}(r)$, $\lambda x.r$, $\mathrm{p}(r,s)$, $\mathrm{sup}(r,s)$, $\mathrm{i}(r)$, $\mathrm{j}(r)$, $\mathrm{P}(r,s,(x,y)t)$, $\mathrm{Ap}(r,s)$, $\mathrm{p}_0(r)$, $\mathrm{p}_1(r)$, $\mathrm{R}(r,(x,y,z)s)$, $\mathrm{D}(r,(x)s,(x')t)$, $\widehat{\Pi}x \in r.s$, $\widehat{\Sigma}x \in r.s$, $\widehat{\mathrm{W}}x \in r.s$, $r \widehat{+} s$, $\widehat{\mathrm{I}}(r,s,t)$ *are g-terms; if $n \in \mathbb{N}$ and r, $s_1, \ldots, s_n$ are g-terms, then $\mathrm{C}_n(r, s_1, \ldots, s_n)$ is a g-term.*
Let $\mathrm{Term}_{\mathrm{Cl}}$ *be the set of closed g-terms.*

(e) *The g-types are $\mathbb{N}_k$ ($k \in \omega$), $\mathbb{N}$, U, and, if A, B are g-types, $x \in \mathrm{Var}_{\mathrm{ML}}$, r, s g-terms, then $\Pi x \in A.B$, $\Sigma x \in A.B$, $\mathrm{W}x \in A.B$, $A + B$, $\mathrm{I}(A,r,s)$, $\mathrm{T}(r)$ are* *g-types.*

(f) *A g-context-piece is a string* $x_1 : A_1, \ldots, x_n : A_n$*, where* $n \geq 0$*,* x_i *different variables,* A_i *g-types.*
A g-context is a g-context-piece $x_1 : A_1, \ldots, x_n : A_n$*, s.t.* $\mathrm{FV}(A_i) \subset \{x_1, \ldots, x_{n-1}\}$*.*
A g-judgement is $A = B : \mathrm{type}$ *or* $s = t : A$ *where* A*,* B *are g-types,* s*,* t *are g-terms. A g-dependent judgements is* $\Gamma \Rightarrow \theta$ *where* Γ *is a g-context and* θ *a g-judgement.*

(g) *R-terms (for Russel-terms) are defined as the g-terms, except, that we replace* $\widehat{\Pi}$, $\widehat{\Sigma}$, $\widehat{\mathrm{W}}$, $\widehat{+}$, $\widehat{\mathrm{I}}$ *by* $\Pi, \Sigma, \mathrm{W}, +, \mathrm{I}$ *respectively.*
R-types are defined by the same definition as the g-types, but referring to R-terms instead of g-terms, and replacing that "$\mathrm{T}(r)$ *is a g-type" by "*r *is an R-type for* r *an R-term". R-context-pieces, -contexts, -judgements, and -dependent judgements are defined as the corresponding g-constructions, but referring to R-terms and -types instead of the g-terms and -types.*

(h) *We treat the usual judgements* $A : \mathrm{type}$ *and* $s : A$ *as abbreviations:*
$(A : \mathrm{type}) :\equiv (A = A : \mathrm{type})$*,* $(s : A) :\equiv (s = s : A)$*.*

(i) *We abbreviate* $[\vec{x}/\vec{t}] := [x_1/t_1, \ldots, x_n/t_n]$ *if* $\vec{x} = x_1, \ldots, x_n$ *and* $\vec{t} = t_1, \ldots, t_n$*.*
$[x_1/t_1, \ldots, x_n/t_n] \setminus \{y\} := [\vec{x}/\vec{t}] \setminus \{y\}$ *is the result of omitting in* $[\vec{x}/\vec{t}]$ *the* x_i/t_i *s.t.* $x_i = y$*, and* $[\vec{x}/\vec{t}] \setminus \{y_1, \ldots, y_m\} := (\cdots(([\vec{x}/\vec{t}] \setminus \{y_1\}) \setminus \{y_2\}) \cdots \setminus \{y_m\})$*.*

Definition 2.2 *of extensional Martin-Löf Type Theory with* W*-type and one Universe.*
We will define the rules, which are of the form

$$(Rule)\ \frac{\Gamma_1 \Rightarrow \Theta_1 \qquad \cdots \qquad \Gamma_n \Rightarrow \Theta_n}{\Gamma \Rightarrow \Theta}$$

where $\Gamma_1, \ldots, \Gamma_n, \Gamma$ *are g-context-pieces,* $\Theta_1, \ldots, \Theta_n, \Theta$ *are g-judgements (*$n = 0$ *is allowed) in the version à la Tarski, and R-context-pieces and R-judgements in the version à la Russell.*
Then we define for $\mathrm{ML} = \mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T}}$ *(the extensional version à la Tarski) or* $\mathrm{ML} = \mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T,U}}$ *(the extensional version à la Tarski with additional rules for the universe),* $\mathrm{ML} \vdash \Gamma \Rightarrow \Theta$ *inductively by:*
If $(Rule)$ *is a rule of* ML *as above,* Δ *is a g-context-piece such that* $\Delta, \Gamma_1, , \ldots, \Delta, \Gamma_n, \Delta, \Gamma$ *are g-contexts, and if* $\mathrm{ML} \vdash \Delta, \Gamma_i \Rightarrow \Theta_i$ *for* $i = 1, \ldots, n$*, then* $\mathrm{ML} \vdash \Delta, \Gamma \Rightarrow \Theta$*.*
Analogously we define $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R}} \vdash \Gamma \Rightarrow \Theta$ *(the extensional version à la Russell) and* $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R,U}} \vdash \Gamma \Rightarrow \Theta$ $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R,U}} \vdash \Gamma \Rightarrow \Theta$ *(the version à la Russell corresponding to* $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T,U}}$*), but referring to R-context-pieces, R-contexts etc.instead of g-context-pieces, g-contexts etc.*
We will write Θ *for* $\Rightarrow \Theta$ *as a premise of a rule.*

In the following, let A, B, A', B' be g-types (or R-types in the formulation à la Russell), $a, b, r, s, t, r_i, s_i, t_i, a', b', r', s', t', r'_i, s'_i, t'_i, t''$ be g-terms (or R-terms) θ, θ' be g-(or R-)judgements Γ' be a g-(R-)context-pieces.
Furthermore, let $x, y, z, u \in \mathrm{Var}_{\mathrm{ML}}$. Additionally assume for all rules, that all substitution mentioned explicitly are allowed. For instance in the rule $(\mathbb{N}^{=}_{\mathrm{S}})$, assume that
$s_1[x/t, y/\mathrm{P}(t, s_0, (x, y)s_1)]$ and $A[z/\mathrm{S}(t)]$ are allowed substitutions.

General Rules

$$(\mathrm{ASS})\ \frac{A : \mathrm{type}}{x : A \Rightarrow x : A} \qquad (\mathrm{THIN})\ \frac{A : \mathrm{type} \qquad \Gamma' \Rightarrow \Theta}{x : A, \Gamma' \Rightarrow \Theta}$$

$$(\mathrm{SYM})\ \frac{t = t' : A}{t' = t : A} \quad \frac{A = B : \mathrm{type}}{B = A : \mathrm{type}} \qquad (\mathrm{SUB})\ \frac{x : A, \Gamma' \Rightarrow \Theta \qquad \Rightarrow t : A}{\Gamma'[x/t] \Rightarrow \Theta[x/t]}$$

$$(\mathrm{TRANS})\ \frac{t = t' : A \qquad t' = t'' : A}{t = t'' : A} \qquad \frac{A = B : \mathrm{type} \qquad B = C : \mathrm{type}}{A = C : \mathrm{type}}$$

$$(\mathrm{REPL1})\ \frac{x : A, \Gamma' \Rightarrow B : \mathrm{type} \qquad \Rightarrow t = t' : A}{\Gamma'[x/t] \Rightarrow B[x/t] = B[x/t'] : \mathrm{type}}$$

$$(\mathrm{REPL2})\ \frac{x : A, \Gamma' \Rightarrow s : B \qquad \Rightarrow t = t' : A}{\Gamma'[x/t] \Rightarrow s[x/t] = s[x/t'] : B[x/t]}$$

$$(\mathrm{REPL3})\ \frac{t = t' : A \qquad A = B : \mathrm{type}}{t = t' : B}$$

$$(\mathrm{ALPHA})\ \frac{x : A, \Gamma' \Rightarrow \theta}{x : A', \Gamma' \Rightarrow \theta} \quad \frac{A : \mathrm{type}}{A = A' : \mathrm{type}} \quad \frac{t : A}{t = t' : A} \quad \textit{(if } A{=_\alpha}A',\ t{=_\alpha}t'\textit{)}$$

$$(\mathbb{N}^{\mathrm{F}}_k)\quad \mathbb{N}_k : \mathrm{type} \quad (k \in \mathbb{N}) \qquad (\mathbb{N}^{\mathrm{F}})\quad \mathbb{N} : \mathrm{type}$$

$$(\Pi^{\mathrm{F}})\ \frac{\begin{array}{c}A = A' : \mathrm{type}\\ x : A \Rightarrow B = B' : \mathrm{type}\end{array}}{\Pi x \in A.B = \Pi x \in A'.B' : \mathrm{type}} \qquad (\Sigma^{\mathrm{F}})\ \frac{\begin{array}{c}A = A' : \mathrm{type}\\ x : A \Rightarrow B = B' : \mathrm{type}\end{array}}{\Sigma x \in A.B = \Sigma x \in A'.B' : \mathrm{type}}$$

$$(\mathrm{W}^{\mathrm{F}})\ \frac{A = A' : \text{type} \qquad x : A \Rightarrow B = B' : \text{type}}{\mathrm{W}x \in A.B = \mathrm{W}x \in A'.B' : \text{type}}$$

$$(+^{\mathrm{F}})\ \frac{A = A' : \text{type} \qquad B = B' : \text{type}}{A + B = A' + B' : \text{type}}$$

$$(\mathrm{I}^{\mathrm{F}})\ \frac{A = A' : \text{type} \qquad t = t' : A \qquad s = s' : A}{\mathrm{I}(A, t, s) = \mathrm{I}(A', t', s'') : \text{type}}$$

Introduction Rules

$$(\mathbb{N}_k^{\mathrm{I}})\quad \mathrm{A}_n^k : \mathbb{N}_k \quad (n < k,\ n, k \in \mathbb{N})$$

$$(\mathbb{N}^{\mathrm{I}})\quad 0 : \mathbb{N} \qquad \frac{t = t' : \mathbb{N}}{\mathrm{S}(t) = \mathrm{S}(t') : \mathbb{N}}$$

$$(\Pi^{\mathrm{I}})\ \frac{x : A \Rightarrow B : \text{type} \qquad x : A \Rightarrow t = t' : B}{\lambda x.t = \lambda x.t' : \Pi x \in A.B}$$

$$(\Sigma^{\mathrm{I}})\ \frac{x : A \Rightarrow B : \text{type} \qquad s = s' : A \qquad t = t' : B[x/s]}{\mathrm{p}(s, t) = \mathrm{p}(s', t') : \Sigma x \in A.B}$$

$$(\mathrm{W}^{\mathrm{I}})\ \frac{x : A \Rightarrow B : \text{type} \qquad r = r' : A \qquad s = s' : B[x/r] \to \mathrm{W}x \in A.B}{\sup(r, s) = \sup(r', s') : \mathrm{W}x \in A.B}$$

$$(+_1^{\mathrm{I}})\ \frac{A : \text{type} \qquad B : \text{type} \qquad s = s' : A}{\mathrm{i}(s) = \mathrm{i}(s') : A + B}$$

$$(+_2^{\mathrm{I}})\ \frac{A : \text{type} \qquad B : \text{type} \qquad s = s' : B}{\mathrm{j}(s) = \mathrm{j}(s') : A + B}$$

$$(\mathrm{I}^{\mathrm{I}})\ \frac{t = t' : A}{\mathrm{r} : \mathrm{I}(A, t, t')}$$

Elimination Rules

$$(\mathbb{N}_k^{\mathrm{E}})\ \frac{x : \mathbb{N}_k \Rightarrow A : \text{type} \qquad t = t' : \mathbb{N}_k \qquad s_i = s_i' : A[x/\mathrm{A}_i^k](i = 0 \ldots k-1)}{\mathrm{C}_k(t, s_0, \ldots, s_{k-1}) = \mathrm{C}_k(t', s_0', \ldots, s_{k-1}') : A[x/t]} \quad (k \in \mathbb{N})$$

$$(\mathbb{N}^{\mathrm{E}})\ \frac{x : \mathbb{N} \Rightarrow A : \text{type} \qquad r = r' : \mathbb{N} \qquad s = s' : A[x/0] \qquad x : \mathbb{N}, y : A \Rightarrow t = t' : A[x/\mathrm{S}(x)]}{\mathrm{P}(r, s, (x, y)t) = \mathrm{P}(r', s', (x, y)t') : A[x/r]}$$

$$(\Pi^{\mathrm{E}})\ \frac{x : A \Rightarrow B : \text{type} \qquad s = s' : \Pi x \in A.B \qquad r = r' : A}{\mathrm{Ap}(s, r) = \mathrm{Ap}(s', r') : B[x/r]}$$

$$(\Sigma_0^{\mathrm{E}})\ \frac{x : A \Rightarrow B : \mathrm{type} \quad r = r' : \Sigma x \in A.B}{\mathrm{p}_0(r) = \mathrm{p}_0(r') : A}$$

$$(\Sigma_1^{\mathrm{E}})\ \frac{x : A \Rightarrow B : \mathrm{type} \quad r = r' : \Sigma x \in A.B}{\mathrm{p}_1(r) = \mathrm{p}_1(r') : B[x/\mathrm{p}_0(r)]}$$

$$(\mathrm{W}^{\mathrm{E}})\ \frac{\begin{array}{c} x : A \Rightarrow B : \mathrm{type} \\ u : \mathrm{W}x \in A.B \Rightarrow C : \mathrm{type} \\ r = r' : \mathrm{W}x \in A.B \\ x : A, y : B \to \mathrm{W}x \in A.B, z : \Pi v \in B.C[u/\mathrm{Ap}(y,v)] \Rightarrow t = t' : C[u/\mathrm{sup}(x,y)] \end{array}}{\mathrm{R}(r,(x,y,z)t) = \mathrm{R}(r',(x,y,z)t') : C[u/r]}$$

$$(+^{\mathrm{E}})\ \frac{\begin{array}{cc} A : \mathrm{type} & B : \mathrm{type} \\ z : A + B \Rightarrow C : \mathrm{type} & r = r' : A + B \\ x : A \Rightarrow s = s' : C[z/\mathrm{i}(x)] & y : B \Rightarrow t = t' : C[z/\mathrm{j}(y)] \end{array}}{\mathrm{D}(r,(x)s,(y)t) = \mathrm{D}(r',(x)s',(y)t') : C[z/r]}$$

$$(\mathrm{I}^{\mathrm{E}})\ \frac{s : A \quad t : A \quad r : \mathrm{I}(A,s,t)}{s = t : A}$$

Equality Rules

$$(\mathbb{N}_k^{=})\ \frac{x : \mathbb{N}_k \Rightarrow A : \mathrm{type} \quad s_i : A[x/\mathrm{A}_i^k](i = 0 \ldots k-1)}{\mathrm{C}_k(\mathrm{A}_n^k, s_0, \ldots, s_{k-1}) = s_n : A[x/\mathrm{A}_n^k]} \quad (n < k,\ n,k \in \mathbb{N})$$

$$(\mathbb{N}_0^{=})\ \frac{x : \mathbb{N} \Rightarrow A : \mathrm{type} \quad s : A[x/0] \quad x : \mathbb{N}, y : A \Rightarrow t : A[x/\mathrm{S}(x)]}{\mathrm{P}(0,s,(x,y)t) = s : A[x/0]}$$

$$(\mathbb{N}_{\mathrm{S}}^{=})\ \frac{\begin{array}{cc} x : \mathbb{N} \Rightarrow A : \mathrm{type} & r : \mathbb{N} \\ s : A[x/0] & x : \mathbb{N}, y : A \Rightarrow t : A[x/\mathrm{S}(x)] \end{array}}{\mathrm{P}(\mathrm{S}(r),s,(x,y)t) = t[x/r, y/\mathrm{P}(r,s,(x,y)t)] : A[x/\mathrm{S}(r)]}$$

$$(\Pi^{=})\ \frac{\begin{array}{c} x : A \Rightarrow B : \mathrm{type} \\ \lambda x.t : \Pi x \in A.B \\ r : A \end{array}}{\mathrm{Ap}(\lambda x.t, r) = t[x/r] : B[x/r]}$$

$$(\Pi^{\eta})\ \frac{x : A \Rightarrow B : \mathrm{type} \quad t : \Pi x \in A.B}{\lambda x.\mathrm{Ap}(t,x) = t : \Pi x \in A.B} \quad \textit{(if } x \notin \mathrm{FV}(t)\textit{)}$$

$$(\Sigma_0^{=})\ \frac{A : \mathrm{type} \quad \mathrm{p}(r,s) : \Sigma x \in A.B}{\mathrm{p}_0(\mathrm{p}(r,s)) = r : A}$$

$$(\Sigma_1^=)\ \frac{x \in A \Rightarrow B : \text{type} \qquad \mathrm{p}(r,s) : \Sigma x \in A.B}{\mathrm{p}_1(\mathrm{p}(r,s)) = s : B[x/r]}$$

$$(\Sigma_2^=)\ \frac{x \in A \Rightarrow B : \text{type} \qquad t : \Sigma x \in A.B}{t = \mathrm{p}(\mathrm{p}_0(t), \mathrm{p}_1(t)) : \Sigma x \in A.B}$$

$$(\mathrm{W}^=)\ \frac{\begin{array}{c} x : A \Rightarrow B : \text{type} \\ u : \mathrm{W}x \in A.B \Rightarrow C : \text{type} \\ r : A \\ s : B[x/r] \to \mathrm{W}x \in A.B \\ x : A, y : B \to \mathrm{W}x \in A.B, z : (\Pi v \in B.C[u/\mathrm{Ap}(y,v)]) \Rightarrow t : C[u/\mathrm{sup}(x,y)] \end{array}}{\mathrm{R}(\mathrm{sup}(r,s),(x,y,z)t) = t[x/r, y/s, z/\lambda v.\mathrm{R}(\mathrm{Ap}(s,v),(x,y,z)t)] : C[u/\mathrm{sup}(r,s)]}$$

(If $v \notin \mathrm{FV}(s) \cup \mathrm{FV}((x,y,z)t)$)

$$(+_0^=)\ \frac{z \in A + B \Rightarrow C : \text{type} \quad r : A \quad x : A \Rightarrow s : C[z/\mathrm{i}(x)] \quad y : B \Rightarrow t : C[z/\mathrm{j}(y)]}{\mathrm{D}(\mathrm{i}(r),(x)s,(y)t) = t[x/r] : C[z/\mathrm{i}(r)]}$$

$$(+_1^=)\ \frac{z \in A + B \Rightarrow C : \text{type} \quad r : B \quad x : A \Rightarrow s : C[z/\mathrm{i}(x)] \quad y : B \Rightarrow t : C[z/\mathrm{j}(y)]}{\mathrm{D}(\mathrm{j}(r),(x)s,(y)t) = t[y/r] : C[z/\mathrm{j}(r)]}$$

$$(\mathrm{I}^=)\ \frac{A : \text{type} \quad s : A \quad t : A \quad r : \mathrm{I}(A,s,t)}{r = \mathrm{r} : \mathrm{I}(A,s,t)}$$

Rules for the Universe (à la Tarski)

Formation Rules for the Universe

$(\mathrm{U}^{\mathrm{I}})\quad \mathrm{U} : \text{type}$

$$(\mathrm{T}^{\mathrm{I}})\ \frac{a = a' : \mathrm{U}}{\mathrm{T}(a) = \mathrm{T}(a') : \text{type}}$$

Introduction Rules for the Universe

$(\widehat{\mathbb{N}}_k^{\mathrm{I}})\quad \widehat{\mathbb{N}}_k : \mathrm{U} \quad (k \in \omega)$

$(\widehat{\mathbb{N}}^{\mathrm{I}})\quad \widehat{\mathbb{N}} : \mathrm{U}$

$$(\widehat{\Pi}^{\mathrm{I}})\ \frac{a = a' : \mathrm{U} \qquad x : \mathrm{T}(a) \Rightarrow b = b' : \mathrm{U}}{\widehat{\Pi} x \in a.b = \widehat{\Pi} x \in a'.b' : \mathrm{U}}$$

$$(\widehat{\Sigma}^{\mathrm{I}})\ \frac{a = a' : \mathrm{U} \qquad x : \mathrm{T}(a) \Rightarrow b = b' : \mathrm{U}}{\widehat{\Sigma} x \in a.b = \widehat{\Sigma} x \in a'.b' : \mathrm{U}}$$

$$(\widehat{\mathrm{W}}^{\mathrm{I}})\ \frac{a = a' : \mathrm{U} \qquad x : \mathrm{T}(a) \Rightarrow b = b' : \mathrm{U}}{\widehat{\mathrm{W}}x \in a.b = \widehat{\mathrm{W}}x \in a'.b' : \mathrm{U}}$$

$$(\widehat{+}^{\mathrm{I}})\ \frac{a = a' : \mathrm{U} \qquad b = b' : \mathrm{U}}{a\widehat{+}b = a'\widehat{+}b' : \mathrm{U}}$$

$$(\widehat{\mathrm{I}}^{\mathrm{I}})\ \frac{a = a' : \mathrm{U} \qquad s = s' : \mathrm{T}(a) \qquad t = t' : \mathrm{T}(a)}{\widehat{\mathrm{I}}(a, s, t) = \widehat{\mathrm{I}}(a', s', t') : \mathrm{U}}$$

Equality rules for the Universe

$$(\widehat{\mathbb{N}}_k^{=})\quad \mathrm{T}(\widehat{\mathbb{N}}_k) = \mathbb{N}_k : \mathrm{type} \quad (k \in \omega) \qquad\qquad (\widehat{\mathbb{N}}^{=})\quad \mathrm{T}(\widehat{\mathbb{N}}) = \mathbb{N} : \mathrm{type}$$

$$(\widehat{\Pi}^{=})\ \frac{a : \mathrm{U} \qquad x : \mathrm{T}(a) \Rightarrow b : \mathrm{U}}{\mathrm{T}(\widehat{\Pi}x \in a.b) = \Pi x \in \mathrm{T}(a).\mathrm{T}(b) : \mathrm{type}}$$

$$(\widehat{\Sigma}^{=})\ \frac{a : \mathrm{U} \qquad x : \mathrm{T}(a) \Rightarrow b : \mathrm{U}}{\mathrm{T}(\widehat{\Sigma}x \in a.b) = \Sigma x \in \mathrm{T}(a).\mathrm{T}(b) : \mathrm{type}}$$

$$(\widehat{\mathrm{W}}^{=})\ \frac{a : \mathrm{U} \qquad x : \mathrm{T}(a) \Rightarrow b : \mathrm{U}}{\mathrm{T}(\widehat{\mathrm{W}}x \in a.b) = \mathrm{W}x \in \mathrm{T}(a).\mathrm{T}(b) : \mathrm{type}}$$

$$(\widehat{+}^{=})\ \frac{a : \mathrm{U} \qquad b : \mathrm{U}}{\mathrm{T}(a\widehat{+}b) = \mathrm{T}(a) + \mathrm{T}(b) : \mathrm{type}}$$

$$(\widehat{\mathrm{I}}^{=})\ \frac{a : \mathrm{U} \qquad t : \mathrm{T}(a) \qquad s : \mathrm{T}(a)}{\mathrm{T}(\widehat{\mathrm{I}}(a, t, s)) = \mathrm{I}(\mathrm{T}(a), t, s) : \mathrm{type}}$$

The rules of $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T}}$ *are all rules mentioned above (using g-terms, g-types etc.).*
The rules of $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T,U}}$ *are all rules of* $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T}}$ *and additionally the following rules:*

$$(\widehat{\Sigma}^{\mathrm{E}})\ \frac{\widehat{\Sigma}x \in a.b : \mathrm{U}}{a : \mathrm{U}} \qquad \frac{\widehat{\Sigma}x \in a.b : \mathrm{U}}{x : \mathrm{T}(a) \Rightarrow b : \mathrm{U}} \qquad (\widehat{\Pi}^{\mathrm{E}})\ \frac{\widehat{\Pi}x \in a.b : \mathrm{U}}{a : \mathrm{U}} \qquad \frac{\widehat{\Pi}x \in a.b : \mathrm{U}}{x : \mathrm{T}(a) \Rightarrow b : \mathrm{U}}$$

$$(\widehat{\mathrm{W}}^{\mathrm{E}})\ \frac{\widehat{\mathrm{W}}x \in a.b : \mathrm{U}}{a : \mathrm{U}} \qquad \frac{\widehat{\mathrm{W}}x \in a.b : \mathrm{U}}{x : \mathrm{T}(a) \Rightarrow b : \mathrm{U}} \qquad (\widehat{+}^{\mathrm{E}})\ \frac{a\widehat{+}b : \mathrm{U}}{a : \mathrm{U}} \qquad \frac{a\widehat{+}b : \mathrm{U}}{b : \mathrm{U}}$$

$$(\widehat{\mathrm{I}}^{=})\ \frac{\widehat{\mathrm{I}}(a, s, t) : \mathrm{U}}{a : \mathrm{U}} \qquad \frac{\widehat{\mathrm{I}}(a, s, t) : \mathrm{U}}{s : \mathrm{T}(a)} \qquad \frac{\widehat{\mathrm{I}}(a, s, t) : \mathrm{U}}{t : \mathrm{T}(a)}$$

The rules of $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R}}$ *are the same as the rules of* $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T}}$*, but referring to R-terms,* *R-types etc. and replacing the Rules for the Universe by:*

Rules for the Universe à la Russell

Formation Rules for the Universe

$(\mathrm{U}^{\mathrm{I}})\quad \mathrm{U} : \mathrm{type}$

$$(\mathrm{T}^{\mathrm{I}})\ \frac{a = a' : \mathrm{U}}{a = a' : \mathrm{type}}$$

Introduction Rules for the Universe

$(\mathbb{N}_k^{\mathrm{U}})\quad \mathbb{N}_k : \mathrm{U} \quad (k \in \omega)$

$(\mathbb{N}_k^{\mathrm{U}})\quad \mathbb{N} : \mathrm{U}$

$$(\Pi^{\mathrm{U}})\ \frac{a = a' : \mathrm{U} \qquad x : a \Rightarrow b = b' : \mathrm{U}}{\Pi x \in a.b = \Pi x \in a'.b' : \mathrm{U}}$$

$$(\Sigma^{\mathrm{U}})\ \frac{a = a' : \mathrm{U} \qquad x : a \Rightarrow b = b' : \mathrm{U}}{\Sigma x \in a.b = \Sigma x \in a'.b' : \mathrm{U}}$$

$$(\mathrm{W}^{\mathrm{U}})\ \frac{a = a' : \mathrm{U} \qquad x : a \Rightarrow b = b' : \mathrm{U}}{\mathrm{W} x \in a.b = \mathrm{W} x \in a'.b' : \mathrm{U}}$$

$$(+^{\mathrm{U}})\ \frac{a = a' : \mathrm{U} \qquad b = b' : \mathrm{U}}{a + b = a' + b' : \mathrm{U}}$$

$$(\mathrm{I}^{\mathrm{U}})\ \frac{a = a' : \mathrm{U} \qquad t = t' : a \qquad s = s' : a}{\mathrm{I}(a, t, s) = \mathrm{I}(a', t', s') : \mathrm{U}}$$

The rules of $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R,U}}$ *are all rules of* $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R}}$ *and additionally the following rules:*

$$(\Sigma^{\mathrm{U,E}})\ \frac{\Sigma x \in a.b : \mathrm{U}}{a : \mathrm{U}} \qquad \frac{\Sigma x \in a.b : \mathrm{U}}{x : a \Rightarrow b : \mathrm{U}}$$

$$(\Pi^{\mathrm{U,E}})\ \frac{\Pi x \in a.b : \mathrm{U}}{a : \mathrm{U}} \qquad \frac{\Pi x \in a.b : \mathrm{U}}{x : a \Rightarrow b : \mathrm{U}}$$

$$(\mathrm{W}^{\mathrm{U,E}})\ \frac{\mathrm{W} x \in a.b : \mathrm{U}}{a : \mathrm{U}} \qquad \frac{\mathrm{W} x \in a.b : \mathrm{U}}{x : a \Rightarrow b : \mathrm{U}}$$

$$(+^{\mathrm{U,E}})\ \frac{a + b : \mathrm{U}}{a : \mathrm{U}} \qquad \frac{a + b : \mathrm{U}}{b : \mathrm{U}}$$

$$(\mathrm{I}^{\mathrm{U,E}})\ \frac{\mathrm{I}(a, s, t) : \mathrm{U}}{a : \mathrm{U}} \qquad \frac{\mathrm{I}(a, s, t) : \mathrm{U}}{s : a} \qquad \frac{\mathrm{I}(a, s, t) : \mathrm{U}}{t : a}$$

3 Intensional Martin-Löf Type Theory and its Embedding into Extensional Type Theory

In this section we introduce the intensional version of Martin-Löf Type Theory, and repeat the standard proof how to interpret it in extensional type theory. The well-ordering proof in [Set98b] was carried out in that theory. We note that [TD88] has a different way of formulating intensional type theory, which we used in [Set93], and which is trivially a sub theory of extensional type theory.

