Identification of weak nonlinearities in MDOF systems based on reconstructed constant response tests

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Abstract:

A novel strategy to characterize and identify structural nonlinearities in MDOF systems based on reconstructing constant response tests from constant excitation tests is developed in this paper. Constant displacement frequency response functions (FRFs) can be measured by a stepped sine test where the displacement is controlled at every frequency of interest. In these FRFs the nonlinear restoring force is effectively linearized and natural frequencies can be estimated by linear modal analysis. Using a series of constant displacement tests, the relationship of equivalent stiffness versus displacement can be established by curve fitting and hence the nonlinear stiffness characterized. This paper proposes a method to reconstruct the constant displacement FRFs from stepped sine tests with constant excitation; this avoids the requirement to control either the response or force amplitude, leads to a faster and more stable testing programme. Similarly, damping nonlinearities in structures can be characterized and identified by constant velocity tests reconstructed in a similar way. This approach of FRF reconstruction is mathematically simple and suitable for structures with weak nonlinearities. The method is demonstrated on a framed structure with unknown weak nonlinearities, and the nonlinear stiffness and damping parameters of the structure are identified and validated. The results demonstrate the feasibility and effectiveness of the approach, and also show the potential for practical applications in engineering.

1 Introduction

Practical engineering structures often exhibit nonlinear dynamic behaviour. In order to accurately predict dynamic nonlinear responses, it is vital to characterize and identify these nonlinearities and construct accurate nonlinear models from measured vibration data. Recently, Noël and Kerschen [1] gave an overview of nonlinear system identification in structural dynamics and an updated literature review over the previous 10 years since the review by Kerschen et al. [2]. Hence only a brief overview, concentrating on modal and FRF methods, will be given here. Identification methods can use a range of measured and predicted responses to estimate uncertain parameters. Nonlinear normal modes (NNMs) are important dynamic features that can be used to understand the nonlinear response of a structure. The updating of finite element models based on nonlinear normal modes (NNMs) has also been recommended [3]. Kurt et al. [4] constructed frequency–energy plots (FEPs) from transient system responses, and the system parameters were characterized and updated by matching the backbone branches of the FEPs with reduced-order models using experimental or computational results. The finite element model updating of nonlinear

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structures using stochastic filtering and batch Bayesian estimation has also been proposed [5],[6]. Wang et al. [7], [8] proposed a new nonlinear structural model updating method based on instantaneous characteristics. They also proposed a procedure to localise nonlinear elements using spatially incomplete measured frequency response data from a structural vibration test [9] and proposed a model updating strategy for structures with localised nonlinearities using frequency response measurements [10]. Ewins et al. [11] and Carri et al. [12] extended modal testing technology to validate models of engineering structures with sparse nonlinearities and divided their approach into three phases: Preparation, Test and Identification, and Verification and Validation. Cooper et al. [13] proposed a pragmatic approach to integrate test-based system identification and FE modelling of a nonlinear structure, with three different phases: the derivation of an Underlying Linear Model (ULM) of the structure, nonlinear identification using measured time series and augmenting the linear FE model, and experimental validation of the nonlinear FE model.

The linearization method has continued to be an important and popular solution to engineering structures because of the maturity and extensive application of linear technology [1]. Recently, sinusoidal excitation and modal analysis techniques have been developed for nonlinear identification [14],[15],[16]. The method, based on equivalent linearization theory, is expected to be applied widely in industry because of its mathematical simplicity and the maturity of linear identification techniques [17]. Zang et al. [18] proposed an effective linearization method to validate the nonlinear structural dynamic model for the Sandia Structural Dynamics Challenge. Schwingshackl et al. [19] investigated the influence of nonlinearity on uncertainty and variability for dynamic models. Based on the simulated response of a three degrees of freedom vibration system with a weak nonlinearity, two dynamic tests, namely a constant-amplitude displacement test and a constant-amplitude velocity test, were exploited to identify the nonlinear stiffness and damping behaviour under sinusoidal force excitation. This method can effectively establish the relationship between displacement and equivalent stiffness, as well as the relationship between velocity and equivalent damping. The unknown nonlinearities may be characterized and their appropriate functional forms obtained. Zhang and Zang [20] successfully applied the constant response test method to a single degree of freedom system. The response comparison between the predicted response and the experimental test indicates that the identified nonlinear model can accurately represent the actual nonlinear behaviour of the structure, although there was a slight difference in the resonance zone. However, difficulties still exist in the application to more complicated multi degree of freedom systems, such the approach to characterize and identify the structural nonlinearities, how to establish the relationship between equivalent stiffness and displacement (or equivalent damping and velocity) to characterize the nonlinear stiffness (or damping) and the model validation process.

This paper develops the strategy for identification of structural nonlinearities based on reconstructed constant response tests for nonlinear model validation. The method to reconstruct the constant response FRFs from stepped sine constant excitation test data is described and approaches for nonlinear characterization and parameter identification are also addressed. The method is demonstrated on a 3 DOFs system with unknown nonlinearities and the experimental results verify the feasibility of the method and its capability to characterize the nonlinearity and estimate the associated parameters.

2 Nonlinear parameter identification in MDOF systems

2.1 Strategy of nonlinear parameter identification

The target structure for the proposed identification methods is a predominantly linear structure with a localized weak nonlinearity. The proposed strategy to identify the nonlinear parameters of the structural system includes several steps. First, the underlying linear model is validated based on experiments with low amplitude excitation forces; this works well for structures with smooth nonlinearities, and care has to be exercised for non-smooth nonlinearities such as friction. Following the linear identification, higher amplitude tests are used for nonlinear detection, nonlinear characterizatio Verification of nonlinear model verification. The framework of this strategy shown in Fig.**1**, where the main steps are to characterize the nonlinear stiffness and damping based on reconstructed constant displacement and velocity tests, and to estimate the nonlinear parameters by equivalent linearization theory.

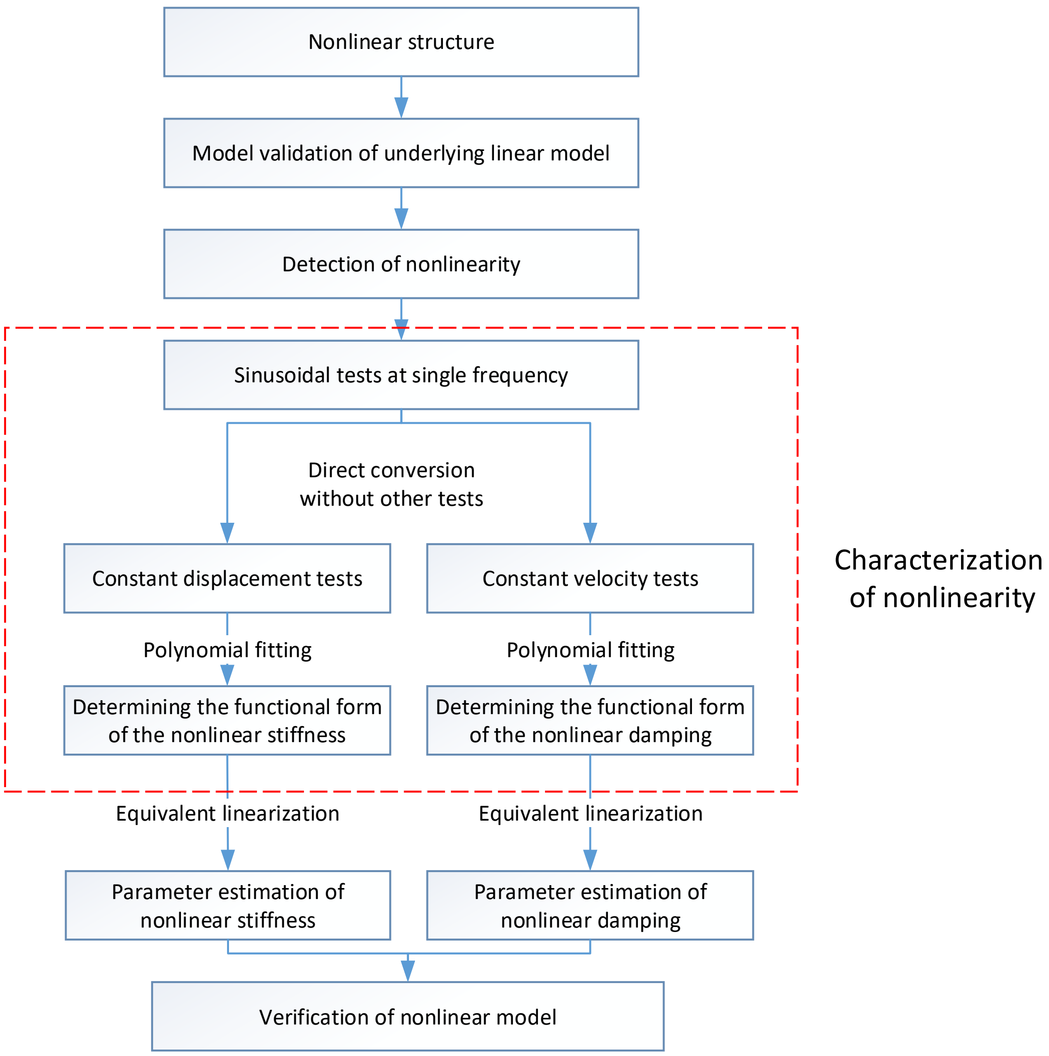


Fig.1. A framework for nonlinear parameter identification

2.2 Model validation of the underlying linear model

The first step of the framework for nonlinear parameter identification is to validate the underlying linear model. The influence of nonlinear elements on the natural frequencies and damping ratios of the linearized system are then analyzed in subsequent steps.

