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# Reconstructing rational stable motivic homotopy theory

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Abstract

Using a recent computation of the rational minus part of $SH(k)$ by Ananyevskiy–Levine–Panin [ALP17], a theorem of Cisinski–Déglise [CD12] and a version of the Röndigs–Østvær [RØ08a] theorem, rational stable motivic homotopy theory over an infinite perfect field of characteristic different from 2 is recovered in this paper from finite Milnor–Witt correspondences in the sense of Calmès–Fasel [CF14].

## 1. Introduction

By the celebrated Serre finiteness theorem [Ser53] the positive stable homotopy groups of the classical sphere spectrum with rational coefficients are zero. It implies that the stable homotopy category of $S^1$-spectra with rational coefficients $SH_{\mathbb{Q}}$ is naturally equivalent to the homotopy category of $H\mathbb{Q}$-modules $\mathrm{Ho}(\mathrm{Mod}\, H\mathbb{Q})$, where $H\mathbb{Q}$ is the Eilenberg–Mac Lane symmetric spectrum of $\mathbb{Q}$. By the Robinson theorem [Rob87] the homotopy category of $HA$-modules $\mathrm{Ho}(\mathrm{Mod}\, HA)$, where $A$ is a ring with identity, is equivalent to the derived category $D(A)$ of $A$. Thus $SH_{\mathbb{Q}}$ is naturally equivalent to the derived category of rational vector spaces $D(\mathbb{Q})$.

In the motivic world the role of a ring is played by a "preadditive category of correspondences" $\mathcal{A}$ whose objects are the smooth algebraic varieties $Sm/k$ over a field $k$. Using the category theory terminology, $\mathcal{A}$ is a ring with several objects, whose objects are those of $Sm/k$. In turn, the role of the classical derived category over a ring is played by the category $DM_{\mathcal{A}}(k)$, which is just an extension of the celebrated Voevodsky triangulated category [Voe00] $DM(k)$ to general correspondences. Since motivic homotopy theory requires the Nisnevich topology and contractibility of the affine line $\mathbb{A}^1$, we require the relevant properties for $\mathcal{A}$ to satisfy (see Section 2 for details).

The rational stable motivic homotopy theory $SH(k)_{\mathbb{Q}}$ splits in two parts: $SH^+(k)_{\mathbb{Q}}$ and $SH^-(k)_{\mathbb{Q}}$. The plus part $SH^+(k)_{\mathbb{Q}}$ is equivalent to Voevodsky's $DM(k)_{\mathbb{Q}}$ (this follows from a theorem of Cisinski–Déglise [CD12, Theorem 16.2.13]). Ananyevskiy–Levine–Panin [ALP17] have computed $SH^-(k)_{\mathbb{Q}}$ as the category of Witt motives with rational coefficients (see Bachmann [Bac18] as well). Using these results, we show in Theorem 4.2 that the rational motivic sphere spectrum $\mathbb{S}\otimes\mathbb{Q}$ is naturally equivalent to the "additive motivic sphere spectrum" $\mathbb{S}^{MW}\otimes\mathbb{Q}$ associated with the additive category of finite Milnor–Witt correspondences $\widetilde{\mathrm{Cor}}$ in the sense of Calmès–Fasel [CF14].

Next, we extend the Röndigs–Østvær theorem [RØ08a, Theorem 1] to the triangulated category $DM_{\mathcal{A}}(k)$ (see Theorem 5.3). *This extension is of independent interest!* For example, it is of great utility to compare various triangulated categories of motives in [Gar18]. The generalised Röndigs–Østvær theorem can also be regarded as a motivic counterpart of the Robinson

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theorem [Rob87].

Theorem 4.2 computing $\mathbb{S}\otimes\mathbb{Q}$ together with generalised Röndigs–Østvær's Theorem 5.3 lead to the proof of the main result of the paper which is formulated as follows (see Theorem 5.5).

THEOREM (Reconstruction). *If $k$ is an infinite perfect field of characteristic not 2, then $SH(k)_{\mathbb{Q}}$ is equivalent to the triangulated category of Milnor–Witt motives with rational coefficients $DM_{MW}(k)_{\mathbb{Q}}$ in the sense of [DF17]. The equivalence preserves the triangulated structure.*

One of the approaches to constructing motivic homotopy theory, pioneered by Voevodsky, is to use various correspondences on smooth algebraic varieties. This approach has many computational advantages. Voevodsky constructed [Voe00] the category of motives $DM(k)$ by using finite correspondences. Later he developed the theory of framed correspondences [Voe01]. One of the aims was to suggest another framework for Morel–Voevodsky's stable motivic homotopy theory $SH(k)$. In [GP14b] the author and Panin use Voevodsky's theory to develop the theory of big framed motives which converts the classical Morel–Voevodsky stable motivic homotopy theory into an equivalent local theory of framed bispectra.

One of the central objects of the theory of (big) framed motives in the sense of [GP14b] is linear framed motives of algebraic varieties. They are explicitly constructed complexes of Nisnevich sheaves with framed correspondences $\mathbb{Z}F(-\times\Delta^\bullet, X)$, where $X\in Sm/k$. As an application of the Reconstruction Theorem we prove the following result comparing motivic complexes with framed and Milnor–Witt correspondences respectively (see Theorem 6.1).

THEOREM (Comparison). *Given an infinite perfect field of characteristic not 2 and a $k$-smooth scheme $X$, each morphism of complexes of Nisnevich sheaves*

$$f_n : \mathbb{Z}F(-\times\Delta^\bullet, X\times\mathbb{G}_m^{\wedge n})\otimes\mathbb{Q}\to\widetilde{\mathrm{Cor}}(-\times\Delta^\bullet, X\times\mathbb{G}_m^{\wedge n})_{\mathrm{nis}}\otimes\mathbb{Q},\quad n\geqslant 0,$$

*is a quasi-isomorphism, where the left complex is the $n$-twisted linear framed motive of $X$ with rational coefficients in the sense of [GP14b].*

Throughout the paper we denote by $Sm/k$ the category of smooth separated schemes of finite type over the base field $k$.

## 2. Additive categories of correspondences

In this section we set up a framework within which we shall work later.

DEFINITION 2.1. We say that a preadditive category $\mathcal{A}$ is a *category of correspondences* if:

(i) Its objects are those of $Sm/k$. Its morphisms are also referred to as $\mathcal{A}$-*correspondences* or just *correspondences*.

(ii) There is a functor $\rho : Sm/k\to\mathcal{A}$, which is the identity map on objects. The image $\rho(f)$ of a morphism of smooth schemes $f : X\to Y$ will be referred to as the *graph of $f$* and denoted by $\Gamma_f$. We have in particular that $\Gamma_{gf}=\Gamma_g\circ\Gamma_f$ and $\Gamma_{\mathrm{id}}=\mathrm{id}$. Thus we have a functor

$$\mathcal{A} : (Sm/k)^{\mathrm{op}}\times Sm/k\to\mathrm{Ab},\quad (X,Y)\mapsto\mathcal{A}(X,Y),$$

such that $\mathcal{A}(1_X,g)=\Gamma_g\circ -$ and $\mathcal{A}(h,1_Y)=-\circ\Gamma_h$.

(iii) For every elementary Nisnevich square

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & X' \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

the sequence of Nisnevich sheaves

$$0 \to \mathcal{A}(-, U')_{\mathrm{nis}} \to \mathcal{A}(-, U)_{\mathrm{nis}} \oplus \mathcal{A}(-, X')_{\mathrm{nis}} \to \mathcal{A}(-, X)_{\mathrm{nis}} \to 0$$

is exact. Moreover, we require $\mathcal{A}(-, \emptyset)_{\mathrm{nis}} = 0$ (corresponding to the "degenerate distinguished square", $\emptyset$, with only one entry in the lower right-hand corner).

(iv) For every $\mathcal{A}$-presheaf $\mathcal{F}$ (i.e. an additive contravariant functor from $\mathcal{A}$ to Abelian groups Ab) the associated Nisnevich sheaf $\mathcal{F}_{\mathrm{nis}}$ has a unique structure of an $\mathcal{A}$-presheaf for which the canonical morphism $\mathcal{F} \to \mathcal{F}_{\mathrm{nis}}$ is a morphism of $\mathcal{A}$-presheaves.

(v) There is an action of $Sm/k$ on $\mathcal{A}$ in the following sense. Given $U \in Sm/k$ there is a homomorphism

$$\alpha_U : \mathcal{A}(X, Y) \to \mathcal{A}(X \times U, Y \times U),$$

functorial in $X$ and $Y$, such that for any morphism $f : U \to V$ in $Sm/k$ the following square of Abelian groups is commutative

$$\begin{tikzcd} \mathcal{A}(X \times V, Y \times V) \arrow[r, "\mathcal{A}(1_X \times f, 1_{Y \times V})"] & \mathcal{A}(X \times U, Y \times V) \\ \mathcal{A}(X, Y) \arrow[u, "\alpha_V"] \arrow[r, "\alpha_U"] & \mathcal{A}(X \times U, Y \times U). \arrow[u, "\mathcal{A}(1_{X \times U}, 1_Y \times f)"'] \end{tikzcd}$$

We require $\alpha_U(\mathrm{id}_X) = \mathrm{id}_{X \times U}$ for all $U, Z \in Sm/k$. By the functoriality of $\alpha_U$ in $X$ we mean that the following square of Abelian groups is commutative for any $Y \in Sm/k$ and any morphism $f : X' \to X$ in $\mathcal{A}$

$$\begin{tikzcd} \mathcal{A}(X \times U, Y \times U) \arrow[r, "\mathcal{A}(f \times 1_U, 1_{Y \times U})"] & \mathcal{A}(X' \times U, Y \times U) \\ \mathcal{A}(X, Y) \arrow[u, "\alpha_U"] \arrow[r, "\mathcal{A}(f, 1_Y)"] & \mathcal{A}(X', Y). \arrow[u, "\alpha_U"'] \end{tikzcd}$$

By the functoriality of $\alpha_U$ in $Y$ we mean that the following square of additive functors is commutative for any $X \in Sm/k$ and any morphism $g : Y \to Y'$ in $\mathcal{A}$

$$\begin{tikzcd} \mathcal{A}(X \times U, Y \times U) \arrow[r, "\mathcal{A}(1_{X \times U}, g \times 1_U)"] & \mathcal{A}(X \times U, Y' \times U) \\ \mathcal{A}(X, Y) \arrow[u, "\alpha_U"] \arrow[r, "\mathcal{A}(1_X, g)"] & \mathcal{A}(X, Y'). \arrow[u, "\alpha_U"'] \end{tikzcd}$$

In other words, we have a functor

$$\boxtimes : \mathcal{A} \times Sm/k \to \mathcal{A}$$

sending $(X, U) \in Sm/k \times Sm/k$ to $X \times U \in Sm/k$ and such that $1_X \boxtimes f = \Gamma_{1_X \times f}$, $(u + v) \boxtimes f = u \boxtimes f + v \boxtimes f$ for all $f \in \mathrm{Mor}(Sm/k)$ and $u, v \in \mathrm{Mor}(\mathcal{A})$.