Definition 3.1 *(a) The symbols and constructors of intensional Martin-Löf type theory are the symbols and constructors of extensional Martin-Löf type theory, but replacing* r *by* refl *of arity* 1 *and adding a constructor* J *of arity 3.*

(b) The set of $\mathrm{b}^{\mathrm{int}}$*-objects,* $\mathrm{g}^{\mathrm{int}}$*-terms,* $\mathrm{g}^{\mathrm{int}}$*-types,* $\mathrm{g}^{\mathrm{int}}$*-context pieces,* $\mathrm{g}^{\mathrm{int}}$*-contexts,* $\mathrm{g}^{\mathrm{int}}$*-judgements,* $\mathrm{g}^{\mathrm{int}}$*-dependent judgements,* $\mathrm{R}^{\mathrm{int}}$*-terms,* $\mathrm{R}^{\mathrm{int}}$*-types,* $\mathrm{R}^{\mathrm{int}}$*-context pieces,* $\mathrm{R}^{\mathrm{int}}$*-contexts,* $\mathrm{R}^{\mathrm{int}}$*-judgements,* $\mathrm{R}^{\mathrm{int}}$*-dependent judgements, are defined as the corresponding b-/g-/*R*-constructions, but replacing* r *by* refl*, adding* J *as a constructor for terms, omitting that* r *is a* $\mathrm{g}^{\mathrm{int}}$*-term, and adding the clause for* refl*,* J *for* $\mathrm{g}^{\mathrm{int}}$*-terms as follows: If* r, s, t, t' *are* $\mathrm{g}^{\mathrm{int}}$*-terms* $x \in \mathrm{Var}_{\mathrm{ML}}$*, then* $\mathrm{refl}(r)$ *and* $\mathrm{J}(r, s, t, (x)t')$ *are* $\mathrm{g}^{\mathrm{int}}$*-terms.*

Definition 3.2 *of intensional Martin-Löf Type Theory with* W*-type and one Universe.*
The rules for $\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{T}}$ *(the intensional version à la Tarski) and* $\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{R}}$ *(the intensional version à la Russell) and their extensions* $\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{T,U}}$ *and* $\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{R,U}}$ *are the rules for* $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T}}$*,* $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R}}$*,* $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T,U}}$*,* $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R,U}}$*, respectively, but referring to* $\mathrm{g}^{\mathrm{int}}$*-objects and* $\mathrm{R}^{\mathrm{int}}$*-objects instead of* g*-objects and* R*-objects, and replacing the rules* $(\mathrm{I}^{\mathrm{I}})$*,* $(\mathrm{I}^{\mathrm{E}})$*, and* $(\mathrm{I}^{=})$ *by the following rules:*

Intensional Equality Rules

$$(\mathrm{I}^{\mathrm{I}})\ \frac{t : A}{\mathrm{refl}(t) : \mathrm{I}(A, t, t)}$$

$$(\mathrm{I}^{\mathrm{E}})\ \frac{\begin{array}{c} A : Set \\ x : A, y : A, z : \mathrm{I}(A, x, y) \Rightarrow C : \mathrm{type} \\ r = r' : A \\ s = s' : A \\ t = t' : \mathrm{I}(A, r, s) \\ z' : A \Rightarrow u = u' : C[x/z', y/z', z/\mathrm{refl}(z')] \end{array}}{\mathrm{J}(r, s, t, (z')u) = \mathrm{J}(r', s', t', (z')u') : C[x/r, y/s, z/t]}$$

$$(\mathrm{I}^{=})\ \frac{\begin{array}{c} A : \mathrm{type} \\ x : A, y : A, z : \mathrm{I}(A, x, y) \Rightarrow C : \mathrm{type} \\ r : A \\ z' : A \Rightarrow u : C[x/z', y/z', z/\mathrm{refl}(z')] \end{array}}{\mathrm{J}(r, r, \mathrm{refl}(r), (z')u) = u[z'/r] : C[x/r, y/r, z/\mathrm{refl}(r)]}$$

Definition 3.3 *We define a translation* ϕ *of* $\mathrm{b}^{\mathrm{int}}$-, $\mathrm{g}^{\mathrm{int}}$- *and* $\mathrm{R}^{\mathrm{int}}$*-constructions into the corresponding respective* b-, g- *and* R*-constructions by recursively replacing*

- $\mathrm{refl}(r)$ *by* r*,*

- $\mathrm{J}(r, s, t, (z')u)$ *by* $u[z'/r]$.

Lemma 3.4 *If* $\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{T}}$, $\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{R}}$, $\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{R,U}}$, *or* $\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{T,U}}$ *proves* $\Gamma \Rightarrow \theta$, *then the corresponding respective extensional version* $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T}}$, $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R}}$, $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R,U}}$, $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T,U}}$, *proves* $\phi(\Gamma) \Rightarrow \phi(\theta)$.

Proof: By induction on the derivation. For all unchanged rules this follows by applying the same rule to the IH. All what remains to show is that for the intensional equality rules, the translation of the conclusion is derivable in extensional type theory from the translated assumptions using the extensional equality rules.
In case of (I^{I}) this is obvious (the translated conclusion is $\mathrm{r} : \mathrm{I}(A, t, t)$). In case of I^{E} one needs to show that the assumptions imply the translated conclusion which is $u[z'/r] = u'[z'/r'] : C[x/r, y/s, z/t]$. From $t = t' : \mathrm{I}(A, r, s)$ one concludes $t : \mathrm{I}(A, r, s)$, therefore $r = s : A$ and therefore $\mathrm{I}(A, r, s) = \mathrm{I}(A, r, r)$. Therefore $t = t' : \mathrm{I}(A, r, r)$, and $t = \mathrm{r} : \mathrm{I}(A, r, s)$. It follows $C[x/r, y/s, z/t] = C[x/r, y/r, z/\mathrm{r}]$. We have $u[z'/r] = u'[z'/r'] : C[x/r, y/r, z/\mathrm{r}]$ and therefore $u[z'/r] = u'[z'/r'] : C[x/r, y/s, z/t]$.
In case of ($\mathrm{I}^{=}$) the translated conclusion is $u[z'/r] = u[z'/r] : C[x/r, y/r, z/\mathrm{r}]$ which follows from the last translated assumption by substitution.

4 Embedding of the Russell Version of Martin Löf Universes into the Tarski Version

In this section we prove, that Martin-Löf Type Theory à la Russell $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R,U}}$ can be embedded into Martin-Löf Type Theory à la Tarski $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T,U}}$. Therefore, the upper bound proved for $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T,U}}$ is an upper bound for $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R}}$ and $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{R,U}}$ as well, and as well for $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W}_{\mathrm{T}}$ (since it is a proper sub theory). By Lemma 3.4 it is as well an upper bound for $\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{T}}$, $\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{T,U}}$, $\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{R}}$ and $\mathrm{ML}_1^{\mathrm{i}}\mathrm{W}_{\mathrm{R,U}}$.

Definition 4.1 *(a) Define for* C *constructors,*

$\psi(\mathbb{N}) :\equiv \widehat{\mathbb{N}}$	$\psi(\Sigma) :\equiv \widehat{\Sigma}$,	$\psi(\mathrm{I}) :\equiv \widehat{\mathrm{I}}$
$\psi(\mathbb{N}_k) :\equiv \widehat{\mathbb{N}}_k$,	$\psi(\mathrm{W}) :\equiv \widehat{\mathrm{W}}$,	$\psi(C) :\equiv C$ *otherwise*
$\psi(\Pi) :\equiv \widehat{\Pi}$,	$\psi(+) :\equiv \widehat{+}$,	

(b) Define ψ : *R-term* $\rightarrow$ *g-term by recursion on the b-objects:*
$\psi(x) :\equiv x$ *(*$x \in \mathrm{Var}_{\mathrm{ML}}$*),*
$\psi(\lambda x.t) :\equiv \lambda x.\psi(t)$,
$\psi(C(t_1, \ldots, t_n)) :\equiv \psi(C)(\phi(t_1), \ldots, \phi(t_n))$ *(for constructors* C*).*

(c) Define the function ρ : *R-type* $\rightarrow$ *g-type by recursion on the g-types:*
$\rho(Sx \in r.s) :\equiv Sx \in \rho(r).\rho(s)$ $(S \in \Sigma, \Pi, \mathrm{W})$,
$\rho(r + s) :\equiv \rho(r) + \rho(s)$, $\quad\rho(\mathrm{I}(r, s, t)) :\equiv \mathrm{I}(\rho(r), \psi(s), \psi(t))$,
$\rho(C) :\equiv C$ *for* $C \in \{\mathbb{N}, \mathbb{N}_k, \mathrm{U}\}$, $\quad\rho(t) :\equiv \mathrm{T}(\psi(t))$, *otherwise.*

(d) If $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ *is a g-context-piece, then*
$\rho(\Gamma) :\equiv x_1 : \rho(A_1), \ldots, x_n : \rho(A_n)$

(e) *If* r, s *are g-terms,* A *is a g-type, then*
$\rho(r = s : A) :\equiv (\psi(r) = \psi(s) : \rho(A))$,
$\rho(A = B : \text{type}) :\equiv (\rho(A) = \rho(B) : \text{type})$.

(f) *Define* μ : *g-type* $\to$ *g-type by recursion on the g-types:*
$\mu(\mathrm{T}(sx \in r.s)) :\equiv Sx \in \mu(\mathrm{T}(r)).\mu(\mathrm{T}(s))$ *(where* $s = \widehat{\Sigma}, \widehat{\Pi}, \widehat{\mathrm{W}}$ *and* $S = \Sigma, \Pi, \mathrm{W}$ *respectively),*
$\mu(\mathrm{T}(r \widehat{+} s)) :\equiv \mu(\mathrm{T}(r)) + \mu(\mathrm{T}(s))$, $\mu(\mathrm{T}(\widehat{\mathrm{I}}(r, s, t)))) :\equiv \mathrm{I}(\mu(\mathrm{T}(r)), s, t)$,
$\mu(\mathrm{T}(\widehat{\mathbb{N}})) :\equiv \mathbb{N}$ $\mu(\mathrm{T}(\widehat{\mathbb{N}}_k)) :\equiv \mathbb{N}_k$
$\mu(r + s) :\equiv \mu(r) + \mu(s)$, $\mu(\mathrm{I}(r, s, t)) :\equiv \mathrm{I}(\mu(r), s, t)$,
$\mu(Sx \in r.s) :\equiv Sx \in \mu(r).\mu(s)$ $(S \in \{\Sigma, \Pi, \mathrm{W}\})$
$\mu(t) :\equiv t$, *otherwise.*

Lemma 4.2 *Assume* r, s, t, s_i *b-objects,* $x_i \in \mathrm{Var}_{\mathrm{ML}}$.

(a) $\mathrm{FV}(t) = \mathrm{FV}(\psi(t)) = \mathrm{FV}(\rho(t)) = \mathrm{FV}(\mu(t))$.

(b) *If* $t[x_1/s_1, \ldots, x_n/s_n]$ *allowed, then* $\psi(t)[x_1/\psi(s_1), \ldots, x_n/\psi(s_n)]$, $\mu(t)[x_1/\psi(s_1), \ldots, x_n/\psi(s_n)]$, $\rho(t)[x_1/\psi(s_1), \ldots, x_n/\psi(s_n)]$ *are allowed.*

(c) *If* t *is an R-term, then* $\rho(t) = \mu(\mathrm{T}(\psi(t)))$.

(d) *If* t, s_i *are b-objects, then* $\psi(t[x_1/s_1, \ldots, x_n/s_n]) = \psi(t)[x_1/\psi(s_1), \ldots, x_n/\psi(s_n)]$.
If t *is a g-type,* s_i *are g-terms, then* $\mu(\mu(t)) = \mu(t)$,
$\mu(t[x_1/s_1, \ldots, x_n/s_n]) = \mu(\mu(t)[x_1/s_1, \ldots, x_n/s_n])$,
If t *is an R-type,* s_i *are R-terms, then*
$\rho(t[x_1/s_1, \ldots, x_n/s_n]) = \mu(\rho(t)[x_1/\psi(s_1), \ldots, x_n/\psi(s_n)])$.

(e) *If* t *is a g -judgement, -context, -context-piece, or dependent judgement, then* $\rho(t)$ *is an R-judgement, -context, -context-piece, or dependent judgement, respectively.*

(f) *If* $r =_\alpha s$*, then* $\psi(r) =_\alpha \psi(s)$, $\rho(r) =_\alpha \rho(s)$, $\mu(r) =_\alpha \mu(s)$.

(g) *If* r *is a g-term and b-term, then* $\psi(r) = r$.
If r *is a g-type and b-type, then* $\rho(r) = r$.

Lemma 4.3 *In all versions of Martin-Löf Type Theory, we have the following useful derived rule:*

$$\frac{\Gamma, x : A, \Gamma' \Rightarrow \theta \quad \Gamma \Rightarrow A = A' : \text{type}}{\Gamma, x : A', \Gamma' \Rightarrow \theta}$$

Proof:
Let y be a fresh variable. Then by (THIN) $\Gamma, y : A', x : A, \Gamma' \Rightarrow \theta$, easily we have $\Gamma, y : A' \Rightarrow y : A$, By (SUB) $\Gamma, y : A', \Gamma'[x/y] \Rightarrow \theta[x/y]$, and by change of the variable we obtain the assertion.

Lemma 4.4 *Let* $\mathrm{ML_T}$ *be* $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W_T}$ *or* $\mathrm{ML}_1^{\mathrm{e}}\mathrm{W_{T,U}}$ *and* Γ, Γ' *be g-context-pieces,* a, b, t *g-terms,* A, B *g-types,* x *a variable. The following applies:*

(a) *If* $\mathrm{ML_T} \vdash \Gamma \Rightarrow s : A$*, or* $\mathrm{ML_T} \vdash \Gamma \Rightarrow s = t : A$ *then* $\mathrm{ML_T} \vdash \Gamma \Rightarrow A : \mathrm{type}$.

(b) *If* $\mathrm{ML_T} \vdash \Gamma \Rightarrow Sx \in A.B : \mathrm{type}$ *(*$S \in \{\Sigma, \Pi, \mathrm{W}\}$*), then* $\mathrm{ML_T} \vdash \Gamma \Rightarrow A$, $\mathrm{ML_T} \vdash \Gamma, y : A \Rightarrow B[x/y] : \mathrm{type}$*, for all* $y \in \mathrm{Var_{ML}} \setminus X$ *for some finite set* X.

(c) *If* $\mathrm{ML_T} \vdash \Gamma \Rightarrow A + B : \mathrm{type}$ *then* $\mathrm{ML_T} \vdash \Gamma \Rightarrow A : \mathrm{type}$*,* $\mathrm{ML_T} \vdash \Gamma \Rightarrow B : \mathrm{type}$.

(d) *If* $\mathrm{ML_T} \vdash \Gamma \Rightarrow \mathrm{I}(A, b, c) : \mathrm{type}$*, then* $\mathrm{ML_T} \vdash \Gamma \Rightarrow b : A$*,* $\mathrm{ML_T} \vdash \Gamma \Rightarrow c : A$.

(e) *If* $\mathrm{ML_T} \vdash \Gamma \Rightarrow \mathrm{T}(b) : \mathrm{type}$ *then* $\mathrm{ML_T} \vdash \Gamma \Rightarrow b : \mathrm{U}$.

(f) *If* $\mathrm{ML_T} \vdash \Gamma \Rightarrow A = B : \mathrm{type}$*, then* $\mathrm{ML_T} \vdash \Gamma \Rightarrow A : \mathrm{type}$, $\mathrm{ML_T} \vdash \Gamma \Rightarrow B : \mathrm{type}$.

(g) *If* $\mathrm{ML_T} \vdash \Gamma, x : A, \Gamma \Rightarrow \theta$ *then* $\mathrm{ML_T} \vdash \Gamma \Rightarrow A : \mathrm{type}$.

Note that (f) is trivial, but will be needed in the following as an additional premise for the induction.

Proof: We first change the calculus, treating $A : \mathrm{type}$ no longer as an abbreviation for $A = A : \mathrm{type}$. For every instance of a rule with conclusion $A = A : \mathrm{type}$ we add a rule with conclusion $A : \mathrm{type}$ where any premise $B = B : \mathrm{type}$ is replaced by $B : \mathrm{type}$, and any premise $s = s : B$ is replaced by $s : B$. Furthermore, add the rules

$$(\mathrm{REPL1})\ \frac{x : A, \Gamma', \Gamma'' \Rightarrow B : \mathrm{type} \qquad \Rightarrow t = t' : A}{\Gamma'[x/t], \Gamma''[x/t'] \Rightarrow B[x/t'] : \mathrm{type}}$$

$$(\mathrm{ALPHA_1})\ \frac{A : \mathrm{type}}{A' : \mathrm{type}} \quad \text{if } A =_\alpha A'$$

$$(\mathrm{REFL})\ \frac{A : \mathrm{type}}{A = A : \mathrm{type}}$$

If for this calculus the theorem is provable, then this calculus is equivalent to the original: If we have a proof in the original calculus, then embed it into the calculus, by applying, whenever we need the removed rules the weak inferences. If we have a proof in the new calculus, the result is a proof in the original calculus, since we only added derived rules.

Now the proof follows by induction on the length of the (new) derivation. The only difficult case is (SUB), where the difficulty are the the second conclusion in the cases (b): let the conclusion be for instance $\Gamma, \Gamma'[x/t] \Rightarrow (\Sigma x' \in A.B)[x/t] : \mathrm{type}$. By IH $\Gamma, \Gamma', y : A \Rightarrow B[x'/y] : \mathrm{type}$ for $y \notin X$, therefore $\Gamma, \Gamma'[x/t], y : A[x/t] \Rightarrow B[x'/y][x/t]$ for $x \neq y$, $x \notin X$ (the substitution is allowed). Then for $y \notin X \cup \{x\}$, if $x = x'$ or $x \notin \mathrm{FV}(B)$, it follows $(\Sigma x' \in A.B)[x/t] = \Sigma x' \in A[x/t].B$ and we have the assertion, otherwise $x' \notin \mathrm{FV}(t)$ and $B[x'/y][x/t] = B[x/t][x'/y] : \mathrm{type}$.

Similarly we argue in (REPLi), the other rules are easy.

Lemma 4.5 *(a)* $\mathrm{ML}_1^e\mathrm{W}_{\mathrm{T,U}} \vdash \Gamma \Rightarrow r = s : \mathrm{U}$ *, then* $\mathrm{ML}_1^e\mathrm{W}_{\mathrm{T,U}} \vdash \Gamma \Rightarrow \mathrm{T}(r) = \mu(\mathrm{T}(r)) : \mathrm{type}$ *and* $\mathrm{ML}_1^e\mathrm{W}_{\mathrm{T,U}} \vdash \Gamma \Rightarrow \mathrm{T}(s) = \mu(\mathrm{T}(s)) : \mathrm{type}$

(b) $\mathrm{ML}_1^e\mathrm{W}_{\mathrm{T,U}} \vdash \Gamma \Rightarrow A = B : \mathrm{type}$ *, then* $\mathrm{ML}_1^e\mathrm{W}_{\mathrm{T,U}} \vdash \Gamma \Rightarrow A = \mu(A) : \mathrm{type}$ *and* $\mathrm{ML}_1^e\mathrm{W}_{\mathrm{T,U}} \vdash \Gamma \Rightarrow B = \mu(B) : \mathrm{type}$

Proof:
(a): Induction on the definition of r being a b-object. If for instance $\mathrm{ML}_1^e\mathrm{W}_{\mathrm{T,U}} \vdash \Gamma \Rightarrow \widehat{\Sigma} x \in a.b : \mathrm{U}$, then by the additional rules of $\mathrm{ML}_1^e\mathrm{W}_{\mathrm{T,U}}$ it follows $\Gamma \Rightarrow a : \mathrm{U}$ and $\Gamma, x : \mathrm{T}(a) \Rightarrow b : \mathrm{U}$, by IH therefore $\Gamma \Rightarrow \mathrm{T}(a) = \mu(\mathrm{T}(a)) : \mathrm{type}$, $\Gamma, x : \mathrm{T}(a) \Rightarrow \mathrm{T}(b) = \mu(\mathrm{T}(b)) : \mathrm{type}$, by type introduction follows the assertion, similarly for the other terms, for which $\mu(\mathrm{T}(t))$ does something.
(b) Similar, using Lemma 4.4 instead of the new rules.

Lemma 4.6 *If* $\mathrm{ML}_1^e\mathrm{W}_{\mathrm{R,U}} \vdash \Gamma \Rightarrow \theta$ *then* $\mathrm{ML}_1^e\mathrm{W}_{\mathrm{T,U}} \vdash \rho(\Gamma) \Rightarrow \rho(\theta)$*.*
Especially, if $\Gamma \Rightarrow \theta$ *is a dependent judgement of both* $\mathrm{ML}_1^e\mathrm{W}_{\mathrm{R,U}}$ *and* $\mathrm{ML}_1^e\mathrm{W}_{\mathrm{T,U}}$*, then we have:*
If $\mathrm{ML}_1^e\mathrm{W}_{\mathrm{R,U}} \vdash \Gamma \Rightarrow \theta$ *then* $\mathrm{ML}_1^e\mathrm{W}_{\mathrm{T,U}} \vdash \Gamma \Rightarrow \theta$*.*

Proof: Induction on the derivation.
In most rules, the assertion follows by the same rules.
Difficult rules: (SUB), (REPLi): Use 4.2 (d), 4.5 and 4.3.
Equality rules and extensional equality rules: use for the substitution part the same argument.
Second and third rule in (U^{I}): we conclude $\mathrm{T}(\psi(A))$, and using 4.5 and an easy argument follows the assertion.
Universe introduction rules (possibly extensional): easy.

Notation 4.7 *In the following we will write* ML *for* $\mathrm{ML}_1^e\mathrm{W}_{\mathrm{T,U}}$*.*

5 Definition of KPI^+

We introduce now the Kripke-Platek set theory KPI^+, in which we will interpret afterwards ML. For more details on it, the reader should refer to [Bar75], [Jäg79], [Jäg83], [JP82], [Jäg86], and [Poh82].