There are usually two ways to obtain the underlying linear model of a nonlinear structure. One is based on the assumption that the nonlinearities in structures are negligible at low amplitudes of excitation. Therefore, if modal testing is performed on the nonlinear structure at a very low excitation level, the nonlinearity in the structure would be insignificant and can be ignored. In this case, a conventional (linear) model updating and validation procedure using linear modal parameters from tests can be undertaken and an accurate mathematical linear model obtained. The other approach is based on curve fitting and extrapolation of modal parameters under different excitation levels of the nonlinear structure. Usually, a series of modal tests of a nonlinear structure at different excitation levels are performed and the modal parameters extracted. Afterwards, the relationship between the different excitation levels and corresponding modal parameters can be built by curve fitting. By extrapolating the fitted curve to the zero excitation level, the corresponding modal parameters can be obtained and these parameters are assumed to represent the underlying linear structure. Generally, the former approach is often sufficient and simpler than the latter, but the latter is more precise to describe the linear part of the nonlinear structure.

After the validation of the underlying linear model, a one-to-one mapping between the equivalent stiffness, k_{eq}, of the nonlinear element and the natural frequency, ω, that is $\omega = \phi\left(k_{eq}\right)$, is established with the help of the underlying linear model simulation. The corresponding relationship between the displacement amplitude, D, and the natural frequencies, that is $\omega = \varphi(D)$, is then found from the subsequent constant displacement tests, and the relationship between the equivalent stiffness and displacement is created as $k_{eq} = \varphi^{-1}\left(\phi(D)\right) = f(D)$. The nonlinear stiffness is then characterized through curve fitting. This procedure is illustrated in Fig.**2**.

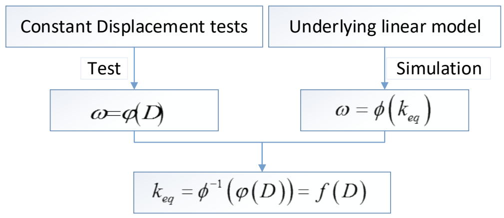


Fig.2. Relationship between displacement response amplitude and equivalent stiffness

Similarly, the one-to-one mapping between the equivalent damping and the structural damping ratio can also be analyzed. In fact, the underlying linear model builds a bridge linking the physical parameters (stiffness, damping) and the modal parameters (natural frequency, damping ratio). A precise linear model is the basis for the identification of the nonlinear parameters.

The following describes how to analyze the influence of the equivalent stiffness on the natural frequencies and the equivalent damping on the damping ratio.

2.2.1 Equivalent linearization of a nonlinear system

For a clear description of the method, only one nonlinear element in the system is considered here, and the extension to multiple nonlinear elements in the system is discussed later. The general equation of motion for a nonlinear system with n DOFs with viscous damping and harmonic excitation is expressed as

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} + \boldsymbol{f}_{nl}\left(\boldsymbol{x}, \dot{\boldsymbol{x}}\right) = \boldsymbol{f}\left(t\right) \tag{1}$$

where $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are the mass, damping and stiffness matrices of the underlying linear model, respectively. The vectors $\boldsymbol{x}$, $\boldsymbol{f}\left(t\right)$ and $\boldsymbol{f}_{nl}\left(\boldsymbol{x}, \dot{\boldsymbol{x}}\right)$ represent the displacements, the external sinusoidal forces and the nonlinear internal force.

Based on the equivalent linearization theory, the nonlinear system can be approximated by

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} + \boldsymbol{K}_{eq}\boldsymbol{x} + \boldsymbol{C}_{eq}\dot{\boldsymbol{x}} = \boldsymbol{f}\left(t\right) \tag{2}$$

where $\boldsymbol{K}_{eq}$ and $\boldsymbol{C}_{eq}$ are the equivalent stiffness and damping matrices resulting from the nonlinear element.

Due to the diversity of nonlinear characteristics, the equivalent stiffness and damping forces, $f_s(\boldsymbol{x})$ and $f_d(\dot{\boldsymbol{x}})$ have different forms. Furthermore, the nonlinear element could be grounded or located between two degrees of freedom; these cases are described below.

（1）If the nonlinear force is related to only one DOF, such as x_i, then

$$\begin{aligned} f_d(\dot{\boldsymbol{x}}) &= k_{eq} x_i \\ f_s(\dot{\boldsymbol{x}}) &= c_{eq} \dot{x}_i \end{aligned} \tag{3}$$

where k_{eq} and c_{eq} are the equivalent stiffness and equivalent damping respectively.

（2）If the nonlinear force is related to multiple DOFs, such as x_i and x_j, then the nonlinear terms are expressed as

$$\begin{aligned} \begin{bmatrix} f_d(\boldsymbol{x}) \\ -f_d(\boldsymbol{x}) \end{bmatrix} &= \begin{bmatrix} k_{eq}(x_i - x_j) \\ -k_{eq}(x_i - x_j) \end{bmatrix} = \begin{bmatrix} k_{eq} & -k_{eq} \\ -k_{eq} & k_{eq} \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix} \\ \begin{bmatrix} f_d(\dot{\boldsymbol{x}}) \\ -f_d(\dot{\boldsymbol{x}}) \end{bmatrix} &= \begin{bmatrix} c_{eq}(\dot{x}_i - \dot{x}_j) \\ -c_{eq}(\dot{x}_i - \dot{x}_j) \end{bmatrix} = \begin{bmatrix} c_{eq} & -c_{eq} \\ -c_{eq} & c_{eq} \end{bmatrix} \begin{bmatrix} \dot{x}_i \\ \dot{x}_j \end{bmatrix} \end{aligned} \tag{4}$$

The nonlinear forces could be assembled into the matrix. The equivalent linearization form could also be written as

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \left(\boldsymbol{C} + \boldsymbol{C}_{eq}\right)\dot{\boldsymbol{x}} + \left(\boldsymbol{K} + \boldsymbol{K}_{eq}\right)\boldsymbol{x} = \boldsymbol{f}(t) \tag{5}$$

2.2.2 The relationship between equivalent stiffness and damping and modal properties

When the number of DOFs in the system is small, the eigenvalues and eigenvectors are easily computed. In this case, assuming the location of the nonlinearity is known, the relationships could be established by repeated calculation of the eigenvalues and eigenvectors. Thus, the natural frequency may be written as a function of the equivalent stiffness as

$$\omega_r = \varphi\left(k_{eq}\right), r = 1, 2, \cdots, n \tag{6}$$

where r is the mode number.

Often in linear systems proportional damping is assumed to make the calculation of damping ratio much easier. However, general linear damping may be analyzed by considering the state space form of the equations of motion and the resulting complex modes [21].

- In either case, repeated calculation establishes the relationship between the equivalent damping and the damping ratio as

$$\zeta_r = \phi\left(c_{eq}\right), r = 1, 2, \cdots, n \tag{7}$$

Note that we have assumed that the natural frequency is only a function of equivalent stiffness, and the damping ratio is only a function of equivalent damping. For lightly damped structures this is a good approximation, although, in general, the damping ratio can have a slight sensitivity to the stiffness properties.

2.3 Nonlinearity detection and characterization

The purpose of nonlinearity detection is mainly to detect whether any nonlinearity exists in the structure over the frequency range of interest. In this case the system response with nonlinearity cannot be represented by a linear model. The detection of nonlinearity is often performed using a range of excitation force levels; for example, a series of sinusoidal sweeps at different excitation levels can be applied and differences in the FRFs indicate nonlinearity. Generally, sinusoidal tests (ST) at single frequencies are simple to set up. Although these tests are time-consuming, the obtained nonlinear dynamical behaviors are accurate and are highlight any nonlinear phenomena present.