Remark 2.2. It follows from Definition 2.1(3) that the canonical morphism

$$\mathcal{A}(-, X)_{\mathrm{nis}} \oplus \mathcal{A}(-, Y)_{\mathrm{nis}} \to \mathcal{A}(-, X \sqcup Y)_{\mathrm{nis}}$$

is an isomorphism of Nisnevich sheaves.

Observe that for any category of correspondences $\mathcal{A}$, an $\mathcal{A}$-presheaf $\mathcal{F}$ and $U \in Sm/k$ the presheaf

$$\underline{\mathrm{Hom}}(U,\mathcal{F}) := \mathcal{F}(- \times U)$$

is an $\mathcal{A}$-presheaf. Moreover, it is functorial in $U$.

For instance $\mathcal{A}$ can be given by the naive preadditive category of correspondences $\mathcal{A}_{naive}$ with $\mathcal{A}_{naive}(X,Y)$ being the free abelian group generated by $\mathrm{Hom}_{Sm/k}(X,Y)$. Non-trivial examples are given by finite correspondences Cor in the sense of Voevodsky [Voe00], finite Milnor–Witt correspondences $\widetilde{\mathrm{Cor}}$ in the sense of Calmès–Fasel [CF14] or $K_0^{\oplus}$ in the sense of Walker [Wal96]. Given a ring $R$ (not necessarily commutative) which is flat as a $\mathbb{Z}$-algebra and a category of correspondences $\mathcal{A}$, we can form an *additive category of correspondences* $\mathcal{A}_R$ with coefficients in $R$. By definition, $\mathcal{A}_R(X,Y) := \mathcal{A}(X,Y) \otimes R$ for all $X, Y \in Sm/k$.

Definition 2.3. We say that a category of correspondences $\mathcal{A}$ is a *$V$-category of correspondences* ("$V$" for Voevodsky) if for any $\mathbb{A}^1$-invariant $\mathcal{A}$-presheaf of abelian groups $\mathcal{F}$ the associated Nisnevich sheaf $\mathcal{F}_{\mathrm{nis}}$ is $\mathbb{A}^1$-invariant. Recall that a Nisnevich sheaf $\mathcal{F}$ of abelian groups is *strictly $\mathbb{A}^1$-invariant* if for any $X \in Sm/k$, the canonical morphism

$$H^*_{\mathrm{nis}}(X,\mathcal{F}) \to H^*_{\mathrm{nis}}(X \times \mathbb{A}^1,\mathcal{F})$$

is an isomorphism. A $V$-category of correspondences $\mathcal{A}$ is a *strict $V$-category of correspondences* if for any $\mathbb{A}^1$-invariant $\mathcal{A}$-presheaf of abelian groups $\mathcal{F}$ the associated Nisnevich sheaf $\mathcal{F}_{\mathrm{nis}}$ is strictly $\mathbb{A}^1$-invariant.

Observe that any (strict) $V$-category of correspondences is a (strict) $V$-ringoid in the sense of [GP14a]. For example Cor and $K_0^{\oplus}$ are $V$-categories of correspondences, which are strict whenever the base field $k$ is perfect (see [Voe00] and [Wal96]). The category $\widetilde{\mathrm{Cor}}$ is a $V$-category of correspondences, which is strict if $k$ is infinite and perfect with char $k \neq 2$ (see [DF17, Kol17]). Observe that if $\mathcal{A}$ is a $V$-category of correspondences then so is $\mathcal{A}_R$ with $R$ commutative flat as a $\mathbb{Z}$-algebra. Moreover, if $R$ is a ring of fractions of $\mathbb{Z}$ like, for example, $\mathbb{Z}[\frac{1}{p}]$ or $\mathbb{Q}$, then $\mathcal{A}_R$ is a strict $V$-category of correspondences whenever $\mathcal{A}$ is.

Let $\mathcal{A}$ be a category of correspondences. Let $Sh(Sm/k)$ (respectively $Sh(\mathcal{A})$) denote the category of Nisnevich sheaves on $Sm/k$ (respectively Nisnevich $\mathcal{A}$-sheaves). Similar to [GP12, Corollary 6.4] $Sh(\mathcal{A})$ is a Grothendieck category such that $\{\mathcal{A}(-,X)_{\mathrm{nis}}\}_{X \in Sm/k}$ is a family of generators of $Sh(\mathcal{A})$. Denote by $D(Sh(Sm/k))$ and $D(Sh(\mathcal{A}))$ the corresponding derived categories of unbounded complexes. Note that $D(Sh(Sm/k)) = D(Sh(\mathcal{A}_{naive}))$.

The category $\mathbb{M}$ of motivic spaces consists of contravariant functors from $Sm/k$ to pointed simplicial sets. We refer the reader to [Jar00, MV99] for the definition of motivic weak equivalences between motivic spaces.

Lemma 2.4. *Given any field $k$, let $\mathcal{A}$ be a category of correspondences. Then the natural map*

$$f : \mathcal{A}(-, X \times \mathbb{A}^1) \to \mathcal{A}(-,X)$$

*is a motivic weak equivalence in the category of motivic spaces $\mathbb{M}$.*

*Proof.* We follow an argument of [RØ08a, p. 694]. As in classical algebraic topology, an inclusion of pointed motivic spaces $g : A \to B$ is an $\mathbb{A}^1$-deformation retract if there exist a map $r : B \to A$

such that $rg = \mathrm{id}_A$ and an $\mathbb{A}^1$-homotopy $H : B \wedge \mathbb{A}^1_+ \to B$ between $gr$ and $\mathrm{id}_B$ which is constant on $A$. Then $\mathbb{A}^1$-deformation retracts are motivic weak equivalences.

There is an obvious map $r : \mathcal{A}(-, X) \to \mathcal{A}(-, X \times \mathbb{A}^1)$ such that $fr = 1$. Since $Sm/k$ naturally acts on $\mathcal{A}$, it follows that $\mathcal{A}(- \times \mathbb{A}^1, X \times \mathbb{A}^1)$ is an $\mathcal{A}$-presheaf.

There is a natural isomorphism

$$\mathrm{Hom}(\mathcal{A}(-, X \times \mathbb{A}^1), \mathcal{A}(- \times \mathbb{A}^1, X \times \mathbb{A}^1)) \cong \mathcal{A}(X \times \mathbb{A}^1 \times \mathbb{A}^1, X \times \mathbb{A}^1),$$

where the Hom-set on the left is taken in the category of $\mathcal{A}$-presheaves. Consider the functor $\rho : Sm/k \to \mathcal{A}$. Denote by $\alpha$ the obvious map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$. We set $h = \rho(1_X \times \alpha)$; then $h$ uniquely determines a morphism of $\mathcal{A}$-presheaves

$$h' : \mathcal{A}(-, X \times \mathbb{A}^1) \to \mathcal{A}(- \times \mathbb{A}^1, X \times \mathbb{A}^1).$$

This morphism can be regarded as a morphism in $\mathbb{M}$, denoted by the same letter. By adjointness $h'$ uniquely determines a map in $\mathbb{M}$

$$H : \mathcal{A}(-, X \times \mathbb{A}^1) \wedge \mathbb{A}^1_+ \to \mathcal{A}(-, X \times \mathbb{A}^1).$$

Then $H$ yields an $\mathbb{A}^1$-homotopy between the identity map and $rf$. We see that $f$ is a motivic weak equivalence, as required. $\square$

By the general localization theory of compactly generated triangulated categories [Nee96] one can localize $D(Sh(\mathcal{A}))$ with respect to the localizing subcategory $\mathcal{L}$ generated by complexes of the form

$$\cdots \to 0 \to \mathcal{A}(-, X \times \mathbb{A}^1)_{\mathrm{nis}} \xrightarrow{pr_X} \mathcal{A}(-, X)_{\mathrm{nis}} \to 0 \to \cdots, \quad X \in Sm/k.$$

The resulting quotient category $D(Sh(\mathcal{A}))/\mathcal{L}$ is denoted by $D_{\mathbb{A}^1}(Sh(\mathcal{A}))$.

If we denote by $DM^{eff}_{\mathcal{A}}(k)$ the full subcategory of $D(Sh(\mathcal{A}))$ consisting of the complexes with strictly $\mathbb{A}^1$-invariant homology sheaves, then similar to a theorem of Voevodsky [Voe00] the composite functor

$$DM^{eff}_{\mathcal{A}}(k) \hookrightarrow D(Sh(\mathcal{A})) \to D_{\mathbb{A}^1}(Sh(\mathcal{A}))$$

is an equivalence of triangulated categories whenever $\mathcal{A}$ is a strict $V$-category of correspondences. Moreover, the functor

$$C_* : D(Sh(\mathcal{A})) \to D(Sh(\mathcal{A})), \quad X \mapsto Tot(X(- \times \Delta^\bullet)),$$

lands in $DM^{eff}_{\mathcal{A}}(k)$. The kernel of $C_*$ is $\mathcal{L}$ and $C_*$ is left adjoint to the inclusion functor

$$i : DM^{eff}_{\mathcal{A}}(k) \hookrightarrow D(Sh(\mathcal{A}))$$

(see [Voe00] for details or [GP14a, Theorem 3.5]).

Let $(\mathbb{G}_m, 1) \in \mathbb{M}$ denote $\mathbb{G}_m$ pointed at 1 and let $\mathbb{G}_m^{\mathcal{A}}$ be the sheaf

$$\mathrm{Coker}(\mathcal{A}(-, pt)_{\mathrm{nis}} \to \mathcal{A}(-, \mathbb{G}_m)_{\mathrm{nis}}),$$

induced by the map $pt \mapsto 1 \in \mathbb{G}_m$ in $Sm/k$. Regarding it as a complex concentrated in zeroth degree, we have an endofunctor

$$- \boxtimes \mathbb{G}_m^{\mathcal{A}} : D_{\mathbb{A}^1}(Sh(\mathcal{A})) \to D_{\mathbb{A}^1}(Sh(\mathcal{A})),$$

induced by the action of $Sm/k$ on $\mathcal{A}$. In more detail, by [AG16, Proposition 3.4] $\mathrm{Ch}(Sh(\mathcal{A}))$ is a Grothendieck category with generators of the form $\{D^n\mathcal{A}(-, U)_{\mathrm{nis}}\}_{n \in \mathbb{Z}, U \in Sm/k}$. Here $D^n\mathcal{A}(-, U)_{\mathrm{nis}}$

is the complex which is $\mathcal{A}(-,U)_{\mathrm{nis}}$ in degrees $n$ and $n-1$ and 0 elsewhere, with interesting differential being the identity map. Every complex $X \in \mathrm{Ch}(Sh(\mathcal{A}))$ is written as a colimit of generators

$$X = \mathrm{colim}_{(D^n\mathcal{A}(-,U)_{\mathrm{nis}} \to X)} D^n\mathcal{A}(-,U)_{\mathrm{nis}}.$$

We set,

$$X \boxtimes \mathbb{G}_m^{\mathcal{A}} := \mathrm{colim}_{(D^n\mathcal{A}(-,U)_{\mathrm{nis}} \to X)} D^n\mathcal{A}(-,U \wedge \mathbb{G}_m^{\wedge 1})_{\mathrm{nis}},$$

where the sheaf $\mathcal{A}(-,U \wedge \mathbb{G}_m^{\wedge 1})_{\mathrm{nis}} := \mathrm{Coker}(\mathcal{A}(-,U \times pt)_{\mathrm{nis}} \to \mathcal{A}(-,U \times \mathbb{G}_m)_{\mathrm{nis}})$.