Definition 5.1 *Definition of Kripke-Platek set theory:*

(a) Let $\mathcal{L}_{KP}$ *be the classical first-order language, with terms being variables, atomic formulas being* $u \in v$, $\neg(u \in v)$, $\mathrm{Ad}(u)$, $\neg\mathrm{Ad}(u)$*. The set of Variables should be* $\mathrm{Var}_{\mathrm{KP}} = \{u_i | i \in \mathbb{N}\}$ *(a meta-set),* $u_i \neq u_j$ *for* $i \neq j$*.*
The formulas are built from atomic formulas by $\wedge$, $\vee$, $\forall$, $\exists$*. We define* $\neg A$ *by the deMorgan's laws. The quantifier in* $\forall x.\phi$ *(*$\exists x.\phi$*) is bounded, if* ϕ *of the form* $x \in v \to B$ *(*$x \in v \wedge B$*) with* $x \neq v$*. A* Δ_0*-formula is a formula with no unbounded quantifier.*
We abbreviate
$A \to B :\equiv ((\neg A) \vee B)$*,*

$\forall x \in v.B :\equiv \forall x.x \in v \to B$,
$\exists x \in v.B :\equiv \exists x.(x \in v \wedge B)$,
$(u = v) :\equiv ((\forall x \in u.x \in v) \wedge (\forall x \in v.x \in u))$,
$u \notin v :\equiv \neg(u \in v)$,
$\mathrm{tran}(u) :\equiv \forall x \in u.\forall y \in x.y \in u$,
$\mathrm{infinite}(u) :\equiv \exists x \in u.(x = x) \wedge \forall x \in u.\exists y \in u.x \in y$.
$\mathrm{Inacc}(x) :\equiv \mathrm{Ad}(x) \wedge \forall y \in x.\exists z \in x.y \in z \wedge \mathrm{Ad}(z)$.
$\mathrm{Inacc}_n(x) :\equiv \exists x_0, \ldots, x_n.\mathrm{Inacc}(x_0) \wedge \mathrm{Ad}(x_1) \wedge \mathrm{Ad}(x_2) \wedge \cdots \wedge \mathrm{Ad}(x_n) \wedge x_0 \in x_1 \wedge x_1 \in x_2 \wedge \cdots \wedge x_{n-1} \in x_n \wedge x = x_n$.

ψ a formula, then ψ^u means the replacing of every unbounded quantifier $\forall v$ by $\forall v \in u$ and $\exists v$ by $\exists v \in u$.
Note, that $\mathrm{Inacc}(x)$ *expresses, that x is an admissible, closed under admissibles, the ordinal of which is an inaccessible, and* $\mathrm{Inacc}_n(x)$*, that x is an admissible, which is at least the nth admissible above an x s.t.* $\mathrm{Inacc}(x)$.

(b) Definition of axiom schemes:

(Ext)	$\forall x.\forall y.\forall z.x = y \to (x \in z \to y \in z) \wedge (\mathrm{Ad}(x) \to \mathrm{Ad}(y))$
(Foud)	$\forall \vec{z}.[\forall x.(\forall y \in x.\phi(y, \vec{z}) \to \phi(x, \vec{z})) \to \forall x.\phi(x, \vec{z})]$ *(ϕ an arbitrary formula)*
(Pair)	$\forall x.\forall y.\exists z.x \in z \wedge y \in z$.
(Union)	$\forall x.\exists z.\forall y \in x.\forall u \in y.u \in z$.
$(\Delta_0 - \mathrm{sep})$	$\forall \vec{z}.\forall w.\exists y.[(\forall x \in y.(x \in w \wedge \phi(x, \vec{z}))) \wedge \forall x \in w.\phi(x, \vec{z}) \to x \in y]$ *(ϕ a Δ_0-formula).*
$(\Delta_0 - \mathrm{coll})$	$\forall \vec{z}.\forall w.[(\forall x \in w.\exists y.\phi(x, y, \vec{z})) \to \exists w'.\forall x \in w.\exists y \in w'.\phi(x, y, \vec{z})]$ *(ϕ a Δ_0-formula).*
(Ad.1)	$\forall x.\mathrm{Ad}(x) \to \mathrm{tran}(x) \wedge \exists w \in x.\mathrm{infinite}(w)$.
(Ad.2)	$\forall x.\forall y.\mathrm{Ad}(x) \wedge \mathrm{Ad}(y) \to x \in y \vee x = y \vee y \in x$.
(Ad.3)	$\forall x.\mathrm{Ad}(x) \to \psi^x$, *($\psi$ an instance of (*Pair*), (*Union*), ($\Delta_0 - \mathrm{sep}$), ($\Delta_0 - \mathrm{coll}$)).*
(Lim)	$\forall x.\exists y.\mathrm{Ad}(y) \wedge x \in y$.
(inf)	$\exists x.\mathrm{infinite}(x)$.
$(+_n)$	$\exists x.\mathrm{Inacc}_n(x)$.

(c) KPI^+ *is the theory*
(Ext) + (Foud) + (Pair) + (Union) + $(\Delta_0 - \mathrm{sep})$ + $(\Delta_0 - \mathrm{coll})$ + (inf) $+(\mathrm{Ad.1} - 3) + \{(+_\mathrm{n}) | \mathrm{n} \in \omega\}$.
So KPI^+ *is a theory, which guarantees the existence of one recursive inaccessible, and of finitely many admissibles above it.*

Definition 5.2 *(a)* Ord *is the class of ordinals.*

(b) $\alpha(a) := \bigcup(a \cap \mathrm{Ord})$.

(c) $\mathrm{Ad}_1 := \bigcap\{x|\mathrm{Ad}(x)\}$, $\mathrm{Ad}_2 := \bigcap\{x|\mathrm{Ad}(x) \wedge \mathrm{Ad}_1 \in x\}$, $\mathrm{Ad}_\mathrm{I} := \bigcap\{x|\mathrm{Inacc}(x)\}$, $\mathrm{Ad}_{\mathrm{I},n} := \bigcap\{x|\mathrm{Inacc}_n(x)\}$. *Note, that* Ad_1, Ad_2, Ad_I, $\mathrm{Ad}_{\mathrm{I},n}$ *can be defined, since there exists* b *s.t.* $\mathrm{Ad}(b)$ *or* $\mathrm{Inacc}(b)$ *or* $\mathrm{Inacc}_n(b)$*, and therefore we can replace the class by* $\{x \in b \cup \{b\}| \cdots\}$.

(d) $\Omega_1 := \alpha(\mathrm{Ad}_1)$, $\mathrm{I} := \alpha(\mathrm{Ad}_\mathrm{I})$, $\mathrm{I}_n := \alpha(\mathrm{Ad}_{\mathrm{I},n})$.

(e) $\mathrm{ad}(u) := \bigcap(\{c \in \mathrm{Ad}_\mathrm{I}|\mathrm{Ad}(c) \wedge u \in c\} \cup \{\mathrm{Ad}_\mathrm{I}\})$,
$\alpha^+(u) := \alpha(\mathrm{ad}(u))$.

Remark 5.3 *In* KPI^+ *we have*

(a) $\mathrm{Ad}(\mathrm{Ad}_1)$, $\mathrm{Ad}(\mathrm{Ad}_2)$, $\mathrm{Inacc}(I)$, $\mathrm{Inacc}_n(\mathrm{I}_n)$.

(b) $u \in \mathrm{Ad}_\mathrm{I} \to \mathrm{ad}(u) \in \mathrm{Ad}_\mathrm{I} \wedge \mathrm{Ad}(\mathrm{ad}(u)) \wedge u \in \mathrm{ad}(u)$.

6 Interpretation of Terms and Types

A type A will be interpreted basically as a set of pairs of closed terms: $\langle t, s\rangle \in A^*$ means that t and s are equal elements of this types. We will define A^* as the set of terms, which are by an introductory rule elements of this type, and close it under the reduction rule. For instance, if A^* and B^* are already defined, then

$$(A + B)^* := +^*(A^*, B^*)$$

where again

$$+^*(u, v) = \mathrm{Compl}(+^{\mathrm{basis}}(u, v))$$
$$+^{\mathrm{basis}}(u, v) := \{\langle \mathrm{i}(a), \mathrm{i}(a')\rangle|\langle a, a'\rangle \in u\} \cup \{\langle \mathrm{j}(b), \mathrm{j}(b')\rangle|\langle b, b'\rangle \in v\}$$
$$\mathrm{Compl}(u) := \{\langle r, s\rangle|\exists r', s'.\langle r', s'\rangle \in u \wedge r \rightarrow_{\mathrm{red}} r' \wedge s \rightarrow_{\mathrm{red}} s'\}$$

Since we only want to interpret finitely many types, namely those types, which occur in a certain derivation, we interpret dependent types as Σ-functions, the arguments of which are represented by the free variables of the type, in such a way, that $A^*[x_1/t_1, \ldots, x_n/t_n] = (A[x_1/t_1, \ldots, x_n/t_n])^*$.
The Π-type has an introductory rule with a premise, where dependency occurs. The intended meaning of the premise $x : A \Rightarrow t = t' : B$ is

$$\forall r, r'.\langle r, r'\rangle \in A^* \to \langle t[x/r], t'[x/r']\rangle \in B^*$$

Furthermore, we know

$$\langle r, r'\rangle \in A^* \Rightarrow B^*[x/r] = B'^*[x/r']$$

Since we have to close it under α-conversion we can therefore define $(\Pi x \in A.B)^* := \Pi^*(A^*, (x)B^*)$, where $\Pi^*(u, f) = \mathrm{Compl}(\Pi^{basic}(u, f))$ and

$$\Pi^{\mathrm{basis}}(u, f) := \{\langle \lambda y.t, \lambda y'.t' \rangle | \forall \langle r, r' \rangle \in u. \langle t[y/r], t'[y'/r'] \rangle \in f(r) \wedge f(r) = f(r')\})$$

The condition $f(r) = f(r')$ has been added for technical reasons.
In order to make proofs about the terms easy, we will have deterministic reduction-rules. We will allow e.g. $\mathrm{Ap}(\lambda x.r, s) \rightarrow_{\mathrm{red}} r[x/s]$ only, if s is in normal form. Furthermore, we do not allow any reductions of $\lambda x.r$, so giving reduction rules generally only for closed terms This simple approach is possible, since, from the definition of $(\Pi x \in A.B)^*$, we see, that, if $\langle \lambda x.t, \lambda x.t \rangle \in (\Pi x \in A.B)^*$, and if $t \rightarrow t'$ in a general sense for open terms, $t[x/r] \rightarrow_{\mathrm{red}} t'[x/r]$ for closed terms r. But now, if B^* is closed under $\rightarrow_{\mathrm{red}}$, we conclude $\langle \lambda x.t, \lambda x.t' \rangle \in (\Pi x \in A.B)^*$.
The interpretation of the W-type, which represents an inductive definition, is done in the usual way: we take some operator on sets and iterate it up to the closure ordinal, which is the next admissible above the components A and B of it. By the introduction rule, we get as the operator F such that

$$\begin{aligned} F_{\mathrm{W}}(u, f)(v) = \quad & \mathrm{Compl}(\{\langle \sup(s, \lambda y.t), \sup(s', \lambda y'.t') \rangle | \langle s, s' \rangle \in u \\ & \qquad \wedge \forall \langle r, r' \rangle \in f(s). \langle t[y/r], t'[y'/r'] \rangle \in v\}) \end{aligned}$$

$(\mathrm{W} x \in A.B)^* := F_{\mathrm{W}}^{\alpha}(A^*, (x)B^*)$. We can choose as α any admissible ordinal s.t. A^* and $(x)B^*$ are elements of L_α. We will take as α I_n, the nth admissible above I, and n is the maximum of $\mathrm{lev}(A)$ and $\mathrm{lev}(B)$, which is the number of nestings of W-types in A and B. Although this ordinal is usually too big, it suffices for our construction.
In the introduction rules for the elements of the universe, e.g.

$$(\widehat{\Pi}^{\mathrm{I}}) \ \frac{a : \mathrm{U} \qquad x : \mathrm{T}(a) \Rightarrow b : \mathrm{U}}{\widehat{\Pi} x : a.b : \mathrm{U}}$$

we introduce simultaneously the elements a of the universe and its interpretation $\mathrm{T}(a)$ as a type. We will therefore first define a set $\widehat{\mathrm{U}}$ of triples $\langle a, A, b \rangle$, where a and b are terms, considered as equal elements of the universe, such that $\mathrm{T}(a)^* = \mathrm{T}(b)^* = A$. Therefore

$$\mathrm{U}^* = \{\langle a, b \rangle | \exists A. \langle a, A, b \rangle \in \widehat{\mathrm{U}}\}$$

and $\mathrm{T}(a)^* = f(a)$, where

$$f = \{\langle a, A \rangle | \exists b. \langle a, A, b \rangle \in \widehat{\mathrm{U}}\}$$

$\widehat{\mathrm{U}}$ is again the fixed point of an operator $\widetilde{\mathrm{U}}$, so $\widehat{\mathrm{U}} = \widetilde{\mathrm{U}}^\alpha$ for some admissible α. Since U is closed under the W-type, in the definition of $\widetilde{\mathrm{U}}$ we need to go to the next admissible. Therefore, α must be closed under the step to the next admissible. We can choose $\alpha := \mathrm{I}$, which is a recursive inaccessible, i.e. an admissible closed under the step to the next admissible. Here we see, why we need the theory KPI^+: We

need, one admissible, closed under admissibles, and finitely many admissibles above it.
We want to interpret an intuitionistic theory in a classical one, using some realisation. Now if we have a realisation interpretation as indices for recursive functions there is naturally a very easy realisation of $\neg\forall x.\exists y.\phi(x,y)$, if $\forall x.\exists y.\phi(x,y)$ is arithmetical formula, such that from x we cannot compute a y such that $\varphi(x,y)$ holds: every term $\lambda x.t$ does it since there is no realisation of $\forall x.\exists y.\phi(x,y)$. Therefore there are false recursive realisations. But proofs carried out in an intuitionistic theory like Martin-Löf Type Theory should not prove false statements. The reason, why it does not prove any, is, that we can apply the realising term $\lambda u.s$ of $\neg\forall x.\exists y.\phi(x,y)$, in some sense to non intuitionistic proofs as well. In order to fix this, we allow to add to our model a constructor, which has a non recursive reduction rule, and gives us the y depending on the x. Then we have a realising term t for $\forall x.\exists y.\phi(x,y)$, and can apply any proof $\lambda u.s$ of $\neg\forall x.\exists y.\phi(x,y)$ to it to get an element of type $\bot$ which is empty. Therefore, adding non recursive constructors, we can achieve that there is no realising term of a false formula.
We want to extend our result even for Π^1_1-sentences. Here again we have the problem, that the powerset of the natural numbers, $\mathbb{N} \to \mathrm{U}$, does not represent all sets in KPI^+. We will not be able to get a result, where we conclude from $\mathrm{ML} \vdash \forall X \in (\mathbb{N} \to \mathrm{U}).\phi(X)$, that $\mathrm{KPI}^+ \vdash \forall x.x \subset \omega \to \phi'(x)$ for the translation ϕ' of ϕ, but only, if we have, that x is an element of the first admissible. (We could easily extend it for x being an element of the first recursive inaccessible admissible without any trouble, but the result is enough to get an upper bound for the provable proof theoretic strength).)
Anyway, this text serves only to motivate the introduction of non recursive constructors. We have to quantify over all possible choices of new constructors. We will have either constructors, that give as a natural number (functions $\omega^l \to \omega$ for some l), or functions, that gives us an element of the universe, in order to get elements of the powerset of $\mathbb{N}$, but we only need the elements $\widehat{\mathbb{N}}_0$ (for is not element) and $\widehat{\mathbb{N}}_1$ (for is an element).

Definition 6.1 *(a) We assume, that we have chosen some Gödel-numbers* $\ulcorner S \urcorner$ *for all symbols* S *of* ML.

(b) A triple $\langle \ulcorner C \urcorner, l, f\rangle$ *is a constructor definition, if* C, l *are natural numbers (*C *is a Gödel number for the constructor), such that* C *is different from the Gödel-numbers for the symbols,* $l > 0$ *and* f *is a function* $f : \omega^{l-1} \to \omega$ *or* $l = 0 \wedge f : \omega \to \{\ulcorner \widehat{\mathbb{N}}_0 \urcorner, \ulcorner \widehat{\mathbb{N}}_1 \urcorner\}$*. In this situation we define* $\mathrm{arity}(C) := \max\{l-1, 0\}$.

(c) A constructor extension set is a finite set of constructor definitions, such that the Gödel-numbers for the constructors are different. We write $\mathrm{CES}(a_0)$ *for* a_0 *being a constructor extension set and* $a_0 \in \mathrm{Ad}_2$.

(d) The a_0*-extended g-terms, g-types, b-objects are defined as the g-terms, g-types, b-objects, but having in addition for each element* $\langle C, l, f\rangle$ *of* a_0 *a term con-*

structor Constr_C *of arity* $\mathrm{arity}(C)$*, and allowing to form a g-term* $\mathrm{Constr}_C(r_1, \dots, r_n)$ *for g-terms* r_i *and* $n = \mathrm{arity}(C)$*. For simplicity we write usually* C *instead of* Constr_c*. Let* Term_{Cl,a_0} *be the set of closed* a_0*-extended g-terms.*

(e) $\forall CES(a_0).\phi(a_0) :\equiv \forall a_0 \in \mathrm{Ad}_2.\mathrm{CES}(a_0) \to \phi(a_0)$

Assumption 6.2 *As long as there is no other* CES *mentioned, let* a_0 *be a* CES *s.t.* $a_0 \in \mathrm{Ad}_2$*. Most of the next definitions will depend on* a_0*, which we will not mention in the following. If we have to mention it, we will add it as a subscript.*

Definition 6.3 *(a) We choose some Gödelization of* a_0*-extended b-terms, but will omit the Gödel-brackets.*

(b) The introductory term constructors are the term constructors 0, r, $\widehat{\mathbb{N}}$, S, i, j, p, sup, $\widehat{+}$, $\widehat{\Pi}$, $\widehat{\Sigma}$, $\widehat{\mathrm{W}}$, $\widehat{\mathrm{I}}$*;* $\widehat{\mathbb{N}}_k$ *for* $k \in \omega$*, and* A_n^k *for* $n < k \in \omega$ *.*

(c) Let $\to_{\mathrm{red,imm}_{a_0}}$ *or short* $\to_{\mathrm{red,imm}}$ *be the relation between closed* a_0*-extended g-terms, defined by*

$\mathrm{p}_0(\mathrm{p}(r,s)) \to_{\mathrm{red,imm}} r$ $\qquad$ $\mathrm{p}_1(\mathrm{p}(r,s)) \to_{\mathrm{red,imm}} s$

$\mathrm{Ap}(\lambda x.r, s) \to_{\mathrm{red,imm}} r'[x/s]$ *where* $r' =_\alpha r$ *s.t.* $r'[x/s]$ *is allowed and* r' *is chosen minimal w.r.t. the choice of variables substituted in lexicographic order.*

$\mathrm{C}_n(\mathrm{A}_i^n, r_1, \dots, r_n) \to_{\mathrm{red,imm}} r_i$

$\mathrm{D}(\mathrm{i}(r), s, t) \to_{\mathrm{red,imm}} s\ r$ $\qquad$ $\mathrm{D}(\mathrm{j}(r), s, t) \to_{\mathrm{red,imm}} t\ r$

(note that we write $r\ s$ *for* $\mathrm{Ap}(r,s)$*)*

$\mathrm{P}(0, s, t) \to_{\mathrm{red,imm}} s$ $\qquad$ $\mathrm{P}(\mathrm{S}(r), s, t) \to_{\mathrm{red,imm}} t\ r\ \mathrm{P}(r,s,t)$

$\mathrm{R}(\mathrm{sup}(r,s), t) \to_{\mathrm{red,imm}} (t\ r\ s\ (\lambda z_i.\mathrm{R}(s\ z_i, t)))$*, where* i *is minimal such that* $z_i \notin \mathrm{FV}(s) \cup \mathrm{FV}(t)$

$C(\mathrm{S}^{n_1}(0), \dots, \mathrm{S}^{n_l}(0)) \to_{\mathrm{red,imm}_{a_0}} \mathrm{S}^{f(n_1,\dots,n_l)}(0)$*, (if* $\langle C, l+1, f\rangle \in a_0$*),*

$C(\mathrm{S}^n(0)) \to_{\mathrm{red,imm}_{a_0}} f(n)$ *(if* $\langle C, 0, f\rangle \in a_0$*, and* $f : \omega \to \{\widehat{\mathbb{N}}_0, \widehat{\mathbb{N}}_1\}$*)*

(d) We define inductively a set of (indices for) terms in normal-form $\mathrm{Term}_{\mathrm{nf}}$*, a subset of the closed* a_0*-extended g-terms:*

If C *is an introductory* n*-ary term constructor,* $t_1, \dots, t_n \in \mathrm{Term}_{\mathrm{nf}}$*, then* $C(t_1, \dots, t_n) \in \mathrm{Term}_{\mathrm{nf}}$*.*

If C *is a* n*-ary term constructor (possibly an extended term constructor) that is not introductory,* $t_1, \dots, t_n \in \mathrm{Term}_{\mathrm{nf}}$*, and there exists no* t *such that* $C(t_1, \dots, t_n) \to_{\mathrm{red,imm}} t$*, then* $C(t_1, \dots, t_n) \in \mathrm{Term}_{\mathrm{nf}}$*.*

If $t \in \mathrm{Term}$*,* $x \in \mathrm{Var}_{\mathrm{ML}}$*,* $\mathrm{FV}(t) \subset \{x\}$*, then* $\lambda x.t \in \mathrm{Term}_{\mathrm{nf}}$*.*

(e) We define for a_0*-extended g-terms* t*, the next reduced term* t^{red}*.*

For $t \in \mathrm{Term}_{\mathrm{nf}}$*.* $t^{red} := t$*.*

If C *is a* n*-ary (possibly extended) term constructor,* $r_i \in \mathrm{Term}_{\mathrm{Cl}}$*,* $\exists i.r_i \notin \mathrm{Term}_{\mathrm{nf}}$*, then* $C(r_1, \dots, r_n)^{red} := C(r_1^{red}, \dots, r_n^{red})$*.*

If $t := C(r_1, \dots, r_n) \notin \mathrm{Term}_{\mathrm{nf}}$*,* $r_i \in \mathrm{Term}_{\mathrm{nf}}$*, then* $t \to_{\mathrm{red,imm}} t'$ *for some* t'*,* $t^{red} := t'$*.*

We define $r \rightarrow_{red} s$ if and only if there exists a sequence $\langle s_0, \ldots, s_n \rangle$ such that $r = s_0$, $s = s_n$ and $\forall i < n.s_{i+1} = (s_i)^{red}$.

Lemma 6.4 *(a)* $\mathrm{KPI}^+ \vdash \forall r, s, s' \in \mathrm{Term}_{\mathrm{Cl}}.(r \rightarrow_{\mathrm{red}} s \wedge r \rightarrow_{\mathrm{red}} s') \rightarrow (s \rightarrow_{\mathrm{red}} s' \vee s' \rightarrow_{\mathrm{red}} s)$.

(b) $\mathrm{KPI}^+ \vdash \forall r, s, s' \in \mathrm{Term}_{\mathrm{Cl}}.(r \rightarrow_{\mathrm{red}} s \wedge r \rightarrow_{\mathrm{red}} s' \wedge s, s' \in \mathrm{Term}_{\mathrm{nf}}) \rightarrow s = s'$.

(c) If C is a n-ary constructor, then

$$\begin{array}{l} \mathrm{KPI}^+ \vdash \forall r_1, \ldots, r_n, r'_1, \ldots, r'_n. \\ \qquad (r_1 \rightarrow_{\mathrm{red}} r'_1 \in \mathrm{Term}_{\mathrm{nf}} \wedge \cdots \wedge r_n \rightarrow_{\mathrm{red}} r'_n \in \mathrm{Term}_{\mathrm{nf}}) \\ \qquad \rightarrow (C(r_1, \ldots, r_n) \rightarrow_{\mathrm{red}} C(r'_1, \ldots, r'_n)) \end{array}$$

(d) $\mathrm{KPI}^+ \vdash \forall t, t', s \in \mathrm{Term}_{\mathrm{Cl}}.(t \rightarrow_{\mathrm{red}} s \wedge t =_\alpha t') \rightarrow \exists s' \in \mathrm{Term}_{\mathrm{Cl}}.t' \rightarrow_{\mathrm{red}} s' \wedge s =_\alpha s'$.

(e) $\mathrm{KPI}^+ \vdash \forall t, t' \in \mathrm{Term}_{\mathrm{Cl}}.t =_\alpha t' \rightarrow (t \in \mathrm{Term}_{\mathrm{nf}} \iff t' \in \mathrm{Term}_{\mathrm{nf}})$.

Definition 6.5 *If F is a Σ function, we define by recursion on $\alpha \in \mathrm{Ord}$*

$$F^\alpha := \begin{cases} \emptyset & \text{if } \alpha = 0, \\ F(F^\beta) & \text{if } \alpha = \beta + 1, \\ \bigcup_{\beta < \alpha, \beta \in \mathrm{Ord}} F^\beta & \text{if } \alpha \in \mathrm{Lim}. \end{cases}$$

Definition 6.6 *(a) Let* Compl *be the Σ-function*

$$\begin{array}{rl} \mathrm{Compl}(u) := & \{\langle r, s \rangle \in \mathrm{Term}_{cl} \times \mathrm{Term}_{cl} | \exists r', s' \in \mathrm{Term}_{a,\mathrm{nf}}. \\ & \qquad r \rightarrow_{\mathrm{red}} r' \wedge s \rightarrow_{\mathrm{red}\, a} s' \wedge \langle r', s' \rangle \in u\} \end{array}$$

(b) $\mathbb{N}_k^{\mathrm{basis}} := \{\langle \mathrm{A}_n^k, \mathrm{A}_n^k \rangle | n < k\}$, $\mathbb{N}_k^{**} := \mathrm{Compl}(\mathbb{N}_k^{\mathrm{basis}})$, *which are Σ-functions, depending on the parameter k.*

(c) $\mathbb{N}^{\mathrm{basis}} := \{\langle \mathrm{S}^n(0), \mathrm{S}^n(0) \rangle | n < \omega\}$, $\mathbb{N}_k^{**} := \mathrm{Compl}(\mathbb{N}^{\mathrm{basis}})$.