2.3.1 Constant displacement tests (CDT) and constant velocity tests (CVT)

Generally, three types of constant dynamic tests are used for structures, with the objective to keep constant either the force, displacement, or velocity, during sinusoidal excitation. The constant force test is most widely used, and aims to keep the amplitude of the sinusoidal excitation force constant as the frequency is sweep over the range of interest. The constant displacement/velocity tests need to adjust the amplitudes of sinusoidal force in order to keep the displacement/velocity response amplitudes unchanged during the frequency sweep. The difficulty in conducting constant displacement/velocity tests is that a controller is required to maintain the constant response amplitudes at the steady state, and this is particularly challenging for lightly damped structures close to resonance. It should be noted that even constant force tests require active control of the excitation level from the analyzers because the interaction between the shaker and structure causes the force level to drop near the resonances for a given excitation voltage to the shaker.

Constant displacement tests use single frequency sinusoidal excitation to measure the frequency response functions (FRFs) where the amplitudes of the displacement response are constant at every frequency of interest by adjusting the amplitude of sinusoidal excitation. These FRFs are called constant displacement FRFs, while the FRFs in traditional stepped sine tests are constant voltage FRFs. At every frequency of the constant displacement FRFs, the equivalent stiffness of the structure is the same. The stiffness nonlinearities are effectively linearized and linear modal analysis techniques can be used to identify the modal parameters at the given amplitude of displacement response. The constant velocity test is similar.

Constant displacement and velocity tests need feedback control and are usually time-consuming as the control algorithm for the force amplitude needs to converge. A direct conversion method without feedback will be introduced reconstruct the constant displacement FRFs and the constant velocity FRFs from the constant voltage FRF test data.

2.3.2 A transformation method from ST to CDT and CVT

The purpose of the CDT and the CVT is to obtain FRFs with constant displacement or velocity. Here, a simple method is given to construct the constant displacement or velocity FRFs from the constant voltage FRFs so that no feedback control is required during testing. However more constant voltage tests will typically be required.

The relationship between the sinusoidal excitation and response amplitudes is affected by nonlinearity at various frequencies, and is generally not linear but remains monotonically increasing for many structures with weak nonlinearities. The transformation method is illustrated in Fig.3.

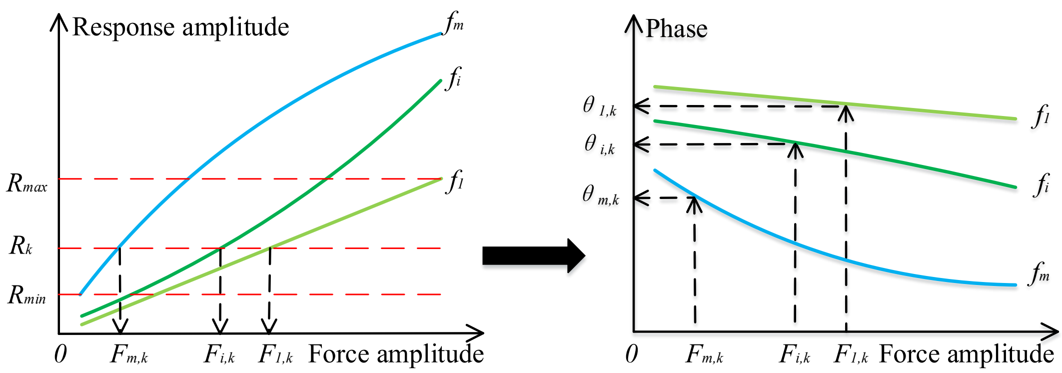


Fig.3. The transformation method from ST to CDT and CVT

The displacement / velocity range is determined by the minimum value $R_{\min}$, and the maximum value $R_{\max}$. Then a set of l points $[R_{\min},\cdots,R_k,\cdots,R_{\max}]$ are generated that are linearly spaced.

To obtain the CDT with displacement R_k as an example, the force amplitude, $F_{i,k}$, is determined by interpolation for each frequency of interest, f_i. Once the force is obtained, the corresponding phase $\theta_{i,k}$ is also obtained by interpolation. This allows the constant displacement FRF to be calculated at this frequency. This is repeated for all frequencies of interest.

It should be noted that this transformation method is usually exploited in the case that the relationship between the excitation and response amplitudes is guaranteed to be monotonic. Difficulties will exist when the system is strongly nonlinear, particularly with jump phenomena.

2.4 Parameter estimation of nonlinearity

2.4.1 Equivalent stiffness and damping of one nonlinear element

One nonlinear element could be placed in an SDOF system to obtain the equivalent linearization. In the case of an SDOF nonlinear system where the nonlinearity is additively separable, the equation of motion can be written as

$$m\ddot{x}+c\dot{x}+kx+f_d\left(\dot{x}\right)+f_s\left(x\right)=F\sin\left(\omega t\right) \tag{8}$$

where $f_d\left(\dot{x}\right)$ represents the weak nonlinearity of the damping behavior and $f_s\left(x\right)$ represents the weak nonlinear stiffness. Based on equivalent linearization theory, $f_s\left(x\right)$ and $f_d\left(\dot{x}\right)$ may be approximated by:

$$f_s\left(x\right)=k_{eq}x \tag{9}$$

$$f_d\left(\dot{x}\right)=c_{eq}\dot{x} \tag{10}$$

where c_{eq} is the equivalent linear damping for the damping nonlinearity and k_{eq} is the equivalent linear stiffness for the stiffness nonlinearity.

Thus, its equivalent linearization is written as [17]:

$$m\ddot{x}+\left(c+c_{eq}\right)\dot{x}+\left(k+k_{eq}\right)x=F\sin\left(\omega t\right) \tag{11}$$

The development of equivalent linearization method is introduced in detail in refs. **Error! Reference source not found.**, [23] and [24]. The equivalent linearization technique dates back to the fundamental work of Caughey[25], which considers statistical linearization. In order to reduce the error and lead to a better approximate solution, Anh [26] proposed a duality method to analyze the responses of nonlinear systems under periodic and random excitations. The harmonic balance method was also used to obtain the analytical expressions of equivalent linear stiffness and equivalent linear damping for nonlinear identification and characterization [16].Considering that the excitations are harmonic forces in the experiment, equivalent linearization by HBM is more concise mathematically.

Generally, the fundamental harmonic of the response are dominant in the experiment for weakly nonlinear systems. Therefore, we mainly consider the first harmonic and neglect the sub-harmonics and the super-harmonics in the harmonic balance method for weakly nonlinear systems. For strongly nonlinear systems, in order to improve the identification accuracy, it is necessary to consider the high order harmonics in the HBM if the influence of higher order harmonics are not negligible.

Assuming that the response to a sinusoidal excitation is a sinusoid at the same frequency, the displacement and velocity can be expressed as

$$x = A\sin\left(\omega t + \varphi\right) \tag{12}$$

$$\dot{x} = \omega A \sin\left(\omega t + \varphi + \frac{\pi}{2}\right) \tag{13}$$

where A is the amplitude of the displacement response at steady state.

The harmonic balance (HB) method applied to the equation of motion yields the equivalent stiffness as

$$k_{eq} = \frac{1}{\pi A}\int_0^{2\pi} f_s\left(A\sin\theta\right)\sin\theta \, \mathrm{d}\theta = f\left(A\right) \tag{14}$$

and the equivalent damping as

$$c_{eq} = \frac{1}{\pi\omega A}\int_0^{2\pi} f_d\left(\omega A\sin\theta\right)\cos\theta \, \mathrm{d}\theta = g(\omega A) \tag{15}$$

According to Eqs. (14) and (15), the equivalent stiffness is a function of displacement amplitude and the equivalent damping is a function of velocity amplitude. Therefore, if the amplitudes of displacement are kept constant through adjusting the amplitude of the sinusoidal force over the frequency range of interest, the influence of the nonlinear stiffness on the measured FRF data can be minimised. This constant displacement test can be used to measure the relationship between the equivalent stiffness and the displacement amplitude. The nonlinear damping behaviour and terms can be also obtained in a similar way from a constant velocity test of the structure.

Afterwards, the identified nonlinear properties can be included into the model to predict the response of nonlinear system.