Stabilizing $D_{\mathbb{A}^1}(Sh(\mathcal{A}))$ in the $\mathbb{G}_m$-direction with respect to this endofunctor, we arrive at the category $D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A}))$. If $\mathcal{A}$ is a strict $V$-category of correspondences, we can likewise stabilize $DM^{eff}_{\mathcal{A}}(k)$ in the $\mathbb{G}_m$-direction. The resulting category is denoted by $DM_{\mathcal{A}}(k)$. The triangulated equivalence $C_* : D_{\mathbb{A}^1}(Sh(\mathcal{A})) \to DM^{eff}_{\mathcal{A}}(k)$ extends to a triangulated equivalence

$$C_* : D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A})) \to DM_{\mathcal{A}}(k).$$

Given a category of correspondences $\mathcal{A}$ and $p > 0$, we shall write $D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A}))[p^{-1}]$ (respectively $D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A}))_{\mathbb{Q}}$) to denote the category $D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A} \otimes \mathbb{Z}\left[\frac{1}{p}\right]))(k)$ (respectively $D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A} \otimes \mathbb{Q}))(k)$). Note that $\mathcal{A} \otimes \mathbb{Z}\left[\frac{1}{p}\right]$ and $\mathcal{A} \otimes \mathbb{Q}$ are categories of correspondences.

Definition 2.5. We say that a category of correspondences $\mathcal{A}$ is *symmetric monoidal* if the usual product of schemes defines a symmetric monoidal structure on $\mathcal{A}$.

The categories Cor, $\widetilde{\mathrm{Cor}}$, $K_0^{\oplus}$ are examples of symmetric monoidal $V$-categories (see [CF14, DF17, Sus03, SV00, Wal96] for more details). $\mathcal{A}_{naive}$ is obviously symmetric monoidal.

Given a symmetric monoidal category of correspondences $\mathcal{A}$, a theorem of Day [Day70] implies that the category of $\mathcal{A}$-presheaves $PSh(\mathcal{A})$ is a closed symmetric monoidal category with a tensor product defined as

$$X \otimes Y = \int^{(U,V)\in\mathcal{A}\times\mathcal{A}} X(U) \otimes Y(V) \otimes \mathcal{A}(-,U \times V).$$

The monoidal unit equals $\mathcal{A}(-,pt)$ with $pt = \operatorname{Spec} k$.

The tensor product is then extended to a tensor product $\widetilde{\otimes}$ on $Sh(\mathcal{A})$. Namely, for all $F,G \in Sh(\mathcal{A})$ we set $F\widetilde{\otimes}G$ to be the sheaf associated with the presheaf $F \otimes G$ defined above. With this tensor product $Sh(\mathcal{A})$ is a closed symmetric monoidal category with $\mathcal{A}(-,pt)_{\mathrm{nis}}$ a monoidal unit. Likewise, $\widetilde{\otimes}$ is extended to chain complexes $\mathrm{Ch}(Sh(\mathcal{A}))$ which also defines a closed symmetric monoidal structure on the derived category $D(Sh(\mathcal{A}))$ with respect to the derived tensor product $\widetilde{\otimes}^L$ (we also refer the reader to [SV00, Section 2] and [CD09, Example 3.3]). It is straightforward to show that the localizing subcategory $\mathcal{L}$ of $D(Sh(\mathcal{A}))$ defined above is closed under the derived tensor product $\widetilde{\otimes}^L$. As a result, one obtains a symmetric monoidal product on $D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A}))$ (and on $DM^{eff}_{\mathcal{A}}(k)$, $DM_{\mathcal{A}}(k)$ if $\mathcal{A}$ is a strict $V$-category).

Remark 2.6. Let $\mathcal{A}$ be a symmetric monoidal strict $V$-category of correspondences. With a little extra care we describe the tensor product in $DM_{\mathcal{A}}(k)$ explicitly as follows. The endofuctor $- \boxtimes \mathbb{G}_m^{\mathcal{A}} : \mathrm{Ch}(Sh(\mathcal{A})) \to \mathrm{Ch}(Sh(\mathcal{A}))$ equals $-\widetilde{\otimes}\mathbb{G}_m^{\mathcal{A}}$. $DM_{\mathcal{A}}(k)$ is equivalent to the homotopy category of the symmetric $\mathbb{G}_m^{\mathcal{A}}$-spectra associated to a monoidal motivic model category structure on $\mathrm{Ch}(Sh(\mathcal{A}))$. We also refer the reader to [DF17], where a monoidal model structure is defined in the case of $MW$-correspondences.

## 3. The additive motivic sphere spectrum $\mathbb{S}^{\mathcal{A}}$

Let $Sp_{S^1,\mathbb{G}_m}(k)$ denote the category of symmetric $(S^1,\mathbb{G}_m)$-bispectra, where the $\mathbb{G}_m$-direction is associated with the pointed motivic space $(\mathbb{G}_m,1)$. It is equipped with a stable motivic model category structure [Jar00]. Denote by $SH(k)$ its homotopy category. The category $SH(k)$ has a closed symmetric monoidal structure with monoidal unit being the motivic sphere spectrum $\mathbb{S}$ (see [Jar00] for details). Given $p>0$, the category $Sp_{S^1,\mathbb{G}_m}(k)$ has a further model structure whose weak equivalences are the maps of bispectra $f: X\to Y$ such that the induced map of bigraded Nisnevich sheaves $f_*:\underline{\pi}^{\mathbb{A}^1}_{*,*}(X)\otimes\mathbb{Z}[\frac{1}{p}]\to\underline{\pi}^{\mathbb{A}^1}_{*,*}(Y)\otimes\mathbb{Z}[\frac{1}{p}]$ is an isomorphism. In what follows we denote its homotopy category by $SH(k)[p^{-1}]$. The category $SH(k)_{\mathbb{Q}}$ is defined in a similar fashion. The corresponding classes of weak equivalences are also called *$p^{-1}$-stable/$\mathbb{Q}$-stable motivic weak equivalences*. We also refer the reader to [RØ08b, Appendix A] for general localization theory of motivic spectra.

It is worth to mention that any other kind of motivic spectra or motivic functors in the sense of [DRØ03] together with the stable motivic model structure lead to equivalent definitions of $SH(k)[p^{-1}]$ and $SH(k)_{\mathbb{Q}}$ respectively.

The isomorphism $\mathrm{tw}:(\mathbb{G}_m,1)\wedge(\mathbb{G}_m,1)\xrightarrow{\cong}(\mathbb{G}_m,1)\wedge(\mathbb{G}_m,1)$ permuting factors is an involution, i.e. $\mathrm{tw}^2=\mathrm{id}$. It gives an endomorphism $\varepsilon:\mathbb{S}\to\mathbb{S}$ such that $\varepsilon^2=\mathrm{id}$. If we denote by $SH(k)[2^{-1}]$ the stable motivic homotopy theory with $\mathbb{Z}\left[\frac{1}{2}\right]$-coefficients, then

$$\varepsilon_+=-\frac{\varepsilon-1}{2}\quad\text{and}\quad\varepsilon_-=\frac{\varepsilon+1}{2}$$

are two orthogonal idempotent endomorphisms of $\mathbb{S}[2^{-1}]$ such that $\varepsilon_++\varepsilon_-=\mathrm{id}$ and $\varepsilon=\varepsilon_--\varepsilon_+$. It follows that

$$\mathbb{S}[2^{-1}]=\mathbb{S}_+\oplus\mathbb{S}_-,$$

where $\mathbb{S}_+$ (respectively $\mathbb{S}_-$) corresponds to the idempotent $\varepsilon_+$ (respectively $\varepsilon_-$).

By [Mor03, Section 6] the stable algebraic Hopf map $\eta:\mathbb{S}\to\mathbb{S}^{-1,-1}$ satisfies $\eta\varepsilon_+=0$, $\eta\varepsilon=\eta$ in $SH(k)[2^{-1}]$. Moreover,

$$\mathbb{S}_-\hookrightarrow\mathbb{S}[2^{-1}]\xrightarrow{\eta}\mathbb{S}^{-1,-1}[2^{-1}]\twoheadrightarrow\mathbb{S}_-^{-1,-1}$$

is an isomorphism in $SH(k)[2^{-1}]$, denoted by the same letter $\eta$. In particular, there is an isomorphism

$$\mathbb{S}_-\cong\mathbb{S}[\eta^{-1},2^{-1}]=\mathrm{hocolim}_{SH(k)[2^{-1}]}(\mathbb{S}\xrightarrow{\eta}\mathbb{S}^{-1,-1}\xrightarrow{\eta}\mathbb{S}^{-2,-2}\xrightarrow{\eta}\cdots).$$

The decomposition $\mathbb{S}[2^{-1}]=\mathbb{S}_+\oplus\mathbb{S}_-$ of the monoidal unit of $SH(k)[2^{-1}]$ implies $SH(k)[2^{-1}]$ is a product of symmetric monoidal triangulated categories

$$SH(k)[2^{-1}]=SH(k)_+\times SH(k)_-,$$

where $\mathbb{S}_+$ and $\mathbb{S}_-$ are monoidal units for $SH(k)_+$ and $SH(k)_-$ respectively.

Consider a category of correspondences $\mathcal{A}$. There is a natural triangulated functor

$$F:SH(k)\to D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A})).$$

In more detail, there is an adjoint pair [GP12, Section 6]

$$SH_{S^1}(k)\rightleftarrows\mathrm{Ho}(\mathrm{Mod}\,\mathcal{A}^{EM}),$$

where $\mathrm{Mod}\,\mathcal{A}^{EM}$ is the category of $\mathcal{A}^{EM}$-modules equipped with the stable projective motivic model structure over the Eilenberg–Mac Lane spectral category $\mathcal{A}^{EM}$ associated with $\mathcal{A}$. Also, there is a zig-zag of triangulated equivalences between $\mathrm{Ho}(\mathrm{Mod}\,\mathcal{A}^{EM})$ and $D_{\mathbb{A}^1}(Sh(\mathcal{A}))$. Then the resulting functor

$$SH_{S^1}(k) \to D_{\mathbb{A}^1}(Sh(\mathcal{A}))$$

is naturally extended to $\mathbb{G}_m$-spectra in both categories.