(d)

$$\begin{array}{rl} \Pi^{\mathrm{basis}}(u, f) := & \{\langle \lambda x.s, \lambda x'.s' \rangle \in \mathrm{Term}_{\mathrm{nf}} \times \mathrm{Term}_{\mathrm{nf}} | \\ & \quad \forall \langle r, r' \rangle \in u. \langle s[x/r], s'[x'/r'] \rangle \in f(r) \wedge f(r) = f(r')\} \end{array}$$

(more precisely we have to write:

$$\begin{array}{rl} \Pi^*(u, f) := & \{\langle t, t' \rangle \in \mathrm{Term}_{\mathrm{nf}} \times \mathrm{Term}_{\mathrm{nf}} | \\ & \quad \exists x, x' \in \mathrm{Var}_{\mathrm{ML}}, r, r' \in \mathrm{Term}.t = \lambda x.s \wedge t' = \lambda x'.s' \wedge \\ & \quad \forall r, r' \in \mathrm{Term}_{\mathrm{Cl}}. \langle r, r' \rangle \in u \rightarrow \\ & \qquad \langle s[x/r], s[x'/r'] \rangle \in f(r) \wedge f(r) = f(r')\} \end{array}$$

similarly in the following definitions)
$\Pi^*(u, f) := \mathrm{Compl}(\Pi^{\mathrm{basis}}(u, f))$.

(e)

$$\begin{aligned}\Sigma^{\mathrm{basis}}(u,f) := \quad & \{\langle \mathrm{p}(r,s), \mathrm{p}(r',s')\rangle \in \mathrm{Term}_{\mathrm{nf}} \times \mathrm{Term}_{\mathrm{nf}} | \\ & \quad \langle r,r'\rangle \in u \wedge \langle s,s'\rangle \in f(r) \wedge f(r) = f(r')\}\end{aligned}$$

$\Sigma^*(u,f) := \mathrm{Compl}(\Sigma^{\mathrm{basis}}(u,f))$.

(f) *Let* $\lambda^*(u) := \{\langle t,u\rangle | t \in \mathrm{Term}_{\mathrm{Cl}}\}$.

(g) $\mathrm{W}^*(u,f,\alpha) := F_{\mathrm{W}}(u,f)^{\alpha}$,
where $F_{\mathrm{W}}(u,f)(v) = \mathrm{Compl}(F^{\mathrm{basis}}_{\mathrm{W}}(u,f)(v))$, *and*

$$\begin{aligned}F^{\mathrm{basis}}_{\mathrm{W}}(u,f)(v) := \quad & \{\langle \sup(r,s), \sup(r',s')\rangle \in \mathrm{Term}_{\mathrm{nf}} \times \mathrm{Term}_{\mathrm{nf}} | \\ & \quad \langle r,r'\rangle \in u \wedge \langle s,s'\rangle \in \Pi^{\mathrm{basis}}(f(r), \lambda^*(v)) \wedge f(r) = f(r')\}\end{aligned}$$

(h)

$$\begin{aligned}+^{\mathrm{basis}}(u,v) := \quad & \{\langle \mathrm{i}(r), \mathrm{i}(r')\rangle \in \mathrm{Term}_{\mathrm{nf}} \times \mathrm{Term}_{\mathrm{nf}} | \langle r,r'\rangle \in u\} \\ \cup \quad & \{\langle \mathrm{j}(r), \mathrm{j}(r')\rangle \in \mathrm{Term}_{\mathrm{nf}} \times \mathrm{Term}_{\mathrm{nf}} | \langle r,r'\rangle \in v\}\end{aligned}$$

$+^*(u,v) := \mathrm{Compl}(+^{\mathrm{basis}}(u,v))$.

(i)

$$\mathrm{I}^{\mathrm{basis}}(u,s,t) := \{\langle \mathrm{r},\mathrm{r}\rangle \in \mathrm{Term}_{\mathrm{nf}} \times \mathrm{Term}_{\mathrm{nf}} | \langle s,t\rangle \in u\}$$

$\mathrm{I}^*(u,s,t,) := \mathrm{Compl}(\mathrm{I}^{\mathrm{basis}}(u,s,t))$.

We will interpret each g-type occurring in a proof of Martin-Löf's type theory as a Σ-function, with arguments represented by the free variables of the type. More precisely, if $\mathrm{FV}(A) = \{z_1, \dots, z_n\}$, ($z_i$ as in the definition 2.1 of $\mathrm{Var}_{\mathrm{ML}}$) the arguments of the interpretation A^* will have arguments given by the variables $\{u_1, \dots, u_n\}$ (u_i as in definition 5.1 (a) of $\mathrm{Var}_{\mathrm{KP}}$). We introduce the following abbreviation:

Definition 6.7 *(a) If A is a Σ function in* KPI^+, *u_i as in the definition 5.1 (a) of* $\mathrm{Var}_{\mathrm{KP}}$ *z_i as in the definition 2.1 of* $\mathrm{Var}_{\mathrm{ML}}$, *$r_1, \dots, r_m$ extended b-objects,*

$$A[z_{i_1}/r_1, \dots, z_{i_n}/r_n] := A[u_{i_1}/r_{j_1}, \dots, u_{i_n}/r_{j_n}],$$

where on the right-hand side we have the real substitution in KPI^+ *in such a way, that, if a variable occurs more than once, only the first occurrence is carried out. If we introducing symbols for Σ-functions, this substitution is the application of the Σ-function to the arguments in the ordering as specified by the definition of the function.*
We will write $A[\vec{x}/\vec{n}]$ for $A[x_1/n_1, \dots, x_n/n_n]$.

(b) In the situation as above let $(z_i)A$ be the Σ-function with the same arguments as A except u_i s.t. $((z_i)A)[\vec{x}/\vec{r}] = \{\langle u, A[z_i/u, \vec{x}/\vec{r}]\rangle | u \in \mathrm{Term}_{\mathrm{Cl}}\}$.

Definition 6.8 *We define for every g-type A the Σ-function A^* together with* $\mathrm{lev}(A) \in \omega$.
If $\mathrm{FV}(A) = \{z_1, \ldots, z_n\}$, *($z_i$ as in definition 2.1 of* $\mathrm{Var}_{\mathrm{ML}}$*), A^* will have arguments given by the variables $\{u_1, \ldots, u_n\}$ (u_i as in definition 5.1 (a) of* $\mathrm{Var}_{\mathrm{KP}}$*). We will define it by giving the values $A^*[\vec{x}/\vec{s}]$.*
Let for $A \subseteq \mathrm{Term}_{\mathrm{nf}} \times \mathrm{Term}_{\mathrm{nf}}$, t possibly depending on r
$\lambda\langle r, \cdot\rangle \in A.t := \{\langle r, t\rangle \mid \langle r, r\rangle \in A\}$.

For $k \in \omega$, $\mathbb{N}_k^*[\,] := \mathbb{N}_k^{**}$, $\mathrm{lev}(\mathbb{N}_k) := 0$.
$\mathbb{N}^*[\,] := \mathbb{N}^{**}$, $\mathrm{lev}(\mathbb{N}) := 0$.
Let A, B be g-types, $m := \max\{\mathrm{lev}(A), \mathrm{lev}(B)\}$.
$\mathrm{lev}(\Pi x \in A.B) := m$,
$(\Pi x \in A.B)^*[\vec{x}/\vec{s}] := \Pi^*(A^*[\vec{x}/\vec{s}], \lambda\langle r, \cdot\rangle \in A^*[\vec{x}/\vec{s}].B^*[\vec{x}/\vec{s}, x/r])$,
$\mathrm{lev}(\Sigma x \in A.B) := m$,
$(\Sigma x \in A.B)^*[\vec{x}/\vec{s}] := \Sigma^*(A^*[\vec{x}/\vec{s}], \lambda\langle r, \cdot\rangle \in A^*[\vec{x}/\vec{s}].B^*[\vec{x}/\vec{s}, x/r])$,
$\mathrm{lev}(\mathrm{W} x \in A.B) := m + 1$,
$(\mathrm{W} x \in A.B)^*[\vec{x}/\vec{s}] := \mathrm{W}^*(A^*[\vec{x}/\vec{s}], \lambda\langle r, \cdot\rangle \in A^*[\vec{x}/\vec{s}].B^*[\vec{x}/\vec{s}, x/r], \mathrm{I}_m)$,
$\mathrm{lev}(A + B) := m$, $(A + B)^*[\vec{x}/\vec{s}] := +^*(A^*[\vec{x}/\vec{s}], B^*[\vec{x}/\vec{s}])$,
$\mathrm{lev}(\mathrm{I}(A, s, t)) := \mathrm{lev}(A)$, $(\mathrm{I}(A, s, t))^*[\vec{x}/\vec{s}] := \mathrm{I}^*(A^*[\vec{x}/\vec{s}], s[\vec{x}/\vec{s}], t[\vec{x}/\vec{s}])$.
$\mathrm{lev}(\mathrm{U}) := 1$, $\mathrm{U}^*[\,] := \sim(\widehat{\mathrm{U}})$,
$\mathrm{lev}(\mathrm{T}(t)) := 0$, $(\mathrm{T}(t))^*[\vec{x}/\vec{s}] := \mathrm{func}(\widehat{\mathrm{U}})(t[\vec{x}/\vec{s}])$,
where $\widehat{\mathrm{U}}$, $\sim(u)$, $\mathrm{func}(u)$ *will be defined in the next definition.*

Definition 6.9

(a) $\sim(u) := \{\langle s, s'\rangle \in \mathrm{Term}_{\mathrm{Cl}} \times \mathrm{Term}_{\mathrm{Cl}} | \exists v \in \mathrm{TC}(u).\langle s, v, s'\rangle \in u\}$.

(b) $\mathrm{func}(u) := \{\langle s, v\rangle \in \mathrm{Term}_{\mathrm{Cl}} \times \mathrm{TC}(u) | \exists s' \in \mathrm{Term}_{\mathrm{Cl}}.\langle s, v, s'\rangle \in u\}$.

(c) $\mathrm{Compl}_{\mathrm{U}}(u) := \{\langle r, b, r'\rangle \in \mathrm{Term}_{\mathrm{Cl}} \times \mathrm{TC}(u) \times \mathrm{Term}_{\mathrm{Cl}} | \exists s, s' \in \mathrm{Term}_{\mathrm{nf}}.$
$r \rightarrow_{\mathrm{red}} s \wedge r' \rightarrow_{\mathrm{red}} s' \wedge \langle s, b, s'\rangle \in u\}$.

(d) $\widetilde{\mathrm{U}}(u) := \mathrm{Compl}_{\mathrm{U}}(\widetilde{\mathrm{U}}^{\mathrm{basis}}(u))$, *where*

$$
\begin{aligned}
&\widetilde{\mathrm{U}}^{\mathrm{basis}}(u) := \\
&\{\langle \widehat{\mathbb{N}}_k, \mathbb{N}_k^{**}, \widehat{\mathbb{N}}_k\rangle \in \mathrm{ad}(u) | k \in \omega\} \\
\cup\ &\{\langle \widehat{\mathbb{N}}, \mathbb{N}^{**}, \widehat{\mathbb{N}}\rangle\} \\
\cup\ &\{\langle \widehat{\Pi} x \in r.s, \Pi^*(b, f), \widehat{\Pi} x' \in r'.s'\rangle | \phi(r, x, s, r', x', s', b, f, u) \wedge b, f \in \mathrm{ad}(u)\} \\
\cup\ &\{\langle \widehat{\Sigma} x \in r.s, \Sigma^*(b, f), \widehat{\Sigma} x' \in r'.s'\rangle | \phi(r, x, s, r', x', s', b, f, u) \wedge b, f \in \mathrm{ad}(u)\} \\
\cup\ &\{\langle \widehat{\mathrm{W}} x \in r.s, \mathrm{W}^*(b, f, \alpha^+(u)), \widehat{\mathrm{W}} x' \in r'.s'\rangle | \\
&\qquad \phi(r, x, s, r', x', s', b, f, u) \wedge b, f \in \mathrm{ad}(u)\} \\
\cup\ &\{\langle r \widehat{+} s, +^*(b, c), r' \widehat{+} s'\rangle \in \mathrm{ad}(u) | \psi_+(r, s, r', s', b, c, u) \wedge b, c \in \mathrm{ad}(u)\} \\
\cup\ &\{\langle \widehat{\mathrm{I}}(r, s, t), \mathrm{I}^*(b, s, t), \widehat{\mathrm{I}}(r', s', t')\rangle \in \mathrm{ad}(u) | \\
&\qquad\qquad \psi_i(r, s, t, r', s', t', b, u) \wedge b \in \mathrm{ad}(u)\}
\end{aligned}
$$

and

$$\begin{aligned}&\phi(r,x,s,r',x',s',b,f,u)\\ &\quad := \ r,r' \in \mathrm{Term}_{\mathrm{nf}} \wedge s,s' \in \mathrm{Term}\\ &\qquad\quad \wedge \mathrm{FV}(s) \subset \{x\} \wedge \mathrm{FV}(s') \subset \{x'\} \wedge \langle r,b,r'\rangle \in u \wedge\\ &\qquad\quad (\forall \langle t,t'\rangle \in b. \langle s[x/t], f(t), s'[x'/t']\rangle \in u)\end{aligned}$$

(note that $f(t) = \bigcup\{c \in \mathrm{TC}(f) | \langle t,c\rangle \in f\}$*)*

$$\psi_+(r,s,r',s',b,c,u) := r,s,r',s' \in \mathrm{Term}_{\mathrm{nf}} \wedge \langle r,b,r'\rangle \in u \wedge \langle s,c,s'\rangle \in u,$$

$$\begin{aligned}\psi_i(r,s,t,r',s',t',b,u) \ := \ & r,s,t,r',s',t' \in \mathrm{Term}_{\mathrm{nf}} \wedge \langle r,b,r'\rangle \in u \wedge\\ & \langle s,s'\rangle \in b \wedge \langle t,t'\rangle \in b,\end{aligned}$$

(e) $\widehat{\mathrm{U}} := \widetilde{\mathrm{U}}^{\mathrm{I}}$.

7 Properties of the Interpretation

Lemma 7.1 *(a)* $\forall v \subset v'. F^*_{\mathrm{W}}(b,f)(v) \subset F^*_{\mathrm{W}}(b,f)(v')$.

(b) $\forall \gamma < \delta. \mathrm{W}^*(b,f,\gamma) \subset \mathrm{W}^*(b,f,\delta)$

(c) $(b \in a \wedge f \in a \wedge \mathrm{Ad}(a)) \to \forall \gamma > \alpha(a). \mathrm{W}^*(b,f,\gamma) = \mathrm{W}^*(b,f,\alpha(a))$.

Proof: (a) immediate, (b) follows from (a) by induction on δ.
(c) It is sufficient to show, with $\alpha := \alpha(u)$, $v := \mathrm{W}^*(b,f,\alpha)$, that $F_{\mathrm{W}}(b,f)(v) \subset v$. Since $\mathrm{Compl}(v) \subset v$ it is sufficient to prove $F^{\mathrm{basis}}_{\mathrm{W}}(b,f)(v) \subset v)$.
Now, if $\langle \sup(r,t), \sup(r',t')\rangle \in F^{\mathrm{basis}}_{\mathrm{W}}(b,f)(v)$, then $t = \lambda x.s$, $t' = \lambda x'.s'$, and $\forall u,u' \in \mathrm{Term}_{\mathrm{Cl}}. \langle u,u'\rangle \in f(r) \to \exists \delta \in \mathrm{Ord} \cap \alpha. \langle s[x/u], s'[x'/u']\rangle \in \mathrm{W}^*(b,f,\delta)$
Since $\mathrm{Ad}(a)$, we have $(\Delta_0 - \mathrm{coll})^a$, therefore for some $\rho < \alpha$, $\forall t,t' \in \mathrm{Term}_{\mathrm{Cl}}. \langle t,t'\rangle \in f(r) \to \exists \delta < \rho. \langle s[x/t], s'[x'/t']\rangle \in \mathrm{W}^*(b,f,\delta)$.
Now it follows $\langle \sup(r,\lambda x.s),\ \sup(r',\lambda x'.s')\rangle \in \mathrm{W}^*(b,f,\rho) \subset v$ and the assertion.

Definition 7.2 *(a)* $\mathrm{equiv}(u) :\Leftrightarrow$

$$\begin{aligned}&\forall r,s,t,r',s' \in \mathrm{Term}_{\mathrm{Cl}}.(\langle r,s\rangle \in u \to \langle s,r\rangle \in u) \wedge\\ &((\langle r,s\rangle \in u \wedge \langle s,t\rangle \in u) \to \langle r,t\rangle \in u).\end{aligned}$$

(note that we do not claim reflexivity)

(b) $\mathrm{equivfun}(f) :\Leftrightarrow \forall x \in \mathrm{dom}(f).\mathrm{equiv}(f(x))$.

(c) $\mathrm{Cor}(u) :\Leftrightarrow$

$$\begin{aligned}&\forall r,r',r'' \in \mathrm{Term}_{\mathrm{Cl}}. \forall b,b'.(\langle r,b,r'\rangle \in u \to [\langle r',b,r\rangle \in u \wedge \mathrm{equiv}(b)\\ &\quad \wedge [\langle r',b',r''\rangle \in u \to (\langle r,b,r''\rangle \in u \wedge b = b')]])\end{aligned}$$

Remark 7.3 *(a)* $(\mathrm{Cor}(u) \wedge \langle r,b,r'\rangle \in u \wedge \langle r,b',r''\rangle \in u) \to (b = b' \wedge \langle r,b,r''\rangle \in u)$.

(b) If $\mathrm{Cor}(u)$ *then with* $\sim := \sim(u)$, $f := \mathrm{func}(u)$ *we have* $\sim$ *is a symmetric and transitive relation,* f *is a function s.t.* $\forall a,b.a \sim b \to f(a) = f(b)$ *and* $\mathrm{equivfun}(f)$.

Lemma 7.4 *(a)* $(\mathrm{equiv}(u) \wedge u \subset \mathrm{Term}_{\mathrm{nf}} \times \mathrm{Term}_{\mathrm{nf}}) \to \mathrm{equiv}(\mathrm{Compl}(u))$.

(b) $(\mathrm{equiv}(u) \wedge \mathrm{equiv}(v) \wedge \mathrm{equivfun}(f) \wedge k \in \omega \wedge s, t \in \mathrm{Term}_{\mathrm{Cl}}) \to$
$(\mathrm{equiv}(\mathbb{N}^{\mathrm{basis}}) \wedge \mathrm{equiv}(\mathbb{N}_k^{\mathrm{basis}}) \wedge \mathrm{equiv}(\Pi^{\mathrm{basis}}(u, f)) \wedge \mathrm{equiv}(\Sigma^{\mathrm{basis}}(u, f)) \wedge$
$\mathrm{equiv}(F_{\mathrm{W}}^{\mathrm{basis}}(u, f)(v)) \wedge \mathrm{equiv}(+^{\mathrm{basis}}(u, v)) \wedge \mathrm{equiv}(\mathrm{i}^{\mathrm{basis}}(u, s, t)))$.

(c) $(\mathrm{equiv}(u) \wedge \mathrm{equiv}(v) \wedge \mathrm{equivfun}(f) \wedge \alpha \in \mathrm{Ord} \wedge k \in \omega \wedge s, t \in \mathrm{Term}_{\mathrm{Cl}}) \to$
$(\mathrm{equiv}(\mathbb{N}^*) \wedge \mathrm{equiv}(\mathbb{N}_k^*) \wedge \mathrm{equiv}(\Pi^*(u, f)) \wedge \mathrm{equiv}(\Sigma^*(u, f))$
$\wedge \mathrm{equiv}(F_{\mathrm{W}}^*(u, f)(v)) \wedge \mathrm{equiv}(\mathrm{W}^*(u, f, \alpha))$
$\wedge \mathrm{equiv}(+^*(u, v)) \wedge \mathrm{equiv}(\mathrm{i}^*(u, s, t)))$.

Lemma 7.5 *Assume* $r, s, t, r', s', t' \in \mathrm{Term}$, $x, x' \in \mathrm{Var}_{\mathrm{ML}}$, b, f, u *sets.*

(a) $(\phi(r, x, s, r', x', s', b, f, u) \wedge \mathrm{Cor}(u)) \to$
$(b \in \mathrm{ad}(u) \wedge \exists f \in \mathrm{ad}(u).\forall \langle t, t' \rangle \in b.f(t) = f(t') = f'(t) = f'(t'))$.

(b) $\psi_+(r, s, r', s', b, c, u) \to b, c \in \mathrm{ad}(u)$.

(c) $\psi_i(r, s, t, r', s', t', b, u) \to b \in \mathrm{ad}(u)$.

(d) $\mathrm{Cor}(u) \to \widetilde{\mathrm{U}}(u) \in \mathrm{ad}(\mathrm{ad}(u))$.

(e) $\forall \gamma \in \mathrm{Ord} \cap \mathrm{I}.\widetilde{\mathrm{U}}^{\gamma} \in \mathrm{Ad}_{\mathrm{I}}$.

Proof:
(a) $b \in \mathrm{TC}(u) \in \mathrm{ad}(u)$.
Let $f' := \{\langle t, c \rangle \in \mathrm{Term}_{\mathrm{Cl}} \times \mathrm{TC}(u) | \langle t, t \rangle \in b \wedge \langle s[x/t], c, s[x/t] \rangle \in u\}$.
$f' \in \mathrm{ad}(u)$. Furthermore, if $\langle t, t' \rangle \in b$, it follows $\langle s[x/t], f(t), s'[x/t'] \rangle \in u$, by $\mathrm{Cor}(u)$ $\langle s[x/t], f(t), s[x/t] \rangle \in u$, $f(t) = f'(t)$, and, since
$\langle s'[x/t], f(t), s[x/t] \rangle \in u \wedge \langle s'[x/t], f(t'), s[x/t'] \rangle \in u$, it follows $f(t) = f(t')$.
(b), (c),(d): easy.
(e): Induction on γ, using (d) and 5.3 (b).

Lemma 7.6 *Assume* $r, s, t, r', s', t', r'', s'', t'' \in \mathrm{Term}$, $x, x', x'' \in \mathrm{Var}_{\mathrm{ML}}$, b, b', f, f', u, u' *sets.*

(a) $\phi(r, x, s, r', x', s', b, f, u) \wedge \mathrm{Cor}(u) \to \phi(r', x', s', r, x, s, b, f, u)$.

(b) $(\phi(r, x, s, r', x', s', b, f, u) \wedge \phi(r', x', s', r'', x'', s'', b', f', u') \wedge \mathrm{Cor}(u \cup u')) \to$
$\phi(r, x, s, r'', x'', s'', b, f, u \cup u') \wedge b = b' \wedge$
$\forall \langle t, t' \rangle \in b.f(t) = f'(t) = f(t') = f'(t')$

(c) $\psi_+(r, s, r', s', b, c, u) \wedge \mathrm{Cor}(u) \to \psi_+(r', s', r, s, b, c, u)$.

(d) $(\psi_+(r, s, r', s', b, c, u) \wedge \psi_+(r', s', r'', s'', b', c', u') \wedge \mathrm{Cor}(u \cup u') \to$
$(\psi_+(r, s, r'', s'', b, c, u \cup u') \wedge b = b' \wedge c = c'))$

(e) $\psi_i(r, s, t, r', s', t', b, u) \wedge \mathrm{Cor}(u) \to \psi_i(r', s', t', r, s, t, b, u)$.

(f) $(\psi_i(r,s,t,r',s',t',b,u) \wedge \psi_i(r',s',t',r'',s'',t'',b',u') \wedge \mathrm{Cor}(u \cup u') \rightarrow$
$(\psi_i(r,s,t,r'',s'',t'',b,u \cup u') \wedge b = b'))$

Lemma 7.7 *(a)* $(\mathrm{Cor}(u) \wedge \sim(u) \subset \mathrm{Term_{nf}} \times \mathrm{Term_{nf}}) \rightarrow \mathrm{Cor}(\mathrm{Compl_U}(u))$.

(b) $\mathrm{Cor}(u) \rightarrow \mathrm{Cor}(\widetilde{\mathrm{U}}(u))$,

(c) $u \subset u' \wedge \mathrm{Cor}(u') \rightarrow \widetilde{\mathrm{U}}(u) \subset \widetilde{\mathrm{U}}(u')$,

(d) $\mathrm{Cor}(\widehat{\mathrm{U}})$.

Lemma 7.8 *If* A *g-type, then* $\mathrm{KPI}^+ \vdash \forall s_1,\ldots,s_n \in \mathrm{Term_{Cl}}.\mathrm{equiv}(A^*[\vec{x}/\vec{s}])$

Proof: Induction on the definition of types.

Definition 7.9 *Let* A, B *g-types,* s, t *g-terms,* $\mathrm{FV}(A)$, $\mathrm{FV}(B)$, $\mathrm{FV}(s)$, $\mathrm{FV}(t) \subset \{x_1,\ldots,x_n\}$, $r_1,\ldots,r_n$, $s_1,\ldots,s_n$ *be extended g-terms.*

(a) $(A = B : \mathrm{type})^*[\vec{x}/\vec{r};\vec{s}] \;:\Leftrightarrow\; (A = B : \mathrm{type})^*[x_1/r_1;s_1,\ldots,x_n/r_n;s_n] \;:\Leftrightarrow$ $(A^*[\vec{x}/\vec{r}] = B^*[\vec{x}/\vec{s}])$.

(b) $(t = t' : A)^*[\vec{x}/\vec{r};\vec{s}] :\Leftrightarrow (t = t' : A)^*[x_1/r_1;s_1,\ldots,x_n/r_n;s_n] :\Leftrightarrow$ $\langle t[\vec{x}/\vec{r}], t'[\vec{x}/\vec{s}]\rangle \in A^*[\vec{x}/\vec{r}]$.

We will not mention the variables $x_1,\ldots,x_n$ explicitly, if they are the variables, mentioned in the context, writing $(A = B : \mathrm{type})^*[\vec{r};\vec{s}]$ and $(t = t' : A)^*[\vec{r};\vec{s}]$.
Note that $s : A$, $A : \mathrm{type}$ abbreviate $s = s : A$, $A = A : \mathrm{type}$, therefore $(s : A)^*[\vec{r};\vec{s}]$, $(A : \mathrm{type})^*[\vec{r};\vec{s}]$ are defined as well.