2.4.2 Parameter estimation of the nonlinearity

Suppose the functional form of equivalent stiffness, $k_{eq,test}$, and displacement amplitude, A, could be established through polynomial curve fitting after the constant displacement test. Thus

$$k_{eq,test} = f(A) = p_1 A + p_2 A^2 + p_3 A^3 + p_4 A^4 \ldots \tag{16}$$

Considering the symmetry of nonlinear forces, the nonlinear stiffness force could be written as

$$f_s = k_1 sign(x) x^2 + k_2 x^3 + k_3 sign(x) x^4 + k_4 x^5 \cdots \tag{17}$$

According to the integral formula of equivalent linearization, Eq. (14), the equivalent stiffness k_{eq} could be written as

$$k_{eq} = q_1 k_1 A + q_2 k_2 A^2 + q_3 k_3 A^3 + q_4 k_4 A^4 \cdots \tag{18}$$

Assuming that

$$k_{eq,test} = k_{eq} \tag{19}$$

where the coefficients $q_1, q_2, q_3, q_4 \ldots$are $\frac{8}{3\pi}, \frac{3}{4}, \frac{32}{15\pi}, \frac{5}{8}, \ldots.$

Thus, the nonlinear stiffness parameters can be expressed as

$$\begin{cases} k_1 = p_1 / q_1 \\ k_2 = p_2 / q_2 \\ k_3 = p_3 / q_3 \\ k_4 = p_4 / q_4 \\ \cdots \end{cases} \tag{20}$$

The estimation of nonlinear damping parameters is similar.

2.5 Verification of the identified nonlinear model

The whole nonlinear model is established after validation of the underlying linear model and identification of the nonlinear stiffness and damping parameters. But the effectiveness of the nonlinear model also needs further verification through a comparison between response prediction and experimental results to evaluate the accuracy of the nonlinear model. This is done by simulating the measured constant voltage FRFs. The identified model with nonlinear parameters can then accurately predict the nonlinear vibration behavior of the structure.

3 Application: 3DOF system with a weakly nonlinear connection

3.1 The 3-DOF test structure with weak nonlinearity

In order to illustrate and verify the identification method for the nonlinear parameters, a series of tests has been conducted on a frame structure with bolted joints shown in Fig.4. This structure represents a 3DOF system by just considering the vibration in the x-direction; the 3 steel plates are equivalent to 3 lumped masses, and the 4 aluminum plates are divided into 3 parts, equivalent to 3 stiffnesses. Four rubber rings are installed between the four aluminum beams and the base plate respectively to introduce unknown nonlinear stiffness and damping behavior.

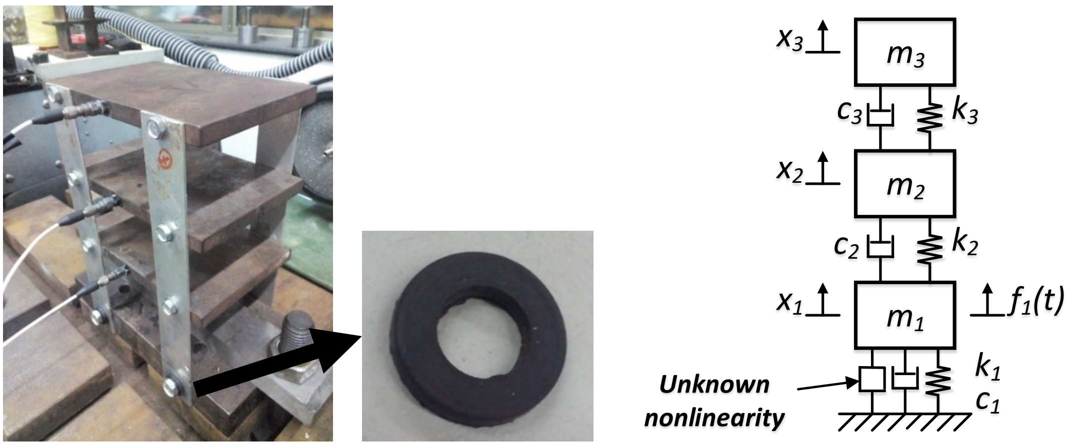


Fig.4. The 3 DOF system with unknown nonlinearities

3.2 Construction of the underlying linear model

3.2.1 Construction of the underlying linear model

The basic equation of motion of the 3DOF system can be written as

$$\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{C}\dot{\boldsymbol{x}}+\boldsymbol{K}\boldsymbol{x}+\boldsymbol{f}_{nl}\left(\boldsymbol{x},\dot{\boldsymbol{x}}\right)=\boldsymbol{f}\left(t\right) \tag{21}$$

where $\boldsymbol{M}=\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \boldsymbol{C}=\begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}, \boldsymbol{K}=\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$

$\boldsymbol{x}=\{x_1,x_2,x_3\}^T$, $\boldsymbol{f}_{nl}(\boldsymbol{x},\dot{\boldsymbol{x}})=\begin{bmatrix} g(x_1,\dot{x}_1) & 0 & 0 \end{bmatrix}^T$ represents the unknown nonlinearity and $\boldsymbol{f}_1(t)=\{f(t) \quad 0 \quad 0\}^T$ represents the external excitation applied on the first DOF, x_1.

The modal data obtained from modal testing with a low excitation level were used to construct the underlying linear spatial model. Thus, the mass, stiffness, and damping matrices can be derived from

$$\boldsymbol{M}=\boldsymbol{\Phi}^{-T}\boldsymbol{\Phi}^{-1} \tag{22}$$

$$\boldsymbol{K}=\boldsymbol{\Phi}^{-T}\begin{bmatrix} \ddots & & \\ & \omega_r^{\ 2} & \\ & & \ddots \end{bmatrix}\boldsymbol{\Phi}^{-1} \tag{23}$$

$$\boldsymbol{C}=\boldsymbol{\Phi}^{-T}\begin{bmatrix} \ddots & & \\ & 2\zeta_r\omega_r & \\ & & \ddots \end{bmatrix}\boldsymbol{\Phi}^{-1} \tag{24}$$

where ω_r, ζ_r denote the rth natural frequency and damping factor, and the matrix $\boldsymbol{\Phi}$ is the modal matrix with columns consisting of mass-normalized mode shape vectors. Modal testing is carried out at a low excitation level. The excitation signal is a chirp and the RMS of the force is 0.85N，as shown in Fig.5**Error! Reference source not found.**. The mode shapes are given in Fig.6. Table 1 shows the modal parameters. Hence the $\boldsymbol{M},\boldsymbol{K},\boldsymbol{C}$ matrices can be obtained and the underlying spatial model can be constructed.

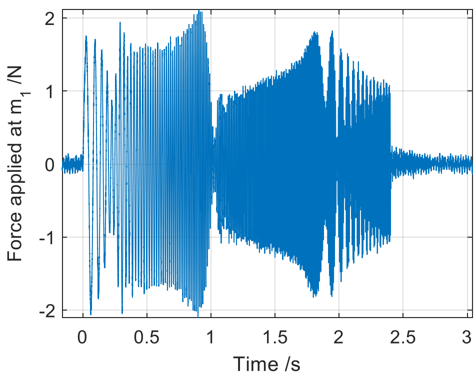


Fig.5. The chirp force applied to mass 1 for modal testing

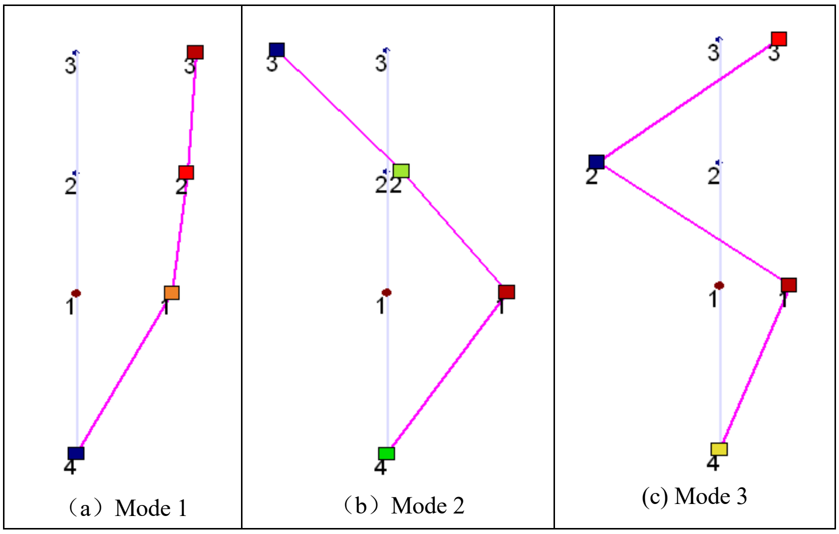


Fig.6. Mode shapes of the 3 DOF system under low excitation level

Table 1 Modal parameters of the 3DOF system under low excitation

Modal parameters	Mode 1	Mode 2	Mode 3
Natural frequency(Hz)	24.47	88.15	155.36
Loss factor (%)	5.20	2.43	0.43

3.2.2 Response prediction of the underlying linear model

The FRFs and time response need to be verified in view of the importance of the underlying linear model. The test FRF and the predicted FRF of the constructed underlying linear model are shown in Fig.**7** and they are in good agreement.