The functor $F$ sends each bispectrum $\Sigma^\infty_{S^1}\Sigma^\infty_{\mathbb{G}_m}X_+$, $X \in Sm/k$, to a $\mathbb{G}_m$-spectrum isomorphic to

$$\mathcal{A}(X)^\infty_{\mathbb{G}_m} := (\mathcal{A}(-,X)_{\mathrm{nis}}, \mathcal{A}(-,X\wedge\mathbb{G}_m^{\wedge 1})_{\mathrm{nis}}, \mathcal{A}(-,X\wedge\mathbb{G}_m^{\wedge 2})_{\mathrm{nis}}, \ldots) \in D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A})).$$

Here each entry is a complex in degree zero, each $\mathcal{A}(-,X\wedge\mathbb{G}_m^{\wedge n})_{\mathrm{nis}}$ is a sheaf associated to the presheaf

$$\mathcal{A}(-,X\times\mathbb{G}_m^{\wedge n})) = \mathcal{A}(-,X\times\mathbb{G}_m^{\times n}))/\sum_{s=1}^{n}(i_s)_*\mathcal{A}(-,X\times\mathbb{G}_m^{\times n-1})),$$

where the natural additive functors $i_s : \mathcal{A}(-,X\times\mathbb{G}_m^{\times(\ell-1)}) \to \mathcal{A}(-,X\times\mathbb{G}_m^{\times\ell})$ are induced by the embeddings $i_s : \mathbb{G}_m^{\times(\ell-1)} \to \mathbb{G}_m^{\times\ell}$ of the form

$$(x_1,\ldots,x_{\ell-1}) \longmapsto (x_1,\ldots,1,\ldots,x_{\ell-1}),$$

where 1 is the $s$th coordinate.

Note that $F$ factors through the stable $\mathbb{A}^1$-derived category $D_{\mathbb{A}^1}(k) := D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A}_{naive}))$ in the sense of Morel [Mor04] (see [CD12, Section 5.3] as well). In what follows we shall denote by $H_{\mathbb{A}^1}\mathbb{Z}$ its monoidal unit. Note that $H_{\mathbb{A}^1}\mathbb{Z}$ is the image of $\mathbb{S}$ under the canonical functor

$$SH(k) \to D_{\mathbb{A}^1}(k).$$

As above, one has decompositions

$$H_{\mathbb{A}^1}\mathbb{Z}[2^{-1}] = H_{\mathbb{A}^1}\mathbb{Z}_+ \oplus H_{\mathbb{A}^1}\mathbb{Z}_-, \quad D_{\mathbb{A}^1}(k)[2^{-1}] = D_{\mathbb{A}^1}(k)_+ \times D_{\mathbb{A}^1}(k)_-.$$

In what follows we shall write $\mathbb{S}^{\mathcal{A}}$ to denote the spectrum $\mathcal{A}(pt)^\infty_{\mathbb{G}_m}$ and call it the *additive motivic $\mathcal{A}$-sphere spectrum*. Taking the Eilenberg–Mac Lane $S^1$-spectra for each sheaf $\mathcal{A}(-,X\wedge\mathbb{G}_m^{\wedge n})_{\mathrm{nis}}$ (see, e.g., [Mor06, §3.2]) we can regard $\mathbb{S}^{\mathcal{A}}$ as an ordinary $(S^1,\mathbb{G}_m)$-bispectrum (and denote it by the same letter if there is no likelihood of confusion).

The canonical triangulated functor

$$F : SH(k) \to D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A}))$$

takes the ordinary motivic sphere $\mathbb{S}$ to a spectrum isomorphic to $\mathbb{S}^{\mathcal{A}}$. $F(\eta)$ induces a morphism

$$\eta_{\mathcal{A}} : \mathbb{S}^{\mathcal{A}} \to (\mathbb{S}^{\mathcal{A}})^{-1,-1}.$$

We also set

$$\mathbb{S}^{\mathcal{A}}[\eta_{\mathcal{A}}^{-1}] := \mathrm{hocolim}_{D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A}))}(\mathbb{S}^{\mathcal{A}} \xrightarrow{\eta_{\mathcal{A}}} (\mathbb{S}^{\mathcal{A}})^{-1,-1} \xrightarrow{\eta_{\mathcal{A}}} (\mathbb{S}^{\mathcal{A}})^{-2,-2} \xrightarrow{\eta_{\mathcal{A}}} \cdots)$$

and $\mathbb{S}^{\mathcal{A}}_- \cong F(\mathbb{S}_-)$, $\mathbb{S}^{\mathcal{A}}_+ \cong F(\mathbb{S}_+)$. Then we have the following relations in $D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A}))$:

$$\mathbb{S}^{\mathcal{A}}[2^{-1}] = \mathbb{S}^{\mathcal{A}}_+ \oplus \mathbb{S}^{\mathcal{A}}_- \quad \text{and} \quad \mathbb{S}^{\mathcal{A}}_- \cong \mathbb{S}^{\mathcal{A}}[\eta_{\mathcal{A}}^{-1}, 2^{-1}].$$

As above, $\eta_{\mathcal{A}}$ annihilates $\mathbb{S}^{\mathcal{A}}_+$ and is an isomorphism on $\mathbb{S}^{\mathcal{A}}_-$.

Remark 3.1. Following an equivalent description of $DM_{\mathcal{A}}(k)$ over a symmetric monoidal strict $V$-category of correspondences in Remark 2.6 in terms of $\mathbb{G}_m^{\mathcal{A}}$-symmetric spectra (in this case

$D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A})), DM_{\mathcal{A}}(k)$ are canonically equivalent), the additive motivic $\mathcal{A}$-sphere spectrum $\mathbb{S}^{\mathcal{A}}$ is nothing but the symmetric sequence

$$(\mathcal{A}(-,pt), \mathbb{G}_m^{\mathcal{A}}, \mathbb{G}_m^{\mathcal{A}} \widetilde{\otimes} \mathbb{G}_m^{\mathcal{A}}, \ldots, (\mathbb{G}_m^{\mathcal{A}})^{\widetilde{\otimes} n}, \ldots),$$

where $\Sigma_n$ acts on $(\mathbb{G}_m^{\mathcal{A}})^{\widetilde{\otimes} n}$ by permutation. It is a commutative monoid in the category of symmetric sequences in $\mathrm{Ch}(Sh(\mathcal{A}))$ (see [Hov01, Section 7]). Moreover, the motivic model category $Sp^{\Sigma}(\mathrm{Ch}(Sh\mathcal{A}), \mathbb{G}_m^{\mathcal{A}})$ of symmetric $\mathbb{G}_m^{\mathcal{A}}$-spectra associated with the motivic model category structure on $\mathrm{Ch}(Sh\mathcal{A})$ is the category of modules in the category of symmetric sequences over the commutative monoid $\mathbb{S}^{\mathcal{A}}$. The homotopy category of $Sp^{\Sigma}(\mathrm{Ch}(Sh\mathcal{A}), \mathbb{G}_m^{\mathcal{A}})$, which is equivalent to $DM_{\mathcal{A}}(k)$, is a closed symmetric monoidal category with $\mathbb{S}^{\mathcal{A}}$ a monoidal unit.

Definition 3.2. Let $\mathcal{A}$ be a category of $V$-correspondences. Following [Voe00, SV00, Sus03] the *$\mathcal{A}$-motive of a smooth algebraic variety $X \in Sm/k$*, denoted by $M_{\mathcal{A}}(X)$, is the complex associated to the simplicial Nisnevich sheaf

$$n \longmapsto \mathcal{A}(- \times \Delta^n, X)_{\mathrm{nis}}, \quad \Delta^n = \operatorname{Spec} k[t_0, \ldots, t_n]/(t_0 + \cdots + t_n - 1).$$

Lemma 3.3. *Let $\mathcal{A}$ be a strict category of $V$-correspondences and $\mathcal{X}$ a motivic $S^1$-spectrum such that its presheaves $\pi_*(\mathcal{X})$ of homotopy groups are homotopy invariant $\mathcal{A}$-presheaves. Then every Nisnevich local fibrant replacement $\mathcal{X}_f$ of $\mathcal{X}$ is motivically fibrant.*

*Proof.* Since $\mathcal{A}$ is a strict category of $V$-correspondences, the sheaves $\pi_*(\mathcal{X})_{\mathrm{nis}}$ are strictly $\mathbb{A}^1$-invariant. Our claim now follows from [Mor06, Theorem 6.2.7]. □

Remark 3.4. It is worth to mention that Lemma 3.3 does not depend on Morel's connectivity theorem [Mor06, Theorem 6.1.8]. Indeed, it easily follows for connected spectra from Brown–Gersten spectral sequence. Then we use the fact that $\mathcal{X}_f = \mathrm{hocolim}_{n \to -\infty}(\mathcal{X}_{\geqslant n})_f$, where $\mathcal{X}_{\geqslant n}$ is the naive $n$th truncation of $\mathcal{X}$.

The spectrum $\mathcal{A}(X)^{\infty}_{\mathbb{G}_m}$ is motivically equivalent to

$$M_{\mathcal{A}}^{\mathbb{G}_m}(X) := (M_{\mathcal{A}}(X), M_{\mathcal{A}}(X \wedge \mathbb{G}_m^{\wedge 1}), M_{\mathcal{A}}(X \wedge \mathbb{G}_m^{\wedge 2}), \ldots).$$

of Nisnevich $\mathcal{A}$-sheaves associated with the simplicial sheaf $n \longmapsto \mathcal{A}(- \times \Delta^n, X \wedge \mathbb{G}_m^{\wedge n})_{\mathrm{nis}}$.

Definition 3.5. Let $\mathcal{A}$ be a category of $V$-correspondences. The *bivariant $\mathcal{A}$-motivic cohomology groups* are defined by

$$H_{\mathcal{A}}^{p,q}(X, Y) := H^p_{\mathrm{nis}}(X, \mathcal{A}(- \times \Delta^{\bullet}, Y \wedge \mathbb{G}_m^{\wedge q})_{\mathrm{nis}}[-q]),$$

where the right hand side stands for Nisnevich hypercohomology groups of $X$ with coefficients in $\mathcal{A}(- \times \Delta^{\bullet}, Y \wedge \mathbb{G}_m^{\wedge q})_{\mathrm{nis}}[-q]$ (the shift is cohomological).

Following [GP18] we say that the bigraded presheaves $H_{\mathcal{A}}^{*,*}(-, Y)$ satisfy the *cancellation property* if all maps

$$\beta^{p,q} : H_{\mathcal{A}}^{p,q}(X, Y) \to H_{\mathcal{A}}^{p+1,q+1}(X \wedge \mathbb{G}_m, Y).$$

induced by the structure maps of the spectrum $M_{\mathcal{A}}^{\mathbb{G}_m}(Y)$ are isomorphisms.