Lemma 7.10 *(Substitution Lemma).*
Let C, D *be g-types,* r, s, t_i, t'_i *g-terms,* $x_i, y_i \in \mathrm{Var_{ML}}$. *Then:*

(a) *If* $r[\vec{x}/\vec{t}]$ *is an allowed substitution,* $\mathrm{FV}(r[\vec{x}/\vec{t}]) \subset \{y_1,\ldots,y_n\}$, *then*
$\mathrm{KPI}^+ \vdash \forall \vec{r} \in \mathrm{Term_{Cl}}.r[\vec{x}/\vec{t}][\vec{y}/\vec{r}] = r[x_1/t_1[\vec{y}/\vec{r}],\ldots,x_n/t_n[\vec{y}/\vec{r}],\vec{y}/\vec{r}]$.
(Note that, if variables occur more than once in $[\vec{y}/\vec{r}]$*, only the first substitution is relevant.)*

(b) *If* $C[\vec{x}/\vec{t}]$ *is an allowed substitution,* $\mathrm{FV}(C[\vec{x}/\vec{t}]) \subset \{y_1,\ldots,y_n\}$, *then*
$\mathrm{KPI}^+ \vdash \forall \vec{r}, r' \in \mathrm{Term_{Cl}}.C[\vec{x}/\vec{t}]^*[\vec{y}/\vec{r}] = C^*[x_1/t_1[\vec{y}/\vec{r}],\ldots,x_n/t_n[\vec{y}/\vec{r}],\vec{y}/\vec{r}]$.

(c) *If* $A[\vec{x}/\vec{t}]$, $B[\vec{x}/\vec{t'}]$ *are allowed substitutions,*
$\mathrm{FV}(A[\vec{x}/\vec{t}]), \mathrm{FV}(B[\vec{x}/\vec{t'}]) \subset \{y_1,\ldots,y_n\}$, *then*
$\mathrm{KPI}^+ \vdash \forall \vec{r}, \vec{s} \in \mathrm{Term_{Cl}}.(A = B : \mathrm{type})^*[\vec{x}/(\vec{t}[\vec{y}/\vec{r}]);(\vec{t'}[\vec{y}/\vec{s}]),\vec{y}/\vec{r};\vec{s}] \iff$
$(A[\vec{x}/\vec{t}] = B[\vec{x}/\vec{t'}] : \mathrm{type})^*[\vec{y}/\vec{r};\vec{s}]$.

(d) *If* $A[\vec{x}/\vec{t}]$, $r[\vec{x}/\vec{t}]$ *are allowed substitutions,* $\mathrm{FV}(A[\vec{x}/\vec{t}]), \mathrm{FV}(r[\vec{x}/\vec{t}]) \subset \{y_1,\ldots,y_n\}$, *then*
$\mathrm{KPI}^+ \vdash \forall \vec{r}, \vec{s} \in \mathrm{Term_{Cl}}.(r : A)^*[\vec{x}/(\vec{t}[\vec{x}/\vec{r}]);(\vec{t}[\vec{x}/\vec{s}]),\vec{x}/\vec{r};\vec{s}] \iff$
$(r[x/t] : A[x/t])^*[\vec{x}/\vec{r};\vec{s}]$.

(e) *If* $A[\vec{x}/\vec{t}]$, $r[\vec{x}/\vec{t}], s[\vec{x}/\vec{t'}]$ *are allowed substitutions,* $\mathrm{FV}(A[\vec{x}/\vec{t}])$, $\mathrm{FV}(r[\vec{x}/\vec{t}])$, $\mathrm{FV}(s[\vec{x}/\vec{t'}]) \subset \{y_1, \ldots, y_n\}$, *then*
$\mathrm{KPI}^+ \vdash \forall \vec{r}, \vec{s} \in \mathrm{Term}_{\mathrm{Cl}}.(r = s : A)^*[\vec{x}/(\vec{t}[\vec{x}/\vec{r}]); (\vec{t'}[\vec{x}/\vec{s}]), \vec{x}/\vec{r}; \vec{s}] \iff$
$(r[\vec{x}/\vec{t}] = s[x/\vec{t'}] : A[x/\vec{t}])^*[\vec{x}/\vec{r}; \vec{s}]$.

Proof by induction on the definition of the terms and types.

Lemma 7.11 *For every g-type* A $\mathrm{FV}(A) \subset \{x_1, \ldots, x_n\}$, *it follows*

(a) $\forall \vec{r}, r, r', s, s' \in \mathrm{Term}_{\mathrm{Cl}}.(r \to_{\mathrm{red}} r') \to (s \to_{\mathrm{red}} s')$
$\to \langle r, s\rangle \in A^*[\vec{x}/\vec{r}] \to \langle r', s'\rangle \in A^*[\vec{x}/\vec{r}]$.

(b) $\forall \vec{r}, r, r' \in \mathrm{Term}_{\mathrm{Cl}}.\langle r, r'\rangle \in A^*[\vec{x}/\vec{r}] \to \exists s, s' \in \mathrm{Term}_{\mathrm{nf}}.r \to_{\mathrm{red}} s \wedge r' \to_{\mathrm{red}} s'$.

Proof: easy, since for each type, Compl was applied to some set.

Definition 7.12 (a) $\mathrm{Stable}(a) := \forall r, s, r', s' \in \mathrm{Term}_{\mathrm{Cl}}.\langle r, s\rangle \in a \to r =_\alpha r' \to s =_\alpha s' \to \langle r', s'\rangle \in a$

(b) *For every g-type* A *with* $\mathrm{FV}(A) = \{x_1, \ldots, x_n\}$ *we define*
$\mathrm{Flex}(A) := \forall r_1, \ldots, r_n, s_1, \ldots, s_n \in \mathrm{Term}_{\mathrm{Cl}}.v(r_1 =_\alpha s_1 \wedge \cdots \wedge r_n =_\alpha s_n) \to$
$A^*[\vec{x}/\vec{r}] = A^*[\vec{x}/\vec{s}]$

Lemma 7.13 *For every g-type* C, D *with* $\mathrm{FV}(C) = \{x_1, \ldots, x_n\}$ *and* $C =_\alpha D$ *we have*

(a) $\mathrm{KPI}^+ \vdash \mathrm{Flex}(C)$

(b) $\mathrm{KPI}^+ \vdash \forall r_1, \ldots, r_n \in \mathrm{Term}_{\mathrm{Cl}}.\mathrm{Stable}(C^*[\vec{x}/\vec{r}])$

(c) $\mathrm{KPI}^+ \vdash \forall r_1, \ldots, r_n \in \mathrm{Term}_{\mathrm{Cl}}.C^*[\vec{x}/\vec{r}] = D^*[\vec{x}/\vec{r}]$.

Proof Easy, simultaneously by induction on the definition of g-types. In the case $C \equiv \mathrm{U}, \mathrm{T}(t)$ we define

$$\begin{aligned}\mathrm{Stable}_{\mathrm{U}}(u) := \quad & \forall s, s', t, t' \in \mathrm{Term}_{\mathrm{Cl}}.\forall b \in \mathrm{TC}(u).\\ & s =_\alpha s' \wedge t =_\alpha t' \wedge \mathrm{pair}(s, b, t) \in u\\ & \to (\mathrm{pair}(s', b, t') \in u \wedge \mathrm{Stable}(b))\end{aligned}$$

We conclude $\mathrm{Stable}_{\mathrm{U}}(u) \to \mathrm{Stable}_{\mathrm{U}}(\widetilde{\mathrm{U}}(u))$ and therefore $\mathrm{Stable}_{\mathrm{U}}(\widehat{\mathrm{U}})$, from which we obtain the assertion for $C = \mathrm{U}$ and $C = \mathrm{T}(t)$. The other cases are straightforward.

In order to state our Main Lemma, we need to express, that, if we assume elements of the types of the context, the interpretation of the conclusion Θ of a judgement of Martin-Löf is valid. Since we need, that this is independent of the choice of equal elements of A_i, we will introduce the following abbreviation:

Definition 7.14 *Let* $\Gamma \equiv x_1 : A_1, \dots, x_k : A_k$ *be a g-context.*

$$\forall \Gamma^{=}(\vec{r}; \vec{s}).\phi :\equiv \forall r_1, \dots, r_k, s_1, \dots, s_k \in \mathrm{Term}_{\mathrm{Cl}}.(\langle r_1, s_1 \rangle \in A_1^*[\ ] \wedge \langle r_2, s_2 \rangle \in A_2^*[x_1/r_1] \wedge \cdots \wedge \langle r_k, s_k \rangle \in A_k^*[x_1/r_1, \dots, x_{k-1}/r_{k-1}]) \to \phi$$

"Assume $\Gamma^{=}(\vec{r}; \vec{s})$*" means:*
"Assume $r_1, \dots, r_k, s_1, \dots, s_k \in \mathrm{Term}_{\mathrm{Cl}}$ *such that* $\langle r_1, s_1 \rangle \in A_1^*[\ ] \wedge$
$\langle r_2, s_2 \rangle \in A_2^*[x_1/r_1] \wedge \cdots \wedge \langle r_k, s_k \rangle \in A_k^*[x_1/r_1, \dots, x_{k-1}/r_{k-1}]$*".*

8 Main Lemma

In this section we prove the Main Lemma, which expresses that if $\mathrm{ML} \vdash r : A$, then $\mathrm{KPI}^+ \vdash \langle r, r \rangle \in A^*$. We have to go through all judgements.

Lemma 8.1 *(***Main Lemma***)*
Let Γ, Δ *be g-context-pieces,* $x, x_i \in \mathrm{Var}_{\mathrm{ML}}$*,* A_i, A, B *g-types,* t, t' *g-terms,* θ *a g-judgement. Assume* $\Gamma = x_1 : A_1, \dots, x_n : A_n$*.*

(a) If $\mathrm{ML} \vdash \Gamma \Rightarrow t = t' : A$*, then*

  (i) $\mathrm{KPI}^+ \vdash \forall \Gamma^{=}(\vec{r}; \vec{s}).(t = t' : A)^*[\vec{x}/\vec{r}; \vec{s}]$.

  (ii) $\mathrm{KPI}^+ \vdash \forall \Gamma^{=}(\vec{r}; \vec{s}).(A : \mathrm{type})^*[\vec{x}/\vec{r}; \vec{s}]$.

(b) If $\mathrm{ML} \vdash \Gamma \Rightarrow A = A' : \mathrm{type}$*, then*

$$\mathrm{KPI}^+ \vdash \forall \Gamma^{=}(\vec{r}; \vec{s}).(A = A' : \mathrm{type})^*[\vec{x}/\vec{r}; \vec{s}].$$

(c) If $\mathrm{ML} \vdash \Gamma, x : A, \Delta \Rightarrow \theta$*, then*

$$\mathrm{KPI}^+ \vdash \forall \Gamma^{=}(\vec{r}; \vec{s}).(A : \mathrm{type})^*[\vec{x}/\vec{r}; \vec{s}].$$

Proof of the Main Lemma:
We proof simultaneously (a) - (c) by induction on the derivation. We write IH 3 for the Induction-hypothesis for the 3rd premise, etc. IH 3(c) for the Induction-hypothesis (c) for the 3rd premise of the rule etc.
If there is more than one rule of one category (as in the case of (TRANS)), we refer to them by $(\mathrm{TRANS})_1$, $(\mathrm{TRANS})_2$, etc.
Let $\Gamma = x_1 : A_1, \dots, x_n : A_n$, $\Gamma' = y_1 : B_1, \dots, y_m : B_m$.
If $\vec{r} = r_1, \dots, r_n$, $i \leq n$, then $\hat{r}_i := r_1, \dots, r_{i-1}$ ($\hat{r}_1$ is empty).
If θ is $t = t' : A$ or $A = B : \mathrm{type}$, let $\theta' = (A : \mathrm{type})$ (the judgement treated in the cases (i) of (a),(b), or which follows from the assertion in (c).
Distinction by the last rule applied.
Assume that lemma is proved for the premises of a rule, as stated in the definition, weakened by the context Γ.
We treat only some examples of the rules, covering the more complicated ones.

Case (SYM)$_1$ Assume $\Gamma^{=}(\vec{r};\vec{s})$. From $\langle r_i, s_i\rangle \in A_i^*[\hat{r}_i]$ it follows $\langle s_i, r_i\rangle \in A_i^*[\hat{r}_i]$ and by IH (a,ii) $\langle s_i, r_i\rangle \in A_i^*[\hat{s}_i]$. By IH (a,i) it follows $\langle t[\vec{s}], t'[\vec{r}]\rangle \in A^*[\vec{s}]$, and by IH (a,ii) $A^*[\vec{r}] = A^*[\vec{s}]$, and by 7.8 it follows $(t' = t : A)^*[\vec{r};\vec{s}]$.
(a,ii) follows from IH (a,ii).
Case (SYM)$_2$ Assume $\Gamma^{=}(\vec{r};\vec{s})$. As for (SYM)$_1$ we have $\langle s_i, r_i\rangle \in A_i^*[\hat{s}_i]$, by IH $A^*[\vec{s}] = B^*[\vec{r}]$ and therefore the assertion.
Case (SUB) Assume $\Gamma^{=}(\vec{r};\vec{r}')$, $\langle s_i, s_i'\rangle \in B_i[x/t]^*[\vec{r}, \hat{s}_i]$. Now by Lemma 7.10

$$B_i[x/t]^*[\vec{r}, \hat{s}_i] = B_i^*[x/t[\vec{r}, \hat{s}_i], \vec{r}, \hat{s}_i] = B_i^*[x/t[\vec{r}], \vec{r}, \hat{s}_i]$$

By IH 2 (a,i) $\langle t[\vec{r}], t'[\vec{r}']\rangle \in A^*[\vec{r}]$, therefore

$$\theta^*[\vec{x}/\vec{r};\vec{r}', x/t[\vec{r}];t'[\vec{r}'], \vec{y}/\vec{s};\vec{s}'],$$

and by Lemma 7.10 $\theta[x/t]^*[\vec{x}/\vec{r};\vec{r}', \vec{s};\vec{s}']$, similarly for θ'.
Proof for (c): If „$y : B$" in Γ, the assertion follows by IH.
If „$y : B$" in $\Gamma'[x/t]$, it follows by IH $B^*[\vec{r}, x/\vec{t}[\vec{r}], \hat{s}_i] = B^*[\vec{r}', x/t[\vec{r}'], \hat{s}_i']$, and by 7.10 the assertion.
Case (REPL1) Assume $\Gamma^{=}(\vec{r};\vec{r}')$, $\langle s_i, s_i'\rangle \in B_i[x/t]^*[\vec{r}, \hat{s}_i]$. We have $\langle t[\vec{r}], t'[\vec{r}']\rangle \in A^*[\vec{r}]$. By 7.10, it follows $B_i^*[\vec{x}/\vec{r}, x/t[\vec{r}], \hat{y}_i/\hat{s}_i] = B_i[x/t]^*[\vec{r}, \hat{s}_i]$. Therefore we have $\langle s_i, s_i'\rangle \in B_i^*[\vec{x}/\vec{r}, x/t[\vec{r}], \hat{y}_i/\hat{s}_i]$. Then by IH 1
$B^*[\vec{x}/\vec{r}, x/t[\vec{r}], \vec{y}/\vec{s}] = B^*[\vec{x}/\vec{r}', x/t'[\vec{r}'], \vec{y}/\vec{s}']$, and by 7.10 follows the assertion.
Proof for (c): From IH 2 it follows, arguing as for the rule (SYM), the assertion for $\Gamma \Rightarrow t' = t : A$ and further, arguing as for (TRANS) the assertion for $\Gamma \Rightarrow t = t : A$, which is the same as for $\Gamma \Rightarrow t : A$ and now the proof follows as in (SUB).
Case (REPL2) Assume $\Gamma^{=}(\vec{r}, \vec{r}')$, $\langle s_i, s_i'\rangle \in B_i[x/t]^*[\vec{r}, \hat{s}_i]$. Then $\langle t[\vec{r}], t'[\vec{r}']\rangle \in A^*[\vec{r}]$, and by IH 1(a,i)

$$(s = s : B)^*[\vec{x}/\vec{r};\vec{r}', x/t[\vec{r}];t'[\vec{r}'], \vec{y}/\vec{s};\vec{s}']$$

and by 7.10 follows the assertion for (a,i). (a,ii) follows as in (REPL1), using that we get the assertion for $\Gamma \Rightarrow t = t : A$, and (c) follows exactly as in (REPL1).
Case (ALPHA): Immediate by the IH since if $A{=_\alpha}A'$, $t{=_\alpha}t'$, $A[\vec{s}] = A'[\vec{s}]$, $t[\vec{r}]{=_\alpha}t'[\vec{r}']$ and $\langle t[\vec{r}], t[\vec{r}']\rangle \in A^*[\vec{r}] \iff \langle t[\vec{r}], t'[\vec{r}']\rangle \in A^*[\vec{r}]$.
Case ($\Pi^{T,=}$) Assume $\Gamma^{=}(\vec{r};\vec{r}')$. By IH (c) $A^*[\vec{r}] = A'^*[\vec{r}']$, and, if $\langle r, s\rangle \in A^*[\vec{r}]$, it follows $\langle r, r\rangle, \langle s, s\rangle \in A^*[\vec{r}]$, therefore by IH $B^*[\vec{r}, x/r] = B'^*[\vec{r}', x/r]$, $B^*[\vec{r}, x/s] = B'^*[\vec{r}', x/s]$, $\Pi x \in A.B^*[\vec{r}] = (\Pi x \in A'.B')^*[\vec{r}']$.
Case ($\mathbb{N}^{\mathrm{I},=}$)$_2$: Assume $\Gamma^{=}(\vec{r};\vec{r}')$. By IH we have for some $k \in \omega$ $t[\vec{r}]{\rightarrow_{\mathrm{red}}}\mathrm{S}^k(0)$ and $t'[\vec{r}']{\rightarrow_{\mathrm{red}}}\mathrm{S}^k(0)$, therefore $\mathrm{S}(t)[\vec{r}]{\rightarrow_{\mathrm{red}}}\mathrm{S}^{k+1}(0)$, $\mathrm{S}(t')[\vec{r}']{\rightarrow_{\mathrm{red}}}\mathrm{S}^{k+1}(0)$, and we have the assertion.
Case ($\Pi^{\mathrm{I},=}$): Assume $\Gamma^{=}(\vec{r};\vec{r}')$, $\langle r, r'\rangle \in A^*[\vec{r}]$. Then by IH (a,i)
$\langle t[x/r, \vec{r}], t'[x/r', \vec{r}']\rangle \in B^*[x/r, \vec{r}] = B^*[\vec{r}, x/r]$, $\langle (\lambda x.t)[\vec{x}/\vec{r}], (\lambda x.t')[\vec{x}/\vec{r}']\rangle \in (\Pi x \in A.B)^*[\vec{r}]$.
(a,ii) follows as in ($\Pi_1^{T,=}$), since from IH (a,ii) follows (b) for $x : A \Rightarrow B : \mathrm{type}$.

Case ($\mathrm{W}^{\mathrm{I},=}$): Let $n := \max\{\mathrm{lev}(A), \mathrm{lev}(B)\}$, $\mathrm{W}x \in A.B^*$.
Assume $\Gamma^=(\vec{r};\vec{r}')$. Let $F := F_{\mathrm{W}}(A^*[\vec{x}/\vec{r}], (x)B^*[\vec{x}/\vec{r}])$. Then by IH $\langle r[\vec{r}], r'[\vec{r}']\rangle \in A^*[\vec{r}]$, $s[\vec{r}]{\to_{\mathrm{red}}}\lambda x.t$, $s'[\vec{r}']{\to_{\mathrm{red}}}\lambda x'.t'$, $B^*[x/r[\vec{r}], \vec{r}] = B^*[x/r'[\vec{r}'], \vec{r}']$, and

$$\forall\langle u, u'\rangle \in B[x/t]^*[\vec{x}/\vec{r}](= B^*[x/t[\vec{r}], \vec{x}/\vec{r}])\exists\gamma < \mathrm{I}_n.\langle t[x/u], t'[x/u']\rangle \in F^\gamma$$

By $(\Delta_0 - \mathrm{coll})$ and $\mathrm{Ad}(\mathrm{L}_{\mathrm{I}_n})$ there exist a $\delta < \mathrm{I}_n$ such that the γ can be chosen to be $< \delta$. Then $\langle \sup(r, s)[\vec{r}], \sup(r, s)[\vec{r}']\rangle \in F^{\gamma+1} \subset \mathrm{W}x \in A.B^*[\vec{r}]$.
(a,ii) follows as in ($\mathrm{W}^{T,=}$).
Case($\mathbb{N}^{E,=}$): Assume $\Gamma^=(\vec{r};\vec{r}')$. Then by IH 1 $\langle r[\vec{r}], r'[\vec{r}']\rangle \in \mathbb{N}^*$, therefore
$r[\vec{r}]{\to_{\mathrm{red}}}\mathrm{S}^n(0)$, $r'[\vec{r}]{\to_{\mathrm{red}}}\mathrm{S}^n(0)$ for some $n < \omega$. Furthermore, by IH 2 and 7.11 (b) exist $\widetilde{s}$, $\widetilde{s}' \in \mathrm{Term}_{\mathrm{nf}}$ such that $s[\vec{r}]{\to_{\mathrm{red}}}\widetilde{s}$, $t[\vec{r}]{\to_{\mathrm{red}}}\widetilde{s}'$, $\langle \widetilde{s}, \widetilde{s}'\rangle \in A[x/0]^*[\vec{r}] = A^*[x/u, \vec{r}]$.
Let $[\vec{x}'/\vec{s}] := [\vec{x}/\vec{r}] \setminus \{x, y\}$, $[\vec{x}'/\vec{s}''] := [\vec{x}/\vec{r}'] \setminus \{x, y\}$.
Let $P_0(r) := \mathrm{P}(r, \widetilde{s}, \lambda x.\lambda y.(t[\vec{x}'/\vec{s}]))$. $P_1(r) := \mathrm{P}(r, \widetilde{s}', \lambda x.\lambda y.(t'[\vec{x}'/\vec{s}']))$. Then

$$\mathrm{P}(r, s, (x, y)t)[\vec{r}]{\to_{\mathrm{red}}}P_0(\mathrm{S}^n(0)),\ \mathrm{P}(r', s', (x, y)t')[\vec{r}']{\to_{\mathrm{red}}}P_1(\mathrm{S}^n(0))$$

We have $A[z/r]^*[\vec{r}] = A^*[z/r[\vec{r}], \vec{r}] = A^*[z/\mathrm{S}^n(0), \vec{r}]$, and therefore assertion (a,i).
We show: $\forall m \in \omega.\langle P_0(\mathrm{S}^m(0)), P_1(\mathrm{S}^m(0))\rangle \in A^*[z/\mathrm{S}^m(0), \vec{r}]$.
If $m = 0$, $P_0(\mathrm{S}^m(0)){\to_{\mathrm{red}}}\widetilde{s}$, $P_1(\mathrm{S}^m(0)){\to_{\mathrm{red}}}\widetilde{s}'$, $\langle \widetilde{s}, \widetilde{s}'\rangle \in A^*[z/0, \vec{r}]$.
If $m = k + 1$, it follows by IH $P_0(\mathrm{S}^k(0)){\to_{\mathrm{red}}}\widetilde{s}$, $P_1(\mathrm{S}^k(0)){\to_{\mathrm{red}}}\widetilde{s}'$,
$\widetilde{s}, \widetilde{s}' \in \mathrm{Term}_{\mathrm{nf}}$, $\langle \widetilde{s}, \widetilde{s}'\rangle \in A^*[z/\mathrm{S}^k(0), \vec{r}] = A[z/x]^*[x/\mathrm{S}^k(0), \vec{r}]$.
$P_0(\mathrm{S}^m(0)){\to_{\mathrm{red}}}t[\vec{x}'/\vec{s}, x/\mathrm{S}^k(0), y/\widetilde{s}]$, $P_1(\mathrm{S}^m(0)){\to_{\mathrm{red}}}t'[\vec{x}'/\vec{s}'', x/\mathrm{S}^k(0), y/\widetilde{s}']$.
Now $\langle \mathrm{S}^k(0), \mathrm{S}^k(0)\rangle \in \mathbb{N}^*$, therefore by IH 3 it follows

$$\langle t[\vec{x}'/\vec{s}, x/\mathrm{S}^k(0), y/\widetilde{s}], t'[\vec{x}'/\vec{s}'', x/\mathrm{S}^k(0), y/\widetilde{s}']\rangle \in A[z/\mathrm{S}(x)]^*[\vec{r}] = A^*[z/\mathrm{S}^m(0), \vec{r}],$$

and the side induction is finished.
(a,ii) is easy.
Case($\Pi^{E,=}$): Assume $\Gamma^=(\vec{r};\vec{r}')$. By IH 1,2 there exist $\widetilde{r}, \widetilde{r}' \in \mathrm{Term}_{\mathrm{nf}}$ such that $r[\vec{r}]{\to_{\mathrm{red}}}\widetilde{r}_1$, $r'[\vec{r}']{\to_{\mathrm{red}}}\widetilde{r}'$ $\langle \widetilde{r}_1, \widetilde{r}'\rangle \in A^*[\vec{r}]$, and there are $t, t' \in \mathrm{Term}$ and Variables $x, x' \in \mathrm{Var}_{\mathrm{ML}}$ such that

$$s[\vec{r}]{\to_{\mathrm{red}}}\lambda x.r,\ s'[\vec{s}]{\to_{\mathrm{red}}}\lambda x'.r',\ \langle \lambda x.r, \lambda x'.r'\rangle \in (\Pi x \in A.B)^{\mathrm{basis}}[\vec{r}].$$