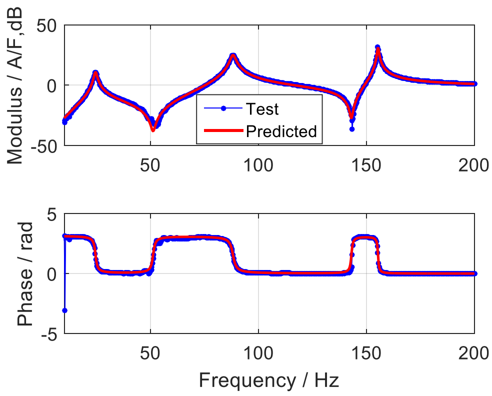


Fig.7. The test FRF and predicted FRF of the underlying linear model

The test and predicted time response of mass m_1 to the low level chirp excitation is also in good agreement, as shown in Fig.8. Hence this model is valid and can be used in the nonlinear identification procedure.

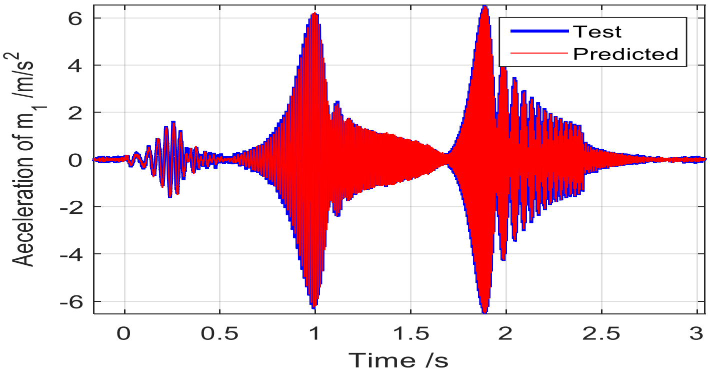


Fig.8. Time response of mass m_1 to the low level chirp excitation

3.2.3 The equivalent linear stiffness and damping

For a non-linear structure, the non-linear elements in the structure can be considered to have equivalent linear stiffness or damping coefficients if the amplitudes of the vibration responses are of fixed steady-state amplitude at each frequency. Then, the values of these equivalent linear stiffness or damping factors can be identified as a linear system. An equivalent equation is described as follows,

$$\boldsymbol{M}\ddot{\boldsymbol{x}}+\left(\boldsymbol{C}+\boldsymbol{C}_{eq}\right)\dot{\boldsymbol{x}}+\left(\boldsymbol{K}+\boldsymbol{K}_{eq}\right)\boldsymbol{x}=\boldsymbol{f}\left(t\right) \tag{25}$$

where $\left(\boldsymbol{C}+\boldsymbol{C}_{eq}\right)=\begin{bmatrix} c_{eq}+c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}$ and $\left(\boldsymbol{K}+\boldsymbol{K}_{eq}\right)=\begin{bmatrix} k_{eq}+k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$

The influence of the equivalent stiffness k_{eq} on the natural frequencies could be analyzed though simulation and is shown in Fig.9 to 11. Thus, the one-to-one mapping between the equivalent stiffness k_{eq} and the natural frequencies has been established.

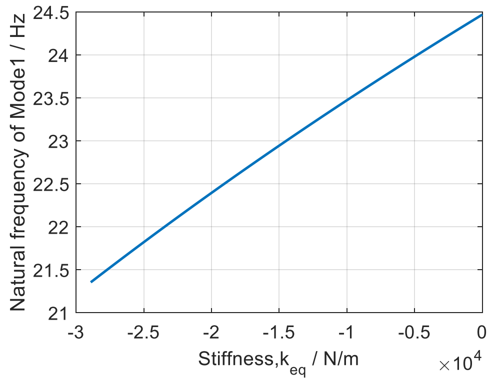


Fig.9. The influence of the equivalent stiffness k_{eq} on the natural frequency of Mode 1

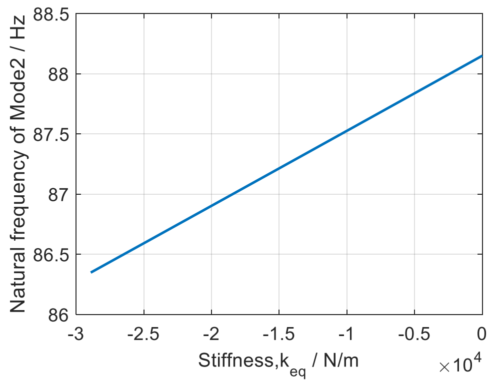


Fig.10. The influence of the equivalent stiffness k_{eq} on the natural frequency of Mode 2

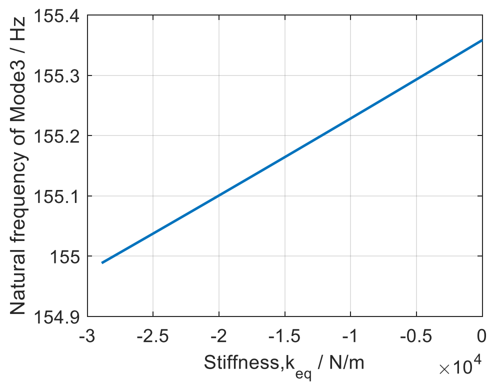


Fig.11. The influence of the equivalent stiffness k_{eq} on the natural frequency of Mode 3

Similarly, the one-to-one mapping between the equivalent damping and the structure damping ratio also can be established as shown in Fig.12.

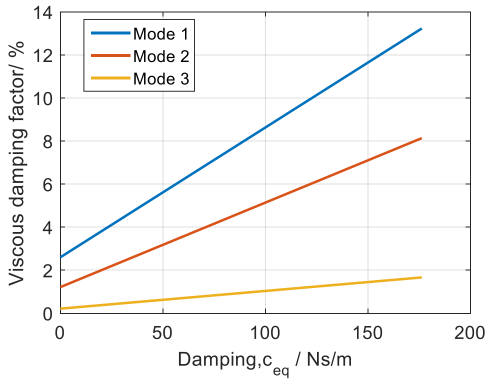


Fig.12. The influence of the equivalent damping c_{eq} on the damping factors

3.3 Nonlinear detection

3.3.1 Overlays of FRFs

To detect the nonlinearity of the 3 DOF system, a series of swept sinusoidal excitation with various force levels from 0.8N to 18.9N are applied to the frame and the acceleration responses are measured. Obviously, the overlay of the measured FRFs in Fig.13 shows the distortion of the acceleration FRFs at higher force levels. With an increase of excitation force level, the resonant frequency shifts downwards, so that it is a softening system. The changes in the natural frequency of mode 1 are clearly shown in Fig.14.

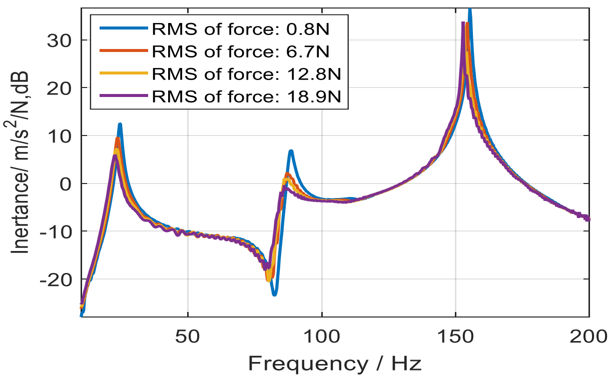
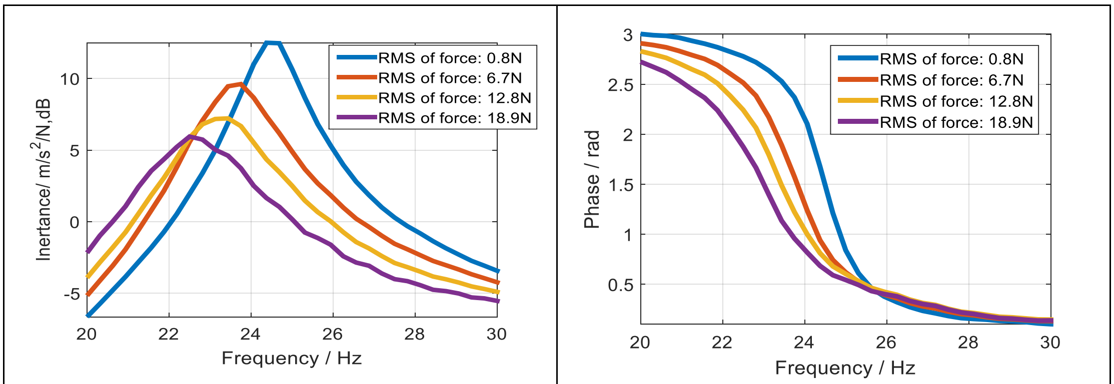