Given $Y \in Sm/k$, denote by

$$M_{\mathcal{A}}^{\mathbb{G}_m}(Y)_f := (M_{\mathcal{A}}(Y)_f, M_{\mathcal{A}}(Y \wedge \mathbb{G}_m^{\wedge 1})_f, M_{\mathcal{A}}(Y \wedge \mathbb{G}_m^{\wedge 2})_f, \ldots),$$

where each $M_{\mathcal{A}}(Y \wedge \mathbb{G}_m^{\wedge n})_f$ is a fibrant Nisnevich local replacement of $M_{\mathcal{A}}(Y \wedge \mathbb{G}_m^{\wedge n})$. It is important to note that each $M_{\mathcal{A}}(Y \wedge \mathbb{G}_m^{\wedge n})_f$ can be constructed within $\mathrm{Ch}(Sh(\mathcal{A}))$ whenever $\mathcal{A}$ is a strict category of $V$-correspondences. (this can be shown similar to [GP12, Theorem 5.12]). Observe as well that $\mathcal{A}(Y)^{\infty}_{\mathbb{G}_m}$ is motivically equivalent to $M^{\mathbb{G}_m}_{\mathcal{A}}(Y)_f$.

LEMMA 3.6. *Suppose $\mathcal{A}$ is a strict $V$-category of correspondences. The bigraded presheaves $H^{*,*}_{\mathcal{A}}(-,Y)$ satisfy the cancelation property if and only if $M^{\mathbb{G}_m}_{\mathcal{A}}(Y)_f$ is motivically fibrant as an ordinary motivic bispectrum.*

*Proof.* Using Lemma 3.3, this is proved similar to [GP18, Lemma 4.5]. □

COROLLARY 3.7. *Suppose $\mathcal{A}$ is a strict $V$-category of correspondences satisfying the cancellation property. Then the presheaves $H^{*,*}_{\mathcal{A}}(-,Y)$ are represented in $SH(k)$ by the bispectrum $M^{\mathbb{G}_m}_{\mathcal{A}}(Y)_f$. Precisely,*

$$H^{p,q}_{\mathcal{A}}(X,Y) = SH(k)(X_+, S^{p,q} \wedge M^{\mathbb{G}_m}_{\mathcal{A}}(Y)_f), \quad p,q \in \mathbb{Z},$$

*where $S^{p,q} = S^{p-q} \wedge (\mathbb{G}_m, 1)^{\wedge q}$.*

Under the assumptions of Corollary 3.7 we can compute $\mathbb{S}^{\mathcal{A}}[\eta_{\mathcal{A}}^{-1}]$ up to an isomorphism in $D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A}))$ as follows.

$$\mathbb{S}^{\mathcal{A}}[\eta_{\mathcal{A}}^{-1}] \cong \operatorname{hocolim}_{DM_{\mathcal{A}}(k)}(M^{\mathbb{G}_m}_{\mathcal{A}}(pt)_f \to \Omega_{(\mathbb{G}_m,1)}(M^{\mathbb{G}_m}_{\mathcal{A}}(pt)_f) \to \Omega_{(\mathbb{G}_m,1)^{\wedge 2}}(M^{\mathbb{G}_m}_{\mathcal{A}}(pt)_f) \to \cdots).$$

Here the maps of the colimit are induced by $\eta_{\mathcal{A}}$. Denote the right hand side by $M^{\mathbb{G}_m}_{\mathcal{A}}(pt)_f[\eta^{-1}]$. It is termwise a spectrum

$$M^{\mathbb{G}_m}_{\mathcal{A}}(pt)_f[\eta^{-1}] = (\Omega^{\infty}_{(\mathbb{G}_m,1)}(M_{\mathcal{A}}(pt)_f), \Omega^{\infty-1}_{(\mathbb{G}_m,1)}(M_{\mathcal{A}}(pt)_f), \Omega^{\infty-2}_{(\mathbb{G}_m,1)}(M_{\mathcal{A}}(pt)_f), \ldots),$$

where each

$$\Omega^{\infty-n}_{(\mathbb{G}_m,1)}(M_{\mathcal{A}}(pt)_f) := \operatorname{hocolim}_{DM^{eff}_{\mathcal{A}}(k)}(M_{\mathcal{A}}(\mathbb{G}_m^{\wedge n})_f \to \Omega_{(\mathbb{G}_m,1)}(M_{\mathcal{A}}(\mathbb{G}_m^{\wedge n})_f) \to \cdots).$$

Since the structure maps of $M^{\mathbb{G}_m}_{\mathcal{A}}(pt)_f[\eta^{-1}]$ are schemewise equivalences by the cancellation property, it follows from the construction of $M^{\mathbb{G}_m}_{\mathcal{A}}(pt)_f[\eta^{-1}]$ that all homotopy sheaves $\pi^{\mathbb{A}^1}_{i,j}(M^{\mathbb{G}_m}_{\mathcal{A}}(pt)_f[\eta^{-1}])$ are concentrated in weight zero only. By [Mor03, Lemma 4.3.11] the canonical map of sheaves

$$\pi^{\mathbb{A}^1}_n(\Omega_{(\mathbb{G}_m,1)}(\Omega^{\infty-n}_{(\mathbb{G}_m,1)}(M_{\mathcal{A}}(pt)_f)) \to \pi^{\mathbb{A}^1}_n(\Omega^{\infty-n}_{(\mathbb{G}_m,1)}(M_{\mathcal{A}}(pt)_f)_{-1}, \quad n \in \mathbb{Z},$$

is an isomorphism, hence the composite map of sheaves is an isomorphism for all $n \geqslant 0$

$$\beta_n : \pi^{\mathbb{A}^1}_{-n,-n}(M^{\mathbb{G}_m}_{\mathcal{A}}(pt)_f[\eta^{-1}]) \to \pi^{\mathbb{A}^1}_{-n-1,-n-1}(\Omega_{(\mathbb{G}_m,1)}M^{\mathbb{G}_m}_{\mathcal{A}}(pt)_f[\eta^{-1}]) \to \\ \to (\pi^{\mathbb{A}^1}_{-n-1,-n-1}(M^{\mathbb{G}_m}_{\mathcal{A}}(pt)_f[\eta^{-1}]))_{-1},$$

where the left map is induced by the structure map.

Denote by $\mathcal{W}^{\mathcal{A}}$ the strictly $\mathbb{A}^1$-invariant sheaf $\pi^{\mathbb{A}^1}_{0,0}(M^{\mathbb{G}_m}_{\mathcal{A}}(pt)_f[\eta^{-1}])$. If we regard it as a complex concentrated in zeroth degree, then the collection of complexes

$$\mathcal{W}^{\mathcal{A}}_{\mathbb{G}_m} := (\mathcal{W}^{\mathcal{A}}, \mathcal{W}^{\mathcal{A}}, \mathcal{W}^{\mathcal{A}}, \ldots)$$

together with isomorphisms $\beta_0 : \mathcal{W}^{\mathcal{A}} \to (\mathcal{W}^{\mathcal{A}})_{-1}$ is an object of $DM_{\mathcal{A}}(k)$, which is $\mathbb{A}^1$-local as an ordinary bispectrum (after taking the Eilenberg–Mac Lane spectrum of each sheaf $\mathcal{W}^{\mathcal{A}}$). Notice that the homotopy module of $\mathcal{W}^{\mathcal{A}}_{\mathbb{G}_m}$ in the sense of [Mor03, Definition 5.2.4] is given by $(M_*, \mu_*)$ with each $M_n = \mathcal{W}^{\mathcal{A}}$ and $\mu_n = \beta_0$, $n \in \mathbb{Z}$. There is a canonical morphism of spectra

$$H : M^{\mathbb{G}_m}_{\mathcal{A}}(pt)_f[\eta^{-1}] \to \mathcal{W}^{\mathcal{A}}_{\mathbb{G}_m}, \tag{1}$$

induced by taking the zeroth homology sheaf of each complex $\Omega^{\infty-n}_{(\mathbb{G}_m,1)}(M_{\mathcal{A}}(pt)_f)$.

Let $\underline{W}$ be the Nisnevich sheaf of Witt rings on $Sm/k$. Following [ALP17, p. 380] we take the model $\underline{W} := \underline{K}_0^{MW}/h$. The isomorphism $\underline{W} \cong \pi^{\mathbb{A}^1}_{n>0,n>0}(\mathbb{S})$ gives the canonical isomorphism of sheaves $\varepsilon : \underline{W} \cong \underline{\mathrm{Hom}}((\mathbb{G}_m, 1), \underline{W})$. More precisely, it takes $w \in \underline{W}(U)$ to $p_1^*(\eta \cdot [t]) \cdot p_2^*(w) \in \underline{W}(U \wedge (\mathbb{G}_m, 1))$, where $t$ is the canonical unit on $\mathbb{G}_m$ and $[t] \in \underline{K}_1^{MW}(\mathbb{G}_m)$ the corresponding section.

Definition 3.8. Suppose $\mathcal{A}$ is a strict $V$-category of correspondences satisfying the cancellation property and $R$ a flat $\mathbb{Z}$-algebra. We say that the spectrum $\mathbb{S}^{\mathcal{A}}[\eta_{\mathcal{A}}^{-1}]$ is *of Witt type with $R$-coefficients* if the zeroth cohomology sheaf $\mathcal{W}_R^{\mathcal{A}} = \pi^{\mathbb{A}^1}_{0,0}(M_{\mathcal{A}}^{\mathbb{G}_m}(pt)_f[\eta^{-1}]) \otimes R$ of the complex $\Omega^\infty_{(\mathbb{G}_m,1)}(M_{\mathcal{A}}(pt)_f) \otimes R$ is the only non-zero cohomology sheaf (the other cohomology sheaves are required to be zero) and $\mathcal{W}_R^{\mathcal{A}}$ is isomorphic to the Nisnevich sheaf $\underline{W}_R = \underline{W} \otimes R$. We also require the diagram

$$\begin{tikzcd} \mathcal{W}_R^{\mathcal{A}} \arrow[r, "\beta_0"] \arrow[d, "\cong"'] & (\mathcal{W}_R^{\mathcal{A}})_{-1} \arrow[d, "\cong"] \\ \underline{W}_R \arrow[r, "\varepsilon"'] & (\underline{W}_R)_{-1} \end{tikzcd}$$

to be commutative. If $R = \mathbb{Z}$ then we just say that $\mathbb{S}^{\mathcal{A}}[\eta_{\mathcal{A}}^{-1}]$ is *of Witt type*.

Lemma 3.9. *Suppose $\mathcal{A}$ is a strict $V$-category of correspondences satisfying the cancellation property and $R$ a ring of fractions of $\mathbb{Z}$. If the spectrum $\mathbb{S}^{\mathcal{A}}[\eta_{\mathcal{A}}^{-1}]$ is of Witt type with $R$-coefficients then it is isomorphic in $SH(k)$ to the bispectrum*

$$\underline{W}_R^{\mathbb{G}_m} := (\underline{W}_R, \underline{W}_R, \ldots),$$

*in which every structure map is induced by $\varepsilon$.*

*Proof.* This immediately follows from Definition 3.8 and the observation that the morphism of spectra (1) is a motivic equivalence. $\square$

We are now in a position to prove the main result of the section.