Therefore

$$\mathrm{Ap}(s, r)[\vec{r}]{\to_{\mathrm{red}}}\mathrm{Ap}(\lambda x.t, \widetilde{r}){\to_{\mathrm{red}}}t[x/\widetilde{r}, \vec{r}],$$
$$\mathrm{Ap}(s', r')[\vec{r}']{\to_{\mathrm{red}}}t'[x'/\widetilde{r}, \vec{r}']$$
$$\langle t[x/\widetilde{r}, \vec{r}], t'[x'/\widetilde{r}', \vec{r}']\rangle \in B^*[x/\widetilde{r}, \vec{r}].$$

As before we conclude

$$\langle r[\vec{r}], r[\vec{r}]\rangle \in A^*[\vec{r}]$$
$$\langle \widetilde{r}, r_1[\vec{r}]\rangle \in A^*[\vec{r}]$$
$$B^*[x/\widetilde{r}, \vec{r}] = B^*[x/r[\vec{r}], \vec{r}] = B[x/r]^*[\vec{r}],$$

and we have IH (a,i).
(a,ii) follows as before
Case($\Sigma^{E,=}$): Assume $\Gamma^{=}(\vec{r};\vec{r}')$. By IH 1 exist $s, s', t, t' \in \mathrm{Term}_{\mathrm{nf}}$ such that

$$r[\vec{r}] \to_{\mathrm{red}} \mathrm{p}(s,t),\ r'[\vec{r}'] \to_{\mathrm{red}} \mathrm{p}(s',t'),\ \langle s,s'\rangle \in A^*[\vec{r}],\ \langle t,t'\rangle \in B^*[x/s,\vec{r}].$$

Then $\mathrm{p}_0(r[\vec{r}]) \to_{\mathrm{red}} s$, $\mathrm{p}_0(r'[\vec{r}']) \to_{\mathrm{red}} s'$, and we are done for the first rule, and $\mathrm{p}_1(r[\vec{r}]) \to_{\mathrm{red}} t$, $\mathrm{p}_1(r'[\vec{r}']) \to_{\mathrm{red}} t'$, and since from $\langle s,s'\rangle \in A[\vec{r}]$, it follows

$$\langle s,s'\rangle \in A[\vec{r}],\ \langle \mathrm{p}_0(r)[\vec{r}], s\rangle \in A[\vec{r}],$$

therefore by IH 2

$$B^*[x/s,\vec{r}] = B^*[x/\mathrm{p}_0(r)[\vec{r}],\vec{r}] = B[x/\mathrm{p}_0(r)]^*[\vec{r}]$$

follows (a,i) for the second rule.
(a,ii) is in $(\Sigma^{E,=})_1$ trivial, in $(\Sigma^{E,=})_2$ we use the proof of $(\Sigma^{E,=})_1$ and argue as before.
Case($\mathrm{W}^{E,=}$): Assume $\Gamma^{=}(\vec{r};\vec{r}')$, $n := \max\{\mathrm{lev}(A), \mathrm{lev}(B)\}$, $F := F_{\mathrm{W}}(A^*[\vec{r}], (x)B^*[\vec{r}])$. By IH $r[\vec{r}] \to_{\mathrm{red}} \widetilde{r}$, $r'[\vec{r}'] \to_{\mathrm{red}} \widetilde{r}'$, $\langle \widetilde{r}, \widetilde{r}'\rangle \in F^{\delta}(\vec{r},\cdot)$. for some $\delta < \alpha$. Let

$$\begin{aligned}[][\vec{x}'/\vec{s}] &:= [\vec{x}/\vec{r}] \setminus \{x,y,z\},\\ [\vec{x}'/\vec{s}'] &:= [\vec{x}/\vec{r}'] \setminus \{x,y,z\},\\ R_0(r) &:= \mathrm{R}(r,(x,y,z)t)[\vec{r}] (= \mathrm{R}(r, \lambda x.\lambda y.\lambda z.(t[\vec{s}])))\\ R_1(r) &:= \mathrm{R}(r,(x,y,z)t')[\vec{r}'].\end{aligned}$$

We show by induction on γ,

$$(+) \qquad \forall \gamma < \alpha. \forall \langle \widetilde{s}, \widetilde{s}'\rangle \in F^{\gamma}. \langle R_0(\widetilde{s}), R_1(\widetilde{s}')\rangle \in C^*[u/\widetilde{s},\vec{r}]$$

Since $C[u/t]^*[\vec{r}] = C^*[u/r[\vec{r}],\vec{r}] = C^*[u/\widetilde{r},\vec{r}] = C^*[u/\widetilde{r}',\vec{r}]$ (using arguments as before), follows the assertion.
The case $\gamma = 0$ is trivial, and if $\gamma \in \mathrm{Lim}$ follows the assertion by IH
Let now
$\gamma = \gamma' + 1$, $u' := F^{\gamma'}$, $\langle \widetilde{s}, \widetilde{s}'\rangle \in F(u')$.
If $\widetilde{s} \to_{\mathrm{red}} s$, $\widetilde{s}' \to_{\mathrm{red}} s'$, $\langle s,s'\rangle \in F^{\mathrm{basis}}(\vec{r}',\cdot)$, $\langle R_0(s), R_1(s')\rangle \in C^*[u/s,\vec{r}]$, it follows $\langle R_0(\widetilde{s}), R_1(\widetilde{s}')\rangle \in C^*[u/s,\vec{r}]$, further, like similar arguments before,

$$C^*[u/s,\vec{r}] = C^*[u/\widetilde{s},\vec{r}] = C^*[u/\widetilde{s}'].$$

We therefore assume $\langle \widetilde{s}, \widetilde{s}'\rangle \in F^{\mathrm{basis}}(\vec{r}, u')$.
Let $\langle \widetilde{s}, \widetilde{s}'\rangle = \langle \sup(a, \lambda x.s), \sup(a', \lambda x'.s')\rangle$, $\langle a,a'\rangle \in A^*[\vec{r}]$. Let $\langle r'', r'''\rangle \in B^*[x/a,\vec{r}]$. Then $r'' \to_{\mathrm{red}} b$, $r''' \to_{\mathrm{red}} b'$ for $\langle b,b'\rangle \in B^*[x/a,\vec{r}]$, $b,b' \in \mathrm{Term}_{\mathrm{nf}}$, and we have $\langle s[x/r''], s'[x'/r''']\rangle \in u'$ and

$$(*) \qquad \langle s[x/b], s'[x'/b']\rangle \in u'$$

Since $u' \subset (\mathrm{W}x \in A.B)^*[\vec{r}]$ it follows from the first of these assertions

$$\langle \lambda x.s, \lambda x'.s' \rangle \in (B \to \mathrm{W}x \in A.B)^*[\vec{r}]$$

Furthermore, for $\langle b, b' \rangle \in B^*[x/r, \vec{r}]$,

$$(R_0((\lambda x.s)v))[v/r''] \to_{\mathrm{red}} R_0(s[x/b])(v \notin \mathrm{FV}(\lambda x.s))$$

$$(R_1((\lambda x.s')v'))[v'/r'''] \to_{\mathrm{red}} R_1(s'[x/b'])(v' \notin \mathrm{FV}(\lambda x.s))$$

and by side IH, it follows

$$\begin{aligned}\langle (R_0((\lambda x.s)v))[v/r''], (R_1((\lambda x'.s')v))[v'/r'''] \rangle &\in C^*[u/(s[x/b]), \vec{r}] \\ &= C^*[u/(s[x/r'']), \vec{r}]\end{aligned}$$

Now we have $\langle r_i, r_i \rangle \in A_i[\hat{r}_i]$, $\mathrm{Ap}(\lambda x.s, r'') \to_{\mathrm{red}} s[x/b]$, and by $(*)$,
$u' \subset (\mathrm{W}x \in A.B)^*[\vec{r}]$, $\mathrm{equiv}((\mathrm{W}x \in A.B)^*[\vec{r}])$ and 7.11 it follows

$$\langle s[x/b], \mathrm{Ap}(\lambda x.s, r'') \rangle \in (\mathrm{W}x \in A.B)^*[\vec{r}]$$

therefore

$$C[u/\mathrm{Ap}(y,v)]^*[v/r'', y/\lambda x.s, \vec{x}/\vec{r}] = C^*[u/\mathrm{Ap}(\lambda x.s, r''), \vec{r}] = C^*[u/(s[x/b]), \vec{r}]$$

further

$$C[u/\mathrm{Ap}(y,v)]^*[v/r'', y/\lambda x.s, \vec{x}/\vec{r}] = C[u/\mathrm{Ap}(y,v)]^*[v/r''', y/\lambda x.s, \vec{x}/\vec{r}],$$

and we have

$$\langle \lambda v.R_0((\lambda x.s)v), \lambda v'.R_1((\lambda x'.s')v') \rangle \in (\Pi v \in B.C[u/\mathrm{Ap}(y,v)])^*[y/\lambda x.s, \vec{x}/\vec{r}]$$

Now by IH 2 it follows

$$\begin{aligned}&\langle t[x/r, y/\lambda x.s, z/\lambda v.R_0((\lambda x.s)v), \vec{r}], t'[x/r', y/\lambda x'.s', z/\lambda v'.R_1((\lambda x.s')v'), \vec{r}']\rangle \\ &\qquad \in C[u/\sup(x,y)]^*[x/r, y/\lambda x.s, \vec{r}]\end{aligned}$$

Since

$$C[u/\sup(x,y)]^*[x/r, y/\lambda x.s, \vec{r}] = C^*[u/\sup(r, \lambda x.s), \vec{r}] = C^*[u/s, s],$$

and

$$\begin{aligned}R_0(s) &\to_{\mathrm{red}} (\lambda x.\lambda y.\lambda z.t[\vec{x}'/\vec{s}])r(\lambda x.s)(\lambda v.R_0((\lambda x.s)v)) \\ &\to_{\mathrm{red}} t[x/r, y/\lambda x.s, z/(\lambda v.R_0((\lambda x.s)v))]\end{aligned}$$

$$R_1(s') \to_{\mathrm{red}} t'[x/r', y/\lambda x.s', z/(\lambda v'.R_1((\lambda x'.s')v'))]$$

follows (+), and we are done. (a,ii) follows as in the case $(\mathbb{N}_k^{E,=})$.

Case $(+^{E,=})$: Assume $\Gamma^{=}(\vec{r};\vec{r'})$. By IH $r[\vec{r}]\rightarrow_{\mathrm{red}}\mathrm{i}(\widetilde{r}) \in \mathrm{Term}_{\mathrm{nf}}$, $r'[\vec{r'}]\rightarrow_{\mathrm{red}}\mathrm{i}(\widetilde{r}') \in \mathrm{Term}_{\mathrm{nf}}$ and $\langle \widetilde{r}, \widetilde{r}'\rangle \in A^*[\vec{r}]$ or $r[\vec{r}]\rightarrow_{\mathrm{red}}\mathrm{j}(\widetilde{r}) \in \mathrm{Term}_{\mathrm{nf}}$, $r'[\vec{r'}]\rightarrow_{\mathrm{red}}\mathrm{j}(\widetilde{r}') \in \mathrm{Term}_{\mathrm{nf}}$ and $\langle \widetilde{r}, \widetilde{r}'\rangle \in B^*[\vec{r}]$. Let $[\vec{x}'/\vec{s}] := [\vec{x}/\vec{r}] \setminus \{x\}$. In the first case we have

$$\mathrm{D}(r,(x)s,(y)t)[\vec{r}]\rightarrow_{\mathrm{red}}(\lambda x.(s[\vec{s}]))\widetilde{r}\rightarrow_{\mathrm{red}}s[x/\widetilde{r},\vec{x}/\vec{r}],$$

$$\mathrm{D}(r',(x)s',(y)t')[\vec{r}]\rightarrow_{\mathrm{red}}s'[x/\widetilde{r}',\vec{x}/\vec{r'}],$$
$$\langle s[x/\widetilde{r},\vec{r}], s'[x/\widetilde{r}',\vec{r'}]\rangle \in C[z/\mathrm{i}(x)]^*[x/\widetilde{r},\vec{r}] = C^*[z/\mathrm{i}(\widetilde{r}),\vec{r}]$$

and using arguments as before

$$C^*[z/\mathrm{i}(\widetilde{r}),\vec{r}] = C^*[z/r[\vec{r}],\vec{r}] = C[z/r]^*[\vec{r}]$$

and we are done. The second assertion follows in the same way.
(a,ii) follows as before.
Case (I^E): Assume $\Gamma^{=}(\vec{r};\vec{r'})$. By IH 1 follows $(\mathrm{I}(A,s,t))^*[\vec{r}] \neq \emptyset$, $\langle s[\vec{r}], t[\vec{r}]\rangle \in A^*[\vec{r}]$. Furthermore, by IH 3 $\langle t[\vec{r}], t[\vec{r'}]\rangle \in A^*[\vec{r}]$, and by $\mathrm{equiv}(A^*[\vec{r}])$ follows (a,i). (a,ii) is trivial.

Case $(\Pi^=)$, $(\Sigma_0^=)$, $(\Sigma_1^=)$: By using the proof for the elimination rules we see, that if the conclusion is $r = s : C$, we conclude assuming $\Gamma^{=}(\vec{r};\vec{r'})$, that $(r = r : C)[\vec{r};\vec{r'}]$, further $(r[\vec{r'}]\rightarrow_{\mathrm{red}}t \in \mathrm{Term}_{\mathrm{nf}}) \rightarrow (s[\vec{r'}]\rightarrow_{\mathrm{red}}t)$, therefore follows $(r = s : C)[\vec{r};\vec{r'}]$.

Case (Π^η):Assume $\Gamma^{=}(\vec{r},\vec{r'})$ By IH we have

$$\langle t[\vec{r}], t[\vec{r'}]\rangle \in (\Pi x \in A.B)^*[\vec{r}],$$

therefore $t[\vec{r}]\rightarrow_{\mathrm{red}}\lambda x.s$, $t[\vec{r'}]\rightarrow_{\mathrm{red}}\lambda x'.s'$,

$$\langle \lambda x.s, \lambda x'.s'\rangle \in (\Pi x \in A.B)^{\mathrm{basis}}[\vec{r}],$$

Assume $\langle r, r'\rangle \in A^*[\vec{r}]$. Then $r\rightarrow_{\mathrm{red}}\widetilde{r}$, $r'\rightarrow_{\mathrm{red}}\widetilde{r}'$, $\langle \widetilde{r}, \widetilde{r}'\rangle \in A^*[\vec{r}]$, $\widetilde{r}, \widetilde{r}' \in \mathrm{Term}_{\mathrm{nf}}$.

$$\mathrm{Ap}(t,x))[\vec{r}][x/r] = \mathrm{Ap}(t[\vec{r}],r)\rightarrow_{\mathrm{red}}\mathrm{Ap}(\lambda x.s,\widetilde{r})\rightarrow_{\mathrm{red}}s[x/\widetilde{r}],$$

and since
$$\langle s[x/\widetilde{r}], s'[x'/\widetilde{r}']\rangle \in B^*[x/\widetilde{r},\vec{r}] = B^*[x/r,\vec{r}],$$

follows
$$\langle \mathrm{Ap}(t,x)[\vec{r}][x/r], s'[x'/\widetilde{r}']\rangle \in B^*[x/r,\vec{r}],$$

therefore
$$\langle \lambda x.\mathrm{Ap}(t,x)[\vec{r}], \lambda x'.s'\rangle \in (\Pi x \in A.B)^*[\vec{r}],$$
$$\langle \lambda x.\mathrm{Ap}(t,x)[\vec{r}], t[\vec{r}]\rangle \in (\Pi x \in A.B)^*[\vec{r}].$$

Case $(\Sigma_2^=)$: Assume $\Gamma^{=}(\vec{r};\vec{r'})$. By IH $t[\vec{r}]\rightarrow_{\mathrm{red}}r$, $t[\vec{r'}]\rightarrow_{\mathrm{red}}r'$ for some $\langle r, r'\rangle \in \Sigma x \in A.B^*[\vec{r}] \cap (\mathrm{Term}_{\mathrm{nf}} \times \mathrm{Term}_{\mathrm{nf}})$, $\mathrm{p}(\mathrm{p}_0(t),\mathrm{p}_1(t))[\vec{r}]\rightarrow_{\mathrm{red}}r$ and we are done.

Case ($\mathrm{I}^=$): Assume $\Gamma^=(\vec{r};\vec{r}')$. By IH we conclude $\langle t_0[\vec{r}], t_0[\vec{r}']\rangle \in \mathrm{I}(A,t_1,t_2)^*[\vec{r}]$, therefore $t_0[\vec{r}']{\rightarrow_{\mathrm{red}}}\mathrm{r}$, $\langle \mathrm{r},\mathrm{r}\rangle \in \mathrm{I}(A,t_1,t_2)^*[\vec{r}]$, $\langle t_0[\vec{r}], \mathrm{r}\rangle \in \mathrm{I}(A,t_1,t_2)^*[\vec{r}]$. (a,ii) is trivial.

Case other equality rules: Let $\widetilde{r} = \widetilde{s} : A$ be the conclusion of the rules. By using several times the general rules, elimination rules and in case $\mathrm{W}^=$ the introduction rules we can conclude $\widetilde{r} = \widetilde{r} : A$, and $\widetilde{s} = \widetilde{s} : A$. (For ($\mathrm{W}^=$) we argue that $\Gamma, v : B[x/t_0] \Rightarrow \mathrm{Ap}(t_1, v) : \mathrm{W}x \in A.B$, by ($\mathrm{W}^{E,=}$) $\Gamma, v : B[x/t_0] \Rightarrow \mathrm{R}(\mathrm{Ap}(s', v), (x, y, z)t') : C[u/\mathrm{Ap}(s', v)]$, by $\Pi^{\mathrm{I},=}$ $\Gamma \Rightarrow \lambda v.\mathrm{R}(\mathrm{Ap}(s', v), (x, y, z)t') : \Pi v \in B.C[u/\mathrm{Ap}(s', v)]$, by (ALPHA) for the z_i, that we need, and it follows $\Gamma \Rightarrow \lambda z_i.\mathrm{R}(\mathrm{Ap}(s', z_i), (x, y, z)t') : \Pi v \in B.C[u/\mathrm{Ap}(s', v)]$, and now by (SUB) follows the assertion). Now, assuming $\Gamma^=(\vec{r},\vec{r}')$, and using the proofs above we can conclude $\langle r[\vec{r}], r[\vec{r}']\rangle \in A^*[\vec{r}]$ and $A^*[\vec{r}] = A^*[\vec{r}']$, so (a,i). In all the cases, we have, if the right side is written as $t[x_1/r_1,\dots,x_n/t_n]$, if x_i corresponds to the type B_i (read off from the rule) follows easily by IH and using the proofs of several rules handled before the assertion for $\Gamma \Rightarrow r_i : B_i$, therefore $r_i[\vec{r}]{\rightarrow_{\mathrm{red}}}\widetilde{r}_i \in \mathrm{Term}_{\mathrm{nf}}$ for some $\widetilde{r}_i$, $\langle \widetilde{r}_i, r_i[\vec{r}]\rangle \in B_i[\vec{r}]$, further $\widetilde{r}[\vec{r}]{\rightarrow_{\mathrm{red}}}t[x_1/\widetilde{r}_1,\cdots x_n\widetilde{r}_n,\vec{r}]$. We conclude

$$\langle t[x_1/\widetilde{r}_1,\dots,x_n/\widetilde{r}_n,\vec{r}], t[x_1/r_1[\vec{r}],\dots,x_n/r_n[\vec{r}],\vec{r}]\rangle \in A^*[\vec{r}].$$

Now using $\mathrm{equiv}(A^*[\vec{r}])$ and Lemma 7.11 we conclude

$$\langle \widetilde{r}[\vec{r}], \widetilde{s}[\vec{r}]\rangle \in A^*[\vec{r}], \langle \widetilde{s}[\vec{r}], \widetilde{s}[\vec{r}']\rangle \in A^*[\vec{r}],$$

and have (a,i).

Case (U^{I}): trivial.

($\mathrm{T}^{\mathrm{I},=}$) we have by IH, assuming $\Gamma^=(\vec{r};\vec{r}')$,

$$\langle a[\vec{r}], a'[\vec{r}']\rangle \in \mathrm{U}^*$$

therefore,

$$\langle a[\vec{r}], b, a'[\vec{r}']\rangle \in \widehat{\mathrm{U}}$$

for some b, by $\mathrm{Cor}_{\mathrm{U}}(\widehat{\mathrm{U}})$,

$$\langle a[\vec{r}], b, a[\vec{r}]\rangle \in \widehat{\mathrm{U}}, \langle a[\vec{r}'], b, a[\vec{r}']\rangle \in \widehat{\mathrm{U}},$$

and

$$\mathrm{T}(a)^*[\vec{r}] = b = \mathrm{T}(a')^*[\vec{r}']$$

Case ($\widehat{\Pi}^{\mathrm{I},=}$): Assume $\Gamma^=(\vec{r};\vec{r}')$. By IH $a[\vec{r}]{\rightarrow_{\mathrm{red}}}\widetilde{a}$, $a'[\vec{r}']{\rightarrow_{\mathrm{red}}}\widetilde{a}'$,

$$\exists\gamma < \mathrm{I}\exists b' \in\ \mathrm{TC}(\widetilde{\mathrm{U}}^\gamma)(\langle \widetilde{a}, b', \widetilde{a}\rangle, \langle \widetilde{a}', b', \widetilde{a}\rangle \in \widetilde{\mathrm{U}}^\gamma),$$

and

$$\forall\langle t, t'\rangle \in b' \rightarrow \exists\delta < \mathrm{I}.\exists c \in\ \mathrm{TC}(\widetilde{\mathrm{U}}^\delta).$$
$$(\langle b[x/t,\vec{r}], c, b'[x/t',\vec{r}']\rangle \in \widetilde{\mathrm{U}}^\delta).$$

Since $\mathrm{Ad}(\mathrm{L_I})$ (here is the central point where we need $(\Delta_0-\mathrm{coll})$ and an admissible a which is closed under the step to the next admissible), and $\mathrm{TC}(\widetilde{\mathrm{U}}^\beta) \in \mathrm{L_I}$ $(\beta < \mathrm{I})$, there is a $\rho < \mathrm{I}$, such that $\gamma < \rho$ and δ can be chosen $< \rho$. There are now b, f such that $([\vec{x}'/\vec{s}] := [\vec{x}/\vec{r}] \setminus \{x\}$, $[\vec{x}'/\vec{s}'] := [\vec{x}/\vec{r}'] \setminus \{x\})$ $\phi(\widetilde{a}, x, b[\vec{s}], \widetilde{a}', x, b[\vec{s}'], b, f, \widetilde{\mathrm{U}}^\rho)$, (note that the c we used above is correct by $\mathrm{Cor}(\widehat{\mathrm{U}})$) and by 7.5 (a) follows

$$\langle \widehat{\Pi} x \in a.b, \Pi^*(b,f), \widehat{\Pi} x \in a'.b' \rangle \in \widetilde{\mathrm{U}}^{\rho+1}.$$

Case $(\widehat{\Pi}^=)$: Assume $\Gamma^=(\vec{r};\vec{r}')$ and chose b', f, ρ as in $(\widehat{\Pi}^{\mathrm{I},=})$. Then $\mathrm{T}(a)^*[\vec{r}'] = b'$, and if $\langle t, t'\rangle \in b'$, $\mathrm{T}(b)^*[x/t, \vec{r}] = f(t) = f(t') = \mathrm{T}(b)^*[x/t', \vec{r}']$. Since we have $\mathrm{Cor}(\widehat{\mathrm{U}})$ (by Lemma 7.5 (d)) it follows

$$\mathrm{T}((\widehat{\Pi} x \in a.b))^*[\vec{r}] = \Pi^*(\mathrm{T}(a)[\vec{r}], (x)\mathrm{T}(b)[\vec{r}]) = (\Pi x \in \mathrm{T}(a).\mathrm{T}(b))^*[\vec{r}].$$

In the case of $(\widehat{\mathrm{W}}^=)$ we conclude as before, that

$$F_{\mathrm{W}}(\mathrm{T}(a)^*[\vec{r}], (x)\mathrm{T}(b)^*[\vec{r}]) = F_{\mathrm{W}}(\mathrm{T}(a)^*[\vec{r}'], (x)\mathrm{T}(b)^*[\vec{r}'])$$

and, since
$\alpha^+(\widetilde{\mathrm{U}}^\rho) < \mathrm{I}$, ($\rho$ chosen as in $(\widehat{\Pi}^{\mathrm{I},=})$) it follows by 7.1
$\mathrm{T}(\widehat{\mathrm{W}} x \in a.b)^*[\vec{r}] = F_{\mathrm{W}}^{\alpha^+(\widetilde{\mathrm{U}}^\rho)}(\mathrm{T}(a)^*[\vec{r}], (x)(\mathrm{T}(b))^*[\vec{r}]) = F_{\mathrm{W}}^{\mathrm{I}}(\mathrm{T}(a)^*[\vec{r}], (x)(\mathrm{T}(b))^*[\vec{r}]) = (\mathrm{W} x \in \mathrm{T}(a).\mathrm{T}(b))^*[\vec{r}']$
Case $(\widehat{\Sigma}^{\mathrm{E}})$: Assume $\Gamma^=(\vec{r};\vec{r}')$. By IH exists c, $\alpha \prec \mathrm{I}$ such that

$$\langle (\widehat{\Sigma} x \in a.b)[\vec{r}], c, (\widehat{\Sigma} x \in a.b)[\vec{r}'] \rangle \in \widetilde{\mathrm{U}}^\alpha.$$

Let α be chosen minimal. Then, $\alpha = \alpha' + 1$, and with $u := \widetilde{\mathrm{U}}^{\alpha'}$ there exist $r, r' \in \mathrm{Term_{nf}}$, c, c', f, f' such that $a[\vec{r}] \to_{\mathrm{red}} r$, $a[\vec{r}'] \to_{\mathrm{red}} r'$, $\langle r, c, r'\rangle \in u$ and (with $[\vec{x}'/\vec{s}] := [\vec{x}/\vec{r}] \setminus \{x\}$, $[\vec{x}'/\vec{s}'] := [\vec{x}/\vec{r}'] \setminus \{x\})$ $\forall \langle t, t'\rangle \in c. \langle s[\vec{s}][x/t], f(t), s[\vec{s}'][x/t'] \rangle \in u$. Therefore $\mathrm{T}(a[\vec{r}])^* = c = \mathrm{T}(a[\vec{r}'])^*$, and for $\langle t, t'\rangle \in \mathrm{T}(a[\vec{r}])^*$, $\langle s[x/t, \vec{r}], s[x/t', \vec{s}'] \rangle \in \mathrm{U}^*[\ ]$.
$(\widehat{\Pi}^{\mathrm{E}})$, $(\widehat{\mathrm{W}}^{\mathrm{E}})$, $(\widehat{+}^{\mathrm{E}})$ are checked in the same way. For $(\widehat{\mathrm{I}}^{\mathrm{E}})$ we observe, that $a[\vec{r}] \to_{\mathrm{red}} \widetilde{a}$, $a[\vec{r}'] \to_{\mathrm{red}} \widetilde{a}'$, $s[\vec{r}] \to_{\mathrm{red}} \widetilde{s}$, $s[\vec{r}'] \to_{\mathrm{red}} \widetilde{s}'$, $t[\vec{r}] \to_{\mathrm{red}} \widetilde{t}$, $t[\vec{r}'] \to_{\mathrm{red}} \widetilde{t}'$, and $\langle \widetilde{a}, c, \widetilde{a}'\rangle \in u$, for some u as before, $\mathrm{T}(a)^*[\vec{r}] = c$, $\langle \widetilde{s}, \widetilde{s}'\rangle \in c$, $\langle \widetilde{t}, \widetilde{t}'\rangle \in c$, and since c is closed under $\to_{\mathrm{red}}$ follows the assertion.