Fig.13. Overlays of acceleration FRFs in the chirp test

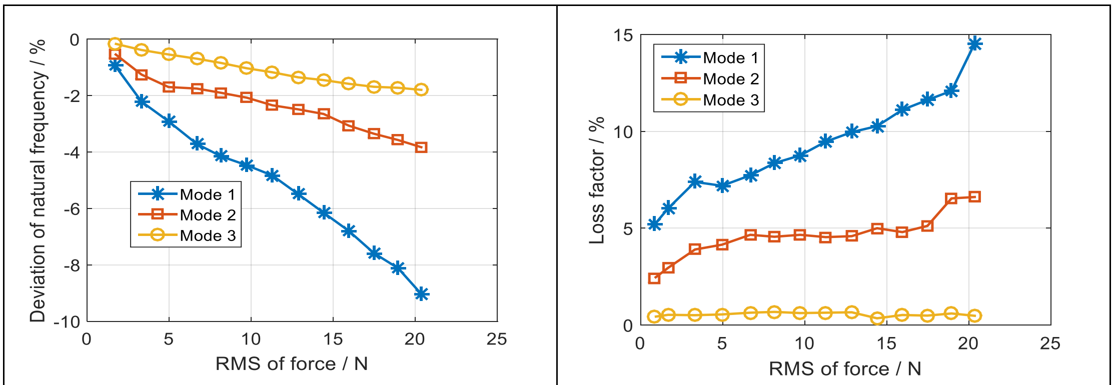


(a) amplitude (b) phase

Fig.14. Acceleration FRFs (20-30Hz) in the chirp test

3.3.2 The change of modal parameters with the excitation level

The modal parameters can be extracted from the FRFs using the circle fit method. It can be seen that the natural frequency of mode 1 decreases by 9% and the loss factor of mode 1 shifts from 5% to 14.5%, when the excitation force is increased from 0.8N to 20N, as shown in Fig.15.



(a) The deviation of natural frequencies at different excitation levels (b) The loss factors at different excitation level

Fig.15. The deviation of natural frequencies and the loss factors at different excitation level

3.4 Sinusoidal tests at a single frequency

20 sets of single frequency sinusoidal test were conducted, and the test parameters are shown in Table 2. A group of sine tests needs about 7 minutes and 20 groups of tests need 140 minutes. This time is relatively long and the temperature of the elastomer may increase during the test, leading to changes in the stiffness and damping properties. However, we will assume that the structure does not change its properties during the tests.

Table 2 The parameter settings for the constant voltage test

Frequency span	Frequency interval	Voltage levels	Sampling Freq	Sampling time
20 Hz—27Hz	0.1Hz	0.05:0.05:1V	2000Hz	4s

3.4.1 The force dropout phenomenon near resonance

In each stepped excitation frequency sinusoidal test, the excitation voltage for each frequency point was kept constant. There exists a force drop-off phenomenon near resonance that is clearly shown in Fig.16. Therefore, feedback control would be required for a constant amplitude force test, which is usually time-consuming. In our test, the purpose is to test the relationship between external excitation and vibration response. Thus, feedback control is not necessary.

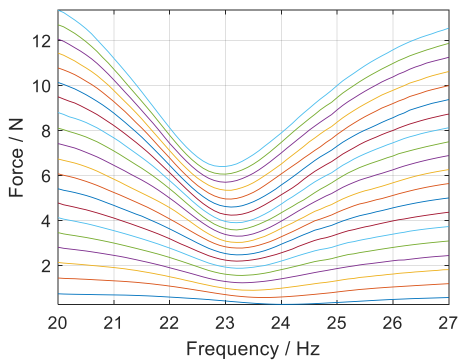


Fig.16. The force dropout near resonance

3.4.2 The FRFs from ST tests

The accelerance FRFs obtained from the sinusoidal tests and their amplitudes and phases against frequency are shown in Fig.17.

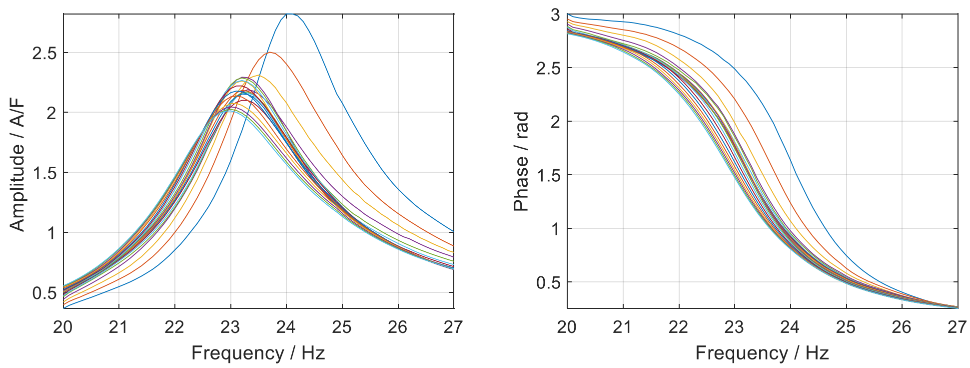


Fig.17. The test acceleration FRFs

If the accelerance FRFs are $H_a(\omega)$, the mobility FRFs $H_v(\omega)$ and the receptance FRFs $H_d(\omega)$ can be written as

$$H_v(\omega) = \frac{H_a(\omega)}{j\omega} \tag{26}$$

$$H_d(\omega) = \frac{H_a(\omega)}{(j\omega)^2} \tag{27}$$

The mobility FRFs and receptance FRFs generated from the acceleration FRFs are plotted in Fig.18 and Fig.19 respectively.

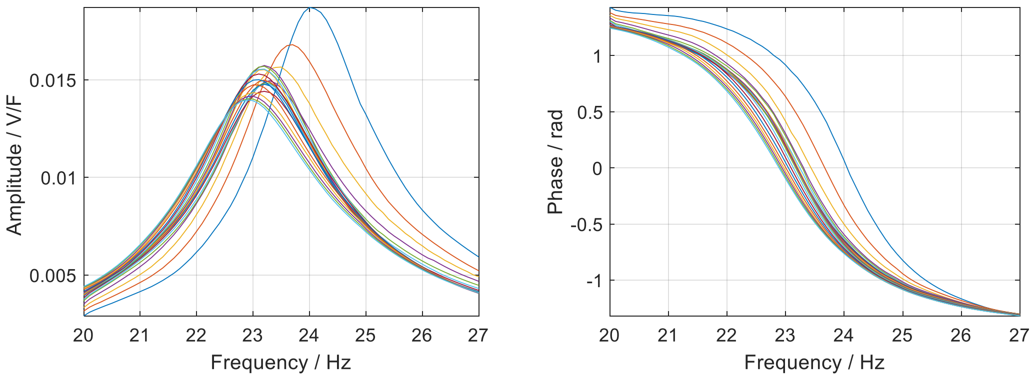


Fig.18. The mobility FRFs generated from the acceleration FRFs

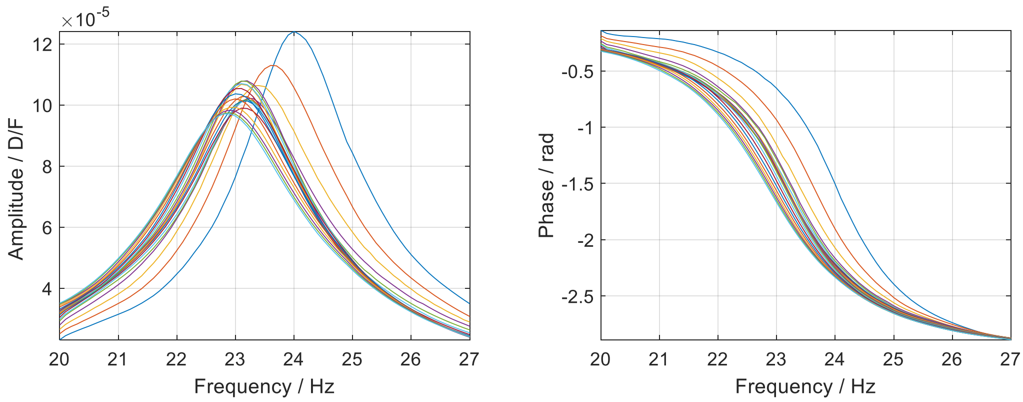


Fig.19. The receptance FRFs generated from the acceleration FRFs

3.4.3 The relationship between response and excitation

The variation of the amplitude and phase of the displacement with the excitation level at different frequencies can be obtained directly as shown in Fig.20 and Fig.21, which are 3D plots.

The right views shown in Fig.22 and Fig.23 represent the relationship between the response and the excitation. A monotonic increasing function between force and displacement amplitudes can be seen clearly while the phase shows a complex function. Similarly, the variations of amplitude and phase of the velocity with excitation level at different frequencies could also be obtained.