Theorem 3.10. *Suppose $\mathcal{A}$ is a strict $V$-category of correspondences satisfying the cancellation property.*

(1) *If the spectrum $\mathbb{S}^{\mathcal{A}}[\eta_{\mathcal{A}}^{-1}]$ is of Witt type with $\mathbb{Q}$-coefficients then the canonical morphism*

$$\mathbb{S}_- \otimes \mathbb{Q} \to \mathbb{S}_-^{\mathcal{A}} \otimes \mathbb{Q}$$

*is an isomorphism in $SH(k)$.*

(2) *If the spectrum $\mathbb{S}^{\mathcal{A}}[\eta_{\mathcal{A}}^{-1}]$ is of Witt type with $\mathbb{Z}\left[\frac{1}{2}\right]$-coefficients then the canonical morphism*

$$H_{\mathbb{A}^1}\mathbb{Z}_- \to \mathbb{S}_-^{\mathcal{A}}$$

*is an isomorphism in $SH(k)$.*

*Proof.* (1). It follows from [ALP17] that the composite morphism

$$\mathbb{S}_- \otimes \mathbb{Q} \to \mathbb{S}_-^{\mathcal{A}} \otimes \mathbb{Q} \to \underline{W}_{\mathbb{Q}}^{\mathbb{G}_m}$$

is an isomorphism in $SH(k)$. By Lemma 3.9 the right morphism is an isomorphism in $SH(k)$, and hence so is the left one.

(2). It follows from [Bac18, Proposition 37] that the composite morphism

$$H_{\mathbb{A}^1}\mathbb{Z}_- \to \mathbb{S}_-^{\mathcal{A}} \to \underline{W}^{\mathbb{G}_m}[2^{-1}]$$

is an isomorphism in $SH(k)$. By Lemma 3.9 the right morphism is an isomorphism in $SH(k)$, and hence so is the left one. $\square$

## 4. The Milnor–Witt sphere spectrum $\mathbb{S}^{MW}$

Throughout this section $k$ is an infinite perfect field with char $k \neq 2$. We refer the reader to [CF14] for basic facts and definitions on the category of finite Milnor–Witt correspondences $\widetilde{\mathrm{Cor}}$. It is a strict $V$-category of correspondences by [DF17]. It follows from [FØ17] that $\widetilde{\mathrm{Cor}}$ has cancellation property. We denote the additive sphere spectrum associated with $\widetilde{\mathrm{Cor}}$ by $\mathbb{S}^{MW}$.

By [CF14, Proposition 5.11] $\widetilde{\mathrm{Cor}}(-,Y)$ is a Zariski sheaf, but not a Nisnevich sheaf in general [CF14, Example 5.12]. However, $\widetilde{\mathrm{Cor}}(-,pt)$ is the Nisnevich sheaf $\underline{K}_0^{MW}$ [CF14, Example 4.5], which is homotopy invariant by [Fas08, Corollaire 11.3.3]. Since $\widetilde{\mathrm{Cor}}$ is a strict additive $V$-category of correspondences by [DF17], we see that the Nisnevich sheaf $\widetilde{\mathrm{Cor}}(-,pt)$ is strictly homotopy invariant. In particular, the normalised complex $M_{MW}(pt)$ associated to the simplicial sheaf $\widetilde{\mathrm{Cor}}(- \times \Delta^\bullet, pt)$ has only one non-trivial homology sheaf $\underline{K}_0^{MW}$. It follows from [CF14, Proposition 5.34] that $\pi_0^{\mathbb{A}^1}(\mathrm{Hom}((\mathbb{G}_m,1)^{\wedge n}, M_{MW}(pt)))$ is isomorphic to the sheaf $\underline{W}$ of Witt rings. Thus the spectrum $\mathbb{S}^{MW}[\eta^{-1}]$ is of Witt type. Theorem 3.10 now implies the following

Proposition 4.1. *The canonical morphisms*

$$\mathbb{S}_- \otimes \mathbb{Q} \to \mathbb{S}_-^{MW} \otimes \mathbb{Q} \quad \text{and} \quad H_{\mathbb{A}^1}\mathbb{Z}_- \to \mathbb{S}_-^{MW}$$

*are isomorphisms in $SH(k)$.*

It follows from properties of finite Milnor–Witt correspondences [CF14] that $\mathbb{S}_+^{MW}$ is isomorphic to $\mathbb{S}^{\mathrm{Cor}}[2^{-1}]$ in $DM_{MW}(k)[2^{-1}] := DM_{\widetilde{\mathrm{Cor}}}(k)[2^{-1}]$. Thus we have a splitting

$$\mathbb{S}^{MW}[2^{-1}] \cong \mathbb{S}^{\mathrm{Cor}}[2^{-1}] \oplus \mathbb{S}_-^{MW}.$$

A theorem of Cisinski–Déglise [CD12, Theorem 16.2.13] shows that the canonical map $\mathbb{S}_+ \otimes \mathbb{Q} \to \mathbb{S}^{\mathrm{Cor}} \otimes \mathbb{Q}$ is an isomorphism in $SH(k)$. Combining this with Proposition 4.1, we have proved the main result of the section:

Theorem 4.2. *Given an infinite perfect field $k$ of characteristic not 2, the canonical morphism of bispectra*

$$\mathbb{S} \otimes \mathbb{Q} \to \mathbb{S}^{MW} \otimes \mathbb{Q}$$

*is an isomorphism in $SH(k)$.*

Let $H_{\mathbb{A}^1}^{*,*}(X)$ be the cohomology theory represented in $SH(k)$ by the bispectrum $H_{\mathbb{A}^1}\mathbb{Z}$. The following statement is a consequence of the preceding theorem and a result of Déglise–Fasel [DF17, Corollary 4.2.6]:

Corollary 4.3. *Given an infinite perfect field $k$ of characteristic not 2, $n \geqslant 0$ and $X \in Sm/k$, there is a natural isomorphism*

$$H_{\mathbb{A}^1}^{2n,n}(X) \otimes \mathbb{Q} \cong \widetilde{CH}^n(X) \otimes \mathbb{Q},$$

*where the right hand side is the $n$-th rational Chow–Witt group of $X$. In particular, if $-1$ is a sum of squares in $k$, then $H_{\mathbb{A}^1}^{2n,n}(X) \otimes \mathbb{Q} \cong CH^n(X) \otimes \mathbb{Q}$, where $CH^n(X) \otimes \mathbb{Q}$ is the $n$-th rational Chow group of $X$.*

## 5. Reconstructing $SH(k)_{\mathbb{Q}}$ from finite Milnor–Witt correspondences

In this section we prove the main result of the paper stating that $SH(k)_{\mathbb{Q}}$ is recovered as $DM_{MW}(k)_{\mathbb{Q}}$ whenever the base field $k$ is infinite perfect of characteristic not 2. To this end, we need to extend Röndigs–Østvær's theorem [RØ08a] to preadditive categories of correspondences. Throughout this section $\mathcal{A}$ is a category of correspondences.

Following [RØ08a, Section 2] define the category $\mathbb{M}^{\mathcal{A}}$ of motivic spaces with $\mathcal{A}$-correspondences as all contravariant additive functors from $\mathcal{A}$ to simplicial abelian groups. A scheme $U$ in $Sm/k$ defines a representable motivic space $\mathcal{A}(-,U) \in \mathbb{M}^{\mathcal{A}}$. Let $\mathcal{U} : \mathbb{M}^{\mathcal{A}} \to \mathbb{M}$ denote the evident forgetful functor induced by the graph $Sm/k \to \mathcal{A}$. It has a left adjoint $\mathbb{Z}^{\mathcal{A}} : \mathbb{M} \to \mathbb{M}^{\mathcal{A}}$ defined as the left Kan extension functor determined by

$$\mathbb{Z}^{\mathcal{A}}((U \times \Delta^n)_+) = \mathcal{A}(-,U) \otimes \mathbb{Z}(\Delta^n).$$

If $X$ is a motivic space, let $X^{\mathcal{A}}$ be short for $\mathbb{Z}^{\mathcal{A}}(X)$.

Similar to [RØ08a, §2.1] we define a projective motivic model category structure on $\mathbb{M}^{\mathcal{A}}$. This model category is denoted by $\mathbb{M}^{\mathcal{A}}_{\mathrm{mot}}$. The projective motivic model category of motivic spaces is denoted by $\mathbb{M}_{\mathrm{mot}}$. We have a Quillen pair

$$\mathbb{Z}^{\mathcal{A}} : \mathbb{M}_{\mathrm{mot}} \rightleftarrows \mathbb{M}^{\mathcal{A}}_{\mathrm{mot}} : \mathcal{U}.$$

Using Definition 2.3(1) and Lemma 2.4, the proof of the following lemma literally repeats [RØ08a, Lemma 9].

Lemma 5.1. *A map between motivic spaces with $\mathcal{A}$-correspondences is a motivic weak equivalence in $\mathbb{M}^{\mathcal{A}}_{\mathrm{mot}}$ if and only if it is so when considered as a map between ordinary motivic spaces.*

Let $\iota : pt = \operatorname{Spec} k \to \mathbb{G}_m$ be the embedding $\iota(pt) = 1 \in \mathbb{G}_m$. The mapping cylinder yields a factorization of the induced map

$$\operatorname{Spec} k_+ \hookrightarrow \mathrm{Cyl}(\iota) \xrightarrow{\simeq} (\mathbb{G}_m)_+$$

into a projective cofibration and a simplicial homotopy equivalence in $\mathbb{M}$. Let $\mathbb{G}$ denote the cofibrant pointed presheaf $\mathrm{Cyl}(\iota)/\operatorname{Spec} k_+$ and $T := S^1 \wedge \mathbb{G}$.

Following [RØ08a, §2.4] we define a motivic spectrum

$$H\mathcal{A} = (\mathcal{U}(pt_+), \mathcal{U}(T^{\mathcal{A}}), \mathcal{U}((T^{\wedge 2})^{\mathcal{A}}), \ldots).$$

The structure maps are induced by morphisms

$$\alpha_T : (T^{\wedge n})^{\mathcal{A}} \to \underline{\mathrm{Hom}}_{\mathbb{M}^{\mathcal{A}}}(T^{\mathcal{A}}, (T^{\wedge n+1})^{\mathcal{A}})$$

(recall that $Sm/k$ acts on $\mathcal{A}$).

Given a symmetric monoidal category of correspondences $\mathcal{A}$, a theorem of Day [Day70] implies that $\mathbb{M}^{\mathcal{A}}$ is a closed symmetric monoidal category with a tensor product defined as

$$X \odot Y = \int^{(U,V) \in \mathcal{A} \times \mathcal{A}} X(U) \otimes Y(V) \otimes \mathcal{A}(-, U \times V).$$

As an example, $\mathcal{A}(-,U) \odot \mathcal{A}(-,V) = \mathcal{A}(-,U \times V)$. The monoidal unit equals $\mathcal{A}(-,pt)$ with $pt = \operatorname{Spec} k$. Similar to [DRØ03, Example 3.4] $H\mathcal{A}$ is a commutative motivic symmetric ring spectrum.