9 Π^1_1-Soundness of the Interpretation of Martin-Löf Type Theory into KPI^+

In this section we want to evaluate the results we have found out to get the proof theoretic strength of Martin-Löf's type theory. We will interpret the language of analysis ($\mathcal{L}_{\mathrm{analysis}}$, introduced in 9.1) in $\mathcal{L}_{ML}$ and $\mathcal{L}_{KP}$ (definition9.6) and prove that it permutes with the interpretation of Martin-Löf's type theory in KPI^+ (Lemmata

9.8 and 9.9). Next we observe that every proof of ML can be interpreted in KPI^+ (Lemma 9.10). This preserves Π^1_1-sentences, where second order quantifiers in Kripke Platek set theory refer to elements of Ad_2. In the next section we will analyse the strength of KPI^+ and obtain the desired upper bound for the type theories in question.

Definition 9.1 *Definition of the language of Peano Arithmetic $\mathcal{L}_{\mathrm{analysis}}$: we have first order variables v_i ($i \in \omega$, $\mathrm{var}_{\mathrm{analysis}} := \{v_i | i \in \omega\}$); second order variables V_i ($i \in \omega$, $\mathrm{VAR}_{\mathrm{analysis}} := \{V_i | i \in \omega\}$); further we have symbols for each primitive recursive function, $=$, $\wedge$, $\vee$, $\to$, $\forall$, $\exists$, $\bot$, and $., ,, (,)$.*
Terms are first-order variables and $f(t_1, \ldots, t_n)$ if t_i are terms and f is a symbol for a n-ary primitive recursive function.
Prime formulas are $\bot$, equations $r = s$, and $r \in X$ for r, s terms, $X \in \mathrm{VAR}_{\mathrm{analysis}}$.
Formulas are prime formulas and $A \to B$, $A \wedge B$, $A \vee B$, $\forall x.A$, $\exists x.A$, if A, B formulas, $x \in \mathrm{var}_{\mathrm{analysis}} \cup \mathrm{VAR}_{\mathrm{analysis}}$.
A Δ^1_0 formula is a formula, not containing bounded second-order quantifiers, and a Π^1_1-formula is $\forall X.\phi$, where ϕ is a Δ^1_0-formula.

Remark 9.2 *We could omit that $A \wedge B$, $A \vee B$, $\exists x.A$ are formulas in the above definition and simplify the correctness theorem below. We keep those sets since the proof of Lemma 9.9 gives some insights about how the interpretation works.*

Assumption 9.3 *After renaming all variables, we assume, we have additional new variables U_i of KPI^+ ($i \in \omega$) and Z_i of ML ($i \in \omega$), s.t. in the step from a g-type to A^*, Z_i becomes U_i and in 6.7 (a), if $x_i = Z_j$, then on the right side we put U_j.*

Definition 9.4 *(a) Let C^{set}_i be new Gödelnumbers for new constructors $\mathrm{Constr}_{\mathrm{C}^{set}_i}$ (for which we write as before for simplicity C^{set}_i).*

(b) Let for a set b, $\mathrm{Embset}_{\mathrm{ML}}(f) := \{\langle n, \widehat{\mathbb{N}}_0\rangle | n \in \omega \setminus b\} \cup \{\langle n, \widehat{\mathbb{N}}_1\rangle | n \in \omega \cap b\}$.

(c) If $\mathrm{CES}(b)$, the Gödel numbers of the constructors in b are $\neq \mathrm{C}^{set}_{i_j}$ ($j = 1, \ldots, m$), $b_1, \ldots, b_m \in \mathrm{Ad}_2$, $b_i \subset \omega$, $Z_{i_j} \neq Z_{i_k}$, ($j \neq k$), then
$\mathrm{CES}^+(b, Z_{i_1}/b_1, \ldots, Z_{i_m}/b_m) := b \cup \{\langle \mathrm{C}^{set}_{i_1}, 0, f_1\rangle, \ldots, \langle \ulcorner \mathrm{C}^{set}_{i_m} \urcorner, 0, f_m\rangle\}$,
where $f_i := \mathrm{Embset}_{\mathrm{ML}}(b_i)$.

Definition 9.5 *Let $\mathcal{P}(\mathbb{N}) := \mathbb{N} \to \mathrm{U}$, $\mathrm{ML}^{\mathrm{i}}_1\mathrm{W}_{\mathrm{T}} \vdash \mathbb{N} \to \mathrm{U} : \mathrm{type}$.*

Definition 9.6 *(a) For each primitive recursive $g : \mathbb{N}^k \to \mathbb{N}$ we define a closed g-term $\mathrm{int}_{\mathrm{PA,ML}}(g)$, (we abbreviate this as $\hat{g} := \mathrm{int}_{\mathrm{PA,ML}}(g)$) such that*

$$\mathrm{ML} \vdash \hat{g} : \underbrace{\mathbb{N} \to \cdots \mathbb{N}}_{k\ times} \to \mathbb{N},$$

and we define a set $\mathrm{int}_{\mathrm{PA,KP}}(g)$ short $\widetilde{g}$ in $\mathcal{L}_{KP}$
such that $\mathrm{KPI}^+ \vdash fun(\widetilde{g}) \wedge \mathrm{dom}(\widetilde{g}) = \mathbb{N}^k \wedge \forall x \in \mathbb{N}^k.\widetilde{g}(x) \in \mathbb{N}$.
Case $g = \mathrm{S}$: $\hat{g} := \lambda x.\mathrm{S}(x)$, $\widetilde{g} := \{\langle x, x+1\rangle | x \in \mathbb{N}\}$.
Case $g = \mathrm{Proj}^n_i$ $\hat{g} := \lambda x_1, \ldots, x_n.x_i$, $\widetilde{g} := \{\langle\langle x_1, \ldots, x_n\rangle, x_i\rangle | x_1, \ldots, x_n \in \mathbb{N}\}$.
Case $g = \mathrm{Cons}^n_c$:

$\hat{g} := \lambda x_1, \ldots, x_n.\mathrm{S}^c(0)$, $\widetilde{g} := \{\langle\langle x_1, \ldots, x_n\rangle, c\rangle | x_1, \ldots, x_n \in \mathbb{N}\}$.
Case $g(x_1, \ldots, x_n) = h(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$*:*
$\hat{g} := \lambda x_1, \ldots, x_n.\hat{h}\ (\hat{g}_1\ x_1 \cdots x_n)\ \cdots\ (\hat{g}_m\ x_1 \cdots x_n)$,
$\widetilde{g} := \{\langle\langle x_1, \ldots, x_n\rangle, h(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))\rangle$
$| x_1, \ldots, x_n \in \mathbb{N}\}$.
Case $g(x_1, \ldots, x_n, 0) = h(x_1, \ldots, x_n)$,
$g(x_1, \ldots, x_n, y+1) = k(x_1, \ldots, x_n, y, g(x_1, \ldots, x_n, y))$*:*
$\hat{g} := \lambda x_1, \ldots, x_n, y.\mathrm{P}(y, \hat{h}\ x_1 \cdots x_n, (u, v)\hat{k}\ x_1 \cdots x_n\ u\ v)$,
define $l(x_1, \ldots, x_n, 0) := \widetilde{h}(x_1, \ldots, x_n)$,
$l(x_1, \ldots, x_n, \mathrm{S}(y)) := \widetilde{k}(x_1, \ldots, x_n, y, l(x_1 \cdots x_n, y))$,
then
$\widetilde{g} := \{\langle\langle x_1, \ldots, x_n, y\rangle, l(x_1, \ldots, x_n, y)\rangle | x_1, \ldots, x_n, y \in \mathbb{N}\}$.

(b) *For each term* t *of analysis we define a g-term* $\mathrm{int}_{\mathrm{PA,ML}}(t)$*, short* $\hat{t}$ *and a term of* $\mathcal{L}_{KP}$ $\mathrm{int}_{\mathrm{PA,KP}}(t)$*, short* $\widetilde{t}$*, such that, if* $\mathrm{FV}(t) = \{v_{i_1}, \ldots, v_{i_n}\}$ *(*$i_1 < \cdots < i_n$*) (*v_i *as in definition 9.1 of* $\mathrm{var}_{\mathrm{analysis}}$*) then* $\mathrm{FV}(\hat{t}) \subset \{z_{i_1}, \ldots, z_{i_n}\}$*,* $\mathrm{FV}(\widetilde{t}) \subset \{u_{i_1}, \ldots, u_{i_n}\}$ *and* $\mathrm{ML} \vdash z_{i_1} : \mathbb{N}, \ldots, z_{i_n} : \mathbb{N} \Rightarrow \hat{t} : \mathbb{N}$*, and*
$\mathrm{KPI}^+ \vdash \forall u_{i_1}, \ldots, u_{i_n} \in \omega.(\widetilde{t} \in \omega)$.
Case $t = v_i$*:* $\hat{t} := z_i$, $\widetilde{t} := u_i$.
Case $t = 0$*:* $\hat{t} := 0$, $\widetilde{t} := 0$.
Case $t = g(t_1, \ldots, t_n)$*:* $\hat{t} := \hat{g}\ \hat{t}_1\ \cdots\ \hat{t}_n$, $\widetilde{t} := \widetilde{g}(\widetilde{t}_1, \ldots, \widetilde{t}_n)$.

(c) *For each formula* A *of analysis we define a g-type* $\mathrm{int}_{\mathrm{PA,ML}}(A)$ *(short* $\widehat{A}$*), and a formula of* $\mathcal{L}_{KP}$ $\mathrm{int}_{\mathrm{PA,KP}}(A)$*, short* $\widetilde{A}$*, such that in* KPI^+ $\widetilde{A}$ *is equivalent to a* Δ_0*-formula, and if* $\mathrm{FV}(A) = \{v_{i_1}, \ldots, v_{i_n}, V_{j_1}, \ldots, V_{j_m}\}$*,* $i_k \neq i_l$*,* $j_k \neq j_l$ *for* $k \neq l$*, then* $\mathrm{FV}(\widehat{A}) \subset \{z_{i_1}, \ldots, z_{i_n}, Z_{j_1}, \ldots, Z_{j_m}\}$*,*
$\mathrm{FV}(\widetilde{A}) \subset \{u_{i_1}, \ldots, u_{i_n}, U_{j_1}, \ldots, U_{j_m}\}$ *and for all versions* ML *of Martin-Löf Type Theory considered in this article we get*
$\mathrm{ML} \vdash z_{i_1} : \mathbb{N}, \ldots, z_{i_n} : \mathbb{N}, Z_{j_1} : \mathcal{P}(\mathbb{N}), \ldots, Z_{j_m} : \mathcal{P}(\mathbb{N}) \Rightarrow \widehat{A} : \mathrm{type}$.
Case $A = (s = t)$*:* $\widehat{A} := I(\mathbb{N}, \hat{s}, \hat{t})$, $\widetilde{A} := (\widetilde{s} = \widetilde{t})$.
Case $A = (t \in V_i)$*:* $\widehat{A} := \mathrm{T}(Z_i\ \hat{t})$ *for the Tarski-version* $\widehat{A} := Z_i\ \hat{t}$ *for the Russell-version,* $\widetilde{A} := \widetilde{t} \in U_i$.
Case $A = (B \wedge C)$*:* $\widehat{A} := (\widehat{B} \times \widehat{C})$, $\widetilde{A} := \widetilde{B} \wedge \widetilde{C}$.
Case $A = (B \vee C)$*:* $\widehat{A} := (\widehat{B} + \widehat{C})$, $\widetilde{A} := (\widetilde{B} \vee \widetilde{C})$.
Case $A = (B \to C)$*:* $\widehat{A} := (\widehat{B} \to \widehat{C})$, $\widetilde{A} := (\widetilde{B} \to \widetilde{C})$.
Case $A = \forall v_i.B$*:* $\widehat{A} := \Pi z_i \in \mathbb{N}.\widehat{B}$, $\widetilde{A} := \forall u_i \in \omega.\widetilde{B}$.
Case $A = \exists v_i.B$*:* $\widehat{A} := \Sigma z_i \in \mathbb{N}.\widehat{B}$, $\widetilde{A} := \exists u_i \in \omega.\widetilde{B}$.
Case $A = \forall V_i.B$*:* $\widehat{A} := \Pi Z_i \in \mathcal{P}(\mathbb{N}).\widehat{B}$, $\widetilde{A} := \forall U_i \in \mathrm{Ad}_2.U_i \subset \omega \to \widetilde{B}$.
Case $A = \exists V_i.B$*:* $\widehat{A} := \Sigma Z_i \in \mathcal{P}(\mathbb{N}).\widehat{B}$, $\widetilde{A} := \exists U_i \in \mathrm{Ad}_2.U_i \subset \omega \wedge \widetilde{B}$.
Case $A = \perp$*:* $\widehat{A} := \mathbb{N}_0$, $\widetilde{A} := (0 \neq 0)$.

Definition 9.7 (a) *We define* $\mathrm{emb} : \omega \to \omega$, $\mathrm{emb}(n) := \mathrm{S}^n(0)(=: \hat{\hat{n}})$ *(or more precisely* $\ulcorner \mathrm{S}^n(0) \urcorner$*), a function definable in* KPI^+*.*

(b) $\langle a, \cdot\rangle := \langle a, a\rangle$.

Lemma 9.8 *(a) If* $g : \mathbb{N}^k \to \mathbb{N}$ *is primitive recursive, then*

$$\mathrm{KPI}^+ \vdash \quad \forall t_1, \ldots, t_k \in \mathrm{Term}_{\mathrm{Cl}}.\forall n_1, \ldots, n_k. \\ (r_1 \rightarrow_{\mathrm{red}} \hat{\hat{n}}_1 \wedge \cdots \wedge r_k \rightarrow_{\mathrm{red}} \hat{\hat{n}}_k) \to \hat{g}\ r_1\ \cdots\ r_k \rightarrow_{\mathrm{red}} \mathrm{emb}(\widetilde{g}(n_1, \ldots, n_k)).$$

(b) If t *is a term of analysis,* $\mathrm{FV}(t) \subset \{v_1, \ldots, v_n\}$*, then*

$$\mathrm{KPI}^+ \vdash \quad \forall r_1, \ldots, r_n \in \mathrm{Term}_{\mathrm{Cl}}.\forall n_1, \ldots, n_k.(r_1 \rightarrow_{\mathrm{red}} \hat{\hat{n}}_1 \wedge \cdots \wedge r_k \rightarrow_{\mathrm{red}} \hat{\hat{n}}_k) \to \\ \hat{t}[z_1/r_1, \ldots, z_n/r_n] \rightarrow_{\mathrm{red}} \mathrm{emb}(\widetilde{t}[u_1/n_1, \ldots, u_n/n_n]).$$

Proof: (a) Case $g = \mathrm{S}$: $\hat{g}\ r_1 \rightarrow_{\mathrm{red}} (\lambda x.\mathrm{S}(x))\ \hat{\hat{n}}_1 \rightarrow_{\mathrm{red}} \mathrm{S}(\hat{\hat{n}}_1) = \mathrm{emb}(\mathrm{S}(n_1))$.
Case $g = \mathrm{Proj}^n_i$: $\hat{g}\ r_1\ \cdots\ r_n \rightarrow_{\mathrm{red}} \hat{\hat{n}}_i = \mathrm{emb}(\widetilde{g}(n_1, \ldots, n_k))$.
Case $g = \mathrm{Cons}^n_c$: trivial.
Case $g(\vec{x}) = h(g_1(\vec{x}), \ldots, g_m(\vec{x}))$, $l_i := \widetilde{g}_i(n_1, \ldots, n_n)$:

$$\hat{g}_i\ r_1\ \cdots\ r_n \rightarrow_{\mathrm{red}} \mathrm{emb}(\widetilde{g}_i(n_1, \ldots, n_n)) =: \mathrm{emb}(l_i),$$

therefore

$$\begin{array}{ll} \hat{g}\ r_1\ \cdots\ r_n & \\ = & \hat{h}\ (\hat{g}_1\ r_1\ \cdots\ r_n))\ \cdots\ (\hat{g}_m\ r_1\ \cdots\ r_n)) \\ \rightarrow_{\mathrm{red}} & \hat{h}\ \hat{l}_1\ \cdots\ \hat{l}_n \\ \rightarrow_{\mathrm{red}} & \mathrm{emb}(\widetilde{h}(l_1, \ldots, l_m)) \\ = & \mathrm{emb}(g(n_1, \ldots, n_n)) \end{array}$$

Case $g(x_1, \ldots, x_n, 0) = h(x_1, \ldots, x_n)$, $g(\vec{x}, x_{n+1}) = k(\vec{x}, y, g(\vec{x}, x_{n+1})))$. Let $\hat{h}\ r_1\ \cdots\ r_n \rightarrow_{\mathrm{red}} \mathrm{emb}(\widetilde{h}(n_1, \ldots, n_n)) =: l_0$. We show by induction on l,

$$\mathrm{P}(\hat{\hat{l}}, l_0, (\lambda u, v.\hat{k}\ r_1\ \ldots\ r_n\ u\ v)) \rightarrow_{\mathrm{red}} \mathrm{emb}(\widetilde{g}(n_1, \ldots, n_n, l))$$

Then follows the assertion, since

$$\begin{array}{ll} g(r_1, \ldots, r_n, r_{n+1}) & \\ \rightarrow_{\mathrm{red}} & \mathrm{P}(\hat{\hat{n}}_{n+1}, l_0, (\lambda u, v.\hat{k}\ r_1\ \cdots\ r_n\ u\ v)) \\ \rightarrow_{\mathrm{red}} & \mathrm{emb}(\widetilde{g}(n_1, \ldots, n_n, n_{n+1})) \end{array}$$

Proof of the statement: If $l = 0$,

$$\begin{array}{ll} \mathrm{P}(\hat{\hat{l}}, l_0, (\lambda u, v.\hat{k}\ r_1\ \cdots\ r_n\ u\ v)) & \\ \rightarrow_{\mathrm{red}} & l_0 \\ = & \mathrm{emb}(\widetilde{g}(n_1, \ldots, n_n, l)) \end{array}$$

If $l = m + 1$,

$$\begin{array}{ll} & \mathrm{P}(\hat{\hat{l}}, l_0, (\lambda u, v.\hat{k}\ r_1,\ \cdots\ r_n\ u\ v)) \\ = & \mathrm{P}(\mathrm{S}(\hat{\hat{m}}), l_0, (\lambda u, v.\hat{k}\ r_1\ \cdots\ r_n\ u\ v)) \\ \to_{\mathrm{red}} & (\lambda u, v.\hat{k}\ r_1\ \cdots\ r_n\ u\ v)\ \hat{\hat{m}}\ \mathrm{P}(\hat{\hat{m}}, l_0, (\lambda u, v.\hat{k}\ r_1\ \ldots\ r_n\ u\ v)) \\ \to_{\mathrm{red}} & (\lambda u, v.\hat{k}\ r_1\ \cdots\ r_n\ u\ v)\ \hat{\hat{m}}\ \mathrm{emb}(g(n_1, \ldots, n_n, m)) \\ \to_{\mathrm{red}} & \hat{k}\ r_1\ \cdots\ r_n\ \hat{\hat{m}}\ \mathrm{emb}(g(n_1, \ldots, n_n, m)) \\ \to_{\mathrm{red}} & \hat{k}\ \hat{\hat{n}}_1, \cdots\ \hat{\hat{n}}_n\ \hat{\hat{m}}\ \mathrm{emb}(g(n_1, \ldots, n_n, m))) \\ \to_{\mathrm{red}} & \mathrm{emb}(\widetilde{k}(n_1, \ldots, n_n, m, g(n_1, \ldots, n_n, m))) \\ = & \mathrm{emb}(\widetilde{g}(n_1, \ldots, n_n, l)) \end{array}$$

(b): If $t = v_i, 0$, this is trivial,
and if $t = g(t_1, \ldots, t_n)$ it follows by IH

$$\hat{t}_i[\vec{r}] \to_{\mathrm{red}} \mathrm{emb}(\widetilde{t}_i[\vec{n}]),$$

by (a) therefore

$$\hat{t}[\vec{r}] = \hat{g}\ \hat{t}_1[\vec{r}]\ \cdots\ \hat{t}_n[\vec{r}] \to_{\mathrm{red}} \widetilde{g}(\mathrm{emb}(\widetilde{t}_1[\vec{n}]), \ldots, \mathrm{emb}(\widetilde{t}_n[\vec{n}])) = \widetilde{t}[\vec{n}]$$

Next task would now be to prove, that, when we first interpret a formula of $\mathcal{L}_{\mathrm{analysis}}$ in $\mathcal{L}_{ML}$ and then use the interpretation, as we have done in section 6, we get an equivalent formula to the one, we get by directly interpreting $\mathcal{L}_{\mathrm{analysis}}$ in $\mathcal{L}_{KP}$. But in this formulation, this is not correct, here is the place, where we need to extend the set of term constructors by non constructive constructors. In order to interpret a true Π^0_2 formula $A = \forall x.\exists y.\phi$ in such a way, that for the false formula $\neg A$ we have $(\neg A)^* = \emptyset$ we need an element of A^*, which gives for the x a witness y of ϕ. But this might be non constructive, so we add here a (possibly) non constructive new constructor. So for every formula we need certain new constructors. Furthermore, we want that Π^1_1-formulas are interpreted correctly as well, that is, we want, that if we have a free set variable V_i, we can replace it in the KPI^+-interpretation by arbitrary subsets $U_i \subset \omega$, $U_i \in \mathrm{Ad}_2$. We achieve this by allowing here arbitrary interpretations for the constructor C^{set}_i.