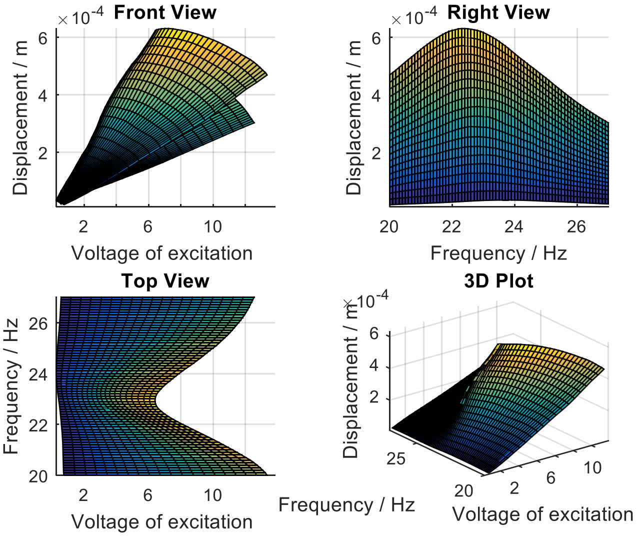


Fig.20. The 3D plot of the displacement amplitude

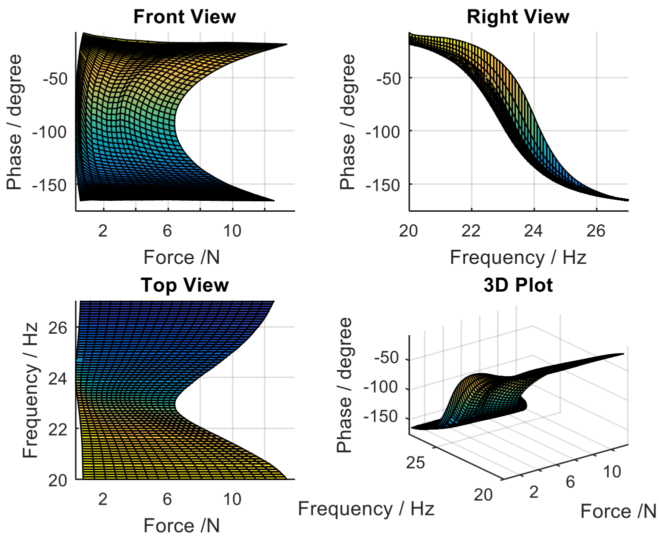


Fig.21. The 3D plot of the displacement phase

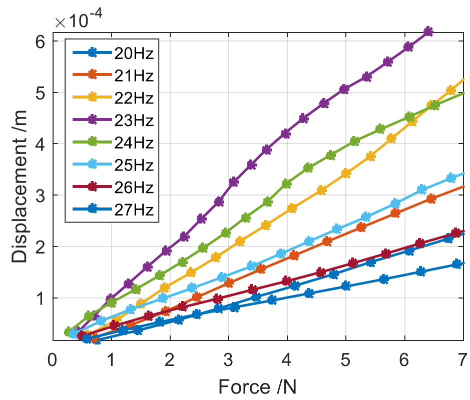


Fig.22. The variation of displacement amplitude with the excitation levels at different frequencies

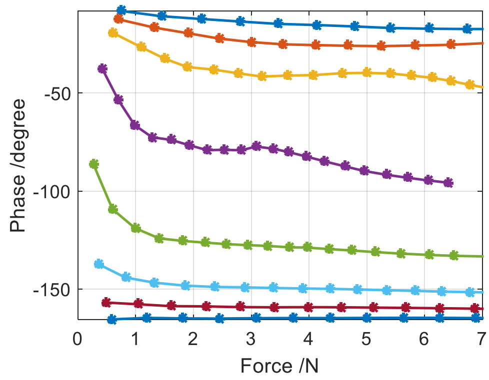


Fig.23. The variation of displacement FRFs phase with the excitation levels at different frequencies

3.5 Reconstructing the constant displacement test

After obtaining the relation between the excitation and the response at each frequency, the excitation and the phase difference between the excitation and the response can be obtained by interpolation at a given displacement level. Then the constant displacement FRFs at the given level can be obtained.

In the constant displacement test, the range of displacements is from 0 to 5×10^{-4} m, and the corresponding excitation forces for each test are shown in Fig.24.

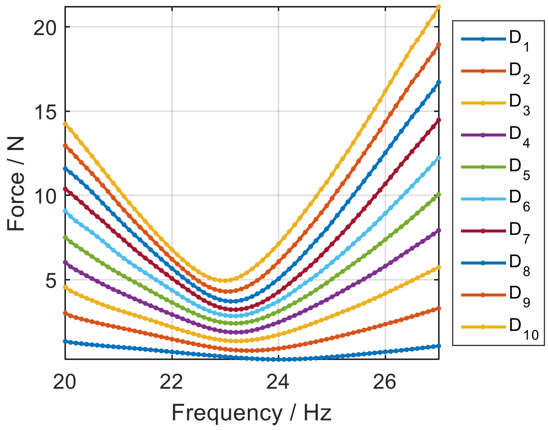


Fig.24. The force for each constant displacement test

The FRFs of the constant displacement tests for the 5 sets of displacement levels are shown in Fig.25.

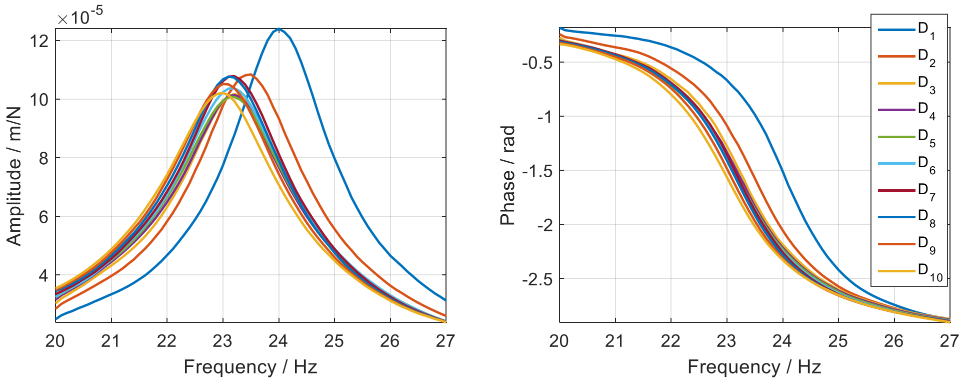
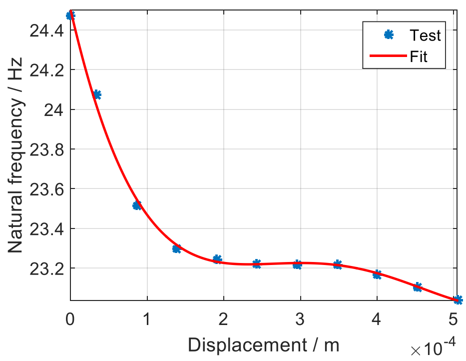


Fig.25. Constant displacement FRFs

These FRFs are effectively linearized at the given displacement level. Hence, linear modal analysis techniques such as the rational fraction method [22], can be used to identify the modal parameters. Then the relation between displacement and natural frequency can be obtained as shown in Fig.26.

The relation between displacement and equivalent stiffness k_{eq} is then established, as shown in Fig.**27**, considering the relation between the equivalent stiffness k_{eq} and the natural frequency given by the underlying linear model.



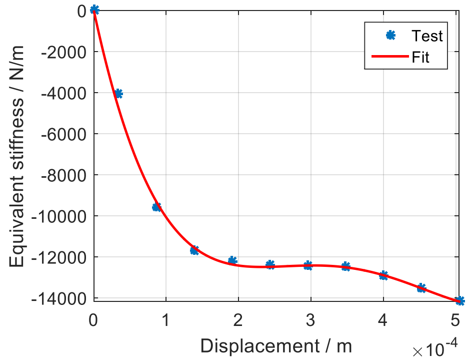


Fig.26. Curve fitting of natural frequencies   Fig.27. Curve fitting of equivalent stiffness

Different functional forms may be tried to fit the curve of equivalent stiffness versus displacement in Fig. 27. A quartic polynomial was found the most suitable function. Thus

$$k_{eq,test} = p_1 D + p_2 D^2 + p_3 D^3 + p_4 D^4 \tag{28}$$

where D is the amplitude of the displacement.

The nonlinear elastic restoring force $f_s(x)$ is then written as

$$f_s(x) = k_1 sign(x) x^2 + k_2 x^3 + k_3 sign(x) x^4 + k_4 x^5 \tag{29}$$

According to the Eq.(20), the nonlinear stiffness parameters identified through constant displacement tests are computed and listed in Table 3.