Suppose $\mathcal{A}$ is a symmetric monoidal category of correspondences. Repeating the proof of [RØ08a, Lemma 10] word for word, the projective motivic model structure on $\mathbb{M}^{\mathcal{A}}_{\mathrm{mot}}$ is symmetric monoidal.

Following [Hov01, RØ08a] one can define the stable monoidal model category of symmetric $T$-spectra $\mathbb{MSS}^{\mathcal{A}}$ associated to $\mathbb{M}^{\mathcal{A}}_{\mathrm{mot}}$ (with projective model structure). The homotopy category of $\mathbb{MSS}^{\mathcal{A}}$ is a model for $D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A}))(k)$. It is as well a model for $DM_{\mathcal{A}}(k)$ whenever $\mathcal{A}$ is a strict $V$-category of correspondences (for this repeat the proof of [RØ08a, Theorem 11] literally).

Below we shall need the following theorem proved by Riou in [LYZ13, Appendix B] (see the proof of [HKØ17, Theorem 5.8] as well).

THEOREM 5.2 (Riou). *Let $k$ be a perfect field. Let $p$ denote the caracteristic exponent of $k$ (i.e., $p > 0$ or $p = 1$ if the characteristic of $k$ is zero). Then, for any smooth finite type $k$-scheme $U$, the suspension spectrum $\Sigma^\infty_T U_+$ is strongly dualisable in $SH(k)[1/p]$.*

We are now in a position to prove the Röndigs–Østvær theorem for $\mathcal{A}$-correspondences. Notice that in all known examples a $V$-category of correspondences is strict whenever the base field $k$ is (infinite) perfect (of characteristic not 2 if $\mathcal{A} = \widetilde{\mathrm{Cor}}$). We also recall the reader that the category $SH(k)[p^{-1}]$ is defined on page 7. It is the homotopy category of the stable model category of motivic functors with weak equivalences being $p^{-1}$-stable motivic equivalences.

THEOREM 5.3 (Röndigs–Østvær). *If $k$ is a perfect field of exponential characteristic $p$ and $\mathcal{A}$ is a symmetric monoidal category of correspondences, then the homotopy category of* $\mathrm{Mod}\, H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]}$ *(respectively* $\mathrm{Mod}\, H\mathcal{A}_{\mathbb{Q}}$*) is equivalent to $D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A}))(k)[\frac{1}{p}]$ (respectively $D^{st}_{\mathbb{A}^1}(Sh(\mathcal{A}))(k) \otimes \mathbb{Q}$). The equivalence preserves the triangulated structure. In particular,* $\mathrm{Ho}(\mathrm{Mod}\, H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]})$ *(respectively* $\mathrm{Ho}(\mathrm{Mod}\, H\mathcal{A}_{\mathbb{Q}})$*) is equivalent to $DM_{\mathcal{A}}(k)[\frac{1}{p}]$ (respectively $DM_{\mathcal{A}}(k)\otimes\mathbb{Q}$) if $\mathcal{A}$ is a symmetric monoidal strict $V$-category of correspondences.*

*Proof.* We verify the theorem for categories with $\mathbb{Z}[\frac{1}{p}]$-coefficients, because the proof of the statement for categories with rational coefficients repeats that for $\mathbb{Z}[\frac{1}{p}]$-coefficients *word for word*. The proof of the theorem for categories with $\mathbb{Z}[\frac{1}{p}]$-coefficients is the same with the original Röndigs–Østvær's theorem [RØ08a]. The only difference is that we shall have to deal somewhere with $p^{-1}$-stable weak equivalences of motivic functors instead of ordinary stable weak equivalences.

We must show that the canonical pair of adjoint (triangulated) functors

$$\Phi : \mathrm{Mod}\, H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]} \rightleftarrows \mathbb{MSS}^{\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]}} : \Psi$$

is a Quillen equivalence ($\Psi$ forgets correspondences).

Similar to [RØ08a, Lemma 43] it suffices to prove that the unit of the adjunction

$$H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]} \wedge U_+ \to \Psi\Phi(H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]} \wedge U_+) \tag{2}$$

is a stable motivic weak equivalence of motivic symmetric spectra for every smooth scheme $U$. Note that $\Psi\Phi(H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]} \wedge U_+)$ is the symmetric spectrum $(\mathcal{A}(-,U)\otimes\mathbb{Z}[\frac{1}{p}], (U_+\wedge T)^{\mathcal{A}\otimes\mathbb{Z}[\frac{1}{p}]}, (U_+ \wedge T^{\wedge 2})^{\mathcal{A}\otimes\mathbb{Z}[\frac{1}{p}]}, \ldots)$.

By Theorem 5.2, $U_+$ is dualizable in $SH(k)[p^{-1}]$ for every $k$-smooth scheme $U$. Suppose $X$ is a motivic functor in the sense of [DRØ03] and $B$ is a cofibrant finitely presentable motivic space such that $-\wedge B$ is dualizable in $SH(k)[p^{-1}]$. When $X$ preserves motivic weak equivalences of cofibrant finitely presentable motivic spaces, then the evaluation of the assembly map

$$X \wedge B \to X \circ (-\wedge B)$$

is a $p^{-1}$-stable weak equivalence between motivic symmetric spectra by [RØ08a, Corollary 56] (though [RØ08a, Corollary 56] is proved within an ordinary stable motivic model structure of motivic functors, it is also true within the $p^{-1}$-stable model structure). We use here notation and terminology of [DRØ03]. Recall that motivic functors give a model for motivic symmetric spectra, and hence for $SH(k)$ [DRØ03].

Consider a motivic functor associated with $H\mathcal{A}$ (denoted by the same letters)

$$H\mathcal{A} : c\mathbb{M} \hookrightarrow \mathbb{M} \xrightarrow{\mathbb{Z}^{\mathcal{A}}} \mathbb{M}^{\mathcal{A}} \xrightarrow{\mathcal{U}} \mathbb{M}.$$

Here $c\mathbb{M}$ is the full subcategory of $\mathbb{M}$ of cofibrant finitely presentable motivic spaces. $\mathbb{Z}^{\mathcal{A}} : \mathbb{M}_{\mathrm{mot}} \to \mathbb{M}^{\mathcal{A}}_{\mathrm{mot}}$ is a left Quillen functor, hence it preserves motivic weak equivalences between cofibrant motivic spaces. By Lemma 5.1, $\mathcal{U}$ preserves weak equivalences in $\mathbb{M}^{\mathcal{A}}_{\mathrm{mot}}$. It follows that $H\mathcal{A}$ preserves motivic equivalences of cofibrant finitely presentable motivic spaces. Hence,

$$H\mathcal{A} \wedge U_+ \to H\mathcal{A} \circ (- \wedge U_+)$$

is a $p^{-1}$-stable weak equivalence between motivic symmetric spectra by [RØ08a, Corollary 56]. Similarly,

$$H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]} \wedge U_+ \to H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]} \circ (- \wedge U_+) \tag{3}$$

is a $p^{-1}$-stable weak equivalence between motivic symmetric spectra. Obviously,

$$\underline{\pi}^{\mathbb{A}^1}_{*,*}(H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]} \circ (- \wedge U_+)) \cong \underline{\pi}^{\mathbb{A}^1}_{*,*}(H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]} \circ (- \wedge U_+)) \otimes \mathbb{Z}[1/p].$$

This is because fibrant replacements of the $T$-spectrum $H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]} \circ (\mathbb{S}_k \wedge U_+)$, where $\mathbb{S}_k$ is the motivic sphere spectrum, can be computed in $\mathbb{MSS}^{\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]}}$.

We claim that

$$\underline{\pi}^{\mathbb{A}^1}_{*,*}(H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]} \wedge U_+) \cong \underline{\pi}^{\mathbb{A}^1}_{*,*}(H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]} \wedge U_+) \otimes \mathbb{Z}[1/p].$$

Indeed, this follows from an isomorphism in $SH(k)$

$$\begin{aligned} H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]} \wedge U_+ &\cong \operatorname{hocolim}(H\mathcal{A} \xrightarrow{p} H\mathcal{A} \xrightarrow{p} H\mathcal{A} \xrightarrow{p} \cdots) \wedge U_+ \\ &\cong \operatorname{hocolim}(H\mathcal{A} \wedge U_+ \xrightarrow{p} H\mathcal{A} \wedge U_+ \xrightarrow{p} \cdots). \end{aligned}$$

We see that (3) is not only a $p^{-1}$-stable weak equivalence between motivic symmetric spectra, but also an ordinary stable motivic equivalence.

Ordinary symmetric $T$-spectra are obtained from motivic spaces by evaluating them at spheres $S^0, T, T^{\wedge 2}, \ldots$ (see [DRØ03, §3.2]). The evaluation of the motivic space $H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]} \wedge U_+$ is the symmetric $T$-spectrum

$$(\mathcal{U}(pt_+), \mathcal{U}(T^{\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]}}), \mathcal{U}((T^{\wedge 2})^{\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]}}), \ldots) \wedge U_+.$$

The evaluation of the motivic space $H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]} \circ (- \wedge U_+)$ is the symmetric $T$-spectrum

$$\Phi\Psi(H\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]} \wedge U_+) = (\mathcal{U}(\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]}(-, U)), \mathcal{U}((U_+ \wedge T)^{\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]}}), \mathcal{U}((U_+ \wedge T^{\wedge 2})^{\mathcal{A}_{\mathbb{Z}[\frac{1}{p}]}}), \ldots).$$

Furthermore, the evaluation of the morphism (3) is the morphism (2). We see that the morphism (2) is a stable motivic equivalence of motivic symmetric spectra, as was to be shown. □

Remark 5.4. Very recently Elmanto and Kolderup [EK17] have suggested another approach to the Röndigs–Østvær theorem for $\mathcal{A} = \widetilde{\mathrm{Cor}}$ that uses Lurie's $\infty$-categorical version of the Barr–Beck theorem.