Lemma 9.9 *For every Δ^1_0-formula A with*

$$\mathrm{FV}(A) \subset \{v_{i_1}, \ldots, v_{i_l}, V_{j_1}, \ldots, V_{j_m}\}$$

with $i_k \neq i_l$, $j_k \neq j_l$, $(k \neq l)$ there is a CES *c not referring to the constructors C^{set}_i $(i \in \omega)$ and a g-term $h \in \mathrm{Term}_{\mathrm{Cl}}$ s.t.* $\mathrm{FV}(h) \subset \{z_{i_1}, \ldots, z_{i_l}\}$*, and with $\vec{z} := z_{i_1}, \ldots, z_{i_l}$, $\vec{Z} := Z_{j_1}, \ldots, Z_{j_m}$, $\vec{u} := u_{i_1}, \ldots, u_{i_l}$, $\vec{U} := U_{j_1}, \ldots, U_{j_m}$,*

$\vec{C^Z} := \mathrm{C}^{set}_{j_1}, \dots, \mathrm{C}^{set}_{j_m}$, $\vec{n} := n_1, \dots, n_l$, *we have*

$$\begin{array}{ll}\mathrm{KPI}^+ \vdash & \forall n_1, \dots, n_l \in \omega . \forall r_1, \dots, r_l \in \mathrm{Term}_{\mathrm{Cl}}. \\ & \quad \forall b_1, \dots, b_m \in \mathrm{Ad}_2 . \forall a_0 \in \mathrm{Ad}_2 . \\ & \quad (b_1 \subset \omega \wedge \cdots \wedge b_m \subset \omega \\ & \quad \;\wedge \mathrm{CES}(a_0) \wedge \mathrm{CES}^+(c, Z_{i_1}/b_1, \dots, Z_{i_m}/b_m) \subset a_0 \\ & \quad \;\wedge (r_1 \rightarrow_{\mathrm{red}} \hat{\hat{n}}_1 \wedge \cdots \wedge r_l \rightarrow_{\mathrm{red}} \hat{\hat{n}}_l)) \\ & \quad \rightarrow ((\exists r \in \mathrm{Term}_{\mathrm{nf}} . h[\vec{z}/\vec{r}] \rightarrow_{\mathrm{red}} r) \\ & \qquad \wedge (\widetilde{A}[\vec{u}/\vec{n}, \vec{U}/\vec{b}] \iff \langle h[\vec{z}/\vec{r}], \cdot \rangle \in \widehat{A}^*[\vec{z}/\vec{r}]_{a_0}) \\ & \qquad \wedge (\widetilde{A}[\vec{u}/\vec{n}, \vec{U}/\vec{b}] \iff \widehat{A}^*[\vec{z}/\vec{r}]_{a_0} \neq \emptyset))\end{array}$$

Proof: by induction on the definition of the formulas.
Note that by Remark 9.2 we could have omitted the formulas $A \wedge B$, $A \vee B$, and $\exists x.A$ and the corresponding cases in the current proof. We kept them because the proofs give some interesting insights how this method works.
As before we will not mention explicitly Variables, that occur in subterms, or do not occur at all.
Case $A = \perp$: Choose $c := \emptyset$, $h := 0$. We have $\neg \widetilde{A}[\vec{n}]$, $\widehat{A}^*[\vec{r}] = \emptyset$.
Case $A = (s = t)$: Choose as $c := \emptyset$, $h := \mathrm{r} \in \mathrm{Term}_{\mathrm{nf}}$. We have, using that for $r \rightarrow_{\mathrm{red}} s \in \mathrm{Term}_{\mathrm{nf}}$ s is unique, and by Lemma 9.8

$$\begin{array}{rcl}\widetilde{A}[\vec{n}] & \iff & \widetilde{s}[\vec{n}] = \widetilde{t}[\vec{n}] \\ & \iff & (\exists n \in \omega . \hat{s}[\vec{r}] \rightarrow_{\mathrm{red}} \mathrm{S}^n(0) \wedge \hat{t}[\vec{r}] \rightarrow_{\mathrm{red}} \mathrm{S}^n(0)) \\ & \iff & \langle \hat{s}[\vec{r}], \hat{t}[\vec{r}] \rangle \in \mathbb{N}^* \\ & \iff & \langle \mathrm{r}, \mathrm{r} \rangle \in \widehat{A}^*[\vec{r}] \\ & \iff & \widehat{A}^*[\vec{r}] \neq \emptyset\end{array}$$

Case $A = (s \in V_i)$: Let $c := \emptyset$, $h := \mathrm{A}^1_0$. Assume $\vec{r}$, $\vec{n}$, b_i, a_0 as in the assumption. Then $\hat{s}[\vec{z}/\vec{r}] \rightarrow_{\mathrm{red}} \mathrm{emb}(\widetilde{s}[\vec{u}/\vec{n}]) = \mathrm{S}^k(0)$ for some k.

$$\begin{array}{rcl}\widehat{A}[\vec{z}/\vec{r}, \vec{Z}/\vec{C^Z}]^*_{a_0} & = & \mathrm{T}(\mathrm{C}^{set}_i(\widetilde{s}[\vec{z}/\vec{r}]))_{a_0} \\ & = & \begin{cases} \mathrm{T}(\widehat{\mathbb{N}}_0) & \text{if } k \notin b_i \\ \mathrm{T}(\widehat{\mathbb{N}}_1) & \text{if } k \in b_i \end{cases} \\ & = & \begin{cases} \emptyset & \text{if } k \notin b_i \\ \mathrm{Compl}(\{\mathrm{A}^1_0\}) & \text{if } k \in b_i \end{cases}\end{array}$$

$\widetilde{A}[\vec{u}/\vec{n}, \vec{U}/\vec{b}] = \widetilde{s}[\vec{u}/\vec{n}] \in b_i$. This implies the assertion.
Case $A = (A_1 \wedge A_2)$: Let c_i, h_i for A_i chosen, $c := c_1 \cup c_2$, $h := \mathrm{p}(h_1, h_2)$. Then for $\vec{n}, \vec{r}, \vec{b}, a_0$ as in the assumption of the assertion there exist $s_1, s_2 \in \mathrm{Term}_{\mathrm{nf}}$, such that $r_i[\vec{r}] \rightarrow_{\mathrm{red}} s_i$, $h[\vec{r}] \rightarrow_{\mathrm{red}} \mathrm{p}(s_1, s_2) \in \mathrm{Term}_{\mathrm{nf}}$.

$$\begin{aligned}
\widetilde{A}[\vec{n}] &\iff \widetilde{B}_1[\vec{n}] \wedge \widetilde{B}_2[\vec{n}] \\
&\iff \langle s_1, \cdot\rangle \in \widehat{B}_1^*[\vec{n}] \wedge \langle s_2, \cdot\rangle \in \widehat{B}_2^*[\vec{n}] \\
&\iff \langle h[\vec{r}], \cdot\rangle \in \widehat{A}^*[\vec{n}] \\
&\iff \widehat{B}_1^*[\vec{r}] \neq \emptyset \wedge \widehat{B}_2^*[\vec{r}] \neq \emptyset \\
&\iff \widehat{A}^*[\vec{n}] \neq \emptyset
\end{aligned}$$

Case $A = (B_0 \vee B_1)$: Let c_i, h_i for B_i chosen. Let

$$f := \{\langle\langle n_1, \ldots, n_l\rangle, i\rangle | (i = 0 \wedge \widetilde{B}_0[\vec{n}]) \vee (i = 1 \wedge \neg\widetilde{B}_0[\vec{n}])\}$$

(note that $\widetilde{B}_0$ is a Δ_0-formula).
Let $\lceil C\rceil$ be a Gödel-number for a new constructor, different from all $\lceil \mathrm{C}^{set}\rceil$, $c := c_1 \cup c_2 \cup \{\langle\lceil C\rceil, l+1, f\rangle\}$, $h := \mathrm{P}(C\ \vec{z}, h_1, (u,v)h_2)$ (u, v new variables). Assume $\vec{n}$, $\vec{r}$, $\vec{b}$, a_0 as in the assumption. Then there exist s_i such that $h_i[\vec{r}] \rightarrow_{\mathrm{red}} s_i \in \mathrm{Term}_{\mathrm{nf}}$

$$C(r_1, \ldots, r_n) \rightarrow_{\mathrm{red}} C(\hat{\hat{n}}_1, \ldots, \hat{\hat{n}}_i) \rightarrow_{\mathrm{red}} \mathrm{S}^i(0)$$

for $i = f(\vec{n}) \in \{0, 1\}$.
We get, if $i = 0$,

$$h[\vec{r}] \rightarrow_{\mathrm{red}} \mathrm{P}(0, s_1, (u,v)h_2[\vec{r}]) \rightarrow_{\mathrm{red}} s_1 \in \mathrm{Term}_{\mathrm{nf}},$$

and if $i = 1$,

$$h[\vec{r}] \rightarrow_{\mathrm{red}} s_2.$$

We have

$$\begin{aligned}
\widetilde{A}[\vec{n}] &\iff \widetilde{B}_1[\vec{n}] \vee (\neg\widetilde{B}_1[\vec{n}] \wedge \widetilde{B}_2[\vec{n}]) \\
&\iff (f(\vec{n}) = 0 \wedge \langle s_1, \cdot\rangle \in \widehat{B}_1^*[\vec{r}]) \vee (f(\vec{n}) = 1 \wedge \langle s_2, \cdot\rangle \in \widehat{B}_2^*[\vec{r}]) \\
&\iff \langle h[\vec{r}], \cdot\rangle \in \widehat{A}^*[\vec{r}] \\
&\iff \widehat{B}_1^*[\vec{r}] \neq \emptyset \vee \widehat{B}_2^*[\vec{r}] \neq \emptyset \\
&\iff \widehat{A}^*[\vec{r}] \neq \emptyset
\end{aligned}$$

Case $A = (B_1 \rightarrow B_2)$. Let c_i, h_i for B_i chosen, $c := c_1 \cup c_2$, $h := \lambda x.h_2$. Assume $\vec{r}$, $\vec{n}$, $\vec{b}$, a_0 as in the assumption. Then $h_2[\vec{r}] \rightarrow_{\mathrm{red}} s_2$ for some $s_2 \in \mathrm{Term}_{\mathrm{nf}}$.
Subcase $\widetilde{A}[\vec{n}]$. If $\widetilde{B}_1[\vec{n}]$ is false, then by IH $\widehat{B}_1^*[\vec{r}] = \emptyset$, therefore

$$\forall\langle r, r'\rangle \in \widehat{B}_1^*[\vec{r}].\langle h_2[x/r, \vec{r}], h_2[x/r', \vec{r}]\rangle \in \widehat{B}_2^*[\vec{r}]$$

therefore $\langle h[\vec{r}], \cdot\rangle \in \widehat{A}^*[\vec{r}]$.
If $\widetilde{B}_1[\vec{n}]$ is true, then $\widetilde{B}_2[\vec{r}]$ is true, therefore $\langle s_2, \cdot\rangle \in \widehat{B}_2^*[\vec{r}]$,

$$\forall\langle r, r'\rangle \in \widehat{B}_1^*[\vec{r}].h_2[x/r, \vec{r}] \rightarrow_{\mathrm{red}} s_2 \wedge h_2[x/r', \vec{r}] \rightarrow_{\mathrm{red}} s_2 \wedge \langle s_2, \cdot\rangle \in \widehat{B}_2^*[\vec{n}],$$

$h[\vec{r}] \in \widehat{A}^*[\vec{n}]$.
Subcase $\neg\widetilde{A}[\vec{n}]$. Then by IH exists s_1 such that $h_1[\vec{r}] \rightarrow_{\mathrm{red}} s_1 \in \mathrm{Term_{nf}}$ and we have $\langle s_1, \cdot\rangle \in \widehat{B}_1^*[\vec{r}]$ and, if we had $\langle s, s'\rangle \in \widehat{A}^*[\vec{r}]$, then $\langle s, \cdot\rangle \in \widehat{A}^*[\vec{r}]$, $s \rightarrow_{\mathrm{red}} \lambda x.t$ for some t, $\langle t[x/s_1], \cdot\rangle \in \widehat{B}_2^*[\vec{r}] = \emptyset$, a contradiction, therefore $\widehat{A}^*[\vec{r}] = \emptyset$.

Case $A = \forall v_i.B$: Let c_1, h_1 for B be chosen, $c := c_1$, $h := \lambda v_i.h_1$. Assume $\vec{n}$, $\vec{r}$, $\vec{b}$, a_0 as in the assertion, $h[\vec{r}] \in \mathrm{Term_{nf}}$.
Assume $\langle r, r'\rangle \in \widehat{A}^*[\vec{r}]$, then $\langle r, r\rangle \in \widehat{A}^*[\vec{r}]$, $r \rightarrow_{\mathrm{red}} \lambda x.t$ and

$$\forall k \in \omega \langle t[x/\hat{k}, \vec{r}], \cdot\rangle . \in \widehat{B}^*[z_i/\hat{k}, \vec{r}],$$

by IH it follows $\forall k \in \omega.\widetilde{B}[u_i/k, \vec{n}]$, therefore $\widetilde{A}[\vec{n}]$.
Assume $\widetilde{A}[\vec{n}]$. Then for all $k \in \omega$ $\widetilde{B}[v_i/k, \vec{r}]$, therefore by IH $\langle h[v_i/r, \vec{r}], \cdot\rangle \in \widehat{B}^*[z_i/r, \vec{r}]$, whenever $r \rightarrow_{\mathrm{red}} \mathrm{S}^k(0)$, therefore $\langle h[\vec{r}], \cdot\rangle \in \widehat{A}[\vec{r}]$.
Case $A = \exists v_i.B$, c_1, h_1 be chosen for B,

$$\begin{aligned} f := \{\langle\langle\vec{n}\rangle, k\rangle| \quad & (\widetilde{B}[u_i/k, \vec{n}] \wedge \forall k' < k.\neg(\widetilde{B}[u_i/k, \vec{n}])) \vee \\ & (k = 0 \wedge \forall k \in \omega.\neg(\widetilde{B}[u_i/k, \vec{n}])) \end{aligned}$$

Let $\ulcorner C \urcorner$ be a new name for a constructor $\neq \mathrm{C}_i^{set}$, $c := c_1 \cup \{\langle\ulcorner C \urcorner, l+1, f\rangle\}$. $h := \mathrm{p}(C(\vec{z}), h_1[z_i/C(\vec{z})])$ Assume $\vec{n}$, $\vec{r}$ $\vec{b}$, a_0 as in the assertion, $k := f(\vec{n})$.

$$C(\vec{r}) \rightarrow_{\mathrm{red}} C(\hat{n}_1, \ldots, \hat{n}_n) \rightarrow_{\mathrm{red}} \mathrm{S}^k(0).$$

By IH we have $h_1[z_i/C(\vec{z})][\vec{r}] = h_1[z_i/C(\vec{r}), \vec{r}] \rightarrow_{\mathrm{red}} t_1$ for some $t_1 \in \mathrm{Term_{nf}}$, therefore $h[\vec{r}] \rightarrow_{\mathrm{red}} \mathrm{p}(\mathrm{S}^k(0), t_1)$.
Assume $\langle r, r'\rangle \in \widehat{A}^*[\vec{r}]$. Then $\langle r, r\rangle \in \widehat{A}^*[\vec{r}]$, $r \rightarrow_{\mathrm{red}} \mathrm{p}(\mathrm{S}^l(0), r'') \in \mathrm{Term_{nf}}$. Then $\langle r'', \cdot\rangle \in \widehat{B}^*[z_i/\mathrm{S}^l(0), \vec{r}]$, by IH $\widetilde{B}[u_i/l, \vec{n}]$, therefore $\widetilde{A}[\vec{n}]$.
Assume $\widetilde{A}[\vec{n}]$. Then by definition $\widetilde{B}[u_i/k, \vec{n}]$ and by IH

$$\langle t_1, \cdot\rangle \in \widehat{B}^*[z_i/\hat{\hat{k}}, \vec{r}] = \widehat{B}^*[z_i/C(\vec{z})[\vec{r}], \vec{r}] = \widehat{B}[z_i/C(\vec{z})]^*[\vec{z}/\vec{r}],$$

therefore $\langle h[\vec{r}], \cdot\rangle \in \widehat{A}^*[\vec{r}]$.

Lemma 9.10 *If ϕ is a Π_1^1-formula,* $\mathrm{ML} \vdash s : \widehat{\phi}$*, then* $\mathrm{KPI}^+ \vdash \widetilde{\phi}$.

Proof: Let $\phi = \forall V_i.B$. $\mathrm{ML} \vdash s : \widehat{\phi}$. By Lemma 8.1 it follows
$\mathrm{KPI}^+ \vdash \forall \mathrm{CES}(b).\langle \hat{s}_b^*, \cdot\rangle \in \widehat{\phi}_b^*$.
$\widehat{\phi}^*[\vec{r}] = \Pi^*(\widehat{\mathbb{N} \rightarrow \mathrm{U}}^*, \lambda\langle r, \cdot\rangle \in \widehat{\mathbb{N} \rightarrow \mathrm{U}}^*.\widehat{B}[\vec{r}, r])$. Assume $b_i \in \mathrm{Ad}_2$, x a variable, c the CES chosen for B as in Lemma 9.9, $a_0 := b_0 \cup \{\langle \mathrm{C}_i^{set}, 0, \mathrm{Embset}(b_i)\rangle\}$. We have $\mathrm{C}_i \in (\mathbb{N} \rightarrow \mathrm{U})_{a_0}^*$. Therefore, under the assumption $b_i \in \mathrm{Ad}_2$, $b_i \subset \omega$, $\mathrm{KPI}^+ \vdash \langle \hat{s}^*\ C_i, \cdot\rangle \in \widehat{B}_{a_0}^*[V_i/\mathrm{C}_i^{set}]$, by Lemma 9.9 $\mathrm{KPI}^+ \vdash \widetilde{B}[U_i/b_i]$, and we have $\mathrm{KPI}^+ \vdash \forall V_i \in \mathrm{Ad}_2.V_i \subset \omega \rightarrow \widetilde{B}$ which is $\widetilde{\phi}$.

10 Main Theorem

In this final section we will prove the result about the proof theoretic strength of the type theories used in this article (Theorem 10.5). We will show that the result about the embedding is sufficient to show that we have an upper bound for the proof-theoretic strength. We have to overcome the fact, that we did only prove, that if $\mathrm{ML} \vdash \mathrm{TI}(\prec)$ ($\mathrm{TI}(\prec)$ for transfinite induction over a primitive recursive relation $\prec$), we get $\mathrm{KPI}^+ \vdash \mathrm{TI}^{\mathrm{Ad}_2}(\prec)$, where $\mathrm{TI}^{\mathrm{Ad}_2}$ means transfinite induction, with the quantifier over subsets of ω, which are elements of Ad_2. But we will see, that this will be sufficient to obtain the result.

Definition 10.1 *We define some formulas in $\mathcal{L}_{KP}$:*

(a) In the following, $(a, \prec)$ will be a pair where a is a set, and $\prec \subset a \times a$. In this context $s \prec t := \langle s, t\rangle \in \prec$, $\forall x \prec t.\phi := \forall x \in a.x \prec t \to \phi$, and $\exists x \prec t.\phi := \exists x \in a.x \prec t \wedge \phi$. Furthermore, $s \preceq t := s \prec t \vee s = t$.

(b) $\mathrm{Wf}^d(a, \prec) := \prec \subset a \times a \wedge \forall x \in d.x \subset a \to x \neq \emptyset \to \exists y \in x.\forall z \prec y.z \notin x$. ($\prec$ is a relation on a which is well-founded, restricted to d).

(c) $\mathrm{Collaps}(a, \prec, f) := Fun(f) \wedge \mathrm{dom}(f) = a \wedge \forall x \in a.f(x) = \{f(y)|y \prec x\}$. ($f$ is a collapsing function on $(a, \prec)$).

Lemma 10.2 *If $\phi(y, y_1, \dots, y_n)$ is a Δ_0-formula with only the free variables mentioned, then*
$\mathrm{KPI}^+ \vdash \mathrm{Wf}^d(a, \prec) \to \mathrm{Ad}(c) \to \forall y_1, \dots, y_n \in c.(\forall x \in a.(\forall y \prec x.\phi(y, y_1, \dots, y_n) \to \phi(x, y_1, \dots, y_n))) \to \forall x \in a.\phi(x, y_1, \dots, y_n)$. *The formula after the second arrow is called principle of restricted induction over $(a, \prec)$.*

Proof: Immediate.

Lemma 10.3 $\mathrm{KPI}^+ \vdash \mathrm{Ad}(c) \to \mathrm{Ad}(d) \to c \in d \to \mathrm{Wf}^d(a, \prec) \to c, d \in a \to \exists f \in c.\mathrm{Collaps}(a, \prec, f)$.

Proof:
As in [Jäg86], Theorem 4.6, but replacing Δ_0-induction by d-induction.

Lemma 10.4 *If $\mathrm{KPI}^+ \vdash \forall x.\mathrm{Ad}(x) \to \phi(x)$ for a Σ_1-formula ϕ^x, then $\mathrm{L}_v \models \phi$, where ϕ^x is the restriction of all unrestricted quantifiers to x, and $v := \psi_{\Omega_1}(\Omega_{\mathrm{I}+\omega})$.*

Proof:
We follow the lines of [Buc92]. First observe, that we can prove as in Theorem 2.9 there, using several applications of $\exists_\kappa$, $(\bigwedge)$, and $\vdash^* \mathrm{Ad}(\mathrm{L}_\kappa)$ for $\kappa \in \mathrm{R}$, and if we have $\lambda \in \mathrm{Lim}$, $(\kappa_i)_{i\in\omega}$ a sequence, s.t. $\kappa_0 \in \mathrm{R}$, $\forall \alpha \in \kappa.\exists \rho \in \kappa.\alpha \in \rho \in \mathrm{R}$ and $\kappa_i \in \kappa_{i+1} \in \mathrm{R} \cap \lambda$ ($i \in \omega$), and if we extend X^* by including as well κ_i ($i \in \omega$) it follows

$$\vdash^*_\lambda (\mathrm{KPI}^+)^\lambda.$$

($\vdash^*_\lambda \phi^\lambda$ for every axiom ϕ of KPI^+). We can adjust Theorem 3.12 of [Buc92] to obtain, if we have in this situation, if $\lambda \in \mathcal{H}$, $\kappa_i \in \mathcal{H}$ ($i \in \omega$) and $\mathcal{H}$ closed under $\xi \mapsto \xi^{\mathrm{R}}$, then:

For each theorem ϕ of KPI^+ exists $k \in \mathbb{N}$ such that $\mathcal{H} \vdash^{\omega^{\lambda+k}}_{\lambda+k} \phi^\lambda$.
Now observe, that $\mathcal{H}_\gamma$ in [Buc92] has the desired properties (with $\lambda := \Omega_{\mathrm{I}+\omega}$, $\kappa_i := \Omega_{\mathrm{I}+i}$) and we conclude that if $\mathrm{KPI}^+ \vdash \forall x.\mathrm{Ad}(x) \to \phi^x$, where ϕ is a Σ-sentence, then $\mathrm{L}_v \models \phi$ for $v := \psi_{\Omega_1}(\Omega_{\mathrm{I}+\omega})$.

Theorem 10.5 $|\mathrm{ML}^{\mathrm{e}}_1\mathrm{W}_{\mathrm{T,U}}|$, $|\mathrm{ML}^{\mathrm{e}}_1\mathrm{W}_{\mathrm{T}}|$, $|\mathrm{ML}^{\mathrm{e}}_1\mathrm{W}_{\mathrm{R,U}}|$, $|\mathrm{ML}^{\mathrm{e}}_1\mathrm{W}_{\mathrm{R}}|$, $|\mathrm{ML}^{\mathrm{i}}_1\mathrm{W}_{\mathrm{T,U}}|$, $|\mathrm{ML}^{\mathrm{i}}_1\mathrm{W}_{\mathrm{T}}|$, $|\mathrm{ML}^{\mathrm{i}}_1\mathrm{W}_{\mathrm{R}}|$, $|\mathrm{ML}^{\mathrm{i}}_1\mathrm{W}_{\mathrm{R,U}}| = \psi_{\Omega_1}(\Omega_{\mathrm{I}+\omega})$, *where the ordinal denotation is as in [Buc92].*

Remark 10.6 *Since intensional type theory using the formulation of [TD88] which was used in [Set93] is a proper subtheory of* $|\mathrm{ML}^{\mathrm{e}}_1\mathrm{W}_{\mathrm{T}}|$, $\psi_{\Omega_1}(\Omega_{\mathrm{I}+\omega})$ *is an upper bound for the proof theoretic strength of the Tarski and Russell formulations of intensional type theory used in [Set93].*

Proof of Theorem 10.5: Let $v := \psi_{\Omega_1}(\Omega_{\mathrm{I}+\omega})$. By [Set98b] we have $|\mathrm{ML}^{\mathrm{i}}_1\mathrm{W}_{\mathrm{T}}| \geq= v$. Since the Russell formulations embed trivially into the Tarski Formulations (replace $\mathrm{T}(r)$ by r and $\widehat{\mathbb{N}}_k, \widehat{\mathbb{N}}, \widehat{\Pi}, \widehat{\Sigma}, \widehat{\mathrm{W}}, \widehat{+}, \widehat{\mathrm{I}}$ by $\mathbb{N}_k, \mathbb{N}, \Pi, \Sigma, \mathrm{W}, +, \mathrm{I}$, respectively), we obtain as well $|\mathrm{ML}^{\mathrm{i}}_1\mathrm{W}_{\mathrm{R}}| \geq v$. Since $\mathrm{ML}^{\mathrm{i}}_1\mathrm{W}_{\mathrm{T,U}}$ is an extension of $\mathrm{ML}^{\mathrm{i}}_1\mathrm{W}_{\mathrm{T}}$, which both embed into $\mathrm{ML}^{\mathrm{e}}_1\mathrm{W}_{\mathrm{T,U}}$ and $\mathrm{ML}^{\mathrm{e}}_1\mathrm{W}_{\mathrm{T}}$, respectively, and $\mathrm{ML}^{\mathrm{i}}_1\mathrm{W}_{\mathrm{R,U}}$ is an extension of $\mathrm{ML}^{\mathrm{i}}_1\mathrm{W}_{\mathrm{R}}$, which both embed into $\mathrm{ML}^{\mathrm{e}}_1\mathrm{W}_{\mathrm{R,U}}$ and $\mathrm{ML}^{\mathrm{e}}_1\mathrm{W}_{\mathrm{R}}$, respectively, v is a lower bound for all theories in question.
Regarding the upper bound, we show $|\mathrm{ML}| \leq v$. Since all the theories can be embedded into $\mathrm{ML} := \mathrm{ML}^{\mathrm{e}}_1\mathrm{W}_{\mathrm{T,U}}$ in such a way, that the principle of transfinite induction remains unchanged (except, that $\mathrm{T}(U_i(t))$ becomes $U_i(t)$ in the version à là Russell), we obtain that v is an upper bound for the proof theoretic strength of all other theories in the theorem as well.
Proof of $|\mathrm{ML}| \leq v$: Assume $\prec$ is a primitive recursive linear ordering on the primitive recursive subset T of ω, $\phi := \forall X.(\forall y.y \in T \to (\forall z.z \prec y \to z \in X) \to y \in X) \to \forall y.y \in T \to y \in X$ and $\mathrm{ML} \vdash \widehat{\phi}$. Then by Lemma 9.10 $\mathrm{KPI}^+ \vdash \widetilde{\phi}$. We follow the proof of [Rat91] Theorem 7.14. Let $a := \{x \in \omega | x \in T\}$, $\prec' := \{\langle x, y\rangle \in \omega \times \omega | x \prec y\}$. Then $\mathrm{KPI}^+ \vdash \mathrm{Ad}(\mathrm{Ad}_1) \wedge \mathrm{Ad}(\mathrm{Ad}_2) \wedge \mathrm{Ad}_1 \in \mathrm{Ad}_2 \wedge \mathrm{Wf}^{\mathrm{Ad}_2}(a, \prec') \wedge a, \prec' \in \mathrm{Ad}_1$, therefore by 10.3 $\mathrm{KPI}^+ \vdash \exists f \in \mathrm{Ad}_1.\mathrm{Collaps}(a, \prec', f)$, $\mathrm{KPI}^+ \vdash \forall x.\mathrm{Ad}(x) \to \exists f \in x.\mathrm{Collaps}(a, \prec', f)$. Therefore $\mathrm{L}_v \models \exists f.\mathrm{Collaps}(a, \prec' o, f)$. Since $\prec$ is linear ordering, it follows that $\mathrm{Image}(f)$ is an ordinal, and, because $v \in \mathrm{Lim}$ we have $\mathrm{Image}(f) \in \mathrm{L}_v$, $\mathrm{ordertype}(\prec) = \mathrm{Image}(f) < v$.

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