Table 3 The identified nonlinear stiffness parameters

k_1	k_2	k_3	k_4
-1.9280×10^{8}	1.0290×10^{12}	-2.2566×10^{15}	1.6973×10^{18}

3.6 Reconstructing the constant velocity test

Similar to the nonlinear stiffness identification, the range of velocity is from 0 to 0.07 m/s, and the corresponding velocity and excitation force for each test are shown in Fig.28.

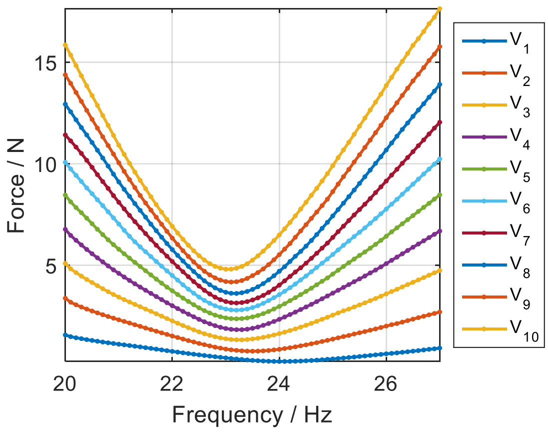


Fig.28 The force for each constant velocity test

The amplitude and phase characteristics of the constant velocity FRFs within the frequency range from 22 to 27Hz are plotted in Fig.29.

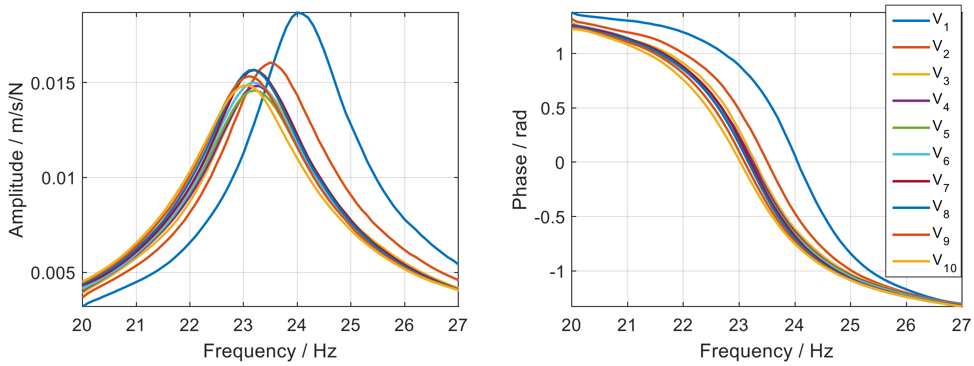


Fig.29. Constant velocity FRFs

These FRFs are effectively linearized at the given velocity level. So the linear modal analysis technique can be used to identify the modal parameters. Then the relation between velocity and damping ratio can be obtained and is shown in Fig.30. The relationship between velocity and equivalent damping c_{eq} is then established, as shown in Fig.31.

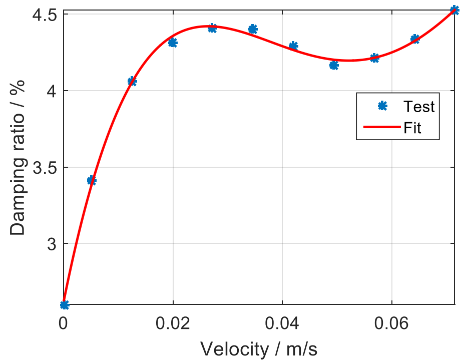


Fig.30. Curve fitting of damping ratio

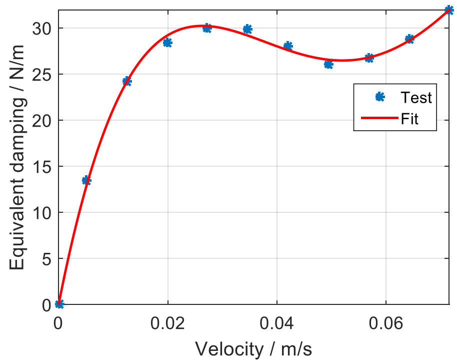


Fig.31. Curve fitting of equivalent damping

The equivalent damping could also be fitted by a quartic polynomial. Thus

$$c_{eq,test} = p_1 V + p_2 V^2 + p_3 V^3 + p_4 V^4 \tag{30}$$

where $V = \omega D$ is the amplitude of the velocity and D is the amplitude of the displacement.

The nonlinear damping force f_d is then written as

$$f_d = c_1 sign(\dot{x})\dot{x}^2 + c_2\dot{x}^3 + c_3 sign(\dot{x})\dot{x}^4 + c_4\dot{x}^5$$

According to Eq. (20), the linear and nonlinear damping parameters identified through the constant velocity tests are computed and listed in Table 4 .

Table 4 The identified nonlinear stiffness parameters

c_1	c_2	c_3	c_4
3.5383×10^3	-1.3491×10^5	1.9320×10^6	-8.8839×10^6

3.7 Verification of the identified nonlinear model

The complete nonlinear model is constructed using the underlying linear model and the identified nonlinear stiffness and damping parameters. Comparisons between the prediction and the measurement under the same low and high excitation levels were taken to verify the identified nonlinear model, especially for the linear and nonlinear behaviors. The Harmonic Balance Method (HBM) is used to predict the fundamental response. The overlay of the predicted and measured FRFs at a low level force of 0.2 N and a high level force of 2.2 N are plotted in Fig.32. It can be seen that the resonance frequency shifts downwards and the amplitude decreases with the increase of the excitation level. The results are consistent with only a slight discrepancy near the resonance for the nonlinear case. The constructed nonlinear model is able to predict the linear and nonlinear behaviors of the structure under different excitation levels.

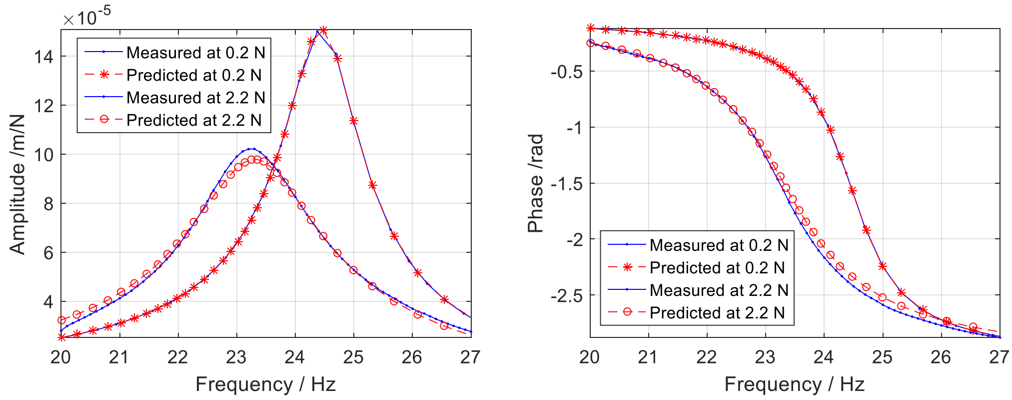


Fig.32. The predicted and measured constant voltage tests

4 Conclusion and discussion

The relationship of equivalent stiffness and displacement of the MDOF system with weak nonlinearity can be established by reconstructing constant displacement tests with the proposed method. A similar approach gives the relationship of equivalent damping and velocity by reconstructing constant velocity tests. These relationships can be curve fitted to achieve nonlinear characterization and can be used to construct the functional forms of the nonlinearities of the system. The equivalent linearization theory used in this approach is concise in mathematical form. The method effectively linearizes the constant displacement FRFs and constant velocity FRFs at the given response level so that the standard linear modal analysis techniques may be used to extract the modal parameters. This is a convenient approach for application in industry, and builds on existing knowledge of modal testing. The stepped sine test data can fully characterize the nonlinearities in the structure and can identify the nonlinear parameters effectively, although it is time-consuming.

The theoretical analysis of the equivalent linearization and the harmonic balance method in this approach are used to establish the relationships of both equivalent stiffness vs displacement and equivalent damping vs velocity. Establishing the relationship using analytical formulas for MDOF nonlinear systems is still a challenge. The identified nonlinear parameters of the MDOF nonlinear system may need further updating. In addition, the case of a nonlinear system with strong nonlinearities and small damping, especially a nonlinear system with jump phenomena, is not considered here because of the hugely different responses near and far from the resonance and it is rather difficult to ensure that the response amplitude at each frequency can be kept constant.

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Conflict of Interest Statement

The authors declared that they have no conflicts of interest to this work.

We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted

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