Theorem 5.5 (Reconstruction). *If $k$ is an infinite perfect field with* $\operatorname{char} k \neq 2$*, then $SH(k)_{\mathbb{Q}}$ is equivalent to $DM_{MW}(k)_{\mathbb{Q}}$. The equivalence preserves the triangulated structure.*

*Proof.* $SH(k)_{\mathbb{Q}}$ is equivalent to $D_{\mathbb{A}^1}(k)_{\mathbb{Q}}$ (see [Mor04]). By Theorem 5.3 the latter is equivalent to the homotopy category of $\mathbb{S}^{\mathcal{A}_{naive}} \otimes \mathbb{Q}$-modules. $\mathbb{S}^{\mathcal{A}_{naive}} \otimes \mathbb{Q}$ is motivically equivalent to the commutative monoid spectrum $\mathbb{S}\otimes\mathbb{Q}$. By [SS00, Theorem 4.3] the homotopy category of $\mathbb{S}^{\mathcal{A}_{naive}}\otimes \mathbb{Q}$-modules is equivalent to the homotopy category of $\mathbb{S} \otimes \mathbb{Q}$-modules. By Theorem 4.2 $\mathbb{S} \otimes \mathbb{Q}$ is motivically equivalent to the commutative monoid spectrum $\mathbb{S}^{MW} \otimes \mathbb{Q}$. By [SS00, Theorem 4.3] the homotopy category of $\mathbb{S} \otimes \mathbb{Q}$-modules is equivalent to the homotopy category of $\mathbb{S}^{MW} \otimes \mathbb{Q}$-modules. Wee see that $SH(k)_{\mathbb{Q}}$ is equivalent to the homotopy category of $\mathbb{S}^{MW} \otimes \mathbb{Q}$-modules. By Theorem 5.3 the latter category is triangle equivalent to $DM_{MW}(k)_{\mathbb{Q}}$, as was to be shown. □

Remark 5.6. The triangulated equivalence of Theorem 5.5 is in fact symmetric monoidal. The main point here is that the natural functor between categories of correspondences $\mathcal{A}_{naive} \to \widetilde{\mathrm{Cor}}$ is extended to a symmetric monoidal triangulated functor $D_{\mathbb{A}^1}(k) \to DM_{MW}(k)$ (see [DF17, Section 3.3] as well). Consider a commutative diagram of natural triangulated functors

$$\begin{tikzcd}
DM_{MW}(k)_{\mathbb{Q}} \arrow[r] \arrow[d] & D_{\mathbb{A}^1}(k)_{\mathbb{Q}} \arrow[d] \\
\mathrm{Ho}(\mathrm{Mod}-\mathbb{S}^{MW} \otimes \mathbb{Q}) \arrow[r] & \mathrm{Ho}(\mathrm{Mod}-\mathbb{S}^{naive} \otimes \mathbb{Q})
\end{tikzcd}$$

The proof of the preceding theorem implies that the lower and the vertical functors are equivalences. We see that the upper functor is an equivalence. It is right adjoint to the functor $D_{\mathbb{A}^1}(k)_{\mathbb{Q}} \to DM_{MW}(k)_{\mathbb{Q}}$. It follows that the latter functor is an equivalence, too. It is plainly symmetric monoidal.

## 6. Comparing motivic complexes with framed and $MW$-correspondences

In this section we apply the Reconstruction Theorem 5.5 to compare rational motives with framed and Milnor–Witt correspondences respectively. Throughout this section the base field $k$ is infinite perfect of characteristic different from 2.

It is shown in [GP14b] that the suspension bispectrum $\Sigma^\infty_{S^1}\Sigma^\infty_{\mathbb{G}} X_+ \in SH(k)$ of a $k$-smooth algebraic variety $X$ is stably equivalent to the bispectrum

$$M^{\mathbb{G}}_{fr}(X) = (M_{fr}(X), M_{fr}(X \times \mathbb{G}_m^{\wedge 1}), M_{fr}(X \times \mathbb{G}_m^{\wedge 2}), \ldots),$$

each term of which is a twisted framed motive of $X$. Since the functor $\widetilde{\mathbb{Z}} : \mathcal{X} \mapsto \widetilde{\mathbb{Z}}[\mathcal{X}]$ respects stable weak equivalences of bispectra, it follows that bispectrum $\widetilde{\mathbb{Z}}[\Sigma^\infty_{S^1}\Sigma^\infty_{\mathbb{G}} X_+] \in SH(k)$ is stably equivalent to the bispectrum

$$\mathbb{Z}M^{\mathbb{G}}_{fr}(X) = (\mathbb{Z}M_{fr}(X), \mathbb{Z}M_{fr}(X \times \mathbb{G}_m^{\wedge 1}), \mathbb{Z}M_{fr}(X \times \mathbb{G}_m^{\wedge 2}), \ldots).$$

By [GPN16, Theorem 1.2] the latter bispectrum is stably equivalent to the bispectrum

$$LM^{\mathbb{G}}_{fr}(X) := (LM_{fr}(X), LM_{fr}(X \times \mathbb{G}_m^{\wedge 1}), LM_{fr}(X \times \mathbb{G}_m^{\wedge 2}), \ldots)$$

consisting of twisted linear framed motives in the sense of [GP14b]. If we take a levelwise Nisnevich local fibrant replacement of $LM_{fr}(X \times \mathbb{G}_m^{\wedge n})$, we get a bispectrum

$$LM^{\mathbb{G}}_{fr}(X)_f := (LM_{fr}(X)_f, LM_{fr}(X \times \mathbb{G}_m^{\wedge 1})_f, LM_{fr}(X \times \mathbb{G}_m^{\wedge 2})_f, \ldots),$$

where each $LM_{fr}(X\times\mathbb{G}_m^{\wedge n})_f$ is a Nisnevich local fibrant replacement of the $S^1$-spectrum $LM_{fr}(X\times \mathbb{G}_m^{\wedge n})$. It follows from the Cancellation Theorem for linear framed motives [AGP16] that $LM^{\mathbb{G}}_{fr}(X)_f$

is a motivically fibrant bispectrum. In particular, $(\mathbb{S}\wedge X_+)\otimes\mathbb{Q}$ is computed locally in the Nisnevich topology as the bispectrum

$$LM_{fr}^{\mathbb{G}_m}(X)\otimes\mathbb{Q}=(LM_{fr}(X)\otimes\mathbb{Q},LM_{fr}(X\times\mathbb{G}_m^{\wedge 1})\otimes\mathbb{Q},\ldots)$$

consisting of twisted rational linear framed motives of $X$. Each $S^1$-spectrum $LM_{fr}(X\times\mathbb{G}_m^{\wedge n})$ is the Eilenberg–Mac Lane spectrum associated with the simplicial Nisnevich sheaf $\mathbb{Z}F(-\times\Delta^\bullet,X\times\mathbb{G}_m^{\wedge n})$ defined in terms of the category of linear framed correspondences $\mathbb{Z}F_*(k)$ and stabilised in the $\sigma$-direction (see [GP14b] for details).

It is natural to compare twisted complexes defined by various categories of correspondences. There is constructed a functor in [DF17]

$$F:Fr_*(k)\to\widetilde{\mathrm{Cor}}.$$

It induces morphisms of twisted complexes

$$f_n:\mathbb{Z}F(-\times\Delta^\bullet,X\times\mathbb{G}_m^{\wedge n})\to\widetilde{\mathrm{Cor}}(-\times\Delta^\bullet,X\times\mathbb{G}_m^{\wedge n})_{\mathrm{nis}},\quad n\geqslant 0.$$

A question, originally due to Calmès and Fasel, is whether the $f_n$-s are quasi-isomorphisms of complexes of Nisnevich sheaves. The following theorem answers this question in the affirmative with rational coefficients.

Theorem 6.1 (Comparison). *Given an infinite perfect field of characteristic not 2 and a k-smooth scheme $X$, each morphism of complexes of Nisnevich sheaves*

$$f_n:\mathbb{Z}F(-\times\Delta^\bullet,X\times\mathbb{G}_m^{\wedge n})\otimes\mathbb{Q}\to\widetilde{\mathrm{Cor}}(-\times\Delta^\bullet,X\times\mathbb{G}_m^{\wedge n})_{\mathrm{nis}}\otimes\mathbb{Q},\quad n\geqslant 0,$$

*is a quasi-isomorphism.*

*Proof.* We defined the bispectrum $M_{MW}^{\mathbb{G}_m}(X)$ on p. 9. Taking levelwise Nisnevich local fibrant replacements, we get a bispectrum $M_{MW}^{\mathbb{G}_m}(X)_f$. The canonical morphism of bispectra $\Sigma_{S^1}^\infty\Sigma_{\mathbb{G}}^\infty X_+\to M_{MW}(X)_f$ factors as

$$\Sigma_{S^1}^\infty\Sigma_{\mathbb{G}}^\infty X_+\xrightarrow{\ell}LM_{fr}^{\mathbb{G}_m}(X)_f\xrightarrow{F}M_{MW}^{\mathbb{G}_m}(X)_f.$$

As we have shown above, the left arrow is rationally a stable motivic equivalence. $F$ is a map between fibrant bispectra which are both locally given by twisted complexes with linear framed and finite Milnor–Witt correspondences respectively. It follows that the morphisms of the corollary are quasi-isomorphisms if and only if $(F\circ\ell)\otimes\mathbb{Q}$ is an isomorphism in $SH(k)$. But the latter follows from the Reconstruction Theorem 5.5. □

Remark 6.2. Bachmann and Ananyevskiy pointed out recently to the author that Theorem 6.1 cannot be true with integer coefficients even for $X=pt$. Moreover, it is not true with $\mathbb{Z}[\frac{1}{p_1},\ldots,\frac{1}{p_s}]$-coefficients for any finite collection of primes $p_1,\ldots,p_s$. Therefore the quasi-isomorphism of the Comparison Theorem is genuinely rational.

## 7. Concluding remarks

The methods developed in the previous sections should also be applicable to compute $SH(k)_{\mathbb{Q}}$ in terms of the hypothetical category of "Hermitian correspondences" $K_0^h$. Its objects are those of $Sm/k$ and morphisms are given by certain bimodules with duality with/without coefficients in some line bundles. $K_0^h$ is expected to be a symmetric monoidal strict $V$-category of correspondences satisfying cancellation property. It is as well expected that

$$\mathbb{S}^{K_0^h}[2^{-1}] \cong \underline{W}^{\mathbb{G}_m}_{\mathbb{Z}[\frac{1}{2}]} \oplus \mathbb{S}^{K_0^{\oplus}}[2^{-1}].$$

Suslin's theorem [Sus03] together with [ALP17, Theorem 3.4] and [CD12, Theorem 16.2.13] then would imply that $\mathbb{S} \otimes \mathbb{Q}$ is isomorphic to $\mathbb{S}^{K_0^h} \otimes \mathbb{Q}$. The proof of the Reconstruction Theorem 5.5 then would be the same for showing that $SH(k)_{\mathbb{Q}}$ is equivalent to $DM_{K_0^h}(k)_{\mathbb{Q}}$.

The Suslin theorem [Sus03] comparing Grayson's cohomology with motivic cohomology is then extended to finite Milnor–Witt correspondences as follows. It states that there is a natural functor between categories of $V$-correspondences

$$K_0^h \to \widetilde{\mathrm{Cor}}$$

such that the induced morphisms of twisted complexes of Nisnevich sheaves

$$K_0^h(- \times \Delta^\bullet, \mathbb{G}_m^{\wedge n})_{\mathrm{nis}} \to \widetilde{\mathrm{Cor}}(- \times \Delta^\bullet, \mathbb{G}_m^{\wedge n})_{\mathrm{nis}}$$

is locally a quasi-isomorphism (at least over infinite perfect fields of characteristic not 2). The extension of the Suslin theorem should be reduced to the original Suslin theorem.

We invite the interested reader to construct the category of "Hermitian correspondences" $K_0^h$ with the desired properties.


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